Unit III

Random Variate Generation

Unit III

- Random Variate Generation:
  - Location
  - Scale and Shape Parameters
  - Discrete and Continuous Probability Distributions
  - Inverse Transformation Method
  - Composition and Acceptance-Rejection Methods

Introduction

Probabilistic Modeling and Uncertainty

Representation of uncertainty:

Uncertainty $\neq$ Lack of knowledge

Uncertainty represented as lack of determinism:

stochastic or random in contrast with fixed.

What might be represented as random phenomena?

Arrivals of passengers to an airport

Arrivals of jobs to a computer operating system

Repair times of a machine

Time between user logons to an e-commerce system

Representing Uncertainty: Probabilistic Modeling

How do we characterize random phenomena?

1. Collect data on the source of uncertainty (the random phenomenon).

2. Fit the collected data to a probability distribution.

3. Estimate the parameters of the selected probability distribution.

Random Variable Vs Random Variate

A *random variable*, usually written *X*, is a variable whose possible values are numerical outcomes of a random phenomenon.

$X = \{1, 2, 3, 4, 5, 6\}$

A random variate is a particular outcome or sample value of a random variable.

$X=5$

A random variable is defined as a quantity whose values are random and to which a probability distribution is assigned.

Outcome(x_i):	1	2	3	4	5	6
Probability(p_i):	0.1	0.3	0.2	0.2	0.1	0.1

Random Number Gens. Vs.Random Variate Gens.

Random Number Generators

Produces values ~ U(0,1).

Random Variate Gens.

Deals with the production of random values (e.g., 26, 54, 71, 10, ...) for a given random variable in such a way that the values produced form the probability distribution of the random variable.

Random Number Gens. Vs.Random Variate Gens.

Random Number Generators

A sequence of random numbers distributed uniformly between 0 and 1 is obtained U(0,1).[$R_i \sim U(0.1)$]

Approaches for generating Random Numbers:

Mid-Square Generator

Linear Congruential Generator(LCG)

Multiplicative LCG

Combined LCG

Random Variate Gens.

The sequence Random numbers is transformed to produce a sequence of random values which satisfy the desired distribution.

Approaches for Generating Random Variates:

Inverse- Transformation Method

Composition Method

Acceptance Rejection Method

What is Random Variate Generators?

Refers to the generation of variates whose probability distribution is different from the uniform distribution on the interval (0,1).

Discrete Probability Distributions		Continuous Probability Distributions	
1.	Bernoulli Distribution	1.	Uniform Distribution
2.	Binomial Distribution	2.	Exponential Distribution
3.	Geometric Distribution	3.	Normal Distribution
4.	Discrete Poisson Distribution	4.	Weibull Distribution

Why do we need random Variate Generator?

Methods such as linear congruential generators, Combined LCG and so on are used to generate random numbers that have the properties of a random sample from a U(0,1).

The output from such generators are subjected to various statistical tests for their quality, before they are considered as if they are samples from U(0,1).

The output of these generators are usually referred to as pseudo-random numbers.

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Discrete Probability Distribution for Random Variables

1. **Bernoulli Distribution**

   Context:

   Random events with two possible values

   Yes/No, True/False, Success/Failure

2. **Binomial Distribution**

   Context:

   Number of successes in a series of n trials.

3. **Geometric Distribution**

   Context:

   It is used to represent random time until a first success occurs (transition occurs).

4. **Discrete Poisson Distribution**

   Context:

   Number of events occurring in a fixed period of time

Continuous Probability Distribution for Random Variables

1. Uniform Distribution

Context:
Any situation in which every outcome in a sample space is equally likely

2. Exponential Distribution

Context:
events occur continuously and independently at a constant average rate. used to model the time until something happens in the process.

3. Normal Distribution

Context:
When we repeat an experiment numerous times and average our results. Widely used for making statistical inferences in both the natural and social sciences.

4. Weibull Distribution

Context:
models a linearly increasing failure rate, where the risk of wear-out failure increases steadily over the product's lifetime.

Random Variate Generators

Once we have obtained / created and verified a quality random number generator for U[0,1), we can use that to obtain random values in other distributions , e.g. Exponential, Normal, etc.

Sequence of Uniform Random Numbers: $R_i \sim U(0.1)$

$\downarrow$

Sequence of Non-Uniform Random Variates: $R_i \sim NU(0.1)$

Why do we need random Variate Generator?

Usually a programming language allows generating a uniform pseudo-random number in a specific range.

So clearly if you want some other distribution you need to use some method:

Inverse Transformation Method

Composition Method

Acceptance-Rejection Method

In a professional simulation, Software used will have predefined functions for all of these variates.

However, it is good to know some of the theory for how they are derived.

Uniform Random Number Generation and Non-Uniform Random Variate Generation.

We use the term non-uniform random variates to represent all distributions other than U(0,1)

To generate random samples from non-uniform distributions such as normal, exponential and so on, we require sufficient number of U(0,1) random numbers.

Given these U(0,1) random numbers, we generate samples from non-uniform distributions.

This is accomplished by various methods such as Inverse transform method, Composition Methods and Acceptance-Rejection method and so on.

Random Variate Generations

Algorithms to produce observations ("variates") from some desired input distribution (Exponential, Normal etc.)

Formal algorithm: depends on desired distribution.

But all algorithms have the same general form:

Generate One or more IID U(0,1) Random Numbers	→	Transformation (depends on the desired distribution)	→	Return X ~ desired distribution

Random Variate Generation

☐ **Bernoulli(p):**
- Return 1 with probability p,
- Return 0 with probability 1-p

$$\begin{cases} u = random(\ ); \\ \text{if } (u < 1-p) \text{ return } 0; \\ \text{else return } 1; \end{cases}$$

☐ **Geometric(p):** $f(x) = p^k \cdot (1-p)$
- Number of Bernoulli trials until first '0')

$$\begin{cases} u = random(\ ); \\ \text{return } \log(1.0-u)/\log(p); \end{cases}$$

☐ **Uniform (a,b):** equally likely to select an integer in interval [a,b]

$$\begin{cases} u = random(\ ); \\ \text{return } a + (u \cdot (b-a+1)); \end{cases}$$

Random Variate Generation

- Exponential distribution with mean $\mu = -1/\lambda$

$$\begin{cases} u = random(\ ); \\ \text{return } -\mu \cdot \log(1-u); \end{cases}$$

- Weibull Distribution with shape, α, and scale, β

$$\begin{cases} u = random(); \\ Return\ X = \beta[-\ln(1-u)]^{1/\alpha} \end{cases}$$

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Random Variate Generation

Inverse Transformation Method

Generates random samples from those distribution which have closed mathematical formula for their CDF's.

Composition

n CDFs are composed together to form the desired CDF

If CDF is given as:

$$F(x) = \sum_{i=1}^{n} p_i F_i(x)$$

Acceptance Rejection Method

Let X has a CDF F and PDF f

If F is hard (or impossible) to invert, too messy ...

But we have access to f

What to do?

Let's see!!!

Where PDF is :

Probability Density Function f(x)

Where CDF is :

Cumulative Distribution Function F(x)

Closed Form mathematical formula/solution

A closed form mathematical formula/solution is an expression for an exact solution given with a finite amount of data.

This is not a closed form solution:

$$y = 4x + 6x^2 + \frac{22}{3}x^3 + \frac{95}{12}x^4 + \cdots$$

because making it exact requires infinitely many terms.

Random Variate Generation

Inverse Transformation Method

The inverse transform method is used to generate random samples from those distribution which have closed mathematical formula for their CDF's.

Not all distributions can be easily transformed.

Some may not have a closed form inverse.

For example:

the cdf for a normal distribution is complex and cannot be inverted in a closed form.

Idea is that some function F^{-1} will map values from U[0,1) into the desired distribution.

Random Variate Generation

Inverse Transformation Method

Idea is that some function F^{-1} will map values from $U[0,1)$ into the desired distribution.

$$F(x) = Pr\ (X \leq x)$$

Typical usage of $F(x)$:

Given x calculate $F(x)=Pr\ (X \leq x)$

For simulation purposes, we want:

Given $F(x)=Pr\ (X \leq x)$ calculate x

Random Variate Generation

Inverse Transformation Method

Inverse of $F(x)$ written as F^{-1}:

implies

If $u=F(x)$ then $x=F^{-1}(u)$

Where,

$F^{-1}(u)$ is defined as the value of x at which $F(x)=u$.

Thus it follows that we can write $X=F^{-1}(u)$

$\boxed{\text{One } u} \rightarrow \boxed{\text{One } X}$

Random Variate Generation

Inverse Transformation Method

The Concept:

For CDF, F(X)

1. Generate a random Number u from Uniform distribution U(0,1)

2. Find X such that F(X)=u and

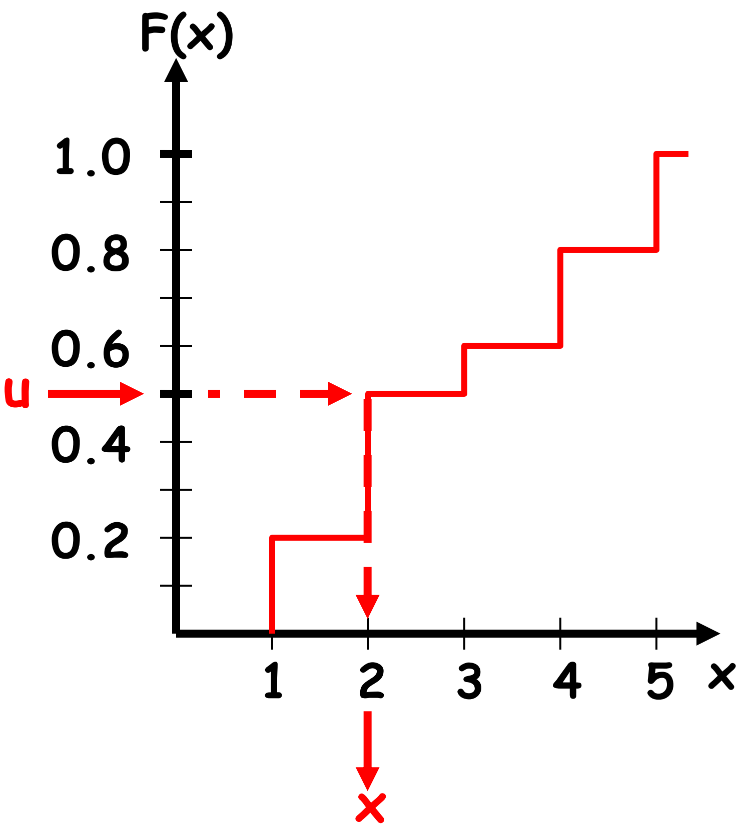
3. Return this value X.

Step 2: involves solving the equation F(X)=u for X: solution is written as $X=F^{-1}(u)$, e..g., we must invert the CDF F.

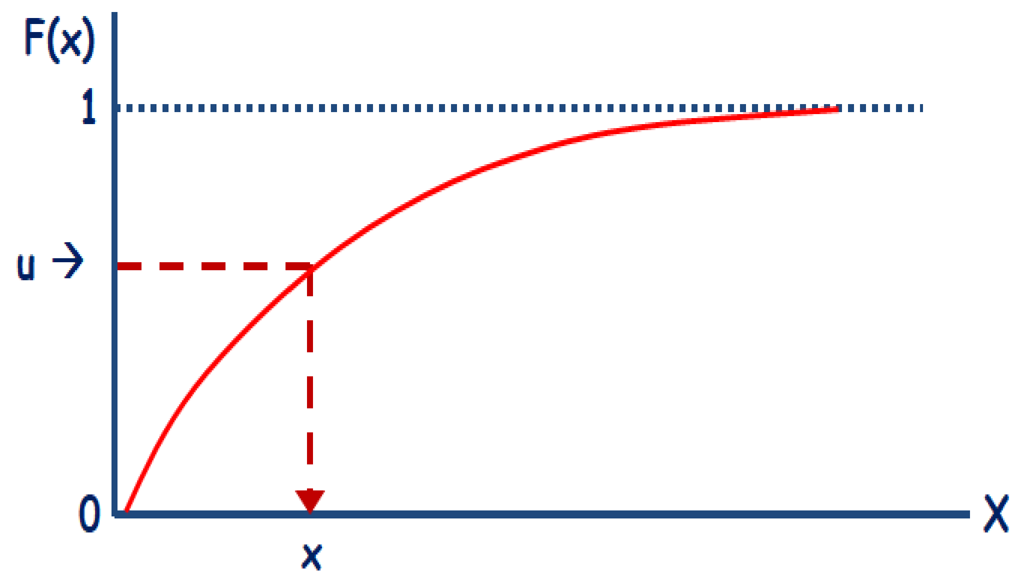
Random Variate Generation

Inverse Transformation Method

Discrete Case



Continuous Case



Random Variate Generation

Inverse Transformation Method

For those probability distributions the inverse $F(x)$ of which can be calculated, the following procedure can be used:

1. Generate a proper random number U over [0,1].
2. Set $U = F(x) = Pr\ (X \leq x)$
3. Solve for x.
4. Deliver x as the random variate.

Random Variate Generation

Inverse Transformation Method

Example : Continuous Distribution

Developing the algorithm for generation of exponentially distributed random variates.

Exponential Cumulative Distribution Function (CDF):

$$F(x) = 1 - e^{-\mu x} \text{ for } x \geq 0.$$

Mean = $1 / \mu$ Variance = $1 / \mu^2$

Random Variate Generation

Inverse Transformation Method

Example : Continuous Distribution

Step 1: Generate a proper random number Z over [0, 1]

Step 2: Set $Z = F(x) = 1 - e^{-\mu x}$

Step 3: Solve for x.

$$e^{-\mu x} = 1 - Z$$

If Z is a random number over [0, 1], then $1 - Z$ is also a random number over [0, 1].

Therefore, let $1 - Z$ be denoted by U.

Take logarithm of each side:

Random Variate Generation

Inverse Transformation Method

Example : Continuous Distribution

$\ln(e^{-\mu x}) = \ln(U) \rightarrow -\mu x \ln(e) = \ln(U)$

Since $\ln(e) = 1$, we have

$x = -(1/\mu)\ln(U)$ **NOTE:** $U \neq 0$ since $\ln(0) = -\infty$

Step 4: Deliver x as the random variate.

Random Variate Generation

Inverse Transformation Method

Algorithm for Generation of Exponentially Distributed Random Variates:

Algorithm (Input: MEAN = $1 / \mu$)

Step 1: Generate a proper random number U over [0, 1]

Step 2: Set $x = -$ MEAN * $\ln(U)$, where $U \neq 0$ since $\ln(0) = -\infty$

Step 3: Deliver x as the random variate.

Random Variate Generation

Inverse Transformation Method

Table Look-up Generator

In those cases where the probability distribution of the random variable cannot be identified as well known AND the volume of collected data is sufficient (large sample size > 100), an "empirical" table look-up generator can be developed.

Random Variate Generation

Inverse Transformation Method

Table Look-up Generator

Procedure

1. Generate $U \sim U(0,1)$
2. Find the smallest x_i such that $U \leq F(x_i)$
3. Set $X = x_i$

Random Variate Generation

Inverse Transformation Method

Table Look-up Generator

Example: Discrete Case

The daily demand for a commodity, X, takes on values 1, 2, and 3 with probabilities 0.3, 0.5, and 0.2.

$$X = \begin{cases} 1 & \text{with probability } 0.3 \\ 2 & \text{with probability } 0.5 \\ 3 & \text{with probability } 0.2 \end{cases}$$

X	P(X=x)	F(x)
1	0.3	0.3
2	0.5	0.8
3	0.2	1.0

In this case, F(1) = 0.3, F(2) = 0.8, and F(3) =1.

Random Variate Generation

Inverse Transformation Method

Table Look-up Generator

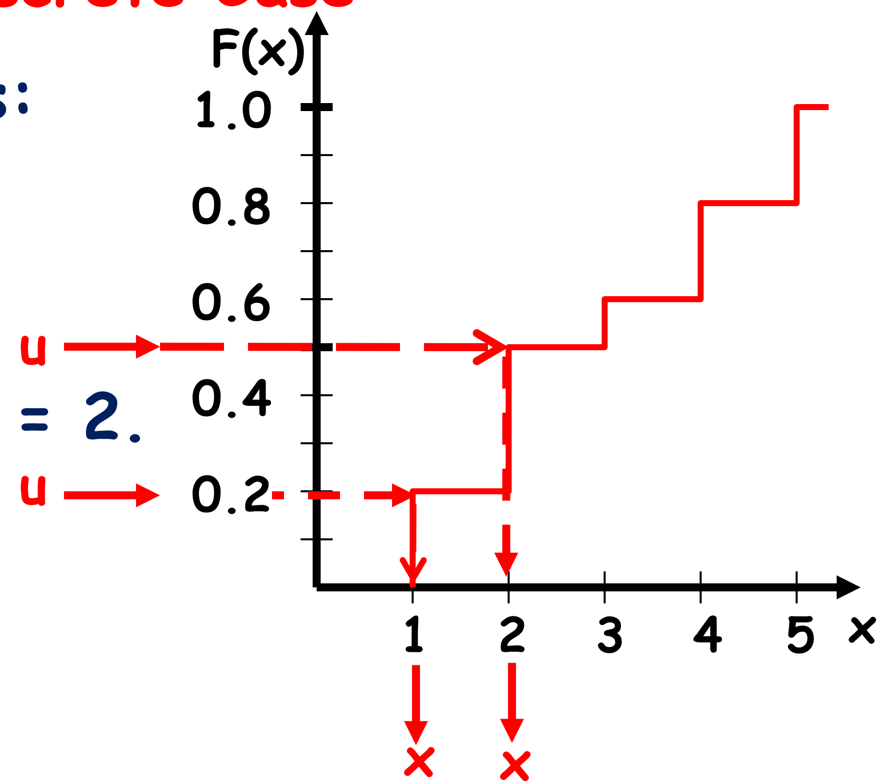
Example: Discrete Case

Then, X is generated as follows:

1. Generate $U \sim U(0,1)$
2. If $U \leq 0.3$, set $X = 1$.

   If $0.3 < U \leq 0.8$, set $X = 2$.

   Otherwise, set $X = 3$.



Random Variate Generation

Inverse Transformation Method

Table Look-up Generator

Example: Discrete Case

Algorithm for RVG from Collected Data

Step 1: Generate a proper random number U over $[0, 1]$.

Step 2: If $0 \le U \le F(x_1)$ deliver x_1 otherwise

If $F(x_1) < U \le F(x_2)$ deliver x_2 otherwise

If $F(x_2) < U \le F(x_3)$ deliver x_3 otherwise

...

If $F(x_{n-2}) < U \le F(x_{n-1})$ deliver x_{n-1} otherwise

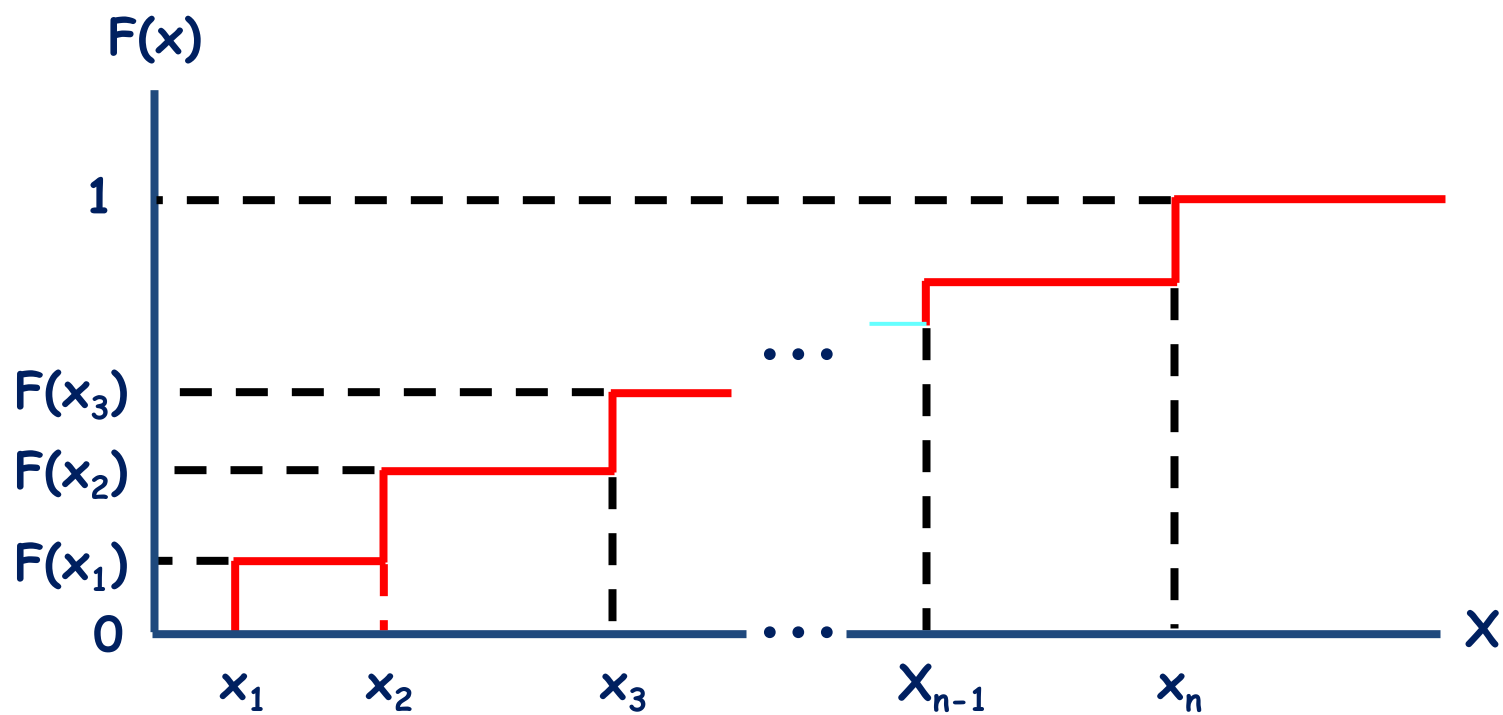
If $F(x_{n-1}) < U \le 1$ deliver x_n

Random Variate Generation

Inverse Transformation Method

Table Look-up Generator

Algorithm for RVG from Collected Data



Random Variate Generation

Inverse Transformation Method

Example: Discrete

$$X = \begin{cases} -1 & \text{with probability } 0.6 \\ 2.5 & \text{with probability } 0.3 \\ 4 & \text{with probability } 0.1 \end{cases}$$

X	P(X=x)	F(x)	U(0,1)
-1	0.6	0.6	[0.0, 0.6]
2.5	0.3	0.9	(0.6, 0.9]
4	0.1	1.0	(0.9, 1.0]

Now, generate a Random Number u, say u=0.63. We take X=2.5.

Written as $X=F^{-1}(u)$

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Random Variate Generation

Inverse Transformation Method

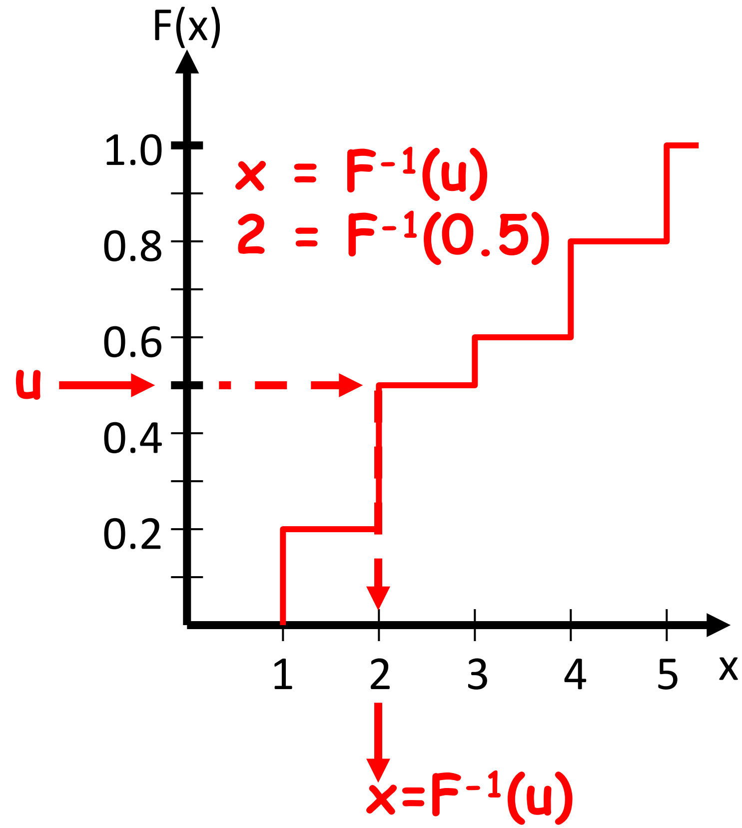
Example: Discrete

PDF:

x	f(x)
1	0.2
2	0.3
3	0.1
4	0.2
5	0.2

CDF:

x	f(x)	F(x)
1	0.2	0.2
2	0.3	0.5
3	0.1	0.6
4	0.2	0.8
5	0.2	1.0



Random Variate Generation

Inverse Transformation Method

The inverse transform method is used to generate random samples from those distribution which have closed-form expressions for their CDF's.

Random Variate Generation

Composition Method

Can be used if CDF F(x) = Weighted sum of n other CDFs.

$$F(x) = \sum_{i=1}^{n} p_i F_i(x)$$

n CDFs are composed together to form the desired CDF. Hence the name.

Random Variate Generation

Composition Method

Suppose a RV actually comes from two RV's:

E.g., your plane can leave the airport gate late for two reasons :

air traffic delays and

maintenance delays

which compose the overall delay time.

Random Variate Generation

Composition Method

Consider a rv X *that takes on two, or more, other rv's at* random.

Suppose you're equally likely to choose roads 1 and 2 every day.

Suppose travel times on roads 1 and 2, X_1 *and* X_2, are exponentially distributed with rates λ_1 and λ_2

Then, your travel time is a composition or a "mixture" of X_1 and X_2.

The distribution function of X *can be written as*

$$F_x(x)=p_1F(x_1)+p_2F(x_2)$$

$$F_X(x) = 0.5F_{x1}(x) + 0.5F_{X2}(x)$$

$$F_X(x) = 0.5(1-e^{-\lambda_1 x}) + 0.5(1-e^{-\lambda_2 x})$$

Random Variate Generation

Composition Method

Exponential Distribution

$$F(x)=1-e^{-\lambda x}$$

Let $F(x)=U$

$1-e^{-\lambda x}=U$

$e^{-\lambda x}=1-U$

$\ln(e^{-\lambda x})=\ln(1-U)$

$-\lambda x=\ln(1-U)$

$x=(-1/\lambda)\ln(1-U)$

[both U and (1-U) are U(0,1)]

$x=(-1/\lambda)\ln U$

The algorithm is as follows:

1. Generate $U_1 \sim U(0,1)$
2. Generate $U_2 \sim U(0,1)$.

If $U_1 < 0.5$, set $X = -(1/\lambda_1)\ln(U_1)$.
Otherwise, set $X = -(1/\lambda_2)\ln(U_2)$.

Random Variate Generation

Acceptance- Rejection Method

Let X has a CDF F and PDF f.

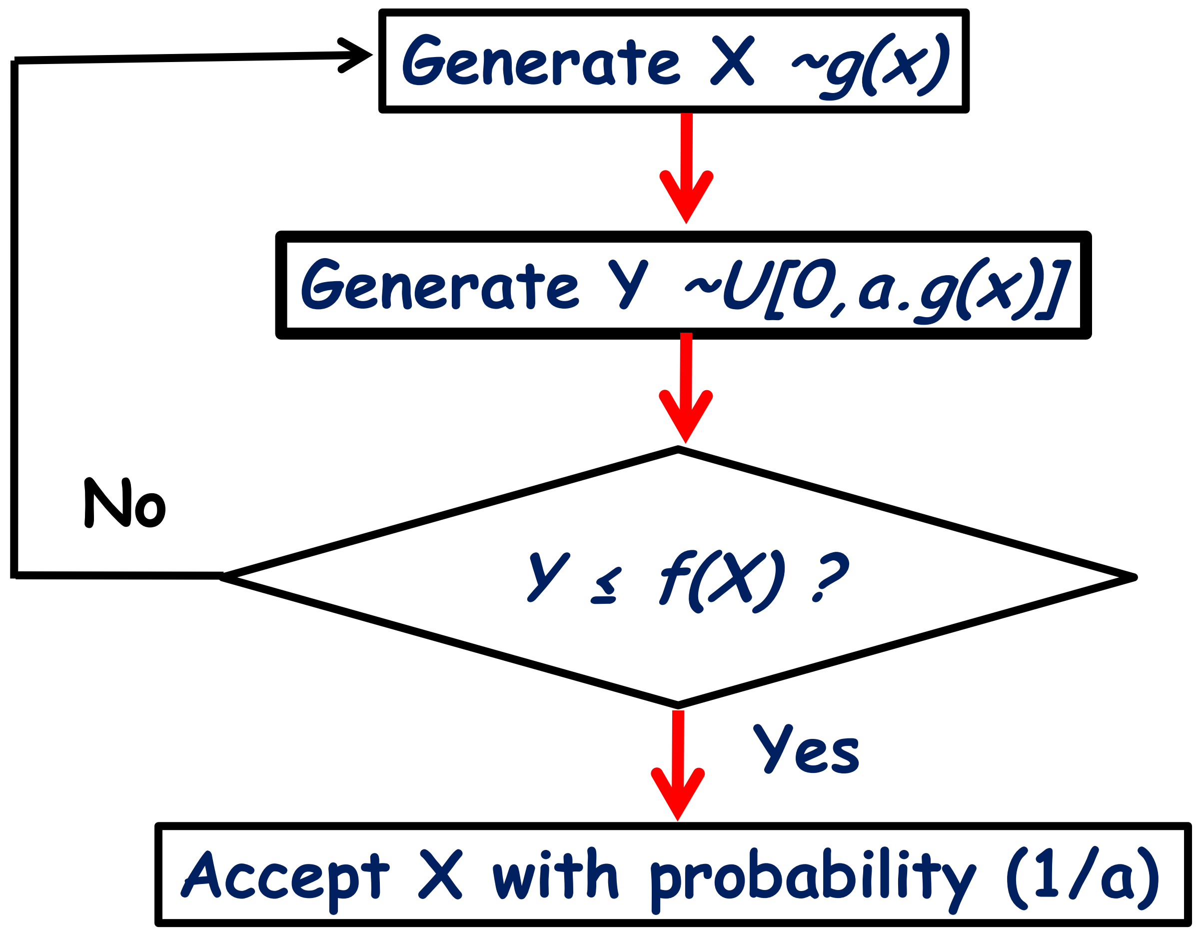
If F is hard (or impossible) to invert, too messy ...

But we have access to f.

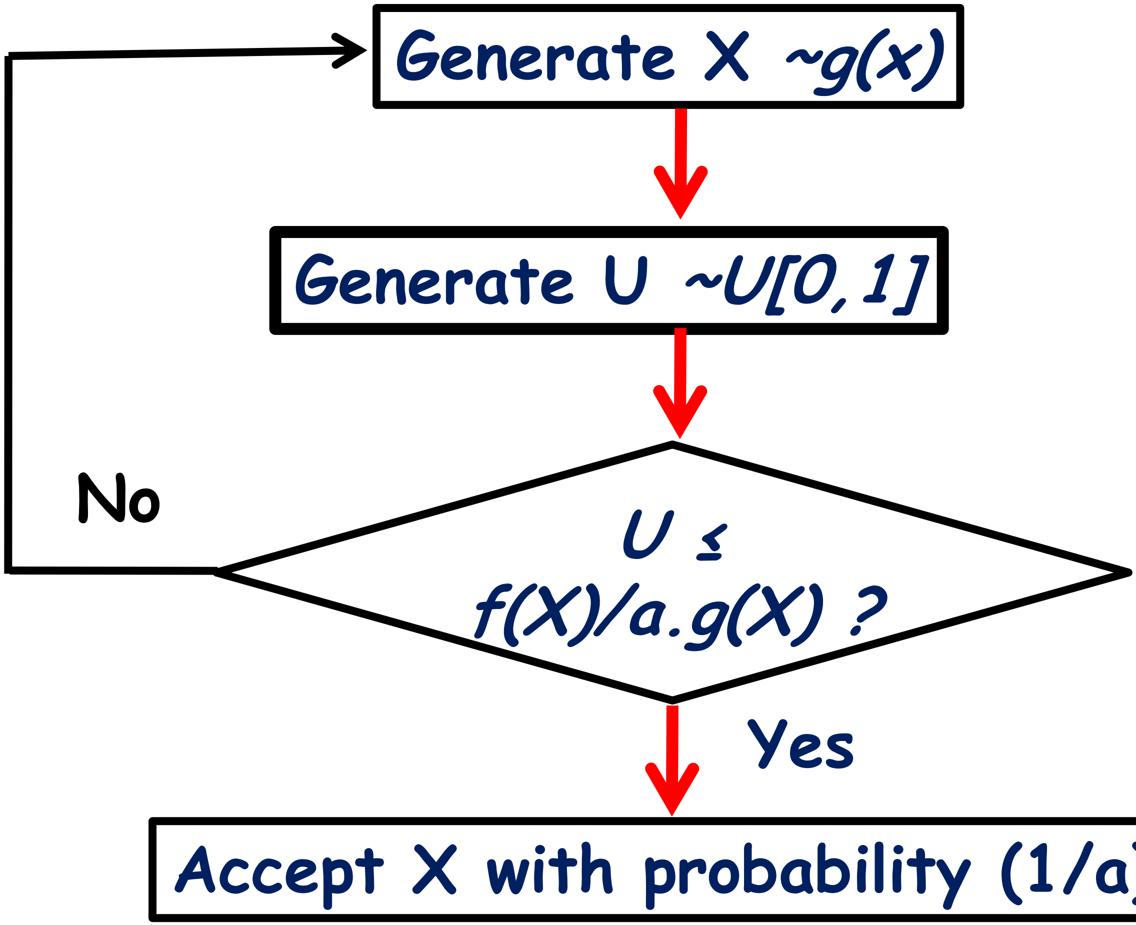
What to do?

Generate Y from a more manageable distribution and accept as coming from f with a certain probability.

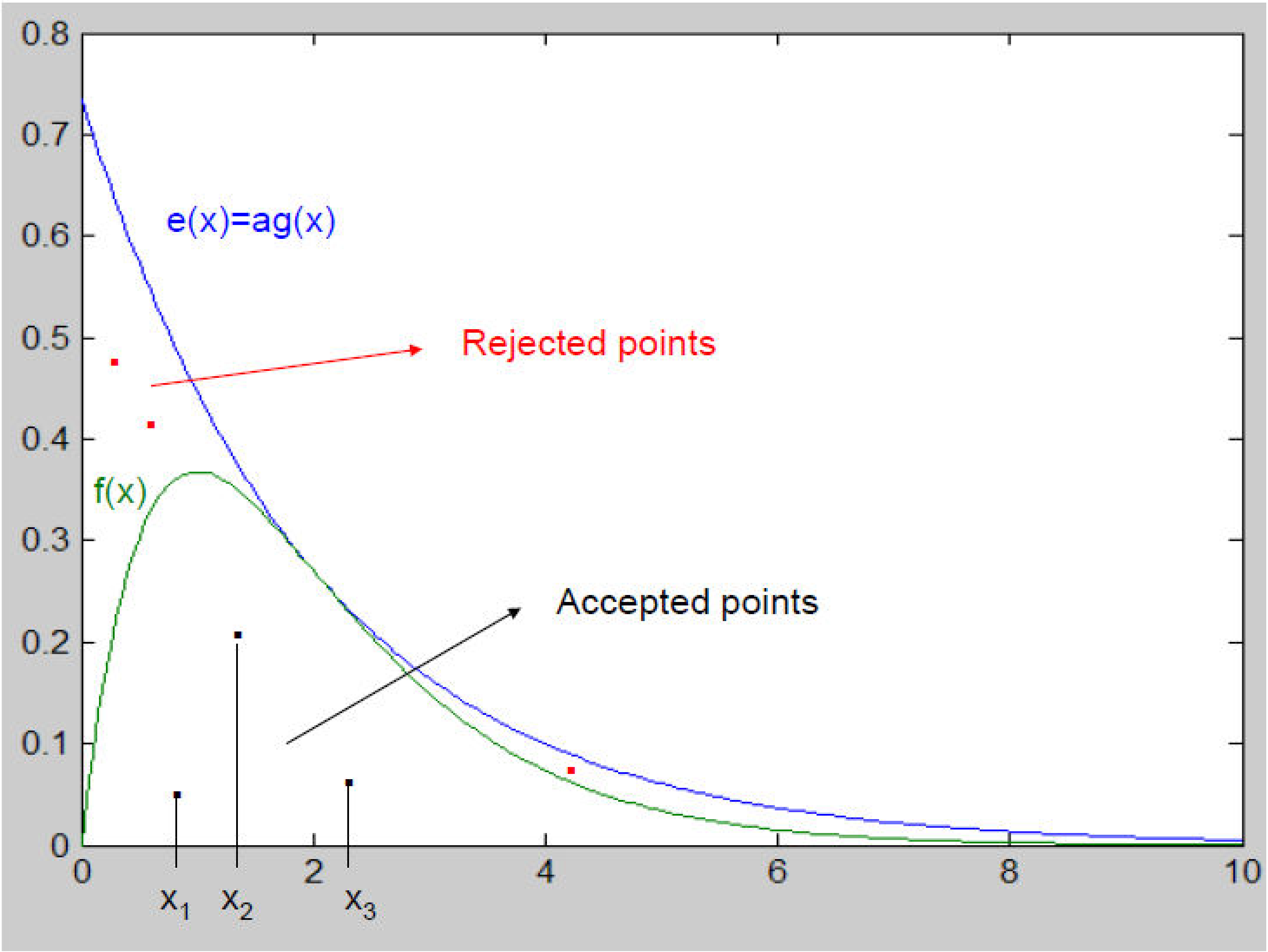
Generalized Acceptance-Rejection Method



Generalized Acceptance-Rejection Method



Generalized Acceptance-Rejection Method



Acceptance-Rejection Method

If X is a random variate with pdf $f(x)$ and cdf $F(x)$ without analytical form ($\Rightarrow$ Inverse transformation methods fail to be applied).

There exists $e(x) : e(x) \geq f(x), \forall x$

Majorant function $e(x)$ to be efficient:

$e(x)$ and $f(x)$ are 'close' in the region area

$e(x) = a\ g(x)$ where $g(x)$ is pdf (probability density function) easy/cheap to generate

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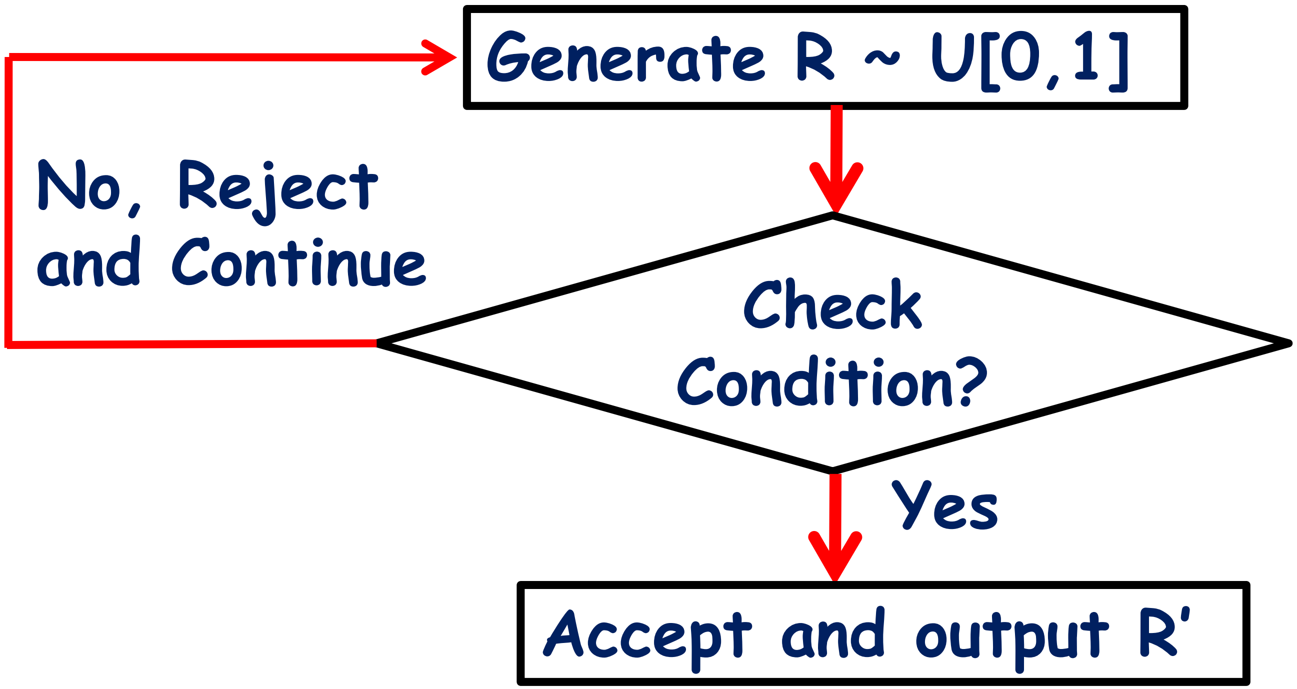
Random Variate Generation

Acceptance- Rejection Method



Acceptance-Rejection Method

Illustration:

To generate random variates, $X \sim U(1/4,\ 1)$

Procedures:

Step 1. Generate $R \sim U[0,1]$

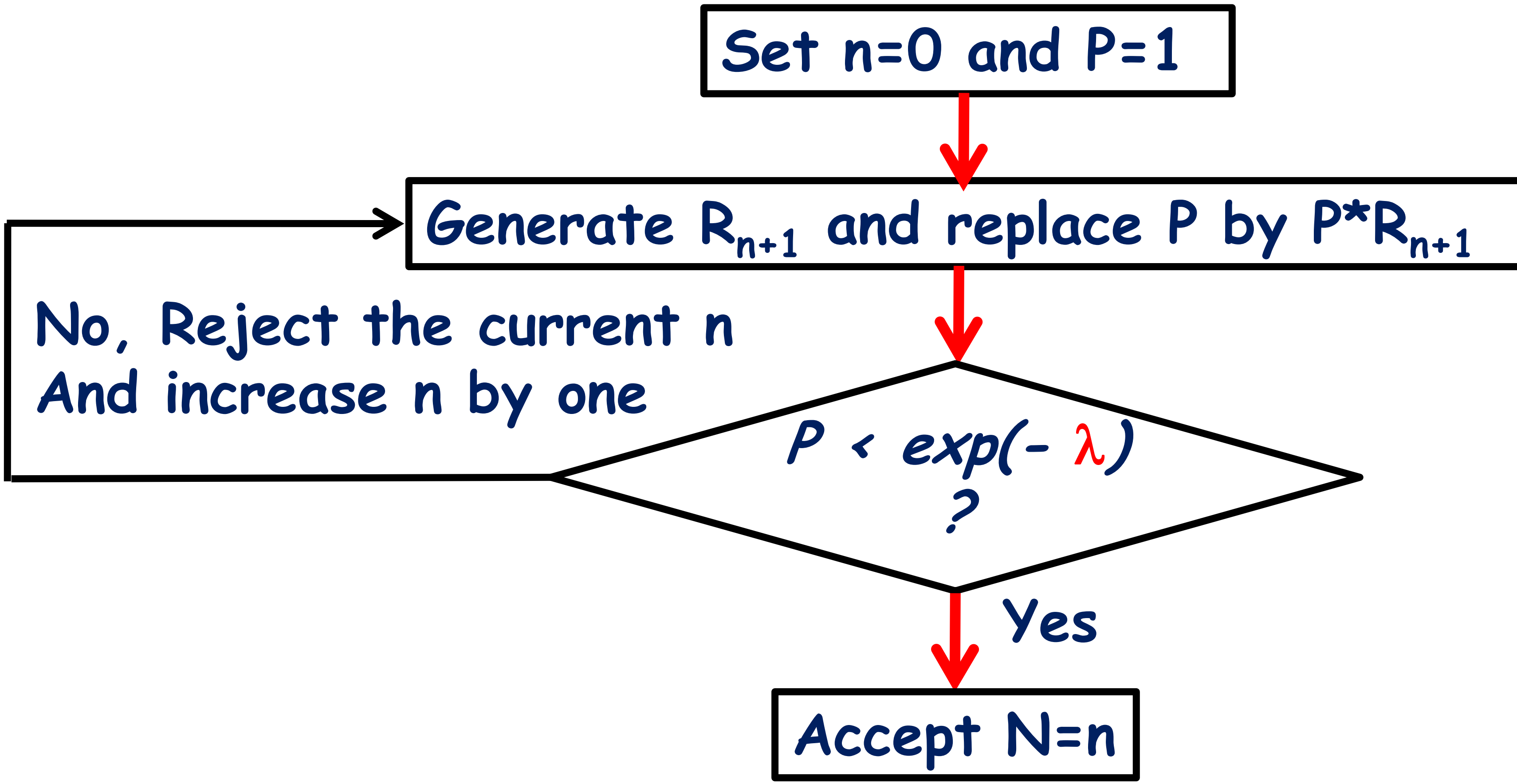
Step 2. If $R >= \frac{1}{4}$, accept $X=R$.

Step 3. If $R < \frac{1}{4}$, reject R, return to Step 1

R does not have the desired distribution, but R conditioned (R') on the event $\{R \geq \frac{1}{4}\}$ does.

Acceptance-Rejection Method

Poisson Distribution



Acceptance-Rejection Method

Procedure of generating a Poisson random variate N

Step 1: Set $n=0$, $P=1$

Step 2: Generate a random number R_{n+1}, and replace P by $P * R_{n+1}$

Step 3: If $P < \exp(-\lambda)$, then accept $N=n$

Otherwise, reject the current n, increase n by one, and return to step 2.

Acceptance-Rejection Method

Example:

Generate three Poisson variates with mean $\alpha = e^{-\lambda} = 0.2$, exp(-0.2) = 0.8187.

Variate 1:

Step 1: Set $n = 0$, $P = 1$

Step 2: $R_1 = 0.4357$, $P = 1 \times 0.4357$

Step 3: Since $P = 0.4357 < exp(-0.2)$,

accept $N = 0$

Acceptance-Rejection Method

Example:

Variate 2:

Step 1: Set $n = 0$, $P = 1$

Step 2: $R_1 = 0.4146$, $P = 1 \times 0.4146$

Step 3: Since $P = 0.4146 < \exp(-0.2)$,

accept $N = 0$

Acceptance-Rejection Method

Example:

Variate 3:

Step 1: Set $n = 0$, $P = 1$

Step 2: $R_1 = 0.8353$, $P = 1 \times 0.8353$

Step 3: Since $P = 0.8353 > \exp(-0.2)$, reject $n = 0$ and return to Step 2 with $n = 1$

Step 2: $R_2 = 0.9952$, $P = 0.8353 \times 0.9952 = 0.8313$

Step 3: Since $P = 0.8313 > \exp(-0.2)$, reject $n = 1$ and return to Step 2 with $n = 2$

Step 2: $R_3 = 0.8004$, $P = 0.8313 \times 0.8004 = 0.6654$

Step 3: Since $P = 0.6654 < \exp(-0.2)$, accept $N = 2$

Acceptance-Rejection Method

Example:

N	R_{n+1}	P	Accept/Reject		Result
0	0.4357	0.4357	$P<\exp(-a)$	Accept	N=0
0	0.4146	0.4146	$P<\exp(-a)$	Accept	N=0
0	0.8353	0.8353	$P\geq\exp(-a)$	Reject	
1	0.9952	0.8313	$P\geq\exp(-a)$	Reject	
2	0.8004	0.6654	$P<\exp(-a)$	Accept	N=2

It took five random numbers to generate three Poisson variates

In long run, the generation of Poisson variates requires some overhead!

Random Variate Generation

☐ **Bernoulli(p)**:

- Return 1 with probability p,
- Return 0 with probability 1-p

$$\begin{cases} u = random(\ ); \\ \text{if } (u < 1 - p) \text{ return } 0; \\ \text{else return } 1; \end{cases}$$

☐ **Geometric(p)**: $f(x) = p^k \cdot (1-p)$

- Number of Bernoulli trials until first '0')

$$\begin{cases} u = random(\ ); \\ \text{return } \log(1.0 - u)/\log(p); \end{cases}$$

☐ **Uniform (a,b)**: equally likely to select an integer in interval [a,b]

$$\begin{cases} u = random(\ ); \\ \text{return } a + (u \cdot (b - a + 1)); \end{cases}$$

Random Variate Generation

- **Exponential distribution with mean μ= *-1/λ***

$$\begin{cases} u = random(\ \ ); \\ \text{return } -\mu \cdot \log(1-u); \end{cases}$$

- **Weibull Distribution with shape, α, and scale, β**

$$\begin{cases} u = random(); \\ Return\ X = \beta[-\ln(1-u)]^{1/\alpha} \end{cases}$$

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Necessary Factors to Be Considered/decided for generating RVGs

Statistical Capabilities w.r.t generating RVGs

Requirement from RVGs

1. **Exactness**

   As far as possible, use methods that results in random variates with exactly (not approximately) the desired distribution.

2. **Efficiency:**

   Low storage: Fast (setup and marginal time)

3. **Complexity(Simplicity):**

   Refers to both algorithmic simplicity and implementation simplicity.

   Understand and implement (often tradeoff against efficiency).

4. **Robustness**

   An algorithm is efficient regardless of parameter values.

Questions

- What are random Variates. Give necessary factors for deciding correct algorithm for generating random variates?
- What is the use of Random numbers? Explain Statistical Capabilities with respect to generating random variates.

Questions

- What are all the acceptance rejection techniques? Briefly explain them.
- Explain in detail the Acception-Rejection test for generating random variates.
- Give expressions for Binomial, Poisson and Normal distributions. Under what conditions Binomial distribution is approximated by Poisson distribution?

Questions

- Describe the importance of Discrete Probability Functions.
- Discuss in details, the discrete probability function. How it is different from continuous probability function?
- Explain the following:
  - Continuous probability functions
  - Discrete Probability Functions
- What is stochastic variable? How does it help in simulation?

Questions

- What is an exponential distribution? Explain with an example.
- Explain any two terms:
  - Maximum Density
  - Weibull continuous distribution
  - Chi square test
- Explain in detail the Inverse Transform method with diagram.

Questions

- What do you understand by Parameterization of continuous distribution? Explain any three parameters in detail.
- Name any four types of discrete probability distribution. Explain normal continuous distribution in detail.
- Explain any two types of continuous probability distribution with examples.
- What do you understand by Composition method of generating random variate.

Acceptance Rejection Method

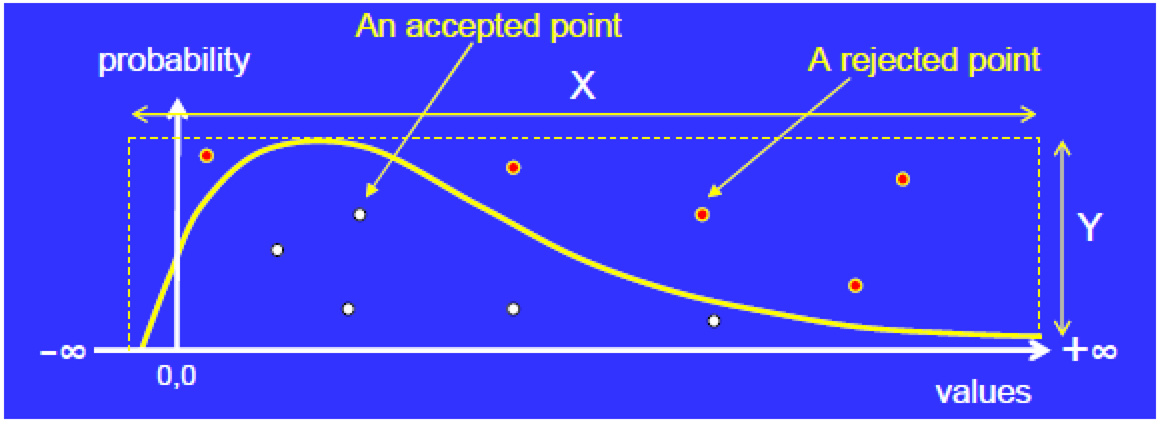
- Suppose that we need to sample from a distribution whose inverse function is hard to solve.
- In that case, acceptance-rejection method can be used.

Acceptance Rejection Method

- Generate a random point (X,Y) on the graph.
- If (X,Y) lies under the graph of f(X) then

  Accept X
- Otherwise

  Reject X

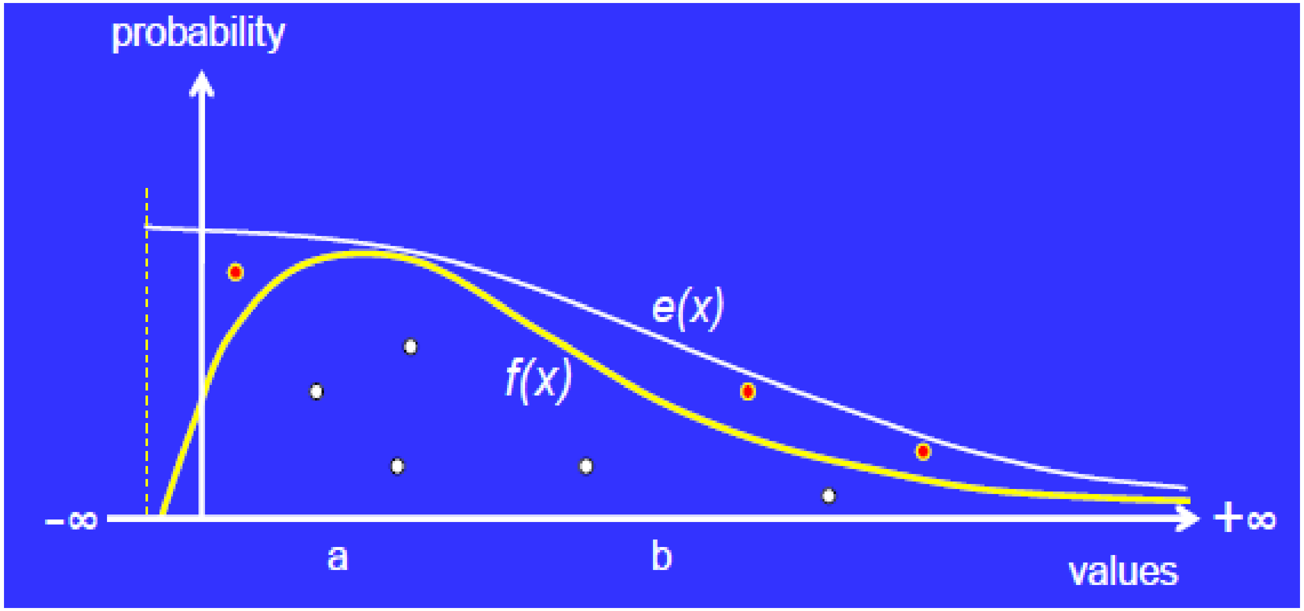


Acceptance Rejection Method(drawback)

- **Trials ratio:** Average number of points (X,Y) needed to produce one accepted X.
- We need to make trial ratio close to 1.
- Else generator may not be efficient enough because of wasted computing effort.

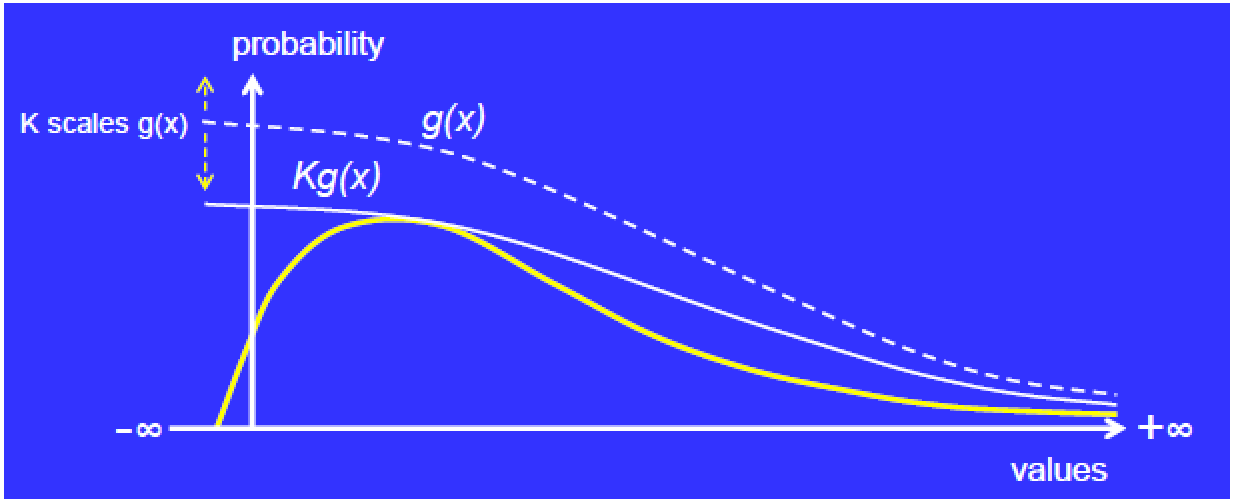
Acceptance Rejection Method (Making more Efficient)

- One way to make generator efficient is:
  - To generate points uniformly scattered under a function *e(x), where area between* the graph of *f and e be small.*



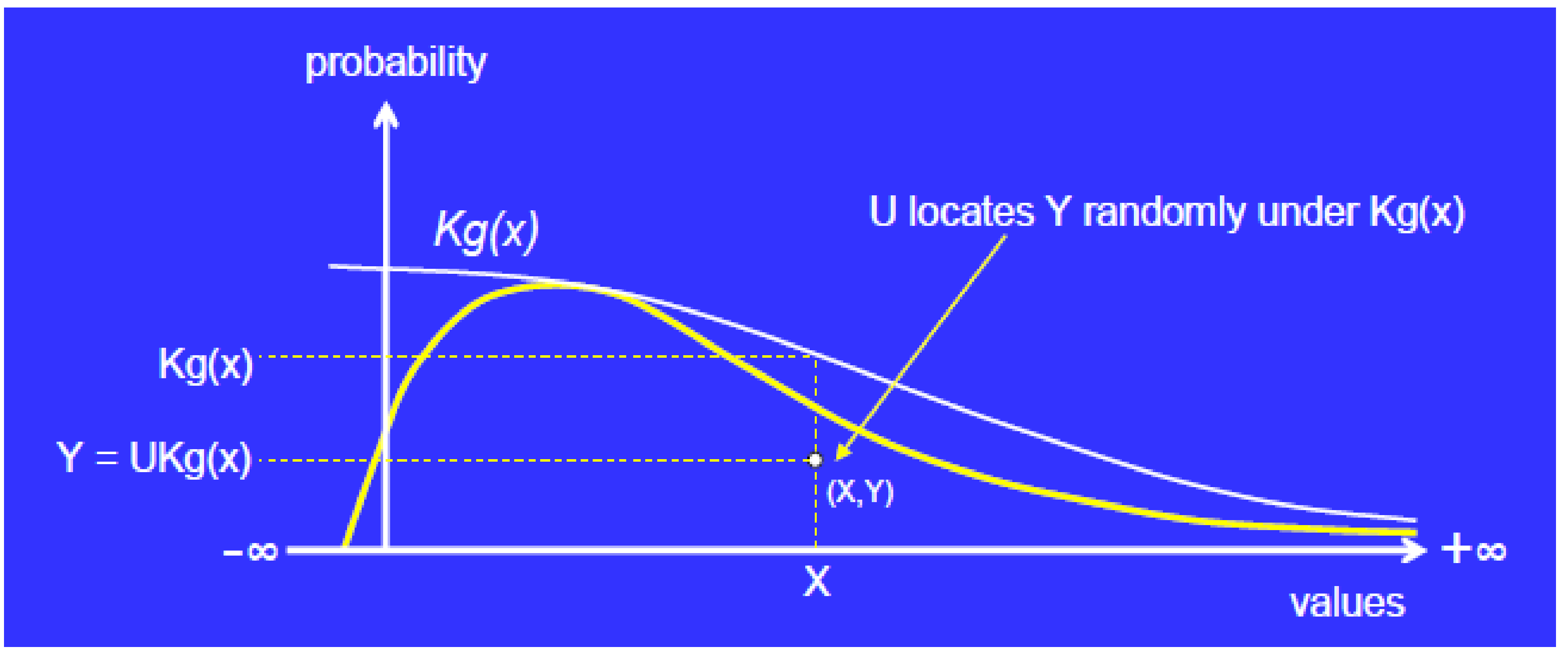
Acceptance Rejection Method[Constructing e(x))]

- Take *e(x) = Kg(x)*
- *g(x) = density function of a distribution for which an easy way of generating variates already exists.*
- *K = scale factor*



Acceptance Rejection Method[Producing (X,Y)]

- Let X = a variate produced from Kg(x)
- Let U = *RN(0,1)*
- (X,Y) = (X, UKg(X))



Acceptance Rejection Method[Producing (X,Y)]

- Assume that a RVG algorithm exists for a probability distribution g(x), which covers the entire f(x), for which we want to develop a RVG algorithm. Algorithm:
  - 1. Generate a random point (x, y) under g(x) using the known RVG algorithm for g(x).
  - 2. If the random point (x, y) falls under f(x), then accept the random point and deliver x as a random variate; otherwise reject the random point and go to Step 1.

Discrete versus Continuous Random Variables

Discrete Random Variable	Continuous Random Variable
Finite Sample Space e.g. {0, 1, 2, 3}	Infinite Sample Space e.g. [0,1], [2.1, 5.3]
Probability Mass Function (PMF) $p(x_i) = P(X = x_i)$ 1. $p(x_i) \geq 0$, for all i 2. $\sum_{i=1}^{\infty} p(x_i) = 1$	Probability Density Function (PDF) $f(x)$ 1. $f(x) \geq 0$, for all x in R_X 2. $\int_{R_X} f(x)dx = 1$ 3. $f(x) = 0$, if x is not in R_X

Cumulative Distribution Function (CDF) $p(X \leq x)$

$p(X \leq x) = \sum_{x_i \leq x} p(x_i)$	$p(X \leq x) = \int_{-\infty}^{x} f(t)dt = 0$ $p(a \leq X \leq b) = \int_{a}^{b} f(x)dx$

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Probability Distributions For Random Numbers

Discrete

Bernoulli Distribution:

$$f(x)=P(X = 1) = p$$
$$f(x)=P(X = 0) = 1 - p$$

Binomial Distribution:

$$f(x) = P(X=k)=\binom{n}{k}p^k(1-p)^{n-k} \text{ for } k=0,1,2,\ldots,n.$$

Geometric Distribution

$$f(x)=(1-p)^{x-1}p \text{ for } x=0,1,2,3,4,\ldots\ldots$$

Poisson Distribution

$$f(x) = p(X=x)=p(x)=\frac{e^{-\lambda}\lambda^x}{x!}, \quad x=0,1,2,\ldots.$$

Continuous

Uniform Distribution

$$\text{PDF:} \quad f(x)=\begin{cases}\dfrac{1}{b-a}, & a \le x \le b\\ 0, & \text{otherwise}\end{cases}$$

$$\text{CDF:} \quad F(x)=\begin{cases}0, & x \prec a\\ \dfrac{x-a}{b-a}, & a \le x \prec b\\ 1, & x \ge b\end{cases}$$

Exponential Distribution

$$\text{PDF:} \quad f(x)=\begin{cases}\lambda \cdot \exp(-\lambda \cdot x), & x \ge 0\\ 0, & \text{otherwise}\end{cases}$$

$$\text{CDF:} \quad F(x)=\begin{cases}0, & x < 0\\ \int_0^x \lambda e^{-\lambda t}dt = 1-e^{-\lambda x}, & x \ge 0\end{cases}$$

Normal Distribution

$$\text{PDF:} f(x)=\frac{1}{\sigma \cdot \sqrt{2\pi}} \cdot \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$$

$$\text{CDF:} F(x)=\frac{1}{2}\cdot\left(1+erf\left(\frac{x-\mu}{\sigma\cdot\sqrt{2}}\right)\right)$$

Weibull Distribution

$$f(x)=\begin{cases}\dfrac{\beta}{\alpha}\left(\dfrac{x-\nu}{\alpha}\right)^{\beta-1}\exp\left[-\left(\dfrac{x-\nu}{\alpha}\right)^{\beta}\right], & x \ge \nu\\ 0, & \text{otherwise}\end{cases}$$

$$f(x)=\frac{\beta}{\alpha^\beta}x^{\beta-1}e^{-(\frac{x}{\alpha})^\beta}$$

$$F(X)=1-e^{-(\frac{x}{\alpha})^\beta}$$

Generating Discrete Random Variates

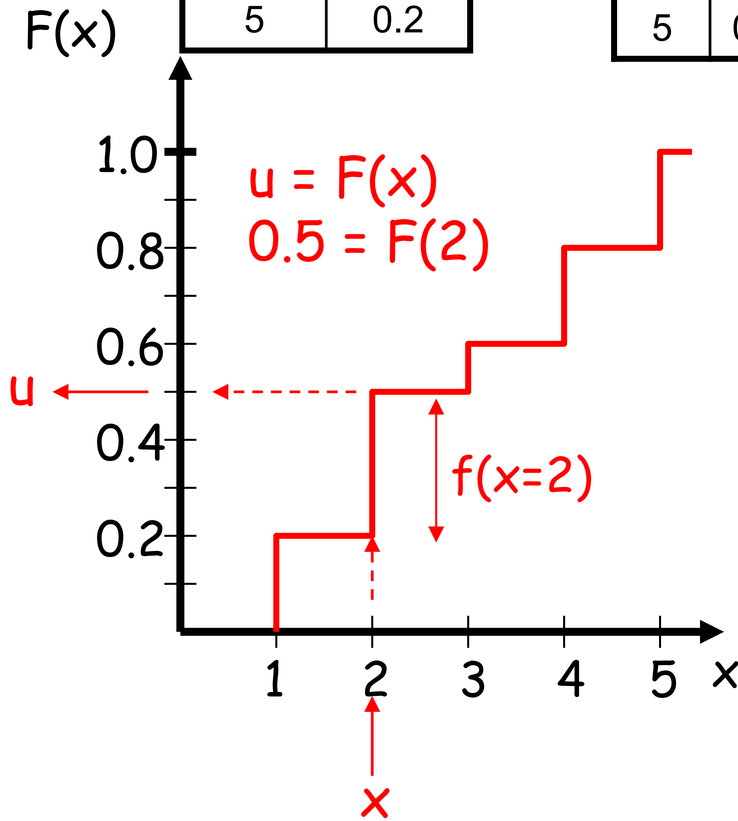
PDF:

x	f(x)
1	0.2
2	0.3
3	0.1
4	0.2
5	0.2

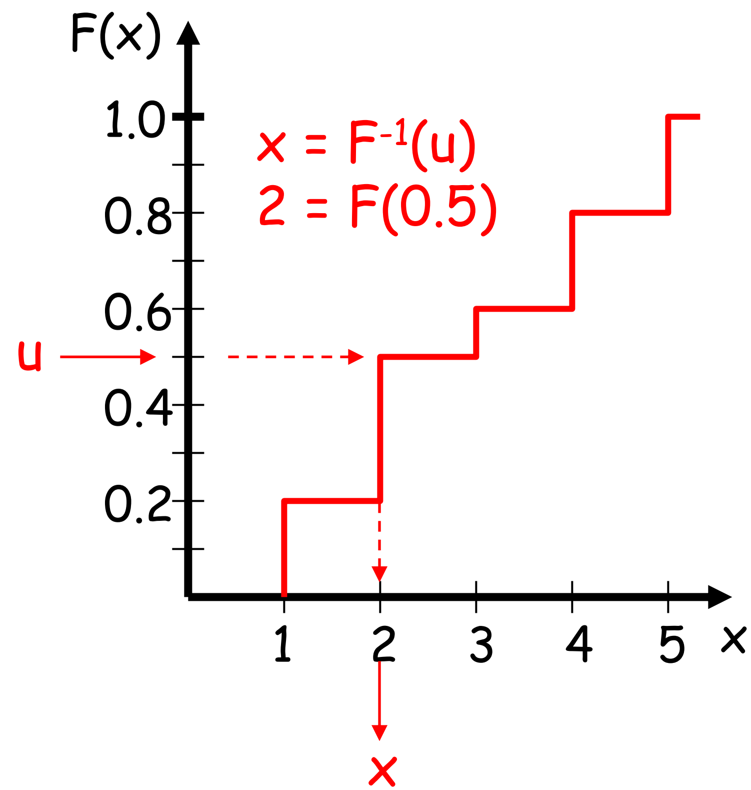
CDF:

x	f(x)	F(x)
1	0.2	0.2
2	0.3	0.5
3	0.1	0.6
4	0.2	0.8
5	0.2	1.0

Random variate generation:
1. Select u, uniformly distributed (0,1)
2. Compute $F^{-1}(u)$; result is random variate with distribution f()



Cumulative Distribution Function of X:
$F(x) = P(X \le x)$



Inverse Distribution Function (idf) of X:
$F^{-1}(u) = \min \{x: u < F(x)\}$

END