Probabilistic Reasoning

course notes 2018

UU – ICS Master Programmes:
Computing Science
Artificial Intelligence

Universiteit Utrecht

Probabilistic reasoning with Bayesian networks

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Prerequisites:	probability theory & graph theory
Literature:	syllabus & slides & studymanual
Form:	lectures & exercises (formative self assessment) (tip: discuss exercises on Blackboard forum)
Grading:	practical assignments & written exam
Additional info:	see course website: `http://www.cs.uu.nl/docs/vakken/prob/`

Chapter 1:

Introduction

Reasoning under uncertainty

In numerous application areas of knowledge-based decision-support systems we have

- uncertainty concerning the general domain knowledge;
- problem-specific information that is often uncertain, incomplete and even contradictory.

A decision-support system should be capable of dealing with these types of knowledge.

Application of probability theory

Consider a discrete joint probability distribution $\Pr$ on a set of random variables $\boldsymbol{V} = \{V_1, \ldots, V_n\}$. In general we have that:

- the representation of $\Pr$ requires exponential space

  consider e.g. $n = 2$ binary-valued variables, or $n = 40$; what if they have 5 values each? (and how do you get the numbers?)

- calculating the (conditional) probability of a value of a variable by conditioning and marginalisation requires exponential time

  consider e.g. computing $\Pr(V_1 = \text{true})$ from $\Pr(\boldsymbol{V})$, or $\Pr(V_1 = \text{true} \mid V_2 = \text{true})$

This cannot be improved without additional knowledge about the probability distribution.

Diagnosis problem: pioneering in the 1960s

Let $\boldsymbol{H} = \{h_1, \ldots, h_n\}, n \geq 1$, be a set of hypotheses, and let $\boldsymbol{E} = \{e_1, \ldots, e_m\}, m \geq 1$, be a set of relevant findings (evidence).

Determine the 'best' diagnosis given findings $\boldsymbol{e} \subseteq \boldsymbol{E}$.

The approach: Compute for each $\boldsymbol{h} \subseteq \boldsymbol{H}$ the probability

$$\Pr(\boldsymbol{h} \mid \boldsymbol{e}) = \frac{\Pr(\boldsymbol{e} \mid \boldsymbol{h}) \Pr(\boldsymbol{h})}{\Pr(\boldsymbol{e})}$$

Drawback: An exponential number of probabilities need to be computed; storage is also exponential.

Pioneering in the 1960s

Determine the diagnosis given findings $\boldsymbol{e} \subseteq \boldsymbol{E}$.

The approach: Assume $h_i \in \boldsymbol{H}$ mutually exclusive, and collectively exhaustive: $\cup_{i=1}^{n}\{h_i\} = \Omega$.

Then, compute for each $h_i \in \boldsymbol{H}$:

$$\Pr(h_i \mid \boldsymbol{e}) = \frac{\Pr(\boldsymbol{e} \mid h_i)\Pr(h_i)}{\Pr(\boldsymbol{e})} = \frac{\Pr(\boldsymbol{e} \mid h_i)\Pr(h_i)}{\sum_{k=1}^{n}\Pr(\boldsymbol{e} \mid h_k)\Pr(h_k)}$$

Drawback: We compute only $n-1$ probabilities, but computation still requires an exponential number of probabilities.

Pioneering in the 1960s

Determine the diagnosis given findings $\boldsymbol{e} = \{e_p, \ldots, e_q\}$, $1 \leq p, q \leq m$.

<u>**The approach**</u>: Assume *in addition* that all findings $e_1, \ldots, e_m$ are conditionally independent given h_i, $i = 1, \ldots, n$. Then:

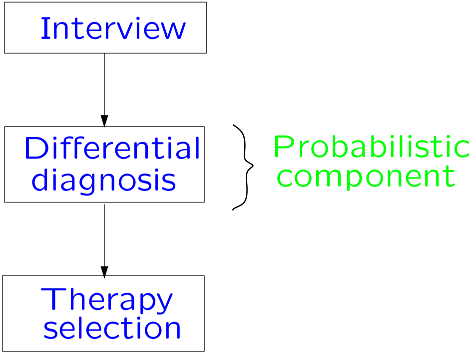
$$\begin{aligned}
\Pr(h_i \mid \boldsymbol{e}) &= \frac{\Pr(e_p, \ldots, e_q \mid h_i) \Pr(h_i)}{\sum_{k=1}^{n} \Pr(e_p, \ldots, e_q \mid h_k) \Pr(h_k)} \\
&= \frac{\Pr(e_p \mid h_i) \cdot \ldots \cdot \Pr(e_q \mid h_i) \Pr(h_i)}{\sum_{k=1}^{n} \Pr(e_p \mid h_k) \cdot \ldots \cdot \Pr(e_q \mid h_k) \Pr(h_k)}
\end{aligned}$$

<u>**Benefit**</u>: Only $m \cdot n$ conditional probabilities and $n - 1$ prior probabilities are required for the computation.

GLADYS

GLADYS (GLASGOW DYSPEPSIA SYSTEM) is a system for diagnosing dyspepsia.

The global structure of the system:



developed with data collected from $\pm$ 1200 patients.

D.J. Spiegelhalter, R.P. Knill-Jones (1984). *Statistical and knowledge-based approaches to clinical decision-support systems with an application in gastroenterology,* Journal of the Royal Statistical Society (Series A), vol. 147, pp. 35-77.

Symptoms and diseases

Context: patients with an Ulcer. Question: which type?

		duodenal ulcer ($n = 248$)	gastric ulcer ($n = 43$)
Sex:	male	169	17
	female	79	26
Age:	< 26	43	1
	26 - 40	82	5
	41 - 55	87	19
	> 55	36	18
Daily pain:	yes	21	11
	no	214	27
Effect food on pain:	worsens	44	11
	no effect	82	9
	relieves	104	17
probability		0.85	0.15

The idea

Let $\Pr$ be a joint distribution on the diagnosis search space including hypothesis h and observed findings e.

The prior odds for h, and posterior odds for h given $\boldsymbol{e}$, are defined by

$$O(h) = \frac{\Pr(h)}{1 - \Pr(h)} = \frac{\Pr(h)}{\Pr(\neg h)}, \text{ and } \quad O(h \mid \boldsymbol{e}) = \frac{\Pr(h \mid \boldsymbol{e})}{\Pr(\neg h \mid \boldsymbol{e})}$$

Assume that all findings $e_i \in \boldsymbol{e}$ are conditionally independent given h, then

$$O(h \mid \boldsymbol{e}) = \frac{\Pr(\boldsymbol{e} \mid h) \cdot \Pr(h)}{\Pr(\boldsymbol{e} \mid \neg h) \cdot \Pr(\neg h)} = \prod_i \frac{\Pr(e_i \mid h)}{\Pr(e_i \mid \neg h)} \cdot O(h)$$

Now consider the following transformation: $10 \cdot \ln O(h \mid \boldsymbol{e}) \ldots$

The idea (cntd)

Applying the transformation $10 \cdot \ln$ to

$$O(h \mid \boldsymbol{e}) = \prod_i \lambda_i \cdot O(h), \quad \text{where} \quad \lambda_i = \frac{\Pr(e_i \mid h)}{\Pr(e_i \mid \neg h)}$$

results in a score s:

$$s \;=\; 10 \cdot \ln O(h \mid e) \;=\; 10 \cdot \ln O(h) + \sum_i 10 \cdot \ln \lambda_i \;=\; w_0 + \sum_i w_i$$

where w_i is a weight for finding e_i.

The probability $\Pr(h \mid \boldsymbol{e})$ is now computed from

$$\Pr(h \mid \boldsymbol{e}) = \frac{O(h \mid \boldsymbol{e})}{1 + O(h \mid \boldsymbol{e})} = \frac{e^{\frac{s}{10}}}{1 + e^{\frac{s}{10}}} \;=\; \frac{1}{1 + e^{-\frac{s}{10}}}$$

A scoring system

	h: duodenal ulcer (du) $(n = 248)$	$\neg h$: gastric ulcer (gu) $(n = 43)$
male (m)	169	17
female (f)	79	26

Calculation of probabilities, likelihood ratios and weights:

$$\Pr(\mathsf{m} \mid \mathsf{du}) = \frac{169}{248} \sim 0.68,\ \Pr(\mathsf{m} \mid \mathsf{gu}) \sim 0.40 \ \Rightarrow$$

$$\lambda_{\mathsf{m}} = \frac{\Pr(\mathsf{m} \mid \mathsf{du})}{\Pr(\mathsf{m} \mid \mathsf{gu})} = \frac{0.68}{0.40} \sim 1.7 \implies \mathsf{w}_{\mathsf{m}} = 10 \cdot \ln \lambda_{\mathsf{m}} \sim 5$$

$$\Pr(\mathsf{f} \mid \mathsf{du}) = \frac{79}{248} \sim 0.32,\ \Pr(\mathsf{f} \mid \mathsf{gu}) \sim 0.60 \ \Rightarrow$$

$$\lambda_{\mathsf{f}} = \frac{\Pr(\mathsf{f} \mid \mathsf{du})}{\Pr(\mathsf{f} \mid \mathsf{gu})} = \frac{0.32}{0.60} \sim 0.53 \implies \mathsf{w}_{\mathsf{f}} = 10 \cdot \ln \lambda_{\mathsf{f}} \sim -6$$

Symptoms and their weights

		duodenal ulcer ($n = 248$)	gastric ulcer ($n = 43$)	weight
Sex:	male	169	17	5
	female	79	26	–6
Age:	< 26	43	1	18
	26 - 40	82	5	10
	41 - 55	87	19	–2
	> 55	36	18	–10
Daily pain:	yes	21	11	–12
	no	214	27	3
Effect food on pain:	worsens	44	11	–4
	no effect	82	9	4
	relieves	104	17	0
prior		0.85	0.15	17

An example diagnosis

A 30 year old woman reports to the clinic. She has pain in the abdominal area, but not on a daily basis; the pain worsens as soon as she eats.

Calculation of the score:

• the initial score:	$+17$
• the patient is female:	$-\ 6$
• her age is 30:	$+10$
• she is in pain, but not every day:	$+\ 3$
• food intake worsens the pain:	$-\ 4$
	$+20$

Given that the patient has one of the two diseases, duodenal ulcer and gastric ulcer, she has with probability

$$(1 + e^{-\frac{20}{10}})^{-1} \approx 1.14^{-1} \approx 0.88$$

a duodenal ulcer and a gastric ulcer with probability 0.12.

Reviewing ‘Idiot’s Bayes’

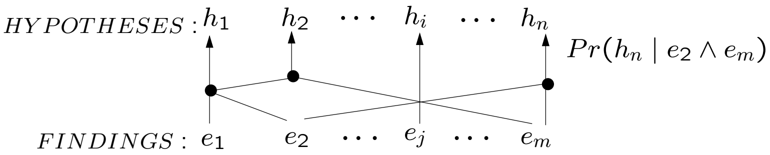
The naive Bayes approach is

- mathematically correct, and
- computationally easy.

However

- underlying assumptions usually unacceptable;
- and, *at the time*, for larger applications
  - # of hypotheses often large $\rightarrow$ undoable to compute each $\Pr(h_i \mid \boldsymbol{e})$;
  - often not enough information for reliable probability assessments.

History: diagnosis in the 1970s



The most likely hypothesis given observed findings is determined as follows:

- prune the search space using heuristic rules;
- approximate the missing probabilities required, for example with:

$$\Pr(e_i \wedge e_j) = \min\{\Pr(e_i), \Pr(e_j)\};$$

- select the hypothesis with the highest probability.

Reviewing the quasi-probabilistic models

The quasi-probabilistic models are

- computationally easy, and
- easy to use,

even for larger applications.

However, these models are

- mathematically incorrect, and
- even as an approximation model not convincing.

The rehabilitation of probability theory in the 1980s

Judea Pearl introduces Bayesian belief networks as representational device

- + algorithms for inferring (computing) ’beliefs’ from those represented
- first for trees and polytrees (singly connected graphs)
- then for multiply-connected graphs
- for the latter, the algorithm by Steffen Lauritzen & David Spiegelhalter was the first to find wide-spread use.

Also see “Inference in Bayesian Networks: a Historical Perspective”, by Adnan Darwiche

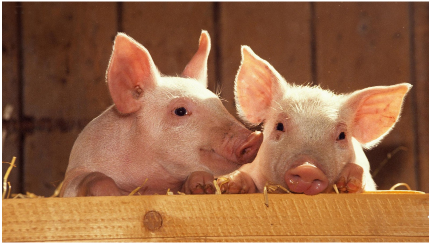
The Bayesian network framework

A Bayesian network is a very compact representation of a joint probability distribution $\Pr$. Such a network comprises

- qualitative knowledge of $\Pr$: a graphical representation of the independences between the variables involved;
- quantitative knowledge of $\Pr$: conditional probability distributions that describe $\Pr$ ‘locally’ per group of variables.

Associated with a Bayesian network are algorithms for computing probabilities and for processing evidence.

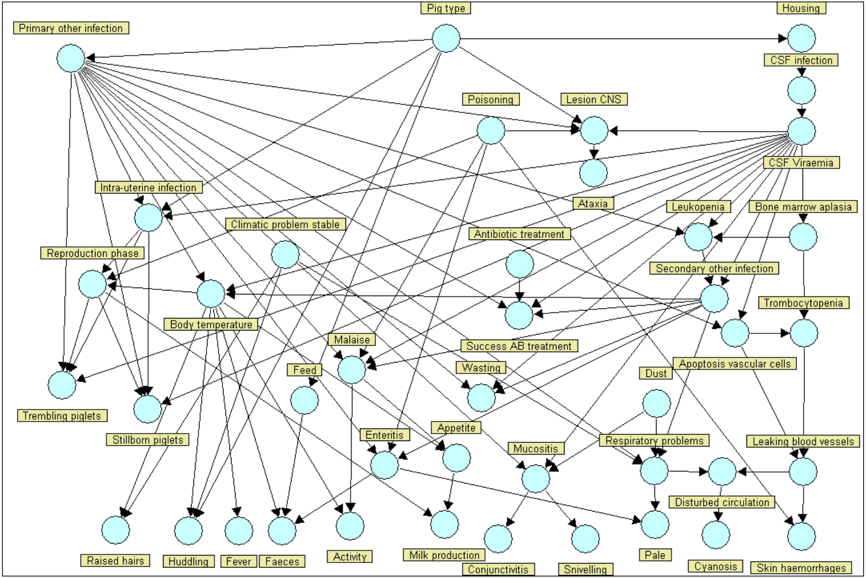
An example: Classical Swine Fever (CSF)



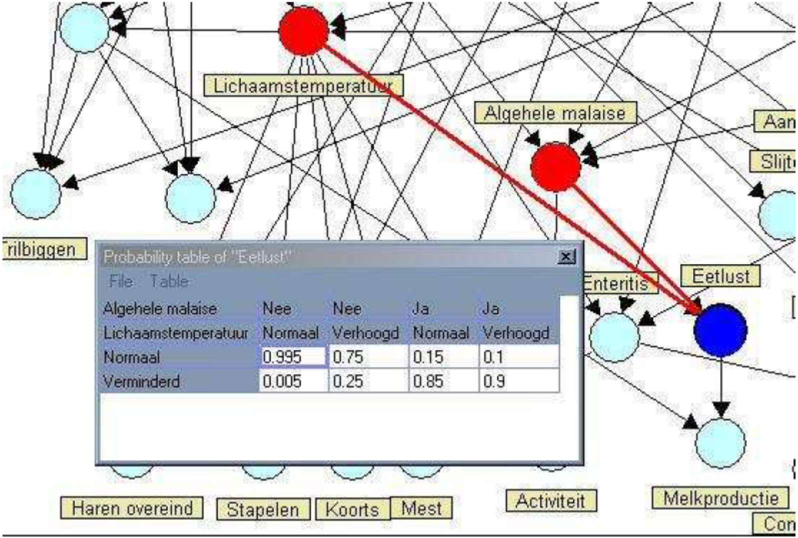
The classical swine fever network is a decision-support system for the early detection of classical swine fever (varkenspest).

- early detection of CSF is important, but hard;
- the network has been developed in cooperation with 2 veterinarians of the Central Veterinary Institute of Wageningen UR;
- part of european EPIZONE project;
- veterinarians all over the country collected data with PDAs

The Classical swine fever network: initial graphical structure



The Classical swine fever network: probability tables



$$\Pr(Appetite \mid BodyTemp \wedge Malaise)$$

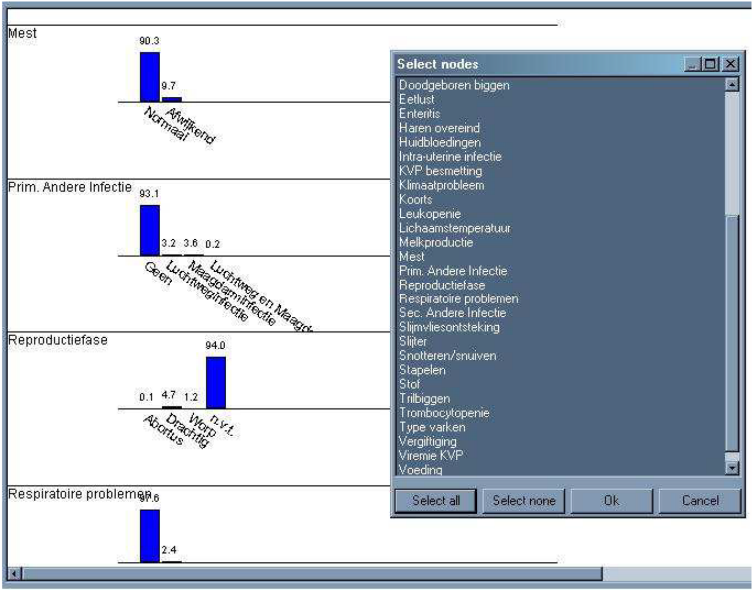
Classical swine fever: prior probabilities

Faeces

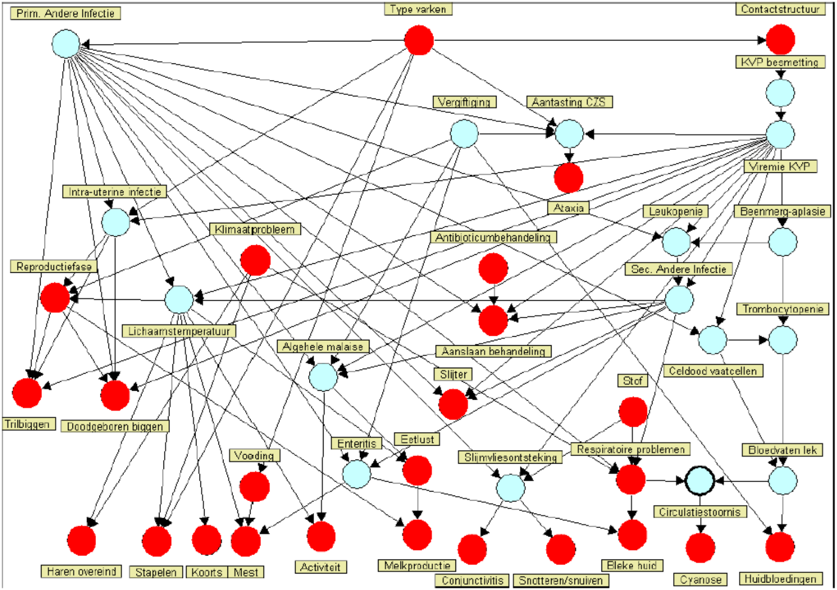
Prim. Other Infection

Reproduction phase

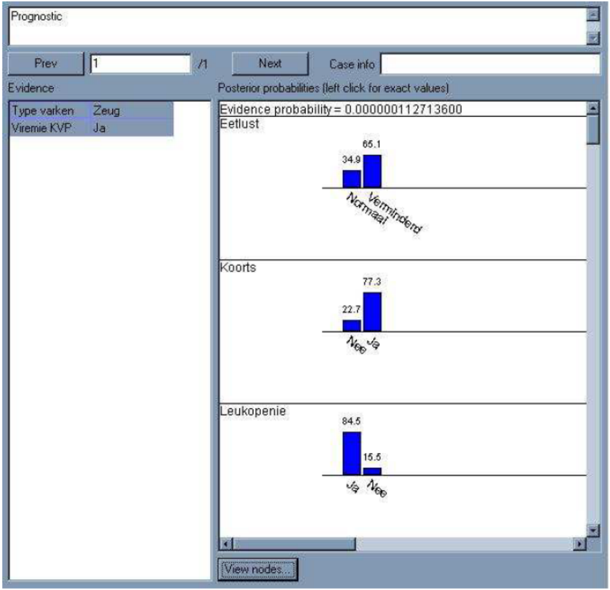
Respiratory problems



Classical swine fever: diagnostic reasoning



Classical swine fever: prognostic reasoning



A Bayesian network: necessary ingredients

Definition:
A Bayesian network is a pair $\mathcal{B} = (G, \Gamma)$ such that

- G is an *acyclic directed graph* with nodes representing a set of *random variables* $\boldsymbol{V}$;
- $\Gamma = \{\gamma_{V_i} \mid V_i \in \boldsymbol{V}\}$ is a set of *assessment functions*.

Property:

$$\Pr(\boldsymbol{V}) = \prod_{V_i \in \boldsymbol{V}} \gamma_{V_i}(V_i \mid \boldsymbol{\rho}(V_i))$$

defines a *joint probability distribution* $\Pr$ on $\boldsymbol{V}$ such that G is a *directed I-map* for the *independence relation* $I_{\Pr}$ of $\Pr$.

About this course ...

The following subjects will be addressed in this course:

- the syntactics and semantics of a Bayesian network;
- algorithms for reasoning with a Bayesian network;
- methods for constructing a Bayesian network for a domain of application;
- methods for evaluating a Bayesian network's performance and behaviour;
- algorithms for controlling reasoning;

Overview of subjects