Substitution:-

Recall: The chain rule gives:

$$\frac{d}{dx}\left(f(g(x))\right) = f'(g(x)) \cdot g'(x).$$

Then antiderivative of $f'(g(x)) \cdot g'(x)$ is $f(g(x))$.

ie, $\int f'(g(x)) g'(x) dx = f(g(x)) + \text{Constant}.$ ———— ①

Let's take $\boxed{u = g(x)}$ ———————— ✷

Then, $u' = g'(x)$, ie, $\frac{du}{dx} = g'(x)$

$$\Rightarrow \boxed{du = g'(x) dx}$$ ———— ✷✷

Now, substituting ✷ & ✷✷ in ① we get,

$$\int f'(g(x)) g'(x) dx = f(g(x)) + c \quad \text{as}$$

$$\int f'(u) du = f(u) + c$$

Steps: ① Choose proper substitution for u.
② Convert the problem in terms of u.
③ Compute the integrals with respect to u.
④ Return back the substitutions.

Eg (1). Compute $\int (7x+9)^8 dx$.

Let, $u = 7x + 9$

$\therefore du = 7 dx \Rightarrow dx = \frac{1}{7} du$

So, $\int u^8 \cdot \frac{1}{7} \, du = \frac{1}{7} \int u^8 \, du$

$$= \frac{1}{7} \cdot \frac{u^9}{9} + C$$

$$= \frac{u^9}{63} + C$$

$$= \frac{(7x+9)^9}{63} + C$$

Therefore, $\int (7x+9)^8 \, dx = \frac{(7x+9)^9}{63} + C$.

Eg (2). Compute $\int \frac{7}{7x+5} \, dx$

Let, $u = 7x+5$ $\Rightarrow$ $du = 7 \, dx$
$\Rightarrow$ $dx = \frac{1}{7} \, du$

$$\int \frac{7}{7x+5} \, dx = \int \frac{7}{u} \cdot \frac{1}{7} \, du = \int \frac{du}{u}$$

$$= \ln|u| + C$$

$$= \ln|7x+5| + C$$

Eg (3). Compute $\int \cos^2\theta \sin\theta \, d\theta$

$u = \cos\theta$ $\Rightarrow$ $du = -\sin\theta \, d\theta$ $\Rightarrow$ $-du = \sin\theta \, d\theta$.

$$\int \cos^2\theta \sin\theta \, d\theta = \int u^2 (-du)$$

$$= -\int u^2 \, du$$

$$= -\frac{u^3}{3} + C$$

$$= -\frac{\cos^3\theta}{3} + C$$

Substitution on Definite Integral:

Find $\displaystyle\int_0^1 \frac{3}{\sqrt{6x+5}}\,dx$

Method (1). First compute $\displaystyle\int \frac{3}{\sqrt{6x+5}}\,dx$ (without the constant)

Let $u = 6x+5 \implies du = 6\,dx \implies dx = \frac{1}{6}\,du$

Now, $\displaystyle\int \frac{3}{\sqrt{6x+5}}\,dx = \int \frac{3}{\sqrt{u}}\cdot\frac{1}{6}\,du = \int \frac{1}{2}\,u^{-1/2}\,du$

$$= \frac{1}{2}\frac{u^{1/2}}{1/2}$$

$$= u^{1/2}$$

$$= \sqrt{6x+5}$$

Then, $\displaystyle\int_0^1 \frac{3}{\sqrt{6x+5}}\,dx = \left[\sqrt{6x+5}\right]_{x=0}^1$

$$= \sqrt{6(1)+5} - \sqrt{6(0)+5}$$

$$= \sqrt{11} - \sqrt{5}$$

Method (2). Changing Integral limits.

Let $u = 6x+5 \implies du = 6\,dx \implies dx = \frac{1}{6}\,du$

&

x	0	1
u	$6(0)+5$ $=5$	$6(1)+5$ $=11$

Then, $\displaystyle\int_{x=0}^1 \frac{3}{\sqrt{6x+5}}\,dx = \int_{u=5}^{11} \frac{3}{\sqrt{u}}\cdot\frac{1}{6}\,du = \frac{1}{2}\int_5^{11} u^{-1/2}\,du$

$$= \frac{1}{2}\cdot\frac{u^{1/2}}{1/2}\Big|_5^{11} = \sqrt{11} - \sqrt{5}$$