Fuzzy Logic

Lecture 5

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Recap

- Fuzzy Set: Basic Concepts Example
- α-Cut and Strong α-cut Concept
- The Support and Core of a Fuzzy Set
- Height of Fuzzy Set A
- Important Property of Fuzzy Set –Convexity
- Scalar Cardinality
- Extension Principle for Fuzzy Sets

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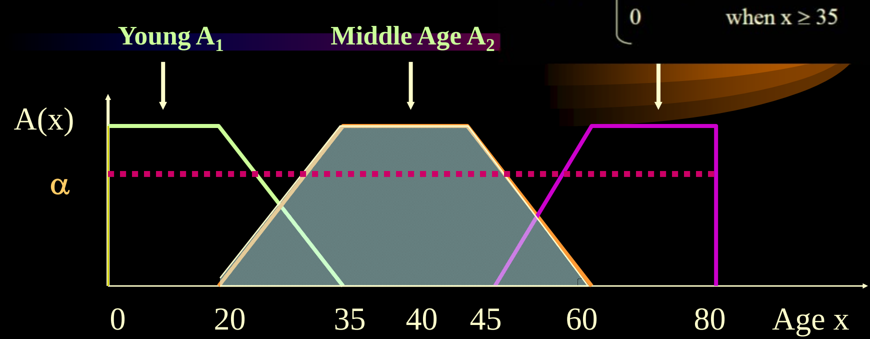
Membership Function A_1

$$A_1(x) = \begin{cases} 1 & \text{when } x \le 20 \\ (35-x)/15 & \text{when } 20 < x < 35 \\ 0 & \text{when } x \ge 35 \end{cases}$$



${}^0A_1 = {}^0A_2 = {}^0A_3 = [0,80] = X$

${}^\alpha A_1 = [0, 35-15\alpha]$, ${}^\alpha A_2 = [15\alpha+20, 60-15\alpha]$, ${}^\alpha A_3 = [15\alpha+45, 80]$ for all $\alpha \in (0,1]$

${}^{\alpha+}A_1 = (0, 35-15\alpha)$, ${}^{\alpha+}A_2 = (15\alpha+20, 60-15\alpha)$, ${}^{\alpha+}A_3 = (15\alpha+45, 80)$ for all $\alpha \in [0,1)$

${}^{1+}A_1 = {}^{1+}A_2 = {}^{1+}A_3 = \varnothing$

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Inferencing with Rules

- *Backward Chaining:* [Goal Driven]
  - Start with a goal. Look for a rule that has that goal as its conclusion. Check the premise of this first rule. Check the assertions (facts) first. If that fails, find another rule whose conclusion is the same as the premise of the first rule. Try to satisfy this second rule. Continue until the first rule is satisfied, or all possibilities have been checked.

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Inferencing with Rules

- ***Forward Chaining:*** [Data Driven]
  - Start from the assertions (facts) and try to infer conclusions. If a new conclusion can be inferred, then assert it. Using the now expanded assertion base, try to infer some more conclusions. Continue until all the possible conclusions have been inferred.

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Backward Chaining Example

- Consider the following question:
  - Should an investor invest in IBM stock?
- ***The Facts:***
  - she is 25 years old
  - she has $10,000 to invest

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Backward Chaining Example (Contd.)

- The following variables are involved:
  - A. Have $10,000
  - B. Younger than 30
  - C. Education at college level
  - D. Annual income > $40,000
  - E. Invest in securities
  - F. Invest in growth stocks
  - G. Invest in IBM stock

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Backward Chaining Example (Contd.)

- ***The Rules:***
  - R1: IF she has $10,000
    and has a college degree,
    THEN she should invest in securities
  - R2: IF her annual income is at least $40,000
    and she has a college degree,
    THEN she should invest in growth stocks
  - R3: IF she is younger than 30
    and invests in securities,
    THEN she should invest in growth stocks
  - R4: IF she is younger than 30,
    THEN she has a college degree
  - R5: IF she wants to invest in a growth stock,
    THEN the stock should be IBM

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Backward Chaining Example (Contd.)

The Starting Point:

- Goal: Should she invest in IBM stock?
- only rule R5 satisfies the goal as its conclusion

R1: IF she has $10,000
and has a college degree,
THEN she should invest in securities

R2: IF her annual income is at least $40,000
and she has a college degree,
THEN she should invest in growth stocks

R3: IF she is younger than 30
and invests in securities,
THEN she should invest in growth stocks

R4: IF she is younger than 30,
THEN she has a college degree

R5: IF she wants to invest in a growth stock,
THEN the stock should be IBM

given facts
- **she is 25 years old**
- **she has $10,000 to invest**

asserted facts

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Backward Chaining Example (Contd.)

Step 1:

- try to *accept* or *reject* our Goal
  - establish if "she wants to invest in a growth stock" is true
- if we check our assertion base (ie. the given and asserted facts), we see that:
  - "she is 25 years old" and "she has $10,000 to invest" are both true
  - rule R5 cannot be fired (invoked) yet

R1: IF she has $10,000
and has a college degree,
THEN she should invest in securities

R2: IF her annual income is at least $40,000
and she has a college degree,
THEN she should invest in growth stocks

R3: IF she is younger than 30
and invests in securities,
THEN she should invest in growth stocks

R4: IF she is younger than 30,
THEN she has a college degree

R5: IF she wants to invest in a growth stock,
THEN the stock should be IBM

given facts
- **she is 25 years old**
- **she has $10,000 to invest**

asserted facts

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Backward Chaining Example (Contd.)

Step 2:

- to establish if "she wants to invest in a growth stock" is true:
- note that "she wants to invest in a growth stock" is the conclusion of both rule R2 and rule R3
- suppose we look at the premises of rule R2 (arbitrarily) first

R1: IF she has $10,000
and has a college degree,
THEN she should invest in securities

R2: IF her annual income is at least $40,000
and she has a college degree,
THEN she should invest in growth stocks

R3: IF she is younger than 30
and invests in securities,
THEN she should invest in growth stocks

R4: IF she is younger than 30,
THEN she has a college degree

R5: IF she wants to invest in a growth stock,
THEN the stock should be IBM

given facts
- **she is 25 years old**
- **she has $10,000 to invest**

asserted facts

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Backward Chaining Example (Contd.)

Step 3:

- to fire rule R2 requires both premises be true
- but "her annual income is at least $40,000" is neither a given fact nor is it the conclusion of any rule
- either more information is required (ie a fact about "her annual income is at least $40,000" must be established)
- or we can return to rule R3
- [this backing away from one rule and looking at an alternative is called backtracking]

R1: IF she has $10,000
and has a college degree,
THEN she should invest in securities

R2: IF her annual income is at least $40,000
and she has a college degree,
THEN she should invest in growth stocks

R3: IF she is younger than 30
and invests in securities,
THEN she should invest in growth stocks

R4: IF she is younger than 30,
THEN she has a college degree

R5: IF she wants to invest in a growth stock,
THEN the stock should be IBM

given facts

- she is 25 years old
- she has $10,000 to invest

asserted facts

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Backward Chaining Example (Contd.)

Step 4:

- to fire rule R3 requires both "she is younger than 30" and "invests in securities" be true
- we know "she is younger than 30" is true - it is a given fact ("she is 25 years old")
- to establish if "invests in securities" is true, we note that it is the conclusion for rule R1

R1: IF she has $10,000
and has a college degree,
THEN she should invest in securities

R2: IF her annual income is at least $40,000
and she has a college degree,
THEN she should invest in growth stocks

R3: IF she is younger than 30
and invests in securities,
THEN she should invest in growth stocks

R4: IF she is younger than 30,
THEN she has a college degree

R5: IF she wants to invest in a growth stock,
THEN the stock should be IBM

given facts

- she is 25 years old
- she has $10,000 to invest

asserted facts

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Backward Chaining Example (Contd.)

Step 5:

- to fire rule R1, we require "she has $10,000" and "has a college degree" to be true
- "she has $10,000" is true - it is a given fact
- to test "has a college degree" , we note that it is the conclusion of rule R4

R1: IF she has $10,000
and has a college degree,
THEN she should invest in securities

R2: IF her annual income is at least $40,000
and she has a college degree,
THEN she should invest in growth stocks

R3: IF she is younger than 30
and invests in securities,
THEN she should invest in growth stocks

R4: IF she is younger than 30,
THEN she has a college degree

R5: IF she wants to invest in a growth stock,
THEN the stock should be IBM

given facts

- **she is 25 years old**
- **she has $10,000 to invest**

asserted facts

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Backward Chaining Example (Contd.)

Step 6:

- to fire rule R4, we require "she is younger than 30" to be true
- "she is younger than 30" is true - it is a given fact ("she is 25 years old")
- therefore rule R4 fires and "has a college degree" becomes a fact and is added to the assertion base

R1: IF she has $10,000
and has a college degree,
THEN she should invest in securities

R2: IF her annual income is at least $40,000
and she has a college degree,
THEN she should invest in growth stocks

R3: IF she is younger than 30
and invests in securities,
THEN she should invest in growth stocks

R4: IF she is younger than 30,
THEN she has a college degree

R5: IF she wants to invest in a growth stock,
THEN the stock should be IBM

given facts

- **she is 25 years old**
- **she has $10,000 to invest**

asserted facts

- **she has a college degree**

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Backward Chaining Example (Contd.)

Step 7:

- moving forward again along the rule chain:
- we can use our newly asserted fact to fire rule R1 to establish "she should invest in securities" as a fact

R1: IF she has $10,000
and has a college degree,
THEN she should invest in securities

R2: IF her annual income is at least $40,000
and she has a college degree,
THEN she should invest in growth stocks

R3: IF she is younger than 30
and invests in securities,
THEN she should invest in growth stocks

R4: IF she is younger than 30,
THEN she has a college degree

R5: IF she wants to invest in a growth stock,
THEN the stock should be IBM

given facts
- **she is 25 years old**
- **she has $10,000 to invest**

asserted facts
- **she has a college degree**
- **she should invest in securities**

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Backward Chaining Example (Contd.)

Step 8:

- moving forward again along the rule chain:
- we can now fire rule R3 to assert "she should invest in growth stocks" as a fact

R1: IF she has $10,000
and has a college degree,
THEN she should invest in securities

R2: IF her annual income is at least $40,000
and she has a college degree,
THEN she should invest in growth stocks

R3: IF she is younger than 30
and invests in securities,
THEN she should invest in growth stocks

R4: IF she is younger than 30,
THEN she has a college degree

R5: IF she wants to invest in a growth stock,
THEN the stock should be IBM

given facts
- **she is 25 years old**
- **she has $10,000 to invest**

asserted facts
- **she has a college degree**
- **she should invest in securities**
- **she should invest in growth stocks**

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Backward Chaining Example (Contd.)

Step 9:

- moving forward again along the rule chain:
- we can now fire rule R5 to assert "she should invest in IBM stock" as a fact

R1: IF she has $10,000
and has a college degree,
THEN she should invest in securities

R2: IF her annual income is at least $40,000
and she has a college degree,
THEN she should invest in growth stocks

R3: IF she is younger than 30
and invests in securities,
THEN she should invest in growth stocks

R4: IF she is younger than 30,
THEN she has a college degree

R5: IF she wants to invest in a growth stock,
THEN the stock should be IBM

given facts
- **she is 25 years old**
- **she has $10,000 to invest**

asserted facts
- **she has a college degree**
- **she should invest in securities**
- **she should invest in growth stocks**
- **the stock should be IBM**

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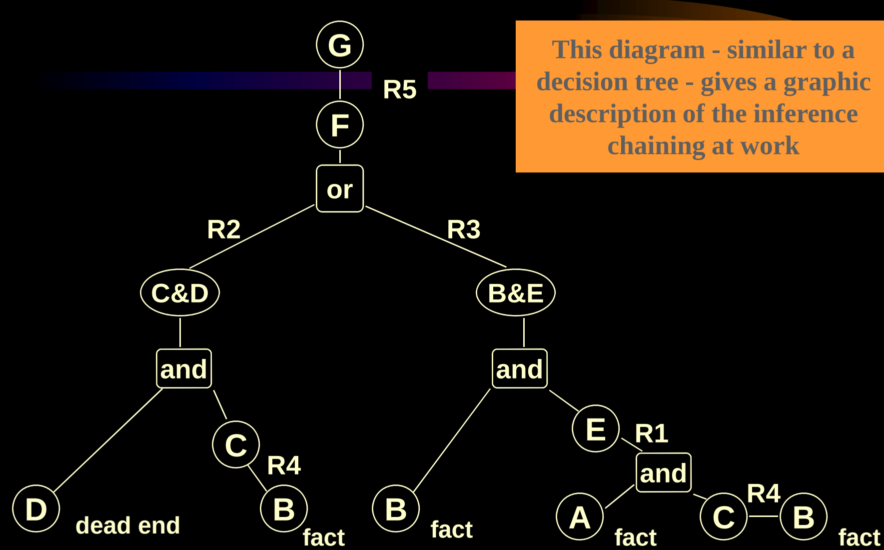
Inference Tree

This diagram - similar to a decision tree - gives a graphic description of the inference chaining at work



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Forward Chaining Example

- The Facts:
  - she is 25 years old
  - she has $10,000 to invest
- The Rules:
  - R1: IF she has $10,000
    and has a college degree,
    THEN she should invest in securities
  - R2: IF her annual income is at least $40,000
    and she has a college degree,
    THEN she should invest in growth stocks
  - R3: IF she is younger than 30
    and invests in securities,
    THEN she should invest in growth stocks
  - R4: IF she is younger than 30,
    THEN she has a college degree
  - R5: IF she wants to invest in a growth stock,
    THEN the stock should be IBM
- **What conclusions can we draw?**

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Forward Chaining Example (Contd.)

The Starting Point:

- Known: “she has $10,000” and “she is younger than 30” are true
- rule R1 (arbitrarily) contains “she has $10,000” in its IF side
- rule R1 includes “she should invest in securities” in its conclusion (THEN side)

R1: IF she has $10,000
and has a college degree,
THEN she should invest in securities

R2: IF her annual income is at least $40,000
and she has a college degree,
THEN she should invest in growth stocks

R3: IF she is younger than 30
and invests in securities,
THEN she should invest in growth stocks

R4: IF she is younger than 30,
THEN she has a college degree

R5: IF she wants to invest in a growth stock,
THEN the stock should be IBM

given facts

- **she is 25 years old**
- **she has $10,000 to invest**

asserted facts

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Forward Chaining Example (Contd.)

Step 1:

- attempt to verify "she should invest in securities"
- since "she has $10,000" is known (fact)
- try to find "she has a college degree" in the assertion base - it's not there
- because "she has a college degree" is not asserted, find a rule with "she has a college degree" in the THEN side
- this is rule R4

R1: IF she has $10,000
and has a college degree,
THEN she should invest in securities

R2: IF her annual income is at least $40,000
and she has a college degree,
THEN she should invest in growth stocks

R3: IF she is younger than 30
and invests in securities,
THEN she should invest in growth stocks

R4: IF she is younger than 30,
THEN she has a college degree

R5: IF she wants to invest in a growth stock,
THEN the stock should be IBM

given facts

- **she is 25 years old**
- **she has $10,000 to invest**

asserted facts

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Forward Chaining Example (Contd.)

Step 2:

- test rule R4
- "she has a college degree" is true because "she is younger than 30" (fact) is in the assertion base
- assert "she has a college degree" as true

R1: IF she has $10,000
and has a college degree,
THEN she should invest in securities

R2: IF her annual income is at least $40,000
and she has a college degree,
THEN she should invest in growth stocks

R3: IF she is younger than 30
and invests in securities,
THEN she should invest in growth stocks

R4: IF she is younger than 30,
THEN she has a college degree

R5: IF she wants to invest in a growth stock,
THEN the stock should be IBM

given facts

- **she is 25 years old**
- **she has $10,000 to invest**

asserted facts

- **she has a college degree**

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Forward Chaining Example (Contd.)

Step 3:

- rule R1 now fires and “she should invest in securities” is asserted
- this leads to rule R3 where “she should invest in securities” is in the IF side

R1: IF she has $10,000
and has a college degree,
THEN she should invest in securities

R2: IF her annual income is at least $40,000
and she has a college degree,
THEN she should invest in growth stocks

R3: IF she is younger than 30
and invests in securities,
THEN she should invest in growth stocks

R4: IF she is younger than 30,
THEN she has a college degree

R5: IF she wants to invest in a growth stock,
THEN the stock should be IBM

given facts
- **she is 25 years old**
- **she has $10,000 to invest**

asserted facts
- **she has a college degree**
- **she should invest in securities**

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Forward Chaining Example (Contd.)

Step 4:

- since “she is younger than 30” (fact) is in the assertion base
- and “she should invest in securities” (asserted) is in the assertion base
- rule R3 fires and “she should invest in growth stocks” is asserted

R1: IF she has $10,000
and has a college degree,
THEN she should invest in securities

R2: IF her annual income is at least $40,000
and she has a college degree,
THEN she should invest in growth stocks

R3: IF she is younger than 30
and invests in securities,
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R5: IF she wants to invest in a growth stock,
THEN the stock should be IBM

given facts
- **she is 25 years old**
- **she has $10,000 to invest**

asserted facts
- **she has a college degree**
- **she should invest in securities**
- **she should invest in growth stocks**

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Fuzzy Arithmetic

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Fuzzy Numbers

- Various types of fuzzy sets are defined on the set R of real numbers. Membership functions of these sets are given by

$$A : R \rightarrow [0,1]$$

- To qualify as a fuzzy number, a fuzzy set A on R must have at least the following three properties.

(i) A must be normal fuzzy set

(ii) ${}^{\alpha}A$ must be closed interval for every $\alpha \in (0,1]$

(iii) The support of A, ${}^{0+}A$ must be bounded.

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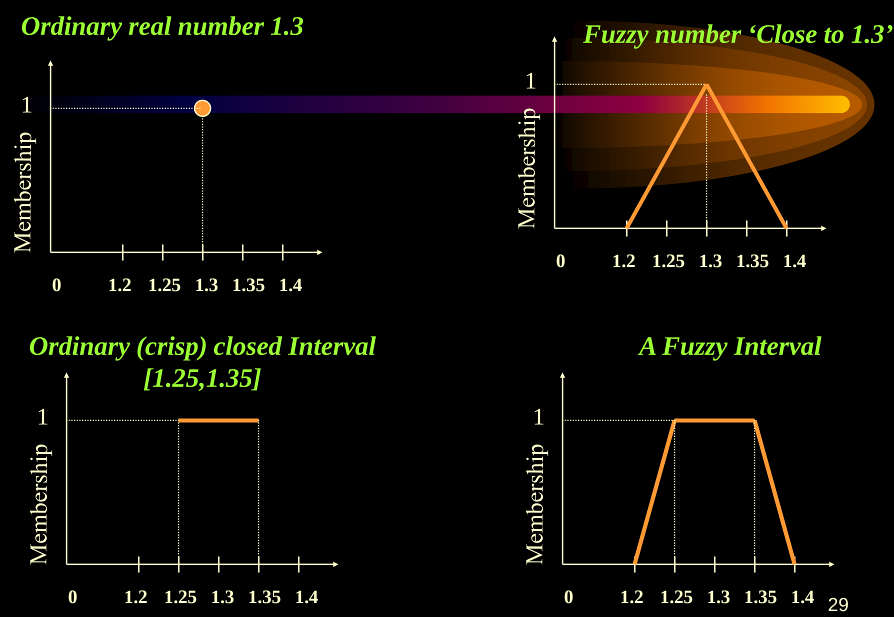
Comparison



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Membership functions of fuzzy number need not be symmetric

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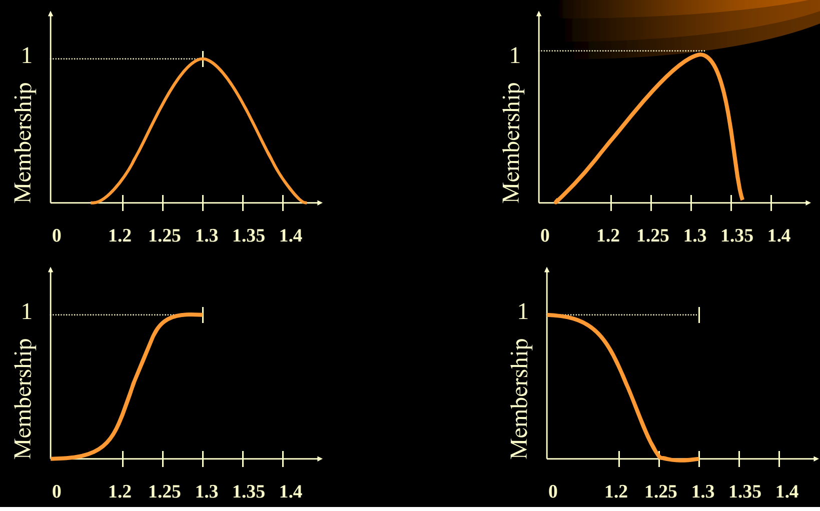
Basic types of fuzzy numbers



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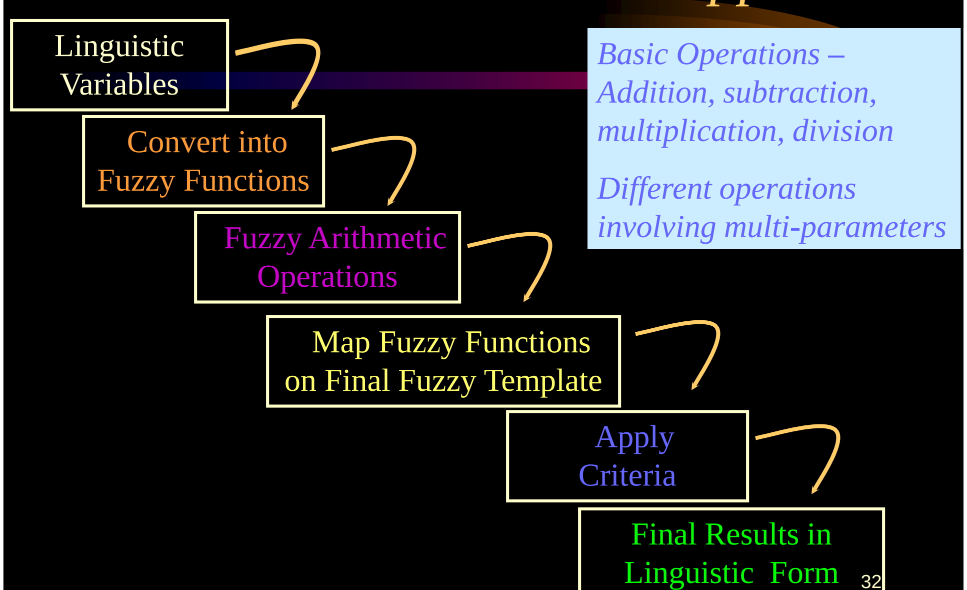
General Approach



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Identification of Linguistic Variable

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Linguistic Variables

- Expressed by linguistic terms interpreted as specific fuzzy numbers – known as ***base variable***
- ***Base Variables –***

  Physical variables – temperature, pressure, speed, voltage, humidity, etc.

  Numerical variables – age, interest rate, performance, salary, blood count, probability, reliability etc.
- In linguistic variable, linguistic terms representing approximate values of a a base variable for a particular application, are represented by appropriate fuzzy numbers

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Linguistic Variables

- Each linguistic variable is fully characterized by a quintuple ***(v, T, X, g, m)***
- ***v*** name of the variable
- ***T*** set of linguistic terms of ***v*** that refer to base variable whose values range over a universal set ***X***
- ***g*** is syntactic rule (a grammar) for generating linguistic terms
- ***m*** is a semantic rule that assigns to each linguistic term.

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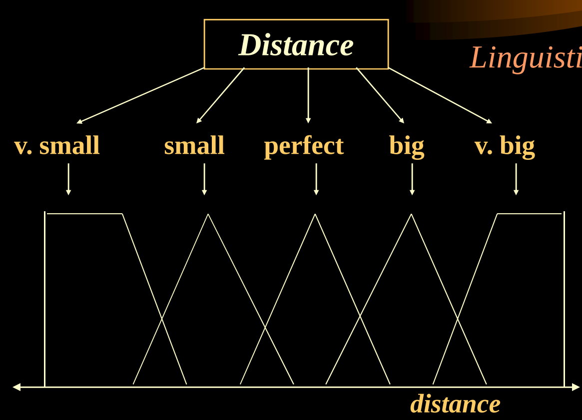
Example of Linguistic Variable

Linguistic variable

Linguistic values (states)



Semantic Rule

Fuzzy restrictions

Base variable

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General Approach

1. Linguistic Variables
2. Convert into Fuzzy Functions
3. Fuzzy Arithmetic Operations
4. Map Fuzzy Functions on Final Fuzzy Template
5. Apply Criteria
6. Final Results in Linguistic Form

Basic Operations – Addition, subtraction, multiplication, division

Different operations involving multi-parameters

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Arithmetic Operations on Intervals

- Fuzzy Arithmetic is based on two methods

(I) Each fuzzy set, and thus also each fuzzy number, can fully be represented by its α-cuts

(II) α-cuts of each fuzzy number are closed intervals of real number for all $\alpha \in (0,1]$

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Arithmetic Operations on Intervals

- Previously discussed two properties permit – in defining arithmetic operations on fuzzy numbers in terms of arithmetic operations on their α-cuts (i.e. arithmetic operations on closed intervals) → **interval analysis – Well established area of classical mathematics.**

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Arithmetic Operations on Closed Intervals

- Let * denote any of the following operations
  - Addition +
  - Subtraction –
  - Multiplication .
  - Division /
- Then

[a, b] * [d, e] = {f * g | a ≤ f ≤ b, d ≤ g ≤ e}

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Arithmetic Operations on Closed Intervals

[a, b] * [d, e] = {f * g | a ≤ f ≤ b, d ≤ g ≤ e}

Is a general property of all arithmetic operations on closed intervals, except that [a, b]/ [d, e] is not defined when 0 ∈ [d, e]

- The result of an arithmetic operation on closed intervals is again a closed interval.

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Arithmetic Operations on Closed Intervals

- [a, b] + [d, e] = [a + d, b + e]
- [a, b] – [d, e] = [a – e, b – d]
- [a, b] . [d, e] = [min(ad, ae, bd, be), max (ad, ae, bd, be)]
- If 0 ∉ [d, e], then
  [a, b] / [d, e] = [a, b] . [1/e, 1/d] =
  [min(a/d, a/e, b/d, b/e), max (a/d, a/e, b/d, b/e)]

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Samples

- [2,5] + [1,3] = [3,8]
- [0,1] + [-6,5] = [-6,6]
- [2,5] – [1,3] = [-1,4]
- [0,1] – [-6,5] = [-5.7]
- [-1,1].[-2,-0.5] = [-2,2]
- [3,4].[2,2] = [6,8]
- [-1,1]/[-2,-0.5] = [-2,2]
- [4,10]/[1,2] = [2,10]

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Some properties – $A=[a_1, a_2]$, $B=[b_1, b_2]$, $C=[c_1, c_2]$, $\mathbf{0} = [0,0]$, $\mathbf{1} = [1,1]$

- A + B = B + A, A . B = B . A (Commutativity)
- (A +B) + C = A + (B + C)

  (A . B) . C = A . (B. C) (Associativity)
- A = **0** + A = A + **0**

  A = **1** . A = **1** . A (Identity)

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Some properties – $A=[a_1, a_2]$, $B=[b_1, b_2]$, $C=[c_1, c_2]$, $\mathbf{0} = [0,0]$, $\mathbf{1} = [1,1]$

- A . (B + C) $\subseteq$ A . B + A . C (Subdistributivity)
- If b . c $\geq$ 0 for every b $\in$ B and c $\in$ C,

  then A . (B +C) = A . B + A . C (Distributivity)
- Furthermore, if A = [a, a],

  then a . (B + C) = a . B + a . C
- 0 $\in$ A – A and 1 $\in$ A/A

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Some properties – $A=[a_1, a_2]$, $B=[b_1, b_2]$, $C=[c_1, c_2]$, $\mathbf{0} = [0,0]$, $\mathbf{1} = [1,1]$

- If A $\subseteq$ E and B $\subseteq$ F, then

  A + B $\subseteq$ E + F

  A – B $\subseteq$ E – F

  A . B $\subseteq$ E . F

  A / B $\subseteq$ E / F (inclusion monotonicity)

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Arithmetic Operations on Fuzzy Number

- Two methods – (a) Based on interval arithmetic as discussed in previous slides (b) Extension principles – operations on real numbers are extended to operations on fuzzy numbers.
- Assumption – fuzzy numbers are represented by continuous membership function

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Arithmetic Operations on Fuzzy Number (Contd.)

- A and B Fuzzy Number
- * denotes four basic arithmetic operations
- We define a fuzzy set **R**, A*B**,** by defining its α-cut, ${}^{\alpha}(A*B)$ as

  ${}^{\alpha}(A*B) = {}^{\alpha}A * {}^{\alpha}B$

  for any $\alpha \in (0, 1]$

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Demonstrative Example

- Consider Two triangular-shape fuzzy set numbers A and B defined as

$$A(x) = \begin{cases} 0 & \text{for } x \le -1 \text{ and } x > 3 \\ (x+1)/2 & \text{for } -1 < x \le 1 \\ (3-x)/2 & \text{for } 1 < x \le 3 \end{cases}$$

$$B(x) = \begin{cases} 0 & \text{for } x \le 1 \text{ and } x > 5 \\ (x-1)/2 & \text{for } 1 < x \le 3 \\ (5-x)/2 & \text{for } 3 < x \le 5 \end{cases}$$

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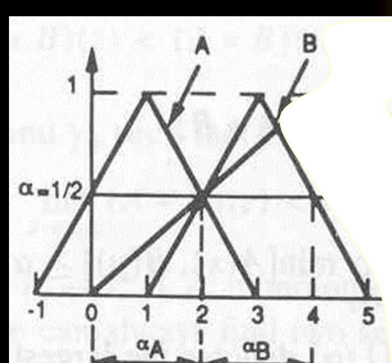
Demonstrative Example



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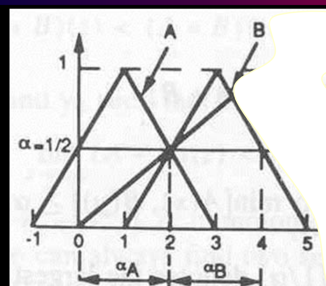
Demonstrative Example



α - cuts are:

$^{\alpha}A = [\,2\alpha - 1,\ 3 - 2\alpha]$

$^{\alpha}B = [2\alpha + 1,\ 5 - 2\alpha]$

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Demonstrative Example

$^{\alpha}(A + B)$ = $[4\alpha,\ 8 - 4\alpha]$ for $\alpha \in (0,1]$

$^{\alpha}(A - B)$ = $[4\alpha - 6,\ 2 - 4\alpha]$ for $\alpha \in (0,1]$

$^{\alpha}(A \,.\, B)$ = $[-4\alpha^2 + 12\alpha - 5,\ 4\alpha^2 - 16\alpha + 15]$ for $\alpha \in (0,0.5]$
$[-4\alpha^2 - 1,\ 4\alpha^2 - 16\alpha + 15]$ for $\alpha \in (0.5, 1]$

$^{\alpha}(A / B)$ = $[(2\alpha - 1)/(2\alpha + 1),\ (3 - 2\alpha)/(2\alpha + 1)]$ for $\alpha \in (0,0.5]$
$[(2\alpha - 1)/(5 - 2\alpha),\ (3 - 2\alpha)/(2\alpha + 1)]$ for $\alpha \in (0.5,1]$

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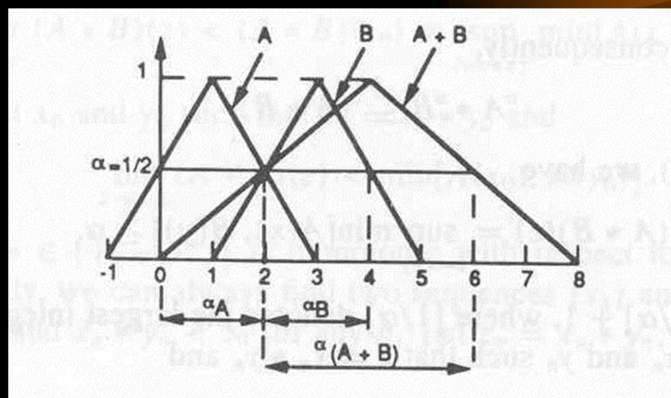
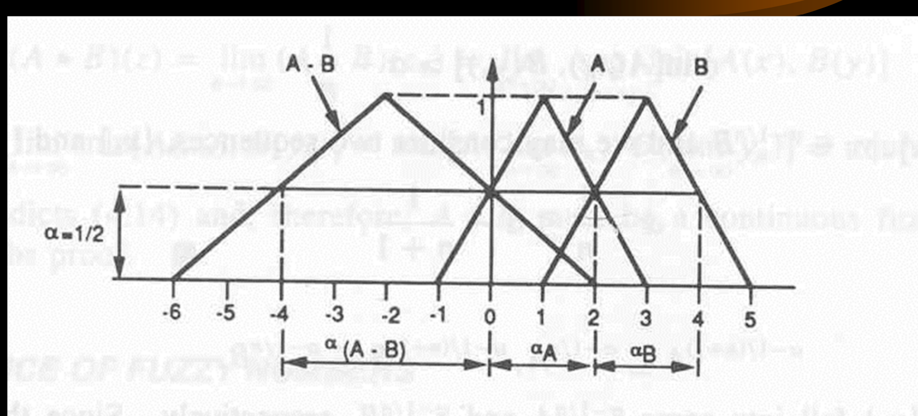
$A + B$


A
B
A + B
1
α = 1/2
-1 0 1 2 3 4 5 6 7 8
αA
αB
α(A + B)

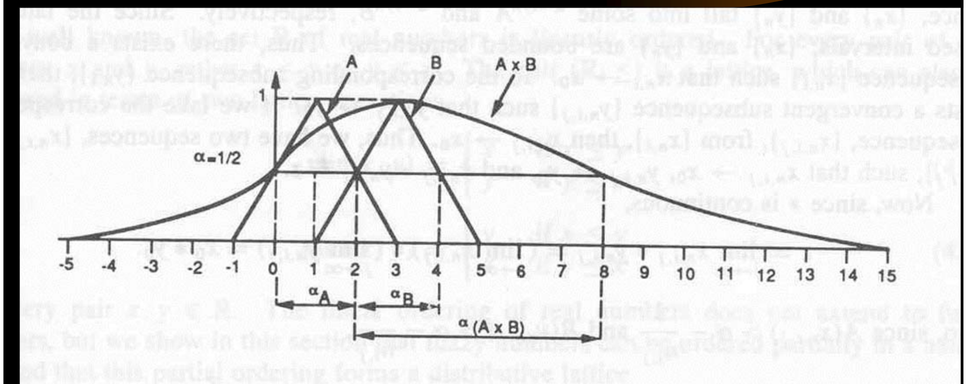
53

$A - B$


A - B
A
B
1
α = 1/2
-6 -5 -4 -3 -2 -1 0 1 2 3 4 5
α(A - B)
αA
αB

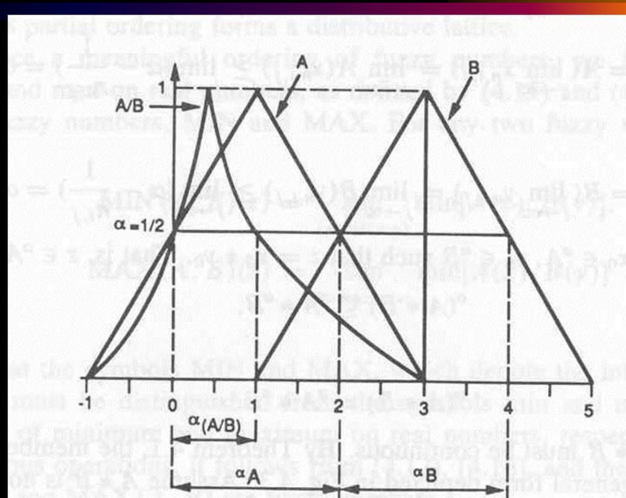
54

A . B


A
B
A x B
1
α=1/2
-5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15
α_A
α_B
α (A x B)

55

A/B


A
B
A/B
1
α=1/2
-1 0 1 2 3 4 5
α(A/B)
αA
αB

56

Fuzzy Equation

A + X = B

and

A. X = B

Home-work

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