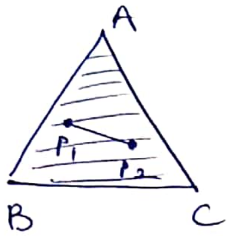
UNIT 4

Linear Programming

Convex Set A non-empty set S of points is called convex if any line segment joining any two points in S also lies entirely in S.



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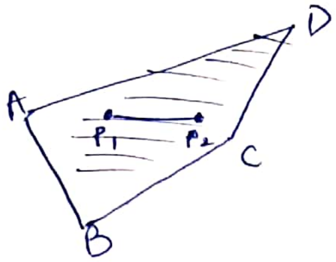

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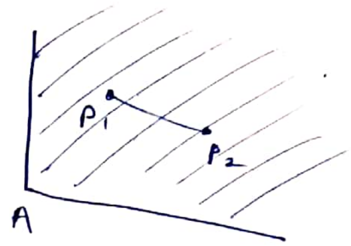
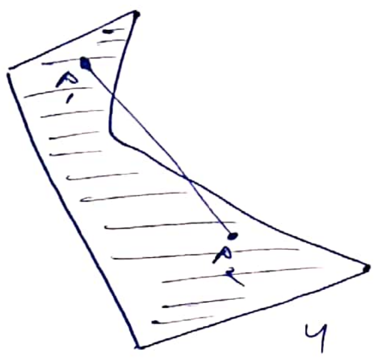


3

Shaded region in fig are all examples of convex set.



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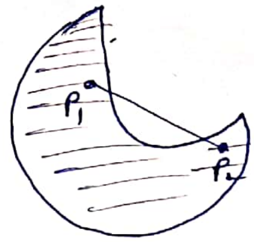

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The shaded regions in fig. do not rep. convex sets since the lines joining two included points P_1 & P_2 do not lie entirely in the set.

Extreme points in fig 1 A, B, C
fig 2 A B C, D
fig 3 A

these are the points which are not in the interior point. ~~but they are~~

Bounded Convex Set is one which can be enclosed by a sufficiently big rectangle

Unbounded C. Set : is one which can not be enclosed by any sufficiently large rectangle.

fig 1, 2 are bounded

fig 3, 4, 5, 6 are unbounded.

~~Thus~~ for solving a LPP in two variable graphically, we should proceed as follows

(i) Sketch the feasible region

(ii) find all the extreme points of S

(iii) Evaluate objective fn at each extreme point

(iv) choose an extreme point at which obj. fn is largest.

General linear programming Problem

Find the values of variables $x_1, x_2 - - - x_n$ which maximize or minimize the obj. fn.

$$Z = C_1 x_1 + C_2 x_2 + \cdots + C_n x_n$$

Subject to the constraints

$$a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \leq b_1$$

$$a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \leq b_2$$

$$\vdots$$

$$a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \leq b_m$$

along with $x_1 \geq 0$, $x_2 \geq 0$ .... $x_n \geq 0$.

The set of values of $x_1, x_2 - - x_n$ which satisfies the constraints is called sol. of the L.P.P. Further the sol. of the LPP which satisfy the non-negativity restrictions is called its feasible sol.

★ Any feasible sol which maximize or minimize the obj fn. is called its optimal sol.

Q:- A photographic firm develops two types of films fine and superfine every day using two material solutions A & B.

The fine film consumes 2ml of sol A and 1ml of sol B, while the super fine films consumes 1 ml of sol A and 2ml of sol. B.

The profit on the fine film is 8Rs and 10Rs on super fine film.

if the firm has 50 ml of sol A and 70ml of sol B available on a particular day, how many fine films and how many super fine films be made on that day so that profit is max.

Sol Let x & y be no. of fine and super fine films resp. on that day.

Total amt. of Sol A is req. $2x + y$

Total . . . . . . B . . . . . $x + 2y$

But the firm has only 50 ml of Sol A and 70 ml of Sol B, on that day.

$\therefore$ constraints are $2x + y \leq 50$

$x + 2y \leq 70.$

Obviously $x \geq 0, y \geq 0.$

The profit is the objective

$\therefore$ obj. fn. is $Z = 8x + 10y.$

The LPP is mathematically formulated as find the values of x and y which will maximize the objective function

$$Z = 8x + 10y.$$

Subject to the constraints

$$2x + y \leq 50$$
$$x + 2y \leq 70$$
$$x \geq 0,\ y \geq 0.$$

the inequalities $x \geq 0$, $y \geq 0$ rep. the area in the 1st quadrant of xy-plane.

$2x + y = 50$. (1)

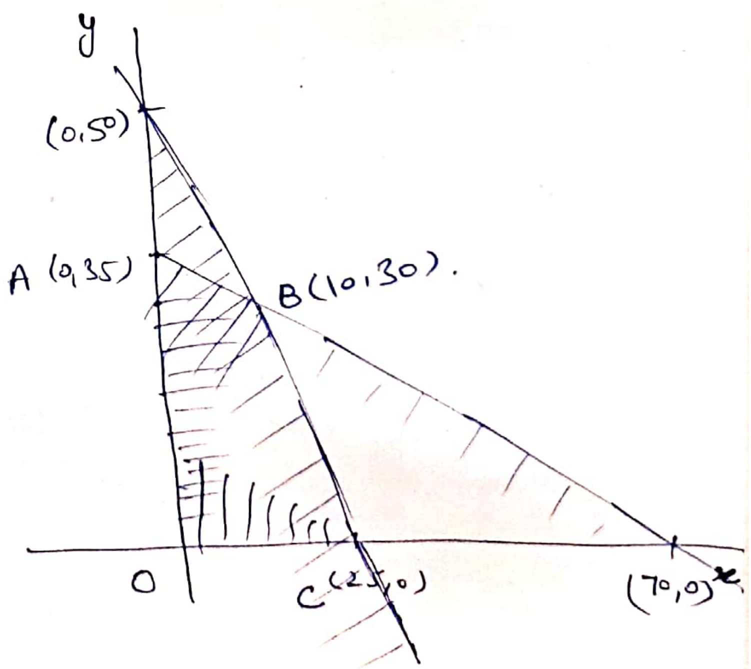
$x + 2y = 70$. (2)

$x = 0 \quad y = 50.$

$y = 0 \quad x = 25$

$(0, 50)$

$(25, 0)$



$x + 2y = 70$

$x = 0 \quad y = 35$

$y = 0 \quad x = 70.$

$(0, 35) \quad (70, 0)$

Intersection pt.

$x = 70 - 2y$ put in (1)

$140 - 4y + y = 50$

$-3y = 50 - 140 = -90$

$y = 30,\ x = 10$

The constraint $2x+y \le 50$ rep the area region on and below the line $2x+y=50$.

and constraint $x+2y \le 70$ stands for the area on and below the line $x+2y=70$.

$\therefore$ shaded area is bdd. by the lines

$x=0$, $y=0$, $2x+y=50$ and $x+2y=70$.

is the feasible region

shaded region is bdd convex set.

with extreme points O, A, B, C

obj fn is $Z = 8x+10y$.

At O $\quad Z = 0$.

At A $\quad Z = 350$

At B $\quad Z = 380$

At C $\quad Z = 200$.

Hence $\max Z = 380$ at B.

$\therefore$ at B we have optimal sol.

Q:- obtain the set of points that maximize the obj. fn. $Z = 8x+10y$ subject to the constraints $2x+3y \le 6$

$x+2y \ge 6$ $\quad x \ge 0, y \ge 0$.

Sol $\quad 2x+3y=6$ $\qquad (0,2)$

$x=0 \; y=2$, $y=0 \; x=3$ $\qquad (3,0)$

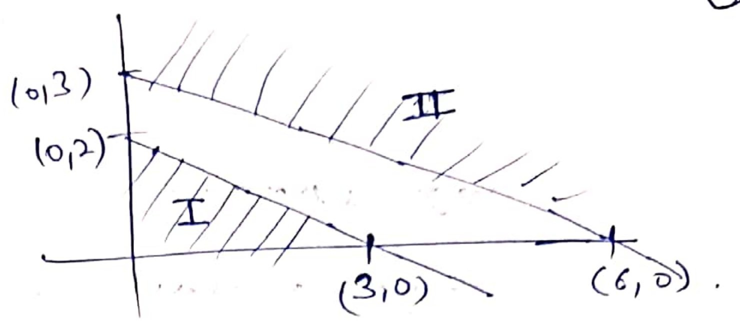
$x + 2y = 6.$

$x = 0 \quad y = 3$

$y = 0 \quad x = 6$

$(0, 3)$

$(6, 0)$



Constraints $x \geq 0$, $y \geq 0$, $2x + 3y \leq 6$ indicates region I

Constraints $x \geq 0$, $y \geq 0$, $x + 2y \geq 6$. indicates region II

There are no points satisfying all the four constraints (No common shaded area).

$\therefore$ there is no feasible sol.

Q:- A dietician prepares a menu that includes food A and B. Suppose that one kg of food A has 2 units of protein, 1 unit of iron and 1 unit of carbohydrates, while 1 kg. of food B has 1 unit of protein, 1 unit of iron and 3 units of carbohydrates. If each kg of food A costs Rs. 30 and each kg of food B costs Rs. 40, the dietician wants the meal to provide at least 12 units of protein,

(8)

at least 9 units of iron and at least 15 units of carbohydrates. How many kgs. of each food should be used to minimize the cost of the meals.

Sol:- Let x kg of food A and y kg. of food B be used to prepare the meals.

Mathematically problem is

for protein $\quad 2x + y \geq 12$

iron $\quad x + y \geq 9$

carbo. $\quad x + 3y \geq 15$

$x \geq 0$, $y \geq 0$ $\quad$ (1)

The cost of meal $30x + 40y$.

$\min Z = 30x + 40y$

Subject to the restrictions (1).

$2x + y = 12$

$x = 0 \quad y = 0$

$y = 12 \quad x = 6$

$x + y = 9$

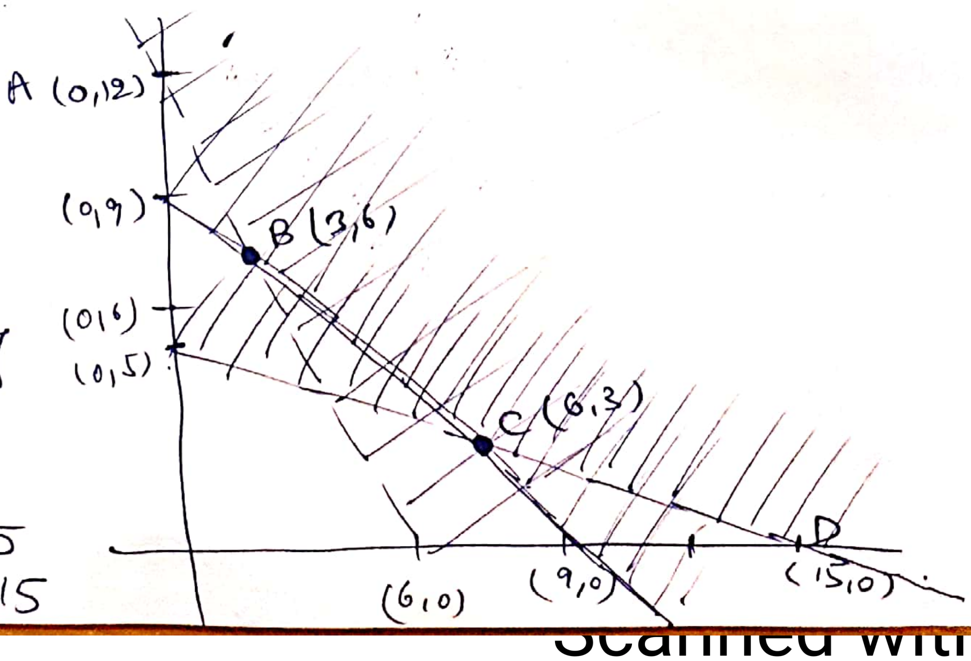
$x = 0 \quad y = 9$

$y = 0 \quad x = 9$

$x + 3y = 15$

$x = 0 \quad y = 5$

$y = 0 \quad x = 15$



$$\left.\begin{aligned} 2x + y &= 12 \\ x + y &= 9 \end{aligned}\right\} \text{Sub} \qquad \begin{aligned} x &= 3 \\ y &= 6. \end{aligned} \qquad (3, 6).$$

$$\left.\begin{aligned} x + 3y &= 15 \\ x + y &= 9 \end{aligned}\right\} \text{Sub.} \qquad 2y = 6 \qquad \begin{aligned} y &= 3 \\ x &= 6. \end{aligned} \qquad (6, 3).$$

feasible sol. lies in the convex set rep. by shaded area.

But the area is unbounded, but has extreme points A, B, C, D.

(feasible because common shaded area by all constraints).

$$Z = 30x + 40y.$$

at A	$Z = 480$
B	$Z = 330$
C	$Z = 300$
D.	$Z = 450$

Z is min at C

$\therefore$ $x = 6$, $y = 3$ gives Z as min.

General L.P. Problem

A L.P.P involving (non-basic) variables $x_1, x_2 --- x_n$ may be expressed as follows.

Determine the values of $x_1, x_2 - x_n$ which maximize (or minimize) the objective function

$$Z = c_1x_1 + c_2x_2 + ----+ c_nx_n \quad -(1)$$

Subject to the constraints

$$(2)\left\{\begin{array}{l} a_{11}x_1 + a_{12}x_2 + ---- + a_{1n}x_n \le b_1 \\ a_{21}x_1 + a_{22}x_2 + ---- + a_{2n}x_n \le b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + ----+ a_{mn}x_n \le b_m \end{array}\right.$$

and satisfy the restrictions

$$x_1, x_2, --- x_n \ge 0. \quad —— (3)$$

Solution of the L.P.P :-

The set of values $x_1, x_2 - - x_n$ satisfying the constraints (2) is called the sol. of the L.P.P.

feasible solution :-

Any sol. to the L.P.P which satisfies (3) of the problem is called ~~the~~ its feasible sol.

Optimal sol :- Any feasible sol sol. which maximize or minimize the objective fn. Z is called its optimal sol.

Actually in (2) some may be equalities, some may be inequalities of ≤ type or remaining ones may be ~~of~~ ≥ type.

The inequality constraints can be changed to equalities by adding or subtracting non-negative variables to (from) the left hand sides of such constraints.

If the constraints of a general L.P.P. are

$$\sum_{j=1}^{n} a_{ij} x_i \leq b_i \quad (i = 1, 2, \dots k)$$

then the non-negative variable 's_i' which satisfies

$$\sum_{j=1}^{n} a_{ij} x_i + s_i = b_i \quad (i = 1, 2 \dots k)$$

are called slack variables.

And, if the constraints of a general L.P.P are

$$\sum_{j=1}^{n} a_{ij} x_i \geq b_i \quad (i = k, k+1 \dots)$$

then the non(-ve) variables, s_i which satisfies

$\sum_{j=1}^{n} a_{ij} x_j - s_i = b_i$, $i = 1 \text{ to } m$

are called surplus variables

The Canonical and Std. forms of L.P.P

Canonical form :-

The general L.P.P can always be expressed in the foll. form

maximize $Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$
subject to

$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \le b_i$

$i = 1, 2, \dots, m$

& $x_1, x_2, \dots, x_n \ge 0$

by applying some elementary transformation.

(iv, ...) This form of L.P.P is called its canonical form and has the foll. characteristics

(i) objective fn is of maximization type,
(ii) all the constraints are of $\leq$ type.
(iii) all the var. x_i are non-negative.

Use of Canonical form is found in the duality concept.

<u>Std. form</u>:- The general L.P.P can also be expressed in the foll. form:-

maximize $Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$

Subject to

$$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n = b_i$$

$$i = 1, 2 \text{ --- } m$$

$$x_1, x_2 \text{ --- } x_n \geq 0.$$

This form of L.P.P is called Std. form and has the fol[lowing] characteristics:

(i) Objective fn. is of maximization t[ype]

(ii) all the constraints are expressed as equations,

(iii) the right hand side of each constraint is non -ve, i.e.

$b_1, b_2 \text{———} b_n \geq 0.$

(iv) All the variables are non - ve.

Note

(1) In the L.P.P if we are to min[imize] $Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$, it [is] equivalent to write maximize $z'(=-z) = -c_1x_1 - c_2x_2 \dots$

Thus the objective fn can always be expressed in the maximization [form]

2. The inequality constraints can always be converted to equalities by slack and surplus variables.

3. Sometimes the decision variables $x_1, x_2 — x_n$ could also be zero or -ve. If a variable is -ve, it can always be expressed as the difference of two non-ve variables

for eg. a variable x_i can always be written as $x_i = x_i' - x_i''$ where $x_i', x_i'' \geq 0$.

Q. Convert the foll. L.P.P into std. form

$\max.\ Z = 2x_1 + 3x_2 + 6x_3$

subject to $x_1 - 2x_2 \leq 7$, $3x_1 + 4x_2 - 6x_3 \geq 10$

$4x_1 + 3x_3 \leq 5$, $x_1, x_2 \geq 0$.

Sol Since x_3 is unrestricted,

let $x_3 = x_3' - x_3''$ where $x_3', x_3'' \geq 0$

$x_1 - 2x_2 \leq 7$, $3x_1 + 4x_2 - 6x_3' + 6x_3'' \geq 10$

$4x_1 + 3x_3' - 3x_3'' \leq 5$, $x_1, x_2, x_3', x_3'' \geq 0$

Std. form is

$$\text{Max } Z = 2x_1 + 3x_2 + 6x_3' - 6x_3''$$

Subj. to

$$x_1 - 2x_2 + s_1 = 7$$

$$3x_1 + 4x_2 - 6x_3' + 6x_3'' - s_2 = 10$$

$$4x_1 + 3x_3' - 3x_3'' + s_3 = 5$$

$$x_1, x_2, x_3', x_3'' \geq 0, \quad s_1, s_2 \geq 0.$$

Simplex Method :-

Let there be m constraints and n decision variables $x_1, x_2 \ldots x_n$ and m slack variables, where $m < n$: the starting sol. is obtained by setting n var. equal to zero and then solving the remaining m equations, provided the sol. exists and is unique.

The n zero var. are called non-basic var. while the remaining m var. are called basic variable. and they form a basic sol.

In L.P.P the variables must be non-ve. If some of the basic sols. contain -ve var., such sols. are called basic infeasible sols. and should be rejected.

For this we start with non-ve basic sol. The next basic sol. should also be non-ve. This is ensured by the feasibility cond. Such a sol. is known as basic feasible sol.

Non-degenerate sol:- If all the vars. in the basic feasible sol.(s) are non-zero, it is called non. d.

Degenerate sol:- If any one of the (b.) basic vars. become zero, it is called a degenerate basic feasible sol.

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Steps

Let us assume that an initial basic feasible sol. to the L.P.P exists, then the optimal sol. of the L.P.P can be obtained as follows:-

Step 1 Check whether the obj fn is to be maximized or minimized.
If $Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$ is to minimized then convert it in to maximization problem by writing $\min Z = \max(-Z)$

check all b's are +ve.
if -ve, we multiply both sides of that constraint by -1 so as to make the R.H.S +ve.

Step 2 Express the prob. in to std. form.

Step 3 find an initial basic feasible sol.

[If there are m eqns. involving n unknowns then assign zero values to any (n-m) of the variables for finding a sol.]

Starting with a basic sol for which $x_1, x_2, \ldots x_{n-m}$ are each zero, find all s_i.

If all $s_i \geq 0$ the basic sol. is feasible and non-degenerate

If one or more of the s_i are zero, then the sol. is degenerate.

This whole information is expressed in the foll. simplex table.

	C_j	c_1	c_2	c_3	0	0	0	
C_B	Basis	x_1	x_2	x_3	s_1	s_2	s_3	b
0	s_1	a_{11}	a_{12}	a_{13}	1	0	0	b_1
0	s_2	a_{21}	a_{22}	a_{23}	0	1	0	b_2
0	s_3	a_{31}	a_{32}	a_{33}	0	0	1	b_3
		body matrix			unit matrix			

s_1 s_2 s_3 are called basic var.
x_1 x_2 x_3 . . . . . . non-basic var.

Basis means basic var. s_1, s_2, s_3 . .
$C_B \rightarrow$ ~~denotes the~~ coef. denotes the

coeff. of the basic variables only in the objective fn.

Step 4 Now apply the optimality test.

find $C_j = c_j - z_j$

where $z_j = \sum c_B a_{ij}$

This C_j-row is called net evaluation row and indicates the per unit increase in the objective fn. if the var. heading the col. is brought in to the sol.

If all C_j are -ve then the initial basic feasible sol. is optimal. otherwise we proceed to next step.

Step 5

identify incoming and outgoing var.

Step 5 If there are more than one positive C_j, then the incoming var. is the one that heads the col containing max. C_j.
that col. is called key col. that is marked with an arrow at the bottom.

Now divide the elts. under the b-col by the corresponding elts. of the key col. and choose the row containing the min +ve ratio θ. Then replace the corresponding basic var. (by making its value zero). This is called outgoing var. This row is called key row.

The elt. at the intersection of the key row & col. is called key elt.

If all these ratio's are ≤ 0, the incoming var. can be made as large as we please without violating feasibility cond. and no further iteration should be done

Q:- Max. $Z = 4x_1 + 3x_2 + 6x_3$

Subject to

$$2x_1 + 3x_2 + 2x_3 \le 440$$
$$4x_1 + 3x_3 \le 470$$
$$2x_1 + 5x_2 \le 430$$
$$x_1, x_2, x_3 \ge 0.$$

Step 1. The obj fn Z is to maximized and all b_i's are positive.

Step 2 $\max Z = 4x_1 + 3x_2 + 6x_3 + 0s_1 + 0s_2 + 0s_3$

Subj to

s_i's are slack var.

$$2x_1 + 3x_2 + 2x_3 + s_1 + 0s_2 + 0s_3 = 440$$
$$4x_1 + 0x_2 + 3x_3 + s_2 = 470$$
$$2x_1 + 5x_2 + 0x_3 + s_3 = 430$$

Step 3 eqns. $m = 3$

unknowns $n = 3$.

we find an initial basic feasible sol.

The basic (non-degenerate) feasible sol. is

$x_1 = x_2 = x_3 = 0$, (non-basic)

hence

$$\left.\begin{array}{l} s_1 = 440 \\ s_2 = 470 \\ s_3 = 430 \end{array}\right\} \text{basic}$$

Since all $s_i \geq 0$ the basic sol. is feasible and non-degenerate.

Initial basic feasible sol. is given by the table.

	c_j	4	3	6	0	0	0		
C_B	Basis	x_1	x_2	x_3	s_1	s_2	s_3	b	θ
0	s_1	2	3	2	1	0	0	440	
0	s_2	4	0	3	0	1	0	470	
0	s_3	2	5	0	0	0	1	430	

$+ 0s_3$

s_i's are slack var.

$C_B \times a_{ij}$ then Σ i.e. $0\times2 + 0\times4 + 0\times2$ $\quad 0\times3+0\times0+0\times5$

$0\times2+0\times3+0\times0$

$z_j = \Sigma C_B a_{ij}$

z_j	0	0	0	0	0	0
$c_j - z_j = C_j$	4	3	6	0	0	0

↑ incoming

step 4 here all C_j is +ve, the initial basic feasible sol is not optimal and we proceed to the next test or step.

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Step 5 Identify incoming & outgoing vars. From the above table it is clear that x_3 is incoming and s_2 is outgoing var.

($\because$ 6 is the +ve no. that is highest in all C_j so. ↑ x_3 is incoming

x_3	θ
2	$440/2 = 220$
s_2 3	$470/3 = 156.6$, min +ve value
0	$430/0 = \infty$.
6	
↑	

$x_3 \rightarrow$ Incoming
$s_2 \rightarrow$ outgoing
(3) $\rightarrow$ key elt.

make it zero

	2	3	2	1	0	0	440
Key Rows	4/3	0	1	0	1/3	0	470/3
	2	5	0	0	0	1	430

already zero

$440 - \frac{940}{3}$

$\frac{1320-940}{3} = \frac{380}{3}$

$R_1 \rightarrow R_1 - 2R_2$

-2/3 3 0 1 -2/3 0 380/3

C_B
0
6
0

$C_j - Z_j$

$\frac{1880}{9}$

C_B
3
6
0

$-\frac{1900}{9} + 430$
$= \frac{-1900+3870}{9}$
$\frac{1970}{9}$

	C_j	4	3	6	0	0	0		
C_B	Basis	x_1	x_2	x_3	s_1	s_2	s_3	b	θ
0	s_1	$-2/3$	3	0	1	$-2/3$	0	$380/3$	$380/9$
6	x_3	$4/3$	0	1	0	$1/3$	0	$470/3$	∞
0	s_3	2	5	0	0	0	1	430	86
	Z_j	8	0	6	0	2	0	940	
$c_j - Z_j$	$= C_j$	-4	3	0	0	-2	0		

if all C_j -ve stop.

$R_3 \to R_3 - 5R_1$

~~380~~

$-2/9$ 1   0 $1/3$ $-2/9$ 0 $380/9$

2   5   0   0   0   1   430

$\frac{10+18}{9}$

	C_j	4	3	6	0	0	0	
C_B	Basis	x_1	x_2	x_3	s_1	s_2	s_3	b θ
3	x_2	$-2/9$	1	0	$1/3$	$-2/9$	0	$380/9$
6	x_3	$4/3$	0	1	0	$1/3$	0	$470/3$
0	s_3	$28/9$	0	0	$-5/3$	$10/9$	1	$1970/9$
	Z_j	$22/3$	3	6	1	$4/3$	0	$3200/3$
	C_j	$-10/3$	0	0	-1	$-4/3$	0	

$\frac{-1900}{9} + 430$

$= \frac{-1900 + 3870}{9}$

$\frac{1970}{9}$

Since each $C_j \leq 0$

$\therefore$ it is optimal sol:

$x_1 = 0 \quad x_2 = \frac{380}{9} \quad x_3 = \frac{470}{3}$

$\max Z = \frac{3200}{3}$ Rs.

Q:- $\max Z = 107x_1 + x_2 + 2x_3$

$14x_1 + x_2 - 6x_3 + 3x_4 = 7$ — C_B: 0

$16x_1 + \frac{x_2}{2} - 6x_3 \le 5$ — 0

$3x_1 - x_2 - x_3 \le 0$ — 0

$x_1, x_2, x_3, x_4 \ge 0.$

Step 1 objective fn. maximized, all b's are +ve.

Step 2 Std. form of L.P.P.

Here x_4 is already slack var.

$\max Z = 107x_1 + x_2 + 2x_3 + 0x_4 + 0s_1 + 0s_2$

$0s_1 + 0s_2$, $\frac{14}{3}x_1 + \frac{1}{3}x_2 - 2x_3 + x_4 + \ = \frac{7}{3}$ — C_B: 0

$0x_4 + 0s_2$ $16x_1 + \frac{x_2}{2} - 6x_3 + s_1 = 5$ — 0

$0x_4 + 0s_1$ $3x_1 - x_2 - x_3 + s_2 = 0$ — 107

$x_1, x_2, x_3, x_4, s_1, s_2 \ge 0$

find initial basic feasible sol.

$x_1 = x_2 = x_3 = 0$ (non basic)

$x_4 = \frac{7}{3}$ $s_1 = 5$ $s_2 = 0.$ (basic)

C_B	C_j	107	1	2	0	0	0		
	Basis	x_1	x_2	x_3	x_4	s_1	s_2	b	θ
0	x_4	$\frac{14}{3}$	1/3	−2	1	0	0	$\frac{7}{3}$	$\frac{7}{3}\times\frac{3}{14}$
0	s_1	16	1/2	−6	0	1	0	5	$\frac{5}{16}$
0	s_2	3	−1	−1	0	0	1	0	$\frac{0}{3}$ ←
	$z_j = \sum C_B a_{ij}$	0	0	0	0	0	0	0	
	$C_j = c_j - z_j$	107	1	2	0	0	0		
		↑ incoming							

x_1 = incoming

s_2 = outgoing

Basic feasible sol is not optimal since C_j's are +ve.

we identify x_1 is incoming & s_2 is outgoing var

3 is the key elt.

Now dual basic feasible sol is given in the table

C_B	C_j	107	1	2	0	0	0		
	Basis	x_1	x_2	x_3	x_4	s_1	s_2	b	θ
0	x_4	0	17/9	−4/9	1	0	−4/9	7/3	$-\frac{7}{3}\times\frac{9}{4}$
0	s_1	0	35/6	−2/3	0	1	−16/3	5	−15/2
107	x_1	1	−1/3	−1/3	0	0	1/3	0	0
	Z_j	107	−107/3	−107/3	0	0	107/3		min
	C_j	0	$\frac{110}{3}$	$\frac{113}{3}$	0	0	$\frac{-107}{3}$		
				↑					

x_3 – incoming

x_1 – outgoing

As C_j are +ve sol is not optimal

Here x_3 is incoming var. but all values of θ being ≤ 0 x_3 will not enter the basis

$\Rightarrow$ sol. is unbounded.

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Duality in Linear Programming

Formulation of the Dual Problem :-

Consider the foll L.P.P

maximize $z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$

subject to
$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m$$
$$x_1, x_2 - x_n \geq 0.$$

For constructing the dual problem, we adopt the foll- guidelines.

(i) The maximization problem in The primal becomes minimization in the dual & vice versa.

(ii) $\leq$ type constraints in primal becomes $\geq$ type in the dual & vice versa.

(iii) The coeff — $c_1, c_2 - c_n$ in obj fn of primal become the $b_1, b_2 - b_m$ in the obj fn. of the dual.

(iv) The constraints $b_1, b_2 - b_m$ of the primal becomes $c_1, c_2 - c_n$ as the constraints of the dual.

(v) If primal has n variables and m constraints, the dual will have m vari and n constraints. In other words transpose of the body matrix of the primal gives the body matrix of the dual.

(vi) The var. in both the primal & dual must be non-ve.

Thus dual problem for above primal will be

minimize $W = b_1 y_1 + b_2 y_2 + \cdots + b_m y_m$

Subject to

$$a_{11} y_1 + a_{21} y_2 + \cdots + a_{m1} y_m \geq c_1$$
$$a_{12} y_1 + a_{22} y_2 + \cdots + a_{m2} y_m \geq c_2$$
$$\vdots$$
$$a_{1n} y_1 + a_{2n} y_2 + \cdots + a_{mn} y_m \geq c_n$$
$$y_1, y_2 - - y_m \geq 0.$$

Q:- Obtain the dual problem of foll. L.P.P

maximize $Z = 2x_1 + 5x_2 + 6x_3$

subject to
$$5x_1 + 6x_2 - x_3 \le 3$$
$$-2x_1 + x_2 + 4x_3 \le 4$$
$$4x_1 - 5x_2 + 3x_3 \le 1$$
$$-3x_1 - 3x_2 + 7x_3 \le 6$$
$$x_1, x_2, x_3 \ge 0.$$

$n = 3$ var.
$m = 4$ constraints.

$Z = CX$
$= [c_1 c_2 c_3] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Sol:- $AX \le B$ $\quad X \ge 0.$

$$A = \begin{bmatrix} 5 & 6 & -1 \\ -2 & 1 & 4 \\ 4 & -5 & 3 \\ -3 & -3 & 7 \end{bmatrix} \quad B = \begin{bmatrix} 3 \\ 4 \\ 1 \\ 6 \end{bmatrix} \quad C = [2\ 5\ 6]$$

$$A' = \begin{bmatrix} 5 & -2 & 4 & -3 \\ 6 & 1 & -5 & -3 \\ -1 & 4 & 3 & 7 \end{bmatrix}$$

minimize $W = 3y_1 + 4y_2 + y_3 + 6y_4$

subject to
$$5y_1 - 2y_2 + 4y_3 - 3y_4 \ge 2$$
$$6y_1 - y_2 - 5y_3 - 3y_4 \ge 5$$
$$-y_1 + 4y_2 + 3y_3 + 7y_4 \ge 6$$
$$y_1, y_2, y_3, y_4 \ge 0.$$

$n = 4$ var.
$m = 3$ constraints.

When Primal has equality constraints

Q:- Obtain the dual problem of the foll - L.P.P.

maximize $Z = x_1 - 2x_2 + 3x_3$

subj to

$$-2x_1 + x_2 + 3x_3 = 2$$
$$2x_1 + 3x_2 + 4x_3 = 1.$$
$$x_1, x_2, x_3 \geq 0$$

Sol:- The given constraints can be rewritten as

$$\left.\begin{aligned} -2x_1 + x_2 + 3x_3 &\leq 2 \\ 2x_1 - x_2 - 3x_3 &\leq -2 \end{aligned}\right]$$
$$\left.\begin{aligned} 2x_1 + 3x_2 + 4x_3 &\leq 1 \\ -2x_1 - 3x_2 - 4x_3 &\leq -1 \end{aligned}\right]$$

equality can be written as inequality

$\left.\begin{aligned} -2x_1 + x_2 + 3x_3 &\geq 2 \\ \text{or } -2x_1 + x_2 + 3x_3 &\leq 2 \end{aligned}\right]$

$\left.\begin{aligned} -2x_1 + x_2 + 3x_3 &\leq 2 \\ 2x_1 - x_2 - 3x_3 &\leq -2 \end{aligned}\right]$

In matrix form,

$$\begin{bmatrix} -2 & 1 & 3 \\ 2 & -1 & -3 \\ 2 & 3 & 4 \\ -2 & -3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \leq \begin{bmatrix} 2 \\ -2 \\ 1 \\ -1 \end{bmatrix}$$

The dual of the given problem is

Minimize $W = 2y_1' - 2y_1'' + y_2' - y_2''$

subject to
$$\begin{bmatrix} -2 & 2 & 2 & -2 \\ 1 & -1 & 3 & -3 \\ 3 & -3 & 4 & -4 \end{bmatrix} \begin{bmatrix} y_1' \\ y_1'' \\ y_2' \\ y_2'' \end{bmatrix} \geq \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

$$-2y_1' + 2y_1'' + 2y_2' - 2y_2'' \geq 1$$

$$y_1' - y_1'' + 3y_2' - 3y_2'' \geq -2$$

$$3y_1' - 3y_1'' + 4y_2' - 4y_2'' \geq 3$$

Now writing $y_1' - y_1'' = y_1$

$y_2' - y_2'' = y_2$

Min $W = 2y_1 + y_2$

$$\left.\begin{aligned} -2y_1 + 2y_2 &\geq 1. \\ y_1 + 3y_2 &\geq -2 \\ 3y_1 + 4y_2 &\geq 3. \end{aligned}\right\}$$

where y_1, y_2 are unrestricted in sign.

Write the dual of L.P.P

$\min Z = 3x_1 - 2x_2 + 4x_3$

Subj to

$3x_1 + 5x_2 + 4x_3 \geq 7$

$6x_1 + x_2 + 3x_3 \geq 4$

$7x_1 - 2x_2 - x_3 \leq 10$?

$x_1 - 2x_2 + 5x_3 \geq 3$

$4x_1 + 7x_2 - 2x_3 \geq 2$

$x_1, x_2, x_3 \geq 0$

Since the problem is of minimisation all constraints should be of $\geq$ type

$\therefore \quad -7x_1 + 2x_2 + x_3 \geq -10$

Now dual is

$\max W = 7y_1 + 4y_2 - 10y_3 + 3y_4 + 2y_5$

Subject to

$3y_1 + 6y_2 - 7y_3 + y_4 + 4y_5 \leq 3$

$5y_1 + y_2 + 2y_3 - 2y_4 + 7y_5 \leq -2$

$4y_1 + 3y_2 + y_3 + 5y_4 - 2y_5 \leq 4$

$y_1, y_2, y_3, y_4, y_5 \geq 0$.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix}' = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

Using duality solve the foll. problem

$x_1 \geq 4$
$x_2 \geq 6$
$x_1 + 2x_2 \geq 20$
$x_1 + x_2 \geq 18$.

$\min Z = 0.7x_1 + 0.5x_2$ $\quad x_1, x_2 \geq 0$

Subject to

$x_1 \geq 4, \quad x_2 \geq 6 \quad x_1 + 2x_2 \geq 20 \quad 2x_1 + x_2 \geq 18$

Sol Dual is

$\max W = + 4y_1 + 6y_2 + 20y_3 + 18y_4$

Subject to

$y_1 + y_3 + 2y_4 \leq 0.7$

$y_2 + 2y_3 + y_4 \leq 0.5$

$y_1, y_2, y_3, y_4 \geq 0.$

Step 1 Express in std. form.

slackvar.

The dual problem in the std. form becomes

$\max W = 4y_1 + 6y_2 + 20y_3 + 18y_4 + 0s_1 + 0s_2$

$y_1 + 0y_2 + y_3 + 2y_4 + s_1 + 0s_2 = 0.7$

$0y_1 + y_2 + 2y_3 + y_4 + 0s_1 + s_2 = 0.5$

$y_1, y_2, y_3, y_4 \geq 0.$

Step 2 find an initial basic feasible sol.

Sets all non-basic equal to zero

$y_1 = y_2 = y_3 = y_4 = 0$ (non-basic)

$s_1 = 0.7$ } basic
$s_2 = 0.5$

Since $s_1, s_2 > 0$ the initial basic solution is feasible and non-degenerate.

	C_j	4	6	20	18	0	0		
C_B	Basis	y_1	y_2	y_3	y_4	s_1	s_2	b	θ
0	s_1	1	0	1	2	1	0	0.7	$\frac{0.7}{1}$
0	s_2	0	1	2	1	0	1	0.5	$\frac{0.5}{2}$
	Z_j	0	0	0	0	0	0	0	
	C_j	4	6	20	18	0	0		
				↑					

y_3 incoming s_2 is outgoing

	C_j	4	6	20	18	0	0		
C_B	Basis	y_1	y_2	y_3	y_4	s_1	s_2	b	θ
0	s_1	1	$-1/2$	0	$3/2$	1	$-1/2$	$9/20$	$3/10$ ←
20	y_3	0	$1/2$	1	$1/2$	0	$1/2$	$1/4$	$1/2$
	Z_j	0	10	20	10	0	10	5	
	C_j	4	-4	0	8	0	-10		
					↑				

Richa

	C_j	4	6	20	18	0	0	
C_B	Basis	y_1	y_2	y_3	y_4	s_1	s_2	b
18	y_4				1			3/10
20	y_3				0			1/10
	z_j				18			74/10
	C_j	$-4/3$	$-4/3$	0	0	$-16/13$	$-22/13$	

all $C_j \leq 0$ ∴ this table gives optimal sol

$$y_4 = \frac{3}{10} \qquad y_3 = \frac{1}{10}$$

$$\max W = 20y_3 + 18y_4$$

$$= \cancel{20}^{2} \times \frac{1}{\cancel{10}} + \cancel{18}^{9} \times \frac{3}{\cancel{10}_{5}}$$

$$2 + \frac{27}{5} = \frac{37}{5} = 7.4 \text{ Ans}$$

New value = old value − $\frac{\text{corr. col. value} \times \text{corr. row value}}{\text{key element}}$

$$1 - \frac{1 \times 2}{2}$$

$$0$$

$$0 - \frac{1 \times 1}{2}$$

Working procedure for dual simplex method

Step 1. Convert the problem to max. if it is not.

Convert $\geq$ type to $\leq$ type
express the problem in std. form by introducing slack var.

Step 2 find the initial basic sol and make dual simplex table.

Step 3 Test nature of $C_j = c_j - z_j$

if all $C_j \leq 0$ and all $b_i \geq 0$ then optimal basic feasible sol. has been attained.

if all $C_j \leq 0$ and at least one $b_i < 0$. then Cont. with step 4.

if any $C_j \geq 0$ the method fails

Step 4 Mark the outgoing var. Select the rows that contains the most $-ve$ b_i. This will be the

key ...
basic v...

Step 5 test...

if all ...
does ...

if at ...
ratios ...
rows ...
choose ...
ratio ...
the ...
associ...

Step 6 ...
feas...
elt.

as in ...
metho...
until ...
sol is ...
indica...
feas...

key rows and the corresponding basic var. is outgoing var.

Step 5 test nature of key rows elts.

if all elts are ≥ 0 the problem does not have a feasible sol.

if at least one < 0, find the ratios of the correspondig elts of C_j row to these elts.
Choose the smallest of those ratios. The corresponding col. is the key col. and the associated var. is the incoming var.

Step 6 Iterate towards optimal feasible sol. Make the key elt. unity. Perform row operations as in the regular simplex method and repeat iterations until either an optimal feasible sol is attained or there is an indication of non-existence of a feasible sol.

Q. Using dual simplex method

$\max . Z = -3x_1 - 2x_2$

Subject to.

$$x_1 + x_2 \geq 1$$
$$x_1 + x_2 \leq 7$$
$$x_1 + 2x_2 \geq 10$$
$$x_2 \leq 3$$
$$x_1, x_2 \geq 0.$$

<u>Step 1</u>

$$-x_1 - x_2 \leq -1$$
$$x_1 + x_2 \leq 7$$
$$-x_1 - 2x_2 \leq -10$$
$$x_2 \leq 3$$

Std. for

$$\max Z = -3x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$$

$$-x_1 - x_2 + s_1 = -1$$
$$x_1 + x_2 + s_2 = 7$$
$$-x_1 - 2x_2 + s_3 = -10$$
$$x_2 + s_4 = 3.$$

$$x_1, x_2, s_1, s_2, s_3, s_4 \geq 0$$

Step 2 find initial basic sol.

Setting decision var = 0.

$x_1 = x_2 = 0.$

$$\left.\begin{array}{l} s_1 = -1 \\ s_2 = 7 \\ s_3 = -10 \\ s_4 = 3 \end{array}\right\} \text{Basic Sol.}$$

	Cj	-3	-2	0	0	0	0	
C_B	Basis	x_1	x_2	s_1	s_2	s_3	s_4	b
0	s_1	-1	-1	1	0	0	0	-1
0	s_2	1	1	0	1	0	0	7
0	s_3	-1	-2	0	0	1	0	-10 ←
0	s_4	0	1	0	0	0	1	3
	z_j	0	0	0	0	0	0	
	C_j	-3	-2	0	0	0	0	
		$-3/-1=3$	↑ $-2/-2=1$					

Since all $C_j \le 0$ and $b_1 = -1$ $b_3 = -10.$
Sol. is not feasible.

Mark outgoing var.
Since b_3 is -ve and numerically largest the third row is key row

and s_3 is the outgoing var.

Calculate ratios of C_j - rows to the -ve elts of key rows

ratios → $\frac{-3}{-1} = 3$, $\frac{-2}{-2} = 1$.

(neglect ratios corresponding to +ve or zero elts of key rows)

Since smaller ratio is 1.

$\therefore$ x_2 is incoming var.

key elt is (-2)

Drop s_3 and introduce x_2.

Convert the key elt. to unity and make all other elts. of the key col. zero.

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	C_j	-3	-2	0	0	0	0	
C_B	Basis	x_1	x_2	s_1	s_2	s_3	s_4	b
0	s_1	-1/2	0	1	0	-1/2	0	4
0	s_2	1/2	0	0	1	1/2	0	2
-2	x_2	1/2	1	0	0	-1/2	0	5
0	s_4	-1/2	0	0	0	1/2	1	-2 ←
	Z_j	-1	-2	0	0	1	0	-10
	C_j	-2 ↑	0	0	0	-1	0	

$\frac{-2}{-1/2} = 4$

Since all $C_j \leq 0$ and $b_4 = -2$.

not feasible.

outgoing var. → s_4

x_1 → incoming var.

key elt → = $-1/2$

	C_j	-3	-2	0	0	0	0	b
C_B	Basis	x_1	x_2	s_1	s_2	s_3	s_4	6
0	s_1	0	0	1	0	-1	-1	0
0	s_2	0	0	0	1	1	1	3
-2	x_2	0	1	0	0	0	1	4
-3	x_1	1	0	0	0	-10	-2	-18
	Z_j	-3	-2	0	0	3	4	
	C_j	0	0	0	0	-3	-7	

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all $C_j \leq 0$ and all $b's \geq 0$.

$\therefore$ this sol. is optimal & feasib.

$x_1 = 4 \quad x_2 = 3$

$Z_{max} = -18.$

Q $\min Z = 2x_1 + 2x_2 + 4x_3$

subject to

$2x_1 + 3x_2 + 5x_3 \geq 2$

$3x_1 + x_2 + 7x_3 \leq 3$

$x_1 + 4x_2 + 6x_3 \leq 5$

$x_1, x_2, x_3 \geq 0$

$x_1 = 0$

$x_2 = 2/3$

$x_3 = 0$

$\max Z' = -4/3$

$\min Z = 4/3$

Transportation Problem

This is a type of linear programming problem in which the obj. f is to transport a single commodity from various origins to different destinations at the min. cost.

Let there be

plant locations (origins) = m
dist. centres (destinations) = n
production capacity of ith plant = a_i
no. of units req. at jth ~~plant~~ destination = b_j

transportation cost of one unit from ith plant to jth destination = c_{ij}

Our objective = is to determine the no of units to be transported so that the total transportation cost is min.

Some definitions

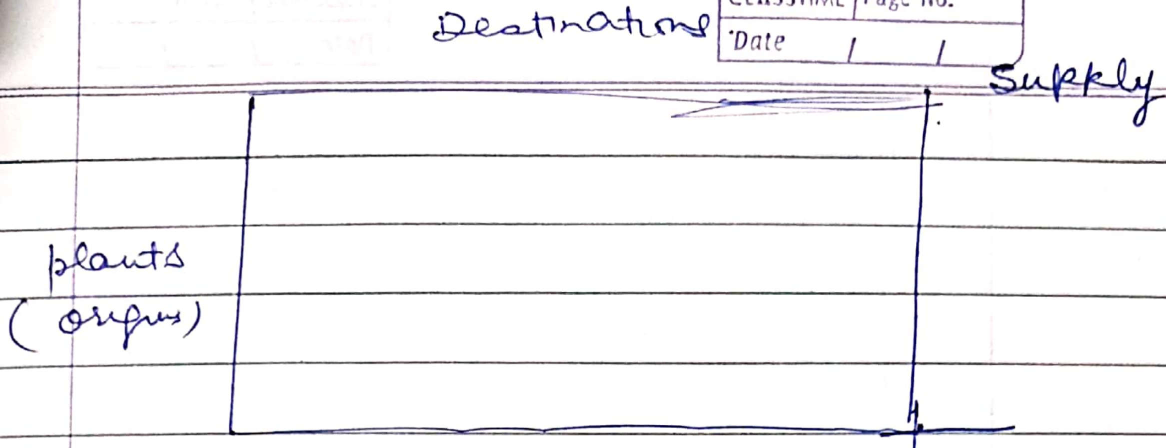
1. balanced transportation problem

if supply & demand are equal, the problem is balanced.

2. A feasible sol. to a transp. problem is said to be a basic feasible sol. if it contains at the most (m+n-1) strictly positive allocations, otherwise the sol. will degenerate.

If the total no. of +ve (non-zero) allocations is exactly (m+n-1), then the basic feasible sol. is said to be non-degenerate.

3. Optimal Sol :- A feasible sol which minimizes the transportation cost is called an optimal sol.



$\sum a_i = \sum b_j$

supply = demand

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Solve the transportation problem

		A	B	C	D		
Source	I	21	16	25	13	11	Availability (supply)
	II	17	18	14	23	13	
	III	32	27	18	41	19	
Requirement (demand)		6	10	12	15	43	

Here demand & supply being the same i.e. 43, the problem is balanced.

find the initial basic feasible solution.
Following VAM,
The difference b/w the smallest and next to the smallest costs in each row and each col. are computed and displayed with in brackets against the resp. rows and col.

find largest of those differences i.e. 10 here, associated with the 4th col.

Now find min of 4th col. i.e. 13 (min cost)
find min (11, 15) = 11
11 is placed in box of 13

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Table 1.

min cost in this col.

21	16	25	[11] 13	11 (3)
17	18	14	23	13 (3)
32	27	18	41	19 (9)
6	10	12	15	
(4)	(2)	(4)	(10) (max. difference)	

min (11, 15) = 11

Since $C_{14} = 13$ is the min cost, we allocate $x_{14} = \min(11, 15) = 11$. This exhausts the availability of first row and ∴ we cross it.

17	18	14	[4] 23	13 (3)
32	27	18	41	19 (9)
6	10	12	4	
(15)	(9)	(4)	(18) max. different	

min (15−11)

max. difference is 18 which is against the fourth col. Since C_{14} (=23) min cost we allocate min (13, 4) = 4. (col. delete)

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So we cross the fourth col & proceed in the same way.

13−4=9

17 [6]	18	14	9 (3)
32	27	18	19 (9)
6	10	12	
(15)	(9)	(4)	

max. difference is 15. in 1st col.

Since min in this col is 17 we allocate

min (6, 9) = 6. (∴ col. delete).

So we delete this col. 1st.

(9−6) min

[3] 18	14	3 (4)
27	18	19 (9)
10	12	
(9)	(4)	

max. diff is 9 corresponding to 1st col. 18 is min cost.

min (3, 10) = 3 (Row delete).

27	[12] 18	19
7	12	

Thus the initial basic feasible sol: is

21	16	25	[11] 13
[6] 17	[3] 18	14	[4] 23
32	[7] 27	[12] 18	41

12

Optimality check :- As no. of allocations $m+n-1 = 6$, we can apply Modi test.

$u_i \backslash v_j$	$v_1=17$	$v_2=18$	$v_3=9$	$v_4=23$
$u_1=-10$				
$u_2=0$				
$u_3=9$				

MODI – Modified dist method.

$u_2+v_1 = 17$

$u_2+v_2 = 18$

$u_3+v_2 = 27$

$u_3+v_3 = 18$

$u_1+v_4 = 13$

$u_2+v_4 = 23$

allocated cells = to cost of that cell

Starting initially with some $u_i = 0$.

Let $u_2 = 0$.

$\Rightarrow v_1 = 17$

$v_2 = 18$

$u_3 = 9$

$v_3 = 9$

$v_4 = 23$

$u_1 = -10$.

Net evaluations

$w_{ij} = (u_i+v_j) - c_{ij}$ for all empty cells.

$w_{11} = (u_1+v_1) - c_{11} = -10+17-21 = -14$

$w_{12} = u_1+v_2-c_{12} = -8$

$w_{13} = \quad = -26$

$w_{23} = \quad = -5$

$w_{31} = \quad = -6$

$w_{34} = \quad = -9$

Since all the net evaluations are -ve. the current sol is optimal. Hence the optimal allocation is given by

$x_{14} = 11$
$x_{21} = 6$
$x_{22} = 3$
$x_{24} = 4$
$x_{32} = 7$
$x_{33} = 12$

∴ The optimal (min) cost of transportation is

$= 11\times13 + 6\times17 + 3\times18$
$+ 4\times23 + 7\times27 + 12\times18$
$= 796$ Rs.

Q:-

	1	2	3	4
1	2	3	11	7
2	1	0	6	1
3	5	8	15	9

Determine the optimal dist. for the company so as to minimize the total transportation cost.

$u_i \backslash v_j$					
	$\boxed{1}$ 2	$\boxed{5}$ 3	11	7	6
	1	0	6	$\boxed{1}$ 1	1
	$\boxed{6}$ 5	8	$\boxed{3}$ 15	$\boxed{1}$ 9	10
	7	5	3	2	

cost
$100[1\times2 + 5\times3 + 1\times1 + 6\times5 + 3\times15 + 1\times9]\times$
$=1020$

$u_1 + v_1 = 2.$ $\quad u_1 = 0$
$u_1 + v_2 = 3.$ $\quad v_1 = 2.$
$u_2 + v_4 = 1$ $\quad v_2 = 3$
$u_3 + v_1 = 5$ $\quad u_3 = 3$
$u_3 + v_3 = 15$ $\quad v_3 = 12$
$u_3 + v_4 = 9$ $\quad v_4 = 6$
$u_2 = -5$

$u_i \backslash v_j$	$v_1=2$	$v_2=3$	$v_3=12$	$v_4=6$
$u_1=0$				
$u_2=-5$				
$u_3=3$				

Net evaluations:—

$w_{ij} = (u_i + v_j) - c_{ij}$

$w_{13} = u_1 + v_3 - C_{13}$

$w_{14} = u_1 + v_4 - C_{14}$

$w_{21} = u_2 + v_1 - C_{21}$

$w_{22} = u_2 + v_2 - C_{22}$

$w_{23} = u_2 + v_3 - C_{23}$

$w_{32} = u_3 + v_2 - C_{32}$

$w_{13} = 12 - 11 = 1$

$w_{14} = 6 - 7 = -1$

$w_{21} = -3 - 1 = -4$

$w_{22} = -2 - 0 = -2$

$u_{23} = 7 - 6 = 1$

$w_{32} = 6 - 8 = -2$

u_i \ v_j	2	3	12	6
0	$\boxed{1}$ 2	$\boxed{5}$ 3	(+) 11	(−) 7
−5	(−) 1	(−) 0	(+) 6	$\boxed{1}$ 1
3	$\boxed{6}$ 5	(−) 8	$\boxed{3}$ 15	$\boxed{1}$ 9

$\left.\begin{matrix}\sqcup\\ \sqcup\end{matrix}\right\}$ Basic

Next basic feasible sol:—

→ Choose the unoccupied cell with the max. w_{ij}. In case of a tie select the one with lower original cost.

cell (1,3) and cell (2,3) each is 1
(2,3) has lower original cost 6
∴ we take this as next basic cell and note θ in it.

→ Draw a closed path begining and ending at θ-cell.

add and subtract θ, alternately to and from the transition cells of the loop subject the rim requ.

Assign a max. value to θ so that one basic var. becomes zero, and other basic variables remain ≥ 0.

Now the basic cell whose allocation has been reduced to zero, leaves the basic

This gives 2nd basic feasible sol.

[1] 2	[5] 3	11	7
1	0	θ 6	[1] −θ 1
[6] 5	8	[3] −θ 15	[1] +θ 9

[1] 2	[5] 3	11	7
1	0	[θ=1] 6	[1−1] 1
[6] 5	8	[3−1] 15	[1+1] 9

∴ Transportation cost of this sol is

$= [1\times2 + 5\times3 + 1\times6 + 6\times5 + 2\times15 + 2\times9]\times100 = 10100$ Rs.

$u_i \backslash v_j$	2	3	12	6
0	[1] 2	[5] 3	(+) 11	(−) 7
−6	(−) 1	(−) 0	[1] 6	(−) 1
3	[6] 5	(−) 8	[2] 15	[2] 9

$u_1+v_1=2 \quad u_1+v_2=3$

$u_2+v_3=6 \quad u_3+v_1=5$

$u_3+v_3=15 \quad u_3+v_4=9$

for previous table

$u_1=0 \quad v_1=2$

$u_2=-6 \quad v_2=3$

$u_3=3 \quad v_3=12$

$v_4=6$

$w_{13}=u_1+v_3-c_{13}=12-11=1$

$w_{14}=u_1+v_4-c_{14}=6-7=-1$

$w_{21}=u_2+v_1-c_{21}=-4-1=-5$

$w_{22}=u_2+v_2-c_{22}=-3-0=-3$

$w_{24}=u_2+v_4-c_{24}=0-1=-1$

$w_{32}=u_3+v_2-c_{32}=6-8=-2$

Now $w(1,3)$ is +ve procedure.

[1−1] −θ 2	[5] 3	[θ=1] θ 11	7
1	0	[1] 6	1
[6+1] θ 5	8	[2−1] −θ 15	[2] 9

Now drop the cell (1,1). Thus next basic feasible sol is in next table.

2	[5] 3	[1] 11	7
1	0	[1] 6	1
[7] 5	8	[1] 15	[2] 9

→

$u_i \backslash v_j$	1	3	11	5
0	(−) 2	[5] 3	[1] 11	(−) 7
−5	(−) 1	(−) 0	[1] 6	(−) 1
1	[7] 5	(−) 8	[1] 15	[2] 9

optimality check.

min optimal transportation cost $= [5\times3 + 1\times11 + 1\times6 + 7\times5 + 1\times15 + 2\times9]\times100 = 10{,}000$

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Assignment Problem :-

An assignment problem is a special type of transportation problem in which the objective is to assign a no. of origins to an equal no. of destinations at a min cost.

Q:- Four jobs are to be done on four different machines. The cost of producing ith job on the jth machine is given below.

		Machines		
	M_1	M_2	M_3	M_4
J_1	15	11	13	15
Jobs J_2	17	12	12	13
J_3	14	15	10	14
J_4	16	13	11	17

Assign the jobs to different machines so as to minimize the total cost.

Subtract smallest elts of from all rows.

4	0	2	4
5	0	0	1
4	5	0	4
5	2	0	6

Col. 1 & 4 donot contain zero. Subtract smaller elt of each from ~~4 cost~~ both cols.

0	0	2	3
1	0	0	0
0	5	0	3
1	2	0	5

Row 4 has single unmarked zero in col. 3. Encircle it and cross all other zeros in col. 3.

0	0	2	3
1	0	$\cancel{0}$	0
0	5	$\cancel{0}$	3
1	2	0	5

Row 3 has single unmarked zero in col 1. encircle it and cross others.

$\cancel{0}$	0	2	3
1	0	$\cancel{0}$	0
0	5	$\cancel{0}$	3
1	2	0	5

Row 1 has single unmarked zero in col 2

$\cancel{0}$	0	2	3
1	$\cancel{0}$	$\cancel{0}$	0
0	5	$\cancel{0}$	3
1	2	0	5

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finally rows 2 in col 4.

	M_1	M_2	M_3	M_4
J_1	$\cancel{0}$	0	2	3
J_2	1	$\cancel{0}$	$\cancel{0}$	0
J_3	0	5	$\cancel{0}$	3
J_4	1	2	0	5

Since we have one encircled zero in each rows and each col, this gives the optimal sol.

∴ The optimal assignment policy is
job1 to machine 2
job 2 to - - - 4
job 3 - - - - 1
job4 - - - 3

min assignment cost = 11+13+14+11
= 49 Rs.

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Assignment

An assignment problem is a particular case of transportation problem in which a no. of operations are to be assigned to an equal no. of operators, where each operator performs only one operation. The objective is to maximize overall profit or minimize overall cost for a given assignment schedule.

Difference between a transportation and an assignment problem.

An assignment problem is a special case of transportation problem in which $m=n$, all the a_i and b_j are unity, and each x_{ij} is limited to either '0' or '1'.

$$\text{Min } Z = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} c_{ij}$$

s.t. $\sum_{i=1}^{n} x_{ij} = 1, \quad j = 1, 2, \ldots n$

$\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, 2 \ldots n$

∵ each operator performs only one operation at a time.

Hungarian method for assignment Problem

1. Prepare a square matrix. If not, make it square by adding suitable no. of dummy rows (or columns) with zero cost element.
2. Subtract the smallest element in each row from every element of that row.

3 Repeat step 2 in each column. ②

4 Examine one by one each row and put a box '□' to cell containing '0' to rows having single zero. Cross all other zeros, in the column in which the assignment has been made.

5 Repeat the procedure for columns.

6 If all 'n' assignments have been made, stop. otherwise go to step 7. Draw horizontal and vertical lines covering all zeros by following rules.

7 (1) Mark (✓) to rows having no assignment and
(2) mark (✓) to columns having '$\emptyset$' in marked row.
(3) mark (✓) to rows having an assignment in marked column.
(4) Draw horizontal and vertical lines to unmarked rows and marked columns.

8 Select the smallest element among all uncovered elements and then subtract this value from all elements in the matrix not covered by lines and add this value to all elements that lie at the intersection of the two lines.

9 Repeat the procedure, untill optimal solution is obtained.

Type I.

There are 4 machines and 4 operaters with their respective Charges as shown. Assign one operator to each machine, so that overall payment is minimum.

	I	II	III	IV
1	6	7	7	8
2	7	8	9	7
3	8	6	7	6
4	8	7	6	9

↓ Step-2
Step-3
Step-4

	I	II	III	IV
1	$\boxed{0}$	1	1	2
2	$\cancel{0}$	1	2	$\boxed{0}$
3	2	$\boxed{0}$	1	$\cancel{0}$
4	2	1	$\boxed{0}$	3

∴

operator	Machine	Charges
1	I	6
2	IV	7
3	II	6
4	III	6

∴ Total min. cost = Rs. 25/—

Type II. (4)

Q2 Find the optimal assignment for the problem with the following cost matrix.

	I	II	III	IV
A	5	3	1	8
B	7	9	2	6
C	6	4	5	7
D	5	7	7	6

Sol Step 2

4	2	0	7
5	7	0	4
2	0	1	3
0	2	2	1

2

Step 3
Step 4
Step 5
Step 6
Step 7

4	2	$\boxed{0}$	6	3
5	7	$\cancel{0}$	3	1.
2	$\boxed{0}$	1	2	
$\boxed{0}$	2	2	$\cancel{0}$	

Step 8 min {4, 2, 6, 5, 7, 3} = 2

2	$\cancel{0}$	$\cancel{0}$	4	1
3	5	$\boxed{0}$	1	3
2	$\boxed{0}$	3	2	3
$\boxed{0}$	2	4	$\cancel{0}$	

2 2

min (2, 4, 3, 1, 2, 2) = 1

1	$\cancel{0}$	$\boxed{0}$	3
2	5	$\cancel{0}$	$\boxed{0}$
1	$\boxed{0}$	3	1
$\boxed{0}$	3	5	$\cancel{0}$

optimal assignment

	min cost
A → III	1.
B → IV	6.
C → II	4
D - I	5
	Rs 16/-

④ Problems having alternate solutions Type III ⑤

Assign 4 trucks for 3 destinations so that distance travelled is minimized. The matrix shows the distance in Km.

	T_1	T_2	T_3	T_4
A	18	24	28	32
B	8	13	17	19
C	10	15	19	22
Dummy Dest	0	0	0	0

Note: ∴ Rows < columns, a dummy destination has been added.

②				
$\boxed{0}$	6	10	14	③
$\cancel{0}$	5	9	11	①
$\cancel{0}$	5	9	12	①
$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	

min = 5

②	②			
$\boxed{0}$	1	5	9	③
$\cancel{0}$	$\boxed{0}$	4	6	③
$\cancel{0}$	$\cancel{0}$	4	7	①
5	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	

min = 4

$\boxed{0}$	1	1	5
$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	2
$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	3
9	4	$\cancel{0}$	$\boxed{0}$

Select a zero arbitrarily in dotted lines ∴ optimal assignment is

	Distance
$A \to T_1$	18
$B \to T_2$	13
$C \to T_3$	19
	50 Km

Alternate assignment is

	Distance
$A \to T_1$	18
$B \to T_3$	17
$C \to T_2$	15
	50 Km

Type. IV

Restrictions on Assignment

Places

	A	B	C	D	E
m_1	4	6	10	5	6
machines m_2	7	4	–	5	4
m_3	–	6	9	6	2
m_4	9	3	7	2	3

For some reasons m_2 cannot be assigned to C and m_3 can not be assigned to A.

Adding dummy machine and assigning cost ∞ to restricted places

4	6	10	5	6
7	4	∞	5	4
∞	6	9	6	2
9	3	7	2	3
0	0	0	0	0

↓

$\boxed{0}$	2	6	1	2
3	$\boxed{0}$	∞	1	$\cancel{0}$
∞	4	7	4	$\boxed{0}$
7	1	5	$\boxed{0}$	1
$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$

optimal assignment has been made.

	Cost
$m_1 \to A$	4
$m_2 \to B$	4
$m_3 \to E$	2
$m_4 \to D$	2
	Rs 12/–

Type 5

Maximization Problem

A company has a team of 4 salesmen and there are 4 districts. The company estimates that profit per day in rupees that each salesman can yield is

	1	2	3	4
A	16	10	14	11
B	14	11	15	15
C	15	15	13	12
D	13	12	14	15

Find the assignment of salesmen to various districts which will yield maximum profit.

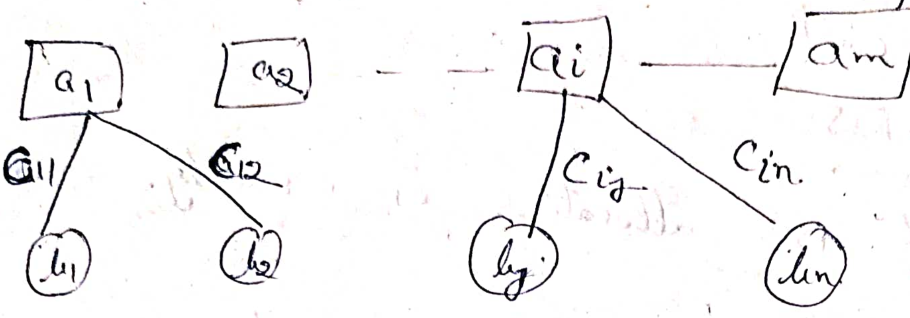
Sol To apply Hungarian method $\max z = -\min z'$, $z' = -z$

$$\begin{bmatrix} -16 & -10 & -14 & -11 \\ -14 & -11 & -15 & -15 \\ -15 & -15 & -13 & -12 \\ -13 & -12 & -14 & -15 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{0} & 6 & 2 & 5 \\ 1 & 4 & \boxed{0} & \cancel{0} \\ \cancel{0} & \boxed{0} & 2 & 3 \\ 2 & 3 & 1 & \boxed{0} \end{bmatrix}$$

optimal assignment is

	Profit
A-1	16
B-3	15
C-2	15
D-4	15
	61/-

Transportation



Sources $i = 1, \ldots m$

$a_i \Rightarrow$ availability in i^{th} source

$b_j \Rightarrow$ Requirement in j^{th} destination.

$c_{ij} \Rightarrow$ Cost of transportation from i^{th} source to j^{th} destination.

The problem is to determine non-negative values of 'x_{ij}' so as to

	D_1	D_2	D_3	—	—	D_n	
S_1	c_{11}	c_{12}	c_{13}	c_{14}		c_{1n}	a_1
S_2							
S_m	c_{m1}	c_{m2}	c_{m3}	c_{m4}		c_{mn}	a_m
	b_1	b_2	—			b_n	

$$\min z = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} c_{ij}$$

s.t.
$$\sum_{j=1}^{n} x_{ij} = a_i \quad \text{for } i = 1, 2, \ldots m$$

$$\sum_{i=1}^{m} x_{ij} = b_j \quad \text{for } j = 1, 2, \ldots n$$

Balanced Transportation Problem.

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

Definitions

Basic Feasible Solution :- A feasible solution of an $m \times n$ transportation problem is said to be B.F.S. if the total no. of +ve allocations x_{ij} is exactly equal to $m+n-1$.

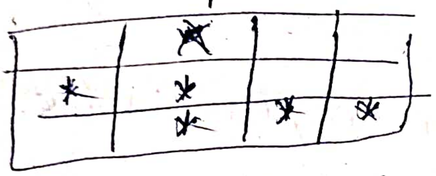
Optimal Solution :- If it minimizes the total transportation cost

Non-degenerate B.F.S. :-

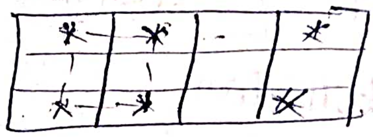
(i) total no. of +ve allocations is exactly equal to $(m+n-1)$.

(ii) these allocations are in independent positions.

Independent positions



Non-Independent positions



Initial basic feasible solution.

I North West Corner Rule

Working Rule

(i) Maximum possible amount is allocated in the cell (1,1) of the transportation table.

(ii) Move to right hand cell (1,2) if there is still any available quantity left otherwise move to down cell (2,1).

(iii) Repeat step (2) again and continue untill all the available quantity is exhausted.

N, W, E, S

Q

From \ To	W_1	W_2	W_3	Availability
F_1	[2] 5	[7] ×	[4] ×	~~5~~
F_2	[3] 2	[3] 6	[1] ×	~~8~~
F_3	[5] ×	[4] 3	[7] 4	~~7~~
F_4	[1] ×	[6] ×	[2] 14	14
Requirement	~~7~~ ~~2~~	~~9~~ ~~3~~	18	34

$$T.C = 5(2)+2(3)+6(3)+3(4)+4(7)+14(2) = 102$$

Lowest Cost Entry Method.

We choose the cell with lowest cost and allocate as much as possible. If such cell of lowest cost is not unique, we select the cell where we can allocate more amount.

To

From	W_1	W_2	W_3	
F_1	[2] ×	[1] 2	[4] 3	5
F_2	[3] ×	[3] ×	[1] 8.	~~8~~
F_3	[5] ×	[4] 7	[7] ×	~~7~~
F_4	[1] 7	[6] ×	[2] 7	~~14~~ ~~7~~
	7	9/2	~~18~~/~~10~~ 3	34

T.C = Rs 831 —

2(1) + 3(4) + 8(1) + 7(4) + 7(2) + 7(1)

Method - III

Unit Cost Penalty Method (Vogal Approximation Method) VAM.

(1) Write difference of the smallest and second smallest cost in each row and column below the corresponding column and on right of corresponding row

(2) Row or column having this largest penalty is allocated by maximum possible amount to the cell with lowest cost. In case of tie, choose arbitrarily or preferably where you can allocate more amount or min cost cell among tied ro…

(3) Rewrite the lowest cost differences in rows and columns and make the allocations.

(4) Continue till all allocations are made.

Note :- Final T.C. may vary when allocations are made arbitrarily in a tie.

			Supply	
[2] 3	[1] 2	[4] ×	5	(2) (2)(3)(
[3] ×	[3] ×	[1] 8	~~8~~	(2) ×
[5] ×	[4] 7	[7] ×	7	(1)(1)(1)(1)
[1] 4	[6] ×	[2] 10	~~14~~ 4	(1)(1)(5)
Demand ~~7~~ ~~3~~ (1)	~~9~~ 7 (1)	~~18~~ 10 (1)		
(1)	(2)	(2)		
(1)	(2)	×		
(3)	(3)			

$$T.C. = 3(2) + 2(7) + 8(1) + 7(4) + 4(1) + 10(2)$$
$$= Rs\,80/-$$

Or By VAM

[2] 5		
	[3] 2	[1] 6
	[4] 7	
[1] 2		[2] 12

T.C = Rs 76/-

Transportation Method or MODI (Modified Distribution) method for an optimal solution of a transportation problem.

S-1 Find an initial B.F.S. using any appropriate method, preferably VAM.

S-2 Find set of numbers u_i ($i = 1, 2 - - m$) and v_j ($j = 1, 2 - - n$) for each occupied cell (r, s) s.t. $C_{rs} = u_r + v_s$.

S-3 Enter C_{ij} for each unoccupied cell (i, j) in the upper left corner, and enter $(u_i + v_j)$ for same cell at upper right corner.

S-4. Enter $d_{ij} = C_{ij} - (u_i + v_j)$ for each unoccupied cell (i, j) at lower right corner.

S-5 Examine d_{ij} for unoccupied cells.

(i) If all $d_{ij} > 0$, solution is optimal and unique.

(ii) If all $d_{ij} \geq 0$, solution is optimal and an alternative optimal solution exists.

(iii) If at least one $d_{ij} < 0$, solution is not optimal.

S-6 Form a new B.F.S by giving maximum allocation to cell for which d_{ij} is most -ve by forming a loop with occupied cells and vaccating one of them. (minimum of those where value is to subtract)

S-7 Repeat step 2 to 5, till optimal solution is attained.

Q B.F.S. by V.A.M. of previous Question.

S-2

[2] 3	[7] 2		$1 = u_1$
		[1] 8	$-1 = u_2$
	[4] 7		$-2 = u_3$
[1] 4		[2] 10	$0 = u_4$
$V_1 = 1$	$V_2 = 6$	$V_3 = 2$	

Calculation of u_i and v_j's using I.B.F.S

degen... generac...

S-3

S-4

0	0	[4] [3] [1]	1
[3] [0] [3]	[3] [5] 0 [-2]	0	-1
[5] [-1] [6]	0	[7] [0] [7]	-2
0	[6] [6] [0]	0	0
1	6	2	

S-5 $d_{22} = -2 < 0$ ∴ Solution is not optimal.

S-6 (1,1) (3+2) (1,2) (2-2) (2,2) 0+2 (2,3) (8-2) (4,1) (4-2) (4,3) (10+2)

S-7 The New B.F.S is

[2] 5			2
	[3] 2	[1] 6	0
	[4] 7		1
[1] 2		[2] 12	1
0	3	1	

New cost = 10 + 6 + 6 + 28 + 2 + 24 = Rs 76/-

	[7] [5] [2]	[4] [3] [1]	2
[3] [0] [3]			0
[5] [1] [4]		[7] [2] [5]	1
	[6] [4] [2]		1
0	3	1	

All $d_{ij} > 0$ ∴ solution is optimal.

Degeneracy In Transportation Problems

Degeneracy occurs in transportation problems, whenever no. of occupied cells is less than m+n-1.

Degeneracy in transportation problems can occur in two ways.

1. Basic feasible solution may be degenerate from initial stage onward.
2. They may become degenerate at any intermediate stage.

Resolution of Degeneracy During Initial Stage.

Rule: An extremely small quantity usually denoted by Δ or ϵ, is introduced in the least cost independent cell. If necessary, two or more Δ's can be introduced in the least and second least cost independent cells.

Q. Determine your transportation plan for the given table. Give minimum distribution cost.

	X	Y	Z	
A	8	7	3	60
B	3	8	9	70
C	11	3	5	80
	50	80	80	210

Solving Using V.A.M

	X	Y	Z	
A	[8] ×	[7] ×	[3] 60	60 (4) (4)
B	[3] 50	[8] ×	[9] 20	~~70~~ 20 (5) (1)
C	[11] ×	[3] 80	[5] ×	80 (2) (2)
	50 (5) ×	~~80~~ (4) (4) ×	80 (2) (2)	

Total no. of occupied cells = 4 < 5 (m+n−1)

∴ Allocating Δ to least cost independent empty cell (3,3) (loop should not be formed)

And checking for optimality.

•	•	[3] 60	$-2 = u_1$
[3] 50		[9] 20	$4 = u_2$
	[3] 80	[5] Δ	$0 = u_3$
$v_1 = -1$	$v_2 = 3$	$v_3 = 5$	

To check d_{ij}:

[8] [−3] [11]	[7] [1] [6]	•	−2
•	[8] [7] [1]	•	4
[11] [−1] [12]	•	•	0
−1	3	5	

Note:- Δ plays only an auxiliary role and has no significance. It is just introduced to find values of u_i and v_j which otherwise is not possible to find.

All $d_{ij} > 0$ ∴ Given solution is optimal.

min cost = 60(3) + 50(3) + 20(9) + 80(3)

= Rs 750/—

...tion of Degeneracy During Solution Stage:

happens when most favourable quantity is allocated to the empty cell having largest negative cell-evaluation, simultaneously 2 or more currently occupied cells are vacated.

To resolve degeneracy, allocate δ to one or more of recently vacated cell, so that no. of occupied cells is $m+n-1$ in new solution.

Unbalanced Transportation Problem.

If $\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$

(1) $\sum a_i > \sum b_j$

Add a dummy destination.

(2) $\sum b_j > \sum a_i$

Add a dummy source.

Example Find the minimum cost transportation schedule.

	A	B	C	Supply
F1	26	32	28	6
F2	19	27	16	9
F3	39	21	32	7
F4	18	24	23	5
Demand	8	7	9	

$\sum a_i = 27$, $\sum b_j = 24$

Supply > Demand $\therefore$ Create a dummy destination D having cost '0' in every cell.

	A	B	C	D	
F1	26	32	28	0	6
F2	19	27	16	0	9
F3	39	21	32	0	7
F4	18	24	23	0	5
	8	7	9	3	27

I.B.F.S. (initial basic feasible solution by VAM)

	A	B	C	D	
F1	[26] 3	[32] ×	[28] ×	[0] 3	~~6~~ 3 (26) ~~(2)~~ (2) (2)
F2	[19] ×	[27] ×	[16] 9	[0] ×	~~9~~ (16) (3) (3)
F3	[39] ×	[21] 7	[32] ×	[0] ×	~~7~~ (21) (11) ×
F4	[18] 5	[24] ×	[23] ×	[0] ×	~~5~~ (18) (5) (5)
	~~8~~ (1) (1) (1)	~~7~~ (3) (3) ×	~~9~~ (7) (7) (7)	~~3~~ (0) ×	

m+n−1 = 7 occupied = 5

∴ allocating Δ to F2 D and F3 (D) (Do not allocate to F4 D ∵ of loop formation)

	A	B	C	D	
F1	[26] 3			[0] 3	$0 = u_1$
F2			[16] 9	[0] Δ	$0 = u_2$
F3		[21] 7		[0] Δ	$0 = u_3$
F4	[18] 5				$-8 = u_4$
	$v_1 = 26$	$v_2 = 21$	$v_3 = 16$	$v_4 = 0$	