Probability Distributions (A Different Perspective)

Srijit Mukherjee

Indian Statistical Institute

June 9, 2021

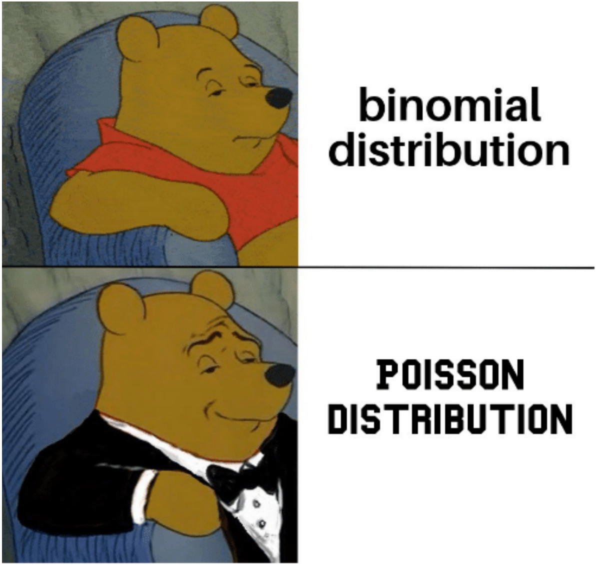
Meme


TH...TH...
he's about to say his first words
VIA 9GAG.COM
THE SUM OF N INDEPENDENT GEOMETRIC
RANDOM VARIABLES FOLLOW
A NEGATIVE BINOMIAL DISTRIBUTION
imgflip.com
VIA 9GAG.COM

Meme


IF HOTNESS WAS A NORMAL
DISTRIBUTION
YOU WOULD BE 3 STANDARD DEVIATIONS
ABOVE THE MEAN

Meme



Discrete Random Variables

- Bernoulli Random Variable
- Binomial Random Variable
- Poisson Random Variable
- Geometric Random Variable
- Negative Binomial Random Variable
- Hyper Geometric Random Variable

Continuous Random Variables

- Exponential Random Variable
- Gamma Distribution
- Beta Random Variable

Bernoulli Trials

Bernoulli(p)

A random variable which takes the value 1 with probability p and the value 0 with probability $q = 1 - p$.

Model

A model for the set of possible outcomes of any single experiment that asks a yes–no question.

Bernoulli Process

Srijit Mukherjee

Memes

Probability Distributions

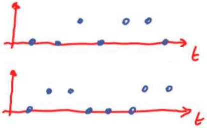
Bernoulli Process

Poisson Process

Summary

Bernoulli Process

A Bernoulli process is a sequence of independent identically distributed Bernoulli trials.



Examples

- Sequence of lottery wins/losses
- Arrivals (each second) to a bank
- Arrivals (at each time slot) to server
- Sampling with Replacement

Five Questions in a Bernoulli Process

The Questions

- How many arrivals(successes) there will be among the first n Bernoulli trials?
- How many trials it will take to get the first arrrival (success)?
- How many trials it will take to get the r^{th} arrrival (success) from the $(r-1)^{th}$ arrrival (success) ?
- How many trials it will take to get the first r arrrivals (successes)?
- Given that there are M successes among N trials, how many of the first n trials are successes?

Binomial Distribution

$X_1, X_2, \cdots X_n, \cdots$ is a Bernoulli Process.

The Question

How many arrivals(successes) there will be among the first n Bernoulli trials?

The Reply

- The number of arrivals(successes) there will be among the first n Bernoulli trials $= S_n = \sum_{i=1}^{n} X_i$
- S_n follows Binomial(n, p)

Geometric Distribution

$X_1, X_2, \cdots X_n, \cdots$ is a Bernoulli Process.

The Question

How many trials it will take to get the first arrrival (success)?

The Reply

- The number of trials it will take to get the first arrrival (success)? $= T_1 = \min\{i : X_i = 1\}$
- T_1 follows Geometric(p)

Geometric Distribution

$X_1, X_2, \cdots X_n, \cdots$ is a Bernoulli Process.

The Question

How many trials it will take to get the r^{th} arrrival (success) from the $(r-1)^{th}$ arrrival (success) ?

The Reply

- The number of trials trials it will take to get the r^{th} arrrival (success) from the $(r-1)^{th}$ arrrival (success)? $= T_{(r-1,r)}$
- $T_{(r-1,r)} = T_1$ follows Geometric(p)

The Memoryless Property

- $X_{N+1}, X_{N+2}, \cdots$ is a Bernoulli Process.
- $X_{T_1+1}, X_{T_1+2}, \cdots$ is a Bernoulli Process.

Negative Binomial Distribution

$X_1, X_2, \cdots X_n, \cdots$ is a Bernoulli Process.

The Question

How many trials it will take to get the first r arrrivals (successes)?

The Reply

- The number of trials it will take to get the first r arrrivals (success)? $= T_r$
- $T_r = \sum_{i=1}^{r} T_{(i-1,1)}$ follows Negative Binomial(r, p)
- $T_{(i-1,1)}$ follows Geometric(p)

Hyper Geometric Distribution

$X_1, X_2, \cdots X_n, \cdots$ is a Bernoulli Process.

The Question

Given that there are M successes among N trials, how many of the first n trials are successes?

The Reply

- Given that there are M successes among N trials, the number of successes out of the first n trials ? $= H_{(N,M,n)}$
- $H_{(N,M,n)}$ follows Hyper Geometric(N, M, n)
- $N \to \infty$, $M/N \to p$, and n is held constant, then $H_{(N,M,n)} \to \text{Bin}(n, p)$

Poisson Process $N(t)$

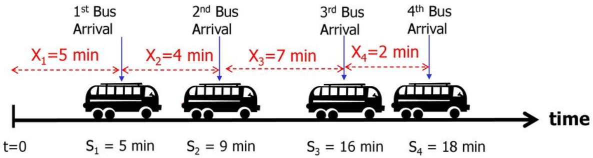
$N(t)$ follows Poi(λt)

The counting process $\{N(t), t \geqslant 0\}$ is said to be a Poisson process having rate $\lambda, \lambda > 0$, if

- $N(0) = 0$
- The process has independent increments.
- The number of events in any interval of length t is Poisson distributed with mean λt. That is, for all $s, t \geqslant 0$

$$P\{N(t+s) - N(s) = n\} = e^{-\lambda t}\frac{(\lambda t)^n}{n!}, \quad n = 0, 1, \ldots$$

Poisson Process $N(t)$



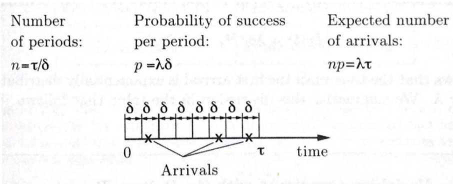
$N(t)$ follows Poi(λt)

The counting process $\{N(t), t \geqslant 0\}$ is said to be a Poisson process having rate $\lambda, \lambda > 0$, if

- $N(0) = 0$
- The process has independent increments.
- $P\{N(h) = 1\} = \lambda h + o(h)$
- $P\{N(h) \geqslant 2\} = o(h)$

Bernoulli Process → Poisson Process

- $N(\tau)$ be the number of the arrivals of $\boldsymbol{X}$ in an interval of length τ.
- Partition the interval into $n \gg 1$ small intervals of length $\left(\frac{\tau}{n}\right)$.
- Let N_i be the number of the arrivals in the i th small interval.



Bernoulli Process → Poisson Process

- The number of the arrivals of $\boldsymbol{X}$ in a small interval is essentially Bernoulli $(\lambda\delta)$.
- $N_i \sim$ Bernoulli $\left(\lambda\left(\frac{\tau}{n}\right)\right)$ and $N(\tau) = N_1 + \cdots + N_n$. Hence

$$N(\tau) \sim \text{ binomial } \left(n, \lambda\left(\frac{\tau}{n}\right)\right) \xrightarrow{n\to\infty} \text{ Poisson } (\lambda\tau)$$

- How many trials → How much time?

Five Questions in a Poisson Process

The Questions

- How many events (arrivals) there will be in the time t?
- How much time it will take to get the first event (arrival)?
- How much time it will take to get the r^{th} event (arrrival) from the $(r-1)^{th}$ event (arrival) ?
- How much time it will take to get the first r events (arrivals)?
- Given that $\alpha + \beta$ events have occurred in a time interval, then what is the fraction of that interval until the α^{th} event occurs?

Poisson Distribution

$\{N(t), t \geqslant 0\}$ is a Poisson process having rate $\lambda, \lambda > 0,$.

The Question

How many events (arrivals) there will be in the time t?

The Reply

- The number of arrivals(successes) in the time $t = N(t)$.
- $S_n = Binomial(n, p = \frac{t}{n}) \overset{np \to \lambda t}{\longrightarrow} N(t) = Poi(\lambda t)$

Exponential Distribution

$\{N(t), t \geqslant 0\}$ is a Poisson process having rate $\lambda, \lambda > 0$,.

The Question

How much time it will take to get the first event (arrival)?

The Reply

- The amount time it will take to get the first arrrival (success)? $= T_1 = \min\{i : N(t) = 1\}$
- T_1 follows Expoential(λ)
- $Geometric(p) \overset{np \to \lambda; x = \frac{k}{n}}{\longrightarrow} Exponential(\lambda)$

Exponential Distribution

$\{N(t), t \geqslant 0\}$ is a Poisson process having rate $\lambda, \lambda > 0$,.

The Question

How much time it will take to get the r^{th} event (arrrival) from the $(r-1)^{th}$ event (arrival) ?

The Reply

- The amount of time it will take to get the r^{th} arrrival (success) from the $(r-1)^{th}$ arrrival (success)? $= T_{(r-1,r)}$
- $T_{(r-1,r)} = T_1$ follows Exponential(p)

The Memoryless Property

- $\{N(t), t \geqslant k\}$ is a Poisson process having rate $\lambda, \lambda > 0$,.
- $\{N(t), t \geqslant T_1\}$ is a Poisson process having rate $\lambda, \lambda > 0$,.

Gamma Distribution

$\{N(t), t \geqslant 0\}$ is a Poisson process having rate $\lambda, \lambda > 0$,.

The Question

How much time it will take to get the first r events (arrivals)?

The Reply

- The amount of time it will take to get the first arrrival (success)? $= T_r$
- $T_r = \sum_{i=1}^{r} T_{(i-1,1)}$ follows Gamma(r, λ)
- $T_{(i-1,1)}$ follows Exponential(λ)

Beta Distribution

$\{N(t), t \geqslant 0\}$ is a Poisson process having rate $\lambda, \lambda > 0$,.

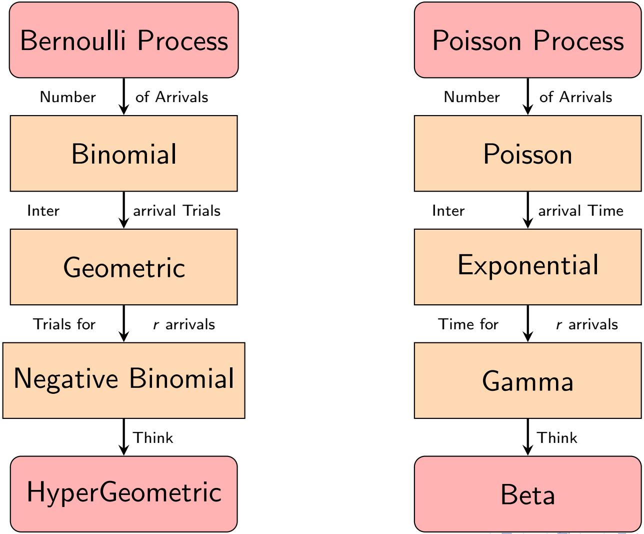
The Question

Given that $\alpha + \beta$ events have occurred in a time interval, then what is the fraction of that interval until the α^{th} event occurs?

The Reply

- It follows Beta(α, β).

Summary


Bernoulli Process
Number of Arrivals
Binomial
Inter arrival Trials
Geometric
Trials for r arrivals
Negative Binomial
Think
HyperGeometric
Poisson Process
Number of Arrivals
Poisson
Inter arrival Time
Exponential
Time for r arrivals
Gamma
Think
Beta