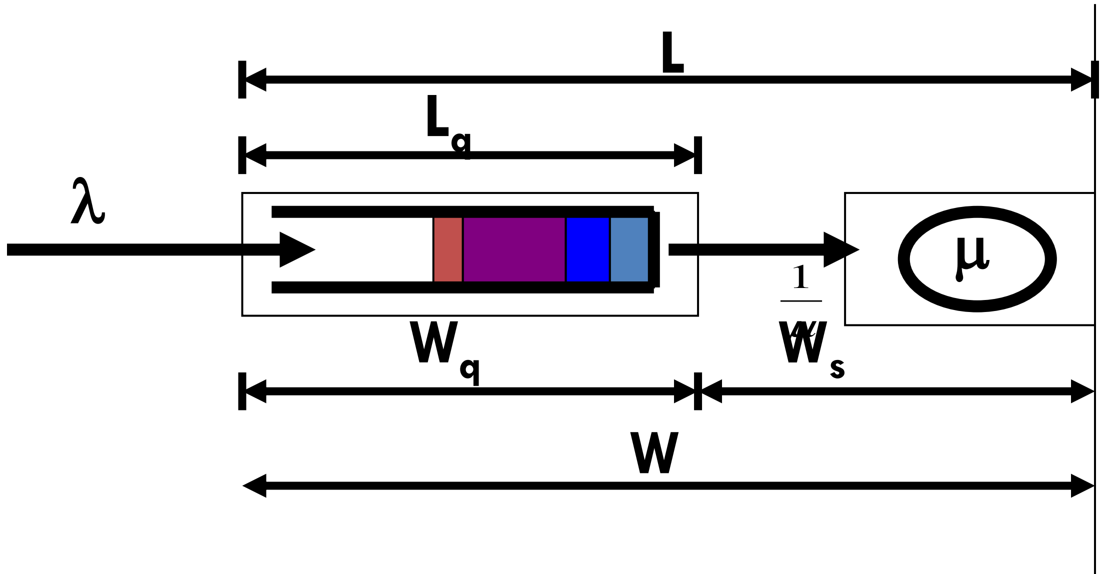
Unit IV

Queuing Models

Unit IV: Syllabus

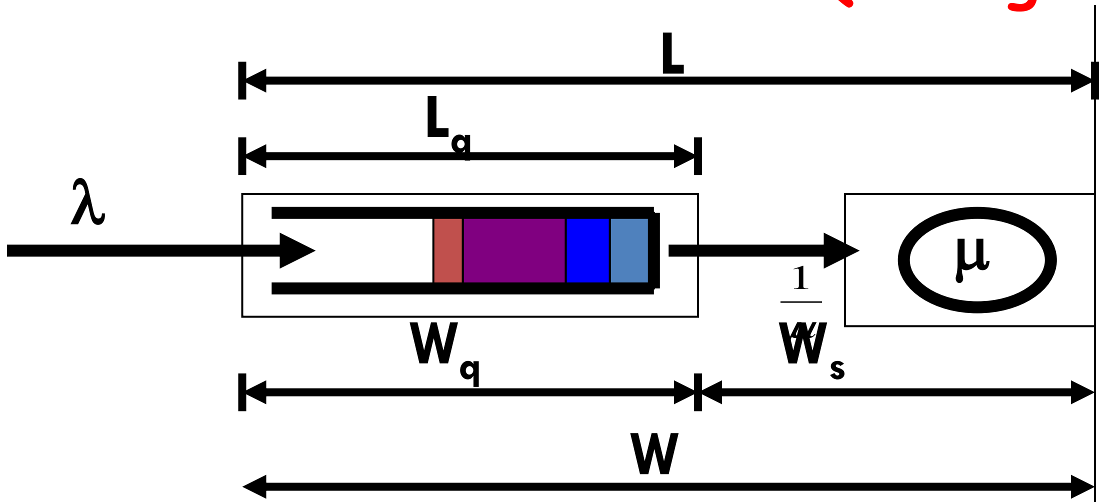
- Queuing Models:
  - Little's Theorem
  - Analytical Results for
    - M/M/1
    - M/M/1/N
    - M/M/c
    - M/G/1 and
  - Other Queuing Models

State Performance Measures of Queuing System



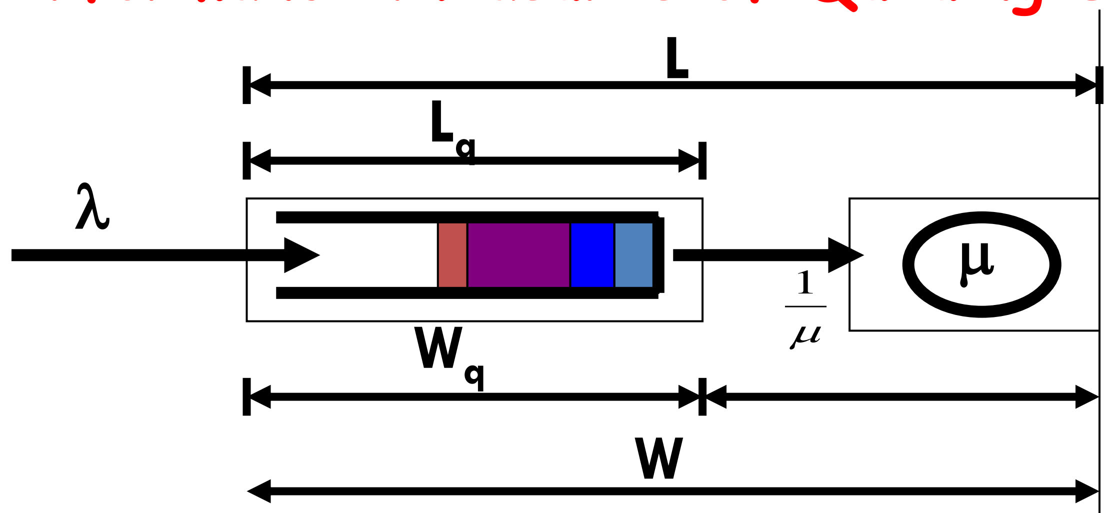
P_0	= Probability that there are no customers in the system.
P_n	= Probability that there are "n" customers in the system.
L	= Average number of customers in the system.
L_q	= Average number of customers in the queue.

State Performance Measures of Queuing System



W = Average time a customer spends in the system.
W_q = Average time a customer spends in the queue.
P_w = Probability that an arriving customer must wait for service.
ρ = Utilization rate for each server (the percentage of time that each server is busy).

State Performance Measures of Queuing System



λ = average number of arrivals *entering* the system per unit time.

1/λ = mean inter arrival time, time between arrivals.

μ = mean service rate per server = average number of units that a server can process per period.

1/μ = mean service time

Learn M & S : From B K Sharma

M / M /1 Queue - Performance Measures

$$P_0 = 1 - (\lambda/\mu)$$

$$P_n = [1 - (\lambda/\mu)](\lambda/\mu)^n$$

$$L = \lambda / (\mu - \lambda)$$

$$L_q = \lambda^2 / [\mu(\mu - \lambda)]$$

$$W = 1 / (\mu - \lambda)$$

$$W_q = \lambda / [\mu(\mu - \lambda)]$$

$$P_w = \lambda / \mu$$

$$\rho = \lambda / \mu$$

The probability that a customer waits in the system more than "t" is $P(X>t) = e^{-(\mu - \lambda)t}$

λ is constant and $\mu > \lambda$ (average service rate > average arrival rate).

M / M /1 Queue

Example

New Delhi Railway Station has a single ticket counter. During the rush hours, customers arrive at the rate of 10 per hour. The average number of customers that can be served is 12 per hour. Find out the following:

1. Probability that the ticket counter is free.
2. Average number of customers in the queue.

Given $\lambda = 10/\text{hour}$, $\mu = 12/\text{hour}$

Probability that the counter is free $P_0 = 1 - (\lambda/\mu) = 1 - \frac{10}{12} = 1/6$

Average number of customers in the queue $(Lq) = \lambda^2/[\mu(\mu - \lambda)] = \frac{(10)^2}{12\,(12 - 10)} = 25/6$

M / M /1 Queue

Example

Students arrive at the head office of www.universalteacher.com according to a Poisson input process with a mean rate of 40 per hour. The time required to serve a student has an exponential distribution with a mean of 50 per hour. Assume that the students are served by a single individual, find the average waiting time of a student.

Given λ = 40/hour, μ = 50/hour

Average waiting time of a student before receiving service

$$W_q = \lambda / [\mu(\mu - \lambda)] = \frac{40}{50(50 - 40)} = 4.8 \text{ minutes}$$

M/M/1//N/ Queue - Performance Measures

For $\rho = (\lambda/\mu) \neq 1$

$$P_0 = \frac{1-\rho}{1-\rho^{N+1}}$$

$$P_n = \rho^n P_0 = \frac{1-\rho}{1-\rho^{N+1}} \cdot \rho^n$$

$$L = \frac{\rho}{1-\rho} - \frac{(N+1)\rho^{N+1}}{1-\rho^{N+1}}$$

$$\mathrm{L_q} = \mathrm{L} - (1 - \mathrm{P_0})$$

$$\mathrm{W} = \mathrm{L}/\bar{\lambda}$$

$$\mathrm{W_q} = \mathrm{L_q}/\bar{\lambda}$$

Where $\bar{\lambda} = \sum_{n=0}^{\infty} \lambda_n P_n$

M / M /1//N/ Queue

Example

Students arrive at the head office of www.universalteacher.com according to a Poisson input process with a mean rate of 30 per day. The time required to serve a student has an exponential distribution with a mean of 36 minutes. Assume that the students are served by a single individual, and queue capacity is 9. On the basis of this information, find the following:

1. The probability of zero unit in the queue.
2. The average line length.

$$\lambda = \frac{30}{60 \times 24} = (1/48) \text{ students per minute}$$

$\rho = 36/48 = 0.75$

$N = 9$

$\mu = 1/36$ students per minute

M / M /1//N/ Queue

Example

$$P_0 = \frac{1-\rho}{1-\rho^{N+1}} = \frac{1 - 0.75}{1 - (0.75)^{9+1}} = 0.26$$

$$L = \frac{\rho}{1-\rho} - \frac{(N+1)\rho^{N+1}}{1-\rho^{N+1}} = \frac{0.75}{1 - 0.75} - \frac{(9+1)(0.75)^{9+1}}{1 - (0.75)^{9+1}}$$

= 2.40 or 2 students.

M/M/c Queue - Performance Measures

Allows for c identical servers working independently from each other.

$$P_0 = \left(\sum_{n=0}^{c-1} \frac{(\lambda/\mu)^n}{n!} + \frac{(\lambda/\mu)^c}{c!} \cdot \frac{1}{1-(\lambda/(c\mu))} \right)^{-1}$$

$$P_n = \begin{cases} \dfrac{(\lambda/\mu)^n}{n!} P_0 & \text{for} \quad n = 1,2,\ldots,c \\ \dfrac{(\lambda/\mu)^n}{c!c^{n-c}} P_0 & \text{for} \quad n = c+1, c+2, \ldots \end{cases}$$

$$L_q = \sum_{n=c}^{\infty} (n-c) P_n = \ldots = \frac{(\lambda/\mu)^c \rho}{c!(1-\rho)^2} P_0$$

Little's Formula $\Rightarrow$ $W_q = L_q/\lambda$

M/M/c Queue - Performance Measures

$W = W_q + (1/\mu)$

Little's Formula $\Rightarrow$ $L = \lambda W = \lambda(W_q + 1/\mu) = L_q + \lambda/\mu$

Steady State Condition:

$\rho = (\lambda / c\mu) < 1$

M / M /c Queue

Example

The Silver Spoon Restaurant has only two waiters. Customers arrive according to a Poisson process with a mean rate of 10 per hour. The service for each customer is exponential with mean of 4 minutes. On the basis of this information, find the following:

1. The probability of having to wait for service.

2. The expected percentage of idle time for each waiter.

λ = 10 per hour or 1/6 per minute μ = 1/4 per minute c=2 ρ = 1/3

$$\frac{1}{P_0} = \sum_{n=0}^{C-1} \frac{(\lambda/\mu)^n}{n!} + \frac{(\lambda/\mu)^c}{c!} \times \frac{1}{1-\rho}$$

$$\frac{1}{P_0} = \sum_{n=0}^{1} \frac{(2/3)^n}{n!} + \frac{(2/3)^2}{2!} \times \frac{1}{1-1/3}$$

M / M /c Queue

Example

$$\frac{1}{P_0} = 1 + \frac{2}{3} + \frac{1}{3}$$

$$P_0 = \frac{1}{2}$$

The expected percentage of idle time for each waiter

$1 - \rho = \quad 1 - 1/3 = 2/3 = 67\%$

M/G/1 Queue - Performance Measures

- $P_0 = 1 - \rho$

- $L = \rho + \{\lambda^2 (\mu^{-2} + \sigma^2)\} / \{2 (1 - \rho)\}$
  $= \rho + \{\rho^2 (1 + \sigma^2 \mu^2)\} / \{2 (1 - \rho)\}$

- $L_q = \{\lambda^2 (\mu^{-2} + \sigma^2)\} / \{2 (1 - \rho)\}$
  $= \{\rho^2 (1 + \sigma^2 \mu^2)\} / \{2 (1 - \rho)\}$

M/G/1 Queue - Performance Measures

- $W = \mu^{-1} + \{\lambda\ (\mu^{-2} + \sigma^2)\} / \{2\ (1 - \rho)\}$

- $W_q = \{\lambda\ (\mu^{-2} + \sigma^2)\} / \{2\ (1 - \rho)\}$

- $\rho = \lambda / \mu$

M/G/1 Queue

Example

There are two workers competing for a job. Able claims an average service time which is faster than Baker's, but Baker claims to be more consistent, if not as fast. The arrivals occur according to a Poisson process at a rate of λ= 2 per hour(1/30 per minute). Able's statistics are an average service time of 24 minutes with a standard deviation of 20 minutes. Baker's service statistics are an average service time of 25 minutes, but a standard deviation of only 2 minutes. If the average length of the queue is the criterion for hiring, which worker should be hired?

M/G/1 Queue

Example

For Able,

$$\lambda = 1/30 \text{ (per min)}, \quad \mu^{-1} = 24 \text{ (min)},$$

$$\rho = \lambda / \mu = 24/30 = 4/5$$

$$\sigma^2 = 20^2 = 400(\text{min}^2)$$

$$L_q = \{\lambda^2 (\mu^{-2} + \sigma^2)\} / \{2 (1 - \rho)\}$$

$$= \{(1/30)^2 (24^2 + 400)\} / \{2 (1-4/5)\}$$

$$= 2.711 \text{ (customers)}$$

M/G/1 Queue

Example

For Baker,

$\lambda = 1/30$ (per min), $\mu^{-1} = 25$ (min),

$\rho = l\ /\ \mu = 25/30 = 5/6$

$\sigma^2 = 2^2 = 4(min^2)$

$L_q = \{\lambda^2 (\mu^{-2} + \sigma^2)\} / \{2 (1 - \rho)\}$

$= \{(1/30)^2 (25^2 + 4)\} / \{2 (1-5/6)\}$

$= 2.097$ (customers)

M/G/1 Queue

Example

Although working faster on the average, Able's greater service variability results in an average queue length about 30% greater than Baker's.

On the other hand, the proportion of arrivals who would find Able idle and thus experience no delay is $P_0 = 1 - \rho = 1 / 5 = 20\%$, while the proportion who would find Baker idle and thus experience no delay is $P_0 = 1 - \rho = 1 / 6 = 16.7\%$.

On the basis of average queue length, L_q, Baker wins.

Questions

- Explain Little Theorem in the queuing models with the help of an example.
- Explain Little's Law of queuing theory.
- List and explain various measures of performance for queuing systems.
- Describe the techniques used for queuing and also discuss queuing discipline.

Questions

- What are queuing Models? Explain the factors that affect Queuing Models. Write down the performance measures for M/M/C.
- Arrival of self-service gasoline pump occurs in an exponential fashion at a rate of 12/hr. Service time has distributed that average 14 minutes. What is expected number of vehicles in the system?
- Compare and Contrast M/M/1 and M/M/1/N queuing model with respect to their performance measures factors.

Questions

- What do you mean by generation of arrival pattern? Briefly explain it.
- How various arrival patterns are generated for the queues? Explain with examples.
- Discuss Kendall's notation for specifying the characteristics of a queue with an example.