

# Regularization Algorithms for Transition Matrices

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Both estimating portfolio credit risk and pricing credit risky securities require transition matrices for arbitrary time horizons. Simply computing the root of the annual transition matrix is unacceptable because the resulting matrices often contain negative elements. A similar situation exists when taking the logarithm of the annual transition matrix to compute a generator. This paper develops regularization algorithms for obtaining transition matrices and generators that give rise to close approximations of a given annual transition matrix. In our approach, the root or logarithm of the annual transition matrix is transformed, on a row-by-row basis, into a valid transition matrix or generator by projecting each row onto an appropriate set in a Euclidean space. Our methods compare favourably with other known regularization algorithms.

Credit risk models typically assume that, over time, a counterparty's credit rating migrates among a set of possible credit states. Mathematically, this process can be modelled as a finite Markov chain, which assumes that the credit rating changes successively from one state to another, at given time intervals, with a certain probability. The probabilities of credit migration form a transition matrix. Annual transition matrices, which specify the probability of changing states over a one-year period, can be obtained from several agencies that specialize in credit data.

Risk management practices and the reduced form pricing of instruments are based on the finite Markov chain model. For example, Jarrow et al. (1997), Das and Tufano (1996), and Lando (2000) discuss the pricing and hedging of over the counter derivatives with counterparty risk and corporate debt with embedded options, relying on a Markov model of credit migration. For both continuous and discrete-time pricing models, knowledge of the annual transition matrix is insufficient for pricing since most instruments have maturities or cash flows that

occur on a non-annual basis. For example, pricing a credit risky instrument that matures in six months requires a six-month transition matrix. Thus, in general, we need to obtain transition matrices over arbitrary time horizons. Note that if the transition process is time homogeneous, then the  $n$ -period transition matrix can be obtained by simply raising the one-period transition matrix to the power  $n$ . Therefore, time homogeneity implies that a one-year transition matrix determines all transitions over any integer number of years.

Although this feature of the model is very attractive, it creates some conceptual problems. A six-month transition matrix, for example, should be a square root of the annual transition matrix. Unfortunately, this may not be possible; raising the annual transition matrix to a power less than one typically results in a matrix with negative elements, which is not a transition matrix. Moreover, even when it does exist, a six-month transition matrix may not be unique (see, e.g., Kingman 1962, Carrette 1995). If a given annual transition matrix gives rise to several possible six-month transition matrices, the

natural question is: Which matrix does one use for pricing purposes?

Thus, computing the fractional roots of an annual transition matrix is an ill-posed problem; the result may not be a valid transition matrix or it may not be unique. When faced with such a problem, one possibility is to reformulate it in a manner that allows it to be more readily solved. In this paper, we use this process, known as regularization, to obtain transition matrices for arbitrary time horizons that closely approximate the roots of an annual transition matrix.

One approach that obtains transition matrices for periods of arbitrary length involves embedding the discrete-time Markov chain into a continuous-time Markov process (Kingman 1962). For a continuous-time Markov process, any transition matrix (i.e., for a period of any length) can be expressed as the exponential of a so-called **generator** matrix. Thus, solving the **embedding problem** essentially allows one to find a generator that is consistent with the annual transition matrix of the discrete-time Markov chain. However, computing the generator of an existing transition matrix by taking its logarithm still raises the issues of existence and uniqueness. In fact, as we note in this paper, empirically observed transition matrices typically have properties that preclude the existence of a generator (see Israel et al. 2001). Alternatively, more than one generator may give rise to the same transition matrix (Kingman 1962, Carrette 1995). To avoid these difficulties, the preferred approach is to estimate the generator directly and then use it to construct transition matrices as necessary. Unfortunately, agencies that provide data on credit migration deal with annualized transition matrices rather than with their generators.

There are a number of examples of using generators for pricing credit risky instruments. In Jarrow et al. (1997), Das and Tufano (1996) and Lando (2000), the discrete-time and continuous-time arbitrage-free models assume the existence and uniqueness of an equivalent martingale measure. To construct this measure in the framework of Markov credit migration models, one has to determine the risk-neutral transition

probabilities from market prices and historical transition probabilities. Thus, one must consider the conditions that guarantee uniqueness of the equivalent martingale measure. The framework suggested by Jarrow et al. (1997) and Das and Tufano (1996) simply assumes the existence of a generator matrix governing credit migration. In Lando (1999), however, the generator is computed from the transition matrix, which potentially raises the issue of uniqueness. If the generator of a transition matrix is not unique then, from a pricing perspective, this may lead to arbitrage. Therefore, the arbitrage-free approach should ideally be based on estimating the generator directly, as in Jarrow et al. (1997) and Lando (2000).

In this paper, we also present a regularization algorithm for computing the generator that can be used for the best approximation of a given transition matrix. Thus, we provide a means to obtain either a root of a transition matrix for discrete-time credit risk models or a generator for continuous-time models. Using our approach, one finds the transition matrix or generator that is closest, in a Euclidean distance sense, to the given root or logarithm, respectively, of the annual transition matrix.

This paper is organized as follows. The first section provides relevant background material. It reviews Markov chains and transition matrices in the context of credit models. The next section describes the embedding problem that is closely related to the regularization of transition matrices. Then, we discuss the issue of uniqueness and examine the properties of a typical transition matrix. In the next section we formulate regularization problems for the root and generator matrices and present algorithms for their solution. We illustrate the regularization procedures by finding a six-month transition matrix and then we compare them on a broader sample of transition matrices. The final section contains our conclusions and suggestions for further research.

## Markov chains and transition matrices

A finite Markov chain in discrete time  $S(t)$ ,  $t = 0, 1, 2, \dots$ , is defined by the following elements:

- A finite set of states  $S = \{1, 2, \dots, n\}$
- An initial probability distribution,  $q(0) = (P\{S(0) = 1\}, P\{S(0) = 2\}, \dots, P\{S(0) = n\})$  satisfying  $P\{S(0) = i\} \geq 0$ ,  $i = 1, 2, \dots, n$  and

$$\sum_{i=1}^n P\{S(0) = i\} = 1$$

- A family of transition matrices  $A(t, t+1) = \|a_{ij}(t, t+1)\|$ , where  $a_{ij}(t, t+1) = P\{S(t+1) = j | S(t) = i\}$ . The transition probabilities satisfy

$$\sum_{j=1}^n a_{ij}(t, t+1) = 1 \quad \text{for } i = 1, 2, \dots, n \quad (1)$$

$$a_{ij}(t, t+1) \geq 0 \quad \text{for } i, j = 1, 2, \dots, n$$

We say that the Markov chain  $S(t)$  is time homogeneous if the transition probabilities do not depend on  $t$ , that is,

$$a_{ij}(t, t+1) = a_{ij}(0, 1) = a_{ij}.$$

In credit risk modelling, a one-year transition matrix is the only source of information regarding the transition probabilities. Furthermore, rating agencies do not provide time-dependent forecasts of transition matrices.<sup>1</sup> For these reasons, it is practical to assume a time-homogeneous Markov chain when simulating credit events. Note that this also allows the above notation to be simplified: we denote a one-period transition matrix by  $A = \|a_{ij}\|$  and, more generally, a  $t$ -period (i.e., for the time interval  $[0, t]$ ) transition matrix by  $A(t) = \|a_{ij}(t)\|$ .

The transition probabilities  $a_{ij}(t)$  satisfy the relation

$$a_{ij}(t) = \sum_{k=1}^n a_{ik}(m) a_{kj}(t-m), \quad m = 1, 2, \dots, t-1$$

or, in matrix form,  $A(t) = A(m)A(t-m)$ . The latter formula, usually called *the semi-group property*, implies

$$A(t) = A^t \quad (2)$$

It is easily verified that any integer power of a

transition matrix is a transition matrix as well. From Equation 2, it follows that the probability distribution of the Markov chain at time  $t$  satisfies

$$q(t) = q(0)A^t$$

where

$$q(t) = (P\{S(t) = 0\}, P\{S(t) = 1\}, \dots, P\{S(t) = n\}).$$

### Regular Credit Migration Model

Finite Markov chains used in credit risk models have one special state that represents a default event. This state is absorbing; once reached, the chain will remain there indefinitely. In what follows, we assume that the default state is numbered  $n$ . Then we have

$$a_{nn}(t) = 1 \quad \text{and} \quad a_{nj}(t) = 0 \quad \text{for all } j \neq n, t = 1, 2, \dots$$

To distinguish from the more general case described in the previous section, we define a **regular credit migration model** (RCMM) to be a Markov chain that also satisfies the following three properties:

- There is one and only one absorbing state.
- There exists  $t_p \geq 1$  such that  $a_{in}(t_p) > 0$  for all  $i = 1, 2, \dots, n$ .
- The determinant of the annual transition matrix  $A$  is not equal to zero, and the eigenvalues are distinct (this allows us to compute the logarithm of  $A$ ).

Empirically, we observe that values  $2 \leq t_p \leq 5$  are common. Thus, even a counterparty with the highest credit rating has a positive probability of defaulting within two to five years. Furthermore, for an RCMM, the transition matrix  $A(t)$  satisfies (Snell 1988)

$$A(t) \rightarrow D \quad \text{as} \quad t \rightarrow \infty \quad (3)$$

where

$$D = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

Equation 3 implies that default will eventually occur regardless of the initial credit rating; the default state is accessible from every credit state. However, the average time to default may be very large for some initial credit states.

Analysis of empirical annual transition matrices indicates that the diagonal probabilities,  $a_{ii}$ , dominate in corresponding rows (i.e.,  $a_{ii} > a_{ij}$  for  $i \neq j$ ). However, this relation is not satisfied for arbitrary time  $t > 1$ , as is apparent from the matrix  $\mathbf{D}$  in Equation 3, for example. For this reason, we do not include this condition in the definition of an RCMM and are unable to use the properties of diagonally dominant matrices in our analysis.

### Embedding problem

Computation of the fractional roots of transition matrices is closely related to the embedding problem of Markov chains. Given a finite Markov chain in discrete time,  $S(t)$ , the embedding problem can be posed as follows:

*Is it possible to construct a Markov process  $\{S(\tau), \tau \geq 0\}$  in continuous time  $\tau$  such that the probability distribution of  $S(\tau)$  at time  $\tau = 1, 2, \dots$  is identical to the distribution of  $S(t)$ ?*

The embedding problem was studied by Kingman (1962), Carrette (1995), Iwanik and Shiflett (1986), and Frydman (1983). The main approach developed in these papers is based on the notion of the generator of a Markov process.

A generator is a matrix  $\mathbf{G}$  whose elements satisfy

$$\sum_{j=1}^n g_{ij} = 0 \quad \text{for } i = 1, 2, \dots, n, \quad (4)$$

$$g_{ij} \geq 0 \quad \text{for } i, j = 1, 2, \dots, n \text{ and } i \neq j.$$

Generators form a semi-group in the space of matrices, which means that the sum of two generators is also a generator. Note that if  $\mathbf{A}$  is a transition matrix then  $\mathbf{A} - \mathbf{I}$  is a generator, where  $\mathbf{I}$  is the identity matrix of the same dimensions as  $\mathbf{A}$ . Moreover, if  $\mathbf{G}$  is a generator then  $e^{\mathbf{G}}$  is a transition matrix (Snell 1988).

Let  $\tilde{\mathbf{A}}(\tau)$  denote the transition matrix of the Markov process  $\tilde{S}(\tau)$ . The matrix  $\tilde{\mathbf{G}} = \|\tilde{g}_{ij}\|$  is called the generator of  $\tilde{S}(\tau)$  if  $\tilde{\mathbf{A}}(\tau)$  satisfies

$$\frac{d}{d\tau} \tilde{\mathbf{A}} = \tilde{\mathbf{A}} \tilde{\mathbf{G}}$$

In this case, we obtain

$$\tilde{\mathbf{A}}(\tau) = e^{\tau \tilde{\mathbf{G}}}, \tau > 0$$

Therefore, if the distribution of the Markov process  $\tilde{S}(\tau)$  is identical to that of the Markov chain  $S(t)$  at time  $\tau = 1, 2, \dots$  then the transition matrix  $\mathbf{A}$  must satisfy

$$\mathbf{A} = e^{\tilde{\mathbf{G}}} \quad (5)$$

If the matrix  $\tilde{\mathbf{G}}$  exists and is unique, then one can define a continuous-time extension,  $\tilde{\mathbf{A}}(\tau)$ , of the transition matrix  $\mathbf{A}$  as follows

$$\tilde{\mathbf{A}}(\tau) = e^{\tau \tilde{\mathbf{G}}} \quad (6)$$

Note that since  $\tau$  is continuous, Equation 6 allows us to obtain a transition matrix for a period of arbitrary length.

While we can calculate  $\tilde{\mathbf{G}} = \ln(\mathbf{A})$  from Equation 5, there is no guarantee that the matrix  $\ln(\mathbf{A})$  satisfies Equation 4. In general, the question of when a transition matrix  $\mathbf{A}$  can be represented in the form  $\mathbf{A} = e^{\mathbf{G}}$ , where  $\mathbf{G}$  is a generator, is technically very difficult. There are several known results related to this problem. For  $2 \times 2$  transition matrices, the generator exists if the determinant of the matrix is positive (see, e.g., Kingman 1962). However, for the  $3 \times 3$  case, studied by Iwanik and Shiflett (1986) and Frydman (1983), no general conditions were found for the existence of the generator. Note that for transition matrices of arbitrary dimensions, inferring such existence conditions from the characterization of their eigenvalues (see, e.g., Karpelevitch 1951) is an interesting possibility.

Finally, we note that the problem of computing

the logarithm and exponential of a matrix is nontrivial (Moler and Van Loan 1978 and Dieci 1996) and a detailed discussion is beyond the scope of this paper. Our implementation is based on Schur decomposition and the Parlett method (Moler and Van Loan 1978).

### Non-uniqueness problem

In this section, we address the problem of non-uniqueness of the resulting transition matrix when one computes credit transition probabilities over time horizons of less than one year. We start with an instructive example that demonstrates the non-uniqueness effect. Consider the family of transition matrices

$$A(0.5) = \begin{bmatrix} 0 & p & \dots & 0 & 0 & 1-p \\ x & 0 & \dots & 0 & 0 & 1-x \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & p & 1-p \\ 0 & 0 & \dots & x & 0 & 1-x \\ 0 & 0 & \dots & 0 & 0 & 1 \end{bmatrix}$$

Let us impose the condition  $px = a$ , where  $0 < p < 1$  and  $0 < x < 1$ . Then, the annual transition matrix,  $A = A(0.5) \cdot A(0.5)$  is

$$A = \begin{bmatrix} a & 0 & 0 & \dots & 0 & 1-a \\ 0 & a & 0 & \dots & 0 & 1-a \\ 0 & 0 & a & \dots & 0 & 1-a \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a & 1-a \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

However, there exists another transition matrix  $A^*(0.5)$  that is also the square root of  $A$ :

$$A^*(0.5) = \begin{bmatrix} \sqrt{a} & 0 & \dots & 0 & 1-\sqrt{a} \\ 0 & \sqrt{a} & \dots & 0 & 1-\sqrt{a} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \sqrt{a} & 1-\sqrt{a} \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

Note that  $A^*(0.5)$  is the solution that would be obtained using a generator, as discussed in the previous section.

This example demonstrates that there can be infinitely many square roots of the transition matrix, each of which has its own financial

interpretation. It is, therefore, natural to ask: What conditions guarantee the uniqueness the resulting transition matrix? A partial answer to this question can be found in Theorem 1 and Theorem 2 below, from the paper by Israel et al. (2001).

**Theorem 1:** Suppose that all diagonal elements dominate in the transition matrix  $A$  (i.e.,  $a_{ii} > \frac{1}{2}$ ).

Then the eigenvalues of  $A$  belong to the disc  $\|\lambda - 1\| \leq 1$  in the complex plane, and the Taylor series expansion for the logarithm of the transition matrix converges geometrically quickly to a real matrix,  $Q$ , that satisfies  $\exp(Q) = A$ . Furthermore, under this condition, the transition matrix can have, at most, one generator: if a generator exists, then it is unique.

Note that Theorem 1 does not say that diagonal dominance guarantees existence of the generator. In our analysis of 32 empirical transition matrices, only 10 had dominant diagonal elements and, among those, only one matrix had a generator.

Theorem 2 uses the following definition: a state  $j$  is accessible from a state  $i$  if there is a sequence of states  $k_0 = i, k_1, k_2, \dots, k_m = j$  such that

$$\prod_{l=0}^{m-1} a_{k_l k_{l+1}} > 0$$

**Theorem 2:** Suppose that the transition matrix,  $A$ , satisfies one of the conditions.

(1)  $\det A < 0$ ,

(2)  $\det A > \prod_{i=1}^n a_{ii}$

(3) there are states  $i$  and  $j$  such that  $j$  is accessible from  $i$ , but  $a_{ij} = 0$

Then there does not exist an exact generator for  $A$ .

	Aaa	Aa	A	Baa	Ba	B	C	Default
Aaa	85.88%	9.76%	0.48%	0	0.03%	0	0	0
Aa	0.92%	84.87%	9.64%	0.36%	0.15%	0.02%	0	0.04%
A	0.08%	2.24%	86.24%	6.09%	0.77%	0.21%	0	0.02%
Baa	0.08%	0.37%	6.02%	79.16%	6.48%	1.3%	0.11%	0.19%
Ba	0.03%	0.08%	0.46%	4.02%	76.76%	7.88%	0.47%	1.4%
B	0.01%	0.04%	0.16%	0.53%	5.86%	76.07%	2.74%	6.6%
C	0	0	0	1%	2.79%	5.38%	56.74%	25.35%
Default	0	0	0	0	0	0	0	100%

**Table 1:** Moody's average rating transition matrix of all corporates, 1980–1999

The first and the second conditions of Theorem 2 were not satisfied by any of the 32 empirical transition matrices we examined, while the third condition was satisfied in the majority of the cases. There is a natural explanation of why the third condition should be satisfied by an RCM. Usually, the one-year transition probability from the highest credit grade to default is zero. However, the default state is accessible from any credit state of the transition matrix. Therefore, we can conclude that, in the majority of practical cases, an exact generator does not exist. However, we can also not deny the possibility of an annual transition matrix having several generators, which can lead potentially to arbitrage when pricing credit risky securities.

### Example of a transition matrix

Annual transition matrices can be obtained from various sources. The major providers are Moody's Investor Service (Moody's) and Standard & Poor's rating agencies. Another source is the credit models that forecast transition migration and represent results as a one-year transition matrix (i.e., a KMV model). Most transition matrices have eight credit states, although more detailed matrices (i.e., having a greater number of states) are also used. For example, Table 1 shows a transition matrix published in January 2000 by Moody's in the Global Credit Research.

Annual transition matrices are obtained by counting the number of transitions among a set

of credit states for a given pool of companies over a one-year period. It is important to note that the published data are not complete because information is lost about companies that were withdrawn from the rating pool due to mergers or the repayment of their debt (e.g., observe that the rows of the matrix in Table 1 do not necessarily sum to one). In order to obtain a valid transition matrix in this case, it is necessary to make certain assumptions about the companies that have been removed from the sample and then adjust the matrix accordingly. For example, one can modify the diagonal elements as necessary to transform a matrix  $\mathbf{A}'$  that does not satisfy Equation 1 into a matrix  $\mathbf{A}$  that does:

$$a_{ii} = a'_{ii} + \left( 1 - \sum_{j=1}^{n-1} a'_{ij} \right) \quad i, j = 1, 2, \dots, n-1$$

This method was used to create the adjusted matrix in Table 2. Note that the default rates themselves are not adjusted.

In both of the above matrices, the diagonal elements are dominant. They represent the probabilities that a counterparty's credit rating stays unchanged during a one-year period. Thus, diagonal elements with high values suggest that the level of credit migration is low. In general, Standard & Poor's matrices tend to reflect a slightly lower migration level than those of Moody's, while KMV transition matrices exhibit the highest level of migration.



	Aaa	Aa	A	Baa	Ba	B	C	Default
Aaa	89.73%	9.76%	0.48%	0	0.03%	0	0	0
Aa	0.92%	88.87%	9.64%	0.36%	0.15%	0.02%	0	0.04%
A	0.08%	2.24%	90.59%	6.09%	0.77%	0.21%	0	0.02%
Baa	0.08%	0.37%	6.02%	85.45%	6.48%	1.3%	0.11%	0.19%
Ba	0.03%	0.08%	0.46%	4.02%	85.66%	7.88%	0.47%	1.4%
B	0.01%	0.04%	0.16%	0.53%	5.86%	84.06%	2.74%	6.6%
C	0	0	0	1%	2.79%	5.38%	65.48%	25.35%
Default	0	0	0	0	0	0	0	100%

Table 2: Adjusted transition matrix, 1980–1999

In most cases, the probabilities are also unimodal, that is, they decay monotonically in both directions from the diagonal. However, unimodality does not always hold, and it may be more likely for a given obligor to default or be downgraded by several grades, rather than by a single grade. For instance, in Table 1, the probability of a B-rated obligor defaulting is higher than that of moving to grade C.

### The regularization problem

Pricing credit risky securities requires the computation of transition probabilities over time intervals of less than one year. The time homogeneity assumption in this case leads to the problem of finding the transition matrix  $\mathbf{X}$  such that

$$\mathbf{X}^t = \mathbf{A}, \quad t > 1$$

where  $\mathbf{A}$  is the annual transition matrix and  $t$  is the number of time periods per year (e.g.,  $t = 12$  for a monthly transition matrix).

We define the set of transition matrices,  $TM(n)$ , to consist of all matrices of dimension  $n \times n$  that satisfy Equation 1.

Calculating  $\mathbf{X} = \mathbf{A}^{1/t} = \exp\left(\frac{1}{t} \ln(\mathbf{A})\right)$  may result in a matrix that has negative entries and, thus,  $\mathbf{X}$  may not belong to the set  $TM(n)$ . Note that if there is a generator  $\tilde{\mathbf{G}}$ , satisfying  $e^{\tilde{\mathbf{G}}} = \mathbf{A}$ , then

$\tilde{\mathbf{X}} = \exp\left(\frac{1}{t} \tilde{\mathbf{G}}\right)$  is a member of the set  $TM(n)$ .

However, in the vast majority of practical cases, the annual transition matrix  $\mathbf{A}$  does not have a generator. Here, we introduce a framework that allows one to solve this problem by regularization.

The regularization problem can be described in this way: Find a transition matrix  $\mathbf{X}$  that, when raised to the power  $t$ , most closely matches the annual transition matrix  $\mathbf{A}$ . In mathematical terms, this problem may be formally stated as follows:

**Problem BAM: Best approximation of the annual transition matrix**

Find  $\tilde{\mathbf{X}} \in TM(n)$  such that

$$\|\tilde{\mathbf{X}}^t - \mathbf{A}\| = \min_{\mathbf{X} \in TM(n)} \|\mathbf{X}^t - \mathbf{A}\|$$

where  $\|\cdot\|$  is a suitable norm in the space of  $n \times n$  matrices.

Since  $\tilde{\mathbf{X}}$  is raised to a power greater than one, Problem BAM is a high-dimensional, constrained non-linear optimization problem whose solution is computationally intensive.

One heuristic approach that avoids these computations is based on the following simplification of the problem:

**Problem QOM: Quasi-optimization of the root matrix**

Find  $\hat{X} \in TM(n)$  such that

$$\|\hat{X} - A^{1/t}\| = \min_{X \in TM(n)} \|X - A^{1/t}\|$$

Thus, problem QOM finds the transition matrix that is as close as possible to the fractional root of the annual transition matrix, as given by

$$A^{1/t} = \exp\left(\frac{1}{t} \ln(A)\right).$$
 Comparing problems BAM

and QOM suggests that  $\hat{X}$  and  $\tilde{X}$  should be close to each other; for this reason, it is natural to call  $\hat{X}$  a quasi-solution to problem BAM.

The second heuristic approach uses the generator as the object of regularization. First, define the set of generator matrices,  $G(n)$ , consisting of all matrices of dimension  $n \times n$  that satisfy Equation 4. Consider the problem:

*Problem QOG: Quasi-optimization of the generator*

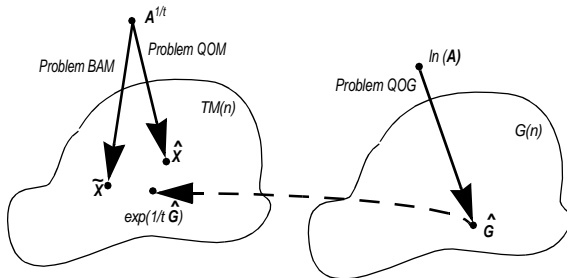
Find  $\hat{G} \in G(n)$  such that

$$\|\hat{G} - \ln(A)\| = \min_{X \in G(n)} \|X - \ln(A)\|$$

Problems BAM and QOG are related under the assumption that  $\exp(\frac{1}{t} \hat{G})$  is close to  $\tilde{X}$ , and thus

the matrix  $\hat{G}$  can also be viewed as a quasi-solution to problem BAM. Again, problem QOG is much more attractive than problem BAM in a computational sense.

Figure 1 illustrates the relationships among the above three problems.



**Figure 1:** Relationship of problems BAM, QOM and QOG

When  $A^{1/t}$  is not a valid transition matrix, problems BAM and QOM find solutions in  $TM(n)$  that are as close as possible to the root matrix. Similarly, when  $\ln(A)$  is not a valid generator, problem QOG finds the closest possible generator matrix  $\hat{G}$ . Exponentiation of the generator then yields a valid transition matrix that is close to  $A^{1/t}$ .

## Solving the quasi-optimization problems

In this section, we present fast algorithms for solving problems QOM and QOG.

### Solving problem QOM

To solve problem QOM, we use the fact that the set of transition matrices,  $TM(n)$ , can be represented as a Cartesian product of  $n$  identical  $n$ -dimensional simplices. That is, each row of the transition matrix satisfies Equation 1 and thus it belongs to the  $n$ -dimensional simplex,  $Sim(n)$ , defined as follows:

$$Sim(n) = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n, \sum_{i=1}^n x_i = 1, x_i \geq 0 \right\} \quad (7)$$

Furthermore, note that  $Sim(n)$  is contained in the hyperplane  $H(n)$

$$H(n) = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n, \sum_{i=1}^n x_i = 1 \right\}$$

Suppose that we use the Euclidean norm to measure the distance between any two points  $x$  and  $y$  in  $\mathbb{R}^n$ :

$$dist(x, y) = \sqrt{\sum_{i=1}^n (y_i - x_i)^2}, \quad x, y \in \mathbb{R}^n$$

Then problem QOM can essentially be solved on a row-by-row basis by projecting a point  $a \in \mathbb{R}^n$  (i.e., a row of the matrix  $A^{1/t}$ ) onto the simplex defined in Equation 7. That is, problem QOM can be reduced to  $n$  independent instances of the following distance minimization problem:



*Problem DMPM: Distance minimization problem for the root matrix*

For a given point  $\mathbf{a} \in \mathbb{R}^n$ ,  $\mathbf{a} = (a_1, \dots, a_n)$ , find  $\mathbf{x}^* \in \text{Sim}(n)$  such that

$$\text{dist}(\mathbf{a}, \mathbf{x}^*) = \min_{\mathbf{x} \in \text{Sim}(n)} \text{dist}(\mathbf{a}, \mathbf{x})$$

To the best of our knowledge, an algorithm for solving problem DMPM has not been previously published. The following algorithm was suggested in Merkoulouvitsh (2000), where the geometrical proof of convergence is given.

**Step 1.** Find the projection  $\mathbf{b}$  of the point  $\mathbf{a}$  on the hyperplane  $H(n)$ : set  $b_i = a_i - \lambda$ , where

$$\lambda = \frac{1}{n} \left( \sum_{i=1}^n a_i - 1 \right)$$

**Step 2.** If all the coordinates of  $\mathbf{b}$  are non-negative then stop;  $\mathbf{b}$  is the solution to problem DMPM.

**Step 3.** Let  $\hat{\mathbf{a}} = \pi(\mathbf{b})$ , where  $\pi$  is a permutation that orders the coordinates of  $\mathbf{b}$  in descending sequence.

**Step 4.** Compute  $C_k = \sum_{i=1}^k \hat{a}_i - k \cdot \hat{a}_k$  for  $k = 1, 2, \dots, n$ . The sums  $C_k$  satisfy  $0 \leq C_1 \leq C_2 \leq \dots \leq C_n$ .

**Step 5.** Find  $k^* = \max\{k: k \geq 1, C_k \leq 1\}$ .

**Step 6.** Construct the vector  $\hat{\mathbf{x}} \in \text{Sim}(n)$  as follows. For all  $j > k^*$  set  $\hat{x}_j = 0$ , and for  $j \leq k^*$  set

$$\hat{x}_j = \hat{a}_j + \frac{1}{k^*} \left( 1 - \sum_{i=1}^{k^*} \hat{a}_i \right)$$

**Step 7.** Apply the inverse permutation  $\pi^{-1}$  to  $\hat{\mathbf{x}}$ ;  $\pi^{-1}(\hat{\mathbf{x}})$  is the solution to problem DMPM.

The correctness of the algorithm above follows from the geometrical proof in Merkoulouvitsh (2000) and from analytical arguments in Tuenter (2000). While a detailed proof is beyond the

scope of this paper, it relies on the following key propositions.

**Proposition 1:** Let  $\mathbf{a} = (a_1, \dots, a_n)$  be the initial point and let  $\mathbf{x}^* = (x_1^*, \dots, x_n^*)$  be the optimal solution to problem DMPM. Then if  $a_1 \geq \dots \geq a_n$ , then  $x_1^* \geq \dots \geq x_n^*$ .

Proposition 1 states that the elements of the optimal solution are ordered in the same sequence as those of the initial point. This allows us to consider only the case where the coordinates of  $\mathbf{a}$  are ordered, without loss of generality.

**Proposition 2:** If  $\mathbf{b}$  is the projection of  $\mathbf{a}$  on  $H(n)$  and  $b_k < 0$  some  $k$ , then  $x_j^* = 0$  for  $j = k, \dots, n$ .

Proposition 2 states that, if after projection on the hyperplane, some of the coordinates are negative, then, in the optimal solution these coordinates equal zero. This allows us to reduce the original problem to a discrete optimization problem as follows.

With  $\lambda$  obtained as in Step 1 of the algorithm,

define the function  $f(l) = l \cdot \lambda^2 + \sum_{i=l+1}^n a_i^2$  for

$l = 1, 2, \dots, n$  (note that  $\sum_{i=k}^m a_i^2 \equiv 0$  for  $k < m$ ).

The solution of the distance minimization problem can be obtained from solving:

$$\begin{aligned} \min f(l) \\ \text{s.t.} \\ l \cdot \lambda + \sum_{i=1}^l a_i = 1 \\ \lambda + a_l \geq 0, l = 1, \dots, n \\ l \in \mathbb{Z}^+ \end{aligned}$$

The solution  $l^*$  to this problem determines the optimal number of coordinates  $k^*$  to be equal to zero in Step 5.

**Proposition 3:** The objective function  $f(l)$  is monotonic (i.e.,  $f(l) > f(l+1)$ ).

Proposition 3 follows from the identity

$$f(m) = f(m+1) + \frac{1}{m \cdot (m+1)} \cdot (1 - T_m)^2$$

where

$$T_m = \sum_{i=1}^m a_i - m \cdot a_{m+1}$$

From Proposition 3, it follows that the optimal solution to the above problem is

$$l^* = \max \left\{ l : 1 \leq l \leq n, l a_l \geq \sum_{i=1}^l a_i - l \right\}$$

This yields an  $l^*$  equal to  $k^*$ , as calculated in Step 5 of the algorithm.

Note that problem DMPM can also be solved in an iterative manner. In this case, we simply replace Step 3 by:

**Step 3'.** Fix any negative elements of  $\mathbf{b}$  equal to zero, set  $\mathbf{a} = \mathbf{b}$  and go to Step 1 (do not update any elements once they have been fixed to zero).

The iterative algorithm stops after  $m$  steps where  $m$  does not exceed the size of the vector  $\mathbf{a}$  (Merkoulovitch 2000).

### Solving problem QOG

Problem QOG is different from problem QOM in a geometrical sense. While the space of the transition matrices,  $TM(n)$ , is a Cartesian product of simplices, the space of their generators,  $G(n)$ , is a Cartesian product of  $n$ -dimensional cones. Each row of a generator has the property that its elements sum to zero and non-diagonal elements are non-negative (Equation 4). By permuting the row elements, one can always represent them as a point in a standard cone,  $K(n)$ , defined by

$$K(n) = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n, \right. \\ \left. \sum_{i=1}^n x_i = 0, x_i \leq 0, x_i \geq 0, \text{ for } i \geq 2 \right\} \quad (8)$$

Note that  $K(n)$  is contained in the hyperplane  $\hat{H}(n)$ :

$$\hat{H}(n) = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n, \sum_{i=1}^n x_i = 0 \right\}$$

In a manner similar to problem QOM, problem QOG can be solved on a row-by-row basis by projecting a point  $\mathbf{a} \in \mathbb{R}^n$  (i.e., a row of the matrix  $\ln(\mathbf{A})$ ) onto the cone defined in Equation 8. Thus, problem QOG can be reduced to  $n$  independent instances of the following distance minimization problem:

*Problem DMPG: Distance minimization problem for the generator*

For a given point  $\mathbf{a} \in \mathbb{R}^n$ ,  $\mathbf{a} = (a_1, \dots, a_n)$ , find  $\mathbf{g}^* \in K(n)$  such that

$$\text{dist}(\mathbf{a}, \mathbf{g}^*) = \min_{\mathbf{g} \in K(n)} \text{dist}(\mathbf{a}, \mathbf{g})$$

The optimal solution to problem DMPG can be obtained as follows:

**Step 1.** Let  $\mathbf{b}$  be the projection of  $\mathbf{a}$  on  $\hat{H}(n)$ : set  $b_i = a_i - \lambda$ , where

$$\lambda = \frac{1}{n} \left( \sum_{i=1}^n a_i \right)$$

**Step 2.** Let  $\hat{\mathbf{a}} = \pi(\mathbf{b})$ , where  $\pi$  is a permutation that orders the coordinates of  $\mathbf{b}$  in descending sequence.

**Step 3.** Find  $l^*$ , the smallest integer  $2 \leq l \leq n-1$  that satisfies  $(n-l+1)\hat{a}_{l+1} \geq \hat{a}_l + \sum_{i=0}^{n-(l+1)} \hat{a}_{n-i}$ .

**Step 4.** Let  $\mathfrak{I} = \{i: 2 \leq i \leq l^*\}$ . Construct the vector  $\hat{\mathbf{g}} \in K(n)$  as follows. For all  $i \in \mathfrak{I}$  set  $\hat{g}_i = 0$ , and set  $\hat{g}_i = \hat{a}_i - \frac{1}{(n-l^*+1)} \sum_{j \notin \mathfrak{I}} \hat{a}_j$  otherwise.

**Step 5.** Apply the inverse permutation  $\pi^{-1}$  to  $\hat{\mathbf{g}}$ ;  $\pi^{-1}(\hat{\mathbf{g}})$  is the solution to problem DMPM.

	Aaa	Aa	A	Baa	Ba	B	C	Default
Aaa	94.713%	5.164%	0.114%	-0.005%	0.014%	-0.001%	-0.000%	-0.001%
Aa	0.486%	94.226%	5.090%	0.104%	0.069%	0.006%	0.000%	0.020%
A	0.038%	1.179%	95.092%	3.244%	0.348%	0.094%	-0.002%	0.006%
Baa	0.041%	0.176%	3.203%	92.341%	3.490%	0.623%	0.053%	0.073%
Ba	0.015%	0.039%	0.205%	2.166%	92.436%	4.271%	0.231%	0.636%
B	0.005%	0.020%	0.078%	0.245%	3.166%	91.583%	1.584%	3.319%
C	0.000%	-0.001%	-0.013%	0.554%	1.542%	3.079%	80.887%	13.952%
Default	0	0	0	0	0	0	0	100%

Table 3: Square root of Moody's transition matrix (1999)

The correctness of the above algorithm can be proved in a manner similar to that for the case of DMPM. An iterative implementation is possible in this case as well.

#### Other regularization methods for generators

In this section, we describe regularization methods suggested by Stromquist (1996) and Araten and Angbazo (1997). These methods adjust the matrix  $G = \ln(A)$ , the logarithm of the annual transition matrix, in order to construct a generator  $\hat{G}$ . To satisfy Equation 6, all negative non-diagonal elements of  $G$  are first set to zero, and then selected elements are adjusted to ensure that each row sums to zero. Two such methods are **diagonal adjustment** (DA), which modifies only the diagonal elements, and **weighted adjustment** (WA), which modifies all non-zero elements. The computation of  $\hat{G}$  proceeds as follows:

**Step 1.** Set  $\hat{g}_{ij} = \begin{cases} 0 & \text{if } (i \neq j) \text{ and } g_{ij} < 0 \\ g_{ij} & \text{otherwise} \end{cases}$  for

$i, j = 1, 2, \dots, n$ .

**Step 2a (diagonal adjustment).** Set the diagonal elements to the negative sum of the non-diagonal elements:

$$\hat{g}_{ii} = - \sum_{j=1, j \neq i}^n \hat{g}_{ij} \quad \text{for } i = 1, 2, \dots, n$$

or

**Step 2b (weighted adjustment).** Adjust the non-zero elements according to their relative magnitudes:

$$\hat{g}_{ij} = \hat{g}_{ij} - \left| \hat{g}_{ij} \right| \frac{\sum_{j=1}^n \hat{g}_{ij}}{\sum_{j=1}^n |\hat{g}_{ij}|} \quad \text{for } i, j = 1, 2, \dots, n$$

#### Example: a six-month transition matrix

We now illustrate the regularization algorithms by calculating a six-month transition matrix based on the annual transition matrix ( $A$ ) in Table 2. Observe that  $A^{1/2}$ , the square root of the annual transition matrix, contains negative elements (Table 3) and so it is not a valid transition matrix.

Table 4 shows  $\hat{X}$ , the transition matrix that is as close as possible to  $A^{1/2}$ , which we obtain by solving problem QOM with the iterative row projection algorithm. Note that all elements of  $\hat{X}$  are non-negative.

We also compute generators that approximate

$\ln(A)$  by solving problem QOG, and with the DA and WA methods. When computing the generator in this case, it is necessary to adjust only four of the eight rows comprising  $\ln(A)$ .

	Aaa	Aa	A	Baa	Ba	B	C	Default
Aaa	94.711%	5.164%	0.113%	0.000%	0.012%	0	0	0
Aa	0.486%	94.226%	5.090%	0.104%	0.069%	0.006%	0	0.020%
A	0.038%	1.179%	95.092%	3.244%	0.348%	0.093%	0	0.006%
Baa	0.041%	0.176%	3.203%	92.341%	3.490%	0.623%	0.053%	0.073%
Ba	0.015%	0.039%	0.206%	2.166%	92.436%	4.271%	0.231%	0.636%
B	0.005%	0.020%	0.078%	0.245%	3.166%	91.583%	1.584%	3.319%
C	0	0	0	0.551%	1.539%	3.076%	80.884%	13.949%
Default	0	0	0	0	0	0	0	100%

Table 4: Six-month transition matrix  $\hat{X}$ 

Table 5 compares the distances from  $\ln(\mathbf{A})$ , as measured by the Euclidean norm, for the rows adjusted using the three methods. As expected, the generator obtained by solving problem QOG is closest to  $\ln(\mathbf{A})$  under this measure.

ROW	QOG	WA	DA
1	6.769	7.355	8.898
2	0.032	0.036	0.042
3	1.021	1.122	1.351
7	6.475	7.052	8.651

Table 5: Euclidean distance from  $\ln(\mathbf{A})$  for rows adjusted by generator regularization methods ( $\times 10^{-4}$ )

To evaluate the effects of regularization, we calculate an approximate annual transition matrix  $\hat{\mathbf{A}}$ , using the result of each method (e.g., Table 6 contains the matrix  $\hat{\mathbf{A}} = \hat{X}^2$ ) and compare them to the original transition matrix. Although we use the Euclidean norm in our regularization algorithms, Euclidean distance is somewhat awkward for this purpose; the absolute

deviation between corresponding transition probabilities provides a more intuitive comparison. Thus, we measure the discrepancy between the original transition matrix,  $\mathbf{A}$ , and its approximation,  $\hat{\mathbf{A}}$ , using:

- The maximum absolute deviation (MAX), or infinity norm,

$$\text{MAX}(\hat{\mathbf{A}}, \mathbf{A}) = \|\hat{\mathbf{A}} - \mathbf{A}\|_{\infty} = \max_{i,j} |\hat{a}_{ij} - a_{ij}|$$

- The mean absolute deviation (MAD), or  $L_1$  norm,

$$\text{MAD}(\hat{\mathbf{A}}, \mathbf{A}) = \|\hat{\mathbf{A}} - \mathbf{A}\|_1 = \frac{1}{n} \sum_{i,j} |\hat{a}_{ij} - a_{ij}|$$

Table 7 indicates that solving problem QOM best approximates the original transition matrix under both the MAX and MAD measures. Thus, we conclude that  $\hat{X}$  is the preferred six-month transition matrix. Note that for the generator-based methods, weight adjustment outperforms row projection under the MAX measure in this case.

	Aaa	Aa	A	Baa	Ba	B	C	Default
Aaa	89.73%	9.76%	0.48%	0.01%	0.03%	0.00%	0.00%	0.00%
Aa	0.92%	88.87%	9.64%	0.36%	0.15%	0.02%	0.00%	0.04%
A	0.08%	2.24%	90.59%	6.09%	0.77%	0.21%	0.00%	0.02%
Baa	0.08%	0.37%	6.02%	85.45%	6.48%	1.30%	0.11%	0.19%
Ba	0.03%	0.08%	0.46%	4.02%	85.66%	7.88%	0.47%	1.40%
B	0.01%	0.04%	0.16%	0.53%	5.86%	84.06%	2.74%	6.60%
C	0.00%	0.00%	0.02%	1.00%	2.78%	5.37%	65.48%	25.34%
Default	0	0	0	0	0	0	0	100%

**Table 6:** Approximate annual transition matrix  $\hat{A} = \hat{X}^2$

	QOM	QOG	WA	DA
MAX	2.320	4.599	4.544	6.341
MAD	0.131	0.382	0.395	0.404

**Table 7:** Differences between original (**A**) and approximate ( $\hat{A}$ ) annual transition matrix ( $\times 10^{-4}$ )

#### Robustness of regularization methods

We now briefly examine the robustness of the proposed regularization methods. Specifically, we consider the question: When solving problems QOM and QOG, do similar annual transition matrices give rise to similar six-month transition matrices?

For purposes of this example, we construct a perturbed matrix  $A'$  from the annual transition matrix **A** by setting

$$a'_{ij} = (a_{ij} u_{ij}) \left( \sum_{j=1}^n (a_{ij} u_{ij}) \right)^{-1}$$

for  $i, j = 1, 2, \dots, n$ , where  $u_{ij}$  is a uniformly distributed random variable from the interval  $[0.99, 1.01]$ . We then compute six-month transition matrices  $X'$  and  $\exp\left(\frac{1}{2}G'\right)$  from  $A'$  by solving problems QOM and QOG, respectively, and measure the deviations between the original and perturbed matrices.

Table 8 reports the results of 1,000 repetitions of the above experiment. Based on these results, we conclude that the regularization methods are robust with respect to small changes in the annual transition matrix.

	$\ A - A'\ $	$\ X - X'\ $	$\left\  \exp\left(\frac{1}{2}G\right) - \exp\left(\frac{1}{2}G'\right) \right\ $
MAX	0.872 (1.880)	0.527 (1.163)	0.516 (1.164)
MAD	0.090 (0.175)	0.060 (0.100)	0.050 (0.107)

**Table 8:** Mean differences between original and perturbed transition matrices (maximum in parentheses) ( $\times 10^{-3}$ )

#### Computational experiments

We now compare the performance of the regularization methods on a set of 32 annual transition matrices obtained from various sources. The sample includes:

- seventeen matrices of dimension  $8 \times 8$  from Standard & Poor's the period 1981–1997
- one  $18 \times 18$  matrix from Standard & Poor's for 1999
- one matrix of dimension  $8 \times 8$  from Moody's for 1999
- four  $8 \times 8$  matrices from CreditMetrics

(1997): KMV for 1995, Standard & Poor's for 1996, Moody's for 1995 and aggregated from historical data of Standard & Poor's

- one  $8 \times 8$  matrix from Lando (2000) (reported on the webpage of CreditMetrics on February 7, 2000)
- eight matrices calibrated from credit spreads at Algorithmics (one of dimension  $19 \times 19$ , one of dimension  $8 \times 8$  and six of dimension  $17 \times 17$ ) for 1999–2000

We find that the logarithm of the annual transition matrix is a valid generator (i.e., it satisfies Equation 6) for only one member of the sample. Thus, the remaining 31 cases require regularization in order to obtain transition matrices for credit risk analysis. We evaluate the following methods:

- QOM: Obtain a quarterly transition matrix by solving problem QOM using iterative row projection.
- QOG: Obtain a generator by solving problem QOG using iterative row projection.
- WA: Obtain a generator using weight adjustment.
- DA: Obtain a generator using diagonal adjustment.

Table 9 summarizes the performance of the regularization methods on the 31 matrices that required regularization. As in the previous example, we report the differences between the actual annual transition matrix and the approximate annual transition matrices derived from the regularization methods. QOM regularization yields the best results overall, while QOG provides the best generator. Moreover, we find that QOM, in fact, performs best in all 31 cases.

Since it may be of interest to consider the magnitudes of these differences on a relative scale, we also report the MAX results on a percentage basis (Table 10). Note that while the maximum relative differences are large, this is due to the small sizes of the changed elements (i.e., 0.0135 for QOM and 0.0038 for the remaining cases).

	QOM	QOG	WA	DA
MAX	1.11 (5.69)	1.65 (8.27)	1.93 (9.58)	2.60 (12.45)
MAD	0.11 (0.62)	0.22 (1.22)	0.29 (1.57)	0.33 (1.81)

**Table 9:** Mean differences between original ( $\mathbf{A}$ ) and approximate ( $\hat{\mathbf{A}}$ ) annual transition matrices (maximum in parentheses) ( $\times 10^{-2}$ )

	QOM	QOG	WA	DA
MAX	16.5 (420.1)	39.0 (605.6)	17.0 (594.0)	5.8 (620.8)

**Table 10:** Mean relative differences between original ( $\mathbf{A}$ ) and approximate ( $\hat{\mathbf{A}}$ ) annual transition matrices (maximum in parentheses) (%)

The effect of regularization on the default rates (i.e., the final column of the transition matrix) is of special interest. There are two reasons why default rates are particularly important: first, they are used to price credit risky instruments and, second, they are often considered to be the most accurate information contained in the transition matrix. Table 11 gives the absolute and relative deviations for the default probabilities, as measured by MAX. Again, QOM regularization results in the smallest deviations. In most cases, the largest error is due to the regularization reducing, rather than increasing, one of the default rates. This implies that one may expect a slight underestimation of defaults as a result of regularization.

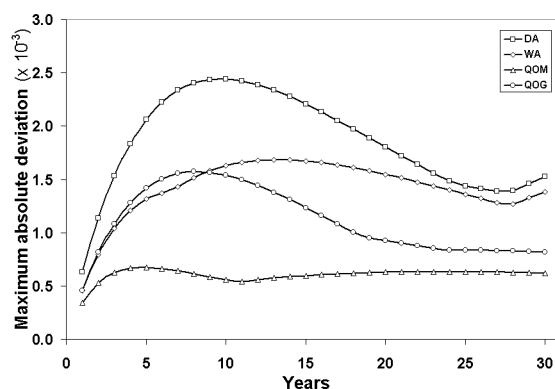
	QOM	QOG	WA	DA
MAX ( $\times 10^{-2}$ )	2.3 (8.7)	2.5 (10.1)	3.7 (19.6)	3.8 (23.5)
MAX (%)	4.9 (248.3)	4.2 (331.2)	4.1 (339.1)	7.8 (339.4)

**Table 11:** Mean differences between default probabilities in original ( $\mathbf{A}$ ) and approximate ( $\hat{\mathbf{A}}$ ) annual transition matrices (maximum in parentheses)



Next, we evaluate the regularization methods over a longer time horizon by computing the differences between  $A^n$  and  $\hat{A}^n$  for  $n = 1, 2, \dots, 30$ . Thus, we are able to identify cases where a particular regularization algorithm works well for a short time period, but gives unacceptable results for a long time horizon. Note that while we raise the approximate annual transition matrix to powers greater than one in this analysis, we do not recommend doing so in practice. Instead, when the time horizon spans several years, it is preferable to use the actual annual transition matrix for an integral number of years, and the regularized transition matrix only for periods of less than one year. Thus, if  $A$  is an annual transition matrix and  $\hat{X}$  is the derived six-month transition matrix, then, it is more accurate, for example, to calculate a 3.5-year transition matrix as  $A^3 \hat{X}$  rather than  $\hat{X}^7$ .

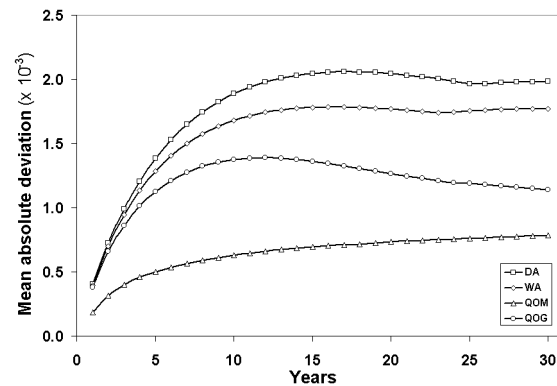
Figure 2 shows MAX for the Moody's transition matrix for 1999 over a 30-year horizon. The QOM regularization results in the lowest error over the entire horizon (recall that we calculate a quarterly transition matrix in this case). Among the generator regularization methods, weighted adjustment performs best for a horizon of up to 10 years, while QOG regularization gives the best results thereafter.



**Figure 2:** MAX for Moody's 1999 transition matrix

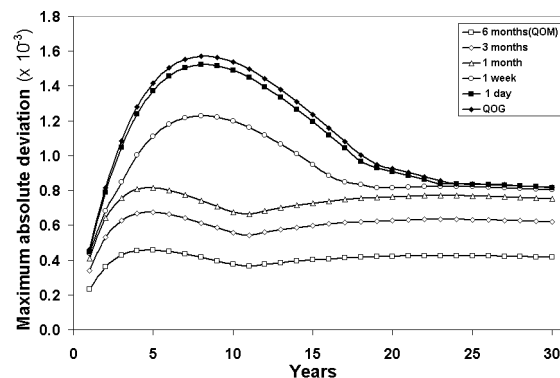
Similarly, when using MAD (Figure 3), QOM regularization again gives the lowest error over the entire 30-year horizon, while QOG

regularization consistently performs best among the generator estimation methods.



**Figure 3:** MAD for Moody's 1999 transition matrix

Figure 4 shows that the error, as measured by MAX, for QOM regularization increases as the period spanned by the transition matrix decreases. In fact, for a daily transition matrix, the performance of QOM regularization and QOG regularization are almost identical. The same effect is observed under the MAD measure as well.



**Figure 4:** MAX for different fractional roots

These results suggest that QOM regularization is the preferred approach for estimating transition matrices for short time periods. However, an important drawback of this method is that the resulting transition matrices cannot be used in continuous-time credit models, which require a generator. Moreover, once a generator is

computed, one can easily calculate a transition matrix for any time interval, no matter how small. Thus, the applicability of QOM regularization is limited to discrete-time models.

Finally, we evaluate the impact of the size and the level of migration in the transition matrix on the performance of QOG regularization. Figure 5 plots MAX when QOG regularization is applied to the following matrices:

- $8 \times 8$ , low migration (Moody's 1999)
- $8 \times 8$ , high migration (KMV 1995)
- $17 \times 17$ , high migration (Algorithmics)
- $18 \times 18$ , low migration (S&P 1999).

The results suggest that the size of the transition matrix does not affect the approximation error. However, the approximation error tends to be larger when transition matrices have a high level of migration.

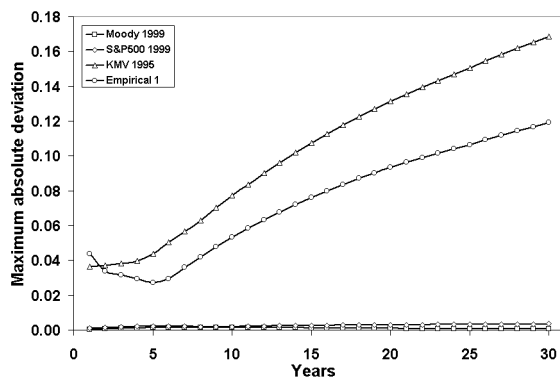


Figure 5: MAX for QOG regularization

## Conclusions

This paper considers the computation of transition matrices for arbitrary time periods based on a given annual transition matrix. Simply calculating roots of the annual transition matrix is not a valid approach because the resulting matrices typically contain negative elements, and, thus, do not represent valid transition matrices. This problem is closely related to the well-known embedding problem for Markov chains. We show that, in most practical cases, the

generator of a particular transitional matrix does not exist and, thus, regularization algorithms are required to find the transition probabilities for time horizons of less than one year.

We demonstrate that the regularization problem can be expressed as a multidimensional, constrained, non-linear optimization problem, whose solution is computationally intensive. We propose an alternative, efficient approach that obtains a valid transition matrix or generator from the root or logarithm, respectively, of the annual transition matrix. Since this transformation is done on a row-by-row basis, the process reduces to solving a set of simple optimization problems.

Evaluating the regularization algorithms on a set of sample problems suggests that approximating the root of the annual transition matrix yields the best results. Given our relatively small number of test cases, we do not draw any formal statistical conclusions in this regard. However, our empirical results do give an indication of the magnitudes of potential approximation errors due to regularization. The results suggest that the use of quasi-optimization is justified by its high precision and computational simplicity. Since the algorithms are very fast, it is feasible to apply all of them, and then select the transition matrix that is most accurate with respect to a chosen error measure.

There are several possible directions for future research. In the transition matrix framework, the most accurate regularization procedure is based on solving problem BAM directly. Thus, solving the corresponding non-linear optimization problem may lead to a better approximation of the transition matrix.

Finally, a large class of credit risk models is based on finite Markov chains with an absorbing state. These models have one significant theoretical disadvantage; namely, the long-term dynamics of the credit states of market participants cannot be described by this type of model because it does not account for the appearance of new companies with a random initial credit state. Therefore, more adequate models can be based on the Markov processes that simulate the birth

of new market players. In this case, the model estimation starts with the parameters of the generator governing the Markov process, and all the difficulties related to the regularization problem are avoided. This is an interesting problem for future discussions.

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## Endnotes

1. It is possible to obtain the set of past historical annual matrices that were published by agencies each year. These data are used extensively by the same sources to construct the averaged annual prediction transition matrices.