Graph Algorithms in VLSI CAD

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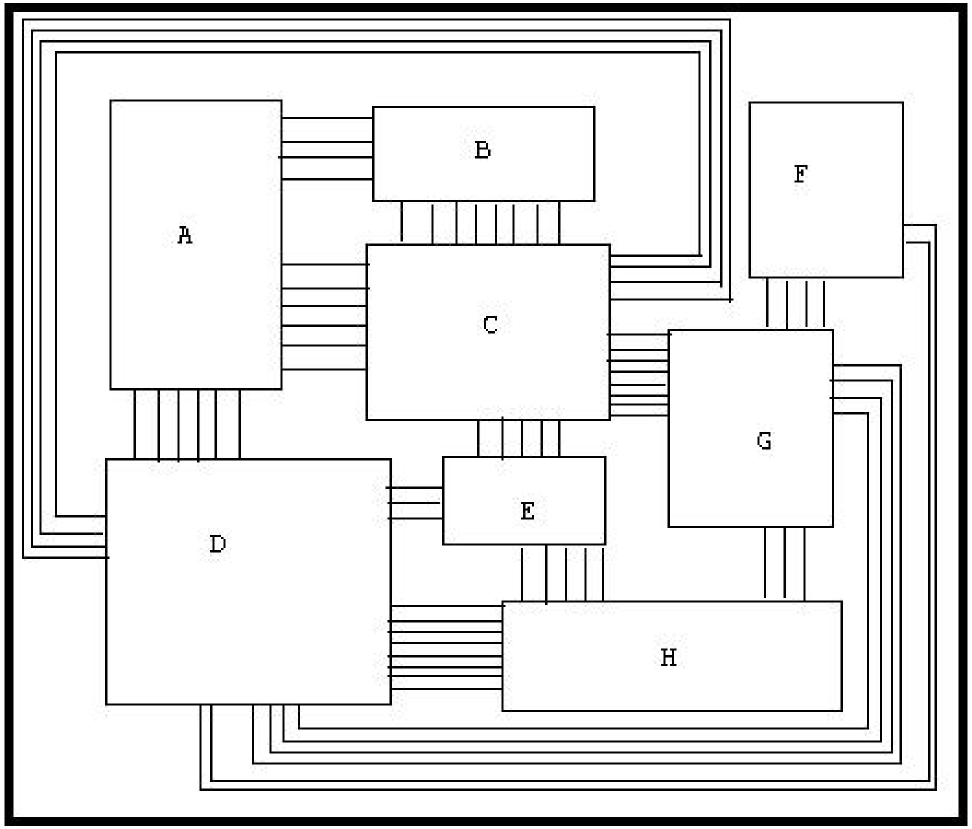
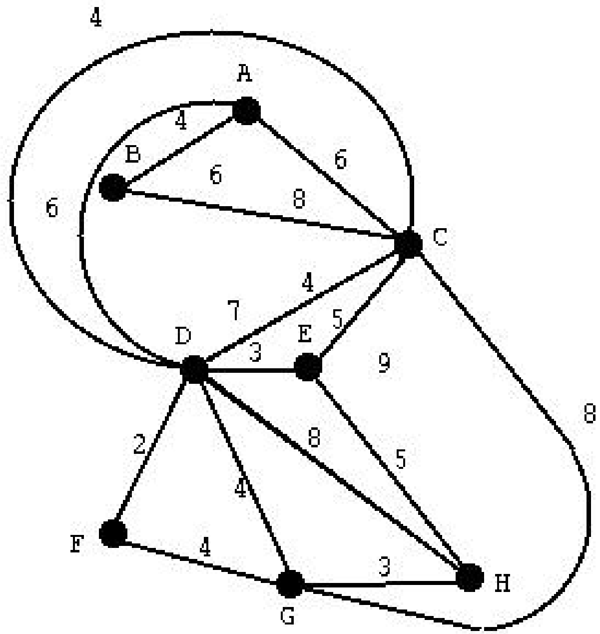
Motivation

- A VLSI layout

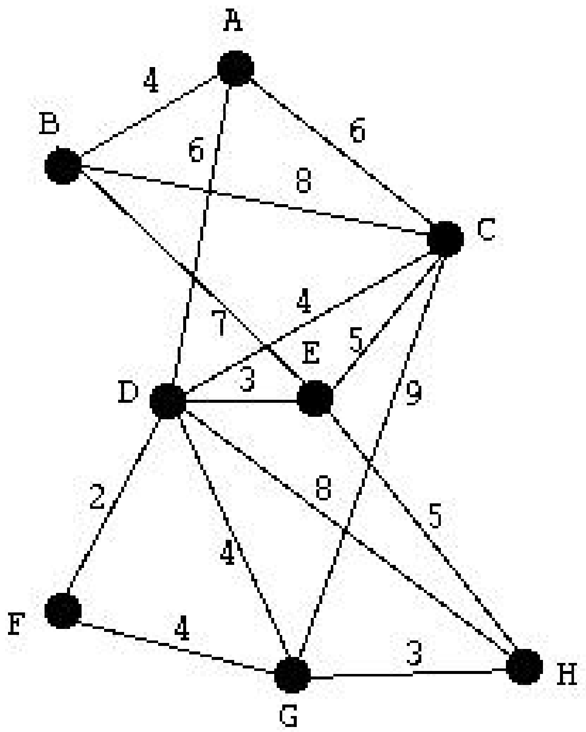


Its graph representation

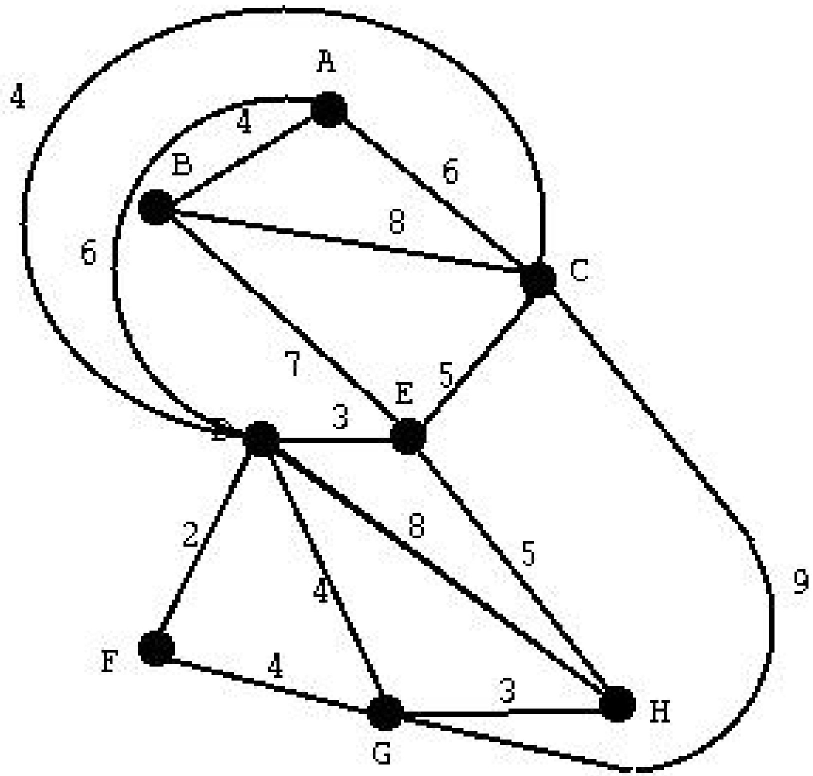


Layout Problem

Interconnection information among circuit modules

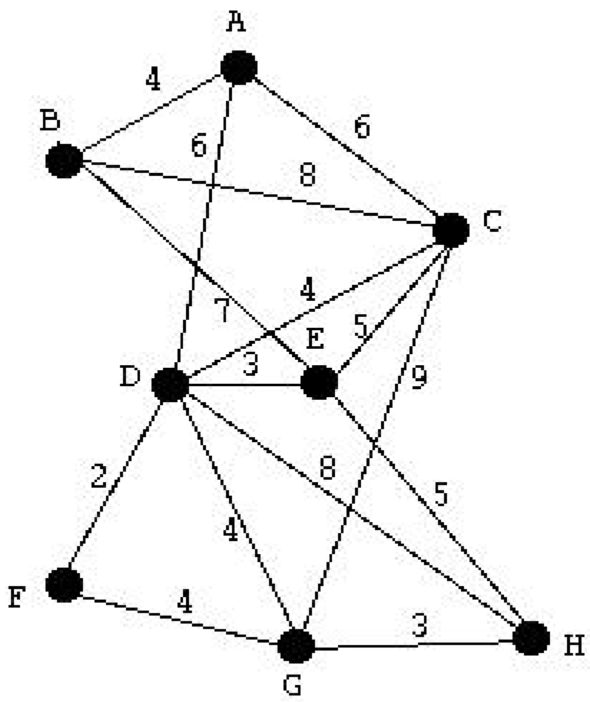


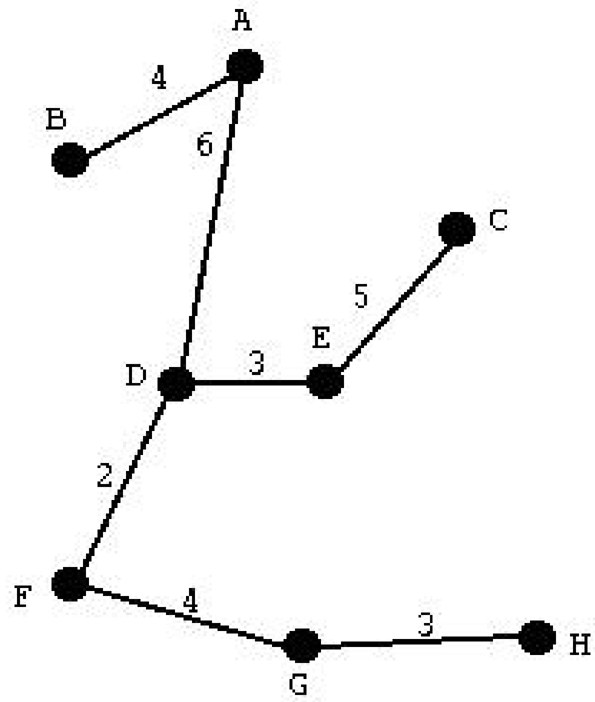
maximal planar subgraph extraction



Routing Problem

- **An electrical signal net connected with a set of modules**
- **Edge costs indicate the cost of connections**
- **Objective : Connect all the modules with minimum cost**





Data Structures and Algorithms for VLSI Design

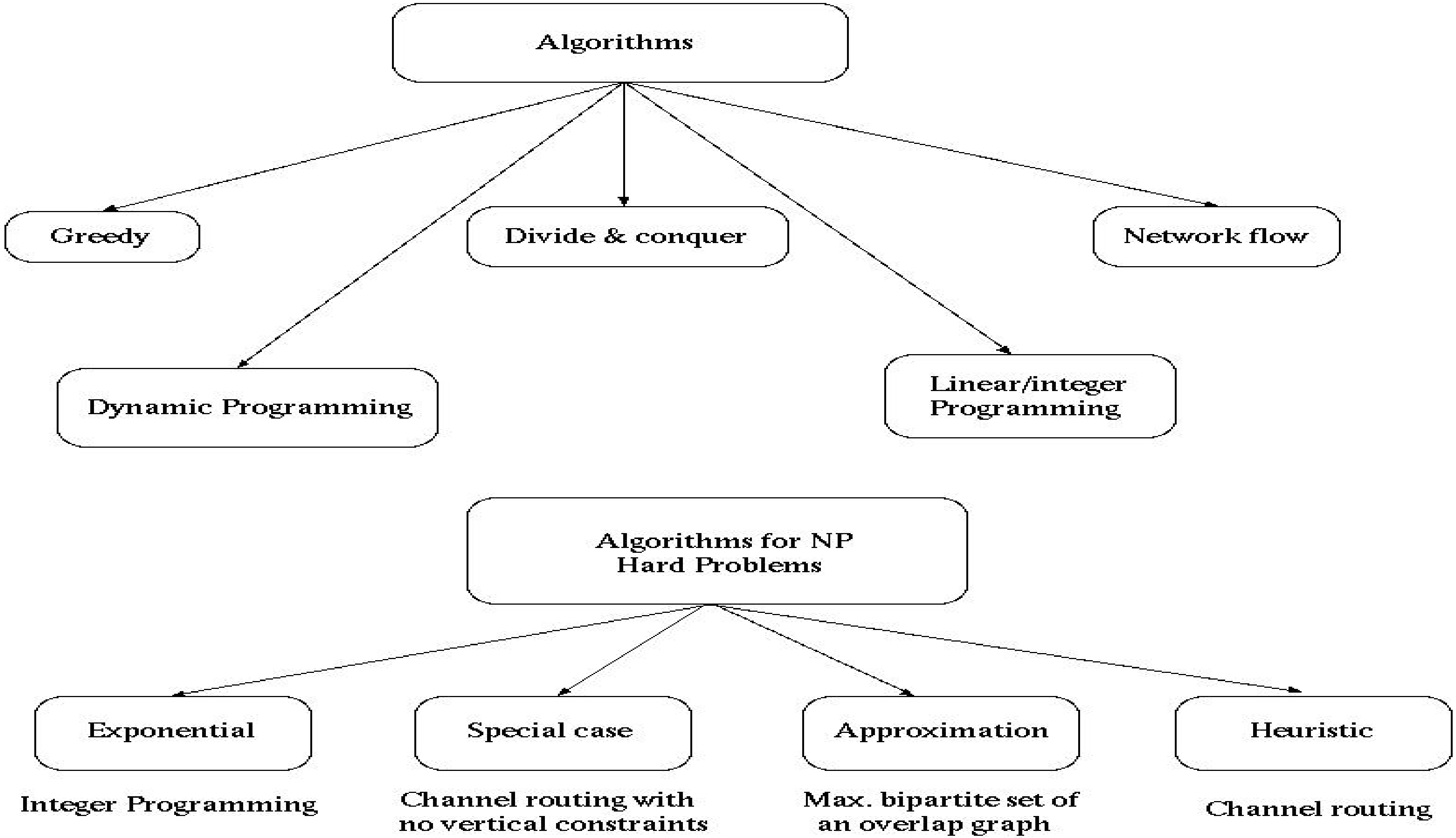
Objectives:

- To understand how a layout is represented and manipulated
- To review basic graph algorithms
- Abstraction of VLSI design problems as an optimization/search problems in appropriate graphs

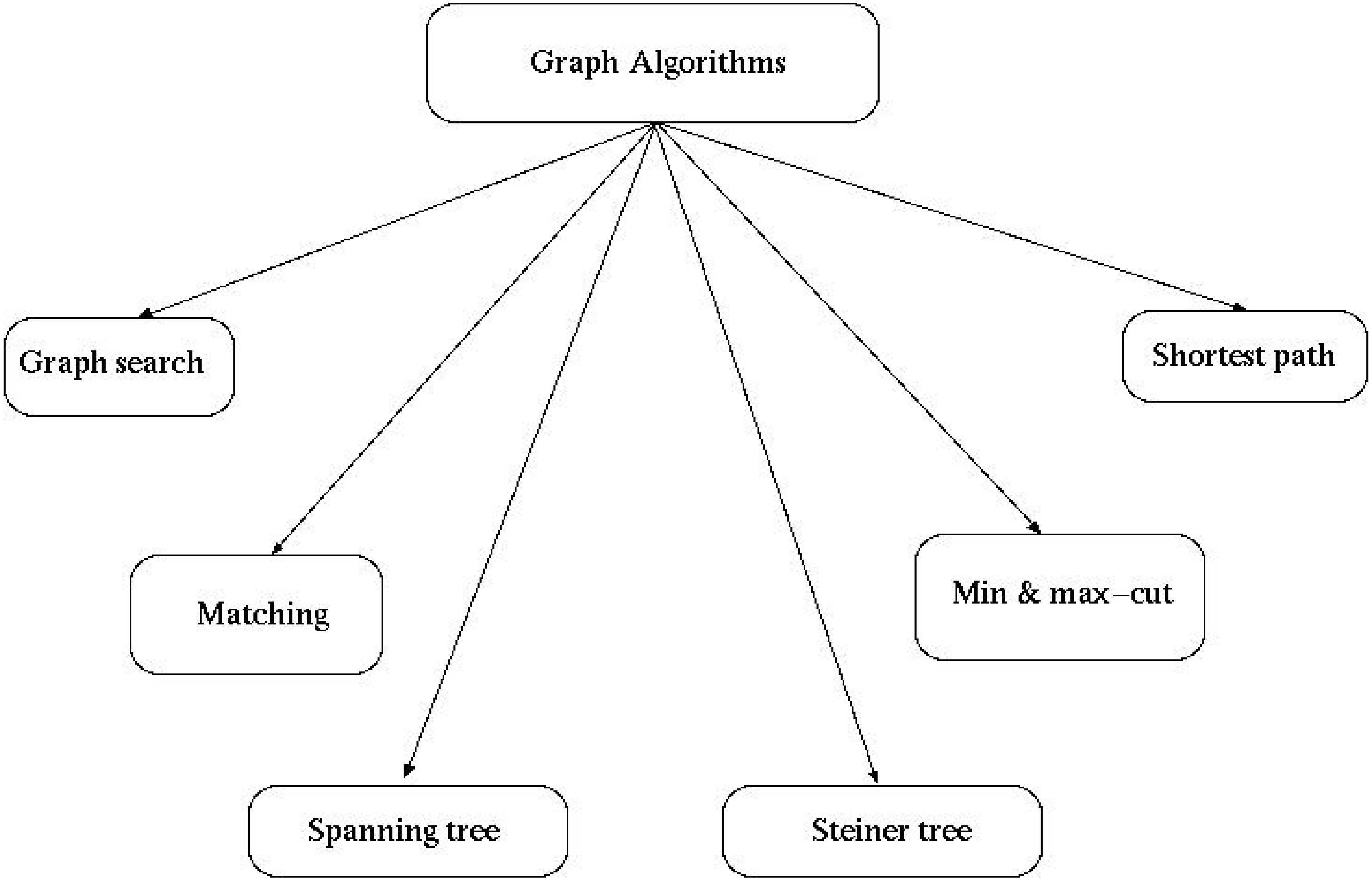
Standard graphs used in studying VLSI design problems

Basic algorithms on those graphs

Classification of Algorithms

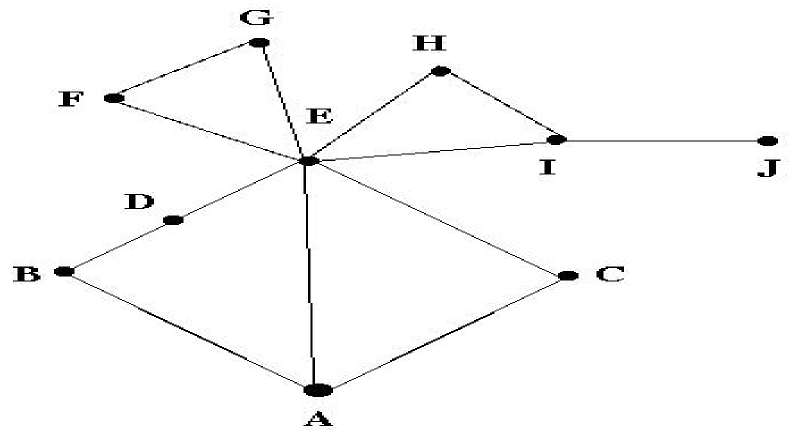


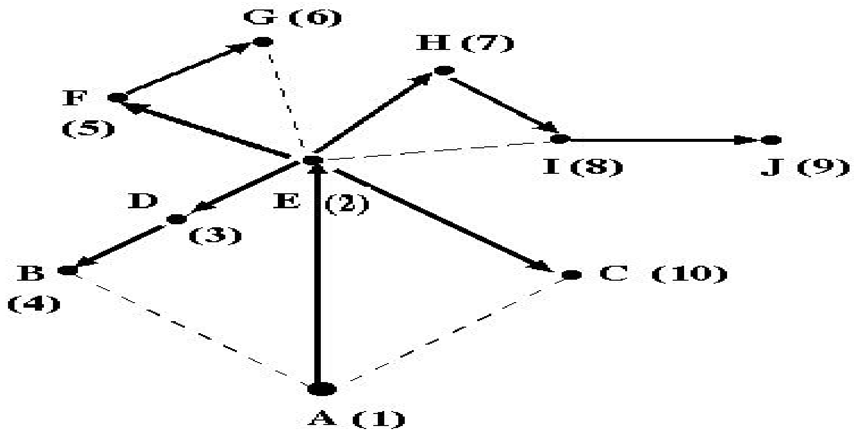
Graph Algorithms



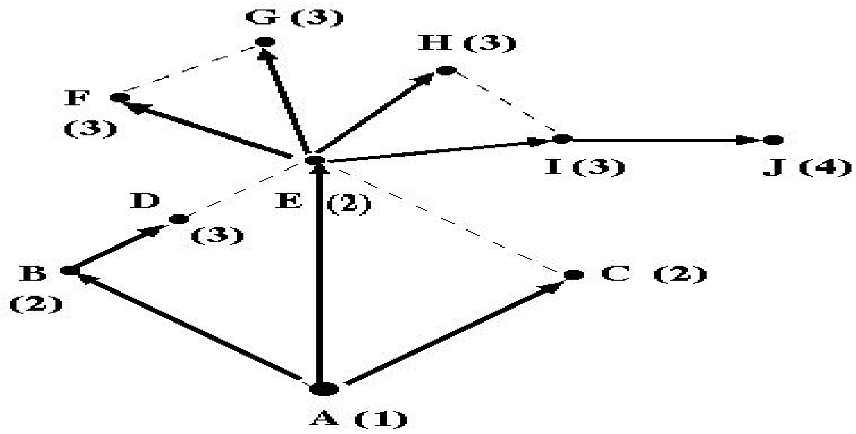
Basic Graph search algorithms

A graph





Depth first search tree



Breadth first search tree

Basic Graph search algorithms

Algorithm DEPTH-FIRST SEARCH(*u*)

begin
MARK(*u*) = 1;
for each vertex *v*, such that $(u,v) \in E$
if MARK(*v*) = 0 **then**
DEPTH-FIRST-SEARCH(*v*);
end.

Algorithm BREADTH-FIRST-SEARCH(*u*)

begin
put the start vertex in *Q*;
while *Q* not empty **do**
u = first element of *Q*
for each vertex *v*, such that $(u,v) \in E$
process *v*;
put *v* in *Q*;
endwhile
end.

Algorithm for MST

- **Instance: A connected weighted undirected graph G(V,E)**
- **Solution space: All trees that span nodes of G**
- **Objective: Minimize l(T) = $\mathbf{S_{e \in T}}$ l(e)**
- **Algorithm:**

A = f;

for each vertex $v \in$ V

    do MAKE-SET (v)

sort the edges of E by non-decreasing weights

for each edge (u,v) $\in$ E (* in order by non-decreasing weight *)

    do if FIND-SET(u) $\neq$ FIND-SET(v)

        then A $\leftarrow$ A$\cup$ {(u,v)}

            UNION (u,v)

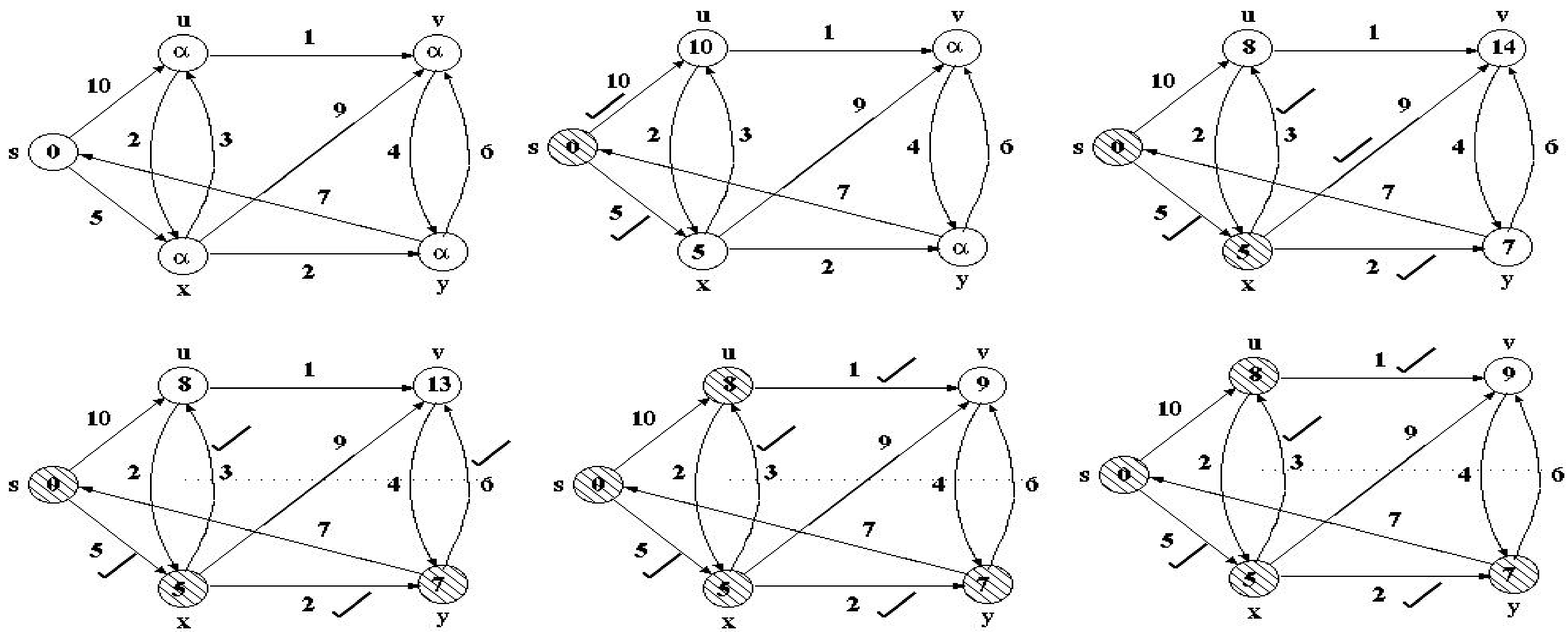
Return A

Single Source Shortest Path

Applications : Routing multi-terminal nets

Input: A directed graph *G*(*V*,*E*) with non-negative edge weights

Output: A set *S* containing the weight of the shortest path of each node from the source node *s*.



DIJKSTRA's Algorithm

Data Structure: Each node is attached with a pointer π to point its predecessor on the Shortest path

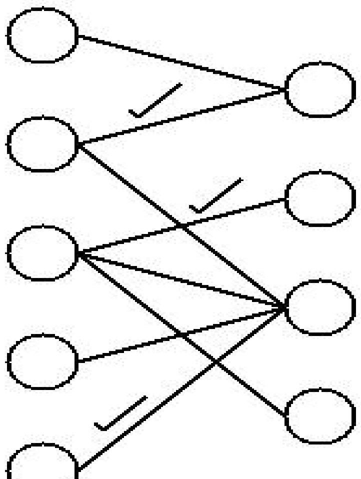
Algorithm DIJKSTRA(G, w, s)
begin
  S = ϕ; $Q = V(G)$; (* $Q \rightarrow$ Priority Queue *)
  while $Q \neq \phi$ **do**
    u = EXTRACT-MIN(Q); $S = S \cup \{u\}$;
    for each vertex $v \in \mathrm{Adj}(u)$ **do**
      RELAX (u,v,w);
  endwhile
end.

Procedure RELAX (u,v,w)
begin
  if $d(v) > d(u) + w(u,v)$ **then**
    $d(v) = d(u) + w(u,v)$;
    $\pi(\mathrm{v}) = u$;
  endif
end.

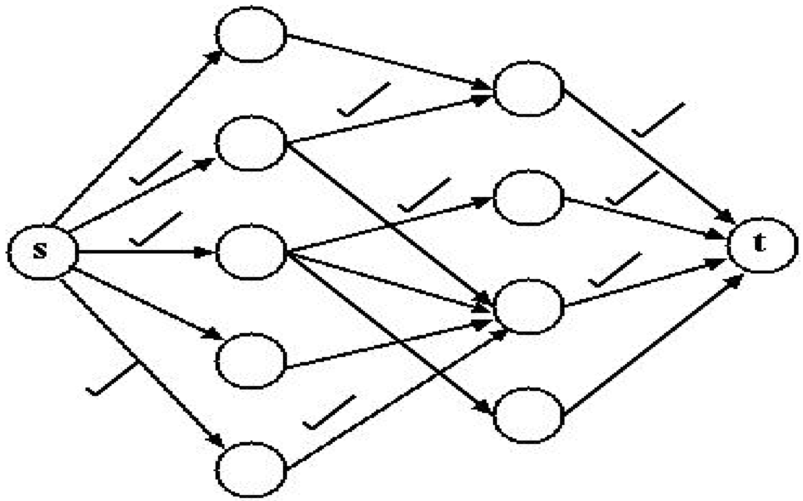
Bipartite Matching

Definition: Given an undirected graph $G(V, E)$,

- A **matching** is a subset of edges $E' \subseteq E$ such that for all vertices $v \in V$, at most one edge of E' is incident on v.
- A **maximum matching** is a matching with maximum cardinality.
- A matching is called **bipartite matching** if the graph G is bipartite.



L R



L R

Transformation
Matching problem $\rightarrow$ Max-flow problem

Given a bipartite graph $G(V, E)$, where $V = L \cup R$,

construct a weighted digraph $G'(V', E')$ as follows:

- $V' = V \cup \{s, t\}$
- $E' = \{(s, u) : u \in L\} \cup \{(u, v) : u \in L, v \in R, (u, v) \in E\} \cup \{(v, t) : v \in R\}$
- Assign unit capacity to each edge.

Result: If M is a matching in G, then there is a integer valued flow f in G' with $|f| = |M|$.

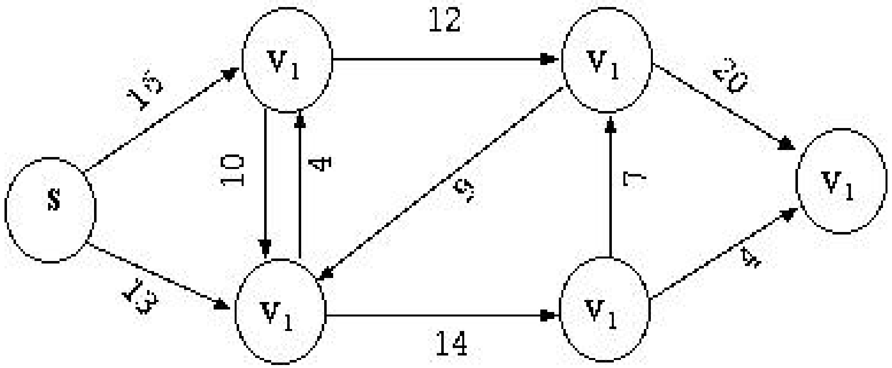
Conversely, if f is an integer valued flow in G', then there is a matching M with cardinality $|M| = |f|$.

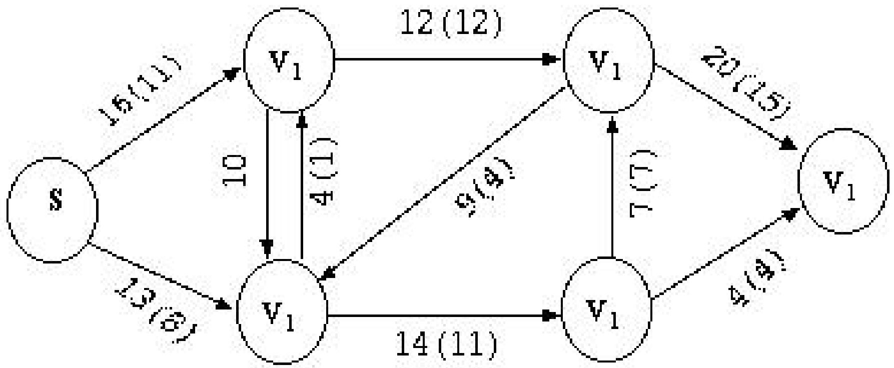
Problem: Find the maximum flow in the network G'.

<u>Matching edges</u> are those edges of G through which <u>flow pass</u>.

Max-flow Min-cut Problem

Given a digraph $G(V, E)$ with each edge having capacity $c(u, v) \geq 0$, and two designated nodes s (source) and t (sink),





Flow $f(u,v)$: a real-valued function (may be "+" / "-"/ 0), and satisfies

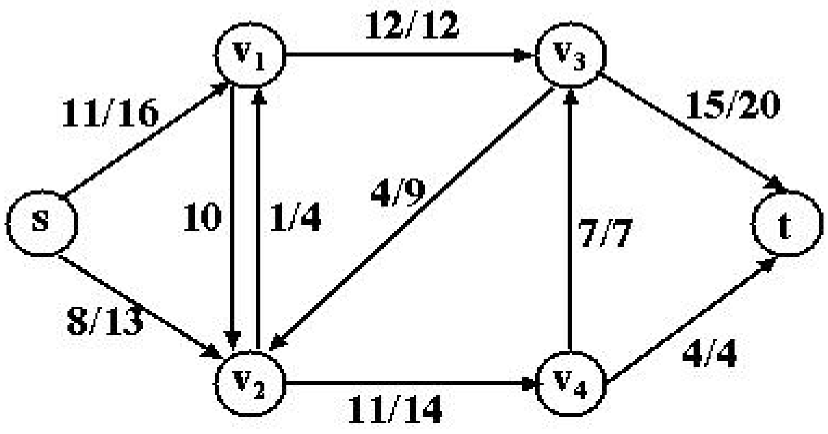
Capacity constraint: For all $u, v \in V$, we require $f(u, v) \leq c(u, v)$.

Skew symmetry: For all $u, v \in V$, we require $f(u, v) = - f(v, u)$.

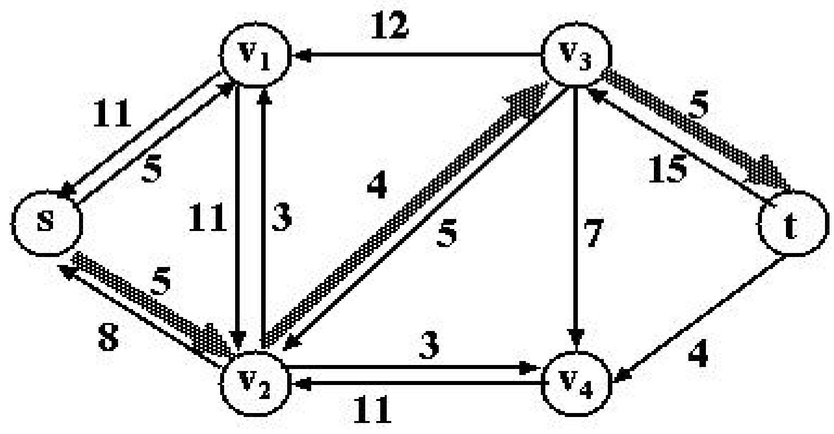
Flow conservation: For all $u \in V - \{s, t\}$, we require $\Sigma_{v \in V} f(u, v) = 0$.

Objective: ***maximize*** value of the flow $|f| = \Sigma_{v \in V} f(s, v)$

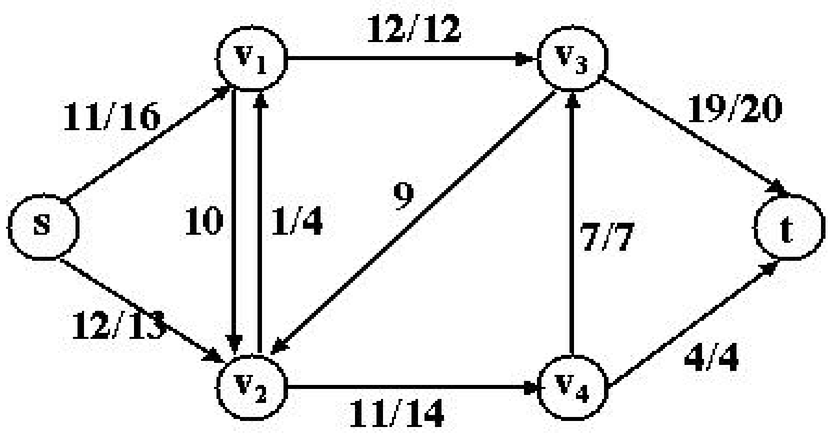
A flow network



A flow augmenting path

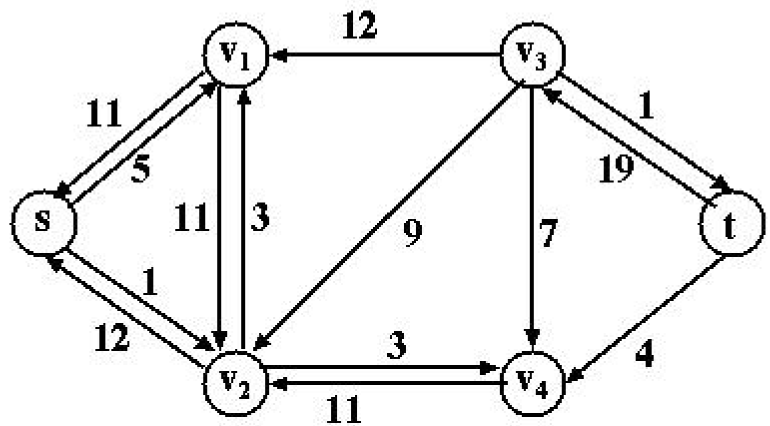


Augmented flow



The residual network

($c(u,v) = c(u,v) - f(u,v)$)



Ford-Fulkerson Method (G, s, t)

(* Initialize flow *f* to 0 *)

for each edge (u,v) $\in$ E

do f[u,v] $\leftarrow$ 0; f[v,u] $\leftarrow$ 0;

while an augmenting path *p* from *s* to *t* exists in the residual network

(* use breadth-first search to find a shortest path from *s* to *t* in residual network *)

do (* augment flow *f* along *p* *)

$c_f(p) \leftarrow \min\{c_f(u,v) : (u,v) \text{ is in } p\}$

for each edge (u,v) in p

do $f[u,v] \leftarrow f[u,v] + c_f(p)$;

$c[u,v] \leftarrow c[u,v] - c_f(p)$;

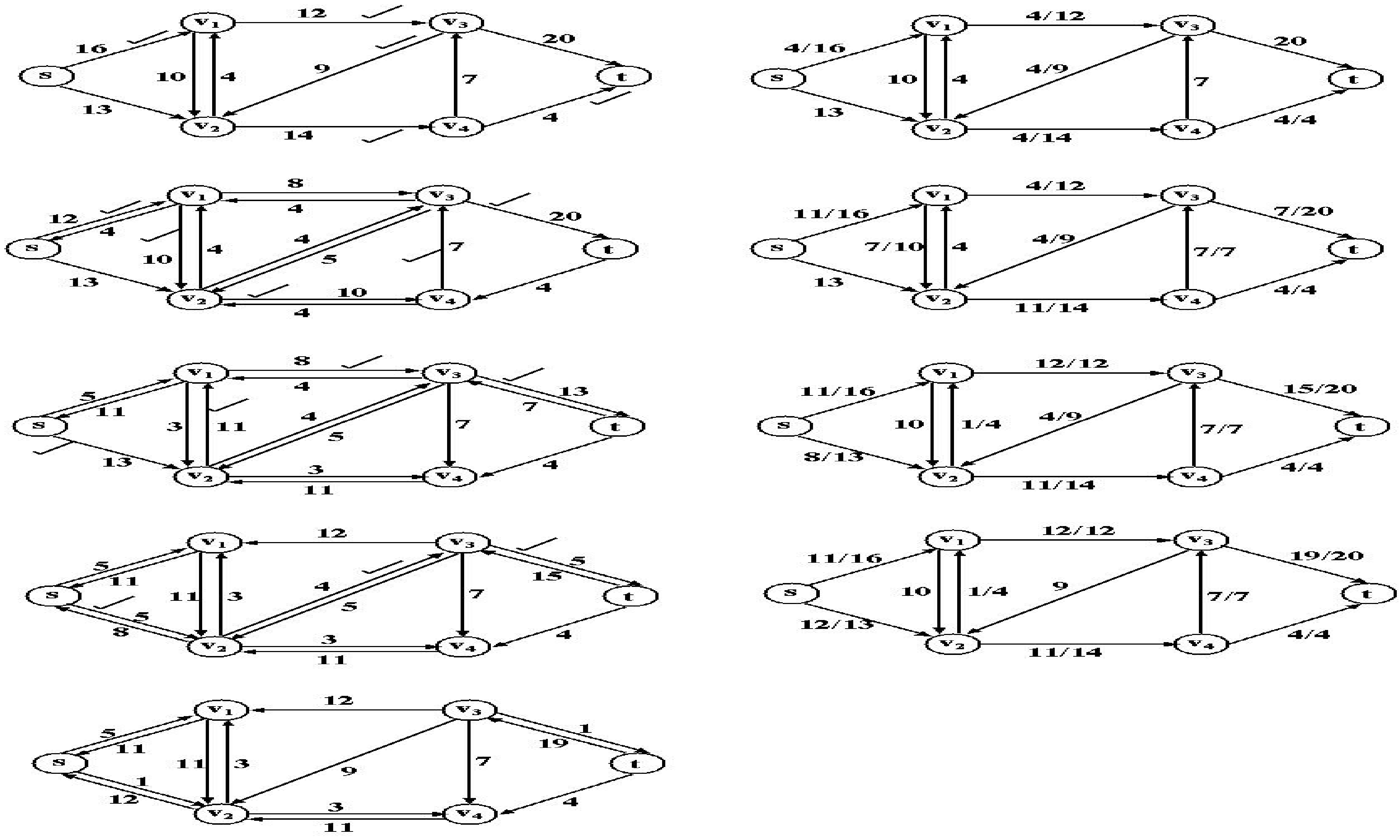
$f[u,v] \leftarrow - f[v,u]$;

return *f*

Time complexity : $O(E\ |f^*|)$

where f* is the maximum flow.

Execution Trace



Significant NP-complete graph problems

Independent set problem

Instance: Graph $G = (V, E)$, and a positive integer $k \leq |V|$.

Question: Does G contain an independent set of size k or more, i.e., a subset

$V' \subset V$ such that no two vertices of V' are adjacent, and $|V'| \geq k$

Clique problem

Instance: Graph $G = (V, E)$, and a positive integer $k \leq |V|$.

Question: Does G contain a clique of size k or more, i.e., a subset $V' \subset V$ such that every pair vertices of V' are adjacent, and $|V'| \geq k$.

Graph k-colorability

Instance: Graph $G = (V, E)$, and a positive integer $k \leq |V|$.

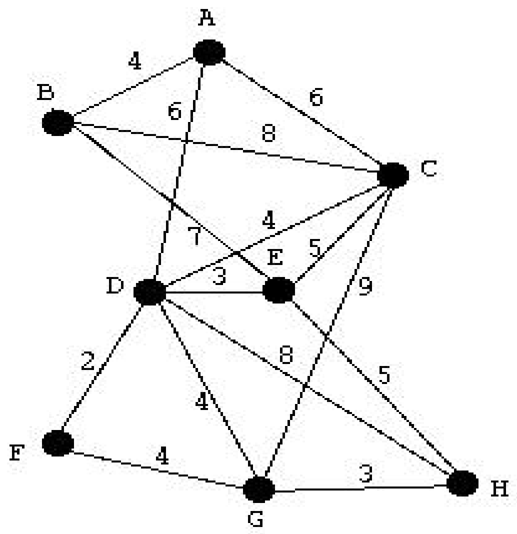
Question: Is G k-colorable, i.e., does there exist a function

$$f : V \rightarrow \{1, 2, \ldots k\} \text{ such that } f(u) \neq f(v) \text{ whenever } \{u,v\} \in E \text{ ?}$$

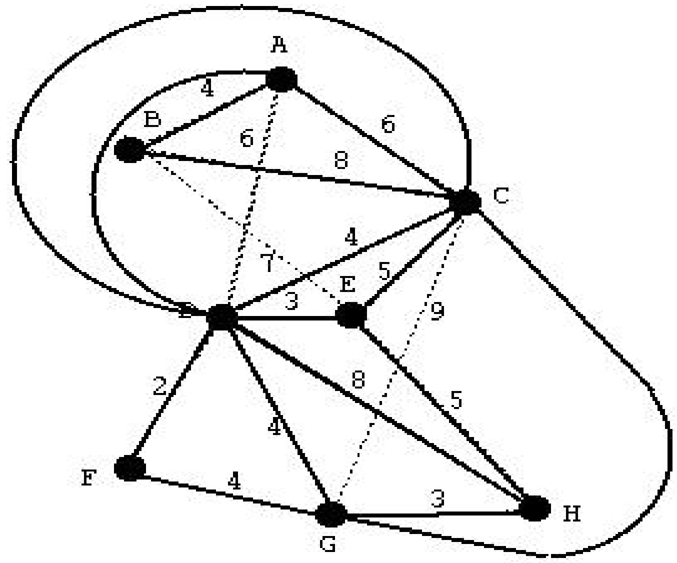
Special Classes of graphs in VLSI CAD

- Planar graphs
- Interval graphs
- Circle graphs
- Permutation graphs

Planar Graphs



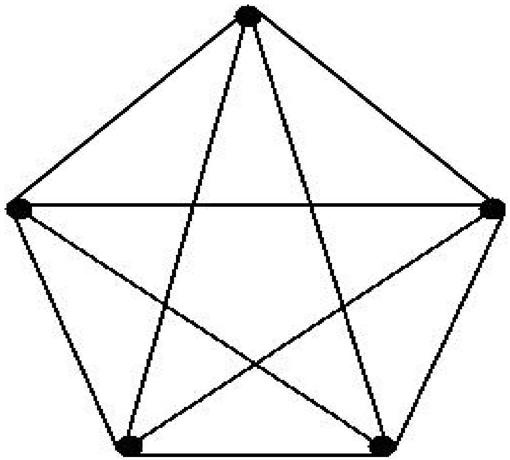
A planar graph

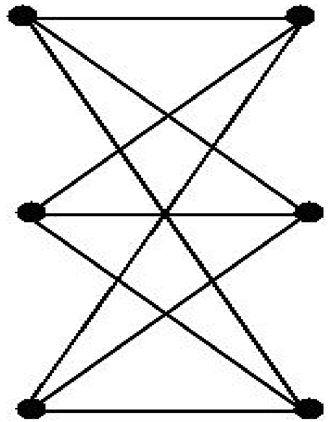


Its planar embedding

- A graph G is planar if it can be drawn in the plane without edges crossing.
- The importance of this type of graphs stems from the fact that several graph problems which are hard in general, are easy if G is known to be planar.
- This class of graphs is useful in circuit layout design

Two smallest non-planar graphs





Kuratowski's Theorem: *G* is planar if and only if *G* is not subgraph-homeomorphic to K_5 or $K_{3,3}$.

Characterization of Planar graphs

Euler's Theorem: If *P* is an arbitrary planar embedding of a connected planar graph *G* with *n* vertices and *m* edges, and if *P* has *f* faces (including outer face), then

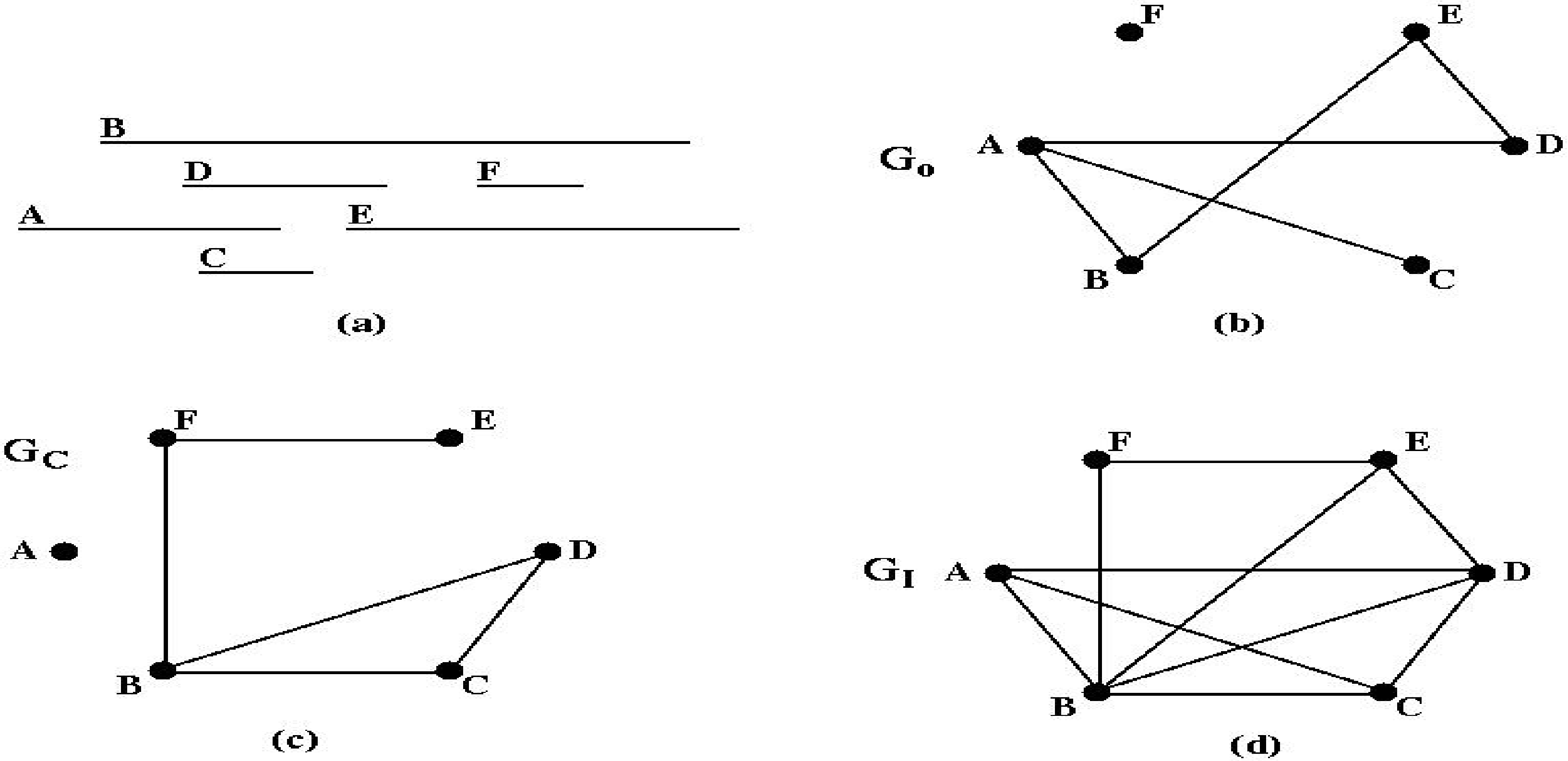
$$n - m + f = 2$$

Results on planar graphs

- If G is planar with n vertices and m edges, then $m \leq 3n$-6. If each face of G consists of 3 edges (excepting the outer face), then $m = 3n$-6. If each face in G has four edges then $m = 2n$-4.
- If G is a planar graph with $n > 4$ vertices, then G has at least four vertices whose degrees are at most 5.
- The time complexity for testing whether a graph is planar or not is $O(n)$, where n is the number of nodes in the graph.
- A planar embedding of a planar graph can be obtained in $O(n)$ time.
- The problem of computing maximum clique, maximum independent set, minimum coloring, etc. are all polynomial time computable.
- The most important problem of finding maximal planar subgraph of an arbitrary undirected graph is NP-hard.

Interval Graphs

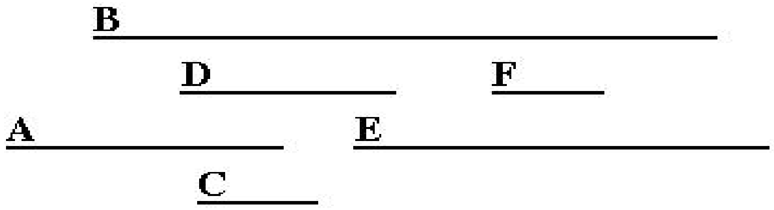
Graphs associated with a set of intervals

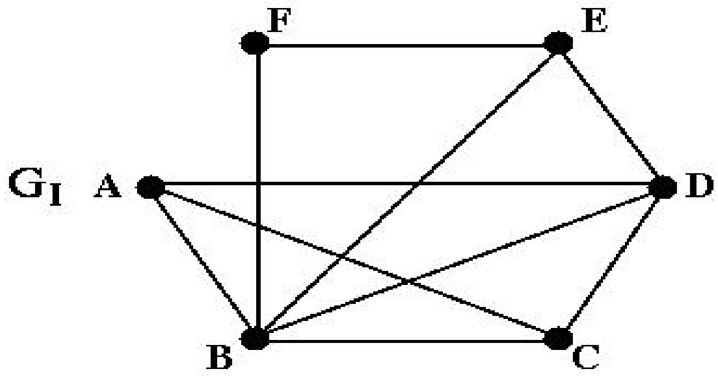


(a) Intervals, (b) overlap graph,

(c) containment graph, (d) interval graph

Definition: A graph G(V,E) is an interval graph if its vertices can be represented by a set of non-empty intervals on the real axis such that edges exist between pairs of vertices if their corresponding intervals overlap.

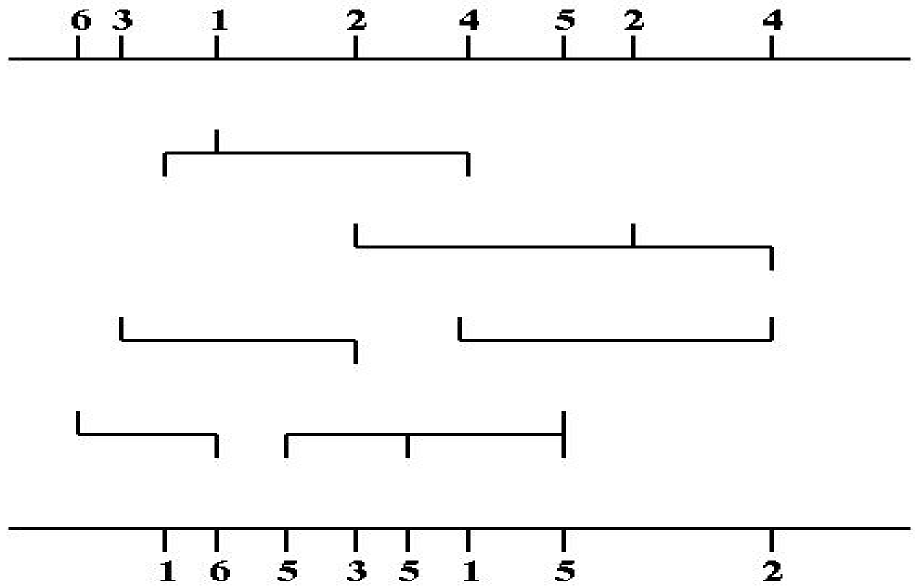




An important application of Interval Graph

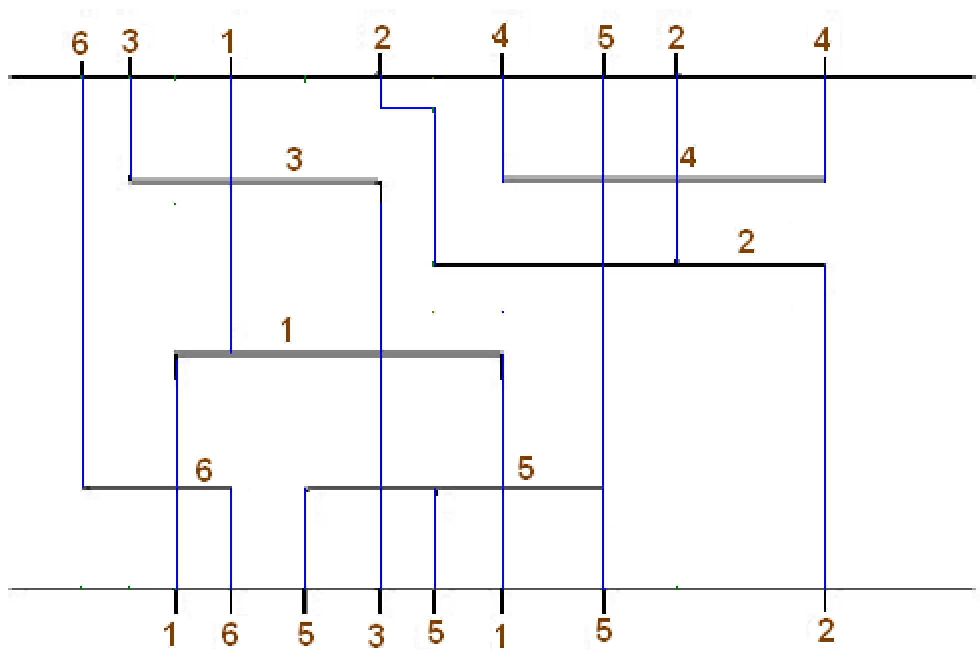
Channel-Width-Minimization problem in the jog free manhattan channel routing model.

A routing instance:

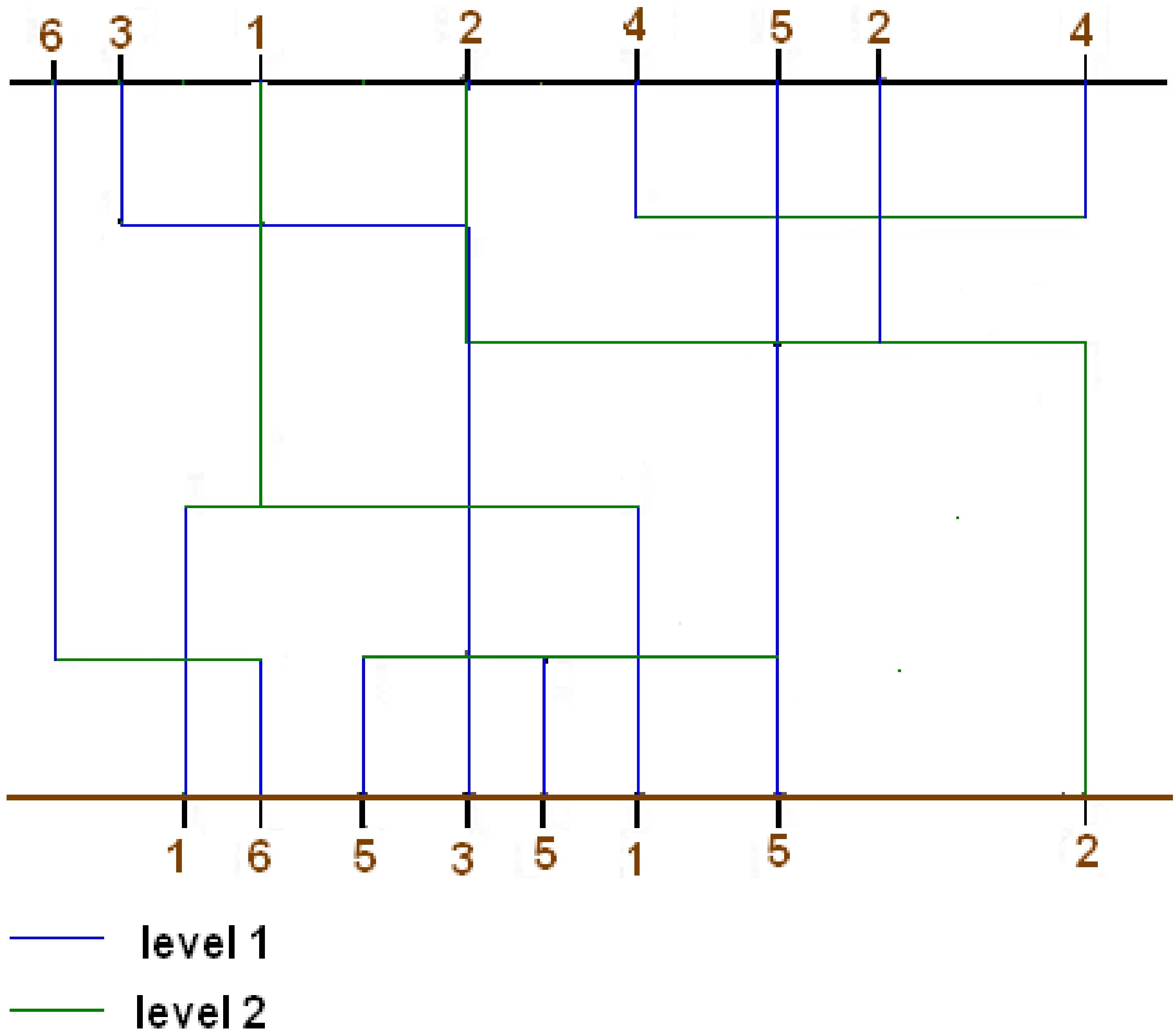


$[l_i, r_i]$ leftmost and rightmost terminals of net n_i.

- Horizontal constraint between n_i and $n_j \Rightarrow$
  $[l_i, r_i] \cap [l_j, r_j] \neq \phi$
- Vertical constraint from net n_i to $n_i \Rightarrow$ one of the two outside terminals of both the nets share a common column.
- In this case, we say that n_i is below n_j or n_j is above n_i.
- $t_i \Rightarrow$ the track assigned to net n_i



6 3 1 2 4 5 2 4
3
4
2
1
6
5
1 6 5 3 5 1 5 2


6 3 1 2 4 5 2 4
1 6 5 3 5 1 5 2
level 1
level 2

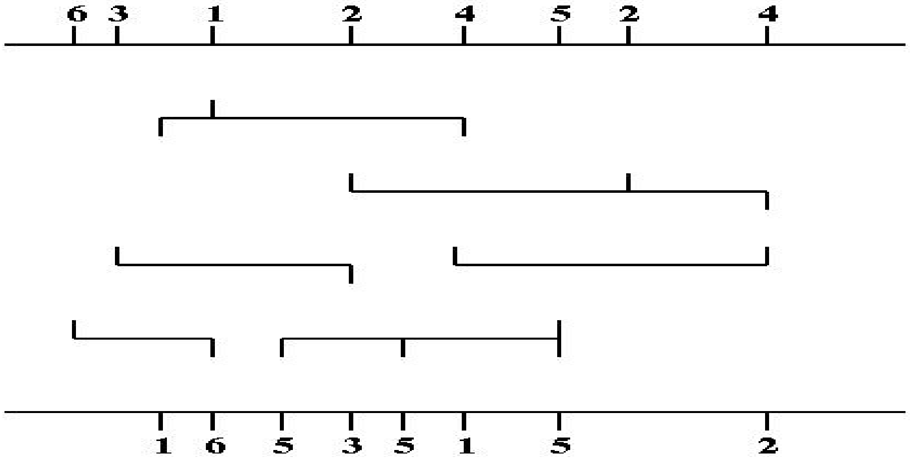
Application of Interval Graph (contd.)

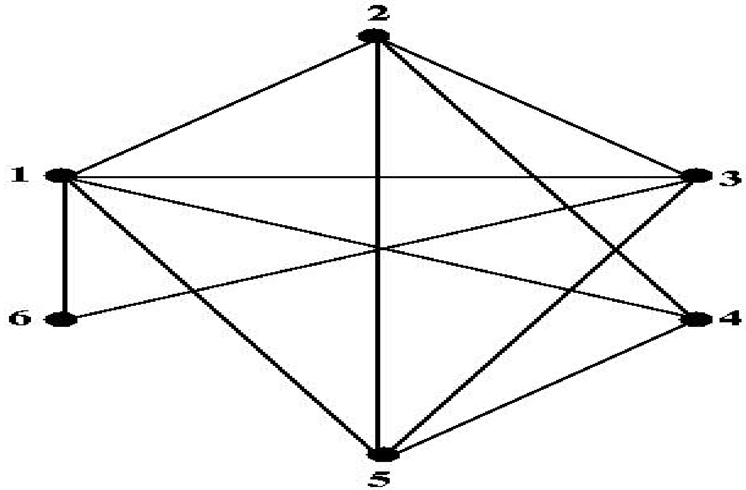
A Jog-free manhattan channel routing model $\Rightarrow$ A routing instance

$I = (t_1, t_2, \ldots, t_n)$ such that

(i) if n_i and n_j overlap then $t_i \neq t_j$ and

(ii) If n_i is below n_j, then $t_i < t_j$





Horizontal constraint graph $G_h(I) = (V_h, E_h) \Rightarrow$ an undirected graph where V_h = Set of nets of I;

$E_h = \{(n_i,n_j)$, if n_i and n_j overlaps$\}$

Optimization Problem: Find the track assignment for the nets with minimum number of tracks

Solution: Find all maximal cliques of the Horizontal constraint graph.

Data structures :

S ⇒ an array containing 2n elements corresponding to $\{(l_i, r_i), i = 1, 2, \ldots n\}$.

Each element *S*[*j*] is attached with two additional fields, called *interval_id* and *tag*

S[*j*].*interval_id* contains = *i* if the j-th element of *S* is equal to l_i or r_i

S[*j*].*tag* = L/R depending on whether S[*j*] corresponds to a left/right end point.

Algorithm:

Sort elements of *S* in increasing order of their values;

for *i* = 1 to 2*n* **do**

    if *S*[*i*].*tag* = L **then**

        insert *S*[*i*].*interval_id* in list *L*; new_clique_flag ← 1;

    if *S*[*i*].*tag* = R **then**

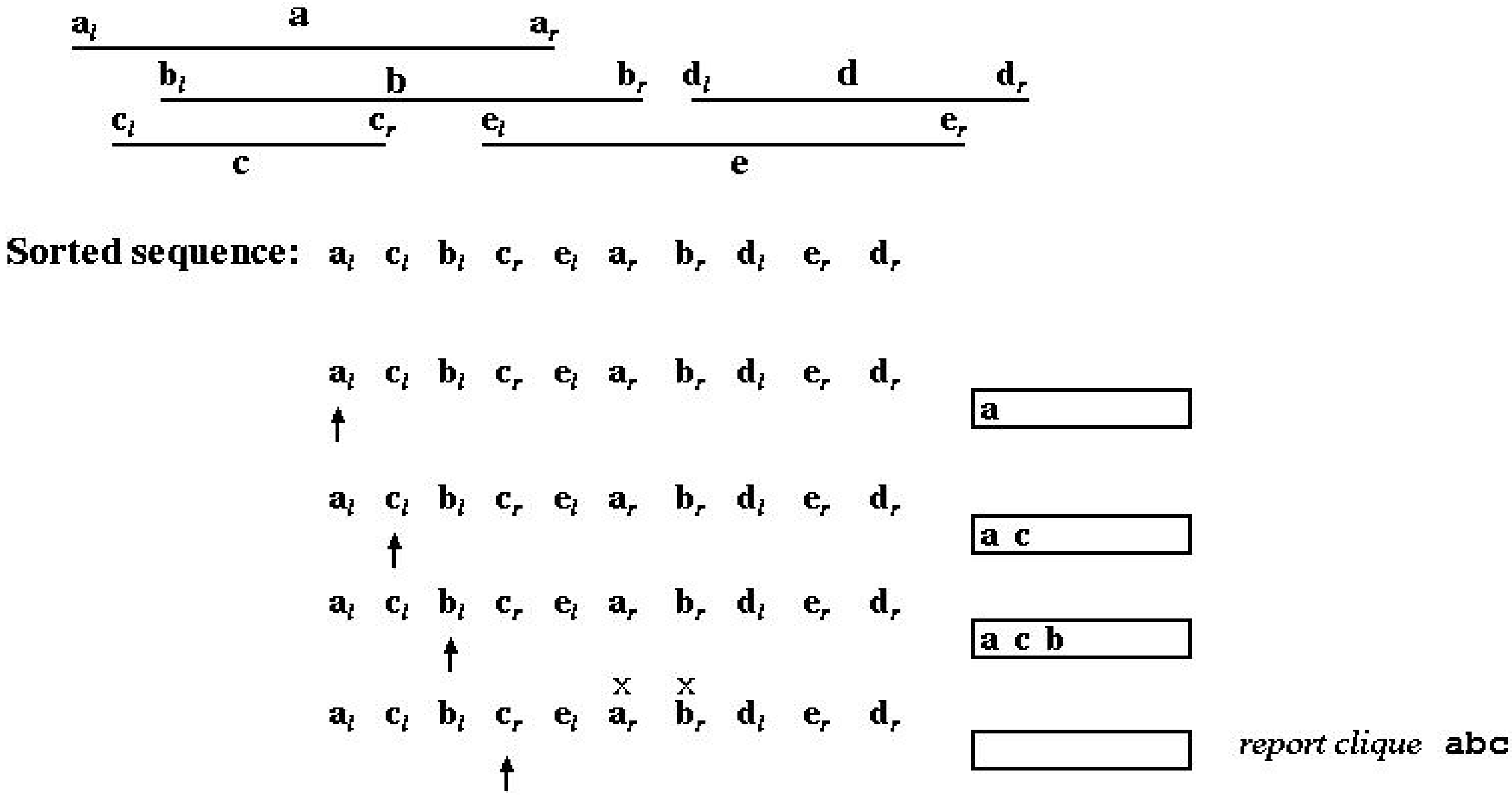
        if new_clique_flag = 1 **then**

            return all the elements in *L* as a clique;

        remove *S*[*i*].*interval_id* from list *L*; set new_clique_flag ← 0;

Size of the largest clique is the minimum track requirement for this routing problem.

Demonstration and Complexity Results



Time complexity results of different problems on Interval Graph

Largest clique : $O(n\log n)$.

Maximum independent set $O(n\log n)$

Permutation Graphs

Π = a permutation $[\pi_1, \pi_2, \quad , \pi_\nu]$ of n integers.

(e.g. [4,3,6,1,5,2], here $\pi_1 = 4$, $\pi_2 = 3$ etc.)

π_i^{-1} = position in the sequence where number *i* is found.

(e.g. $\pi_4^{-1} = 1$, $\pi_3^{-1} = 2$, etc.)

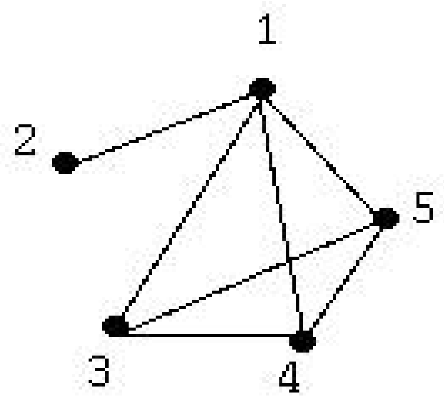
Permutation graph: An undirected graph $G_\pi(V,E)$ such that

$V = \{1, 2, \ldots, n\}$

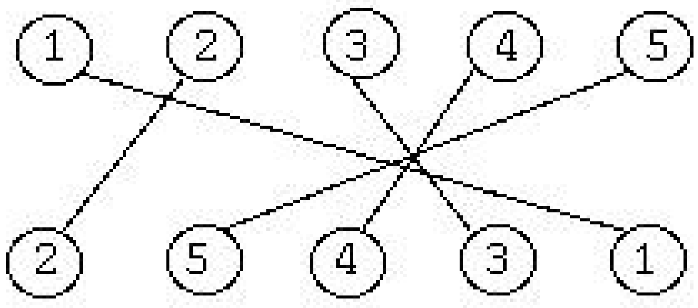
$E = \{(i,j) \mid \text{if } i < j \text{ and } \pi_j^{-1} < \pi_i^{-1}\}$ (* i.e., the larger of the integers appear to the left of the smaller one in π *).

(in other words, $(i,j) \in E$ if $(i-j)\times(\pi_i^{-1} - \pi_j^{-1}) < 0$.)

A graph and its permutation labeling



The graph $G_{[2,5,4,3,1]}$



Important uses:

Recognizing monotone channels (for routing) among a set of rectangular circuit modules on a VLSI floorplan.

Properties of permutation graphs

- **Number of edges : O(n^2).**
- **Transitively orientable**
- **Complement graph is also permutation graph.**

- **Running time Complexities:**
- Chromatic number: O(n log$n$)
- Maximum independent set: O(n log$n$)
- Largest clique: O(n log$n$)

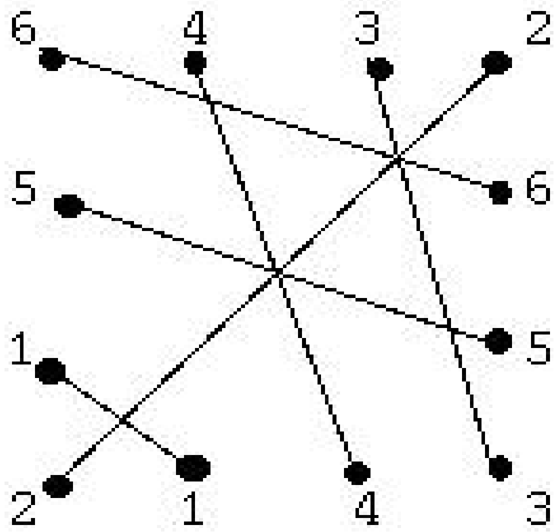
Note: In the permutation labeling,

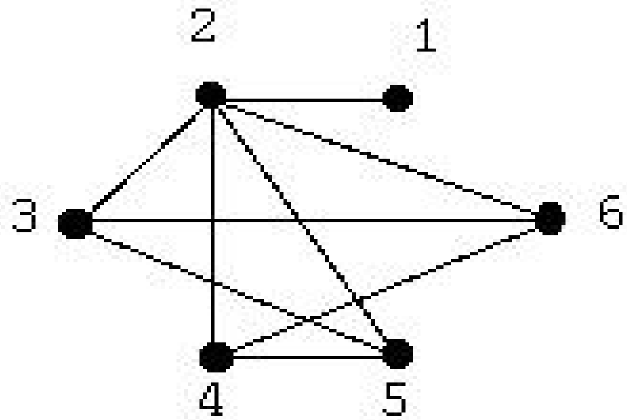
an increasing subsequence represents an independent set,

an decreasing subsequence represents a clique.

Circle Graphs

- An undirected graph G is a circle graph if it is isomorphic to the intersection graph of a finite collection of chords of a circle.

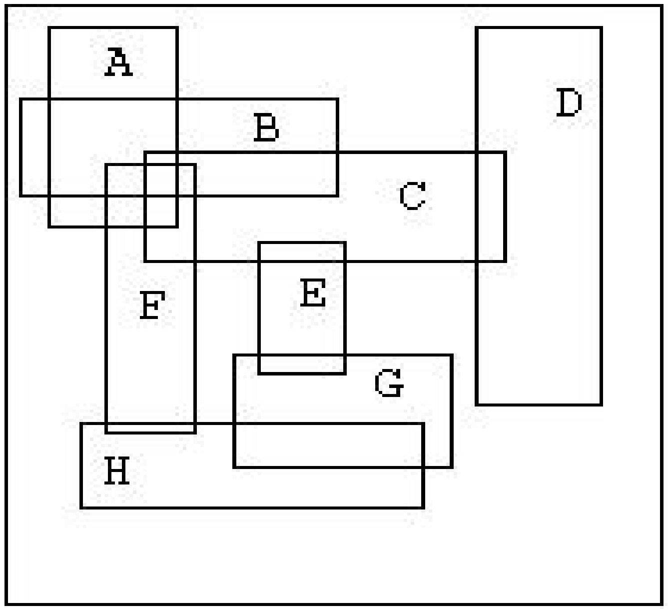


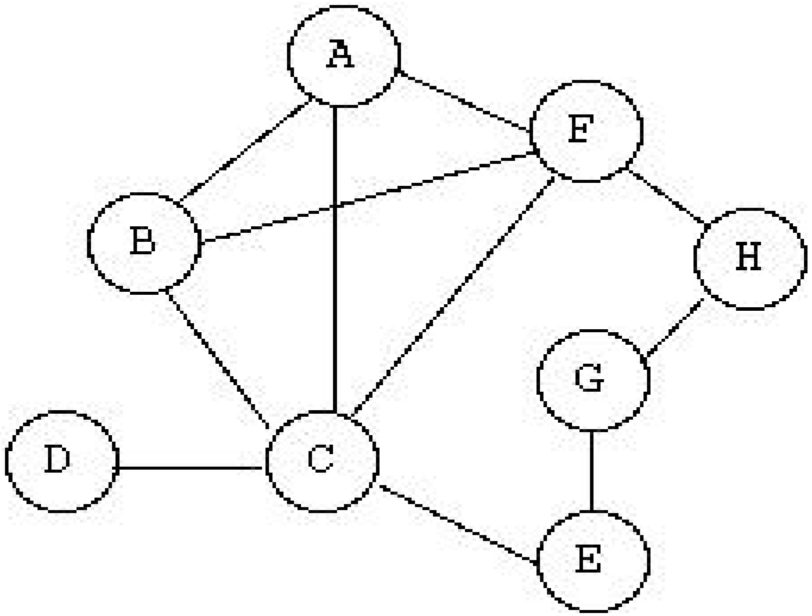


- The graph obtained by considering the intersection of lines in a switchbox is equivalent to a circle graph.

Graphs related to a set of rectangles

- Rectangle intersection graphs





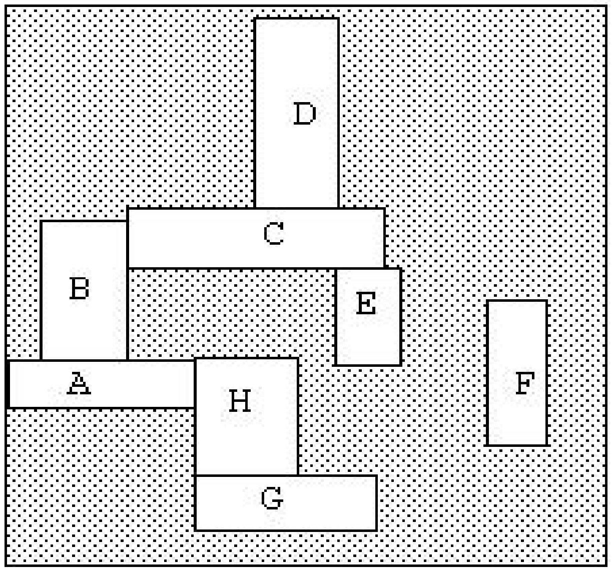
Results on rectangle intersection graphs

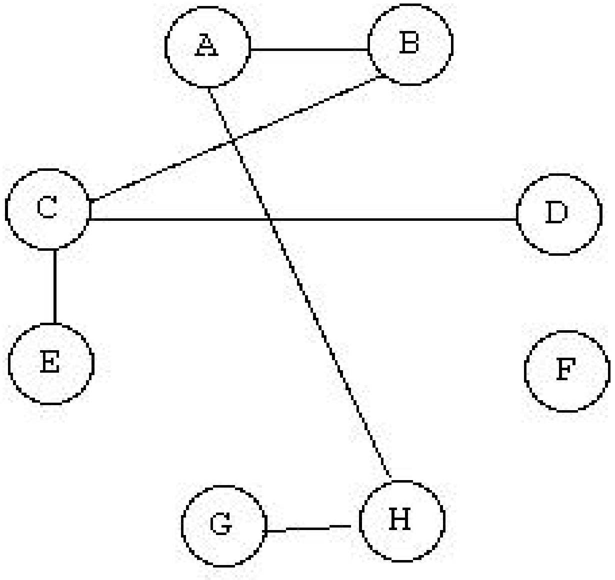
- Satisfies *Helly's Property.*
- All maximal cliques are computable in polynomial time.
- Maximal independent set problem is NP-hard.

- Applications:

- Compaction
- Finding free area for placing a block
- Other geometric optimization problems.

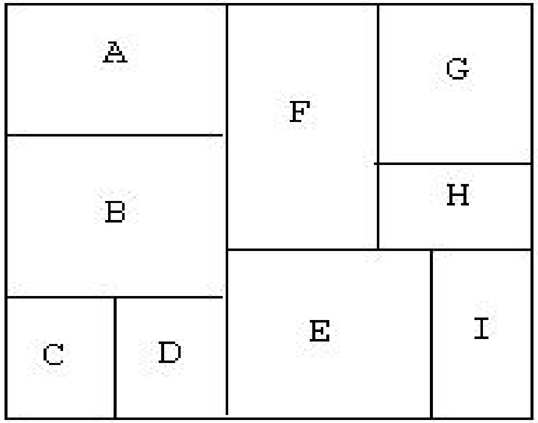
Rectangle Neighborhood Graphs

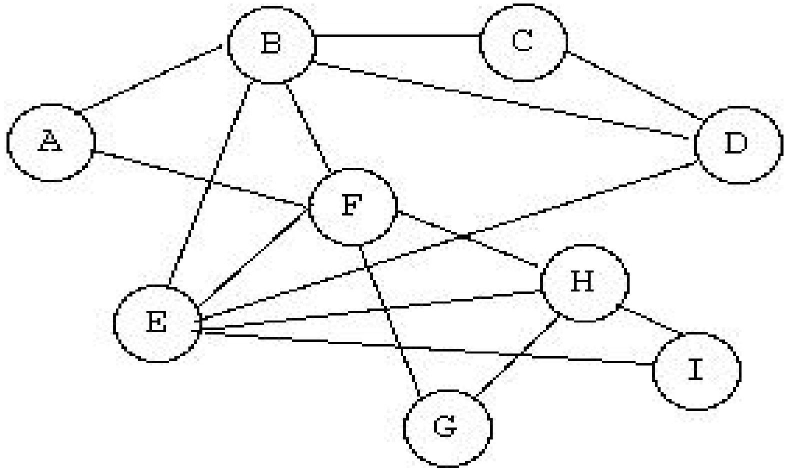




- Useful in global routing for describing physical adjacency relationship among the circuit modules.

Rectangular duals



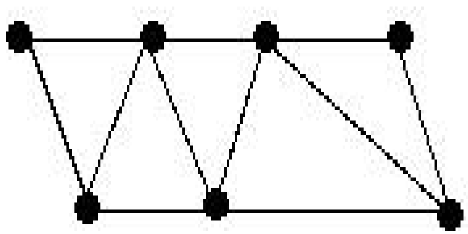


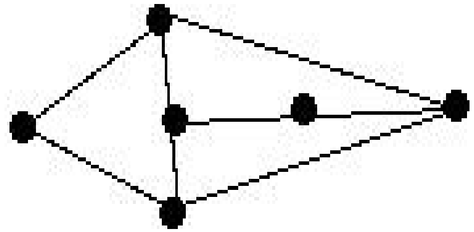
Given a graph G(V,E), its *rectangular dual* is a set of rectangles $R = \{R_1, R_2, \ldots, R_n\}$ where each vertex v_i corresponds to the rectangle $R_i \in R$ and two rectangles are adjacent if the corresponding vertices are adjacent.

- Note : Not all graphs are rectangularly dualizable.
- Use: In floorplanning phase of physical design.

Triangulated Graphs

An undirected graph is called triangulated if every cycle of length strictly greater than 3 possesses a chord.





Characterizing a triangulated graph

A vertex x of G is called *simplicial* if its adjacency set Adj(x) induces a complete subgraph (clique) of G.

A vertex elimination scheme: Repeatedly locate a simplicial vertex and eliminate it from the graph until no other vertex remains, or at some stage no simplicial vertex exists.

Theorem: In the former case, the graph G is triangulated.

In the latter case, the graph G is not triangulated.

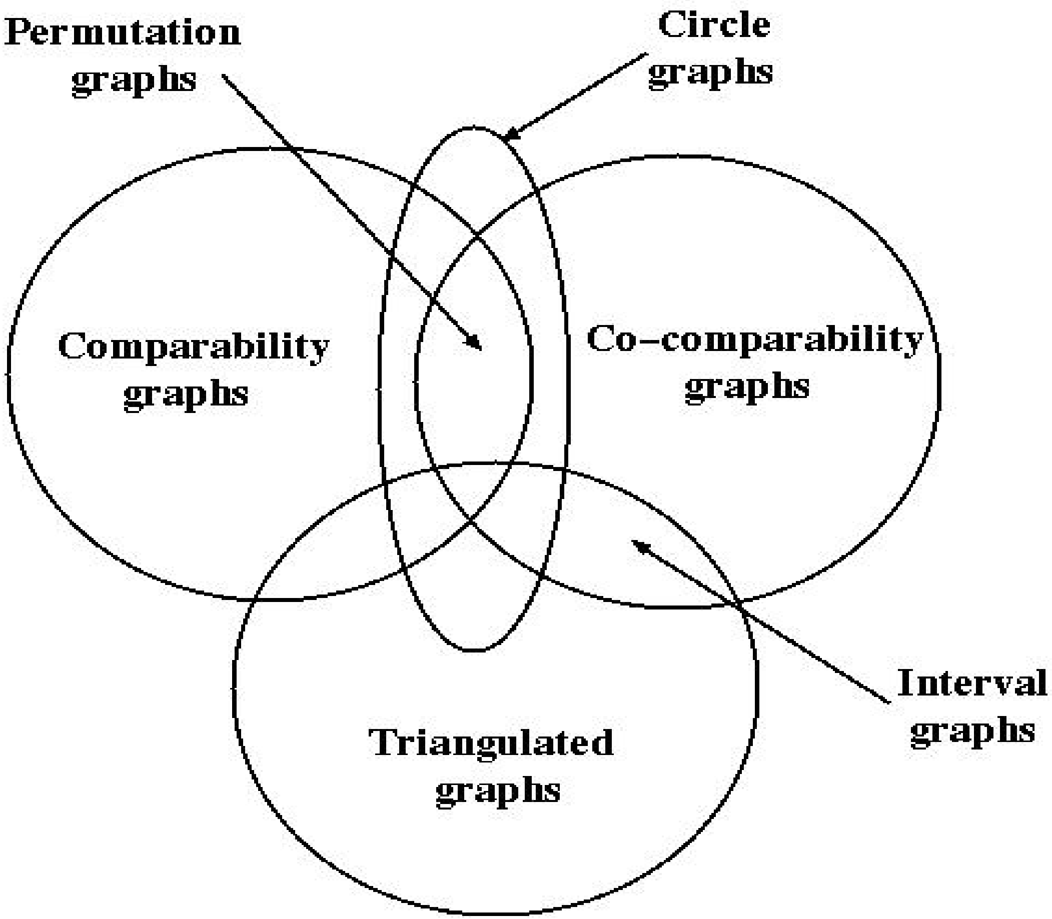
Let G = (V,E) be an undirected graph and let s = [v_1, v_2,…, v_n] be an ordering of the vertices. We say that s is a *perfect vertex elimination scheme* if each v_i is a simplicial vertex of the induced subgraph $G_{\{vi,\, v2,\dots,\, vn\}}$. In other words, each set

$$X_i = \{v_j \in \mathrm{Adj}(v_i) \mid j > i\}$$

is complete.

Theorem: An undirected graph G is triangulated if it has a perfect vertex elimination scheme. Moreover, any simplicial vertex can start a perfect vertex elimination scheme.

Relationship between different graph classes



Perfect Graphs

Consider the following parameters of an undirected graph:

- ♦ w(G) $\Rightarrow$ the *clique number* of the graph *G*: the size of the largest complete subgraph of G.
- ♦ χ(G) $\Rightarrow$ the *chromatic number* of the graph *G*: the fewest number of colors needed to properly color the vertices of G.
- ♦ α(G) $\Rightarrow$ the *size of the maximum independent set* of the graph *G*: the size of the largest subset of vertices such that there exists no arc among any pair of vertices among the members in this set.
- ♦ *k*(G) $\Rightarrow$ the *clique cover number* of the graph *G*: the fewest number of complete subgraphs needed to cover the vertices of G.

Relation among these parameters

- Intersection of a clique and a maximal independent set may be at most one vertex.
- For any graph G, $\omega(G) \leq \chi(G)$.
- For any graph G, $\alpha(G) \leq k(G)$.
- If G^c is the complement of graph G

  then $\alpha(G) = \omega(G^c)$, and $k(G) = \chi(G^c)$.

Perfect graph theorem:

For an undirected graph G=(V,E), the following statements are equivalent:

1. $\omega(G_A) = \chi(G_A)$, for all $A \subseteq V$.
2. $\alpha(G_A) = k(G_A)$, for all $A \subseteq V$.
3. $\omega(G_A) \times \alpha(G_A) \geq |A|$, for all $A \subseteq V$.

Important results on perfect graphs

The following problems are polynomial time solvable for perfect graphs:

- **Minimum COLORING**
- **Maximum CLIQUE**
- **Largest INDEPENDENT SET**
- **Minimum CLIQUE COVER**

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