

Limits at Infinity & Asymptotes:

$$\text{Let } f(x) = 2 + \frac{1}{x}.$$

$$\begin{aligned} \lim_{x \rightarrow \infty} f(x) &= \lim_{x \rightarrow \infty} \left(2 + \frac{1}{x}\right) = 2 + \lim_{x \rightarrow \infty} \frac{1}{x} = 2 + 0 = 2 \\ \lim_{x \rightarrow -\infty} f(x) &= \lim_{x \rightarrow -\infty} \left(2 + \frac{1}{x}\right) = 2 + \lim_{x \rightarrow -\infty} \frac{1}{x} = 2 - 0 = 2 \end{aligned} \left. \vphantom{\lim_{x \rightarrow \infty} f(x)} \right\} \begin{array}{l} \text{finite limit} \\ \text{at infinity} \\ \hline \text{Horizontal} \\ \text{Asymptote.} \end{array}$$

$$\text{Let } f(x) = x^3$$

$$\begin{aligned} \lim_{x \rightarrow \infty} f(x) &= \lim_{x \rightarrow \infty} x^3 = \infty \\ \lim_{x \rightarrow -\infty} f(x) &= \lim_{x \rightarrow -\infty} x^3 = -\infty \end{aligned} \left. \vphantom{\lim_{x \rightarrow \infty} f(x)} \right\} \begin{array}{l} \text{infinite limit} \\ \text{at} \\ \text{infinity.} \end{array}$$

Defⁿ: If f has limit at infinity if $\begin{cases} \lim_{x \rightarrow \infty} f(x) = L \text{ or} \\ \lim_{x \rightarrow -\infty} f(x) = L \end{cases}$

We call $y = L$ line, the horizontal asymptote of f .

Defⁿ: If $\begin{cases} \lim_{x \rightarrow \infty} f(x) = +\infty \text{ or } -\infty \\ \lim_{x \rightarrow -\infty} f(x) = +\infty \text{ or } -\infty \end{cases}$ then f has an infinite limit at infinity.

$$\text{Let } f(x) = 2 + \frac{1}{x}.$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \left(2 + \frac{1}{x}\right) = 2 + \lim_{x \rightarrow 0^+} \frac{1}{x} = 2 + \infty = \infty$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \left(2 + \frac{1}{x}\right) = 2 + \lim_{x \rightarrow 0^-} \frac{1}{x} = 2 - \infty = -\infty$$

Defⁿ: If for a function f any of the following holds,

$$(1) \lim_{x \rightarrow a} f(x) = \pm \infty \text{ or}$$

$$(2) \lim_{x \rightarrow a^+} f(x) = \pm \infty \text{ or}$$

$$(3) \lim_{x \rightarrow a^-} f(x) = \pm \infty$$

then we call $x=a$ line to be the vertical asymptote of f .

- Find the asymptotes for $y = x^k$, where k is an integer.

Case 1: $k > 0$, $f(x) = x^k = \text{monomial / polynomials}$

Then $\lim_{x \rightarrow \pm \infty} x^k = \begin{cases} +\infty, & \text{if } k \text{ is even} \\ -\infty, & \text{if } k \text{ is odd} \end{cases}$ but not a finite value.

Therefore, $f(x) = x^k$, $k > 0$ has no horizontal asymptote.

Also, $\lim_{x \rightarrow a^+ / a^- / a} x^k = a^k$ (a finite value)

So, $f(x) = x^k$, $k > 0$ has no vertical asymptote.

Case 2: $k = 0$, $f(x) = \text{constant}$ i.e., a horizontal line.

Clearly, it doesn't have vertical asymptote.

Also, it is itself the horizontal line, that was supposed to be the horizontal asymptote; so no horizontal asymptote by definition.

Case 3: $k < 0 \Rightarrow k = -r, r > 0$

$$\text{So, } f(x) = x^k = x^{-r} = \frac{1}{x^r}, \quad r > 0.$$

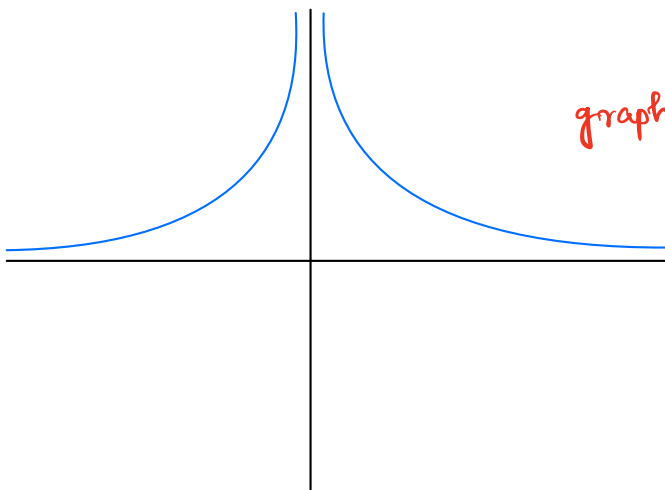
$$\text{Note that } \lim_{x \rightarrow \pm\infty} f(x) = \lim_{x \rightarrow \pm\infty} \frac{1}{x^r} = 0$$

So, $y = 0$ is our horizontal asymptote.

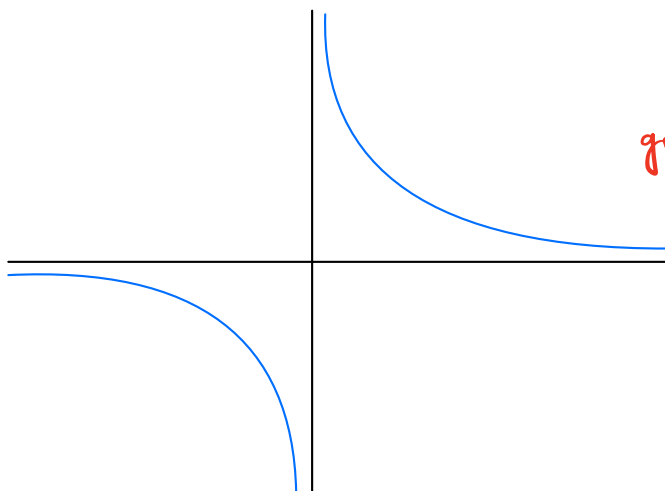
$$\text{Also, } \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{1}{x^r} = +\infty$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{1}{x^r} = \begin{cases} +\infty, & \text{if } r \text{ is even} \\ -\infty, & \text{if } r \text{ is odd.} \end{cases}$$

So, $x = 0$ is the vertical asymptote of $f(x) = x^{-k}, k > 0$.



graph of $f(x) = x^{-k}, k > 0$ & even.



graph of $f(x) = x^{-k}, k > 0$ & odd.

- Find the asymptotes for $f(x) = \frac{x^2 + x - 6}{2x^2 + 6}$.

Since, $2x^2 + 6 \geq 6$ for any real x , therefore the denominator is never 0.

Hence, there is not possibility of getting a vertical asymptote.

$$\begin{aligned}\text{Now, } \lim_{x \rightarrow \infty} f(x) &= \lim_{x \rightarrow \infty} \frac{x^2 + x - 6}{2x^2 + 6} \\ &= \lim_{x \rightarrow \infty} \frac{1 + \frac{x}{x^2} - \frac{6}{x^2}}{2 + \frac{6}{x^2}} \\ &= \frac{1 + 0 - 0}{2 + 0} = \frac{1}{2}\end{aligned}$$

And therefore, $y = \frac{1}{2}$ is the horizontal asymptote.

- Find the asymptotes for $f(x) = \frac{\sqrt{6x^2 + 7}}{x + 7}$.

Note: denominator is 0 when $x = -7$.

$$\text{So if we choose } \lim_{x \rightarrow -7} f(x) = \lim_{x \rightarrow -7} \frac{\sqrt{6x^2 + 7}}{x + 7} = \infty$$

Hence, $x = -7$ is the Vertical Asymptote.

$$\begin{aligned}\text{Also, } \lim_{x \rightarrow \infty} f(x) &= \lim_{x \rightarrow \infty} \frac{\sqrt{6x^2 + 7}}{x + 7} &= \lim_{x \rightarrow \infty} \frac{\sqrt{6 + \frac{7}{x^2}}}{1 + \frac{7}{x}} \\ &= \lim_{x \rightarrow \infty} \frac{\sqrt{x^2(6 + \frac{7}{x^2})}}{x + 7} &= \frac{\sqrt{6 + 0}}{1 + 0} = \sqrt{6}. \\ &= \lim_{x \rightarrow \infty} \frac{x \sqrt{6 + \frac{7}{x^2}}}{x(1 + \frac{7}{x})} &\end{aligned}$$

Hence, $y = \sqrt{6}$ is a Horizontal Asymptote.