Elastic Collisions

During collision of two spherical masses, the total momentum of the system remains conserved, owing to the lack of any external force present on the system.[1] Different types of collision are distinguished based on whether or not they conserve **kinetic energy**.

If the total kinetic energy of the system is conserved, i.e. no energy is dissipated in form of heat, or sound, the collision is said to be **perfectly elastic**.

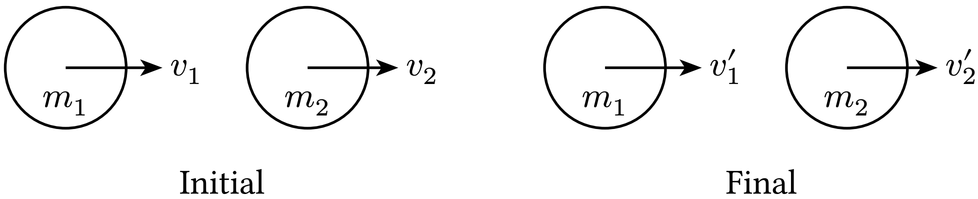
Perfectly Elastic Collisions in Space

Consider two spheres in space having velocities in random directions, such that they collide at some point. Given the initial conditions, we can determine the final velocities of the spheres after the collision.

There are two possible ways in which this can happen:

1. **Head-on Collision** (1D Collision): The spheres collide along the line joining their centers. (The initial velocities of the spheres are along this line.)
2. **Oblique Collision** (2D Collision): The velocities of the spheres are not along the line joining their centers.

One Dimensional Elastic Collision



The two constraints in the collision are kinetic energy conservation and momentum conservation. Using these relations, we can determine the final velocities of the spheres.

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'$$

$$\Rightarrow m_1(v_1 - v_1') = m_2(v_2' - v_2) \tag{1}$$

$$\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}m_1 v_1'^2 + \frac{1}{2}m_2 v_2'^2$$

$$\Rightarrow m_1(v_1^2 - v_1'^2) = m_2(v_2'^2 - v_2^2) \tag{2}$$

[1]Even if any other force is present, it can be neglected in comparison to the force of collision, which is very large, as it causes a finite change in momentum in infinitesimal time.

Dividing (2) by (1), we get:

$$\frac{m_1(v_1^2 - v_1'^2)}{m_1(v_1 - v_1')} = \frac{m_2(v_2'^2 - v_2^2)}{m_2(v_2' - v_2)}$$
$$\Rightarrow v_1 + v_1' = v_2' + v_2$$
$$\Rightarrow v_1' - v_2' = v_2 - v_1$$

Using the above relation, along with (1), we have a system of two linear equations in two variables, which can be solved to find the final velocities of the spheres.

$$v_1' - v_2' = v_2 - v_1 \tag{3}$$

$$m_1 v_1' + m_2 v_2' = m_1 v_1 + m_2 v_2 \tag{4}$$

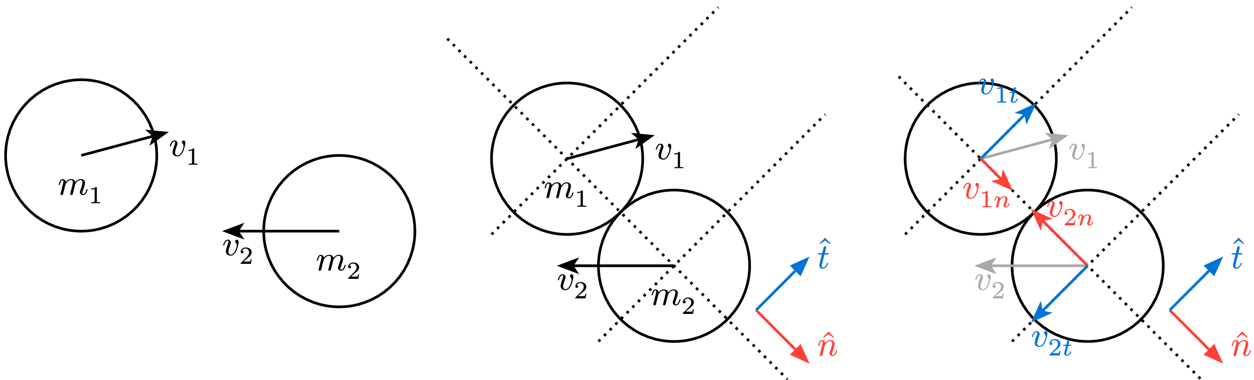
Multiply (3) by m_1 and subtract from (4), on simplification, we get our solutions:

$$\boxed{v_1' = \frac{m_1 - m_2}{m_1 + m_2} v_1 + \frac{2m_2}{m_1 + m_2} v_2}$$

$$\boxed{v_2' = \frac{m_2 - m_1}{m_1 + m_2} v_2 + \frac{2m_1}{m_1 + m_2} v_1}$$

Two Dimensional Elastic Collision

For two non-spinning bodies in two dimensions, to solve for the final velocities, we can resolve the velocities into components along the line of impact ($\hat{n}$) and tangent to the bodies at the point of contact ($\hat{t}$). Since the collision only imparts force along the line of impact, the tangential velocities don't change.



Components of velocity in the $\hat{n}$ direction (along the line of impact) can be resolved by using the formula for one-dimensional elastic collision, whereas velocities in the $\hat{t}$ direction remain unchanged.

$$v_{1n}' = \frac{m_1 - m_2}{m_1 + m_2} v_{1n} + \frac{2m_2}{m_1 + m_2} v_{2n} \qquad v_{1t}' = v_{1t}$$

$$v'_{2n} = \frac{m_2 - m_1}{m_1 + m_2} v_{2n} + \frac{2m_1}{m_1 + m_2} v_{1n} \qquad v'_{2t} = v_{2t}$$

The final velocities $\vec{v'_1}$ and $\vec{v'_2}$ are obtained by the vector sum of the respective $\hat{n}$ and $\hat{t}$ components.

$$\Rightarrow \vec{v'_1} = v'_{1n}\,\hat{n} + v'_{1t}\,\hat{t}$$

$$\vec{v'_1} = \left(\frac{m_1 - m_2}{m_1 + m_2} v_{1n} + \frac{2m_2}{m_1 + m_2} v_{2n}\right)\hat{n} + v_{1t}\,\hat{t}$$

This is the value of $\vec{v'_1}$ in terms of components of initial velocities, and unit vectors along the line of impact and tangent to it. This can be converted into a vector expression, by subtracting the initial velocity vector $\vec{v_1} = v_{1n}\,\hat{n} + v_{1t}\,\hat{t}$ from $\vec{v'_1}$.

Subtracting $\vec{v_1}$ from $\vec{v'_1}$,

$$\begin{aligned}
\vec{v'_1} - \vec{v_1} &= \left(\left(\frac{m_1 - m_2}{m_1 + m_2} v_{1n} + \frac{2m_2}{m_1 + m_2} v_{2n}\right)\hat{n} + \cancel{v_{1t}\,\hat{t}}\right) - \left(v_{1n}\,\hat{n} + \cancel{v_{1t}\,\hat{t}}\right) \\
&= \left(\frac{m_1 - m_2}{m_1 + m_2} v_{1n} + \frac{2m_2}{m_1 + m_2} v_{2n} - v_{1n}\right)\hat{n} \\
&= \left(\frac{-2m_2}{m_1 + m_2} v_{1n} + \frac{2m_2}{m_1 + m_2} v_{2n}\right)\hat{n} \\
&= \frac{2m_2}{m_1 + m_2}\left(v_{2n} - v_{1n}\right)\hat{n}
\end{aligned}$$

Here, we can substitute $v_{1n} = \vec{v_1} \cdot \hat{n}$ and $v_{2n} = \vec{v_2} \cdot \hat{n}$.

$$\vec{v'_1} = \vec{v_1} + \frac{2m_2}{m_1 + m_2}\left(\left(\vec{v_2} - \vec{v_1}\right) \cdot \hat{n}\right)\hat{n}$$

The unit vector along the line of impact, $\hat{n}$, is in the direction of the difference of position vectors of bodies 1 and 2. If $\vec{x}$ denotes the position vector of the centre of the body,

$$\hat{n} = \frac{\vec{x_2} - \vec{x_1}}{|\vec{x_2} - \vec{x_1}|}$$

By substituting all values, we get a vector expression for $\vec{v_1'}$. The expression of $\vec{v_2'}$ can be symmetrically obtained.

$$\boxed{\vec{v_1'} = \vec{v_1} + \frac{2m_2}{m_1 + m_2} \frac{(\vec{v_2} - \vec{v_1}) \cdot (\vec{x_2} - \vec{x_1})}{|\vec{x_2} - \vec{x_1}|^2} (\vec{x_2} - \vec{x_1})}$$

$$\boxed{\vec{v_2'} = \vec{v_2} + \frac{2m_1}{m_1 + m_2} \frac{(\vec{v_1} - \vec{v_2}) \cdot (\vec{x_1} - \vec{x_2})}{|\vec{x_1} - \vec{x_2}|^2} (\vec{x_1} - \vec{x_2})}$$