



Trigonometry

Dipam Sen (Fun Planet)

Degree Measurement of Angles

Usually, we measure angles in degrees, where 360° is one complete rotation.

$$1^\circ = 60'$$

$$1' = 60''$$

Radian Measurement of Angles

Radians is another unit for measuring angles, where a complete rotation is marked as 2π radians.

$$360^\circ = 2\pi$$

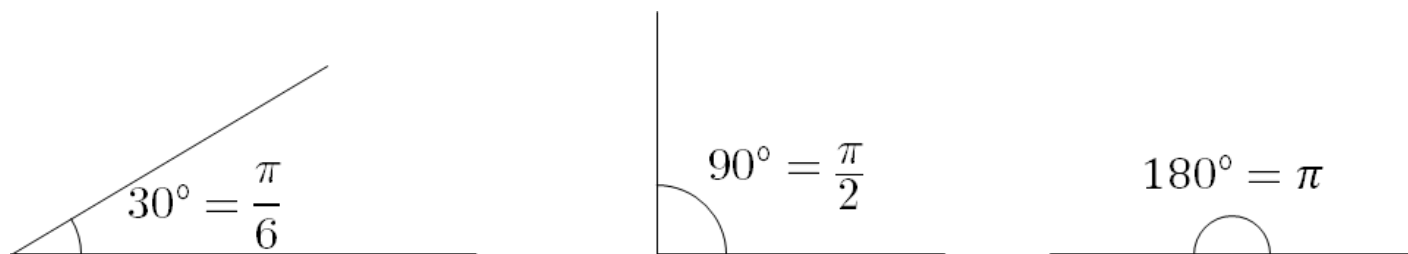


Figure 1: Some angles in Degrees and Radians

- A unit circle is a circle with radius 1 unit.
- The unit 'radian' is often omitted while writing radian angles.



Definition

The angle subtended at the centre by an arc of length x in a unit circle is x rad.

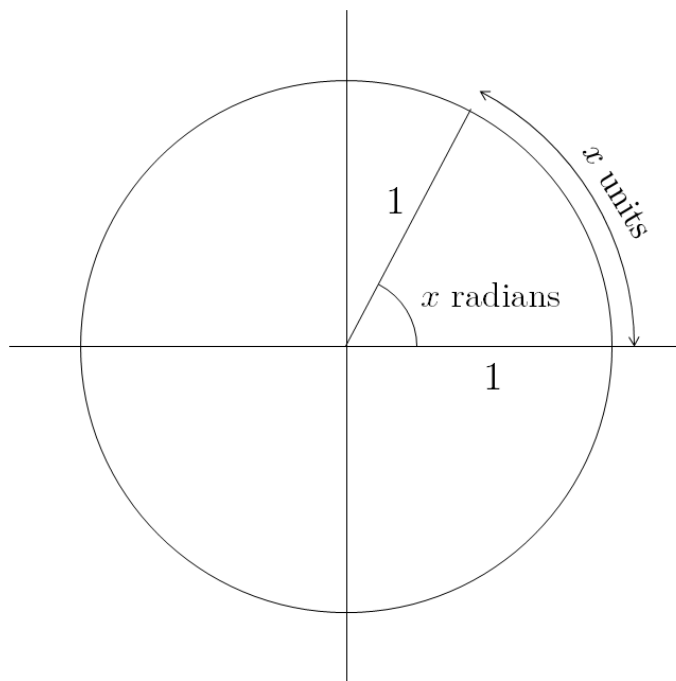


Figure 2: Definition

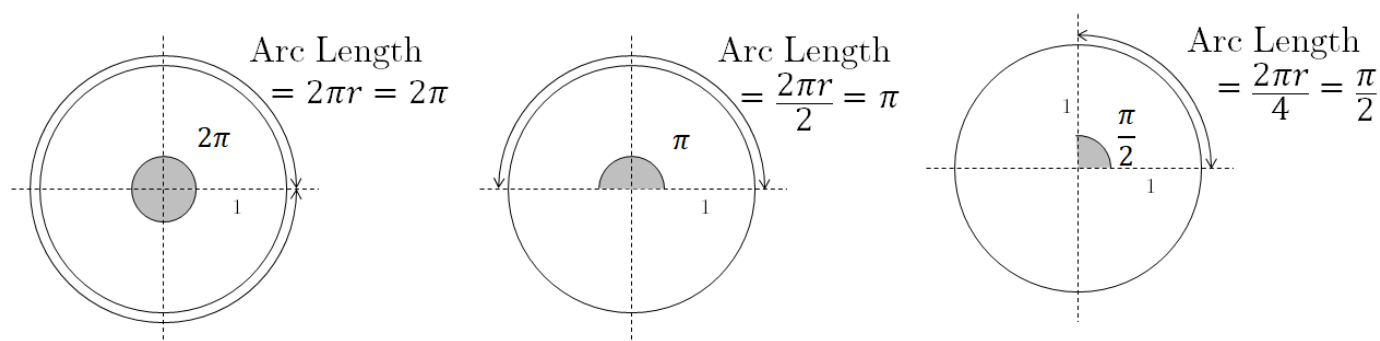


Figure 3: Illustration

$$\text{Radian measure} = \frac{\text{Arc length}}{\text{Radius}}$$



Degree-Radian Conversion

$$180^\circ = \pi$$

$$1 \text{ radian} = \frac{180^\circ}{\pi} = 57^\circ 16'$$

$$1^\circ = \frac{\pi}{180} = 0.01746 \text{ radian}$$

$$\text{Radian measure} = \frac{\pi}{180} \times \text{Degree measure}$$

$$\text{Degree measure} = \frac{180}{\pi} \times \text{Radian measure}$$

Trigonometric Functions

(See Figure 4)

$\sin \theta$ = The **y-coordinate** of the point on the unit circle
 $\cos \theta$ = The **x-coordinate** of the point on the unit circle

Quadrantal Angles: Integral multiples of $\frac{\pi}{2}$

Angle	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin x$	0	1	0	-1	0
$\cos x$	1	0	-1	0	1

Value of 0

$\sin \theta$ is 0 for the values $\theta = 0, \pi, 2\pi, \dots$

$\cos \theta$ is 0 for the values $\theta = \frac{\pi}{2}, 3\frac{\pi}{2}, \dots$

$$\left. \begin{aligned} \therefore \sin \theta = 0 &\implies \theta = n\pi \\ \cos \theta = 0 &\implies \theta = (2n+1)\frac{\pi}{2} \end{aligned} \right\} n \in Z$$

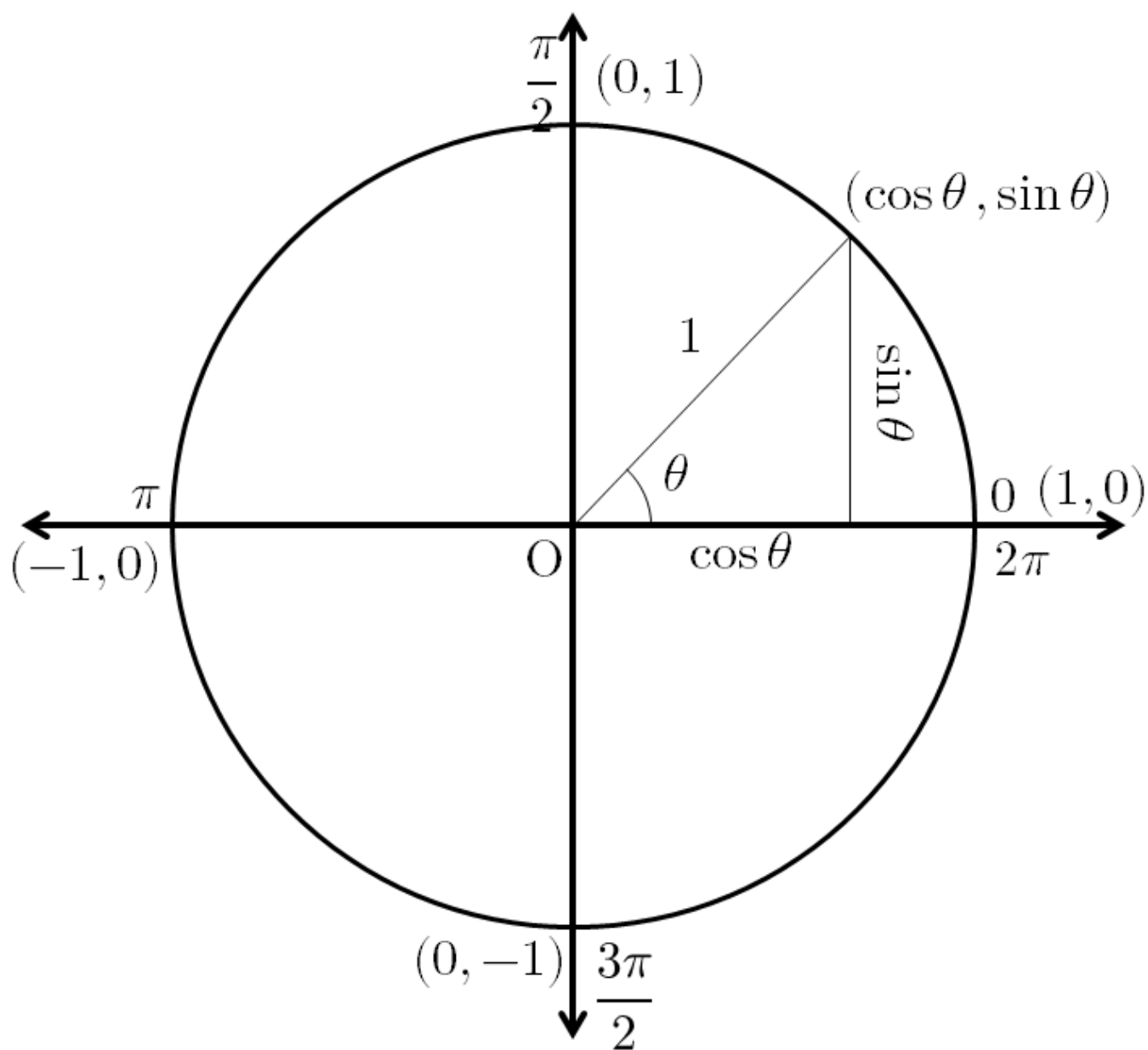


Figure 4: Trigonometric Functions in a unit circle



Other Trigonometric Functions

$$\tan x = \frac{\sin x}{\cos x}$$

$$\cot x = \frac{\cos x}{\sin x}$$

$$\operatorname{cosec} x = \frac{1}{\sin x}$$

$$\sec x = \frac{1}{\cos x}$$

Pythagorean Identities

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$1 + \cot^2 \theta = \operatorname{cosec}^2 \theta$$

Sign of Trigonometric Functions

Negative Angle

From the unit circle, we can determine the value of functions for a negative angle.

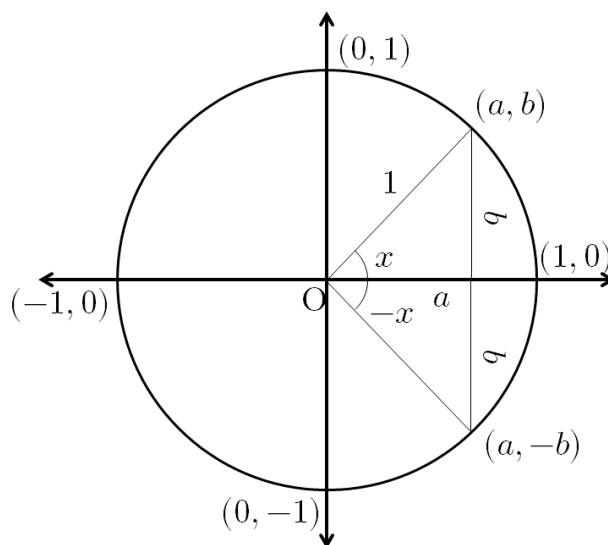


Figure 5: Negative Angle



We can see that $\sin x = b$; $\cos x = a$

For negative value of x (Figure 5), the x coordinate remains unchanged, while the y coordinate becomes negative.

$$\cos(-x) = \cos x$$

$$\sin(-x) = -\sin x$$

Sign in different Quadrants

Quadrant 1: a and b are both positive

Quadrant 2: a is negative and b is positive

Quadrant 3: a and b are both negative

Quadrant 4: a is positive and b is negative

From the signs of $\sin x$ and $\cos x$, we can find out the signs of all other trigonometric functions.

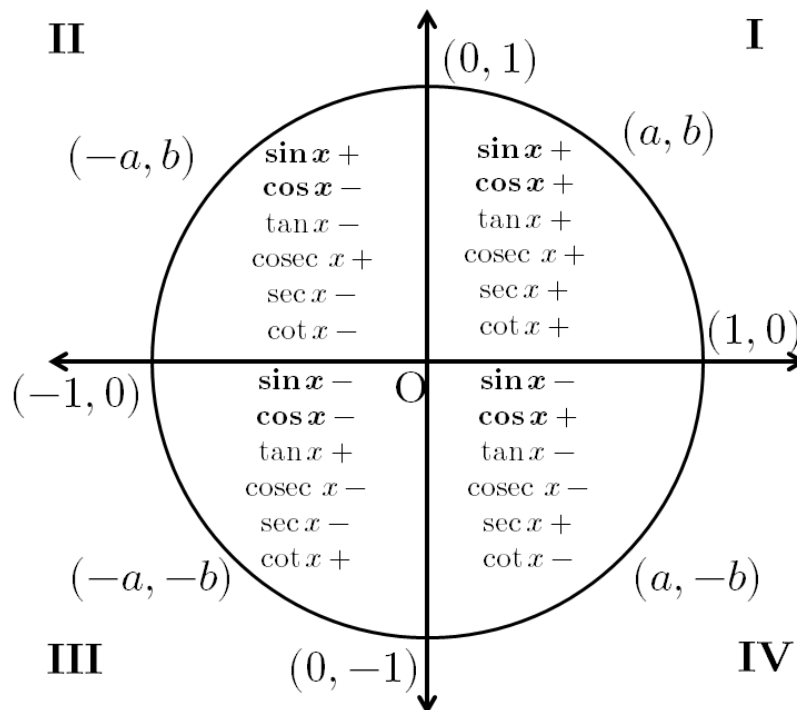


Figure 6: Signs of Trigonometric Functions



Domain and Range

$\sin x$

$$\text{Domain} = R$$

$$\text{Range} = [-1, 1]$$

$\cos x$

$$\text{Domain} = R$$

$$\text{Range} = [-1, 1]$$

$\operatorname{cosec} x$

$$\text{Domain} = \{x : x \in R; x \neq n\pi, n \in Z\}$$

$$\text{Range} = (-\infty, -1] \cup [1, \infty)$$

$\sec x$

$$\text{Domain} = \{x : x \in R; x \neq (2n+1)\frac{\pi}{2}, n \in Z\}$$

$$\text{Range} = (-\infty, -1] \cup [1, \infty)$$

$\tan x$

$$\text{Domain} = \{x : x \in R; x \neq (2n+1)\frac{\pi}{2}, n \in Z\}$$

$$\text{Range} = R$$

$\cot x$

$$\text{Domain} = \{x : x \in R; x \neq n\pi, n \in Z\}$$

$$\text{Range} = R$$

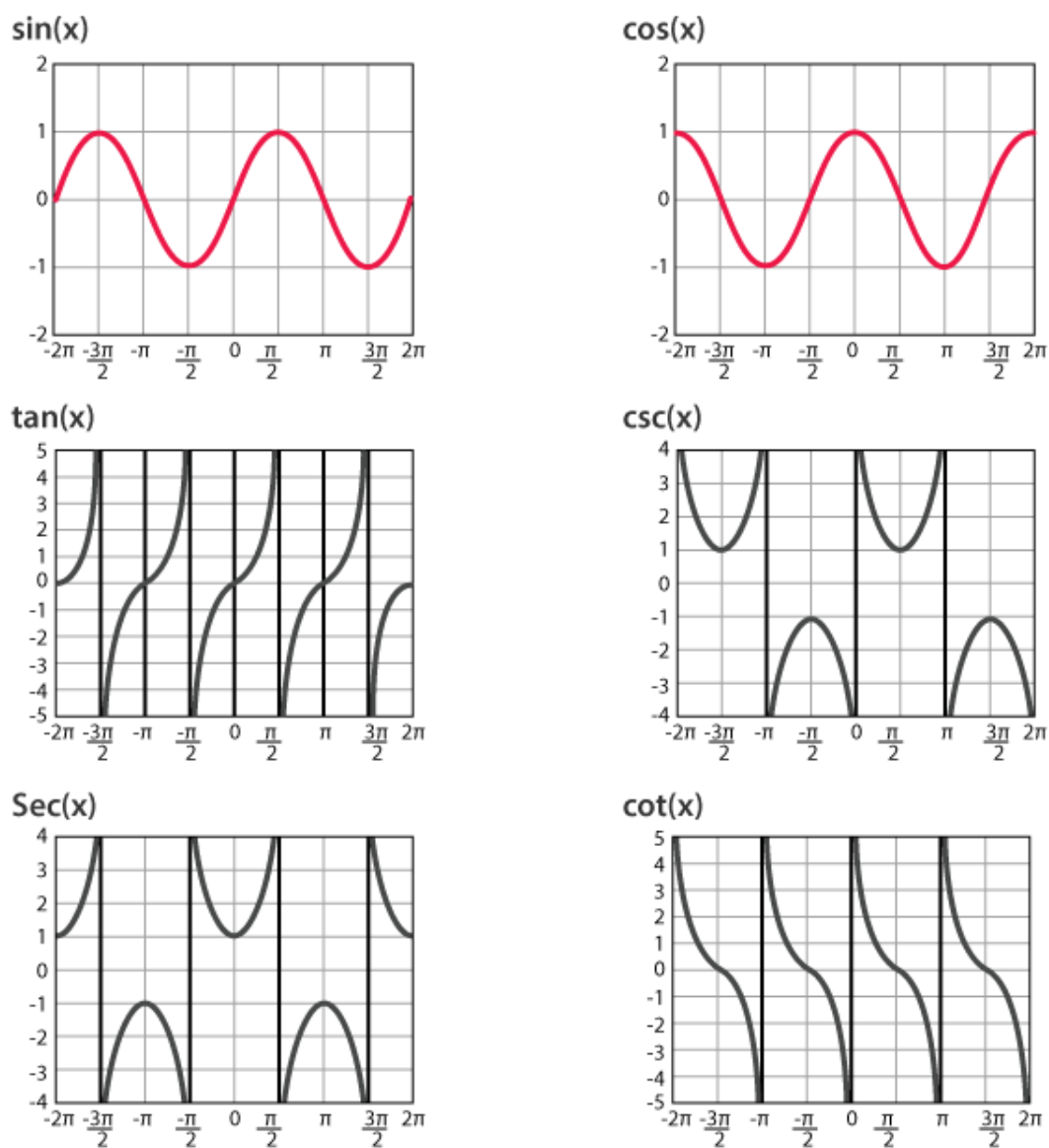


Figure 7: Graphs of Trigonometric Functions



Trigonometric Identities

- **Negative Angle***

1. $\sin(-a) = -\sin a$
2. $\cos(-a) = \cos a$
3. $\tan(-a) = -\tan a$

- **Sum or Difference of Angles**

4. $\sin(a + b) = \sin a \cos b + \cos a \sin b$
5. $\sin(a - b) = \sin a \cos b - \cos a \sin b$
6. $\cos(a + b) = \cos a \cos b - \sin a \sin b$
7. $\cos(a - b) = \cos a \cos b + \sin a \sin b$
8. $\tan(a + b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$
9. $\tan(a - b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}$
10. $\cot(a + b) = \frac{\cot a \cot b - 1}{\cot b + \cot a}$
11. $\cot(a - b) = \frac{\cot a \cot b + 1}{\cot b - \cot a}$

- **Multiples of $\frac{\pi}{2}$ ***

12. $\sin\left(\frac{\pi}{2} - x\right) = \cos x$
13. $\sin\left(\frac{\pi}{2} + x\right) = -\cos x$
14. $\sin(\pi - x) = \sin x$
15. $\sin(\pi + x) = -\sin x$
16. $\sin(2\pi - x) = -\sin x$
17. $\cos\left(\frac{\pi}{2} - x\right) = \sin x$
18. $\cos\left(\frac{\pi}{2} + x\right) = -\sin x$



19. $\cos(\pi - x) = -\cos x$

20. $\cos(\pi + x) = -\cos x$

21. $\cos(2\pi - x) = \cos x$

- **Double Angle**

22. $\sin 2x = 2 \sin x \cos x$

23. $\begin{aligned}\cos 2x &= 2 \cos^2 x - 1 \\ &= 1 - 2 \sin^2 x \\ &= \cos^2 x - \sin^2 x\end{aligned}$

24. $\tan 2x = \frac{2 \tan x}{1 + \tan^2 x}$

- **Triple Angle**

25. $\sin 3x = 3 \sin x - 4 \sin^3 x$

26. $\cos 3x = 4 \cos^3 x - 3 \cos x$

27. $\tan 3x = \frac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}$

- **Sum/Difference of Sines and Cosines**

28. $\cos a + \cos b = 2 \cos \left(\frac{a+b}{2} \right) \cos \left(\frac{a-b}{2} \right)$

29. $\cos a - \cos b = -2 \sin \left(\frac{a+b}{2} \right) \sin \left(\frac{a-b}{2} \right)$

30. $\sin a + \sin b = 2 \sin \left(\frac{a+b}{2} \right) \cos \left(\frac{a-b}{2} \right)$

31. $\sin a - \sin b = 2 \cos \left(\frac{a+b}{2} \right) \sin \left(\frac{a-b}{2} \right)$

- **Product of Sines and Cosines (Inverse of 28-31)**

32. $2 \cos a \cos b = \cos(a+b) + \cos(a-b)$

33. $-2 \sin a \sin b = \cos(a+b) - \cos(a-b)$

34. $2 \sin a \cos b = \sin(a+b) + \sin(a-b)$

35. $2 \cos a \sin b = \sin(a+b) - \sin(a-b)$



- **Zero Values***

$$36. \sin x = 0 \implies x = n\pi, n \in \mathbb{Z}$$

$$37. \cos x = 0 \implies x = (2n + 1)\frac{\pi}{2}, n \in \mathbb{Z}$$

- **Equal Values***

$$38. \sin x = \sin y \implies x = n\pi + (-1)^n y$$

$$39. \cos x = \cos y \implies x = 2n\pi \pm y$$

$$40. \tan x = \tan y \implies x = n\pi + y$$

*These identities can be easily represented by the Unit Circle.