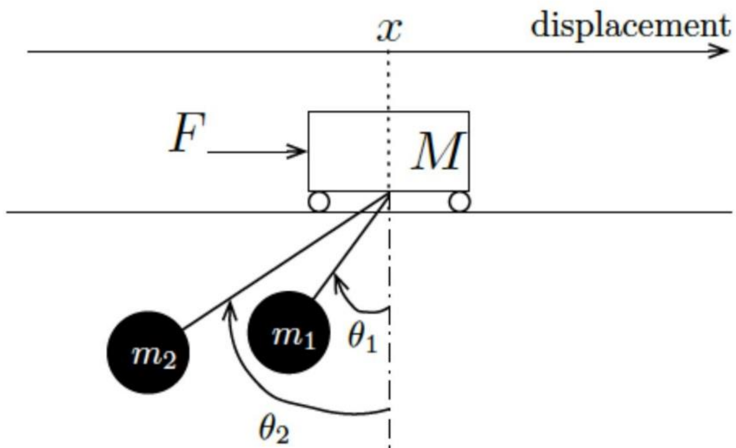
ENPM 667 - Control of Robotic Systems
Final Project

Design of LQR & LQG Controller for Crane & Pendulums



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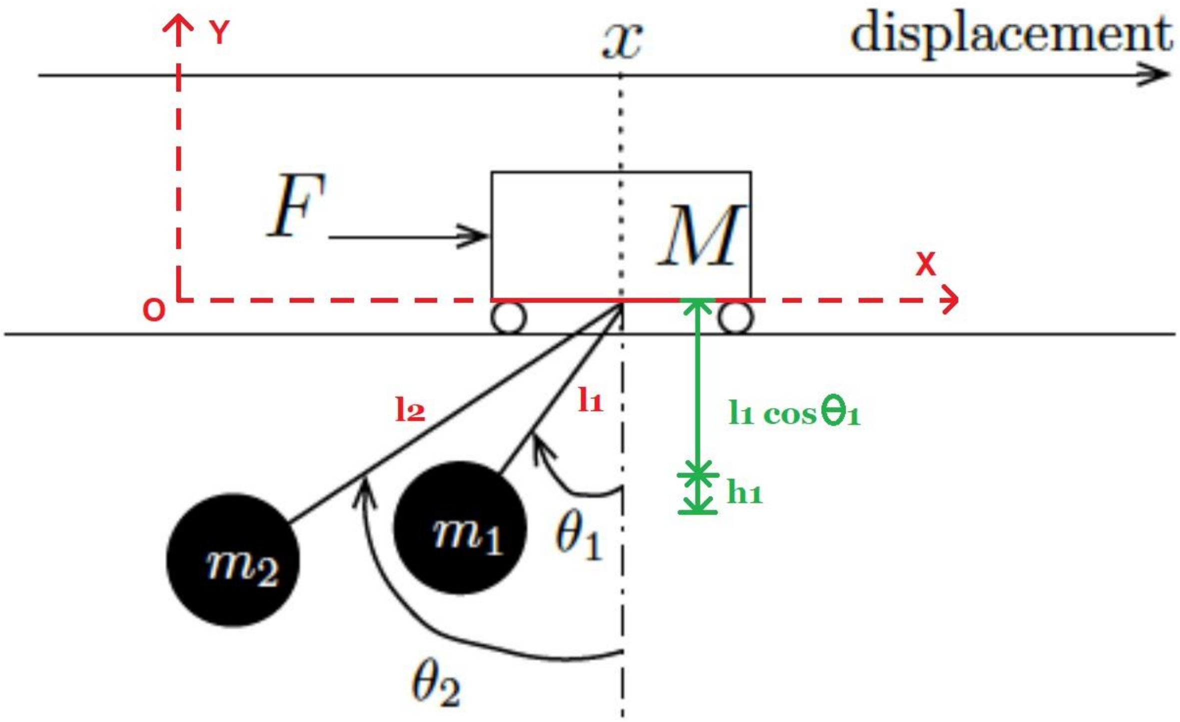
Content

Consider the following figure-



Given,	M	= mass of the block,
	m_1	= mass of load 1,
	m_2	= mass of load 2,
	F	= external force,
	l_1	= length of the cable 1,
	l_2	= length of the cable 2,
	x	= direction of displacement of the crane,
	θ_1	= angle made by m1 with the normal w.r.t the crane,
	θ_2	= angle made by m2 with the normal w.r.t the crane,

Mass m_1

Position

$$x_1 = (x)\,\hat{\imath} - (l_1 sin\theta_1)\,\hat{\imath} + (-l_1 cos\theta_1)\,\hat{\jmath}$$

$$\therefore x_1 = (x - l_1 sin\theta_1)\,\hat{\imath} - (l_1 cos\theta_1)\,\hat{\jmath}$$

Velocity

$$\therefore \dot{x}_1 = (\dot{x} - l_1\dot{\theta}_1 cos\theta_1)\hat{\imath} + (l_1\dot{\theta}_1 sin\theta_1)\,\hat{\jmath}$$

Kinetic Energy

$$\therefore K.E._1 = \frac{1}{2}\, m_1 \left[(\dot{x} - l_1\dot{\theta}_1 cos\theta_1)^2 + (l_1\dot{\theta}_1 sin\theta_1)^2\right]$$

Potential Energy

$$P.E._1 = m_1 g h_1$$

$$\therefore P.E._1 = m_1 g(l_1 - l_1\, cos\theta_1)$$

$$\therefore P.E._1 = m_1 g l_1(1 - cos\theta_1)$$

Mass m_2

Since mass m_2 also has a similar orientation to that of mass m_1, from equations ….., we can write the equations for mass m_2

$$\therefore x_2 = (x - l_2 sin\theta_2)\,\hat{\imath} - (l_2 cos\theta_2)\,\hat{\jmath}$$

$$\therefore \dot{x}_2 = (\dot{x} - l_2\dot{\theta}_2 cos\theta_2)\hat{\imath} + (l_2\dot{\theta}_2 sin\theta_2)\,\hat{\jmath}$$

$$\therefore K.E._2 = \frac{1}{2}\, m_2 \left[(\dot{x} - l_2\dot{\theta}_2 cos\theta_2)^2 + (l_2\dot{\theta}_2 sin\theta_2)^2\right]$$

$$\therefore P.E._2 = m_2 g l_2(1 - cos\theta_2)$$

Mass M

$$K.E. = \frac{1}{2}\, M\dot{x}^2$$

$$P.E. = 0$$

Combining all the Kinetic Energies together, we get-

$$K.E._{Total} = K.E. + K.E._1 + K.E._2$$

$$K.E._{Total} = \frac{1}{2}\, Mx^2 + \frac{1}{2}\, m_1 \left[(\dot{x} - l_1\dot{\theta}_1 cos\theta_1)^2 + (l_1\dot{\theta}_1 sin\theta_1)^2\right] + \frac{1}{2}\, m_2 \left[(\dot{x} - l_2\dot{\theta}_2 cos\theta_2)^2 + (l_2\dot{\theta}_2 sin\theta_2)^2\right]$$

And, combining all the Potential Energies together, we get-

$$P.E._{Total} = P.E. + P.E._1 + P.E._2$$

$$P.E._{Total} = -m_1 g l_1 (1 - cos\theta_1) - m_2 g l_2 (1 - cos\theta_2)$$

A. Equations of Motion and Non-linear State-space Representation

Thus, from the Euler-Lagrange Equation, we get-

$$L = K.E._{Total} - P.E._{Total}$$

$$L = \frac{1}{2} Mx^2 + \frac{1}{2} m_1 \left[(\dot{x} - l_1\dot{\theta}_1 cos\theta_1)^2 + (l_1\dot{\theta}_1 sin\theta_1)^2 \right] + \frac{1}{2} m_2 \left[(\dot{x} - l_2\dot{\theta}_2 cos\theta_2)^2 + (l_2\dot{\theta}_2 sin\theta_2)^2 \right]$$
$$+ m_1 g l_1 \, cos\theta_1 + m_2 g l_2 \, cos\theta_2$$

Using the Euler-Lagrange Equation, we can derive the Equations of Motion as-

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = F$$

$$M\ddot{x} + m_1\ddot{x} - m_1 l_1 \ddot{\theta}_1 cos\theta_1 + m_1 l_1 \dot{\theta}_1{}^2 sin\theta_1 + m_1\ddot{x} - m_2 l_2 \ddot{\theta}_2 cos\theta_2 + m_2 l_2 \dot{\theta}_2{}^2 sin\theta_2 = F$$

$$\boldsymbol{\ddot{x} = \frac{F + m_1 l_1 \ddot{\theta}_1 cos\theta_1 + m_2 l_2 \ddot{\theta}_2 cos\theta_2 - m_1 l_1 \dot{\theta}_1{}^2 sin\theta_1 - m_2 l_2 \dot{\theta}_2{}^2 sin\theta_2}{(M + m_1 + m_2)}}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = 0$$

$$= \frac{1}{2} m_1 [2 l_1{}^2 \ddot{\theta}_1 - 2\ddot{x} l_1 cos\theta_1 + 2\dot{x} l_1 \dot{\theta}_1 sin\theta_1] - m_1 \dot{x} l_1 \dot{\theta}_1 sin\theta_1 - m_1 g l_1 sin\theta_1 = 0$$

$$= m_1 l_1{}^2 \ddot{\theta}_1 - m_1 \ddot{x} l_1 cos\theta_1 + m_1 \dot{x} l_1 \dot{\theta}_1 sin\theta_1 - m_1 \dot{x} l_1 \dot{\theta}_1 sin\theta_1 + m_1 g l_1 sin\theta_1 = 0$$

$$\boldsymbol{\ddot{\theta}_1 = \frac{\ddot{x} cos\theta_1 - g sin\theta_1}{l_1}}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = 0$$

$$= \frac{1}{2}m_2\left[2{l_2}^2\ddot{\theta}_2 - 2\ddot{x}l_2cos\theta_2 + 2\dot{x}l_2\dot{\theta}_2sin\theta_2\right] - m_2\dot{x}l_2\dot{\theta}_2sin\theta_2 - m_2gl_2sin\theta_2 = 0$$

$$= \ m_2{l_2}^2\ddot{\theta}_2 - m_2\ddot{x}l_2cos\theta_2 + m_2\dot{x}l_2\dot{\theta}_2sin\theta_2 - m_2\dot{x}l_2\dot{\theta}_2sin\theta_2 + m_2gl_2sin\theta_2 = 0$$

$$\boldsymbol{\ddot{\theta}_2 = \frac{\ddot{x}cos\theta_2 - gsin\theta_2}{l_2}}$$

From the above results, we can derive the

$$\ddot{x} = \frac{F + m_1l_1\left(\frac{\ddot{x}cos\theta_1 - gsin\theta_1}{l_1}\right)cos\theta_1 + m_2l_2\left(\frac{\ddot{x}cos\theta_2 - gsin\theta_2}{l_2}\right)cos\theta_2 - m_1l_1{\dot{\theta}_1}^2sin\theta_1 - m_2l_2{\dot{\theta}_2}^2sin\theta_2}{(M + m_1 + m_2)}$$

$$\ddot{x} = \frac{F + m_1\ddot{x}cos^2\theta_1 - m_1gsin\theta_1cos\theta_1 + m_2\ddot{x}cos^2\theta_2 - m_2gsin\theta_2cos\theta_2 - m_1l_1{\dot{\theta}_1}^2sin\theta_1 - m_2l_2{\dot{\theta}_2}^2sin\theta_2}{(M + m_1 + m_2)}$$

$$(M + m_1 + m_2)\ddot{x} - m_1\ddot{x}cos^2\theta_1 - m_2\ddot{x}cos^2\theta_2$$
$$= F - m_1gsin\theta_1cos\theta_1 - m_2gsin\theta_2cos\theta_2 - m_1l_1{\dot{\theta}_1}^2sin\theta_1 - m_2l_2{\dot{\theta}_2}^2sin\theta_2$$

Upon solving the above equations, we get-

$$\boldsymbol{\ddot{x} = \frac{F - m_1gsin\theta_1cos\theta_1 - m_2gsin\theta_2cos\theta_2 - m_1l_1{\dot{\theta}_1}^2sin\theta_1 - m_2l_2{\dot{\theta}_2}^2sin\theta_2}{M + m_1sin^2\theta_1 + m_2sin^2\theta_2}}$$

$$\boldsymbol{\ddot{\theta}_1 = \frac{Fcos\theta_1 - m_1gsin\theta_1cos^2\theta_1 - m_1l_1{\dot{\theta}_1}^2sin\theta_1cos\theta_1 - m_2gcos\theta_1sin\theta_2cos\theta_2 - m_2l_2{\dot{\theta}_2}^2cos\theta_1sin\theta_2}{M + m_1sin^2\theta_1 + m_2sin^2\theta_2}}$$

$$\boldsymbol{\ddot{\theta}_2 = \frac{Fcos\theta_2 - m_2gsin\theta_2cos^2\theta_2 - m_2l_2{\dot{\theta}_1}^2sin\theta_1cos\theta_2 - m_1gsin\theta_1cos\theta_1cos\theta_2 - m_2l_2{\dot{\theta}_2}^2sin\theta_2cos\theta_2}{M + m_1sin^2\theta_1 + m_2sin^2\theta_2}}$$

B. State Space Representation of a Linearized System

To achieve the Linearized system from Non-linear system, we approximate the angles to zero and higher order terms are neglected.

$$\boldsymbol{sin\theta \approx 0, \quad sin^2\theta \approx 0, \quad cos\theta \approx 1, \quad cos^2\theta \approx 1,}$$

$$\ddot{x} = \frac{F - m_1 g\theta_1 - m_2 g\theta_2}{M}$$

$$\ddot{\theta}_1 = -\frac{F - m_1 g\theta_1 - m_2 g sin\theta_2 - Mg\theta_1}{l_1 M}$$

$$\ddot{\theta}_2 = -\frac{F - m_1 g\theta_1 - m_2 g sin\theta_2 - Mg\theta_2}{l_2 M}$$

To define the state variables, we consider-

$$x_1 = x,\ x_2 = \dot{x},\ x_3 = \theta_1,\ x_4 = \dot{\theta}_1,\ x_5 = \theta_2,\ x_6 = \dot{\theta}_2$$

The state variables can now be defined as

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \\ \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix}$$

Thus, the non-linear state-space equation can be expressed as-

$$\dot{X} = \begin{bmatrix} \dot{x} \\ \dfrac{F - m_1 g sin\theta_1 cos\theta_1 - m_2 g sin\theta_2 cos\theta_2 - m_1 l_1 \dot{\theta_1}^2 sin\theta_1 - m_2 l_2 \dot{\theta_2}^2 sin\theta_2}{M + m_1 sin^2\theta_1 + m_2 sin^2\theta_2} \\ \dot{\theta}_1 \\ \dfrac{Fcos\theta_1 - m_1 g sin\theta_1 cos^2\theta_1 - m_2 g cos\theta_1 sin\theta_2 cos\theta_2 - m_1 l_1 \dot{\theta_1}^2 sin\theta_1 cos\theta_1 - m_2 l_2 \dot{\theta_2}^2 cos\theta_1 sin\theta_2}{M + m_1 sin^2\theta_1 + m_2 sin^2\theta_2} \\ \dot{\theta}_2 \\ \dfrac{Fcos\theta_2 - m_1 g sin\theta_1 cos\theta_2 - m_2 g sin\theta_2 cos^2\theta_2 - m_1 l_1 \dot{\theta_1}^2 sin\theta_1 cos\theta_2 - m_2 l_2 \dot{\theta_2}^2 sin\theta_2 cos\theta_2}{M + m_1 sin^2\theta_1 + m_2 sin^2\theta_2} \end{bmatrix}$$

And the linear state-space equation for x = θ_1 = θ_1 = 0 can be expressed as-

$$\dot{X} = \begin{bmatrix} \dot{x} \\ \dfrac{F - m_1 g\theta_1 - m_2 g\theta_2}{M} \\ \dot{\theta}_1 \\ \dfrac{F - m_1 g\theta_1 - m_2 g sin\theta_2 - Mg\theta_1}{l_1 M} \\ \dot{\theta}_2 \\ \dfrac{F - m_1 g\theta_1 - m_2 g sin\theta_2 - Mg\theta_2}{l_2 M} \end{bmatrix}$$

The linear state-space equation can be given as-

$$\dot{X} = AX(t) + BU(t)$$

$$Y = CX(t) + DU(t)$$

Thus, the constant matrices are calculated as follows-

$$\boldsymbol{A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-m_1 g}{M} & 0 & \frac{-m_2 g}{M} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-(M+m_1)g}{Ml_1} & 0 & \frac{-m_2 g}{Ml_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-m_2 g}{Ml_2} & 0 & \frac{-(M+m_1)g}{Ml_2} & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{Ml_1} \\ 0 \\ \frac{1}{Ml_2} \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, D = 0}$$

The linearized state space equations can be written as-

$$\dot{X} = \begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-m_1 g}{M} & 0 & \frac{-m_2 g}{M} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-(M+m_1)g}{Ml_1} & 0 & \frac{-m_2 g}{Ml_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-m_2 g}{Ml_2} & 0 & \frac{-(M+m_1)g}{Ml_2} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{Ml_1} \\ 0 \\ \frac{1}{Ml_2} \end{bmatrix} F$$

And the output variable Y is given as-

$$Y = \begin{bmatrix} x \\ \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix}$$

C. Conditions when Linearized System is Controllable

The controllability of the linear system can be determined by the Grammian Controllability. Any linear time invariant system of the form of equation (4), is controllable if the following rule is satisfied

$$Rank\ [B_{\mathrm{k}} \quad AB_{\mathrm{k}} \quad \dots \quad A^{n-1}B_{\mathrm{k}}] = n \text{ where } n \text{ is full rank}$$

If the rank of the above matrix is full rank i.e. n then the given linear time invariant system is controllable. The condition for checking the controllability is-

$$C = Rank\ [B \quad AB \quad A^2B \quad A^3B \quad A^4B \quad A^5B]$$

$$\boldsymbol{C} = \begin{bmatrix} 1 & \frac{1}{M} & 0 & \frac{-g(l_2m_1 + l_1m_2)}{l_1l_2M^2} & \dots \\ \frac{1}{M} & 0 & \frac{-g(l_2m_1 + l_1m_2)}{l_1l_2M^2} & 0 & \dots \\ 0 & \frac{1}{Ml_1} & 0 & \frac{-g(l_2\mathrm{M} + l_2m_1 + l_1m_2)}{{l_1}^2l_2M^2} & \dots \\ \frac{1}{Ml_1} & 0 & \frac{-g(l_2\mathrm{M} + l_2m_1 + l_1m_2)}{{l_1}^2l_2M^2} & 0 & \dots \\ 0 & \frac{1}{Ml_2} & 0 & \frac{-g(l_1\mathrm{M} + l_2m_1 + l_1m_2)}{{l_2}^2l_1M^2} & \dots \\ \frac{1}{Ml_2} & 0 & \frac{-g(l_1\mathrm{M} + l_2m_1 + l_1m_2)}{{l_2}^2l_1M^2} & 0 & \dots \end{bmatrix}$$

Using MATLAB, the rank of the matrix C came out to be 6 which is full rank since there are 6 linearly independent rows, and hence the linearized system is controllable. The relationship between the variables M, m_1, m_2, l_1, l_2 can be found out with the help of the determinant which is

$$\det(C_{\mathrm{on}}) = \frac{-(g^6{l_1}^2 - 2g^6l_1l_2 + g^6{l_2}^2)}{M^6{l_1}^6{l_2}^6} = \frac{-(g^6(l_1 - l_2)^2)}{M^6{l_1}^6{l_2}^6}$$

From the above equation we come to know that the controllability of the system is independent of m_1, m_2. However, the system would become uncontrollable when $l_1 = l_2$ for which the $\det(C) = 0$, otherwise it is controllable.

D. LQR Controller for a Controllable System and Lyapunov's Indirect Methods

Using the values of $M = 1000\, kg,\ m_1 = m_1 = 100\, kg,\ l_1 = 20m,\ l_2 = 10m$ the controllability matrix is given as-

$$C_{\text{on}} = \begin{bmatrix} 0 & 1 & 0 & -0.1472 & 0 & 0.1419 \\ 1 & 0 & -0.1472 & 0 & 0.1419 & 0 \\ 0 & 0.0500 & 0 & -0.0319 & 0 & 0.0027 \\ 0.0500 & 0 & -0.0319 & 0 & 0.0227 & 0 \\ 0 & 0.1000 & 0 & -0.1128 & 0 & 0.1249 \\ 0.1000 & 0 & -0.1128 & 0 & 0.1249 & 0 \end{bmatrix} \times 10^{-3}$$

Using MATLAB, the rank of the matrix C came out to be 6 and hence the system is controllable. And since the system is controllable we can design an LQR controller. The optimal solution to design an LQR controller is given by

$$K = -R^{-1}{B_{\text{k}}}^T P$$

where P is the symmetric positive solution of the following stationary Riccati equation:

$$A^T P + PA - PB_{\text{k}}R^{-1}{B_{\text{k}}}^T P = -Q$$

Using MATLAB, the function 'lqr' solves the Riccati equation for P and gives the gain value K for the controller. The control input is $U = -KX$ and hence we have-

$$\dot{X} = AX + B(-KX)$$

$$= (A - BK)X$$

$$A_{\text{C}} = A - BK \text{ and } B_{\text{C}} = 0.$$

The 6 states represent the position and velocity of the cart and the angle and angular velocity of both the pendulum. The output Y contains the position of the cart and the angle of the pendulums. We want to design the controller so that when a step reference is given to the system, the pendulum after initial displacement should return to its equilibrium position i.e. vertical position. For the LQR controller we determine out state feedback control gain matrix K. For the LQR function, the MATLAB function allows us to choose between two parameters R and Q which will balance the relative importance of the control effort (u) and error. The controller can be tuned by changing the nonzero elements in the Q matrix to achieve a desirable response. For the simulation we have taken non-zero initial conditions

$\boldsymbol{X_0} = [\mathbf{0} \quad \mathbf{0} \quad \mathbf{10} \quad \mathbf{0} \quad \mathbf{15} \quad \mathbf{0}]$ where the angles are in degree.

After trial and error following weights are decided:

$$\boldsymbol{Q(1,1) = 70000000}$$

$$\boldsymbol{Q(3,3) = 8000000000}$$

$$\boldsymbol{Q(5,5) = 9000000000}$$

The resulting gains are

$$\boldsymbol{K = 10^4[0.8366 \quad 1.5580 \quad 8.6606 \quad -2.5918 \quad 4.3927 \quad -8.4377]}$$

Lyapunov Indirect Method:

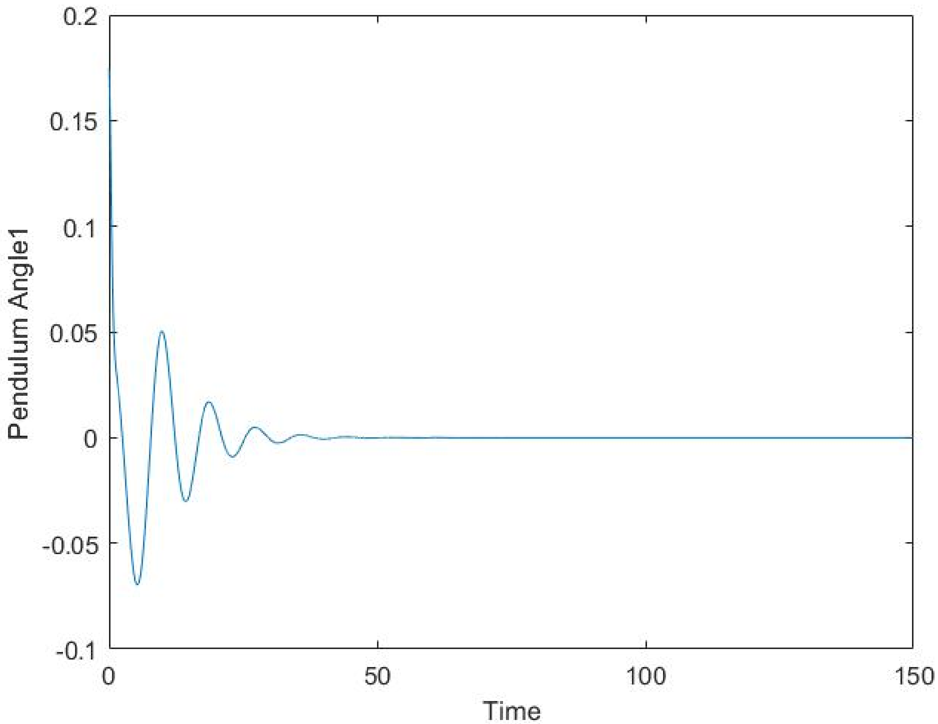
To check the stability of the non-linear system using Lyapunov, we have-

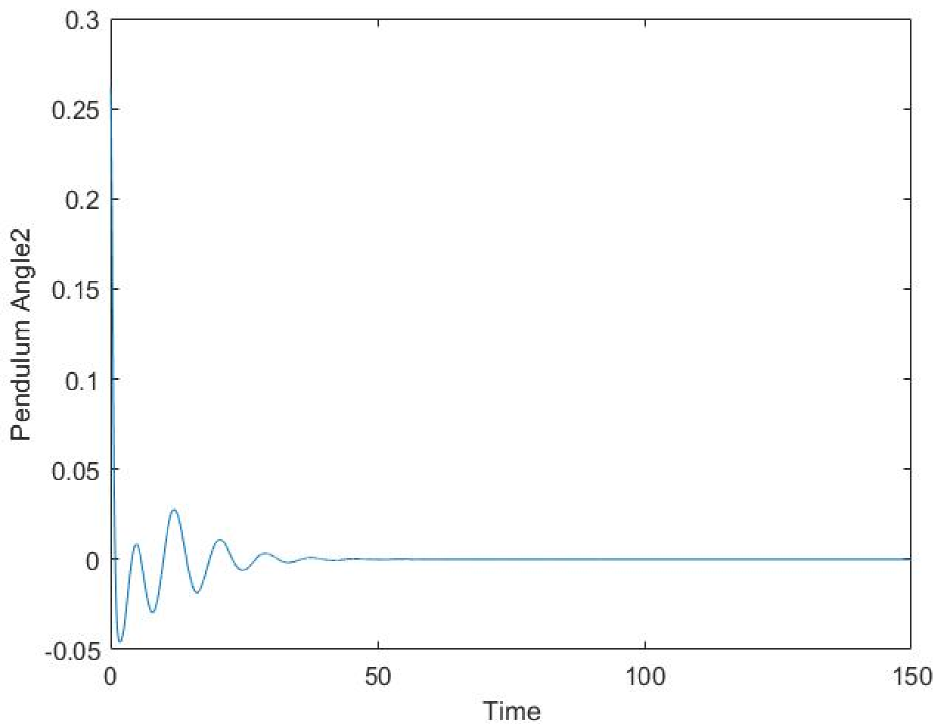
- Linearize the original system around the equilibrium point of interest.
- Check the eigen values of A_{F}.
- The following describes the test:
  - If the eigen values of A_{F} have negative real part, then the original system is at least locally stable around the equilibrium point. In this case a Lyapunov for the linearized system will be valid at least locally.
  - If at least one eigen values of A_{F} is positive, then the original system is unstable around the equilibrium point.
  - If the eigen values of A_{F} have non-positive real part, but at least one I son the imaginary axis then the indirect method is inconclusive.

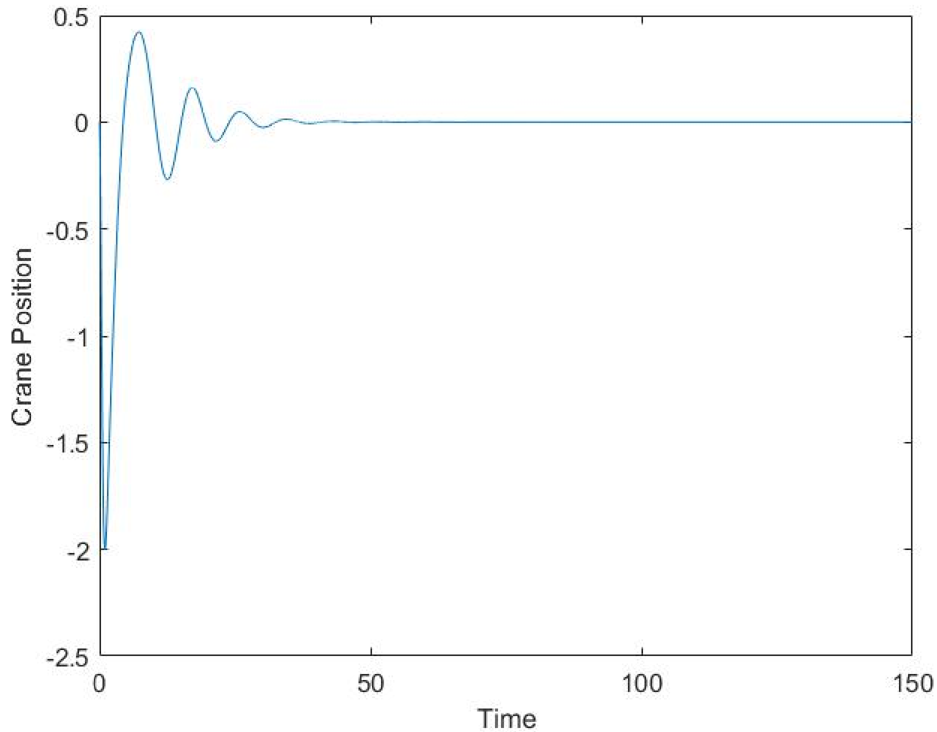
Computing the eigen values of $A_{\mathrm{F}} = A - BK$ we get following result:

$$\begin{gathered}\boldsymbol{-2.5213 + 2.6552i} \\ \boldsymbol{-2.5213 - 2.6552i} \\ \boldsymbol{-0.2509 + 0.6773i} \\ \boldsymbol{-0.2509 - 0.6773i} \\ \boldsymbol{-0.1508 + 0.7434i} \\ \boldsymbol{-0.1508 - 0.7434i}\end{gathered}$$

As we see, all the eigen values have negative real part and hence we can conclude by Lyapunov Indirect Method that the system is locally stable.


Pendulum Angle1
Time
0.2
0.15
0.1
0.05
0
-0.05
-0.1
0
50
100
150


Pendulum Angle2
Time
0.3
0.25
0.2
0.15
0.1
0.05
0
-0.05
0
50
100
150



0.5
0
-0.5
-1
-1.5
-2
-2.5
Crane Position
0
50
100
150
Time

E. Output Vectors for Observable Linearized System

Some of the standard definitions are given as-

State Observation: The concept of state observer involves using current and past values of the input and the output signals to generate an estimate of the current state.

Observability: The linear state equation is called observable on $[t_0, t_f]$ if any initial state $x(t_0) = x_0$ is uniquely determined by the corresponding response $y(t), t \in [t_0, t_f]$.

For the Invariant case the linear state equation is observable if and only if the np x n observability matrix satisfies the below criteria:

$$rank = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$$

In this scenario a system is observable if

$$0 = rank \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \\ CA^4 \\ CA^5 \end{bmatrix} = 6$$

For different cases, we will check the observability of the matrix.

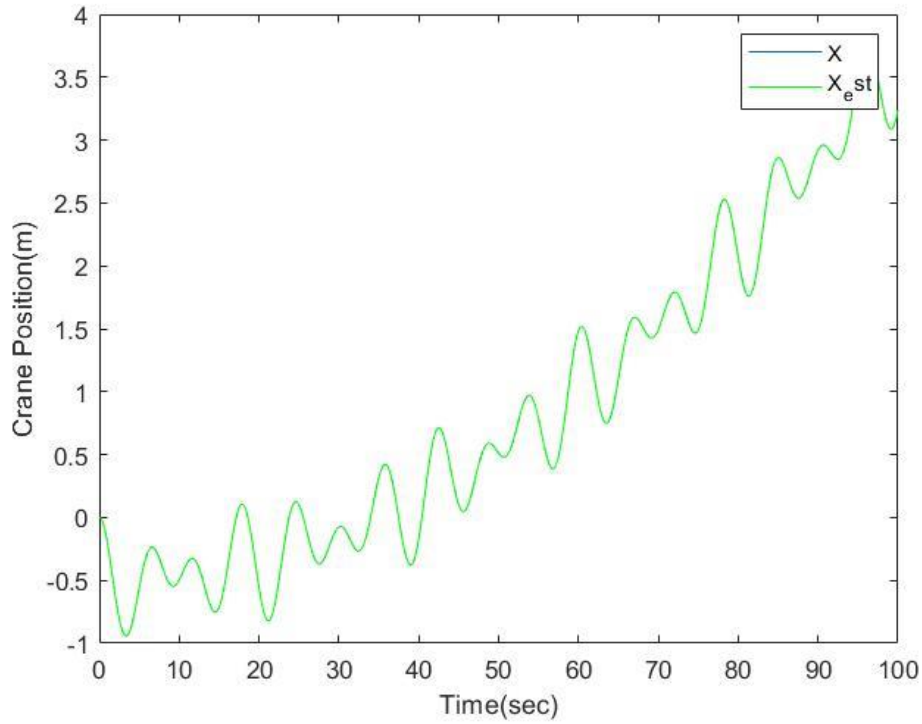
Case 1: $y(t) = x(t)$

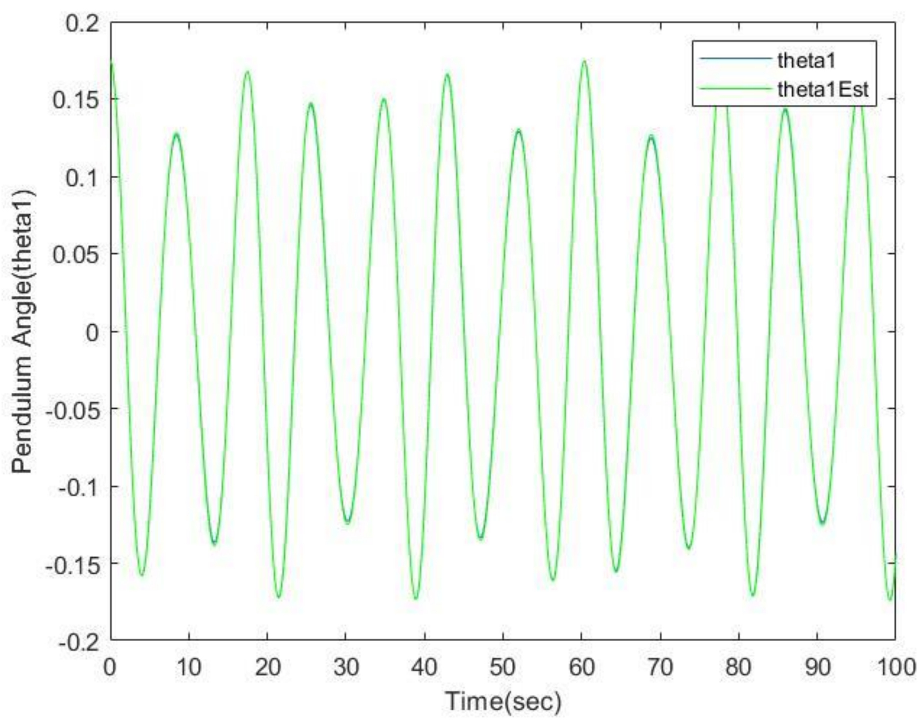
$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

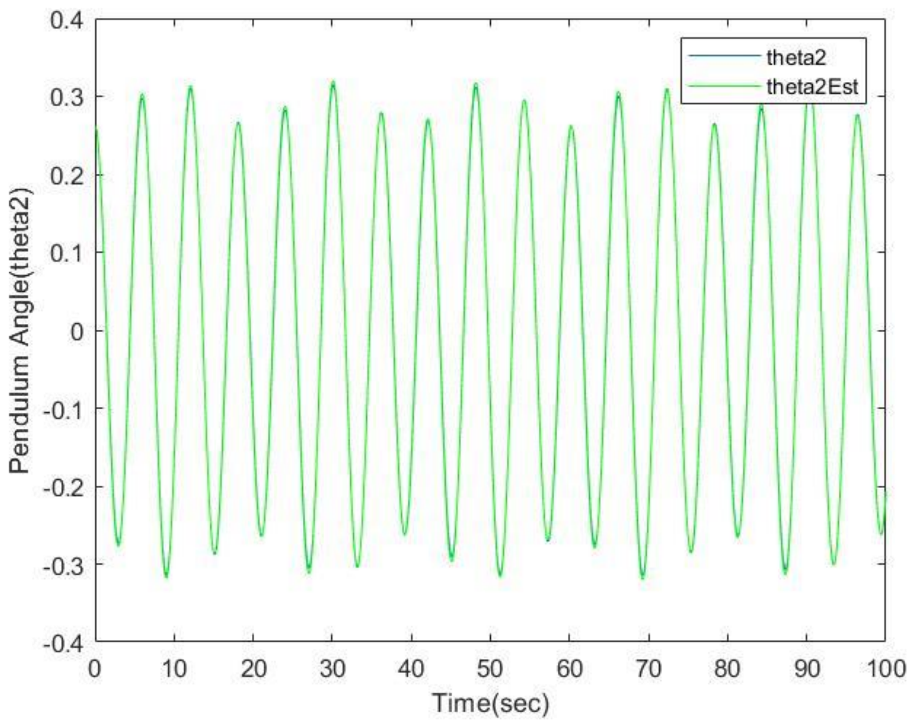
The observability matrix for this case was calculated using MATLAB

$$\boldsymbol{O} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.9810 & 0 & -0.9810 & 0 \\ 0 & 0 & 0 & -0.9810 & 0 & -0.9810 \\ 0 & 0 & 0.6255 & 0 & 1.1067 & 0 \\ 0 & 0 & 0 & 0.6255 & 0 & 1.1067 \end{bmatrix}$$

The rank for this matrix was 6 (using MATLAB), so this observability matrix is observable.


X
X_est
Crane Position(m)
Time(sec)


theta1
theta1Est
Pendulum Angle(theta1)
Time(sec)



Case 2: $y(t) = (\theta_1(t),\ \theta_2(t))$

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The observability matrix for this case was calculated using MATLAB

$$O = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -0.5396 & 0 & -0.0491 & 0 \\ 0 & 0 & -0.0981 & 0 & -1.0791 & 0 \\ 0 & 0 & 0 & -0.5396 & 0 & -0.0491 \\ 0 & 0 & 0 & -0.0981 & 0 & -1.0791 \\ 0 & 0 & 0.2959 & 0 & 0.0794 & 0 \\ 0 & 0 & 0.1588 & 0 & 1.1693 & 0 \\ 0 & 0 & 0 & 0.2959 & 0 & 0.0794 \\ 0 & 0 & 0 & 0.1588 & 0 & 1.1693 \end{bmatrix}$$

The rank for this matrix was 4 (using MATLAB), so this observability matrix is not observable.

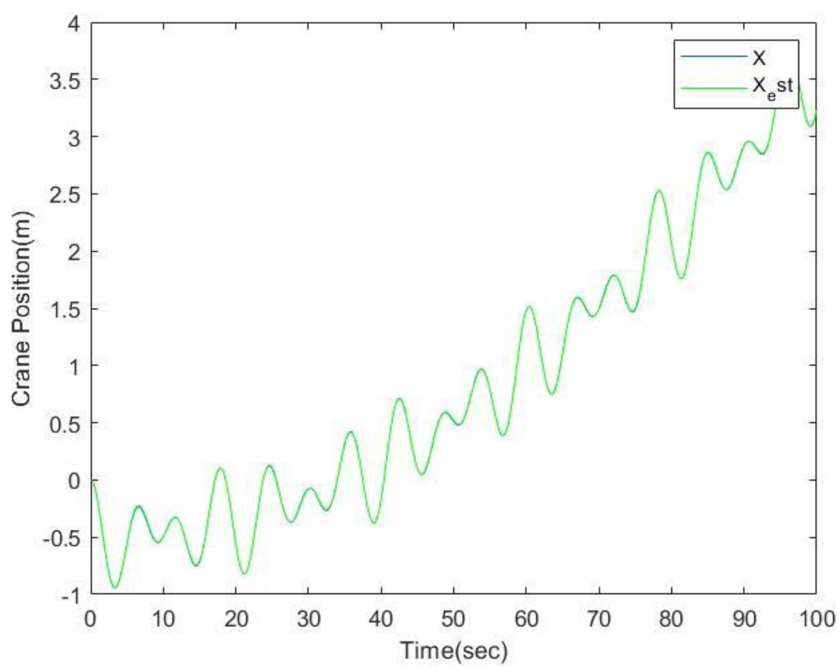
Case 3: $y(t) = (x(t),\ \theta_2(t))$

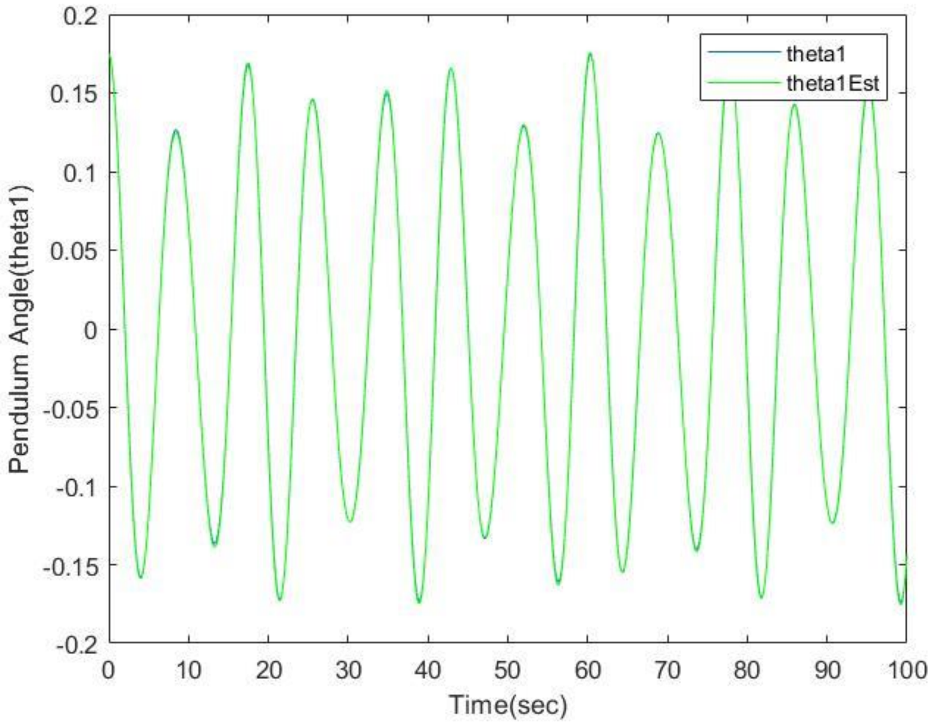
$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

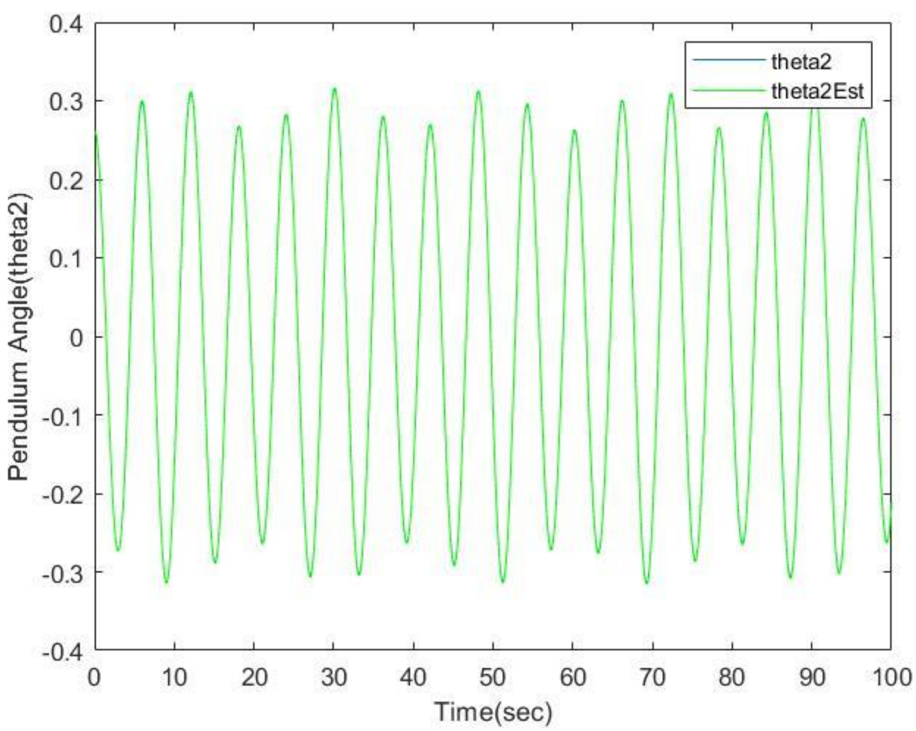
The observability matrix for this case was calculated using MATLAB

$$O = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -0.9810 & 0 & -0.9810 & 0 \\ 0 & 0 & -0.0981 & 0 & -1.0791 & 0 \\ 0 & 0 & 0 & -0.9810 & 0 & -0.9810 \\ 0 & 0 & 0 & -0.0981 & 0 & -1.0791 \\ 0 & 0 & 0.6255 & 0 & 1.1067 & 0 \\ 0 & 0 & 0.1588 & 0 & 1.1693 & 0 \\ 0 & 0 & 0 & 0.6255 & 0 & 1.1067 \\ 0 & 0 & 0 & 0.1588 & 0 & 1.1693 \end{bmatrix}$$

The rank for this matrix was 6 (using MATLAB), so this observability matrix is observable.




theta1
theta1Est
Pendulum Angle(theta1)
Time(sec)
0.2
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
0
10
20
30
40
50
60
70
80
90
100


theta2
theta2Est
Pendulum Angle(theta2)
Time(sec)
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
0
10
20
30
40
50
60
70
80
90
100

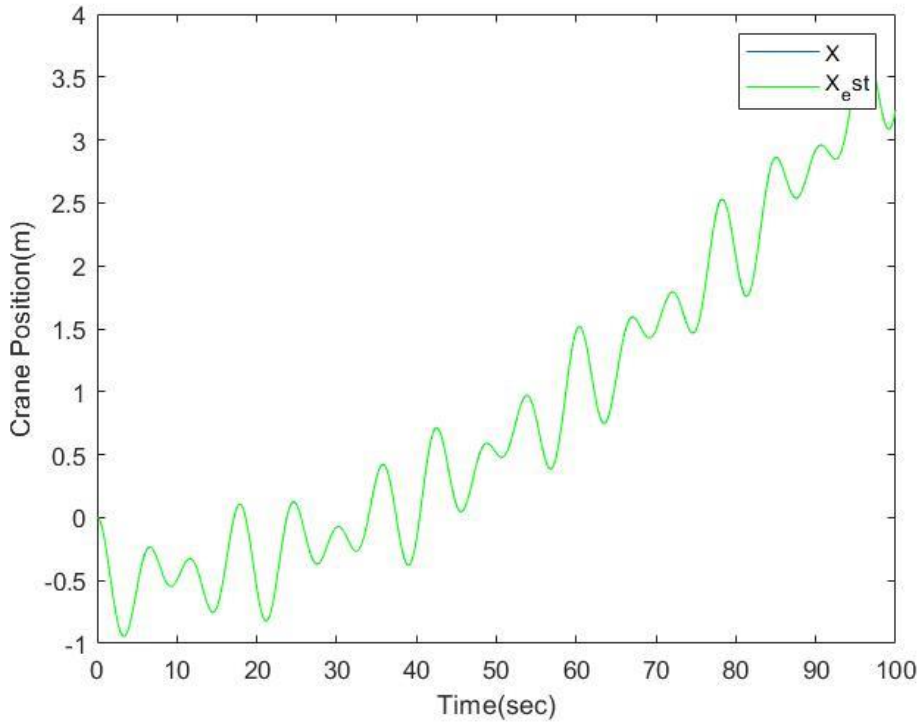
Case 4: $y(t) = (x(t),\ \theta_1(t),\ \theta_2(t))$

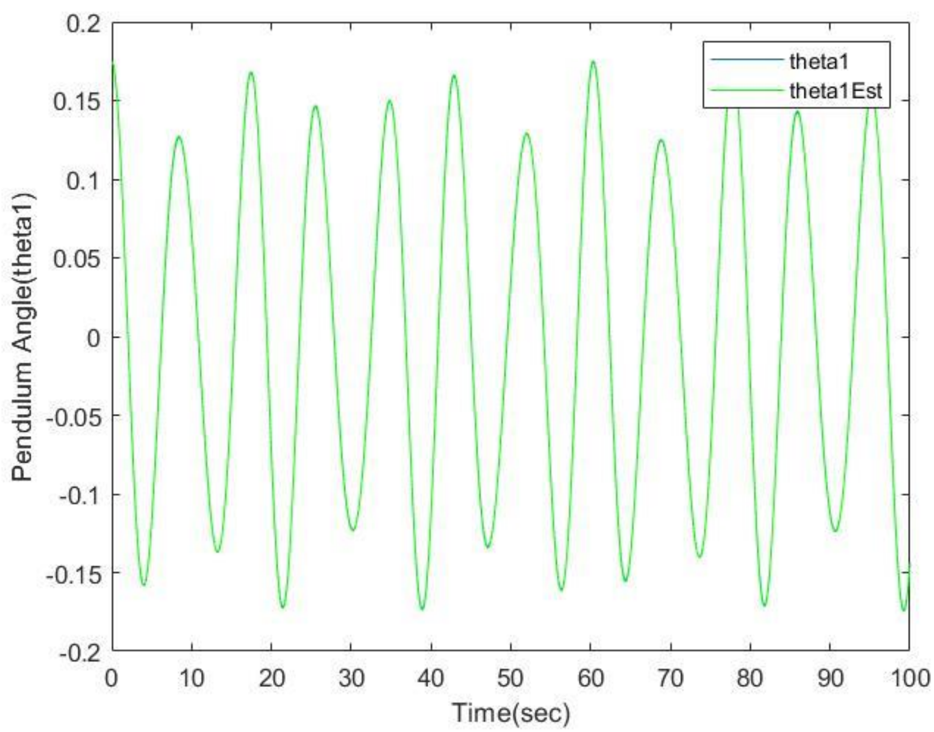
$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

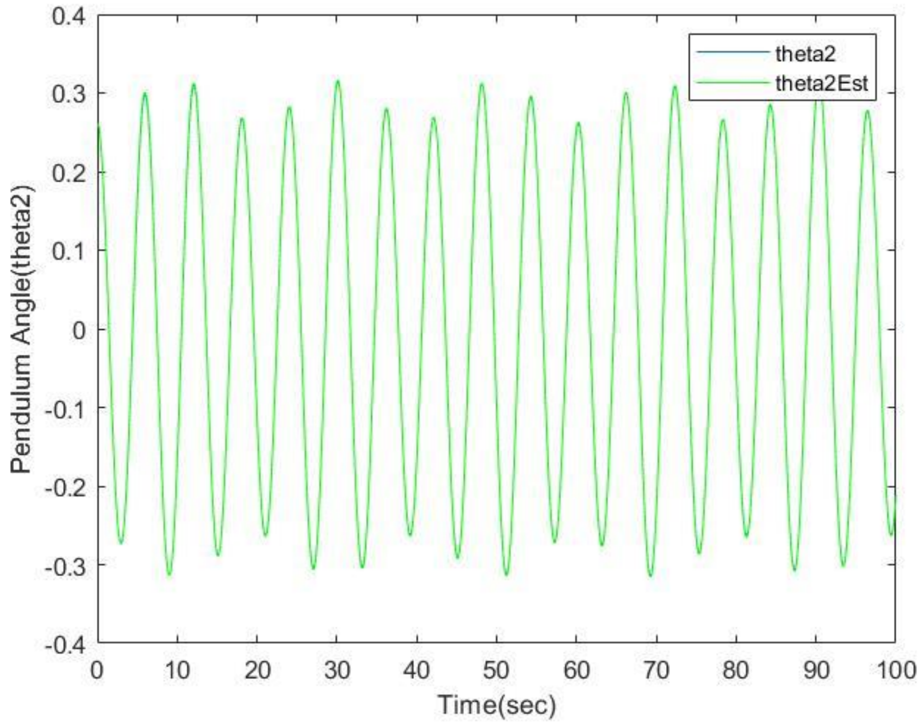
The observability matrix for this case was calculated using MATLAB

$$O = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -0.9810 & 0 & -0.9810 & 0 \\ 0 & 0 & -0.5396 & 0 & -0.0491 & 0 \\ 0 & 0 & -0.0981 & 0 & -1.0791 & 0 \\ 0 & 0 & 0 & -0.9810 & 0 & -0.9810 \\ 0 & 0 & 0 & -0.5396 & 0 & -0.0491 \\ 0 & 0 & 0 & -0.0981 & 0 & -1.0791 \\ 0 & 0 & 0.6255 & 0 & 1.1067 & 0 \\ 0 & 0 & 0.2959 & 0 & 0.0794 & 0 \\ 0 & 0 & 0.1588 & 0 & 1.1693 & 0 \\ 0 & 0 & 0 & 0.6255 & 0 & 1.1067 \\ 0 & 0 & 0 & 0.6255 & 0 & 0.0794 \\ 0 & 0 & 0 & 0.1588 & 0 & 1.1693 \end{bmatrix}$$

The rank for this matrix was 6 (using MATLAB), so this observability matrix is observable.


X
X_est
Crane Position(m)
Time(sec)


theta1
theta1Est
Pendulum Angle(theta1)
Time(sec)


theta2
theta2Est
Pendulum Angle(theta2)
Time(sec)
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
0
10
20
30
40
50
60
70
80
90
100

F. Best Luenberger Observer

The Luenberger observer equation is given by the following state space representation:

$$\dot{\hat{\vec{X}}}(t) = A\hat{\vec{X}}(t) + B_k u_k(t) + L(Y(t) - C\hat{\vec{X}}(t))$$

where, L is the gain matrix and $(Y(t) - C\hat{X}(t))$ is the correction term.

The state space representation in terms of estimation error is given by

$$\overrightarrow{X_e}(t) = X(t) - \hat{X}(t)$$

After simplification, the Luenberger observer equation obtained is

$$\dot{\overrightarrow{X_e}}(t) = (A - LC)\overrightarrow{X_e} - B_D u_D(t)$$

The important points of discussion are-

- The matrix $A - LC$ is stable iff $(A - LC)^T = A^T - C^T L^T$ is stable.
- If (A^T, C^T) is stabilizable, then we say that (A, C) is detectable.
- If (A^T, C^T) is controllable, then we say that (A, C) is observable.

Therefore, the state estimate error dynamics is described by

$$\dot{e} = (A - LC)\, e$$

The error tends to zero if the matrix $(A - LC)$ is stable, i.e. it has negative eigenvalues. The convergence rate of the error means the convergence rate of estimated state to the real state depends upon the placement of the poles matrix $A - LC$. As we plan to use the state estimator as the input to the controller, the state estimator must converge faster than desired closed loop system, which means observer poles must be faster that the controller poles. If the poles for the estimator are too fast than there will be errors in the sensor measurement or the measurement might be corrupted by the noise.

G. MATLAB Code: LQRMatrix.m

```
%% Value of Parameters

m1 = 100; m2 = 100; M = 1000;
l1 = 20; l2 = 10;
g = 9.81;

%% Defining the Matrix A (6 x 6)

A = [0 1 0 0 0 0;
     0 0 (-(m1*g)/M) 0 (-(m2*g)/M) 0;
     0 0 0 1 0 0;
     0 0 (-(m1+M)*g/(l1*M)) 0 (-(m2*g)/(l1*M)) 0;
     0 0 0 0 0 1;
     0 0 (-(m1*g)/(l2*M)) 0 (-(m2+M)*g/(l2*M)) 0];

%% Defining the Matrix B (6 x 1)

B = [0;
     1/M;
     0;
     (1/(M*l1));
     0;
     (1/(M*l2))];

%% Defining the Matrix C (6 x 6)

C = [B (A*B) (A^2*B) (A^3*B) (A^4*B) (A^5*B)];
RL = rank(C);

%% Defining D
D = det(C);
```

LQRController.m

```
%% Value of Parameters

m1 = 100; m2 = 100; M = 1000;
l1 = 20; l2 = 10;
g = 9.81;

%% Substituting the Values of Different Parameters in
Matrices

A = [0 1 0 0 0 0;
     0 0 -(m1*g)/M 0 -(m2*g)/M 0;
     0 0 0 1 0 0;
     0 0 -(M+m1)*g/(M*l1) 0 -(m2*g)/(M*l1) 0;
     0 0 0 0 0 1;
     0 0 -(m1*g)/(M*l2) 0 -(M+m2)*g/(M*l2) 0];

B = [0;
     1/M;
     0;
     1/(M*l1);
     0;
     1/(M*l2)];

C = [1 0 0 0 0 0;
     0 0 1 0 0 0;
     0 0 0 0 1 0];

D = 0;

%% Choosing the Values of Q & R

Q = (C')*(C);

% Assigning the Values in Q using Trial & Error Method

Q(1,1) = 70000000;
Q(3,3) = 8000000000;
Q(5,5) = 9000000000;

R = 1;                    % Ideal Value of R

%% Designing the LQR
```

```
% Calculating the Optimal Gain Matrix K

K = lqr(A, B, Q, R)

% Calculating the Closed Loop Matrix Value of A using
K

ANew = (A - (B*K));     % Values of Other Matrices
remain the same

%% Checking Controllability

RC = rank(C);

% Checking the Controllability

if (RC == min(size(C)))
    disp('System is Controllable');
else
    disp('System is Uncontrollable');
end

%% State Space Representation

States = {'x' 'xDot' 'theta1' 'theta1Dot' 'theta2'
'theta2Dot'};
Inputs = {'r'};
Outputs = {'x'; 'theta1'; 'theta2'};

% Creating the State Space Model

ContSS = ss(ANew, B, C, D, 'statename', States,
'inputname', Inputs, 'outputname', Outputs);

% Displaying the Eigen Values

EigenValues = eig(ANew)
```

LQRLinear.m

```
%% Value of Parameters

m1 = 100; m2 = 100; M = 1000;
l1 = 20; l2 = 10;
g = 9.81;

%% Substituting the Values of Different Parameters in
Matrices

A = [0 1 0 0 0 0;
     0 0 (-(m1*g)/M) 0 (-(m2*g)/M) 0;
     0 0 0 1 0 0;
     0 0 (-(M+m1)*g/(M*l1)) 0 (-(m2*g)/(M*l1)) 0;
     0 0 0 0 0 1;
     0 0 (-(m1*g)/(M*l2)) 0 (-(M+m2)*g/(M*l2)) 0];

B = [0;
     1/M;
     0;
     1/(M*l1);
     0;
     1/(M*l2)];

C = [1 0 0 0 0 0;
     0 0 1 0 0 0;
     0 0 0 0 1 0];

D = 0;

%% Choosing the Values of Q & R

Q = (C')*(C);

% Assigning the Values in Q using Trial & Error Method

Q(1,1) = 70000000;
Q(3,3) = 8000000000;
Q(5,5) = 9000000000;

% Selecting the Ideal Value of R

R = 1;
```

```
%% Designing the LQR

% Calculating the Optimal Gain Matrix K

K = lqr(A, B, Q, R);

% Calculating the New Value of A using K

ANew = (A - (B*K));

% Creating the Observability Matrix

States = {'x' 'x_dot' 'theta1' 'theta1_dot' 'theta2'
'theta2_dot'};
Inputs = {'r'};
Outputs = {'x'; 'phi1'; 'phi2'};

% Creating the State Space Model

ClosSS = ss(ANew, B, C, D,'statename', States,
'inputname', Inputs, 'outputname', Outputs);

%% Initializing Conditions

X0 = [0;
      0;
      10*pi/180;
      0;
      15*pi/180;
      0];

t = 0:0.01:150;

Temp = size(t)
F = zeros(Temp);

% Simulating the Time Response of Dynamic System to
Arbitrary Inputs

[Y, tTemp, XTemp] = lsim(ClosSS, F, t, X0);


%% Plotting the Output
```

```
figure,
plot(t, Y(:,1));
xlabel('Time'); ylabel('Crane Position');

figure,
plot(t, Y(:,2));
xlabel('Time'); ylabel('Pendulum Angle1');

figure,
plot(t, Y(:,3));
xlabel('Time'); ylabel('Pendulum Angle2');
```

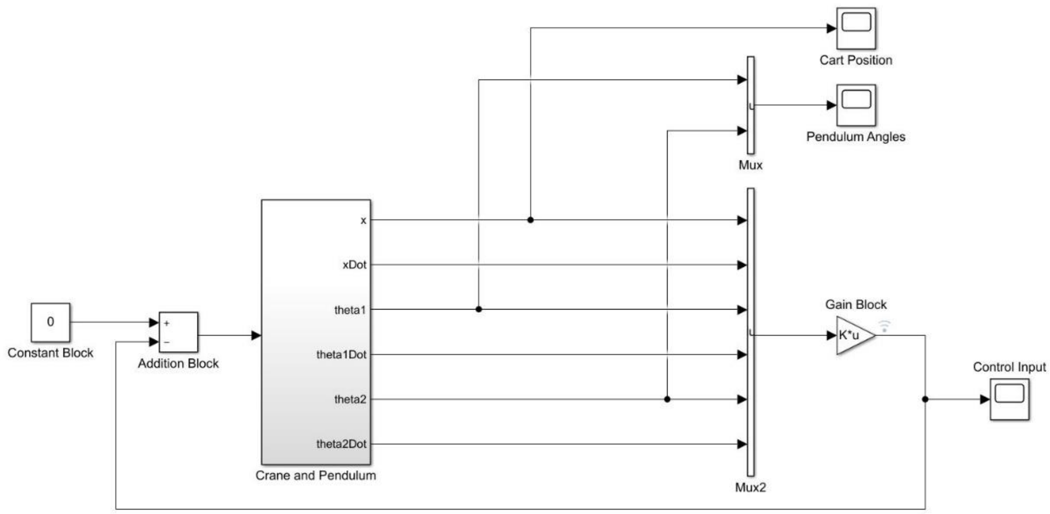
LQRObserver.m

```
%% Value of Parameters

m1 = 100; m2 = 100; M = 1000;
l1 = 20; l2 = 10;
g = 9.81;

%% Substituting the Values of Different Parameters in
Matrices

A = [0 1 0 0 0 0;
     0 0 (-(m1*g)/M) 0 (-(m2*g)/M) 0;
     0 0 0 1 0 0;
     0 0 (-(M+m1)*g/(M*l1)) 0 (-(m2*g)/(M*l1)) 0;
     0 0 0 0 0 1;
     0 0 (-(m1*g)/(M*l2)) 0 (-(M+m2)*g/(M*l2)) 0];

B = [0;
     1/M;
     0;
     1/(M*l1);
     0;
     1/(M*l2)];

% Uncomment one of the Cases and Execute the Code for
Different Conditions

% Case-1
C = [1 0 0 0 0 0];
Outputs = {'x'};

% Case-2
% C = [0 0 1 0 0 0
%        0 0 0 0 1 0];
% Outputs = {'theta1', 'theta2'};

% Case-3
% C = [1 0 0 0 0 0
%        0 0 0 0 1 0];
% Outputs = {'x', 'theta2'};

% Case-4
% C = [1 0 0 0 0 0
```

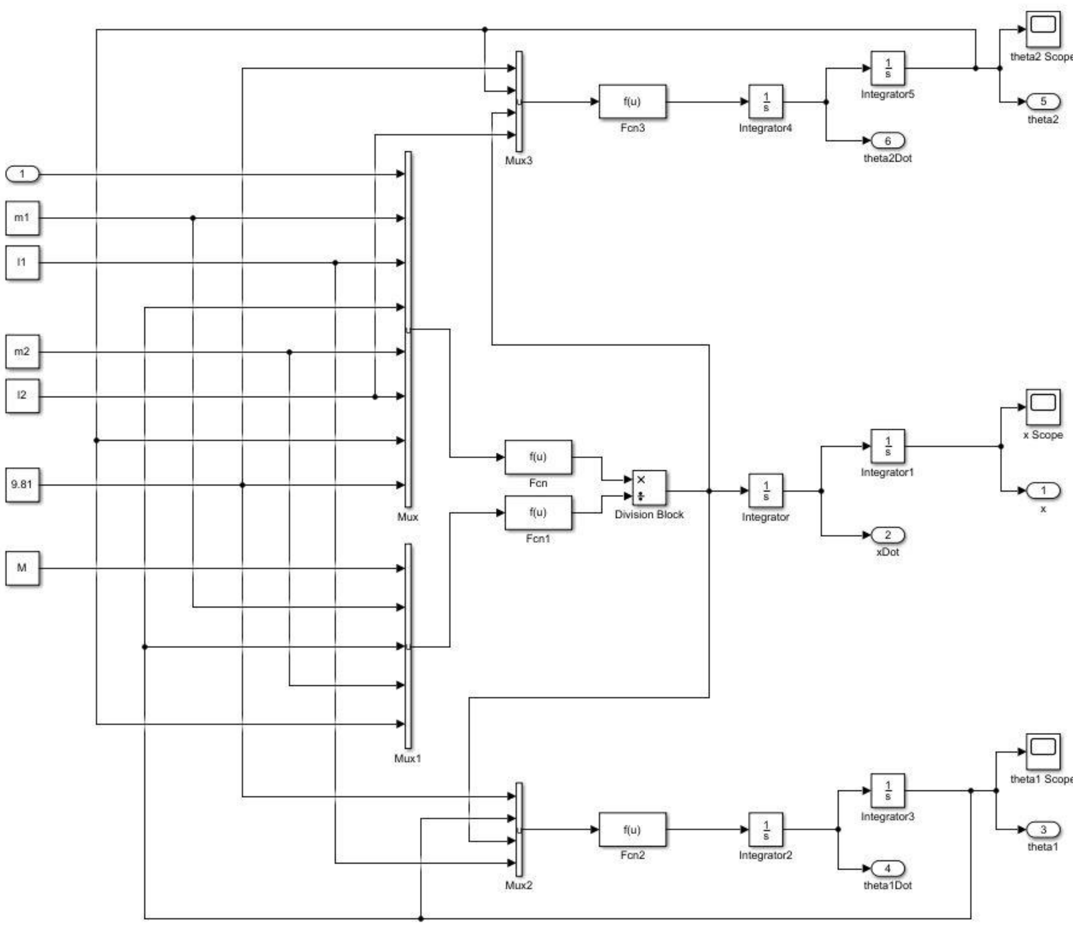
```
%       0  0  1  0  0  0
%       0  0  0  0  1  0];
% Outputs = {'x', 'theta1', 'theta2'};

D = 0;

P = [-0.2 -0.3 -0.4 -0.5 -0.6 -0.7];


X0 = [0;
      0;
      ((10*pi)/180);
      0;
      ((15*pi)/180);
      0];

Xhat = [0;
        0;
        ((10*pi)/180);
        0;
        ((15*pi)/180);
        0];

L = place(A', C', P)';

States = {'X', 'Xdot', 'phi1', 'phi1_dot', 'phi_2',
'phi2_dot'};
Inputs = {'F'};

SystSS = ss(A, B, C, D, 'statename', States,
'inputname', Inputs, 'outputname', Outputs);
O = obsv(SystSS);
RO = rank(O);

%% Checking the Controllability

if (RO == min(size(O)))
    disp('System is Observable');
else
    disp('System is Not Observable');
end

% Simulating the Time Response of Dynamic System to
Arbitrary Inputs
```

```
t = 0:0.01:100;
u = ones(size(t));

[Y,~,X] = lsim(SystSS,u,t,X0);

X_est = Xhat';
k = 2;

for n = 0.01:0.01:100
    dXhat = A * Xhat + B .* u(k) + L * (Y(k,:)' -
C*Xhat);
    Xhat = Xhat + 0.01.*dXhat;
    X_est = [X_est;Xhat'];
    k = k + 1;
end

%% Plotting the Output

figure,
plot(t, X(:,1));
hold on;

plot(t, X_est(:,1),'g');
xlabel('Time(sec)'), ylabel('Crane Position(m)'),
legend('X', 'X_est');
hold off;

figure,
plot(t, X(:,3));
hold on;

plot(t, X_est(:,3), 'g');
xlabel('Time(sec)'), ylabel('Pendulum
Angle(theta1)'),legend('theta1', 'theta1Est');
hold off;

figure,
plot(t, X(:,5));
hold on;

plot(t, X_est(:,5), 'g');
xlabel('Time(sec)'), ylabel('Pendulum Angle(theta2)'),
legend('theta2', 'theta2Est');
```

H. Simulink Code: LQR_Controller.slx

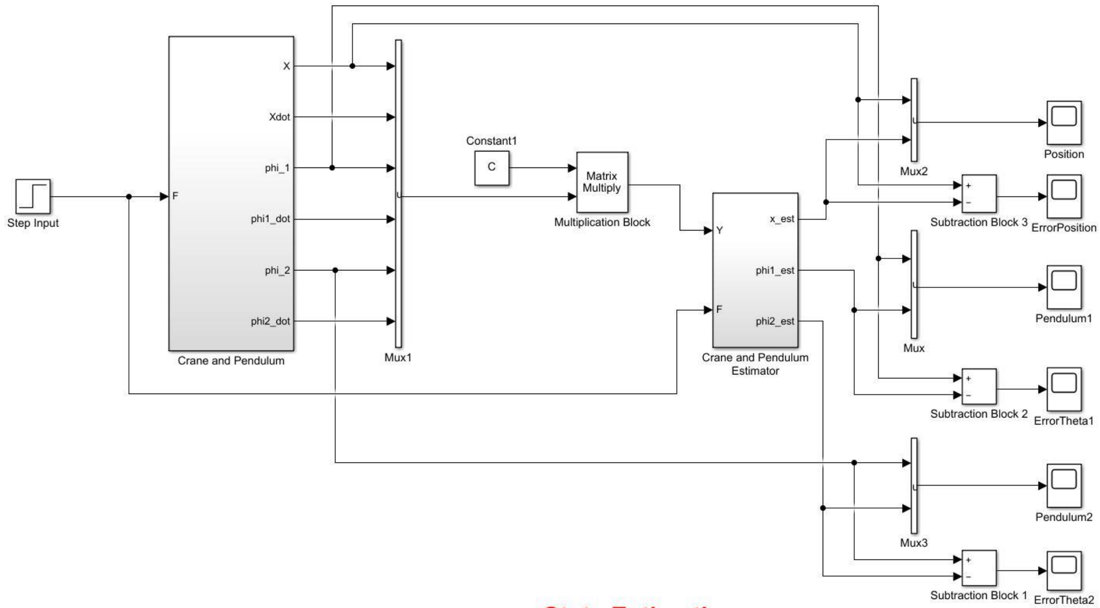


LQR Controller

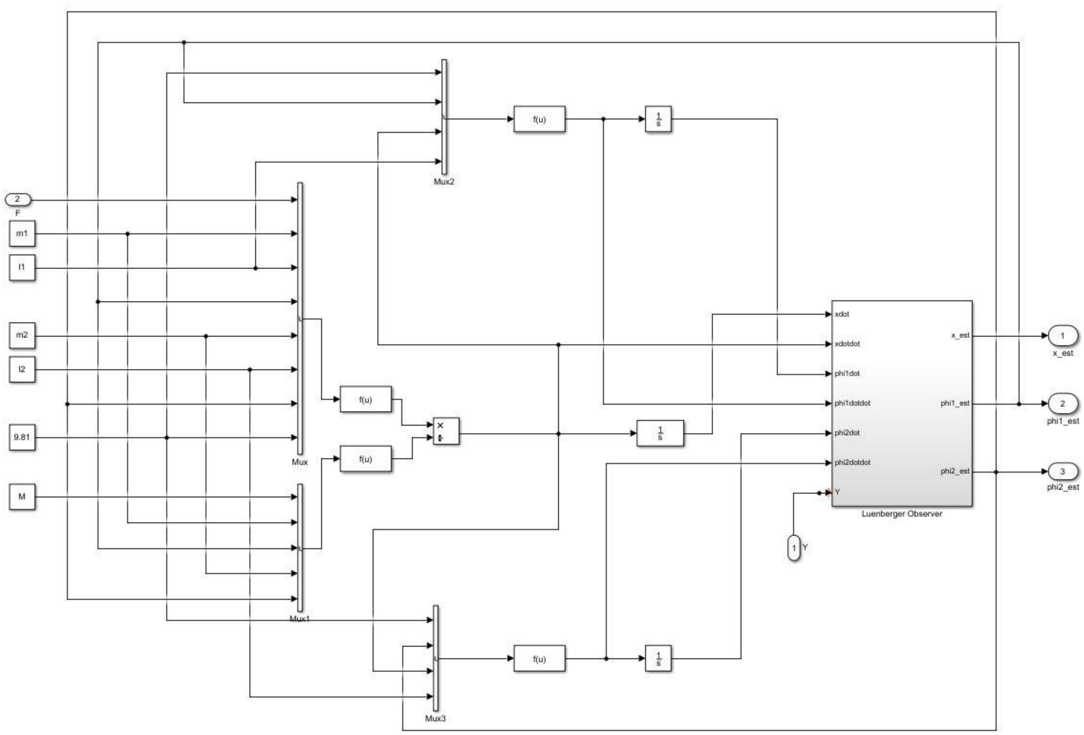


Non-linear Model of the Crane and Pendulum

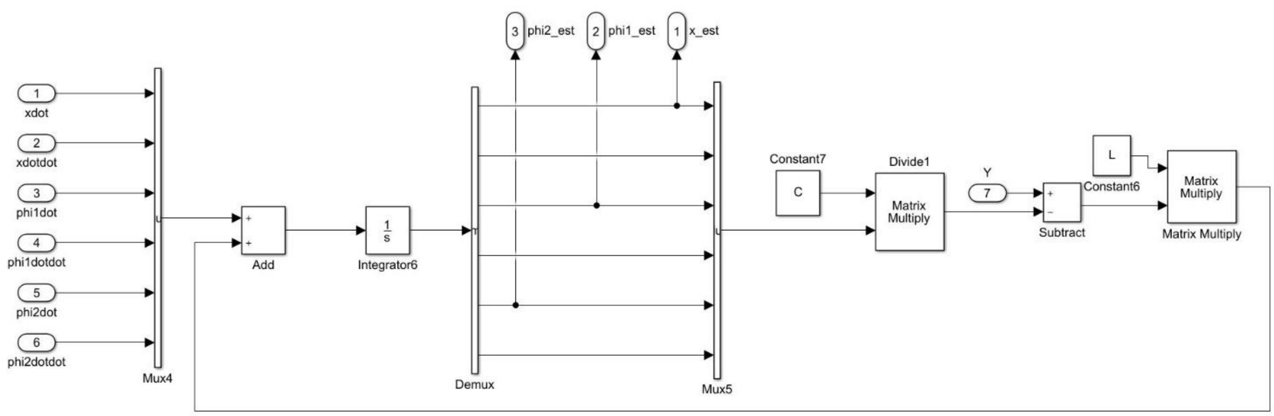
LQR_Observer.slx



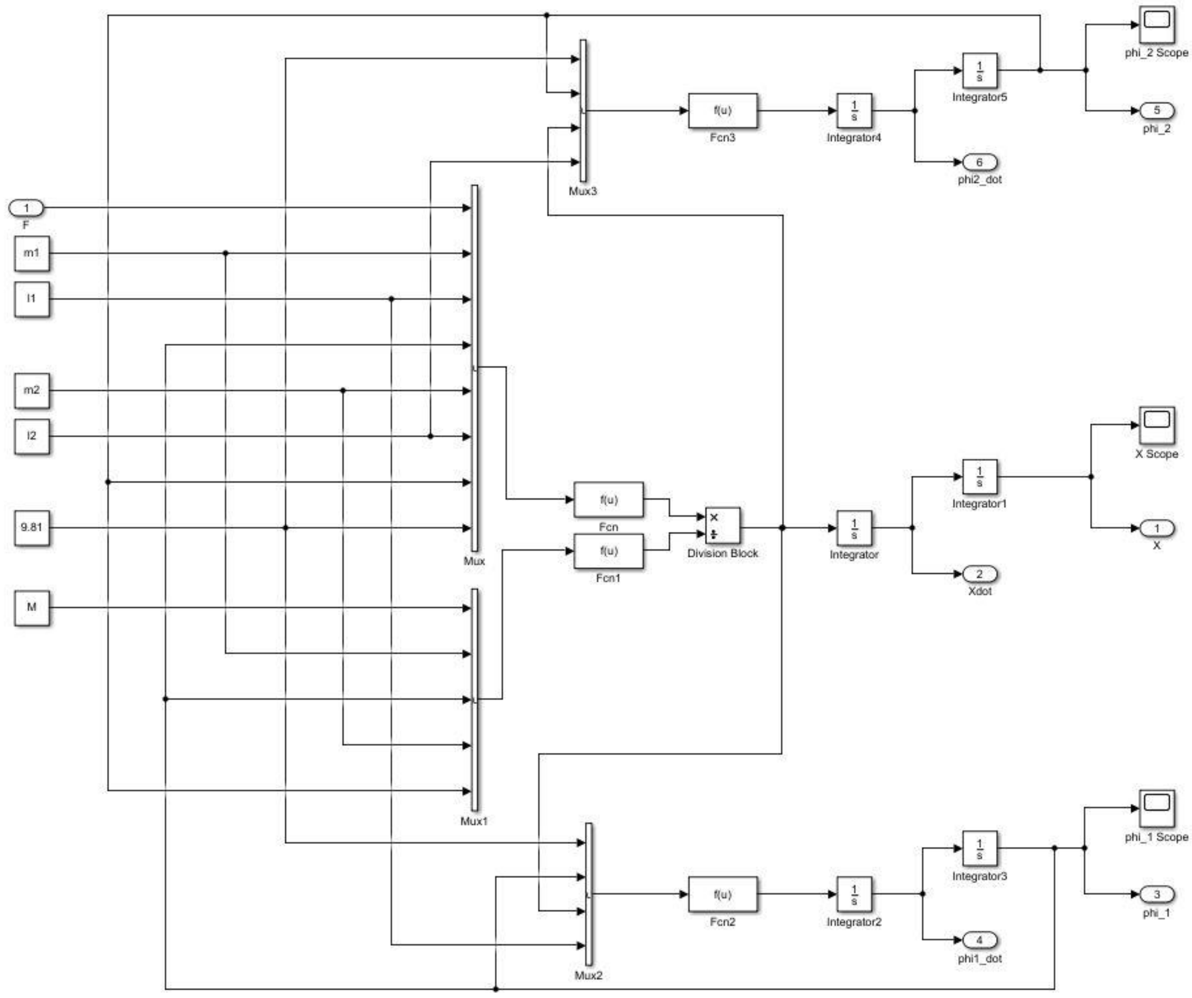
State Estimation



Crane and Pendulum Estimator



Luenberger Observer



Crane and Pendulum