

Derivative of Exponential Functions & Logarithmic Functions.

Exponential function: $f(x) = a^x$, $a > 0$.

Question: Find $f'(x)$.

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{a^{x+h} - a^x}{h} \\ &= \lim_{h \rightarrow 0} \frac{a^x \cdot a^h - a^x}{h} \\ &= \lim_{h \rightarrow 0} \frac{a^x (a^h - 1)}{h} \quad \text{Independent of } h, \text{ so acts as a constant.} \\ &= a^x \cdot \lim_{h \rightarrow 0} \frac{a^h - 1}{h} = \text{constant} \\ &= a^x \cdot \log_e a = a^x (\ln a) \end{aligned}$$

Remark: When $a=e$, then $[e^x]' = [e^x \cdot \ln e] = [e^x]$

Find: $[e^{7x}]'$, $[x^2 e^{-x}]'$, $[2^{\cos x}]'$, $[(e^x + e^{-x})^{-1}]'$

- $[e^{7x}]' = e^{7x} \cdot (7x)' = e^{7x} \cdot 7 = 7e^{7x}$
- $[x^2 e^{-x}]' = (x^2)' \cdot e^{-x} + x^2 \cdot (e^{-x})' = 2x e^{-x} + x^2 (-e^{-x}) = (2x - x^2) e^{-x}$
- $[2^{\cos x}]' = (2^{\cos x} \cdot \ln 2) \cdot (\cos x)' = (2^{\cos x} \ln 2) (-\sin x) = -2^{\cos x} \cdot \ln 2 \cdot \sin x$
- $[(e^x + e^{-x})^{-1}]' = (-1)(e^x + e^{-x})^{-1-1} \cdot (e^x + e^{-x})' = -(e^x + e^{-x})^{-2} \cdot (e^x - e^{-x}) = -\frac{e^x - e^{-x}}{(e^x + e^{-x})^2}$

Logarithmic Functions:

Since Exponential & Logarithmic functions are inverse to each other,

$$a^{\log_a x} = x$$

Let's take derivative on both side (wrt x):

$$[a^{\log_a x} \cdot \ln a] [\log_a x]' = 1$$

$$\Rightarrow [\log_a x]' = \frac{1}{\underbrace{a^{\log_a x}}_x \ln a} = \frac{1}{x \ln a}$$

So, when we replace a by e , we get

$$[\ln x]' = \frac{1}{x}$$

Note: Domain of log functions are always $x > 0$
or $(0, \infty)$

Find. $[\log_5(x^2)]''$, $[\ln(\frac{\sqrt{x+1}}{x^7+1})]'$

$$\begin{aligned} \cdot \log_5(x^2) &= 2 \log_5 x \Rightarrow [\log_5(x^2)]'' = [2 \log_5 x]'' \\ &= 2 [\log_5 x]'' \\ &= 2 \cdot \left[\frac{1}{x \ln 5} \right]' \\ &= \frac{2}{\ln 5} \left[\frac{1}{x} \right]' \\ &= \frac{2}{\ln 5} \left[-\frac{1}{x^2} \right] \end{aligned}$$

$$\begin{aligned} \bullet \quad \ln\left(\frac{\sqrt[7]{x+1}}{x^7+1}\right) &= \ln(\sqrt[7]{x+1}) - \ln(x^7+1) \\ &= \frac{1}{7} \ln(x+1) - \ln(x^7+1) \end{aligned}$$

$$\begin{aligned} \text{So } \left[\ln\left(\frac{\sqrt[7]{x+1}}{x^7+1}\right)\right]' &= \left[\frac{1}{7} \ln(x+1) - \ln(x^7+1)\right]' \\ &= \left[\frac{1}{7} \ln(x+1)\right]' - [\ln(x^7+1)]' \\ &= \frac{1}{7} [\ln(x+1)]' - \frac{1}{x^7+1} \cdot [x^7+1]' \\ &= \frac{1}{7(x+1)} [x+1]' - \frac{7x^6}{x^7+1} \\ &= \frac{1}{7(x+1)} - \frac{7x^6}{x^7+1} \end{aligned}$$