VECTORS

Scalar quantity:-

The physical quantities which have magnitude but not direction are called scalar quantities or scalar. Scalar quantities can be added or substracted according to the rule of algebra.

Example:- Mass, time, distance, speed, density, energy, temperature, pressure, charge, gravitational potential, electric potential energy etc.

Vector quantity:-

The physical quantities which have both magnitude and direction are called vector quantity or vectors. Vector quantities can be added or substracted according to the rule of vector additions.

Example:- displacement, velocity, acceleration, force, area, weight, electric field weight, magnetic field, gravitational field, momentum, torque. etc.

A Vector is graphically represented by a straight line with arrow at one end. The direction of arrow represents the direction of the vector and length of the line represents the magnitude of the vector. A vector is represented in figure below:

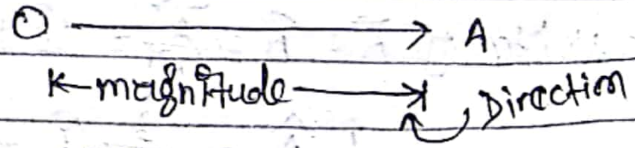

fig: vector

Q. Is it necessary that a quantity having both magnitude and direction be always a vector?

A Soln NO, for a quantity to be vector it should not be added or substracted according to the rule of algebra. But even current has both magnitude and direction, it can be added according to rule of algebra. So, it is a scalar quantity.

Types of Vectors:

(1) Unit Vector:-

A vector having unit magnitude is called unit vector. It is denoted by $\hat{A}$ and given by:

$$\hat{A} = \frac{\vec{A}}{|\vec{A}|} = \frac{\vec{A}}{A}$$ where $A = |\vec{A}|$ = magnitude of vector $\vec{A}$.

It is generally used to indicate the direction of vector.

* Note: If vector $\vec{A} = \hat{i}x + \hat{j}y + \hat{k}z$ then magnitude of vector $\vec{A}$ is denoted by $|\vec{A}|$ or A and given by:

$$|\vec{A}| = A = \sqrt{x^2+y^2+z^2}$$

Q.1) Find the unit vector of $\vec{A} = 3\hat{i} + 4\hat{j} + 5\hat{k}$

A Soln Given; $\vec{A} = 3\hat{i} + 4\hat{j} + 5\hat{k}$

Then, $|\vec{A}| = A = \sqrt{3^2+4^2+5^2} = \sqrt{50} = 5\sqrt{2}$

So, unit vector of $\vec{A} = \hat{A} = \frac{\vec{A}}{|\vec{A}|} = \frac{3\hat{i}+4\hat{j}+5\hat{k}}{5\sqrt{2}}$

$$\therefore \hat{A} = \frac{3}{5\sqrt{2}}\hat{i} + \frac{2\sqrt{2}}{5}\hat{j} + \frac{1}{\sqrt{2}}\hat{k}$$ #

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II) Null vector:-

A vector which has direction but no magnitude (ie. magnitude is zero) is called null Vector. The initial point and terminal point of a null vector coninside (i.e. Same).

III) parallel Vector:-

Two or more vectors having Same direction are called parallel vectors.

$$\xrightarrow{\vec{A}}$$
$$\xrightarrow{\vec{B}}$$

In parallel Vector, the angle between two vectors is zero.

IV) Antiparallel vectors:-

Two or more Vectors having opposite direction are called antiparallel Vectors.

$$\xrightarrow{\vec{A}}$$
$$\xleftarrow{\vec{B}}$$

In antiparallel vector, the angle between 2 vectors is 180°.

V) Equal vectors:

The vectors having Same magnitude and direction are called equal Vectors.

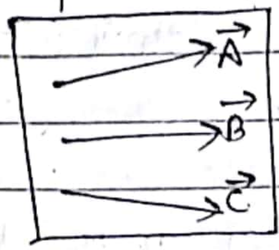
$$\vec{A} \longrightarrow$$
$$\vec{B} \longrightarrow \qquad |\vec{A}| = |\vec{B}|$$

VI) Negative vectors: The vectors having Same ~~direction~~ magnitude but acting in opposite direction are called negative vectors.

$$\xrightarrow{\vec{A}}$$
$$\xleftarrow{\vec{B}}$$

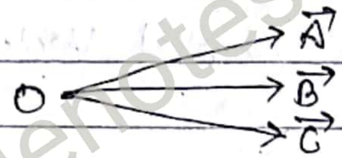
VII) Coplanar vectors:-

Two or more vectors lying on the same plane are called coplanar vectors.



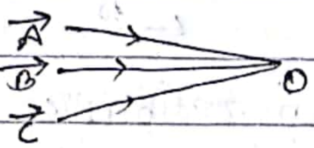
VIII) Co-initial vectors:-

Those vectors whose initial points are same, such vectors are called co-initial vectors.



IX) Co-terminal vectors:

Those vectors whose end points are same, such vectors are called co-terminal vectors.



Imp # Addition of vectors:

(1) Triangle Law of vector Addition:

Its takes that, "If two vectors in magnitude and direction are represented by two sides of triangle taken in order then third side of the triangle represented resultant vector in opposite order."

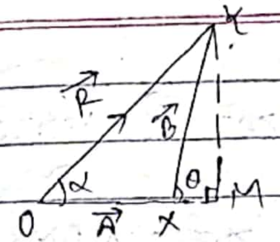


fig: Triangle law of vector addition.

Let; two vectors $\vec{A}$ and $\vec{B}$ are represented by two sides OX and XY of triangle OXY taken in order and their resultant vector $\vec{R}$ is represented by third side OY taken in opposite order.

* Magnitude

To find magnitude, let us produce OX and draw perpendicular on it at M from point Y.

Now, In right angle, ΔMOY,

$$OY^2 = OM^2 + YM^2$$

$$OY^2 = (OX + XM)^2 + YM^2$$

$$R^2 = (A + XM)^2 + YM^2 \quad \text{—— (i)}$$

Again, In Right angle ΔXYM,

$$\sin\theta = \frac{YM}{XY} \quad \text{and,} \quad \cos\theta = \frac{XM}{XY}$$

or, $XM = XY\cos\theta$

or, $YM = XY\sin\theta = B\sin\theta$ —— (ii) or, $XM = B\cos\theta$ —— (iii)

using eqn (ii) and (iii) in eqn (i),

we get;

$$R^2 = (A + B\cos\theta)^2 + (B\sin\theta)^2$$

$$= A^2 + 2AB\cos\theta + B^2\cos^2\theta + B^2\sin^2\theta$$

$$= A^2 + 2AB\cos\theta + B^2(\sin^2\theta + \cos^2\theta)$$

$$= A^2 + 2AB\cos\theta + B^2$$

$R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$ —— (iv) This eqn (iv) gives the magnitude of the resultant of vectors $\vec{A}$ and $\vec{B}$.

* **Direction:-**

Let the resultant $\vec{R}$ makes angle α with vector $\vec{A}$.

Now, In Right angle $\triangle OMX$,

$$\tan\alpha = \frac{XM}{OM} = \frac{XM}{OX+XM} = \frac{XM}{A+XM} \quad \text{---(V)}$$

using eq^n (ii) & (iii) in eq^n (v), we get;

$$\tan\alpha = \frac{B\sin\theta}{A+B\cos\theta}$$

$$\therefore \alpha = \tan^{-1}\left[\frac{B\sin\theta}{A+B\cos\theta}\right] \quad \text{---(vi)}$$

eq^n (vi) gives the direction of resultant $\vec{R}$ of vector $\vec{A}$ & $\vec{B}$ with $\vec{A}$.

If the resultant $\vec{R}$ makes angle α with vector $\vec{B}$, then,

$$\boxed{\alpha = \tan^{-1}\left(\frac{A\sin\theta}{B+A\cos\theta}\right)}$$

(2) **Parallelogram law of vector addition:-**

It states that, "If two vectors in magnitude and direction are represented by two adjacent side of a parallelogram then resultant is represented by diagonal passing through their point of intersection both in magnitude and direction."

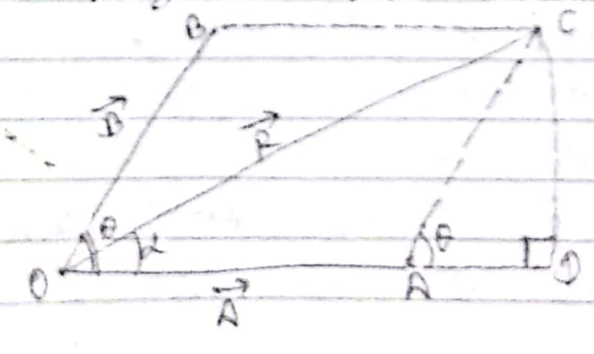


fig: parallelogram law of vector addition.

Let two vectors $\vec{A}$ and $\vec{B}$ are represented by two adjacent side OA and OB of a parallelogram OABCD respectively. Then according to parallelogram law, resultant $\vec{R}$ is represented by diagonal OC as shown in figure above.

* **Magnitude:**

To find magnitude, Let us produce OA and draw a perpendicular on it at D from C.

Now, In Right angle $\triangle ODC$,

$$OC^2 = OD^2 + DC^2 = (OA+AD)^2 + DC^2$$

$$\text{or, } R^2 = (A+AD)^2 + DC^2 \quad \text{---} \text{(i)}$$

Again; In Right angle $\triangle ADC$;

$$\sin\theta = \frac{DC}{AC} \qquad \text{and} \qquad \cos\theta = \frac{AD}{AC}$$

$$\text{or, } AD = AC\cos\theta = B\cos\theta \quad \text{---} \text{(iii)}$$

$$\text{or, } DC = AC\sin\theta = B\sin\theta \quad \text{---} \text{(ii)}$$

using eqn (ii) and (iii) in eqn (i), we get,

$$R^2 = (A+B\cos\theta)^2 + (B\sin\theta)^2$$

$$= A^2 + 2AB\cos\theta + \cos^2\theta + B^2\sin^2\theta$$

$$= A^2 + B^2(\cos^2\theta + \sin^2\theta) + 2AB\cos\theta$$

$$\text{or, } R^2 = A^2 + B^2 + 2AB\cos\theta \quad \text{(iv)}$$

$$\therefore R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$$

This eqn (iv) gives the magnitude of the resultant of vectors $\vec{A}$ & $\vec{B}$

* **Direction:**

Let the resultant $\vec{R}$ makes angle α with vector $\vec{A}$.

Now; In Right angle AOC,

$$\tan\alpha = \frac{CD}{OD} = \frac{CD}{OA+AD} = \frac{CD}{A+AD} \quad \text{---(v)}$$

using eqn (ii) and (iii) in eqn (v) we get;

$$\tan\alpha = \frac{B\sin\theta}{A+B\cos\theta}$$

$$\therefore \alpha = \tan^{-1}\left[\frac{B\sin\theta}{A+B\cos\theta}\right] \quad \text{---(vi)}$$

This eqn (vi) gives the direction of resultant $\vec{R}$ of vectors $\vec{A}$ and $\vec{B}$ with $\vec{A}$.

If the resultant $\vec{R}$ makes angle α with vector $\vec{B}$ then;

$$\alpha = \tan^{-1}\left[\frac{A\sin\theta}{B+A\cos\theta}\right]$$

Special Cases:-

{Case I:} when two vectors acts parallel, i.e. $\theta = 0°$

then, $R = \sqrt{A^2+B^2+2AB\cos\theta} = \sqrt{A^2+B^2+2AB} = \sqrt{(A+B)^2} = A+B$

and

$$\tan\alpha = \frac{B\sin 0°}{A+B\cos 0°} = 0 \qquad \therefore \alpha = 0°$$

Thus, when two vectors acts parallel (i.e. $\theta = 0°$), then the resultant has magnitude equal to the sum of magnitude of 2 vectors and direction along the direction of $\vec{A}$ & $\vec{B}$.

{Case II}:-

☆ When two vectors acts perpendicularly; i.e. $\theta = 90°$

then, $R = \sqrt{A^2+B^2+2AB\cos\theta}$

$= \sqrt{A^2+B^2}$

and,

$\tan\alpha = \frac{B\sin 90°}{A+B\cos 90°} = \frac{B(1)}{A+B(0)} = \frac{B}{A}$ $\therefore \alpha = \tan^{-1}\left(\frac{B}{A}\right)$.

{Case III}:-

☆ When two vectors acts anti-parallel; i.e. $\theta = 180°$

then, $R = \sqrt{A^2+B^2+2AB\cos 180°}$

$= \sqrt{A^2+B^2-2AB}$

$= \sqrt{(A-B)^2}$

$\therefore R = A-B$

If $A=B$ then, $R=0$, i.e.

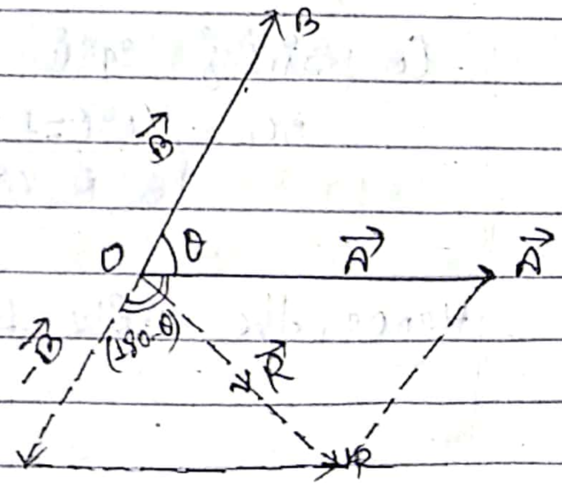
thus, When two vectors are equal and acting antiparallel then their resultant must be zero.

and,

$\tan\alpha = \frac{B\sin 180°}{A+B\cos 180°} = \frac{B(0)}{A+B(-1)} = \frac{0}{A-B} = 0$

$\boxed{\therefore \alpha = 0°}$

Substraction of Vectors:-



Now

$\vec{R} = \vec{A} + (-\vec{B})$

$= \sqrt{A^2+B^2+2AB\cos(180°-\theta)}$

$= \sqrt{A^2+B^2-2AB\cos\theta}$

$$\boxed{\therefore \vec{R} = \vec{A} - \vec{B} = \sqrt{A^2+B^2-2AB\cos\theta}}$$ #

Q.1[A] Two vectors $\vec{A}$ & $\vec{B}$ Such that $\vec{R}=\vec{A}+\vec{B}$ and $R^2=A^2+B^2$. What is angle between $\vec{A}$ & $\vec{B}$?

Soln Given;

$\vec{R}=\vec{A}+\vec{B}$

or, $R^2=A^2+B^2+2AB\cos\theta$ —— (i)

Also;

$R^2=A^2+B^2$ —— (ii)

Comparing eqn (i) and (ii),

$2AB\cos\theta=0$

or, $\cos\theta=0=\cos 90°$

$\boxed{\therefore \theta=90°}$

Hence, the angle between two vector is 90°. #

Q.1[B] Two Vectors $\vec{A}$ & $\vec{B}$ are Such that $\vec{R}=\vec{A}+\vec{B}$ and $R=A+B$. find angle between them?

Soln Given;

$R^2=\vec{A}+B^2$

or, $R^2=A^2+B^2+2AB\cos\theta$ —— (i)

Also; $R=A+B$

or, $R^2=A^2+B^2+2AB$ —— (ii)

Comparing eqn (i) and eqn (ii), we get;

or, $\cos\theta=1=\cos 0°$

$\boxed{\therefore \theta=0°}$

Hence, the angle between two vector is 0°.

Q.1[c] Two vectors $\vec{A}$ & $\vec{B}$ are such that $\vec{R} = \vec{A}+\vec{B}$ and $R = A-B$. Find the angle between them?

★ Soln Given;

$\vec{R} = \vec{A}+\vec{B}$

or, $R^2 = A^2+B^2+2AB\cos\theta$ —— (i)

Also, $R = A-B$

or, $R^2 = A^2+B^2-2AB$ —— (ii)

Comparing eqn (i) & (ii), we get;

~~or, cosθ = 1~~

$\cos\theta = -1 = \cos 180°$

$\Rightarrow \theta = 180°$

Hence, the angle between two vector is 180°.

Q.1[D] Two vectors $\vec{A}$ and $\vec{B}$ are such that $\vec{R} = \vec{A}+\vec{B}$ and $\vec{R} = \vec{A}-\vec{B}$. Find angle between them?

★ Soln Given;

$\vec{R} = \vec{A}+\vec{B}$

or, $R^2 = A^2+B^2+2AB\cos\theta$ —— (i)

Also, $\vec{R} = \vec{A}-\vec{B}$

or, $R^2 = A^2+B^2-2AB\cos\theta$

Comparing eqn (i) & (ii), we get;

$\cos\theta = -\cos\theta$

or, $2\cos\theta = 0$

or, $\cos\theta = 0 = \cos 90°$

$\Rightarrow \theta = 90°$

Hence, the angle between two vectors is 90°. #

Q.[2] Can the sum of two equal vectors be equal to either to the vectors? Explain?

★ Soln We have;

$$R^2 = A^2 + B^2 + 2AB\cos\theta$$

If $A = B = R = x$

then; $x^2 = x^2 + x^2 + 2x^2\cos\theta$

or, $1 = 2 + 2\cos\theta$

or, $2\cos\theta = -1$

or, $\cos\theta = -1/2 = \cos 120°$

$\boxed{\Rightarrow \theta = 120°}$ #

Thus, when the angle between two vectors is equal to 120° the sum of 2 equal vectors be equal to either of the vectors.

Q.1(E) Two vectors $\vec{A}$ & $\vec{B}$ are such that $\vec{A} - \vec{B} = \vec{C}$ & $A - B = C$. Find the angle between them?

★ Soln Given;

$\vec{C} = \vec{A} - \vec{B}$ Also, $C = A - B$

or $C^2 = A^2 + B^2 - 2AB\cos\theta$ —— (i) or, $C^2 = A^2 + B^2 - 2AB$ —— (ii)

Comparing eqn (i) & (ii) we get;

$\cos\theta = 1 = \cos 0°$

$\Rightarrow \theta = 0°$

Hence, the angle between $\vec{A}$ & $\vec{B}$ is $0°$ #

(ii) polygon Law of vector Addition:-

It states that, "If numbers of vectors acting at a point can be represented by side of polygon taken in order, then the resultant is represented by closing side of the polygon taken in opposite order in magnitude and direction.

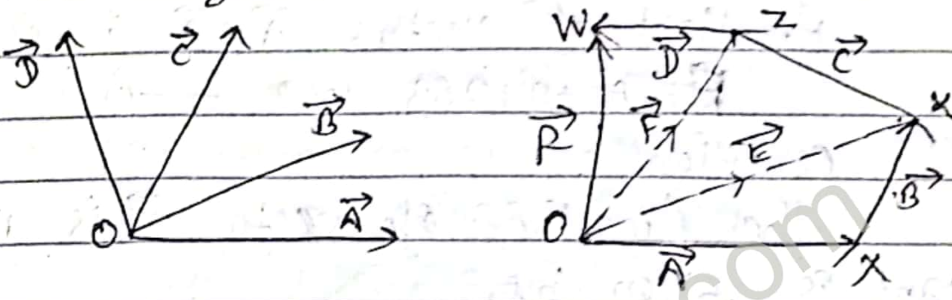


fig:- polygon Law of vector addition:

Let, vectors $\vec{A}, \vec{B}, C, \vec{D}$ acting at a point O are represented by side of polygon OXYZWO. Here closing side WO representeds resultant in both magnitude and direction.

Let us join O to Y and O to Z representing the vector $\vec{E}$ and $\vec{F}$ respectively.

In Δ OXY, using triangle law of vector;

$$\vec{E} = \vec{A} + \vec{B} \text{ --- (i)}$$

Again, In ΔOYZ, using triangle law of vector;

$$\vec{F} = \vec{E} + \vec{C} \text{ --- (ii)}$$

using eqn(i) in eqn(ii)

$$\vec{F} = \vec{A} + \vec{B} + \vec{C} \text{ --- (iii)}$$

Again; IN ΔOZW, using triangle law of vector,

$$\vec{R} = \vec{F} + \vec{D} \text{ --- (iv)}$$

using eqn(iv) in (iii); $\vec{R} = \vec{A} + \vec{B} + \vec{C} + \vec{D}$ --- (v)

this eqn(v) shows that vector $\vec{R}$ is resultant of vectors $\vec{A}, \vec{B}, \vec{C}$ & $\vec{D}$.

Q.[3] Two vectors $\vec{A}$ and $\vec{B}$ has resultant $\vec{R}$. If direction of $\vec{B}$ is reversed then new resultant is $\vec{S}$. Show that $R^2 + S^2 = 2(A^2+B^2)$

☆ Soln

In first condition,

Resultant of vectors $\vec{A}$ & $\vec{B}$ is given by

$$R^2 = A^2 + B^2 + 2AB\cos\theta \quad \text{——(i)}$$

In 2nd condition,

the direction of vector $\vec{B}$ is reversed then resultant is given by,

$$S^2 = A^2 + B^2 - 2AB\cos\theta \quad \text{——(ii)}$$

Now, adding eqn (i) & eqn (ii), we get;

$$R^2 + S^2 = A^2 + B^2 + 2AB\cos\theta + A^2 + B^2 - 2AB\cos\theta$$

OR, $R^2 + S^2 = 2A^2 + 2B^2$

$$\therefore \boxed{R^2 + S^2 = 2(A^2+B^2)}$$ proved.

Resolution of vectors:

The process of spliting of a vector into it's components is called resolution of vectors. Let us consider a vector $\vec{A}$ can be resolved into two components, one is A_y along y-Axis and other is A_x along x-axis. Also let θ be the angle made by vector $\vec{A}$ with x-axis.

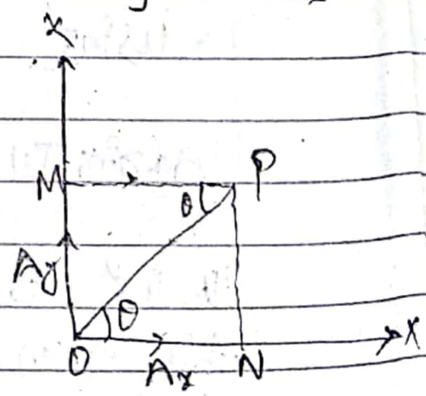


fig: Resolution of vector.

Now; In Right angle A PON,

$$\sin\theta = \frac{PN}{OP} = \frac{OM}{OP} = \frac{A_y}{A}$$

$$\therefore A_y = A\sin\theta \quad \text{---(i)}$$

Again, In right angle A PON

$$\cos\theta = \frac{ON}{OP} = \frac{A_x}{A}$$

$$\therefore A_x = A\cos\theta \quad \text{---(ii)}$$

Also, squaring and ~~addition~~ adding eq^n (i) & (ii)

$$A_x^2 + A_y^2 = A^2\cos^2\theta + A^2\sin^2\theta$$

$$A_x^2 + A_y^2 = A^2(\sin^2\theta + \cos^2\theta)$$

$$\therefore A = \sqrt{A_x^2 + A_y^2} \quad \text{---(iii)}$$

Again, dividing eq^n (i) by (ii)

$$\frac{A\sin\theta}{A\cos\theta} = \frac{A_y}{A_x}$$

$$\boxed{\tan\theta = \frac{A_y}{A_x}} \quad \text{---(iv)}$$

Product of two vectors:-

a) Scalar product or dot product of two vectors:-

The product of two vectors is said to be Scalar product if two vectors multiplied together to give a scalar quantity. The Scalar product or dot product of two vectors $\vec{A}$ & $\vec{B}$ with angle 'θ' between them is denoted by '$\vec{A}.\vec{B}$' and defined by:

$$\vec{A}.\vec{B} = AB\cos\theta \qquad \Rightarrow \theta = \cos^{-1}\left(\frac{\vec{A}.\vec{B}}{AB}\right)$$

* Special cases

(1) When two vectors are parallel, then $\theta = 0°$

$$\therefore \vec{A}\cdot\vec{B} = AB\cos 0° = AB$$

(2) When two vectors are perpendicular, then, $\theta = 90°$

$$\therefore \vec{A}\cdot\vec{B} = AB\cos 90° = 0$$

(3) When two vectors are antiparallel, then $\theta = 180°$

$$\therefore \vec{A}\cdot\vec{B} = AB\cos 180° = -AB$$

* properties of dot product of two vectors

(1) The dot product is commutative

i.e. $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$

* proof: By defination,

$\vec{A}\cdot\vec{B} = AB\cos\theta$ and $\vec{B}\cdot\vec{A} = BA\cos\theta$

$$\therefore \vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$$

which proves that the dot product of two vectors is commutative.

(2) The scalar product is distributive

i.e., $\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$

(3) The scalar product of a vector with itself gives square of it's magnitude, i.e. $\vec{A}\cdot\vec{A} = A^2$

* proof: By def^n $\vec{A}\cdot\vec{B} = AB\cos\theta$

Now, $\therefore \vec{A}\cdot\vec{A} = AA\cos 0° = A^2$

Thus, the scalar product of a vector with itself gives square of it's magnitude.

(b) Vector product or Cross product of two vectors:-

The product of two vectors is said to be vector product if two vectors multiplied together to give a vector quantity. The vector product or cross product of two vectors $\vec{A}$ & $\vec{B}$ with angle θ between them is denoted by $\vec{A} \times \vec{B}$ and defined as

$$\vec{A} \times \vec{B} = AB \sin\theta \, \hat{n}$$

where $\hat{n}$ is unit vector perpendicular to both $\vec{A}$ & $\vec{B}$.

$$|\vec{A} \times \vec{B}| = AB \sin\theta$$

$$\sin\theta = \frac{|\vec{A} \times \vec{B}|}{AB}$$

$$\therefore \theta = \sin^{-1}\left[\frac{|\vec{A} \times \vec{B}|}{AB}\right]$$

* Special Cases:-

1) When two vectors are parallel, then $\theta = 0°$,

$$\therefore \vec{A} \times \vec{B} = AB \sin 0 \, \hat{n}$$
$$= 0$$

2) When two vectors are perpendicular, then $\theta = 90°$,

$$\therefore \vec{A} \times \vec{B} = AB \sin 90° \, \hat{n}$$
$$= AB \, \hat{n}$$

3) When two vectors are antiparallel, then $\theta = 180°$,

$$\therefore \vec{A} \times \vec{B} = AB \sin 180° \, \hat{n}$$
$$= 0$$

* properties of Vector product:-

1) It is anti commutative

i.e, $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

2) It is distributive
i.e. $\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B} + \vec{A}\times\vec{C}$

3) The cross product of a vector with itself gives null vectors.
i.e., $\vec{A}\times\vec{A} = \vec{0}$ (null vector)

☆ proof from defn,
$\vec{A}\times\vec{B} = AB\sin\theta\,\hat{n}$

Now

$\therefore \vec{A}\times\vec{A} = AA\sin 0^\circ\,\hat{n}$
$= 0$

Thus, the cross/vector product of a vector with itself gives null vector.

Q.[4][A] If $\vec{A}\cdot\vec{B}=0$, what is angle between $\vec{A}$ & $\vec{B}$?

☆ Soln:

We have, $\vec{A}\cdot\vec{B} = AB\cos\theta$ —— (i)

Given, $\vec{A}\cdot\vec{B} = 0$

So, eqn (i) becomes,

or, $0 = AB\cos\theta$

or, $\cos\theta = 0 = \cos 90^\circ$

$\Rightarrow \theta = 90^\circ$

Thus, the angle between $\vec{A}$ & $\vec{B}$ is 90°. #

Q.[5][A] If the scalar product of two vectors is equal to the magnitude of their vector product, find the angle between them?

☆ Soln

Given; $\vec{A}\cdot\vec{B} = |\vec{A}\times\vec{B}|$

or, $AB\cos\theta = AB\sin\theta$

or, $\frac{\sin\theta}{\cos\theta} = 1$

or, $\tan\theta = \tan 45°$

$\Rightarrow \theta = 45°$

Thus, the angle between them is 45°. #

Q.5[B] The dot product of two vectors having magnitude or three and 4 is 6. What is angle between them?

Soln Given, $\vec{A}\cdot\vec{B} = 6$

$|\vec{A}| = A = 3$, $|\vec{B}| = B = 4$, $\theta = ?$

We have, $\vec{A}\cdot\vec{B} = AB\cos\theta$

or, $6 = 3\times4\cos\theta$

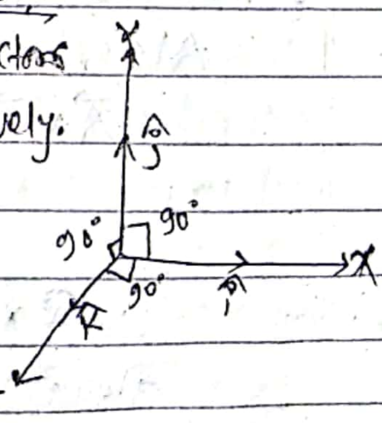
or, $\cos\theta = \frac{6}{12} = \frac{1}{2} = \cos 60°$

$\Rightarrow \theta = 60°$

Thus, the angle between them is 60°. #

Properties of a unit vectors:-

Let, $\hat{i}, \hat{j}$ and $\hat{k}$ be the unit vectors along x-axis, Y-axis and z-axis respectively.



a) Scalar product; (i) $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

(ii) $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$

b) Vector product: (i) $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0$

(ii) $\hat{i}\times\hat{j} = \hat{k}$; $\hat{j}\cdot\hat{k} = \hat{i}$; $\hat{k}\times\hat{i} = \hat{j}$

(iii) $\hat{j}\times\hat{i} = -\hat{k}$; $\hat{k}\times\hat{j} = -\hat{i}$; $\hat{i}\times\hat{k} = -\hat{j}$

Q.6[A] If $\hat{i}, \hat{j}$ and $\hat{k}$ are unit vectors along x, y and z-axis respectively. find $\hat{i}\cdot(\hat{j}\times\hat{k})$

☆ Soln

$\hat{i}\cdot(\hat{j}\times\hat{k}) = \hat{i}\cdot(jk\sin 90^\circ\,\hat{i})$ [Since $\hat{i}$ be the unit vector ⊥ to the plane of $\hat{j}$ & $\hat{k}$]

or, $\hat{i}\cdot(\hat{j}\times\hat{k}) = \hat{i}\cdot\hat{i}$

$= ii\cos 0^\circ$

$= 1$

$\therefore \hat{i}\cdot(\hat{j}\times\hat{k}) = 1$ #

Q.6[B] What does $\vec{A}\cdot\vec{A}$, the scalar quantity of a vector with self gives? What about $\vec{A}\times\vec{A}$, the vector product of vector with self gives?

☆ Soln We have;

$\hat{A}\cdot\hat{A} = AA\cos\theta$

$= AA\cos 0^\circ$ ($\theta = 0^\circ$; angle between $\vec{A}$ & $\vec{A}$)

$= A^2$

Thus, the scalar product of a vector itself gives the square of it's magnitude.

Also,

$\vec{A}\times\vec{A} = AA\sin\theta\cdot\hat{n}$

$= AA\sin 0^\circ\,\hat{n}$

$= 0$

$\therefore \vec{A}\times\vec{A} = \vec{0}$

Thus, the vector product of a vector itself gives null vector (or zero).

Q.6[C]

☆

Q.6[c] A force(in Newton) expressed in vector notation as $\vec{F} = 4\hat{i} + 7\hat{j} - 3\hat{k}$ is applied on a body and produces a displacement(in meter), $\vec{D} = 3\hat{i} - 2\hat{j} - 5\hat{k}$ in 4 seconds. Estimate the power.

☆ Soln Given;

Force $(\vec{F}) = 4\hat{i} + 7\hat{j} - 3\hat{k}$

Displacement $(\vec{D}) = 3\hat{i} - 2\hat{j} - 5\hat{k}$

power $(P) = ?$

We have,

$$\text{Work } (W) = \vec{F} \cdot \vec{D}$$
$$= (4\hat{i} + 7\hat{j} - 3\hat{k}) \cdot (3\hat{i} - 2\hat{j} - 5\hat{k})$$
$$= 12 - 14 + 15$$
$$= 13 \text{ J}$$

Again,

$$P = \frac{W}{t} = \frac{13}{4}$$
$$= 3.25 \text{ W}$$

∴ The Power is 3.25 Watt. #

Note: If $\vec{A} = \hat{i}x_1 + \hat{j}y_1 + \hat{k}z_1$ and $\vec{B} = \hat{i}x_2 + \hat{j}y_2 + \hat{k}z_2$ then, $\vec{A} \cdot \vec{B} = x_1x_2 + y_1y_2 + z_1z_2$.

and,

If $\vec{L} = x\hat{i} + y\hat{j} + z\hat{k}$ then magnitude of $\vec{L}$ or (L) or $|\vec{L}| = \sqrt{x^2 + y^2 + z^2}$.

Numericals

Short questions:

Q.1) The angle between two vectors $\vec{A}$ & $\vec{B}$ is θ. Find the magnitude and direction of $\vec{A}\times\vec{B}$ and $\vec{A}.\vec{B}$.

☆ Magnitude of $\vec{A}\times\vec{B}$ is $AB\sin\theta$ and direction is $\hat{n}$ which is the unit vector $\perp^r$ to the plane containing $\vec{A}$ & $\vec{B}$. Again, magnitude of $\vec{A}.\vec{B}$ is $AB\cos\theta$ and no direction.

Q.2) Given two vectors $\vec{A} = 4.00\hat{i} + 3.00\hat{j}$ and $\vec{B} = 5.00\hat{i} - 2.00\hat{j}$. Find the magnitude of each vector.

☆ Given vectors:

$\vec{A} = 4\hat{i} + 3\hat{j}$ and $\vec{B} = 5\hat{i} - 2\hat{j}$

then, $|\vec{A}| = A = \sqrt{4^2 + 3^2} = 5$ units

then, $|\vec{B}| = B = \sqrt{5^2 + (-2)^2} = \sqrt{29} = 5.38$ units. #

Q.3) Under what condition, the cross product of two vectors equal to zero.

☆ When two vectors are parallel (i.e. $\theta = 0°$) to each other then their cross product is equal to zero.

Q.4) If $\vec{A}$ and $\vec{B}$ are non zero vectors, it is possible for $\vec{A}\times\vec{B}$ and $\vec{A}.\vec{B}$ both to be zero? Explain.

☆ Scalar product of two vectors is zero when they are $\perp^r$ to each other and vector product is zero when they are parallel. But in case of two vectors are non-zero where $\perp^r$ and // must not possible at same time. So, it is not possible for $\vec{A}\times\vec{B}$ and $\vec{A}\cdot\vec{B}$ both to be zero.

Q. 5) Calculate the angle between a two dyne and a three dyne force so that their sum is four dyne.

☆ Solution:

Here; Let, Magnitude of 1st force $(|\vec{A}|) = 2$

" " 2nd " $(|\vec{B}|) = 3$

Angle between $\vec{A}$ and $\vec{B}$ $= \theta$

Magnitude of their sum $(|\vec{R}|) = 4$

Now,

We have, $R = \sqrt{A^2+B^2+2AB\cos\theta}$

or, $(4)^2 = (\sqrt{2^2+3^2+2(2)(3)\times\cos\theta})^2$

or, $16 = 4+9+12\cos\theta$

or, $12\cos\theta = 3$

or, $\cos\theta = \frac{1}{4}$

or, $\cos\theta = \frac{1}{4}$

or, $\theta = \cos^{-1}\left(\frac{1}{4}\right)$

$\Rightarrow \boxed{\theta = 75.52^\circ}$ #

Hence the angle between 2 & 3 dyne force is 75.52°.

Long questions:

Q.1) Two forces of 30N and 40N are inclined to each other at an angle of 60°. What is their resultant? What will be the resultant if the forces are inclined at right angle to each other.

Solution:-

Here; Let, $|\vec{A}| = 30\,N$ and $|\vec{B}| = 40\,N$ where Two forces $\vec{A}$ and $\vec{B}$ are in the form of vector.

angle between them $(\theta) = 60°$

Now, $\boxed{\text{case I:}}$

We have; $R = \sqrt{A^2+B^2+2AB\cos\theta}$

$= \sqrt{(30)^2+(40)^2+2(30)(40)\cos 60°}$

$= \sqrt{900+1600+2400\times \frac{1}{2}}$

$= \sqrt{2500+1200}$

$= \sqrt{3700}$

$\therefore R = 60.82\ N$

Thus, their resultant is 60.82 N #

And, for direction;

We have, $\alpha = \tan^{-1}\left[\frac{B\sin\theta}{A+B\cos\theta}\right]$

or, $\alpha = \tan^{-1}\left[\frac{40\times\sin 60°}{30+40\cos 60°}\right] = \tan^{-1}\left[\frac{20\sqrt{3}}{50}\right] = \tan^{-1}\left[\frac{2\sqrt{3}}{5}\right] = 34.71°$ #

Thus, direction between them $= 34.71°$ #

Again, $\boxed{\text{case II:}}$

$\theta = 90°$ when they are enclined at right angle.

So; Magnitude

$R = \sqrt{(30)^2+(40)^2+2(30)(40)\cos 90°} = \sqrt{2500+0} = 50\ N$ # and direction

$\alpha = \tan^{-1}\left[\frac{B\sin\theta}{A+B\cos\theta}\right] = \tan^{-1}\left[\frac{40\times\sin 90°}{30+40\cos 90°}\right] = \tan^{-1}\left[\frac{40}{30}\right] = \tan^{-1}\left(\frac{4}{3}\right) = 53.13°$ #

ther
will

Q.2) At what angle do the two forces $\vec{A}+\vec{B}$ and $\vec{A}-\vec{B}$ act so that their resultant is $\sqrt{3A^2+B^2}$?

Soln:

Given; their resultant $(R) = \sqrt{3A^2+B^2}$

Given forces $= \vec{A}+\vec{B}$ and $\vec{A}-\vec{B}$

angle between two forces $(\theta) = ?$

Now

according to questions-

$$\sqrt{3A^2+B^2} = \sqrt{(\vec{A}+\vec{B})^2+(\vec{A}-\vec{B})^2+2(\vec{A}+\vec{B})(\vec{A}-\vec{B})\cos\theta}$$

Squaring on both side,

or, $3A^2+B^2 = A^2+2\vec{A}\vec{B}+B^2+A^2-2\vec{A}\vec{B}+B^2+2\cos\theta(A^2-B^2)$

or, $3A^2-2A^2+B^2-2B^2 = 2\cos\theta(A^2-B^2)$

or, $A^2-B^2 = 2\cos\theta(A^2-B^2)$

or, $2\cos\theta = 1$

or, $\cos\theta = \frac{1}{2} = \cos 60°$

$\Rightarrow \boxed{\theta = 60°}$

Hence, angle between two forces $(\vec{A}+\vec{B})$ & $(\vec{A}-\vec{B})$ is $60°$ #

Q.3) Find the scalar product of two vectors $\vec{A} = 4.00\hat{i}+3.00\hat{j}$ and $\vec{B} = 5.00\hat{i}-2.00\hat{j}$. Also find the angle between them?

Soln:-

Given;

$\vec{A} = 4\hat{i}+3\hat{j}$ then $A = \sqrt{4^2+3^2} = 5$ Units

& $\vec{B} = 5\hat{i}-2\hat{j}$ then $B = \sqrt{5^2+(-2)^2} = \sqrt{29}$ Units

Now.

Scalar product of $\vec{A}$ & $\vec{B}$ (i.e. $\vec{A}.\vec{B}$) $= (4\hat{i}+3\hat{j}).(5\hat{i}-2\hat{j})$

$= 20-6$

$\therefore \vec{A}.\vec{B} = 14$ #

Again;

We have, $\vec{A}.\vec{B} = AB\cos\theta$

or, $14 = 5\sqrt{29}\cos\theta$

or, $\cos\theta = \frac{14}{5\sqrt{29}}$

or, $\theta = \cos^{-1}\left[\frac{14}{5\sqrt{29}}\right]$

$\Rightarrow \theta = 58.67°$ #

Thus, angle between $\vec{A}$ & $\vec{B}$ is $58.67°$ #

Q.4) What are the difference between scalar and vector product.

→

Scalar (dot) product	Vector product (cross product)
1) product is said to be scalar if two vector multiplied together to give scalar quantity.	1) product is said to be vector if two vectors multiplied together to give a vector quantity
2) It is denoted by $\vec{A}\cdot\vec{B}$	2) It is denoted by $\vec{A}\times\vec{B}$
3) It is defined by; $\vec{A}\cdot\vec{B} = AB\cos\theta$	3) It is defined by; $\vec{A}\times\vec{B} = AB\sin\theta\,\hat{n}$
4) It is commutative.	4) It is anticommutative.
5) the scalar product of a vector with itself gives square of it's magnitude.	5) The vector product of a vector with itself gives a null vector.
6) When two vectors are parallel then scalar product is AB.	6) When two vectors are parallel then vector product is zero.
7) When two vectors are perpendicular to each other then dot product is 0.	7) When two vectors are perpendicular to each other then vector product is $AB\,\hat{n}$.
8) When antiparallel to each other then scalar product is $-AB$	8) When antiparallel to each other then vector product is 0.