A man have been given a random number from 1-100 and then sent to a room having 100 boxes containing a random number. He can check maximum of 50 boxes to find his number. What is the probability that he will find his number?

Intuitively, the answer to this question is $\frac{1}{2}$, this article tries to mathematically arrive to the answer by using Conditional Probabilities and the Total Probability Theorem. [Disclaimer: This is a very overcomplicated way to arrive to the result.]

Solution

Here, there exist two random elements - selection of number, and arrangement of boxes in the room - making it difficult to perform a probability calculation. Let us fix one of them for now. Say, his randomly chosen number is $N = 10$.

Now, let us assume that the boxes are arranged randomly, and he checks them in order of their positioning. So he checks the first box, then the second, then the third, and so on.

Let A_n denote the event that he finds his number (10) in the n^{th} box.

Before he opens any box, he has no knowledge of any number, so the probabilities are:

$$P(A_1) = \frac{1}{100}, P(A_2) = \frac{1}{100}, P(A_3) = \frac{1}{100}, ..., P(A_{100}) = \frac{1}{100}$$

Once he has opened box 1, he suddenly has gained new information, which is the number in box 1. There are two cases:

Case 1: He has found his number (Success), A_1: Now the probabilities will become:

$$P\left(\frac{A_1}{A_1}\right) = \frac{100}{100}, P\left(\frac{A_2}{A_1}\right) = 0, P\left(\frac{A_3}{A_1}\right) = 0, ..., P\left(\frac{A_{100}}{A_1}\right) = 0$$

Case 2: He has not found his number, A_1': Now the probabilities will become:

$$P\left(\frac{A_1}{A_1'}\right) = 0, P\left(\frac{A_2}{A_1'}\right) = \frac{1}{99}, P\left(\frac{A_3}{A_1'}\right) = \frac{1}{99}, ..., P\left(\frac{A_{100}}{A_1'}\right) = \frac{1}{99}$$

(He will equally distribute the probabilty in all the remaining boxes)

So clearly, the probability values A_n are not static but instead dynamic, i.e. they change as per information gained by the player. These are denoted as conditional probabilities. (Incidentally, this concept is key to understanding many famous probability "paradoxes", like the Monty Hall Problem for example.)

So, to find the overall probability A_2, we can use the Total Probability theoremn:

$$\begin{aligned} P(A_2) &= P(A_1 \cap A_2) + P(A_1' \cap A_2) \\ &= P(A_1)P\left(\frac{A_2}{A_1}\right) + P(A_1')P\left(\frac{A_2}{A_1'}\right) \\ &= \frac{1}{100} \cdot 0 + \frac{99}{100} \cdot \frac{1}{99} \\ &= \frac{1}{100} \end{aligned}$$

Similarly, for A_3 we can write

$$\begin{aligned} P(A_3) &= P(A_2 \cap A_3) + P(A_2' \cap A_3) \\ &= P(A_1 \cap A_2 \cap A_3) + P(A_1' \cap A_2 \cap A_3) + P(A_1 \cap A_2' \cap A_3) + \\ &\quad P(A_1' \cap A_2' \cap A_3) \end{aligned}$$

These are all the possibilities of A_1 and A_2, which might have occured before opening the third box. But, we can clearly see, just like the above case, here the first three terms will become 0. This is because all A_n are disjoint events, and hence they cannot occur simultaneously. So any event of the type $A_i \cap A_j$ will obviously become an impossible event.

So, we can simplify to:

$$\begin{aligned}
P(A_1) &= P(A_1) &&= P(A_1) &&= \frac{1}{100} \\
P(A_2) &= P(A_1' \cap A_2) &&= P(A_1')P\left(\frac{A_2}{A_1'}\right) &&= \frac{99}{100} \cdot \frac{1}{99} \\
P(A_3) &= P(A_1' \cap A_2' \cap A_3) &&= P(A_1')P\left(\frac{A_2'}{A_1'}\right)P\left(\frac{A_3}{A_1' \cap A_2'}\right) &&= \frac{99}{100} \cdot \frac{98}{99} \cdot \frac{1}{98} \\
P(A_4) &= P(A_1' \cap A_2' \cap A_3' \cap A_4) &&= P(A_1')P\left(\frac{A_2'}{A_1'}\right)...P\left(\frac{A_4}{A_1' \cap A_2' \cap A_3'}\right) &&= \frac{99}{100} \cdot \frac{98}{99} \cdot \frac{97}{98} \cdot \frac{1}{97}
\end{aligned}$$

Following the pattern, we can say that $A_n = \frac{1}{100}$, for every n. (We could've directly stated this without doing any calculations, that it is equally likely that he finds his number in any of the boxes, because the order of arrangement is random.)

Now, this result is obviously not specific for $N = 10$, it is true for any N, and all N are equally likely to be picked. So, the final result we will get, is that $A_n = \frac{1}{100}$, meaning each A_n is equally likely.

Finally, to find probability of success,

$$\begin{aligned}
P(\text{Success}) = \frac{\text{Favourable}}{\text{Total}} &= \frac{P(A_1 \cup A_2 \cup A_3 \cup A_4 \cup ... \cup A_{50})}{1} \\
&= 50 \times \frac{1}{100} = \boxed{\frac{1}{2}}
\end{aligned}$$