Complex Numbers

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In the **real** number system, the solution for $x^2 = -1$ is undefined. This is because of the fact that the square of any real number is always non-negative.

In order to solve such equations, we extend the real number system to the **Complex Numbers**.

$$\boxed{\begin{aligned} \sqrt{-1} &= i \\ i^2 &= -1 \end{aligned}}$$

We define a value i to be the square root of -1.

> Here, i is called a *imaginary number*, since it is not present in the real numbers.

Complex Numbers

A Complex Number is a number which has both a real and imaginary component.

$$z = a + ib \qquad (a \in R, b \in R)$$

$$\text{Re } z = a$$

$$\text{Im } z = b$$

Examples: $2 + 5i, \quad 6 + i\sqrt{3}, \quad -\dfrac{1}{11} + i\dfrac{2}{11}$

Equality

$$z_1 = a + ib$$
$$z_2 = c + id$$
$$\text{if } a = c \text{ and } b = d, \implies z_1 = z_2$$

Square Root of a Negative Number

$$\sqrt{-1} = i$$
$$\sqrt{-5} = i\sqrt{5}$$
$$\sqrt{-16} = 4i$$
$$\sqrt{-a} = i\sqrt{a}$$

Powers of i

$$\begin{aligned} i^1 &= i & &= i^{4k} \\ i^2 &= -1 & &= i^{4k+1} \\ i^3 &= -i & &= i^{4k+2} \\ i^4 &= 1 & &= i^{4k+3} \end{aligned}$$

Operations

Addition and Subtraction

$$z_1 = a + ib$$
$$z_2 = c + id$$
$$z_1 + z_2 = (a + c) + i(b + d)$$

Multiplication

$$z_1 = a + ib$$
$$z_2 = c + id$$

$$\begin{aligned} z_1 z_2 &= (a + ib)(c + id) \\ &= ac + adi + bci + bdi^2 \\ &= (ac - bd) + i(ad + bc) \end{aligned}$$

Modulus

Modulus of a complex number is defined as the square root of the sum of the squares of the real and imaginary components.

$$z = a + ib$$
$$|z| = \sqrt{a^2 + b^2}$$

Conjugate

The conjugate of a complex number is derived by inverting the sign of the imaginary component.

$$z = a + ib$$
$$\overline{z} = a - ib$$

Example

$$|-3 + 4i| = \sqrt{(-3)^2 + 4^2} = 5$$
$$\overline{-3 + 4i} = -3 - 4i$$

Inverse (Reciprocal)

For any complex number z, its inverse is defined as:

$$\frac{1}{z} = z^{-1} = \frac{\overline{z}}{|z|^2}$$

Example

$$\frac{1}{-3+4i} = \frac{\overline{-3+4i}}{(-3)^2+4^2} = \frac{-3-4i}{25}$$
$$= \frac{-3}{25} - i\frac{4}{25}$$

Division

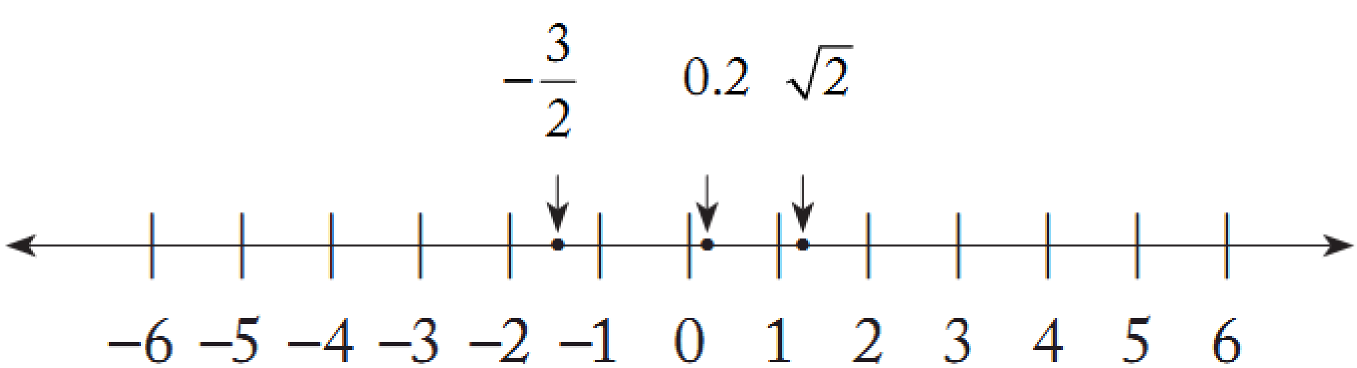
$$z_1 = a + ib$$
$$z_2 = c + id$$
$$\frac{z_1}{z_2} = z_1 \times \frac{1}{z_2}$$

Modulus and Conjugate Properties

- $|z_1 z_2| = |z_1||z_2|$
- $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
- $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$
- $\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}$

Visual Representation

We know that all real numbers can be visually represented using the **real number line**.



All real numbers $(2, -7/5, \sqrt{3}, \pi, \text{ etc.})$ are distinct points on this number line.

When we extend the number system to include imaginary numbers (i), we need to add another dimension to represent *all complex numbers.*

The **Complex Number Plane** is the visual representation of the set of all Complex Numbers, where the x-axis represents *real numbers* and the y-axis represents *imaginary numbers*

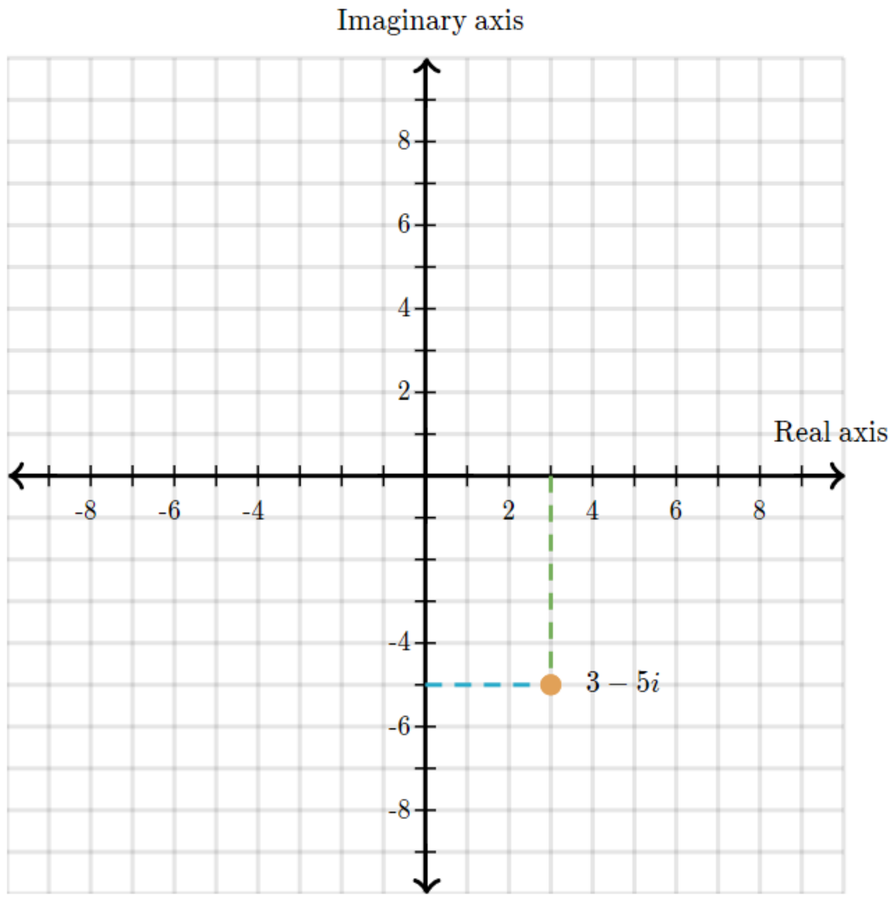
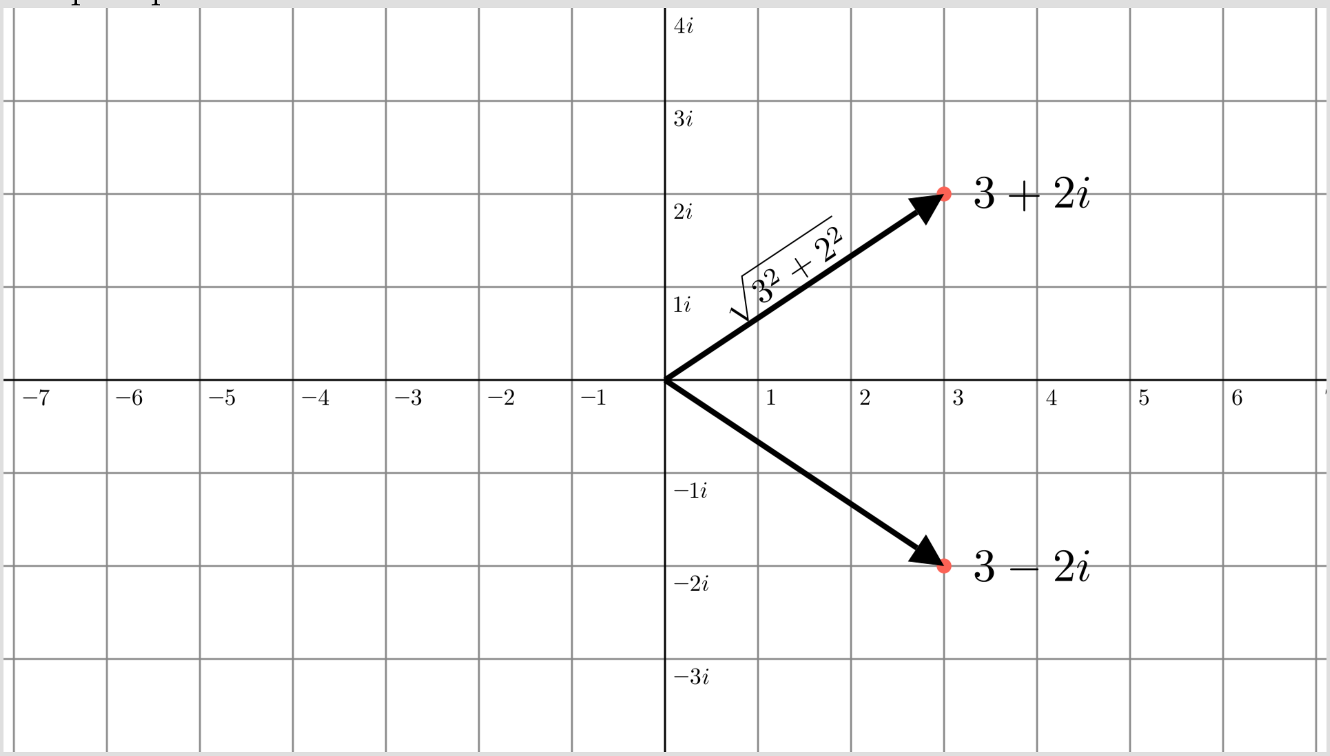


Figure 1: The Complex Number Plane

Each point on this plane represents a specific complex number.

Conversely, all complex numbers $(3+i, \frac{-3}{2}+i\sqrt{2}, 57, \text{ etc.})$ are represented by a point on the plane.

For any complex number, the modulus and conjugate can be visualised on the complex plane.



The modulus of the number is the distance of the point from the origin 0. The conjugate of the number is the mirror image of the number on the Real axis.

Cartesian and Polar Coordinates

In the **Cartesian Coordinate System**, we use two numbers - the x-coordinate and y-coordinate to locate a point on the plane.

Another system of Coordinates is the **Polar Coordinate System**. Here, the two numbers used to represent a point in a plane are:

$$\text{Polar Coordinates: } (r, \theta)$$

$$r = \text{distance from origin}$$

$$\theta = \text{angle with horizontal axis}$$

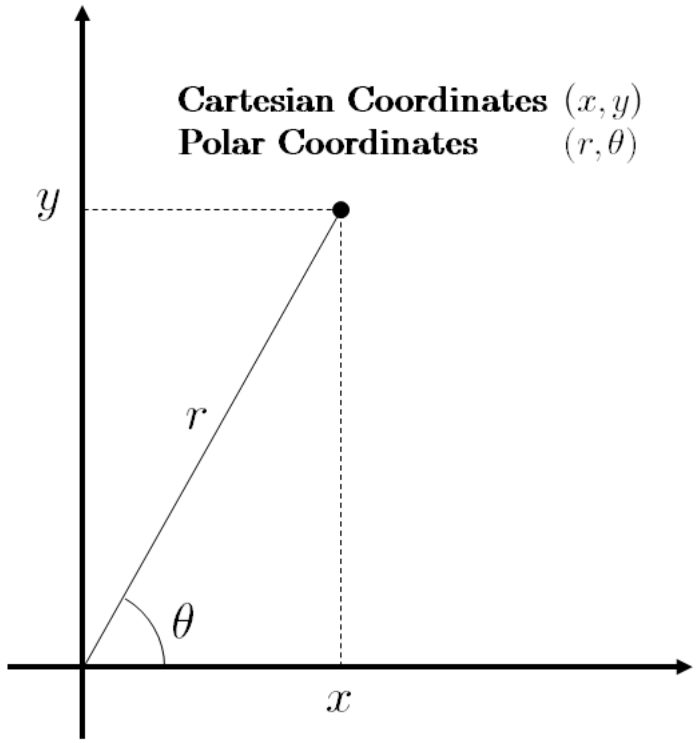


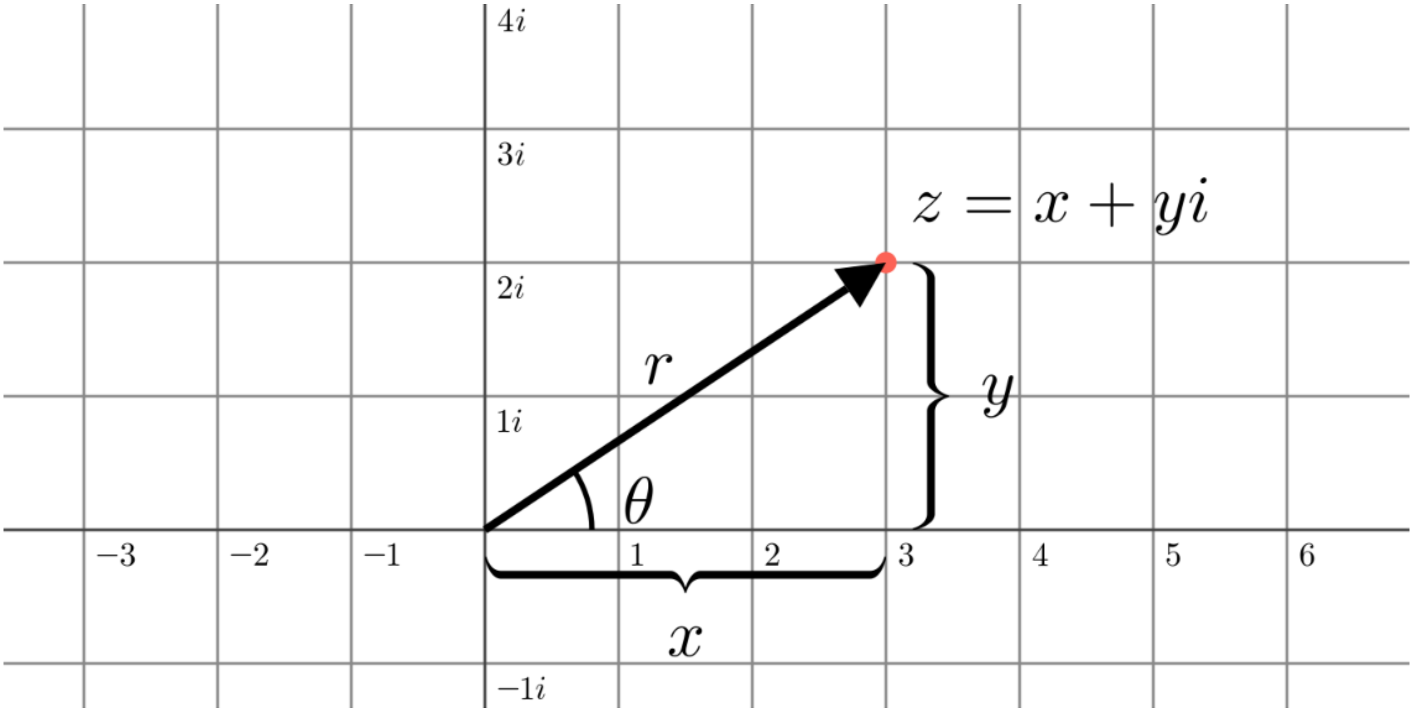
Figure 2: Cartesian and Polar Coordinates

For example, the point $P(5, \pi/3)$ represents a point which is 5 units away from the origin, making a 60° angle with the x-axis.

Complex Numbers in Polar Form

When we represent complex numbers, we use the cartesian form (x, y) where $x = \text{Re } z$ and $y = \text{Im } z$. We can also use the polar system to do so.

Let z be a complex number.



Here, we can calculate the x and y coordinates using **trigonometry**.

$$x = r \cdot \cos\theta$$
$$y = r \cdot \sin\theta$$

$$z = r\cos\theta + ir\sin\theta$$
$$\boxed{z = r(\cos\theta + i\sin\theta)}$$

$$(r = \sqrt{x^2 + y^2} = |z|)$$

This is known as the Polar Representation of a Complex Number.

Here, θ is known as the **argument** of z.

There will be multiple values of θ which satisfy the sin and cos. By convention, we use the value $-\pi < \theta \leq \pi$

Example Find modulus and argument of $1 - i$. Also convert it to the polar form.

$$z = 1 - i$$
$$r = |z| = \sqrt{1^2 + (-1)^2}$$
$$\boxed{r = \sqrt{2}}$$
$$\cos\theta = \frac{\text{Re } z}{r} = \frac{1}{r} = \frac{1}{\sqrt{2}}$$
$$\sin\theta = \frac{\text{Im } z}{r} = \frac{-1}{r} = \frac{-1}{\sqrt{2}}$$

sin is -ve and cos is +ve $\implies$ Angle in Quadrant IV

$$\boxed{\theta = \frac{-\pi}{4}}$$

Polar Form:

$$\boxed{z = \sqrt{2}\left(\cos\frac{-\pi}{4} + i\sin\frac{-\pi}{4}\right)}$$