Coverage of Wireless Sensor Networks

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Abstract. In this paper we study the limiting achievable coverage problems of sensor network. For the sensor networks with uniform distributions we obtain a complete characterization of the coverage probability. For the sensor networks with non-uniform distributions, we derive two different necessary and sufficient conditions respectively in the situations that the density function taking its minimum value on a set with positive Lebesgue measure or at finitely many points. We propose also an economical scheme for the coverage of sensor networks with empirical distributions.

Keywords. coverage, sensor network, large-scale, wireless, uniform distribution.

1 Introduction

Recently, there has been a growing interest in studying large-scale wireless sensor networks. Such a network consists of a large number of sensors which are densely deployed in a certain area. For some reasons (reducing radio interference, limited battery capacity, etc.), these sensors are small in size and they communicate with each other in short distance. There are many fundamental problems that arise in the research of wireless sensor networks. Among them one important issue is that of limiting achievable coverage. A point in an area can be detected by a sensor provided the point is within the distant r of the sensor, where r is the sensing radius of the sensor. The area is said to be covered if every point in the area can be detected by a sensor. In the literature there have been several discussions concerning the minimum sensing radius, depending on the numbers of (active) sensors per unit area, which guarantees that the area is covered in a limiting performance. In [12], the authors considered the problem of covering a square of area A with randomly located circles whose centers are generated by a two-dimensional Poisson point process of density D points per unit area. Suppose that each Poisson point represents a sensor with sensing radius R which may depend on D and A. They proved that, for any $\varepsilon > 0$, if $R = \sqrt{(1+\varepsilon)\ln A/\pi D}$, then $\lim_{A\to\infty} \Pr[\text{square covered}] = 1$. On the other hand, if $R = \sqrt{(1-\varepsilon)\ln A/\pi D}$, then $\lim_{A\to\infty} \Pr[\text{square covered}] = 0$. Therefore the authors observed that, to guarantee that the area is covered, a node must have $\pi[(1+\varepsilon)\ln A/\pi D]D$ or a little more than $\ln A$ nearest neighbors (Poisson point that lie at a distant of R or less from it) on the average. In the paper [14], the authors studied the coverage of a grid-based unreliable sensor network. They derived necessary and sufficient conditions for the random grid network to cover a unit square area. Their result shows that the random grid network asymptotically cover a unit square area if and only if $p_n r_n^2$ is of the order $(\ln n)/n$, where r_n is the sensing radius and p_n is the probability that a sensor is "active" (not failed). In connection with the above two results we mention that Hall[3] has considered the coverage problem of the following model: Circles of radius r are placed in a unit-area disc $\mathcal{D}$ at a Poisson intensity of λ. Let $V(\lambda; r)$ denote the vacancy within $\mathcal{D}$, i.e., $V(\lambda; r)$ is the region of $\mathcal{D}$ not covered by the circles. It has been shown ([3], Theorem 3.11) that

$$\begin{aligned}\frac{1}{20}\min\{1,(1+\pi r^2\lambda^2)e^{-\pi r^2\lambda}\} &< \Pr\{|V(\lambda;r)|>0\}\\ &< \min\{1,3(1+\pi r^2\lambda^2)e^{-\pi r^2\lambda}\},\end{aligned}$$

where $|V(\lambda; r)|$ is the area of $V(\lambda; r)$. Note that by Hall's result, if we set $\lambda = n$ and $r_n = \sqrt{(\ln n + \ln\ln n + a_n)/\pi n}$, then $\lim_{n\to\infty}\Pr[\text{square covered}] = 1$ for $a_n \to +\infty$, and $\lim_{n\to\infty}\Pr[\text{square covered}] < 19/20$ for $a_n \to -\infty$. However, it was not clear whether the limit $\lim_{n\to\infty}\Pr[\text{square covered}] = 0$ for $a_n \to -\infty$. In the literature there are also various other discussions about the coverage problems of wireless sensor networks, see [4 - 11, 13, 15, 16] and reference therein.

In this paper, we employ some results in Aldous[1] to study the coverage problems. Among other results, we show that in the above mentioned Hall's model, if we set $\lambda = n$ and $r_n = \sqrt{(\ln n + \ln\ln n + a_n)/\pi n}$ with $a_n = o(\ln n)$, then

$$\Pr[\text{square covered}] = \exp[-\exp(-a_n)] + o(1),$$

provided the limit of $\{a_n\}$ exists in $[-\infty, +\infty]$ (see Theorem 2.1 and Remark 2.2 below). In particular, if $a_n \to -\infty$, then $\lim_{n\to\infty}\Pr[\text{square covered}] = 0$. Thus we clarify the above mentioned question. Moreover, the above equality tells us something more, that is, if $a_n \to a$ for $a \in (-\infty, +\infty)$, then $\lim_{n\to\infty}\Pr[\text{square covered}] = \exp(-e^{-a}) \in (0, 1)$.

The structure of this paper is as follows. In section 2 we investigate the coverage of sensor networks with uniform distributions, which is asymptotically the same as the sensor networks with sensors located according to a Poisson distribution. In section 3 we investigate the coverage of sensor networks with non-uniform distributions. Our research shows that in non-uniform case the minimum sensing radius in the coverage problem relies mainly on the behavior of the distribution density function around its minimum point. We give first a sufficient condition in Theorem 3.1. We then derive two different necessary and sufficient conditions in Theorem 3.2 and Theorem 3.3, respectively in the situations that the density function taking its minimum value on a set with positive Lebesgue measure or at finitely many points. Finally in Section 4 we propose an economical scheme for the coverage of sensor networks with empirical distributions.

2 Coverage of Sensor Networks with Uniform Distributions

Let $\mathcal{A} = [0,1]^2$ be a unit square. Suppose $\{X_1, \cdots, X_n\}$ are n independent random points with the uniform distribution on $\mathcal{A}$ and r_n is a positive number. We consider $\{r_n; X_1, \cdots, X_n\}$ as a sensor network on the region $\mathcal{A}$ with sensing radius r_n. Let $\mathcal{M}(n, r_n) = \bigcup_{k=1}^{n} B(X_k, r_n)$ be the union of n circles with centers $X_1, \cdots, X_n$ and common radius r_n. Then the region is covered by the sensor network if and only if $\mathcal{A} \subset \mathcal{M}(n, r_n)$. The question that we investigate in this section is to find the smallest r_n that guarantees

$$\lim_{n\to\infty} \Pr[\mathcal{A} \subset \mathcal{M}(n, r_n)] = 1. \tag{2.1}$$

We have the following complete result.

Theorem 2.1 *Let $\{X_1, \cdots, X_n\}$ be n independent random points with the uniform distribution on $\mathcal{A}$. Suppose that $r_n = \sqrt{(\ln n + \ln\ln n + a_n)/\pi n}$, where $\{a_n\}$ is a sequence of real numbers such that $a_n = o(\ln n)$ and has a limit in $[-\infty, +\infty]$. Let $\mathcal{M}(n, r_n) = \bigcup_{k=1}^{n} B(X_k, r_n)$ be as above. Then we have*

$$\Pr[\mathcal{A} \subset \mathcal{M}(n, r_n)] = \exp[-\exp(-a_n)] + o(1). \tag{2.2}$$

In particular, $\lim_{n\to\infty} \Pr[\mathcal{A} \subset \mathcal{M}(n, r_n)] = 1$ if and only if $a_n \to +\infty$, and $\lim_{n\to\infty} \Pr[\mathcal{A} \subset \mathcal{M}(n, r_n)] = 0$ if and only if $a_n \to -\infty$.

Proof. Define a random variable L_n as the radius of the largest circle that lies inside $\mathcal{A}$ containing no points of $\{X_1, \cdots, X_n\}$. Then $\mathcal{A}$ is covered by $\mathcal{M}(n, r_n)$ if and only if $L_n < r_n$. Applying the approximation method used by Aldous in the study of stochastic geometry, we can get ([1], p150, H1d)

$$\Pr[L_n < r_n] \approx \exp\left[-n^2\pi r_n^2 \exp(-n\pi r_n^2)(1-2r_n)^2\right], \tag{2.3}$$

where the relation "$\approx$" is understood as

$$\Pr[L_n < r_n] = \exp\left[-n^2\pi r_n^2 \exp(-n\pi r_n^2)(1-2r_n)^2\right] + O(e^{-n\pi r_n^2}).$$

By the assumption of r_n, we have $n\pi r_n^2 \to \infty$. Therefore,

$$\Pr[\mathcal{A} \subset \mathcal{M}(n, r_n)] = \Pr[L_n < r_n] = \exp[-I(n)] + o(1), \tag{2.4}$$

where

$$I(n) = n^2\pi r_n^2 (1-2r_n)^2 \exp(-n\pi r_n^2).$$

Replacing $n\pi r_n^2$ by $\ln n + \ln\ln n + a_n$ in the expression of $I(n)$, we get

$$\begin{aligned}
(1-2r_n)^{-2} I(n) &= n(\ln n + \ln\ln n + a_n)\exp(-\ln n - \ln\ln n - a_n) \\
&= n(\ln n + \ln\ln n + a_n)(n\ln n)^{-1}\exp(-a_n) \\
&= \exp(-a_n)[1 + \ln\ln n(\ln n)^{-1} + a_n(\ln n)^{-1}]
\end{aligned}$$

By our assumption $a_n = o(\ln n)$, which implies $a_n(\ln n)^{-1} = o(1)$ and $r_n = o(1)$. Therefore, from the above formula we get $I(n) = \exp(-a_n) + o(1)$. Thus by (2.4)

$$\lim_{n\to\infty} \Pr[\mathcal{A} \subset \mathcal{M}(n, r_n)] = \lim_{n\to\infty} \exp[-I(n)] = \lim_{n\to\infty} \exp[-\exp(-a_n)],$$

verifying (2.2). □

Remark 2.2 The conclusion of Theorem 2.1 is equally available in the model that the sensors put in the unit square according to a Poisson point process with intensity n points per unit area. Because the difference of these two schemes is negligible in the limiting procedure. In fact, Aldous's result [1] (H1d) (see (2.3) in this paper) was obtained via the approximation of large number of uniformly distributed points by Poisson point processes with large intensity.

3 Coverage of Sensor Networks with Non-uniform Distributions

In this section we investigate the coverage of sensor networks which are deployed non-uniformly according to a general distribution density function $g(x)$ on the unit square $\mathcal{A}$. We assume that g is a bounded smooth function. Our results show that the minimum sensing radius in the coverage problem relies mainly on the minimum value of g and the behavior of g around its minimum value. We derive first a sufficient condition as follows.

Theorem 3.1 *Suppose $\{X_1, \cdots, X_n\}$ are n independent random points in the unit square $\mathcal{A}$ according to a smooth distribution density function $g(x)$, where $g(x)$ satisfies $0 < \beta \le g(x) \le \gamma < \infty$. Let $r_n = \sqrt{(\ln n + \ln\ln n + a_n)/\beta\pi n}$, where $\{a_n\}$ is a sequence in $\mathbb{R}$ tending to $+\infty$. Then we have*

$$\lim_{n\to\infty} \Pr[\mathcal{A} \subset \mathcal{M}(n, r_n)] = 1. \tag{3.1}$$

(Here and henceforth $\mathcal{M}(n, r_n)$ is the same as defined in the beginning of Section 2.)

Proof. Analogously, we define a random variable L_n as the radius of the largest circle that lies inside $\mathcal{A}$ containing no points. Following (H9c) in Aldous [1] (p160) and its argument, we can get

$$\Pr[L_n < r_n] = \exp\left(-\int_{\mathcal{A}} J_n(x)dx\right) + O(e^{-n\pi r_n^2}), \tag{3.2}$$

where

$$J_n(x) = n^2\pi r_n^2 g(x)^2 \exp(-ng(x)\pi r_n^2).$$

Since $n\beta\pi r_n^2 = \ln n + \ln\ln n + a_n$, we have

$$\begin{aligned} J_n(x) &= ng(x)^2\beta^{-1}(\ln n + \ln\ln n + a_n) \\ &\quad \cdot \exp[-g(x)\beta^{-1}(\ln n + \ln\ln n + a_n)] \\ &\leq n\gamma^2\beta^{-1}(\ln n + \ln\ln n + a_n)\exp[-(\ln n + \ln\ln n + a_n)] \\ &= n\gamma^2\beta^{-1}(\ln n + \ln\ln n + a_n)(n\ln n)^{-1}\exp(-a_n) \\ &= \gamma^2\beta^{-1}[\exp(-a_n) + \ln\ln n(\ln n)^{-1}\exp(-a_n) + a_n(\ln n)^{-1}\exp(-a_n)]. \end{aligned}$$

This yields that $J_n(x)$ tend to 0 as $a_n \to \infty$. By the dominated convergence theorem,

$$\lim_{n\to\infty}\int_{\mathcal{A}} J_n(x)dx = \int_{\mathcal{A}}\lim_{n\to\infty} J_n(x)dx = 0.$$

Thus,

$$\lim_{n\to\infty}\Pr[\mathcal{A}\subset\mathcal{M}(n,r_n)] = \lim_{n\to\infty}\Pr[L_n < r_n] = \lim_{n\to\infty}\exp\left(-\int_{\mathcal{A}} J_n(x)dx\right) = 1. \quad \square$$

In the remainder of this section, we explore the necessary and sufficient condition that guarantees (3.1) holds. It turns out that the behavior of g around its minimum point will effect the coverage rate of r_n. See Theorem 3.2 and Theorem 3.3 for details.

Theorem 3.2 *Let $\{X_1,\cdots,X_n\}$ be n independent random points in the unit square $\mathcal{A}$ according to a smooth distribution density function $g(x)$. Suppose that $g(x)$ satisfies $0<\beta\leq g(x)\leq\gamma<\infty$ and $\lambda(\{x : g(x)=\beta\})>0$, where λ is the Lebesgue measure. Let $r_n = \sqrt{(\ln n + \ln\ln n + a_n)/\beta\pi n}$, where $\{a_n\}$ is a sequence in $\mathbb{R}$ such that $a_n = o(\ln n)$. Then,*

$$\lim_{n\to\infty}\Pr[\mathcal{A}\subset\mathcal{M}(n,r_n)] = 1 \tag{3.3}$$

if and only if a_n tends to $+\infty$ as $n\to\infty$. Moreover,

$$\lim_{n\to\infty}\Pr[\mathcal{A}\subset\mathcal{M}(n,r_n)] = 0 \tag{3.4}$$

if and only if a_n tends to $-\infty$ as $n\to\infty$.

Proof. By the approximation formula (3.2),

$$\Pr[\mathcal{A}\subset\mathcal{M}(n,r_n)] = \exp\left[-I(n)\right] + O(e^{-n\pi r_n^2}), \tag{3.5}$$

where

$$I(n) = \int_{\mathcal{A}} n^2\pi r_n^2 g(x)^2\exp(-ng(x)\pi r_n^2)dx.$$

Letting $J_n(x) = n^2\pi r_n^2 g(x)^2\exp(-ng(x)\pi r_n^2)$ and $S=\{x : g(x)=\beta\}$, we have

$$I(n) = I_1(n) + I_2(n) \tag{3.6}$$

with

$$I_1(n) := \int_S J_n(x)dx = \lambda(S)n^2\pi r_n^2\beta^2 \exp(-n\beta\pi r_n^2) \tag{3.7}$$

and

$$I_2(n) := \int_{\mathcal{A}\setminus S} J_n(x)dx \le \lambda(\mathcal{A}\setminus S)n^2\pi r_n^2\gamma^2 \exp(-n\beta\pi r_n^2). \tag{3.8}$$

Since $\lambda(S) > 0$, we have $I_2(n) \le \alpha I_1(n)$ for some $\alpha \in (0,\infty)$. This implies that

$$I_1(n) \le I(n) \le (1+\alpha)I_1(n). \tag{3.9}$$

Since $n\beta\pi r_n^2 = \ln n + \ln\ln n + a_n$, hence we obtain

$$I_1(n) = \lambda(S)\beta\exp(-a_n) + o(1). \tag{3.10}$$

By (3.5), (3.9) and (3.10), we have

$$\lim_{n\to\infty} \Pr[\mathcal{A} \subset \mathcal{M}(n,r_n)] = 1 \Leftrightarrow \lim_{n\to\infty} I(n) = 0 \Leftrightarrow \lim_{n\to\infty} I_1(n) = 0 \Leftrightarrow a_n \to +\infty.$$

Similarly, we can check that

$$\lim_{n\to\infty} \Pr[\mathcal{A} \subset \mathcal{M}(n,r_n)] = 0 \Leftrightarrow \lim_{n\to\infty} I_1(n) = +\infty \Leftrightarrow a_n \to -\infty. \quad \square$$

In the following we assume that the distribution density function $g(x)$ has finitely many minimum points at which some regular conditions hold. For a real function $g(x) = g(x_1,x_2)$ on $\mathbb{R}^2$ which is secondly differentiable at x^*, we define $B(x^*)$ as the 2×2 matrix with entries $b_{i,j} = \frac{\partial^2 g}{\partial x_i \partial x_j}(x^*)$, $i,j = 1,2$.

Theorem 3.3 *Let $\{X_1,\cdots,X_n\}$ be n independent random points according to a smooth distribution density function $g(x)$ on $\mathcal{A}$. Suppose that $g(x)$ has finitely many minimum points $\{x_i^*\} \subset \mathcal{A}$, $0 < \beta = g(x_i^*) \le g(x) \le \gamma < \infty$, and $B(x_i^*)$ are well defined and strictly positive definite. Let $r_n = \sqrt{(\ln n + a_n)/\beta\pi n}$, where $\{a_n\}$ is a sequence in $\mathbb{R}$ such that $a_n = o(\ln n)$. Then*

$$\lim_{n\to\infty} \Pr[\mathcal{A} \subset \mathcal{M}(n,r_n)] = 1 \tag{3.11}$$

if and only if a_n tends to $+\infty$ as $n\to\infty$. Moreover,

$$\lim_{n\to\infty} \Pr[\mathcal{A} \subset \mathcal{M}(n,r_n)] = 0 \tag{3.12}$$

if and only if a_n tends to $-\infty$ as $n \to \infty$.

Before proving Theorem 3.3, let us prepare several lemmas.

Lemma 3.4 *Suppose B is a d-dimensional matrix which is symmetric and strictly positive definite. Then*

$$\int_{\mathbb{R}} \exp(-\frac{1}{2}x^T B^{-1} x)dx = (2\pi)^{d/2}|B|^{1/2}, \tag{3.13}$$

where $|B|$ is the determinant of the matrix B.

Proof. To prove (3.13), it is enough to note that

$$(2\pi)^{-d/2}|B|^{-1/2}\exp(-\frac{1}{2}x^TB^{-1}x)$$

is the density function of a normal distribution.

Lemma 3.5 *Suppose B is a $d\times d$ strictly positive definite matrix. Let*

$$D_n=\int_{\mathbb{R}^d}\exp(-\frac{n}{2}x^TB\ x)dx$$

and

$$D_n(r)=\int_{|x|\le r}\exp(-\frac{n}{2}x^TB\ x)dx.$$

Then for $r>0$,

$$\lim_{n\to\infty}D_n(r)/D_n=1.$$

Proof. By Lemma 3.4, it can be calculated that

$$D_n=(2\pi)^{d/2}n^{-d/2}|B|^{-1/2}. \tag{3.14}$$

Since B is a strictly positive definite matrix, there exists $\delta>0$ such that $x^TB\,x\ge\delta|x|^2$, where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^d$. Thus,

$$\begin{aligned}
D_n-D_n(r) &= \int_{|x|>r}\exp(-\frac{1}{2}nx^TB\ x)dx\\
&\le \int_{|x|>r}\exp(-\frac{n}{4}\delta r^2-\frac{n}{4}x^TB\ x)dx\\
&\le \exp(-n\delta r^2/4)\int_{\mathbb{R}^d}\exp(-n/4\cdot x^TB\ x)dx\\
&= \exp(-n\delta r^2/4)(2\pi)^{d/2}(n/2)^{-d/2}|B|^{-1/2}\\
&= \exp(-n\delta r^2/4)\cdot 2^{d/2}D_n.
\end{aligned}$$

Therefore,

$$D_n(r)/D_n=1-\exp(-n\delta r^2/4)\cdot 2^{d/2}=1-o(1). \quad \square$$

In what follows for two sequences $\{x_n\}$ and $\{y_n\}$ in $\mathbb{R}^+$, we write $x_n\sim y_n$ to mean that $a\le\liminf_{n\to\infty}x_n/y_n\le\limsup_{n\to\infty}x_n/y_n\le b$ for some $a,b\in(0,+\infty)$. It is clear that "$\sim$" is a equivalence relation.

Lemma 3.6 *Suppose that a real function $g(x)$ on $\mathbb{R}^2$ has a unique minimum points $x=0$ and $B(0)$ is well defined and strictly positive definite. Define*

$$E_n(\Omega)=\int_\Omega\exp[-ng(x)]dx$$

for $\Omega\subseteq\mathbb{R}^2$. Let $F_n(\varepsilon)=E_n(\{x:|x|\le\varepsilon\})$ and $G_n(\alpha)=E_n(\{x:|x|\le n^{-\alpha}\})$ for $\varepsilon,\alpha>0$. Then
(a) for any $\alpha\in(0,1/2]$ and $\varepsilon>0$, $F_n(\varepsilon)\sim G_n(\alpha)$;
(b) for any bounded set Ω containing a neighborhood of $x=0$, $E_n(\Omega)\sim F_n(\varepsilon)$.

Proof. We prove only the assertion (a). The proof of the assertion (b) is similar. Since $B(0)$ is a strictly positive definite matrix, there exits $\delta_1, \delta_2 > 0$ such that

$$\delta_1|x|^2 \le 1/2 \cdot x^T B(0)x \le \delta_2|x|^2.$$

By the Taylor's formula, $g(x) = g(0) + 1/2 \cdot x^T B(0)x + o(|x|^2)$. Moreover, we can choose δ_* and δ^* such that for $|x| \le \varepsilon$,

$$\delta_*|x|^2 \le 1/2 \cdot x^T B(0)x + o(|x|^2) \le \delta^*|x|^2.$$

Therefore,

$$\begin{aligned} F_n(\varepsilon) &= \int_{|x|\le\varepsilon} \exp[-ng(x)]dx \\ &= \int_{|x|\le\varepsilon} \exp[-n(g(0) + 1/2 \cdot x^T B(0)x + o(|x|^2))]dx \\ &\le \int_{|x|\le\varepsilon} \exp[-n(\delta_*|x|^2 + g(0))]dx \\ &= \exp[-ng(0)] \int_{|x|\le\varepsilon} \exp(-n\delta_*|x|^2)dx. \end{aligned}$$

Now we calculate the integral in the above right hand side . Let $x = (\rho cos\theta, \rho sin\theta)$, where $\theta \in [0, 2\pi]$ and $\rho \in [0, \varepsilon]$. Then the Jacobi determinant $|J| = \rho$. Thus,

$$\begin{aligned} & \int_{|x|\le\varepsilon} \exp(-n\delta_*|x|^2)dx \\ = & \int_0^{\varepsilon} \int_0^{2\pi} \exp(-n\delta_*\rho^2)\rho d\rho d\theta \\ = & \frac{\pi}{n\delta_*}[1 - \exp(-n\delta_*\varepsilon^2)]. \end{aligned}$$

Therefore,

$$F_n(\varepsilon) \le \frac{\pi}{n\delta_*}[1 - \exp(-n\delta_*\varepsilon^2)] \exp[-ng(0)]. \tag{3.15}$$

Analogously, for $G_n(\alpha)$ we have

$$\begin{aligned} G_n(\alpha) &= \int_{|x|\le n^{-\alpha}} \exp[-ng(x)]dx \\ &= \int_{|x|\le n^{-\alpha}} \exp[-n(g(0) + \frac{1}{2}x^T B(0)x + o(|x|^2))]dx \\ &\ge \int_{|x|\le n^{-\alpha}} \exp[-n(\delta^*|x|^2 + g(0))]dx \\ &= \exp[-ng(0)] \int_{|x|\le n^{-\alpha}} \exp(-n\delta^*|x|^2)dx. \end{aligned}$$

A similar calculation leads to

$$G_n(\alpha) \geq \frac{\pi}{n\delta^*}\left[1-\exp(-n^{1-2\alpha}\delta^*)\right]\exp[-ng(0)]. \tag{3.16}$$

Comparing (3.15) and (3.16), one obtains

$$\frac{F_n(\varepsilon)}{G_n(\alpha)} \leq \frac{\delta^*}{\delta_*}\cdot\frac{1-\exp(-n\delta_*\varepsilon^2)}{1-\exp(-n^{1-2\alpha}\delta^*)}.$$

Since $\alpha \leq 1/2$, letting $n \to \infty$, we obtain

$$1 \leq \liminf_{n\to\infty} F_n(\varepsilon)/G_n(\alpha) \leq \limsup_{n\to\infty} F_n(\varepsilon)/G_n(\alpha) \leq \frac{\delta^*}{\delta_*}\cdot\frac{1}{1-e^{-\delta^*}} < \infty. \quad \square$$

Proof of Theorem 3.3. Without loss of generality, we assume that $g(x)$ has a unique minimum point x^*. By the approximation formula (3.2),

$$\Pr[\mathcal{A} \subset \mathcal{M}(n, r_n)] = \exp\left[-I(n)\right] + O(e^{-n\pi r_n^2}), \tag{3.17}$$

where

$$I(n) = \int_{\mathcal{A}} n^2\pi r_n^2 g(x)^2 \exp[-ng(x)\pi r_n^2]dx.$$

Since $0 < \beta \leq g(x) \leq \gamma < +\infty$, we have

$$I(n) \sim n^2\pi r_n^2 \int_{\mathcal{A}} \exp[-ng(x)\pi r_n^2]dx. \tag{3.18}$$

Note that $c_n := n\pi r_n^2 \to \infty$. By Lemma 3.6 we get,

$$I(n) \sim nc_n \int_{|x-x^*|\leq c_n^{-1/2}} \exp[-c_n g(x)]dx. \tag{3.19}$$

By the Taylor's formula, $g(x) = g(x^*) + 1/2 \cdot (x-x^*)^T B(x^*)(x-x^*) + o(|x|^2)$. Then the right hand side of (3.19) can be written as

$$nc_n \exp[-c_n g(x^*)] \int_{|x|\leq c_n^{-1/2}} \exp\left[-\frac{c_n}{2}x^T B(x^*)x - c_n o(|x|^2)\right]dx. \tag{3.20}$$

Note that $g(x^*) = \beta$. Applying Lemma 3.4-3.6, one obtains

$$\begin{aligned}
I(n) &\sim nc_n \exp(-c_n\beta) \int_{|x|\leq c_n^{-1/2}} \exp\left[-\frac{c_n}{2}x^T B(x^*)x\right]dx \\
&\sim nc_n \exp(-c_n\beta) \int_{|x|\leq \varepsilon} \exp\left[-\frac{c_n}{2}x^T B(x^*)x\right]dx \\
&\sim nc_n \exp(-c_n\beta) \int_{\mathbb{R}^2} \exp\left[-\frac{c_n}{2}x^T B(x^*)x\right]dx \\
&= nc_n \exp(-c_n\beta)\cdot 2\pi c_n^{-1}|B(x^*)|^{-1/2} \\
&\sim n\exp(-c_n\beta).
\end{aligned}$$

Since $c_n\beta = \pi\beta n r_n^2 = \ln n + a_n$, we have $n\exp(-c_n\beta) = \exp(-a_n)$. Hence,

$$I(n) \sim \exp(-a_n). \tag{3.21}$$

Therefore,

$$\lim_{n\to\infty} \Pr[\mathcal{A} \subset \mathcal{M}(n, r_n)] = 1 \Leftrightarrow \exp\left[-I(n)\right] \to 1 \Leftrightarrow I(n) \to 0 \Leftrightarrow a_n \to +\infty.$$

Thus the first assertion is verified. The second assertion can be checked similarly.
□

4 Coverage of Sensor Networks with Empirical Distributions

From the discussion of the above section, we see that if sensors are deployed non-uniformly, then the minimum sensing radius in the coverage problem relies mainly on the minimum value of $g(x)$, which is certainly not a economical scheme. In this section we propose a more economical scheme for non-uniformly distributed sensor networks. Note that in practice, the density function $g(x)$ can always be approximated by empirical functions. Thus we assume that g is an empirical distribution functions, that is, $g(x) = \sum_{i=1}^K c_i I_{A_i}$, where $\{A_1, \cdots, A_K\}$ is a partition of $\mathcal{A}$, i.e. $\bigcup_{i=1}^K A_i = \mathcal{A}$ and $A_i \bigcap A_j = \phi$ for $i \neq j$. In this situation, we can choose different sensing radius $r_i(n)$ for different A_i. We deploy sensors by the following procedure. (see Fig. 1).

(a) Divide n sensors into K groups. The number n_i of sensors in each group is proportional to $c_i S_i$, where S_i is the area of the region A_i.

(b) Deploy the sensors of each group into the corresponding area A_i independently and according to the uniform distribution on A_i.

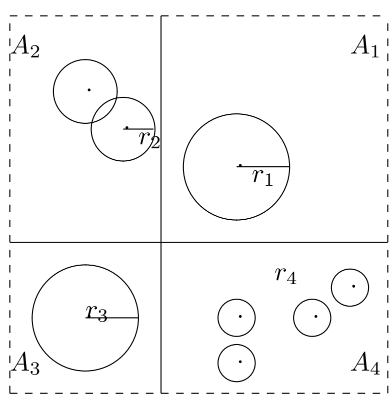

Fig. 1. Sensors with different sensing radii in different regions

With the above scheme we have the following result.

Proposition 4.1 *Let* $\mathcal{A} = [0,1]^2$ *be divided as* $\mathcal{A} = \bigcup_{i=1}^K A_i$ *with* $A_i \bigcap A_j = \phi$ *for* $i \neq j$, $i, j = 1, \cdots, K$, *where each* A_i *is a rectangle with area* $\mu(A_i)$. *For* n *large enough, we put* $m_i = \lceil nc_i\mu(A_i)\rceil$ *sensors in* A_i, *which are located as* m_i *independent random points with uniform distribution on* A_i. *Suppose that each sensor in* A_i *has a sensing radius* $r_i(n) = \sqrt{(\ln n + \ln\ln n + a_i(n))/c_i\pi n}$. *Suppose further that* $a_i(n) = o(\ln n)$ *and* $\lim_{n\to\infty} a_i(n)$ *exists in* $[-\infty, +\infty]$. *Then we have*

$$\Pr[\mathcal{A} \text{ is covered}] = \exp\left[-\sum_{i=1}^K \exp(-a_i(n))\right] + o(1). \tag{4.1}$$

Proof. In the limiting procedure we may ignore the boundary effects. Thus,

$$P[\mathcal{A} \text{ is covered}] = \prod_{i=1}^K P[A_i \text{ is covered}] + o(1).$$

It is enough to check that

$$P[A_1 \text{ is covered}] = \exp\left[-\exp(-a_1(n))\right] + o(1). \tag{4.2}$$

For convenience we assume that $c_1\mu(A_1)n = m_1$ is an integer. Scaling the area and the sensing radius simultaneously, we can apply Theorem 2.1 to obtain

$$P[A_1 \text{ is covered}] = \exp[-I(m_1)] + o(1),$$

where

$$I(m_1) = \frac{1}{\mu(A_1)} m_1^2 \pi r_1(n)^2 \exp\left[-m_1\pi r_1(n)^2 \frac{1}{\mu(A_1)}\right]. \tag{4.3}$$

Under our assumption on $r_1(n)$, we have

$$\begin{aligned}
\pi r_1(n)^2 &= \frac{1}{c_1 n}(\ln n + \ln\ln n + a_1(n)) \\
&= \frac{\mu(A_1)}{c_1 n\mu(A_1)}[\ln(c_1\mu(A_1)n) + \ln\ln(c_1\mu(A_1)n) + a_1(n) + O(1)] \\
&= \frac{\mu(A_1)}{m_1}[\ln m_1 + \ln\ln m_1 + a_1(n) + O(1)]. \qquad (4.4)
\end{aligned}$$

Combining (4.3) and (4.4) we obtain

$$I(m_1) = \exp[-a_1(n)][1 + \ln\ln m_1(\ln m_1)^{-1} + a_1(n)(\ln m_1)^{-1}]. \tag{4.5}$$

Thus $I(m_1) = \exp[-a_1(n)] + o(1)$. Hence (4.2) follows. $\square$

5 Acknowledgment

We thank Yibing Cai for his talk on wireless sensor networks at our seminar. We thank also Qingyang Guan, Xue Cheng, Shaojun Guo, Yongsheng Song for helpful discussions. Zhi-Ming Ma is grateful to the organizers of the Conference CJCDGCGT2005 for inviting him to give a talk. The work is partly supported by NSFC and 973 Project.

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