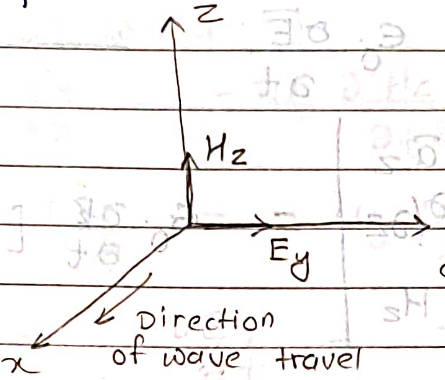
* Wave Equation For Wave In Free Space OR Lossless Medium

For free space $\sigma = 0$



consider electromagnetic field travellingin x dir^n having E_y and H_z component as shown.

consider Maxwell's eq^n derived from Faraday's law

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} = -\mu_0 \frac{\partial \bar{H}}{\partial t} \qquad (1)$$

$$\begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_x & E_y & E_z \end{vmatrix} = -\mu_0 \frac{\partial}{\partial t}\left(H_x \bar{a}_x + H_y \bar{a}_y + H_z \bar{a}_z\right)$$

But existing components are only E_y & H_z

$$\begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & E_y & 0 \end{vmatrix} = -\mu_0 \frac{\partial H_z}{\partial t} \bar{a}_z$$

$$\frac{\partial E_y}{\partial x} = -\mu_0 \frac{\partial H_z}{\partial t} \qquad (2)$$

Consider Maxwell's eq^n derived from Ampere's law

$$\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t} = \sigma \bar{E} + \epsilon_0 \frac{\partial \bar{E}}{\partial t}$$

$$\nabla \times \bar{H} = \epsilon_0 \frac{\partial \bar{E}}{\partial t} \quad \text{———} \quad (3) \qquad \because \sigma = 0$$

$$\therefore \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 0 & H_z \end{vmatrix} = \epsilon_0 \cdot \frac{\partial}{\partial t} [E_y] \, \bar{a}_y$$

$$\therefore \; -\bar{a}_y \frac{\partial}{\partial x} H_z = \epsilon_0 \frac{\partial E_y}{\partial t} \cdot \bar{a}_y$$

$$\therefore \; \frac{-\partial H_z}{\partial x} = \epsilon_0 \frac{\partial E_y}{\partial t} \quad \text{———} \quad (4)$$

Diff. eq^n (4) w.r.t. t

$$-\frac{\partial}{\partial t} \cdot \frac{\partial H_z}{\partial x} = \epsilon_0 \frac{\partial^2 E_y}{d t^2}$$

Interchange order of space & time derivative

$$\therefore \; -\frac{\partial}{\partial x} \cdot \frac{\partial H_z}{\partial t} = \epsilon_0 \frac{\partial^2 E_y}{\partial t^2}$$

From eq^n (2) $\quad \frac{\partial H_z}{\partial t} = \frac{-1}{\mu_0} \cdot \frac{\partial E_y}{\partial x}$

$$\therefore \; -\frac{\partial}{\partial x} \cdot \left(\frac{-1}{\mu_0} \cdot \frac{\partial E_y}{\partial x} \right) = \epsilon_0 \frac{\partial^2 E_y}{\partial t^2}$$

$$\frac{1}{\mu_0} \frac{\partial^2 E_y}{\partial x^2} = \epsilon_0 \frac{\partial^2 E_y}{\partial t^2}$$

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D'Alembert's eqⁿ

$$\therefore \boxed{\frac{\partial^2 E_y}{\partial t^2} = \frac{1}{\mu_0 \epsilon_0} \cdot \frac{\partial^2 E_y}{\partial x^2}}$$

wave eqⁿ in free space or lossless medium in terms of E

Diff. eqⁿ (2) w.r.t. t

$$\frac{\partial}{\partial t} \cdot \frac{\partial E_y}{\partial x} = -\mu_0 \frac{\partial^2 H_z}{\partial t^2}$$

$$\frac{\partial}{\partial x} \cdot \frac{\partial E_y}{\partial t} = -\mu_0 \frac{\partial^2 H_z}{\partial t^2}$$

from eqⁿ (4) $\frac{\partial E_y}{\partial t} = \frac{-1}{\epsilon_0} \cdot \frac{\partial H_z}{\partial x}$

$$\therefore \frac{\partial}{\partial x} \cdot \frac{-1}{\epsilon_0} \cdot \frac{\partial H_z}{\partial x} = -\mu_0 \frac{\partial^2 H_z}{\partial t^2}$$

D'Alembert's eqⁿ

$$\therefore \boxed{\frac{\partial^2 H_z}{\partial t^2} = \frac{1}{\mu_0 \epsilon_0} \frac{\partial^2 H_z}{\partial x^2}}$$ in terms of H

$$\frac{\partial^2 E_y}{\partial t^2} = \frac{1}{\mu \epsilon} \frac{\partial^2 E_y}{\partial x^2}$$

$$\left(\frac{V/m}{sec}\right)^2 = \frac{1}{\mu \epsilon} \left(\frac{V/m}{m}\right)^2$$

$$\Rightarrow \frac{1}{\mu \epsilon} = \frac{m^2}{sec^2}$$

$$\Rightarrow \frac{1}{\sqrt{\mu \epsilon}} = m/sec = \text{velocity}$$

velocity in free space

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = \frac{1}{\sqrt{4\pi \times 10^{-7} \times 8.85 \times 10^{-12}}}$$

$$= 3 \times 10^8 \ m/sec$$

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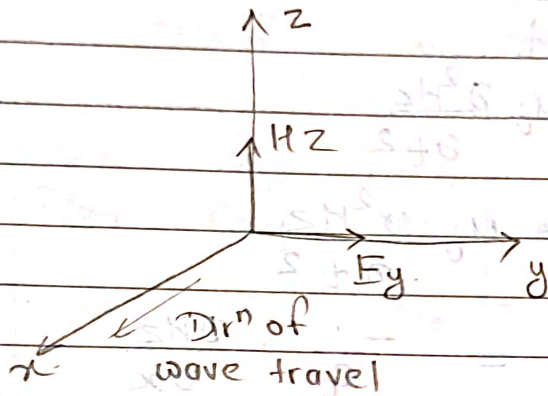
* Wave eqⁿ for wave in lossy medium or conducting medium

i.e. σ = finite



consider EM wave travelling in x-dirⁿ having Ey & Hz component.

consider Maxwell's eqⁿ derived from Faraday's law.

$$\nabla \times \bar{E} = -j\omega\mu \bar{H} \quad \text{——— (1)}$$

$$\begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & E_y & 0 \end{vmatrix} = -j\omega\mu (H_z \bar{a}_z)$$

$$\frac{\partial E_y}{\partial x} \bar{a}_z = -j\omega\mu H_z \bar{a}_z$$

$$\therefore \quad \frac{\partial E_y}{\partial x} = -j\omega\mu H_z \quad \text{——— (2)}$$

consider Maxwell's eq^n derived from Ampere's law

$$\nabla \times \overline{H} = (\sigma + j\omega\epsilon)\overline{E} \quad \text{——— (3)}$$

$$\begin{vmatrix} \overline{a}_x & \overline{a}_y & \overline{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 0 & H_z \end{vmatrix} = (\sigma + j\omega\epsilon) E_y \overline{a}_y$$

$$-\overline{a}_y \cdot \frac{\partial H_z}{\partial x} = (\sigma + j\omega\epsilon) E_y \overline{a}_y$$

$$\therefore \frac{-\partial H_z}{\partial x} = (\sigma + j\omega\epsilon) E_y \quad \text{——— (4)}$$

Diff. eq^n (2) w.r.t. x

$$\frac{\partial^2 E_y}{\partial x^2} = \frac{\partial}{\partial x}(-j\omega\mu H_z)$$

$$\therefore \frac{\partial^2 E_y}{\partial x^2} = -j\omega\mu \cdot \frac{\partial H_z}{\partial x}$$

Sub. in eq^n (4)

$$\boxed{\frac{\partial^2 E_y}{\partial x^2} = j\omega\mu(\sigma + j\omega\epsilon) E_y}$$

$$\boxed{\nabla^2 E = \gamma^2 E}$$ wave eq^n for lossy medium in terms of E

where $\gamma = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)}$ = propogation constant.

Diff. eq^n (4) w.r.t. x

$$\frac{-\partial^2 H_z}{\partial x^2} = (\sigma + j\omega\epsilon) \cdot \frac{\partial E_y}{\partial x}$$

from eq^n (2) $\frac{\partial E_y}{\partial x} = -j\omega\mu H_z$

$$\therefore \boxed{\frac{\partial^2 H_z}{\partial x^2} = j\omega\mu(\sigma + j\omega\epsilon) \cdot H_z}$$ in terms of H

or

$$\boxed{\nabla^2 \bar{H} = \cancel{\gamma}\, \gamma^2 \bar{H}}$$

where $\gamma = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)}$

* **Poynting Vector & Flow of Power**

Electromagnetic energy travels from one pt. to another and there will flow of energy across surface involved. This is explained by Poynting vector.
It states that cross product of $\bar{E}$ & $\bar{H}$ at any point is measure of rate of energy flow per unit area at that point.

i.e. $\bar{P} = \bar{E} \times \bar{H}$ Watt/m^2

where

$\bar{P}$ = Poynting vector

Proof :

consider Maxwell's eq^n derived from Ampere's law

$$\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}$$

$$= \bar{J} + \epsilon \cdot \frac{\partial E}{\partial t}$$

$$\therefore \bar{J} = \nabla \times \bar{H} - \epsilon \cdot \frac{\partial E}{\partial t}$$

By Maxwell's eq^n derived from Faraday's law, Multiplying eq^n by $\bar{E}$

$$\bar{E}\cdot\bar{J} = \bar{E}\cdot(\nabla\times\bar{H}) - \bar{E}\cdot\epsilon\frac{\partial\bar{E}}{\partial t}$$

But, $\bar{E}(\nabla\times\bar{H}) = \bar{H}\cdot(\nabla\times\bar{E}) - \nabla\cdot(\bar{E}\times\bar{H})$

sub. in above eq^n

$$\bar{E}\cdot\bar{J} = \bar{H}\cdot(\nabla\times\bar{E}) - \nabla\cdot(\bar{E}\times\bar{H}) - \bar{E}\,\epsilon\cdot\frac{\partial E}{\partial t}$$

$$= \bar{H}\cdot\left(-\mu\frac{\partial\bar{H}}{\partial t}\right) - \nabla(\bar{E}\times\bar{H}) - \bar{E}\cdot\epsilon\frac{\partial\bar{E}}{\partial t}$$

$$\left[\because \nabla\times\bar{E} = -\frac{\partial\bar{B}}{\partial t} = -\mu\frac{\partial\bar{H}}{\partial t}\right]$$

$$= -\mu\cdot\bar{H}\cdot\frac{\partial\bar{H}}{\partial t} - \epsilon\bar{E}\cdot\frac{\partial\bar{E}}{\partial t} - \nabla\cdot(\bar{E}\times\bar{H})$$

$$\therefore \bar{E}\cdot\bar{J} = \frac{-\partial}{\partial t}\left(\frac{\epsilon E^2}{2} + \frac{\mu H^2}{2}\right) - \nabla\cdot(\bar{E}\times\bar{H})$$

$$\left[\because \mu H\cdot\frac{\partial\bar{H}}{\partial t} = \frac{1}{2}\mu\cdot\frac{\partial H^2}{\partial t}\right]$$

Taking volume integral of above eq^n

$$\underbrace{\int_v \bar{E}\cdot\bar{J}\,dv}_{\text{energy dissipated}} = -\int_v \frac{\partial}{\partial t}\left(\frac{\epsilon E^2}{2} + \frac{\mu H^2}{2}\right)dv - \underbrace{\int_v \nabla\cdot(\bar{E}\times\bar{H})\cdot dv}_{\text{energy flow}}$$

LHS in above eq^n represents energy dissipated in vol^m. The 1st term in RHS indicates energy stored in volume & 2nd term indicates rate at which energy entering in vol^m from outside.

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Applying divergence theorem

$$\int_v \bar{E}\cdot\bar{J}\, dv = -\int \frac{\partial}{\partial t}\left(\frac{\epsilon E^2}{2} + \frac{\mu H^2}{2}\right) dv - \oint \bar{E}\times\bar{H}\, ds$$

$\therefore$ $\bar{E}\times\bar{H}$ represents power flow per unit area.

$\therefore$ $\boxed{\bar{P} = \bar{E}\times\bar{H}}$ Hence proved.