Mathematical Induction

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Mathematical Induction is a principle which is used to prove a proposition P to be true for all natural numbers n.

Steps to Prove a statement

1. **Base Case:** Prove that $P(1)$ is true.
2. **Inductive Step:** Prove that if $P(k)$ is true, then $P(k+1)$ must also be true.

In this way, we end up proving the statement for all natural numbers $n = 1, 2, 3, \cdots$

Example 1: Sum of Natural Numbers

Question: Prove that the sum of the first n natural numbers is equal to $\frac{n(n+1)}{2}$.

Proof

$$P(n) : 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

For $n = 1$,

$$\text{LHS} = 1$$
$$\text{RHS} = \frac{1(2)}{2} = 1$$

$\therefore P(1)$ is true.

Let $P(n)$ be true for $n = k$.

$$P(k) : 1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2} \tag{1}$$

We need to prove that $P(n)$ is also true for $n = k + 1$,

$$P(k+1) : 1 + 2 + 3 + \cdots + k + k + 1 = \frac{(k+1)(k+2)}{2}$$

$$\begin{aligned}
\text{LHS} &= \boxed{1 + 2 + 3 + \cdots + k} + k + 1 \\
&= \frac{k(k+1)}{2} + k + 1 \qquad\qquad \text{(Substituting 1)} \\
&= \frac{k(k+1) + 2(k+1)}{2} \\
&= \frac{(k+1)(k+2)}{2} \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k+1)$ is true whenever $P(k)$ is true.

Hence, from the principle of mathematical induction, $P(n)$ is true for all $n \in N$.

Example 2

Question: Prove that $1 \cdot 3 + 2 \cdot 3^2 + 3 \cdot 3^3 + \cdots + n \cdot 3^n = \dfrac{(2n-1)3^{n+1} + 3}{4}$

Proof

$$P(n) : 1 \cdot 3 + 2 \cdot 3^2 + 3 \cdot 3^3 + \cdots + n \cdot 3^n = \frac{(2n-1)3^{n+1} + 3}{4}$$

For $n = 1$,

$$\text{LHS} = 3$$
$$\text{RHS} = \frac{1(3^2) + 3}{4} = \frac{12}{4} = 3$$

$\therefore P(1)$ is true.

Let $P(n)$ be true for $n = k$.

$$P(k) : 1 \cdot 3 + 2 \cdot 3^2 + 3 \cdot 3^3 + \cdots + k \cdot 3^k = \frac{(2k-1)3^{k+1} + 3}{4} \tag{1}$$

We need to prove that $P(n)$ is also true for $n = k + 1$,

$$\begin{aligned} P(k+1) : 1 \cdot 3 + 2 \cdot 3^2 + 3 \cdot 3^3 + \cdots + k \cdot 3^k + (k+1)3^{k+1} &= \frac{(2(k+1)-1)3^{(k+1)+1} + 3}{4} \\ &= \frac{(2k+1)3^{k+2} + 3}{4} \end{aligned}$$

$$\begin{aligned} \text{LHS} &= \boxed{1 \cdot 3 + 2 \cdot 3^2 + 3 \cdot 3^3 + \cdots + k \cdot 3^k} + (k+1) \cdot 3^{k+1} \\ &= \frac{(2k-1)3^{k+1} + 3}{4} + 3^{k+1}(k+1) && \text{(Substituting 1)} \\ &= \frac{3^{k+1}(2k-1) + 3 + 3^{k+1} \cdot 4(k+1)}{4} \\ &= \frac{3^{k+1}(2k - 1 + 4(k+1)) + 3}{4} \\ &= \frac{3^{k+1}(6k+3) + 3}{4} \\ &= \frac{3^{k+1} \cdot 3(2k+1) + 3}{4} \\ &= \frac{3^{k+2}(2k+1) + 3}{4} \\ &= \text{RHS} \end{aligned}$$

$\therefore P(k+1)$ is true whenever $P(k)$ is true.

Hence, from the principle of mathematical induction, $P(n)$ is true for all $n \in N$.

Example 3

Question: Prove that $n(n+1)(n+5)$ is a multiple of 3 for all $n \in N$.

Proof

$$P(n) : n(n+1)(n+5) = 3m, m \in Z$$

For $n = 1$,

$$\begin{aligned} n(n+1)(n+5) &= 1 \cdot 2 \cdot 6 \\ &= 12 = 3 \times 4 \end{aligned}$$

$\therefore P(1)$ is true.

Let $P(n)$ be true for $n = k$.

$$P(k) : k(k+1)(k+5) = 3m, m \in Z \tag{1}$$

We need to prove that $P(n)$ is also true for $n = k+1$,

$$P(k+1) : (k+1)(k+2)(k+6) = 3q, q \in Z$$

$$\begin{aligned} &= (k+1)(k+2)(k+6) \\ &= (k+1)(k^2+8k+12) \\ &= (k+1)\left((k^2+5k)+(3k+12)\right) \\ &= (k+1)(k^2+5k)+(k+1)(3k+12) \\ &= \boxed{k(k+1)(k+5)} + 3(k+1)(k+4) \\ &= 3m + 3(k+1)(k+4) \\ &= 3\left(m+(k+1)(k+4)\right) \\ &= 3q \end{aligned}$$

$\therefore P(k+1)$ is true whenever $P(k)$ is true.

Hence, from the principle of mathematical induction, $P(n)$ is true for all $n \in N$.