Unit III

Random Variate Generation

Unit III

- Random Variate Generation:
  - Location
  - Scale and Shape Parameters
  - Discrete and Continuous Probability Distributions
  - Inverse Transformation Method
  - Composition and Acceptance-Rejection Methods

Road Map to Learn Unit III

Discrete and Continuous Variables

↓

Random Variables

↓

Probability Distribution of Random Variable

↓

Discrete and Continuous Random Variables

↓

Discrete and Continuous Probability Distributions For Random Variables

- Binomial Distribution
- Bernoulli Distribution
- Geometric Distribution
- Discrete Poisson Distribution

- Uniform Distribution
- Exponential Distribution
- Normal Distribution
- Weibull Distribution

Random Variate Generation Methods

- Inverse Transformation Method
- Composition
- Acceptance Rejection Method

Discrete Variables

Discrete variables are **countable in a finite amount of time.**

For example, we can count the change in our pocket.

We can count the money in our bank account.

We could also count the amount of money in *everyone's* bank account.

It might take you a long time to count that last item, but the point is — it's still countable.

Continuous Variables

Continuous Variables would (literally) take forever to count.

In fact, we would get to "forever" and never finish counting them.

For example, take age.

You can't count "age".

Why not? Because it would literally take forever.

25 years, 10 months, 2 days, 5 hours, 4 seconds, 4 milliseconds, 8 nanoseconds, 99 picosends…and so on.

You *could* turn age into a discrete variable and then you could count it.

For example: A person's age in years.
A baby's age in months.

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Random Variate Generation Methods

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- Composition
- Acceptance Rejection Method

Random Variable

A *random variable*, usually written *X*, is a variable whose possible values are numerical outcomes of a random phenomenon.

A random variable is defined as a quantity whose values are random and to which a probability distribution is assigned.

$$X \equiv \begin{cases} -1 & \text{w.p. } 0.6 \\ 2.5 & \text{w.p. } 0.3 \\ 4 & \text{w.p. } 0.1 \end{cases}$$

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Discrete and Continuous Variables

Random Variables

Probability Distribution of Random Variable

Discrete and Continuous Random Variables

Discrete and Continuous Probability Distributions For Random Variables

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Probability Distribution of a random variable

The probability distribution of a random variable is a list of probabilities associated with each of its possible values.

Suppose a variable X can take the values 1, 2, 3, 4, 5, or 6.

The probabilities associated with each outcome are described by the following table:

Outcome(x_i):	1	2	3	4	5	6
Probability(p_i):	0.1	0.3	0.2	0.2	0.1	0.1

The probabilities p_i must satisfy the following:

1. $0 \leq p_i \leq 1$ for each i
2. $p_1 + p_2 + \ldots + p_k = 1$

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Probability Distribution of Random Variable

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Discrete and Continuous Random Variables

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Discrete Random Variable

Possible values are discrete (countable sample space, Integer values):

$$X : \Omega \rightarrow \{1, 2, 3, 4, \ldots\}$$

Continuous Random Variable

Possible values are continuous (uncountable space, Real values):

$$X : \Omega \rightarrow [1.4, 32.3]$$

Road Map to Learn Unit III

Discrete and Continuous Variables

↓

Random Variables

↓

Probability Distribution of Random Variable

↓

Discrete and Continuous Random Variables

↓

Discrete and Continuous Probability Distributions For Random Variables

Bernoulli Distribution
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Discrete Probability Distribution for Random Variables

1. **Bernoulli Distribution**

   Context:
   Random events with two possible values
   Yes/No, True/False, Success/Failure

2. **Binomial Distribution**

   Context:
   Number of successes in a series of n trials.

3. **Geometric Distribution**

   Context:
   It is used to represent random time until a first success occurs (transition occurs).

4. **Discrete Poisson Distribution**

   Context:
   Number of events occurring in a fixed period of time

Continuous Probability Distribution for Random Variables

1. Uniform Distribution

Context:

Any situation in which every outcome in a sample space is equally likely

2. Exponential Distribution

Context:

events occur continuously and independently at a constant average rate. used to model the time until something happens in the process.

3. Normal Distribution

Context:

When we repeat an experiment numerous times and average our results. Widely used for making statistical inferences in both the natural and social sciences.

4. Weibull Distribution

Context:

models a linearly increasing failure rate, where the risk of wear-out failure increases steadily over the product's lifetime.

Discrete Probability Distribution for Random Variables

1. Bernoulli Distribution

The Bernoulli distribution is a discrete distribution having two possible outcomes labelled by n=0 and n=1 in which n=1 ("success") occurs with probability p and n=0 ("failure") occurs with probability q=1-p , where 0< p <1.

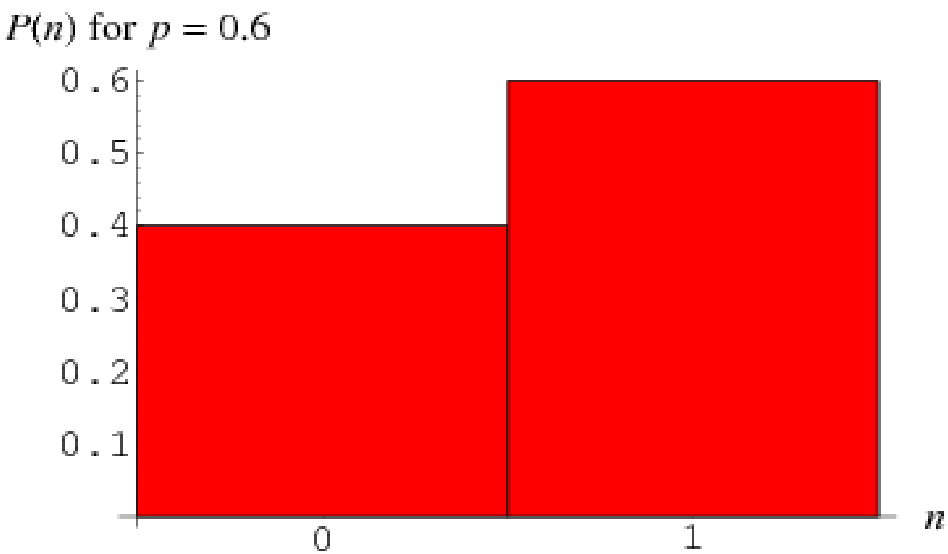
It therefore has probability density function

$$P(n) = \begin{cases} 1-p & \text{for } n=0 \\ p & \text{for } n=1, \end{cases}$$

The distribution of heads and tails in coin tossing is an example of a Bernoulli distribution with p=q=1/2.

Discrete Probability Distribution for Random Variables

1. Bernoulli Distribution



The probability of a failure is labeled on the x-axis as 0 and success is labeled as 1.

In the above Bernoulli distribution, the probability of success (1) is 0.6, and the probability of failure (0) is 0.4.

Discrete Probability Distribution for Random Variables

2. Binomial Distribution

A random variable X is said to follow the Binomial distribution with parameter n and p [written as X ~Bin(n,p) if

Range (X)={0,1,2, 3, ...n) and $0 \le k \le n$

$$P(X = k) = C(n, k)p^k q^{n-k} = \binom{n}{k} p^k q^{n-k}$$

Where, $\binom{n}{k}$ is called Binomial Coefficient. Hence the name Binomial Distribution.

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Discrete Probability Distribution for Random Variables

3. Geometric Distribution

Let X be the number of trials up to the first success.

$$\text{PMF: } p(X = k) = \begin{cases} q^{k-1}p, & k = 0,1,2,\ldots,n \\ 0, & \text{otherwise} \end{cases}$$

$$\text{CDF: } \mathrm{F}(X) = p(X \le k) = 1-(1-p)^{k}$$

$$\text{Expected Value}: E[X] = \frac{1}{p}$$

$$\text{Variance}: V[X] = \sigma^2 = \frac{q}{p^2} = \frac{1-p}{p^2}$$

Notation:

X ~ G(p) Read as X is a random variable with Geometric distribution.

Discrete Probability Distribution for Random Variables

4. Discrete Poisson Distribution

The Poisson distribution is a discrete probability distribution for the counts of events that occur randomly in a given interval of time (or space).

Let X = The number of events in a given interval

The probability of observing x events in a given interval is given by:

$$P(X = x) = e^{-\lambda}\frac{\lambda^x}{x!} \qquad x = 0, 1, 2, 3, 4, \ldots$$

Where,

λ is the mean number of events per interval

Notation:

X ~ Po(λ) Read as X follows the Poisson distribution with parameter λ.

Discrete Probability Distribution for Random Variables

4. Discrete Poisson Distribution

Mean: $\mu = \lambda$

Standard Deviation: $\sigma = \sqrt{\lambda}$

Variance: $\sigma^2 = \lambda$

Continuous Probability Distribution for Random Variables

1. Uniform Distribution
2. Exponential Distribution
3. Normal Distribution
4. Weibull Distribution

Continuous Probability Distribution for Random Variables

1. Uniform Distribution

A random variable X is uniformly distributed on the interval $[a, b]$, $U(a, b)$, if its PDF and CDF are:

PDF: $$f(x) = \begin{cases} \dfrac{1}{b-a}, & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$

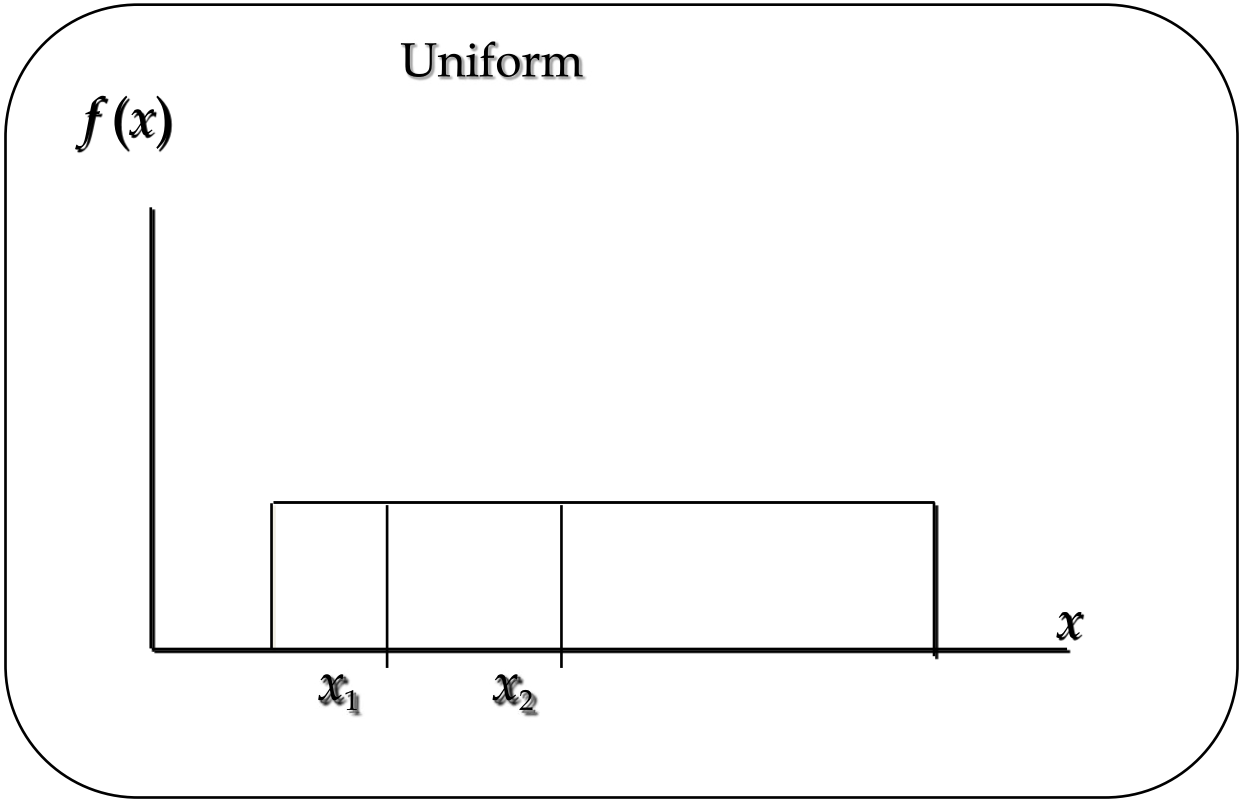
CDF: $$F(x) = \begin{cases} 0, & x \prec a \\ \dfrac{x-a}{b-a}, & a \le x \prec b \\ 1, & x \ge b \end{cases}$$

Expected value: $$E[X] = \frac{a+b}{2}$$

Variance: $$V[X] = \frac{(a+b)^2}{12}$$

Continuous Probability Distribution for Random Variables

1. Uniform Distribution



Continuous Probability Distribution for Random Variables

2. Exponential Distribution

A random variable X is exponentially distributed with parameter $\mu=1/\lambda > 0$ if its PDF and CDF are:

PDF: $$f(x)=\begin{cases}\lambda\cdot\exp(-\lambda\cdot x), & x\geq 0\\ 0, & \text{otherwise}\end{cases} \Rightarrow f(x)=\begin{cases}\frac{1}{\mu}\cdot\exp\left(-\frac{x}{\mu}\right), & x\geq 0\\ 0, & \text{otherwise}\end{cases}$$

CDF: $$F(x)=\begin{cases}0, & x<0\\ \int_0^x \lambda e^{-\lambda t}dt=1-e^{-\lambda x}, & x\geq 0\end{cases} \Rightarrow F(x)=\begin{cases}0, & x<0\\ 1-\exp\left(-\frac{x}{\mu}\right), & x\geq 0\end{cases}$$

Expected value: $E[X]=\frac{1}{\lambda}=\mu$

Variance: $V[X]=\frac{1}{\lambda^2}=\mu^2$

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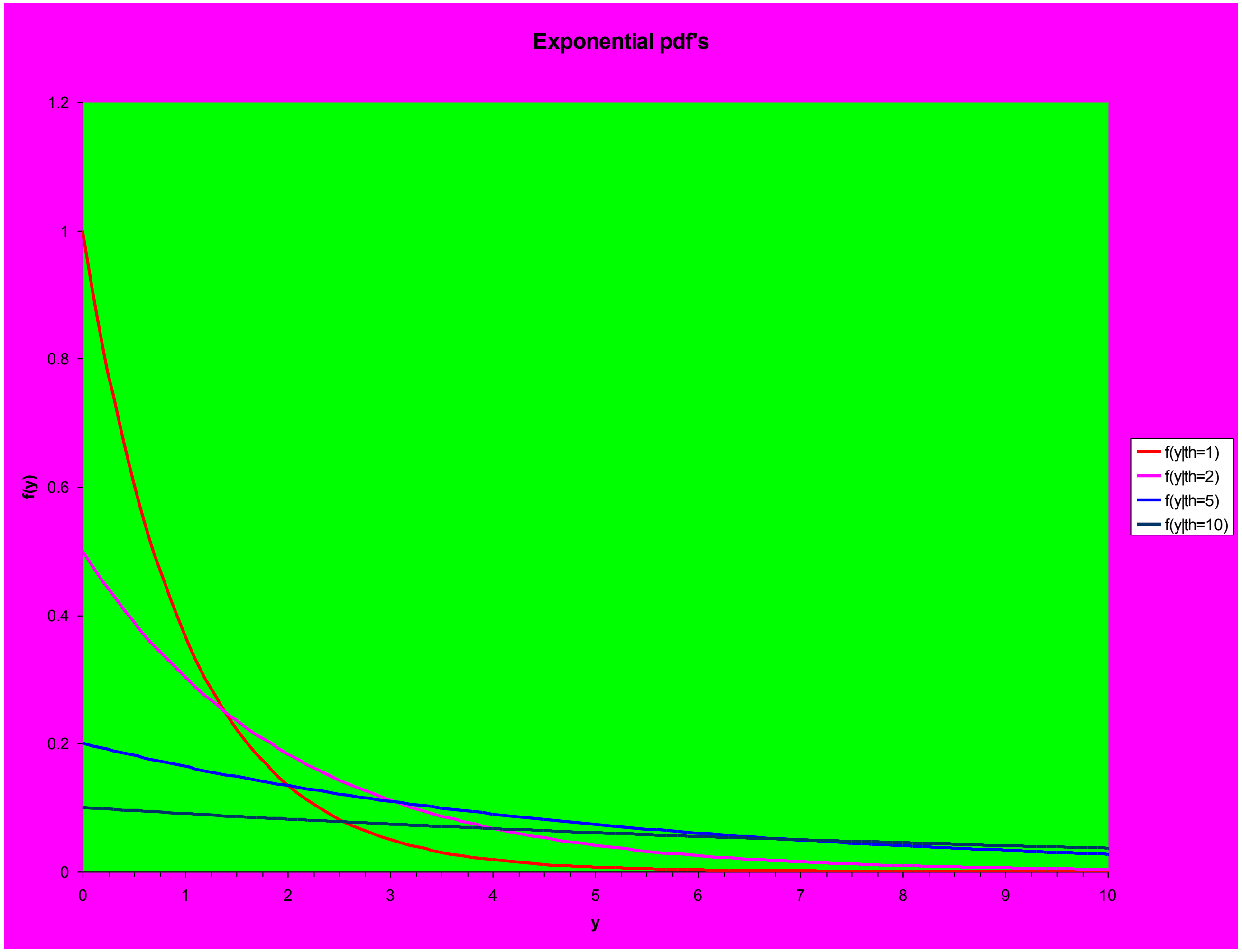
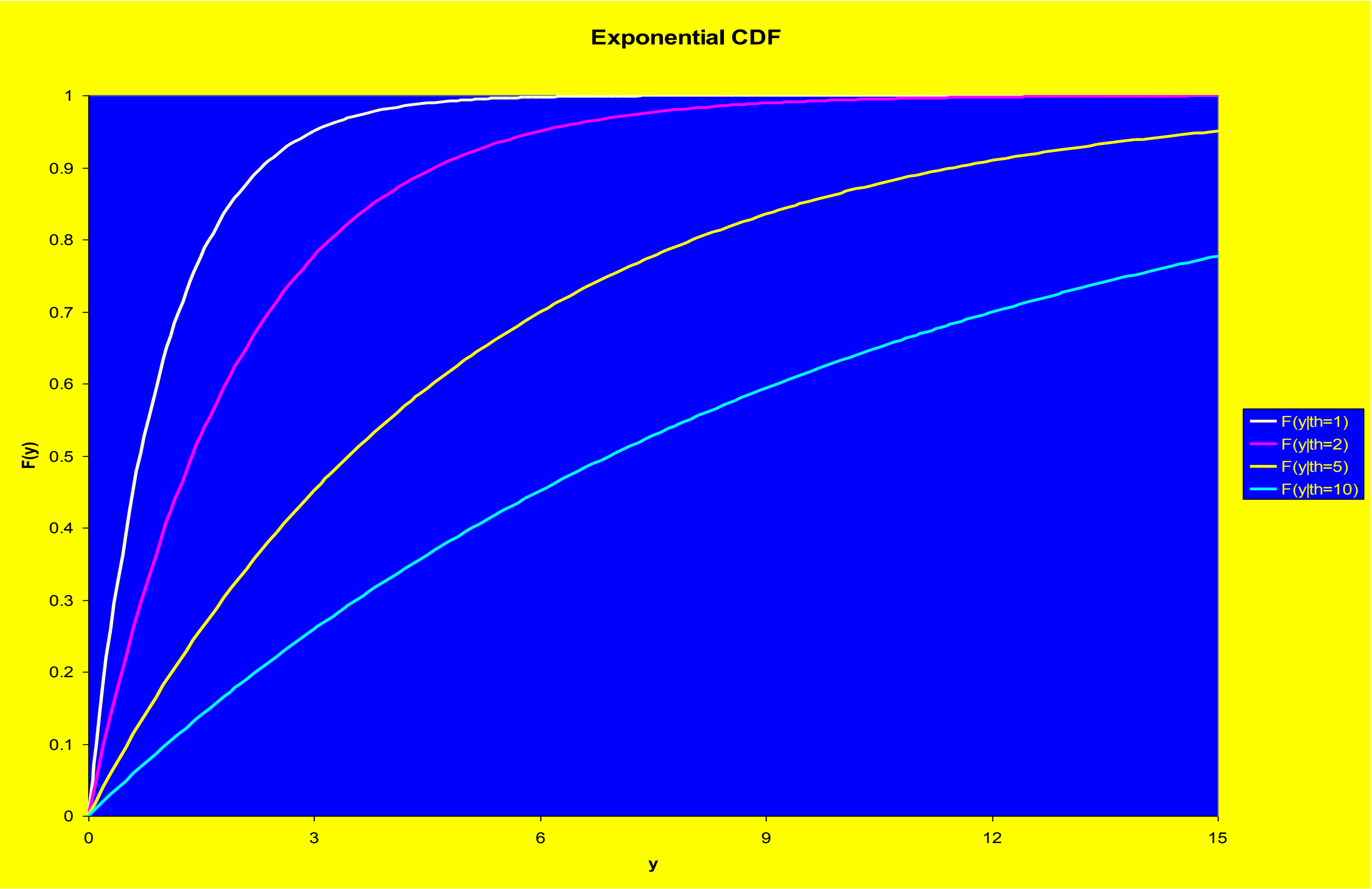
Continuous Probability Distribution for Random Variables

2. Exponential Distribution



Continuous Probability Distribution for Random Variables

2. Exponential Distribution



Continuous Probability Distribution for Random Variables

3. Normal Distribution

A continuous random variable *X*, taking all real values in the range (-∞,+∞) is said to follow a Normal distribution with parameters μ(mean) and σ(variance) if it has the following PDF and CDF:

$$\text{PDF: } f\left(x\right)=\frac{1}{\sigma\cdot\sqrt{2\pi}}\cdot\exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$$

$$\text{CDF: } F\left(x\right)=\frac{1}{2}\cdot\left(1+erf\left(\frac{x-\mu}{\sigma\cdot\sqrt{2}}\right)\right)$$

Where $\text{Error Function: } erf\left(x\right)=\frac{2}{\sqrt{\pi}}\cdot\int_0^x\exp\left(-t^2\right)$

Continuous Probability Distribution for Random Variables

3. Normal Distribution

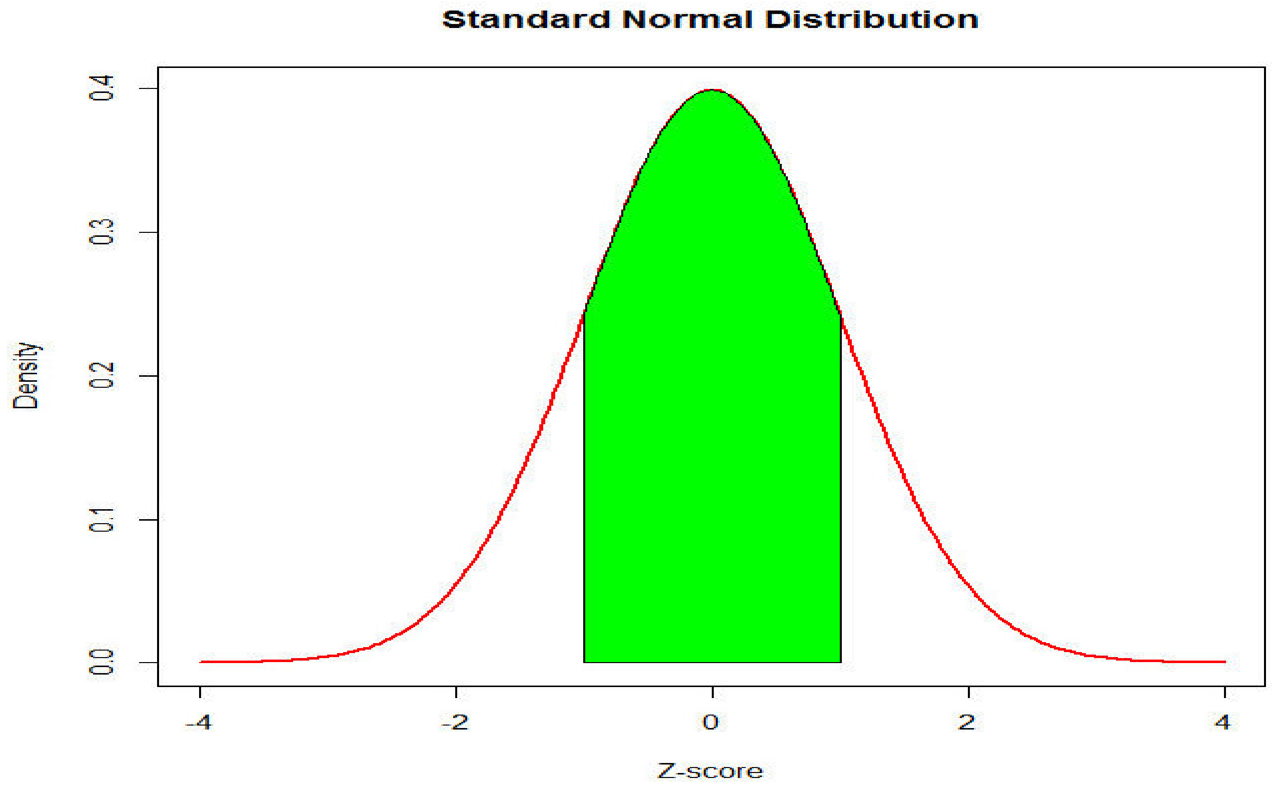
The Standard Normal Distribution is a normal distribution with the special properties that is mean is zero and its standard deviation is one.

$$\mu = 0$$

$$\sigma = 1$$

Continuous Probability Distribution for Random Variables

3. Normal Distribution



Continuous Probability Distribution for Random Variables

4. Weibull Distribution

A random variable X has a Weibull distribution if its PDF and CDF has the form:

$$f(x)=\begin{cases}\frac{\beta}{\alpha}\left(\frac{x-\nu}{\alpha}\right)^{\beta-1}\exp\left[-\left(\frac{x-\nu}{\alpha}\right)^{\beta}\right], & x\geq\nu\\ 0, & \text{otherwise}\end{cases}$$

The cdf of a Weibull rv having parameters α and β is

$$F(X)=1-e^{-\left(\frac{x}{\alpha}\right)^{\beta}}$$

Continuous Probability Distribution for Random Variables

4. Weibull Distribution

Where ,

v is called Location Parameter

β is called Shape Parameter

α is called Scale Parameter

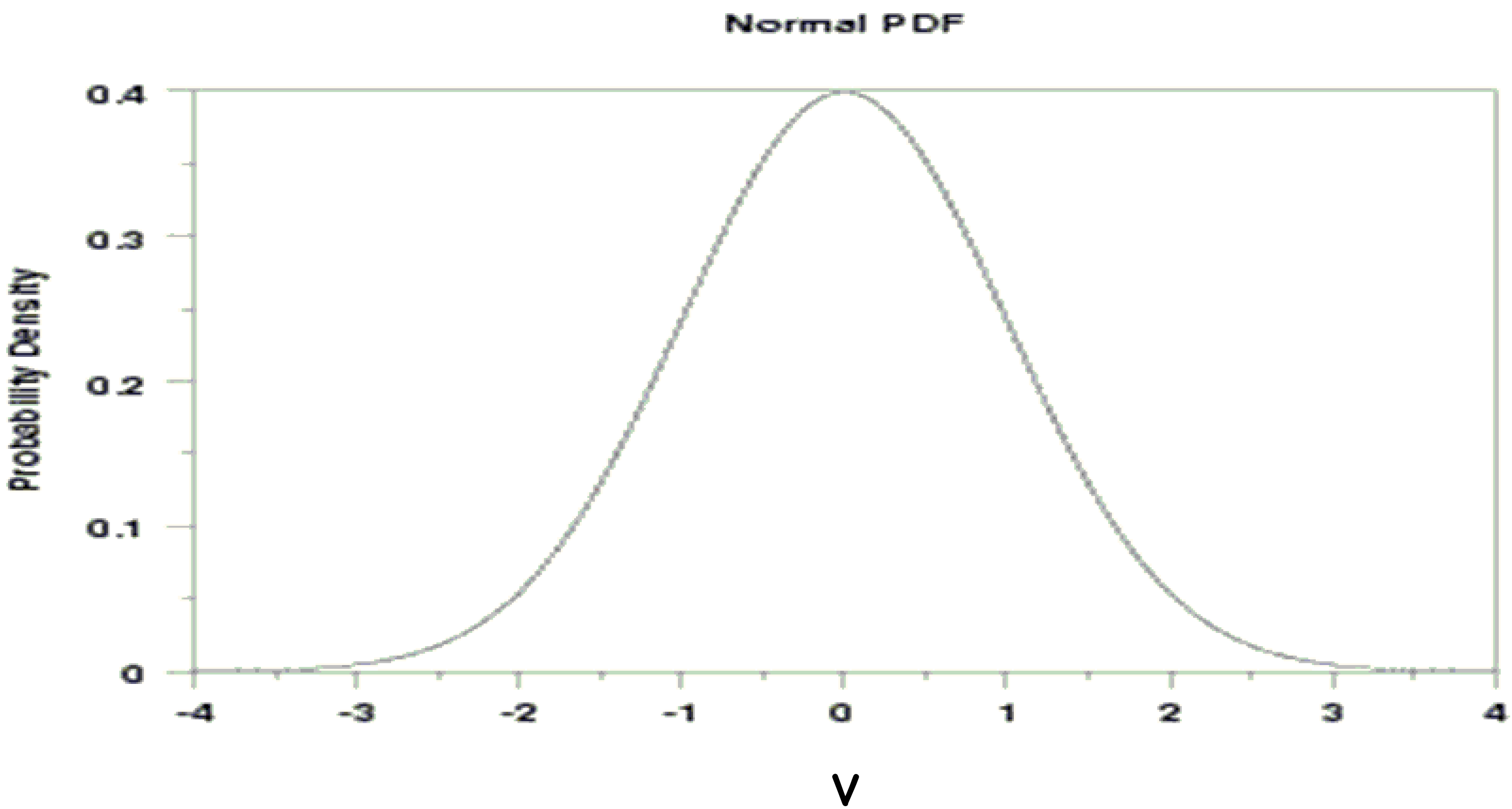
Location Parameter(v): It simply moves the distribution left or right on the horizontal axis.

Shape Parameter(β): It determines the shape

Scale Parameter(α): It simply stretch or squeeze the distribution.

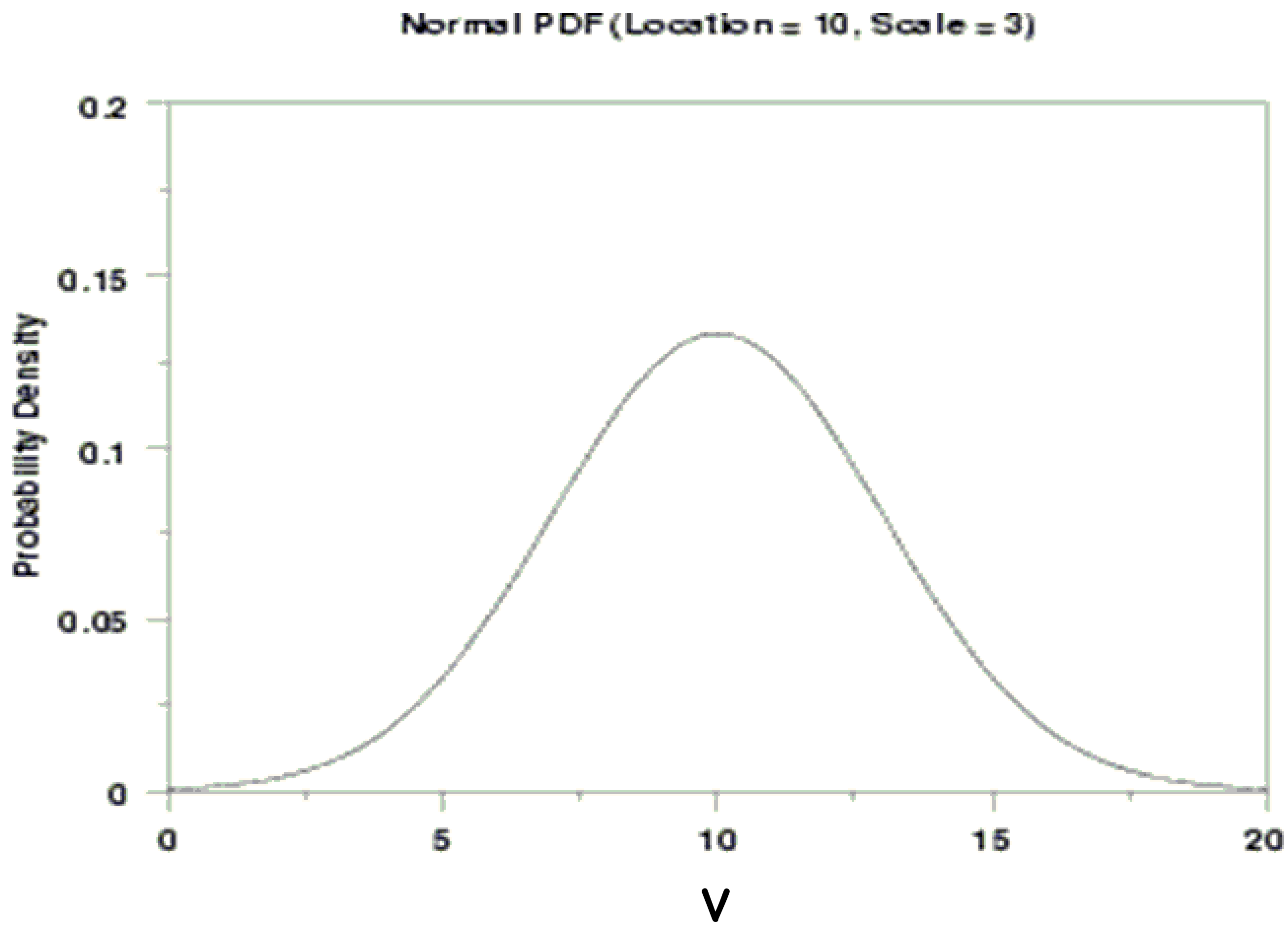
Continuous Probability Distribution for Random Variables

Location parameter equal to zero and Scale parameter equal to one. A location parameter simply shifts the graph left or right on the horizontal axis.



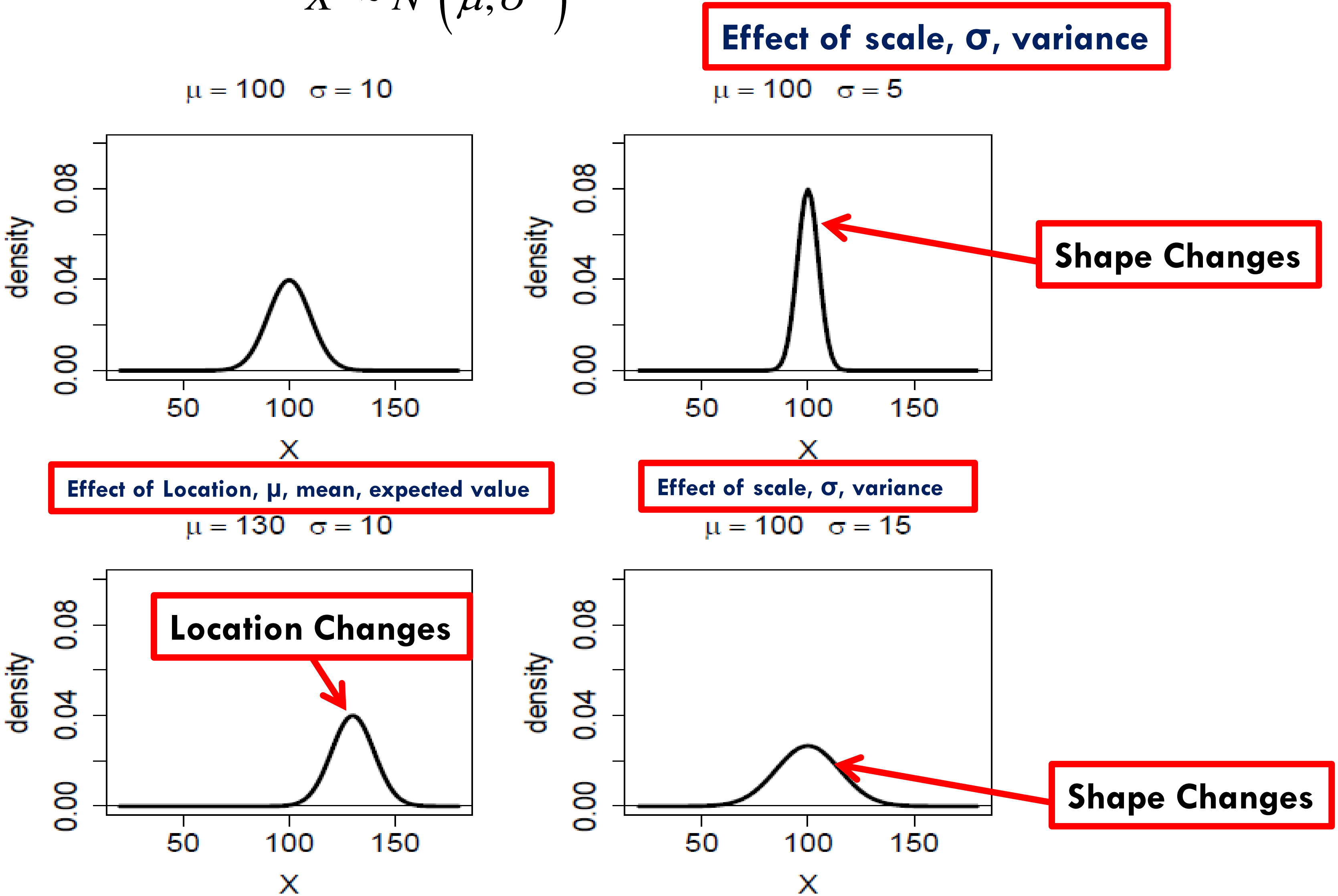
Continuous Probability Distribution for Random Variables

The following graph shows the effect of both a location and a scale parameter. The plot has been shifted right 10 units and stretched by a factor of 3.



Location, Scale and Shape Parameters

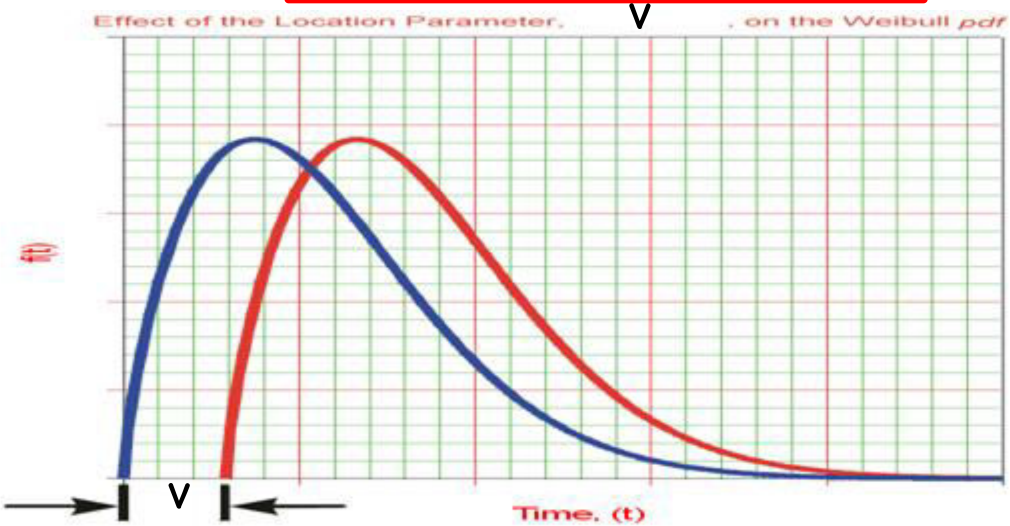
$$X \sim N\left(\mu, \sigma^2\right)$$



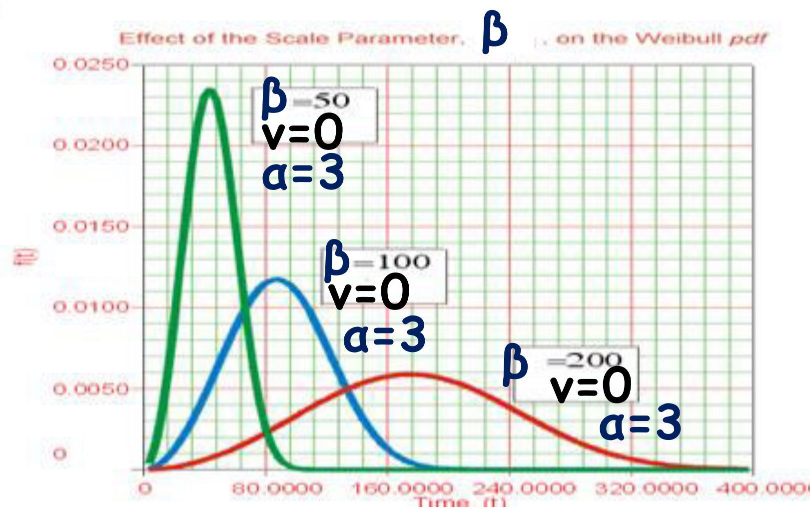
Location, Scale and Shape Parameters

X ~ W(v, α, β)

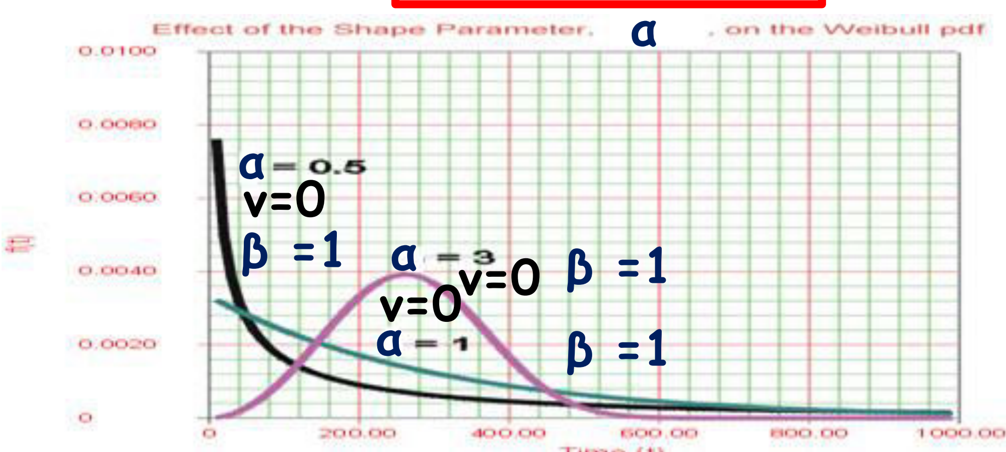
Effect of Location, mean



Effect of Scale, variance



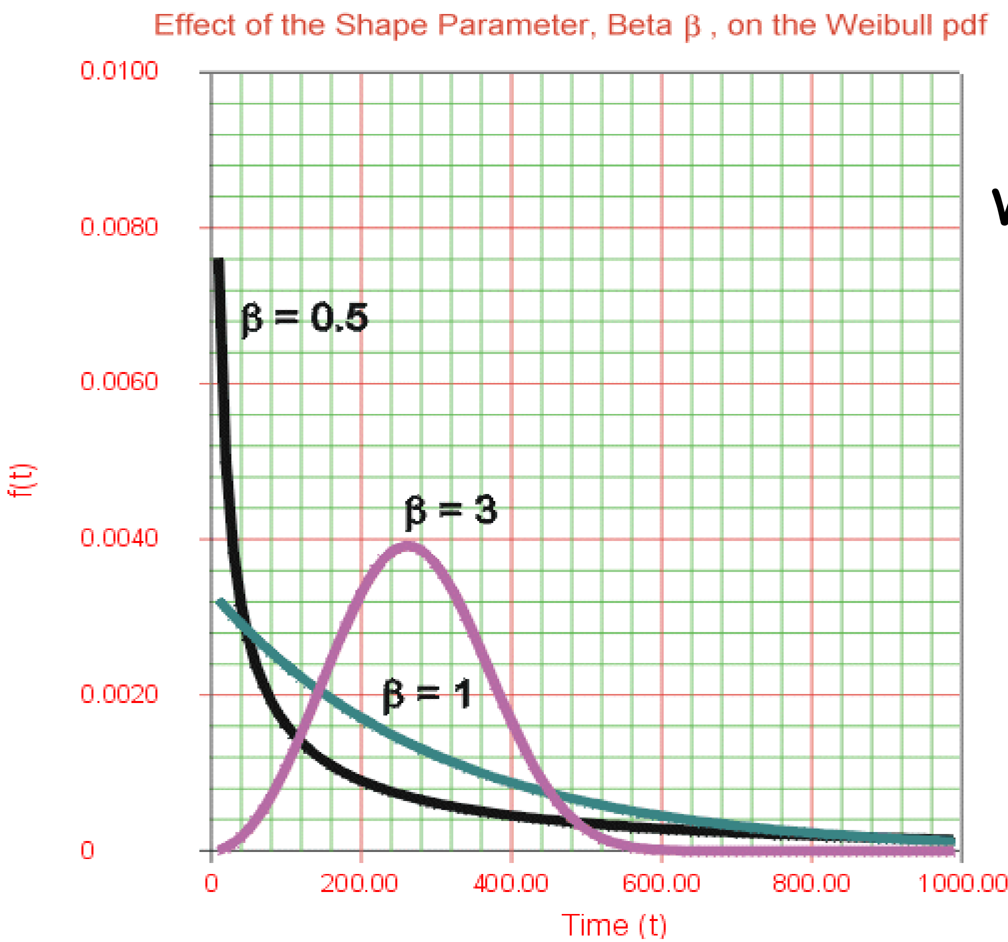
Effect of Shape



Continuous Probability Distribution for Random Variables

This plot demonstrates the effect of the shape parameter, (β), on the Weibull distribution.

Shape Parameter



Only in Weibull Distribution

Discrete versus Continuous Random Variables

Discrete Random Variable	Continuous Random Variable
Finite Sample Space e.g. {0, 1, 2, 3}	Infinite Sample Space e.g. [0,1], [2.1, 5.3]
Probability Mass Function (PMF) $p(x_i) = P(X = x_i)$ 1. $p(x_i) \geq 0$, for all i 2. $\sum_{i=1}^{\infty} p(x_i) = 1$	Probability Density Function (PDF) $f(x)$ 1. $f(x) \geq 0$, for all x in R_X 2. $\int_{R_X} f(x)dx = 1$ 3. $f(x) = 0$, if x is not in R_X

Cumulative Distribution Function (CDF) $p(X \leq x)$

$p(X \leq x) = \sum_{x_i \leq x} p(x_i)$	$p(X \leq x) = \int_{-\infty}^{x} f(t)dt = 0$ $p(a \leq X \leq b) = \int_a^b f(x)dx$

Probability Distributions For Random Numbers

Discrete

Bernoulli Distribution:

$f(x)=P(X=1)=p$

$f(x)=P(X=0)=1-p$

Binomial Distribution:

$$f(x)=P(X=k)=\binom{n}{k}p^k(1-p)^{n-k} \text{ for } k=0,1,2,\ldots,n.$$

Geometric Distribution

$f(x)=(1-p)^{x-1}p$ for $x=0,1,2,3,4,\ldots\ldots$

Poisson Distribution

$$f(x)=p(X=x)=p(x)=\frac{e^{-\lambda}\lambda^x}{x!}, \quad x=0,1,2,\ldots.$$

Continuous

Uniform Distribution

$$\text{PDF:} \quad f(x)=\begin{cases}\frac{1}{b-a}, & a\le x\le b\\ 0, & \text{otherwise}\end{cases}$$

$$\text{CDF:} \quad F(x)=\begin{cases}0, & x\prec a\\ \frac{x-a}{b-a}, & a\le x\prec b\\ 1, & x\ge b\end{cases}$$

Exponential Distribution

$$\text{PDF:} \quad f(x)=\begin{cases}\lambda\cdot\exp(-\lambda\cdot x), & x\ge 0\\ 0, & \text{otherwise}\end{cases}$$

$$\text{CDF:} \quad F(x)=\begin{cases}0, & x<0\\ \int_0^x \lambda e^{-\lambda t}dt=1-e^{-\lambda x}, & x\ge 0\end{cases}$$

Normal Distribution

$$\text{PDF:} f(x)=\frac{1}{\sigma\cdot\sqrt{2\pi}}\cdot\exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$$

$$\text{CDF:} F(x)=\frac{1}{2}\cdot\left(1+erf\left(\frac{x-\mu}{\sigma\cdot\sqrt{2}}\right)\right)$$

Weibull Distribution

$$f(x)=\begin{cases}\frac{\beta}{\alpha}\left(\frac{x-\nu}{\alpha}\right)^{\beta-1}\exp\left[-\left(\frac{x-\nu}{\alpha}\right)^{\beta}\right], & x\ge\nu\\ 0, & \text{otherwise}\end{cases}$$

$$f(x)=\frac{\beta}{\alpha^\beta}x^{\beta-1}e^{-(\frac{x}{\alpha})^\beta}$$

$$F(X)=1-e^{-(\frac{x}{\alpha})^\beta}$$

Location, Scale and Shape Parameters

Normal Distribution $X \sim N\left(\mu, \sigma^2\right)$

$$\text{PDF: } f(x) = \frac{1}{\sigma \cdot \sqrt{2\pi}} \cdot \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$$

$$\text{CDF: } F(x) = \frac{1}{2} \cdot \left(1 + erf\left(\frac{x-\mu}{\sigma \cdot \sqrt{2}}\right)\right)$$

Two parameter:

(i) μ (v) = The mean, expected value of the distribution, is called position (Location) parameter

(ii) σ=The Variance(The standard deviation) of the distribution is called scale parameter

Changing the values of μ and σ alter the positions and shapes of the distributions.

Weibull Distribution X ~ W(v, α, β)

$$f(x) = \begin{cases} \frac{\beta}{\alpha}\left(\frac{x-\nu}{\alpha}\right)^{\beta-1} \exp\left[-\left(\frac{x-\nu}{\alpha}\right)^{\beta}\right], & x \geq \nu \\ 0, & \text{otherwise} \end{cases}$$

Three parameters:

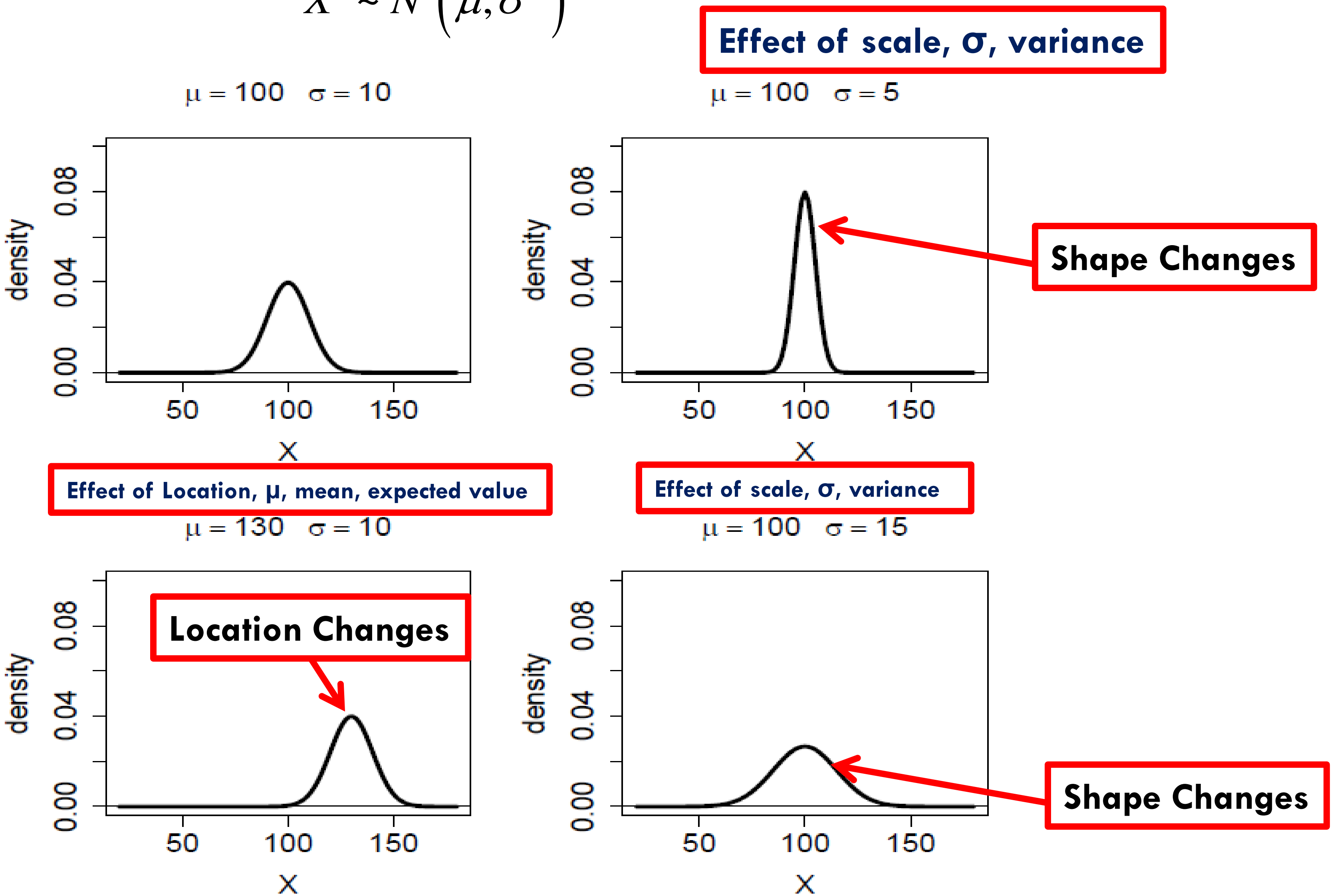
Location parameter: v (gamma), μ

Scale parameter: β ,(eta) (β>0),σ

Shape parameter. α, (beta) (> 0)

Location, Scale and Shape Parameters

$$X \sim N\left(\mu, \sigma^2\right)$$



Questions

- What are random Variates. Give necessary factors for deciding correct algorithm for generating random variates?
- What is the use of Random numbers? Explain Statistical Capabilities with respect to generating random variates.

Questions

- What are all the acceptance rejection techniques? Briefly explain them.
- Explain in detail the Acception-Rejection test for generating random variates.
- Give expressions for Binomial, Poisson and Normal distributions. Under what conditions Binomial distribution is approximated by Poisson distribution?

Questions

- Describe the importance of Discrete Probability Functions.
- Discuss in details, the discrete probability function. How it is different from continuous probability function?
- Explain the following:
  - Continuous probability functions
  - Discrete Probability Functions
- What is stochastic variable? How does it help in simulation?

Questions

- What is an exponential distribution? Explain with an example.
- Explain any two terms:
  - Maximum Density
  - Weibull continuous distribution
  - Chi square test
- Explain in detail the Inverse Transform method with diagram.

Questions

- What do you understand by Parameterization of continuous distribution? Explain any three parameters in detail.
- Name any four types of discrete probability distribution. Explain normal continuous distribution in detail.
- Explain any two types of continuous probability distribution with examples.
- What do you understand by Composition method of generating random variate.

Questions

- How can we write a random variate generator?

END