Trigonometry

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Degree Measurement of Angles

Usually, we measure angles in degrees, where $360°$ is one complete rotation.

$$1° = 60'$$
$$1' = 60''$$

Radian Measurement of Angles

Radians is another unit for measuring angles, where a complete rotation is marked as 2π radians.

$$360° = 2\pi$$

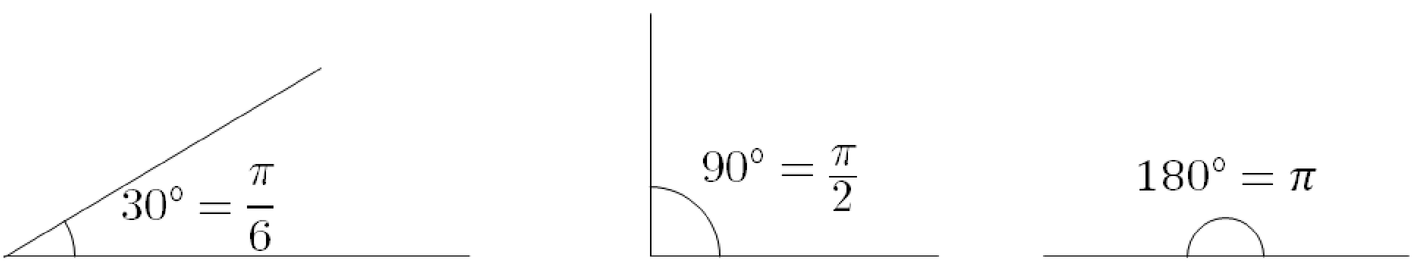

Figure 1: Some angles in Degrees and Radians

- A unit circle is a circle with radius 1 unit.
- The unit ‘radian’ is often omitted while writing radian angles.

Definition

The angle subtended at the centre by an arc of length x in a unit circle is x rad.

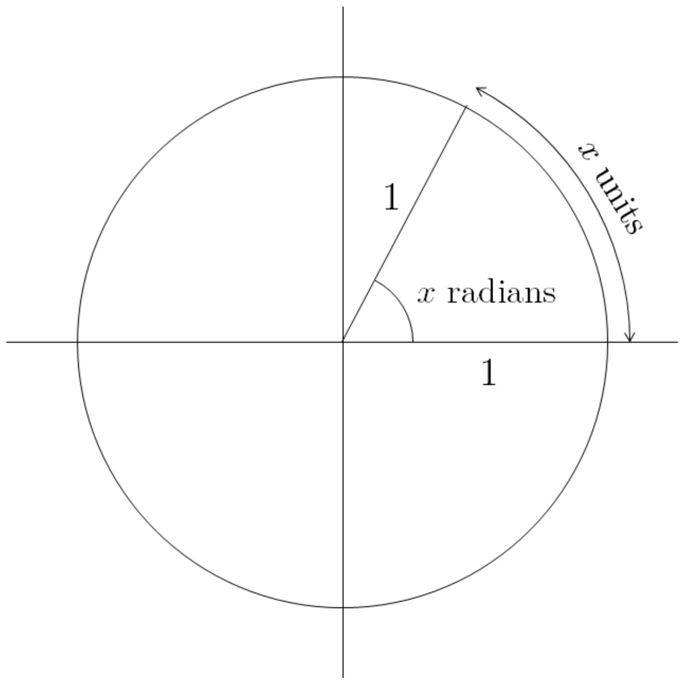

Figure 2: Definition

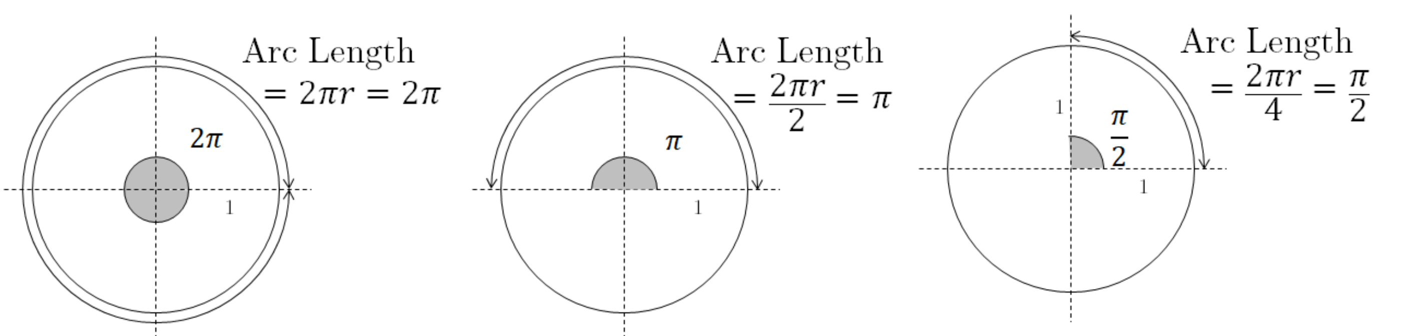

Figure 3: Illustration

$$\boxed{\text{Radian measure} = \frac{\text{Arc length}}{\text{Radius}}}$$

Degree-Radian Conversion

$$180° = \pi$$
$$1 \text{ radian} = \frac{180°}{\pi} = 57°16'$$
$$1° = \frac{\pi}{180} = 0.01746 \text{ radian}$$

$$\boxed{\text{Radian measure} = \frac{\pi}{180} \times \text{Degree measure}}$$

$$\boxed{\text{Degree measure} = \frac{180}{\pi} \times \text{Radian measure}}$$

Trigonometric Functions

(See Figure 4)

$\sin\theta$ = The **y-coordinate** of the point on the unit circle
$\cos\theta$ = The **x-coordinate** of the point on the unit circle

Quadrantal Angles: Integral multiples of $\frac{\pi}{2}$

Angle	**0**	$\boldsymbol{\frac{\pi}{2}}$	$\boldsymbol{\pi}$	$\boldsymbol{\frac{3\pi}{2}}$	$\boldsymbol{2\pi}$
sin x	0	1	0	-1	0
cos x	1	0	-1	0	1

Value of 0

$\sin\theta$ is 0 for the values $\theta = 0, \pi, 2\pi, \cdots$

$\cos\theta$ is 0 for the values $\theta = \frac{\pi}{2}, 3\frac{\pi}{2}, \cdots$

$$\left.\begin{aligned} \therefore \sin\theta = 0 &\implies \theta = n\pi \\ \cos\theta = 0 &\implies \theta = (2n+1)\frac{\pi}{2} \end{aligned}\right\} n \in Z$$

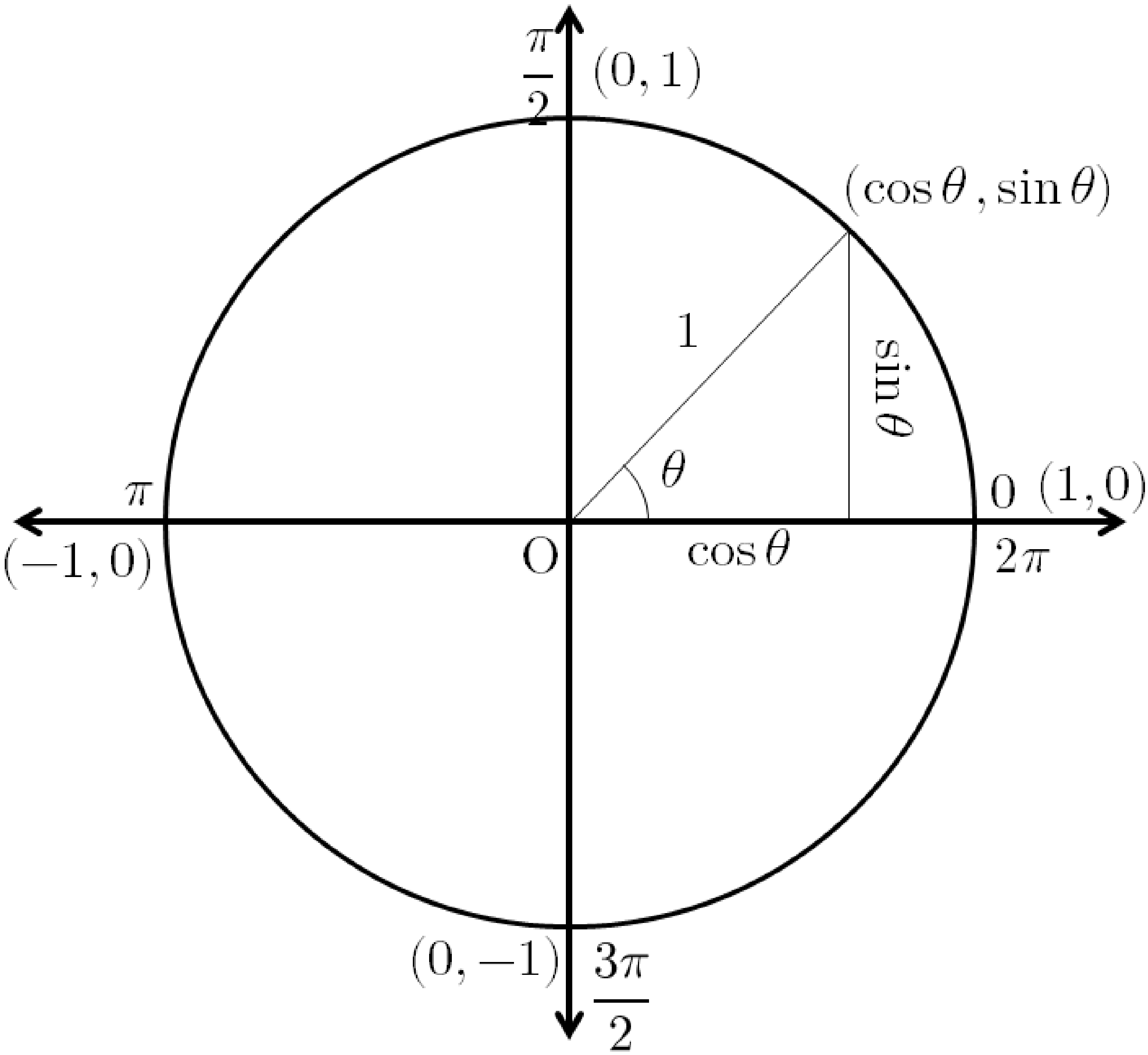

Figure 4: Trigonometric Functions in a unit circle

Other Trigonometric Functions

$$\tan x = \frac{\sin x}{\cos x}$$
$$\cot x = \frac{\cos x}{\sin x}$$
$$\operatorname{cosec} x = \frac{1}{\sin x}$$
$$\sec x = \frac{1}{\cos x}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$
$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \cot^2\theta = \operatorname{cosec}^2\theta$$

Sign of Trigonometric Functions

Negative Angle

From the unit circle, we can determine the value of functions for a negative angle.

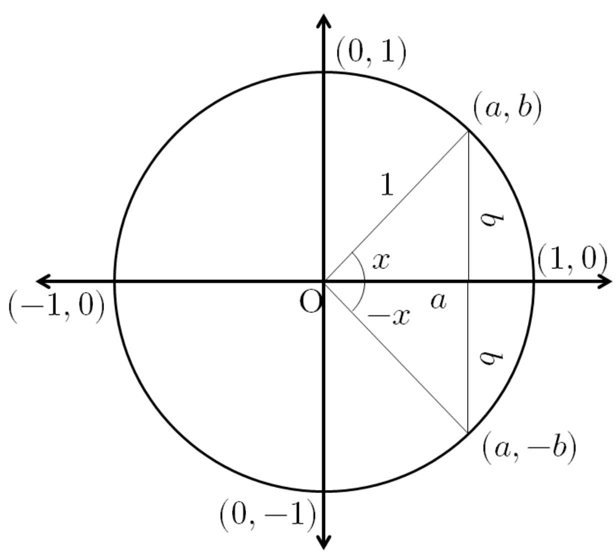

Figure 5: Negative Angle

We can see that $\sin x = b; \cos x = a$

For negative value of x (Figure 5), *the x coordinate remains unchanged, while the y coordinate becomes negative.*

$$\cos(-x) = \cos x$$
$$\sin(-x) = -\sin x$$

Sign in different Quadrants

Quadrant 1: a and b are both positive
Quadrant 2: a is negative and b is positive
Quadrant 3: a and b are both negative
Quadrant 4: a is positive and b is negative

From the signs of $\sin x$ and $\cos x$, we can find out the signs of all other trigonometric functions.

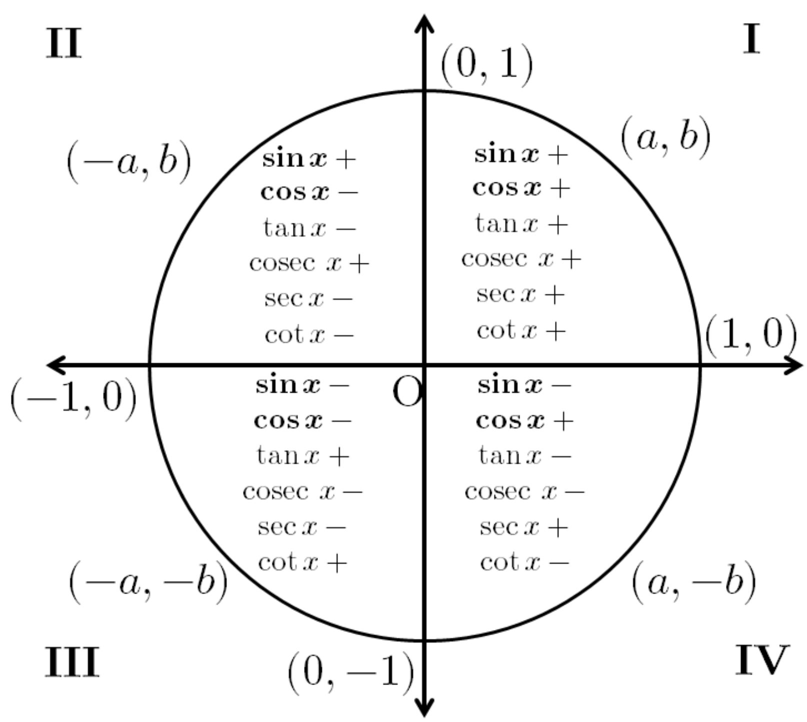

Figure 6: Signs of Trigonometric Functions

Domain and Range

$\sin x$

$$\text{Domain} = R$$
$$\text{Range} = [-1, 1]$$

$\cos x$

$$\text{Domain} = R$$
$$\text{Range} = [-1, 1]$$

$\text{cosec } x$

$$\text{Domain} = \{x : x \in R; x \neq n\pi, n \in Z\}$$
$$\text{Range} = (-\infty, -1] \cup [1, \infty)$$

$\sec x$

$$\text{Domain} = \{x : x \in R; x \neq (2n+1)\frac{\pi}{2}, n \in Z\}$$
$$\text{Range} = (-\infty, -1] \cup [1, \infty)$$

$\tan x$

$$\text{Domain} = \{x : x \in R; x \neq (2n+1)\frac{\pi}{2}, n \in Z\}$$
$$\text{Range} = R$$

$\cot x$

$$\text{Domain} = \{x : x \in R; x \neq n\pi, n \in Z\}$$
$$\text{Range} = R$$

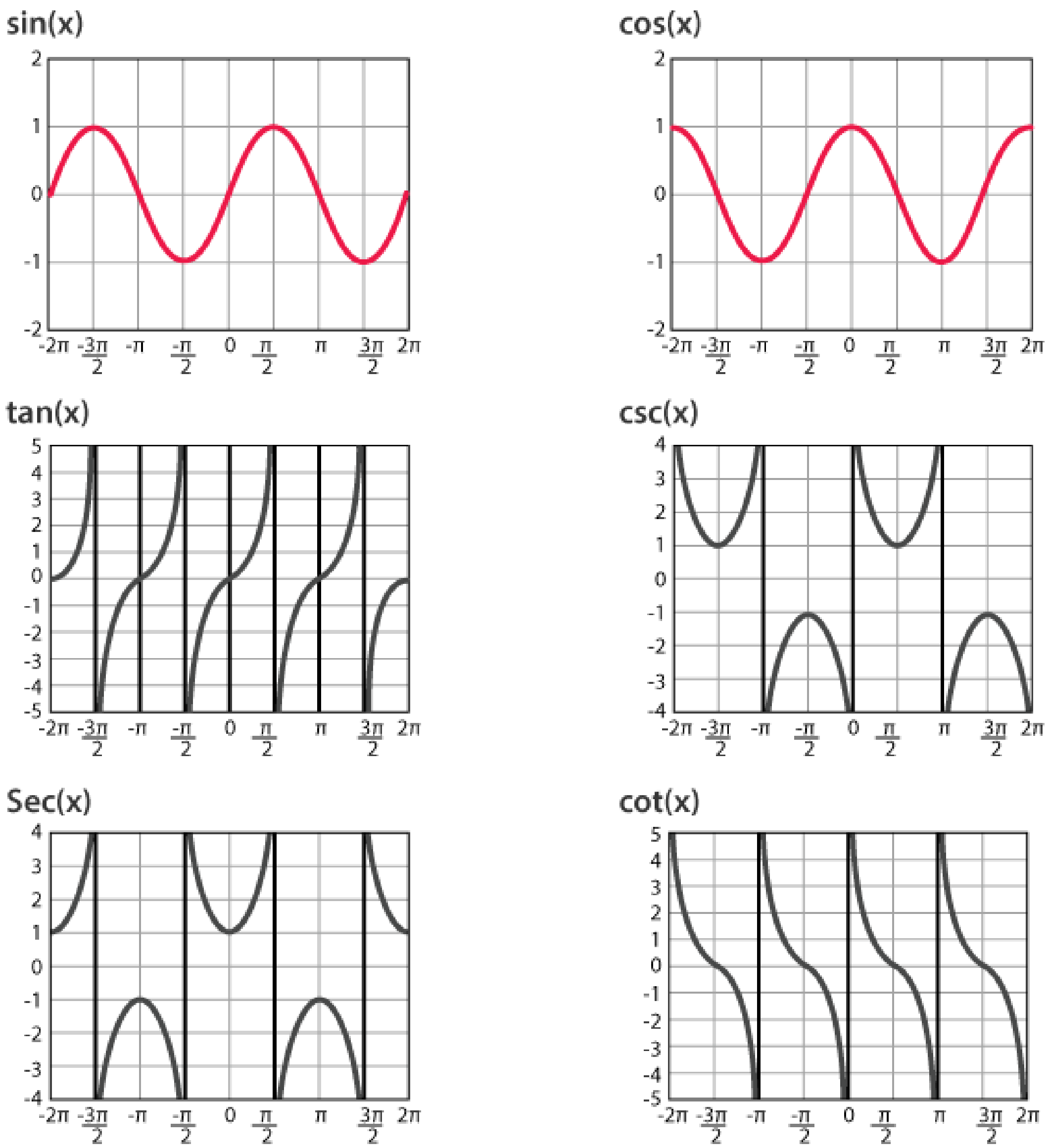

Figure 7: Graphs of Trigonometric Functions

Trigonometric Identities

- **Negative Angle***

1. $\sin(-a) = -\sin a$
2. $\cos(-a) = \cos a$
3. $\tan(-a) = -\tan a$

- **Sum or Difference of Angles**

4. $\sin(a+b) = \sin a \cos b + \cos a \sin b$
5. $\sin(a-b) = \sin a \cos b - \cos a \sin b$
6. $\cos(a+b) = \cos a \cos b - \sin a \sin b$
7. $\cos(a-b) = \cos a \cos b + \sin a \sin b$
8. $\tan(a+b) = \dfrac{\tan a + \tan b}{1 - \tan a \tan b}$
9. $\tan(a-b) = \dfrac{\tan a - \tan b}{1 + \tan a \tan b}$
10. $\cot(a+b) = \dfrac{\cot a \cot b - 1}{\cot b + \cot a}$
11. $\cot(a-b) = \dfrac{\cot a \cot b + 1}{\cot b - \cot a}$

- **Multiples of $\frac{\pi}{2}$***

12. $\sin\left(\dfrac{\pi}{2} - x\right) = \cos x$
13. $\sin\left(\dfrac{\pi}{2} + x\right) = -\cos x$
14. $\sin(\pi - x) = \sin x$
15. $\sin(\pi + x) = -\sin x$
16. $\sin(2\pi - x) = -\sin x$
17. $\cos\left(\dfrac{\pi}{2} - x\right) = \sin x$
18. $\cos\left(\dfrac{\pi}{2} + x\right) = \sin x$

19. $\cos(\pi - x) = -\cos x$
20. $\cos(\pi + x) = -\cos x$
21. $\cos(2\pi - x) = \cos x$

- **Double Angle**

22. $\sin 2x = 2\sin x \cos x$
23. $$\begin{aligned}\cos 2x &= 2\cos^2 x - 1\\ &= 1 - 2\sin^2 x\\ &= \cos^2 x - \sin^2 x\end{aligned}$$
24. $\tan 2x = \dfrac{2\tan x}{1 + \tan^2 x}$

- **Triple Angle**

25. $\sin 3x = 3\sin x - 4\sin^3 x$
26. $\cos 3x = 4\cos^3 x - 3\cos x$
27. $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$

- **Sum/Difference of Sines and Cosines**

28. $\cos a + \cos b = 2\cos\left(\dfrac{a+b}{2}\right)\cos\left(\dfrac{a-b}{2}\right)$
29. $\cos a - \cos b = -2\sin\left(\dfrac{a+b}{2}\right)\sin\left(\dfrac{a-b}{2}\right)$
30. $\sin a + \sin b = 2\sin\left(\dfrac{a+b}{2}\right)\cos\left(\dfrac{a-b}{2}\right)$
31. $\sin a - \sin b = 2\cos\left(\dfrac{a+b}{2}\right)\sin\left(\dfrac{a-b}{2}\right)$

- **Product of Sines and Cosines (Inverse of 28-31)**

32. $2\cos a\cos b = \cos(a+b) + \cos(a-b)$
33. $-2\sin a\sin b = \cos(a+b) - \cos(a-b)$
34. $2\sin a\cos b = \sin(a+b) + \sin(a-b)$
35. $2\cos a\sin b = \sin(a+b) - \sin(a-b)$

- **Zero Values***

36. $\sin x = 0 \implies x = n\pi, n \in Z$
37. $\cos x = 0 \implies x = (2n+1)\dfrac{\pi}{2}, n \in Z$

- **Equal Values***

38. $\sin x = \sin y \implies x = n\pi + (-1)^n y$
39. $\cos x = \cos y \implies x = 2n\pi \pm y$
40. $\tan x = \tan y \implies x = n\pi + y$

*These identities can be easily represented by the Unit Circle.