

Derivative of Exponential functions & Logarithmic functions.

Exponential function: $f(x) = a^x$, $a > 0$.

Question: Find $f'(x)$.

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{a^{x+h} - a^x}{h}$$

$$= \lim_{h \rightarrow 0} \frac{a^x \cdot a^h - a^x}{h}$$

$$= \lim_{h \rightarrow 0} a^x \left(\frac{a^h - 1}{h} \right)$$

Independent of h , so acts as a constant.

$$= a^x \cdot \lim_{h \rightarrow 0} \frac{a^h - 1}{h} = \text{constant}$$

$$= a^x \cdot \log_e a = a^x \cdot (\ln a)$$

Remark: When $a=e$, then $[e^x]' = [e^x \cdot \ln e] = [e^x]$

Find: $[e^{7x}]'$, $[x^2 e^{-x}]'$, $[2^{\cos x}]'$, $[(e^x + \bar{e}^x)^{-1}]'$

$$\bullet [e^{7x}]' = e^{7x} \cdot (7x)' = e^{7x} \cdot 7 = 7e^{7x}$$

$$\bullet [x^2 e^{-x}]' = (x^2)' \cdot e^{-x} + x^2 \cdot (e^{-x})' = 2x e^{-x} + x^2 (-e^{-x}) = (2x - x^2) e^{-x}$$

$$\bullet [2^{\cos x}]' = (2^{\cos x} \cdot \ln 2) \cdot (\cos x)' = (2^{\cos x} \ln 2) (-\sin x) = -2^{\cos x} \cdot \ln 2 \cdot \sin x$$

$$\bullet [(e^x + \bar{e}^x)^{-1}]' = (-1)(e^x + \bar{e}^x)^{-1-1} \cdot (e^x + \bar{e}^x)' = - (e^x + \bar{e}^x)^{-2} \cdot (e^x - \bar{e}^x) = - \frac{e^x - \bar{e}^x}{(e^x + \bar{e}^x)^2}$$

Logarithmic Functions:

Since Exponential & Logarithmic functions are inverse to each other.

$$a^{\log_a x} = x$$

Let's take derivative on both sides (wrt x):

$$[a^{\log_a x} \cdot \ln a] [\log_a x]' = 1$$

$$\Rightarrow [\log_a x]' = \frac{1}{\cancel{a^{\log_a x}} \ln a} = \frac{1}{x \ln a}$$

So, when we replace a by e, we get

$$[\ln x]' = \frac{1}{x}$$

Note: Domain of log functions are always $x > 0$ or $(0, \infty)$

Find. $[\log_5(x^2)]''$, $[\ln\left(\frac{\sqrt[3]{x+1}}{x^7+1}\right)]'$

$$\begin{aligned} \log_5(x^2) &= 2 \log_5 x \Rightarrow [\log_5(x^2)]'' = [2 \log_5 x]'' \\ &= 2 [\log_5 x]'' \\ &= 2 \cdot \left[\frac{1}{x \ln 5} \right]' \\ &= \frac{2}{\ln 5} \left[\frac{1}{x} \right]' \\ &= \frac{2}{\ln 5} \left[-\frac{1}{x^2} \right] \end{aligned}$$

$$\begin{aligned}
\bullet \quad \ln\left(\frac{\sqrt[7]{x+1}}{x^7+1}\right) &= \ln(\sqrt[7]{x+1}) - \ln(x^7+1) \\
&= \frac{1}{7} \ln(x+1) - \ln(x^7+1) \\
\text{So } \left[\ln\left(\frac{\sqrt[7]{x+1}}{x^7+1}\right)\right]' &= \left[\frac{1}{7} \ln(x+1) - \ln(x^7+1)\right]' \\
&= \left[\frac{1}{7} \ln(x+1)\right]' - \left[\ln(x^7+1)\right]' \\
&= \frac{1}{7} [\ln(x+1)]' - \frac{1}{x^7+1} \cdot [x^7+1]' \\
&= \frac{1}{7(x+1)} [x+1]' - \frac{7x^6}{x^7+1} \\
&= \frac{1}{7(x+1)} - \frac{7x^6}{x^7+1}
\end{aligned}$$