

## Limits at Infinity & Asymptotes:

Let  $f(x) = 2 + \frac{1}{x}$ .

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \left(2 + \frac{1}{x}\right) = 2 + \lim_{x \rightarrow \infty} \frac{1}{x} = 2 + 0 = 2$$

$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} \left(2 + \frac{1}{x}\right) = 2 + \lim_{x \rightarrow -\infty} \frac{1}{x} = 2 - 0 = 2$$

finite limit  
at infinity  
Horizontal  
Asymptote.

Let  $f(x) = x^3$

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} x^3 = \infty$$

$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} x^3 = -\infty$$

infinite limit  
at  
infinity.

Def<sup>n</sup>: If  $f$  has limit at infinity if  $\begin{cases} \lim_{x \rightarrow \infty} f(x) = L \text{ or} \\ \lim_{x \rightarrow -\infty} f(x) = L \end{cases}$

We call  $y = L$  line, the horizontal asymptote of  $f$ .

Def<sup>n</sup>: If  $\begin{cases} \lim_{x \rightarrow \infty} f(x) = +\infty \text{ or } -\infty \\ \lim_{x \rightarrow -\infty} f(x) = +\infty \text{ or } -\infty \end{cases}$  then  $f$  has an infinite limit at infinity.

Let  $f(x) = 2 + \frac{1}{x}$ .

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \left(2 + \frac{1}{x}\right) = 2 + \lim_{x \rightarrow 0^+} \frac{1}{x} = 2 + \infty = \infty$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \left(2 + \frac{1}{x}\right) = 2 + \lim_{x \rightarrow 0^-} \frac{1}{x} = 2 - \infty = -\infty$$

Def<sup>n</sup>: If for a function  $f$  any of the following holds,

$$(1) \lim_{x \rightarrow a} f(x) = \pm\infty \text{ or}$$

$$(2) \lim_{x \rightarrow a^+} f(x) = \pm\infty \text{ or}$$

$$(3) \lim_{x \rightarrow a^-} f(x) = \pm\infty$$

then we call  $x=a$  line to be the vertical asymptote of  $f$ .

- Find the asymptotes for  $y=x^k$ , where  $k$  is an integer.

Case 1:  $k > 0$ ,  $f(x)=x^k$  = monomial / polynomials

Then  $\lim_{x \rightarrow \pm\infty} x^k = \begin{cases} +\infty, & \text{if } k \text{ is even} \\ -\infty, & \text{if } k \text{ is odd} \end{cases}$  but not a finite value.

Therefore,  $f(x)=x^k$ ,  $k > 0$  has no horizontal asymptote.

Also,  $\lim_{x \rightarrow a^+/a^-/a} x^k = a^k$  (a finite value)

So,  $f(x)=x^k$ ,  $k > 0$  has no vertical asymptote.

Case 2:  $k=0$ ,  $f(x)=\text{constant}$  ie, a horizontal line.

Clearly, it doesn't have vertical asymptote.

Also, it is itself the horizontal line, that was supposed to be the horizontal Asymptote; so no horizontal asymptote by definition.

Case 3:  $k < 0 \Rightarrow k = -r, r > 0$

So,  $f(x) = x^k = \bar{x}^r = \frac{1}{x^r}, r > 0$ .

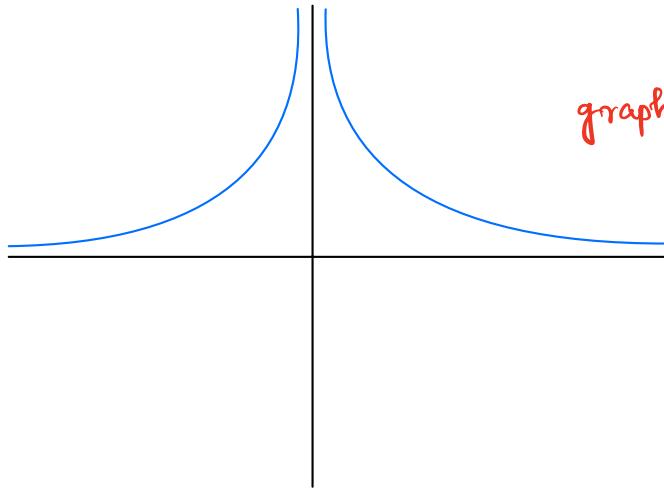
Note that  $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{1}{x^r} = 0$

So,  $y=0$  is our horizontal asymptote.

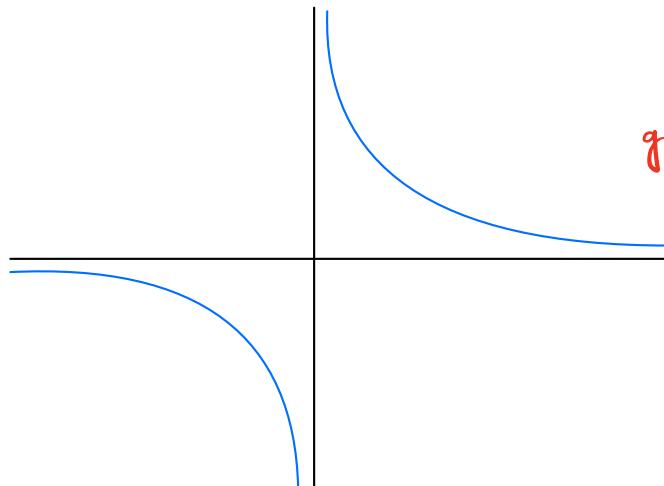
Also,  $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{1}{x^r} = +\infty$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{1}{x^r} = \begin{cases} +\infty, & \text{if } r \text{ is even} \\ -\infty, & \text{if } r \text{ is odd.} \end{cases}$$

So,  $x=0$  is the vertical asymptote of  $f(x) = \bar{x}^k, k > 0$ .



graph of  $f(x) = \bar{x}^k, k > 0$  &  $k$  even.



graph of  $f(x) = \bar{x}^k, k > 0$  &  $k$  odd

- Find the asymptotes for  $f(x) = \frac{x^2+x-6}{2x^2+6}$ .

Since,  $2x^2+6 > 6$  for any real  $x$ , therefore the denominator is never 0.

Hence, there is not possibility of getting a vertical asymptote.

$$\begin{aligned} \text{Now, } \lim_{x \rightarrow \infty} f(x) &= \lim_{x \rightarrow \infty} \frac{x^2+x-6}{2x^2+6} \\ &= \lim_{x \rightarrow \infty} \frac{1 + \frac{x}{x^2} - \frac{6}{x^2}}{2 + \frac{6}{x^2}} \\ &= \frac{1+0-0}{2+0} = \frac{1}{2} \end{aligned}$$

And therefore,  $y = \frac{1}{2}$  is the horizontal asymptote.

- Find the asymptotes for  $f(x) = \frac{\sqrt{6x^2+7}}{x+7}$

Note: denominator is 0 when  $x = -7$ .

$$\text{So if we choose } \lim_{x \rightarrow -7} f(x) = \lim_{x \rightarrow -7} \frac{\sqrt{6x^2+7}}{x+7} = \infty$$

Hence,  $x = -7$  is the Vertical Asymptote.

$$\begin{aligned} \text{Also, } \lim_{x \rightarrow \infty} f(x) &= \lim_{x \rightarrow \infty} \frac{\sqrt{6x^2+7}}{x+7} \\ &= \lim_{x \rightarrow \infty} \frac{\sqrt{x^2(6+\frac{7}{x^2})}}{x+7} \\ &= \lim_{x \rightarrow \infty} \frac{x\sqrt{6+\frac{7}{x^2}}}{x(1+\frac{7}{x})} \quad \left| \begin{array}{l} = \lim_{x \rightarrow \infty} \frac{\sqrt{6+\frac{7}{x^2}}}{1+\frac{7}{x}} \\ = \frac{\sqrt{6+0}}{1+0} = \sqrt{6}. \end{array} \right. \end{aligned}$$

Hence,  $y = \sqrt{6}$  is a Horizontal Asymptote.