Graph Algorithms

CS3104

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Plan for Today

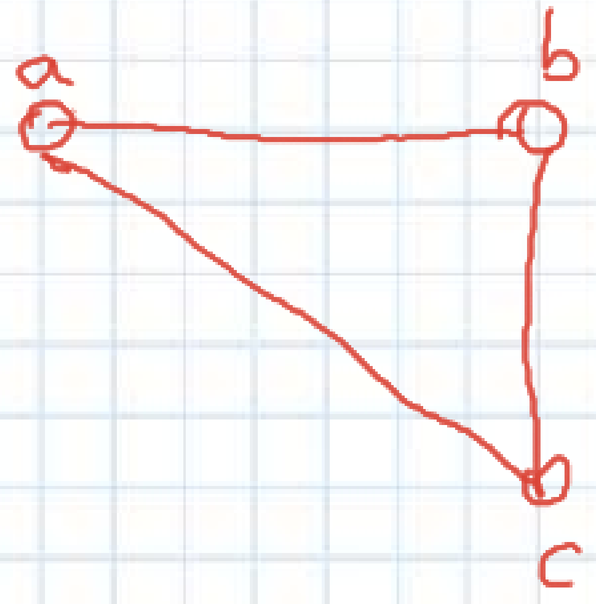
- Graph Terminology

Adjacent Vertices in Undirected Graphs

- Two vertices, ***u*** and ***v*** in an ***undirected graph G*** are called *adjacent* (or neighbors) in ***G***, if ***{u,v}*** is an edge of ***G***.

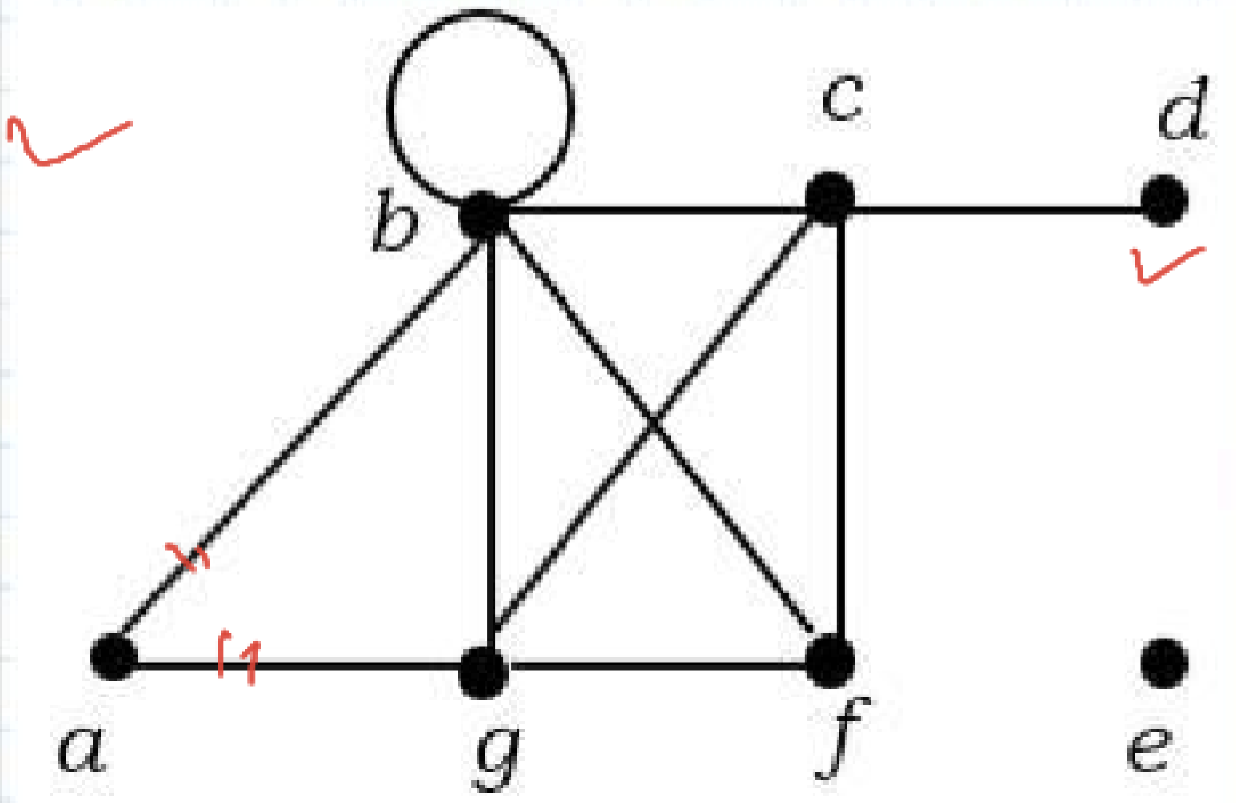
- An edge **e** connecting ***u*** and ***v*** is called *incident* with vertices ***u*** and ***v***, or is said to connect *u* and *v*.

- The vertices *u* and *v* are called *endpoints* of edge {*u*,*v*}.



Degree of a Vertex

- The *degree of a vertex* in an ***undirected graph*** is the number of edges incident with it
  - except that a loop at a vertex contributes twice to the degree of that vertex
- The degree of a vertex ***v*** is denoted by ***deg(v)***.



deg(a) = 2

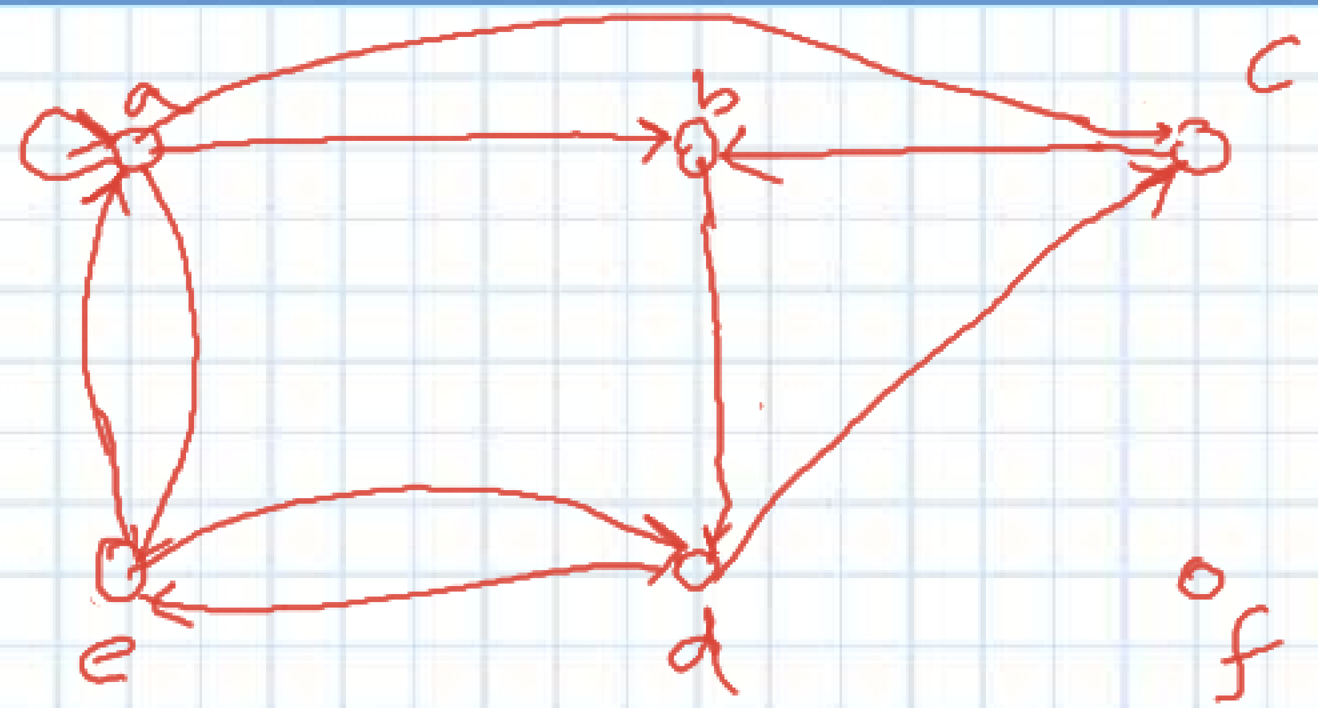
deg(b) = 6

deg(g) = 4

Degree of a Vertex in Directed Graph

- **Adjacent Vertices in Directed Graphs**
  - When ***(u,v)*** is an edge of a directed graph ***G***, *u* is said to be ***adjacent to v*** and ***v*** is said to be ***adjacent from u***.
- ***In-degree*** of a vertex *v*
  - The number of vertices ***adjacent to v*** (the number of edges with *v* as their terminal vertex
  - Denoted by $\mathbf{deg^-(}v\mathbf{)}$
- ***Out-degree*** of a vertex *v*
  - The number of vertices ***adjacent from v*** (the number of edges with *v* as their initial vertex)
  - Denoted by $\mathbf{deg^+(}v\mathbf{)}$
- A loop at a vertex contributes 1 to both the in-degree and out-degree.

Example: in-degree and out-degree



In degrees

$deg^-(a) = 2$ $deg^-(b) = 2$

out degree

$deg^+(a) = 4$ $deg^+(b) = 1$

Theorem

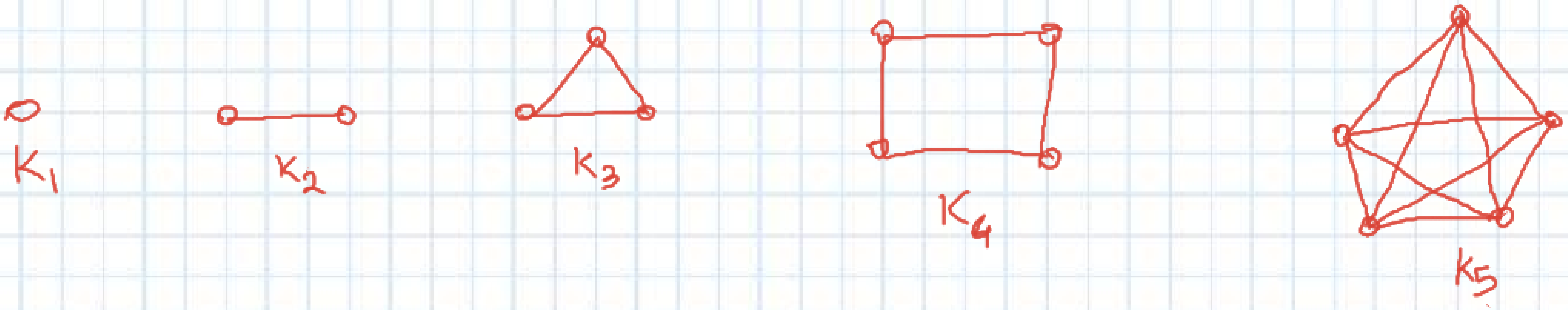
- The sum of the **in-degrees** of all vertices in a digraph = the sum of the **out-degrees** = the number of edges.
- Let $G = (V, E)$ be a graph with directed edges. Then:

$$\sum_{v \in V} deg^{-}(v) = \sum_{v \in V} deg^{+}(v) = |E|$$

Proof: The first sum counts the number of outgoing edges over all vertices and the second sum counts the number of incoming edges over all vertices. It follows that both sums equal the number of edges in the graph.

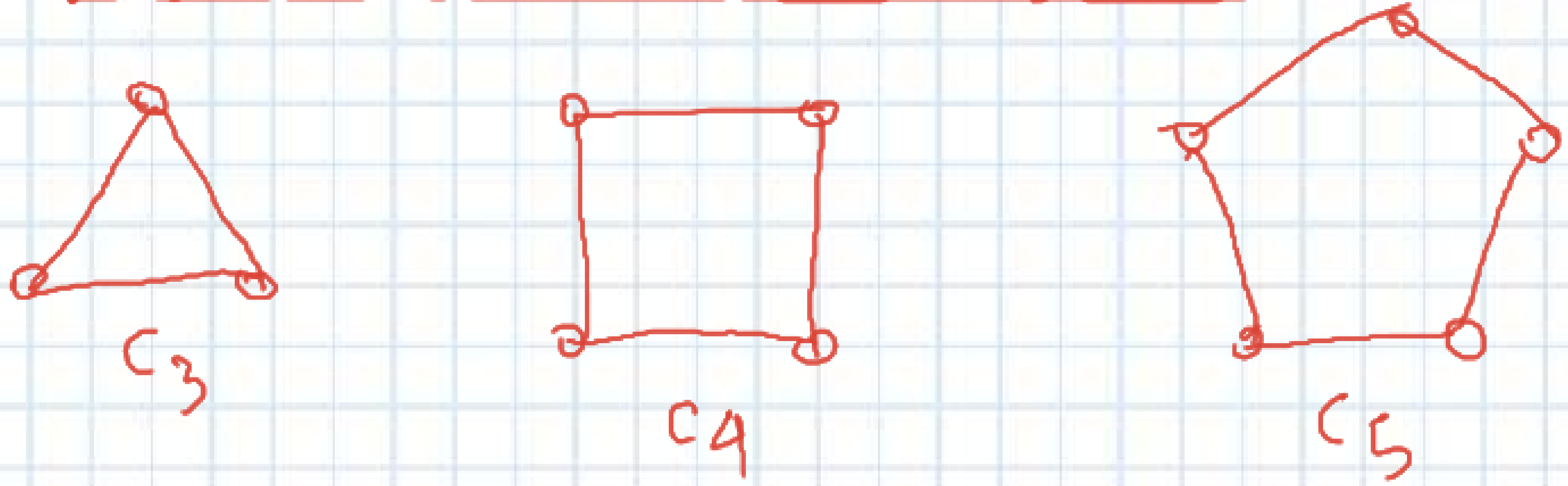
Complete Graph

- **Complete Graph**
  - The *complete graph* on ***n*** *vertices* ($\boldsymbol{K_n}$) is the simple graph that contains exactly one edge between each pair of distinct vertices.



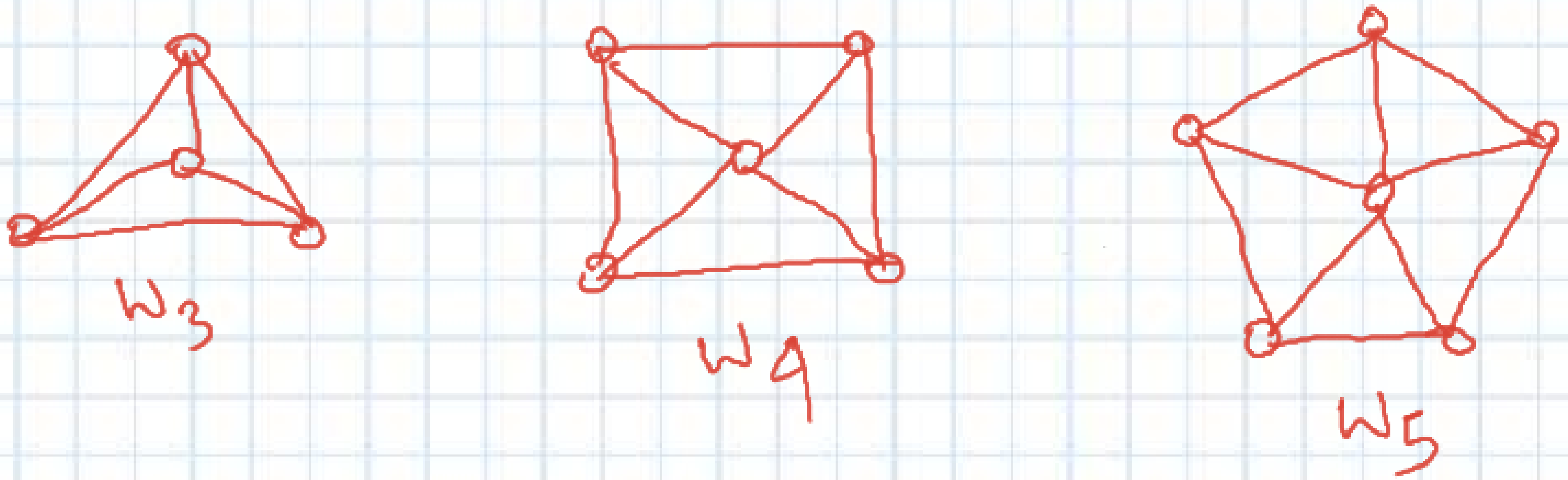
Cycle

- The *cycle* $\mathbf{C}_n$ ($n \geq 3$), consists of n vertices $v_1, v_2, \ldots, v_n$ and edges $\boxed{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \text{ and } \{v_n, v_1\}.}$

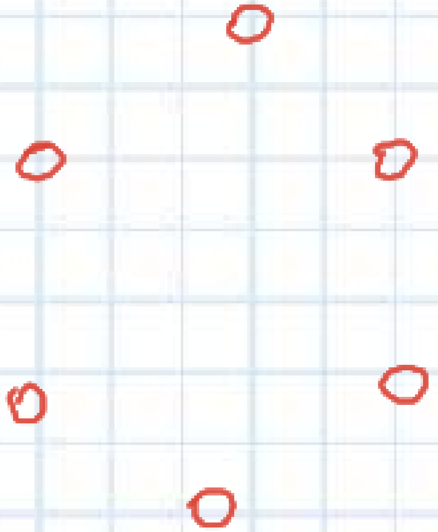


Wheel

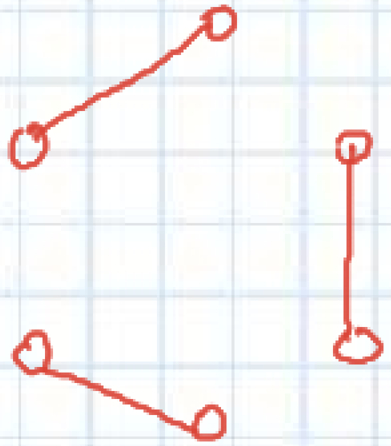
- When a new vertex is added to a cycle C_n and this new vertex is connected to each of the n vertices in C_n, we obtain a ***wheel*** W_n.



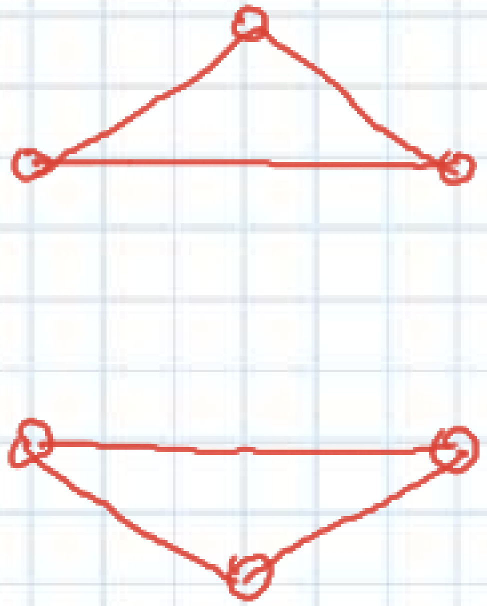
- a **regular graph** is a graph where each vertex has the same number of neighbors; i.e. *every vertex has the same degree*.
- A regular directed graph must also satisfy the stronger condition that the ***in-degree*** and ***out-degree*** of each vertex are equal to each other.
- A regular graph with vertices of degree **k** is called a **k-regular graph** or regular graph of degree **k**.



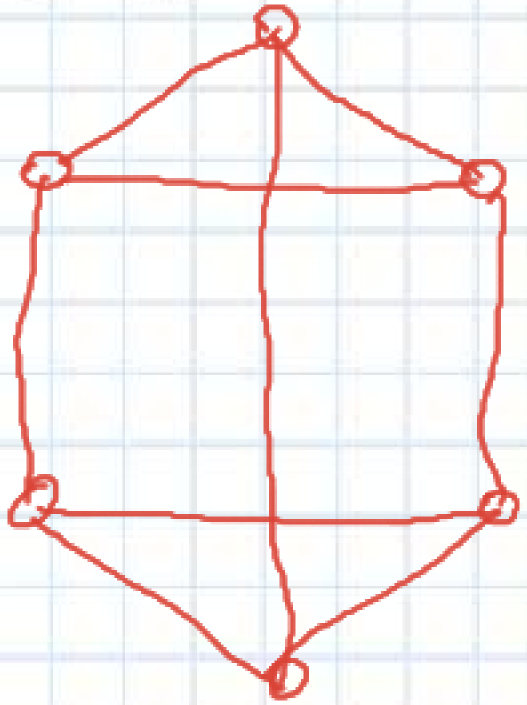
0-regular graph



1-regular graph



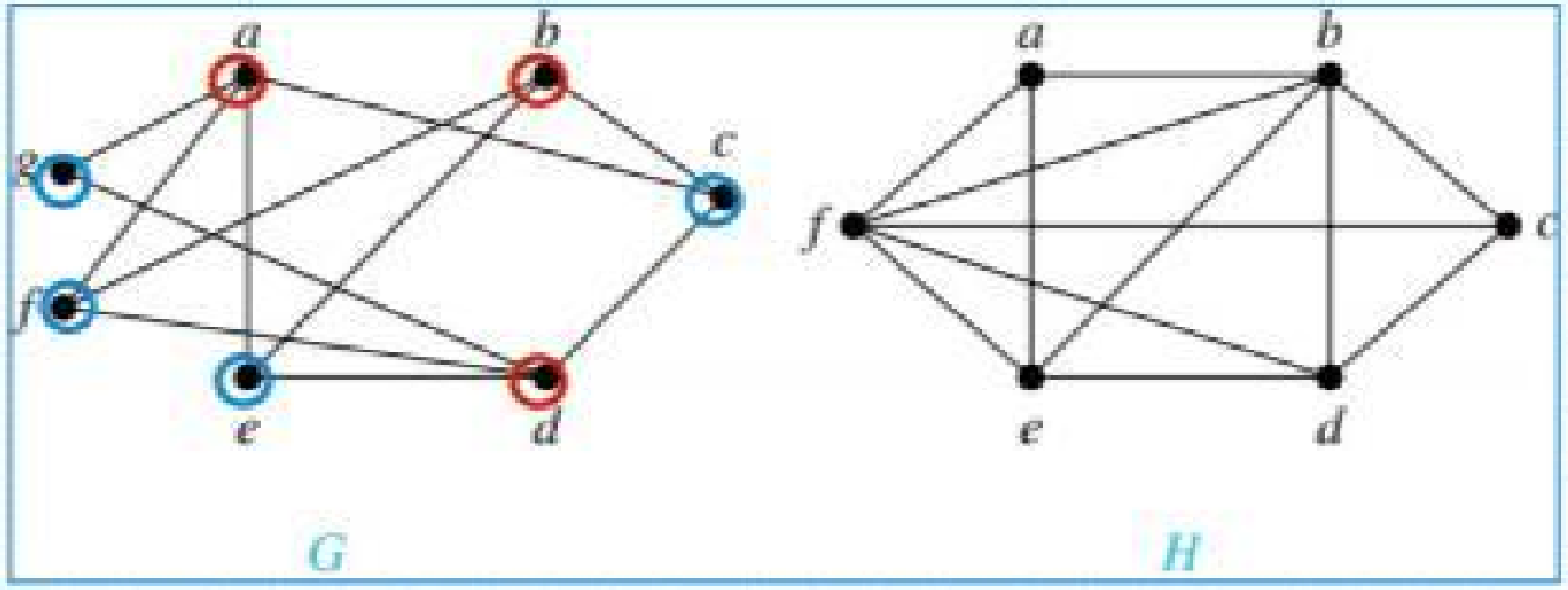
2-regular Graph



3-regular graph

- A simple graph is called *bipartite* if its vertex set V can be partitioned into two disjoint nonempty subsets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2 (so that **no edge in G connects either two vertices in V_1 or two vertices in V_2**).
- It is not hard to show that an equivalent definition of a bipartite graph is a graph where it is possible to **color the vertices red or blue so that no two adjacent vertices are the same color**.

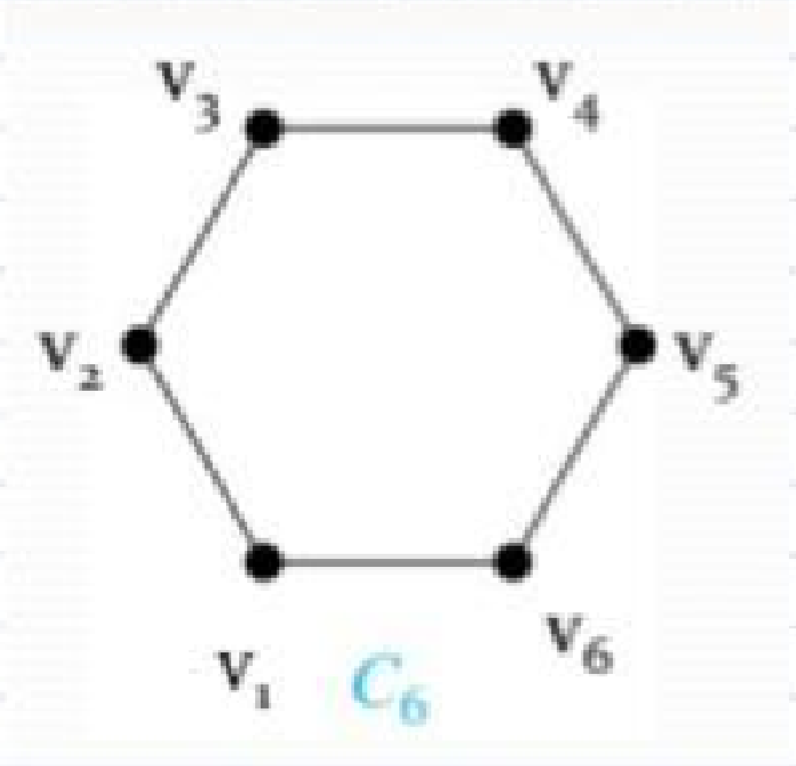
G is bipartite

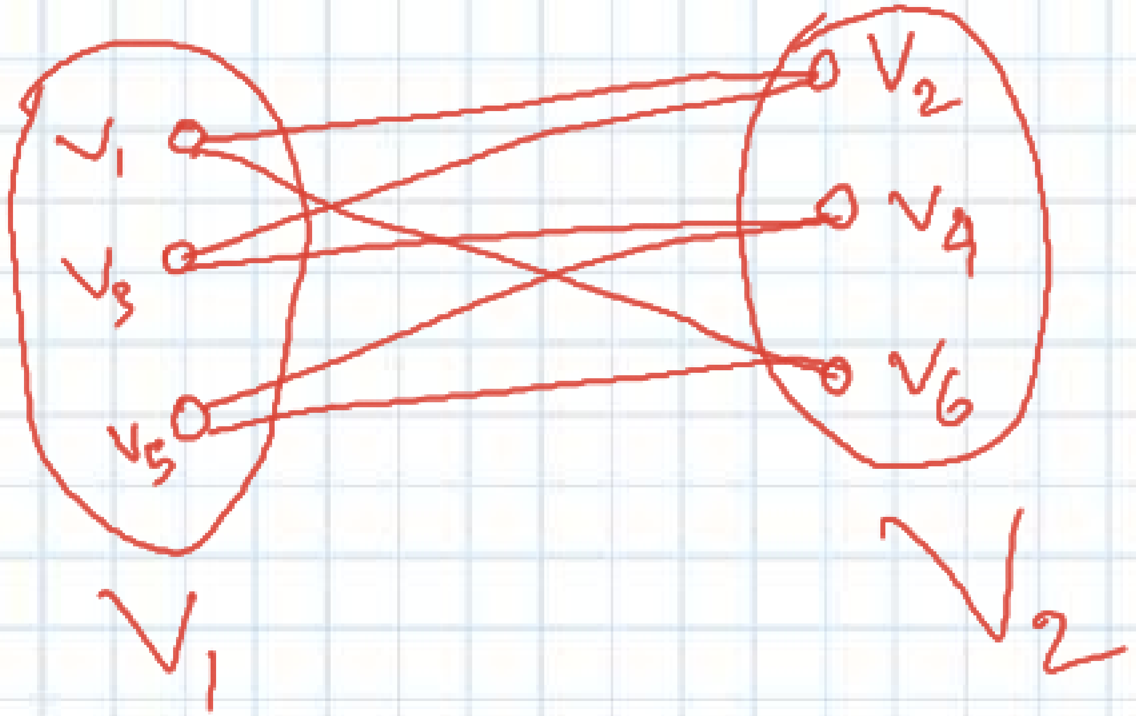


H is not bipartite since if we color a red, then the adjacent vertices *f* and *b* must both be blue.

Bipartite Graphs - Example

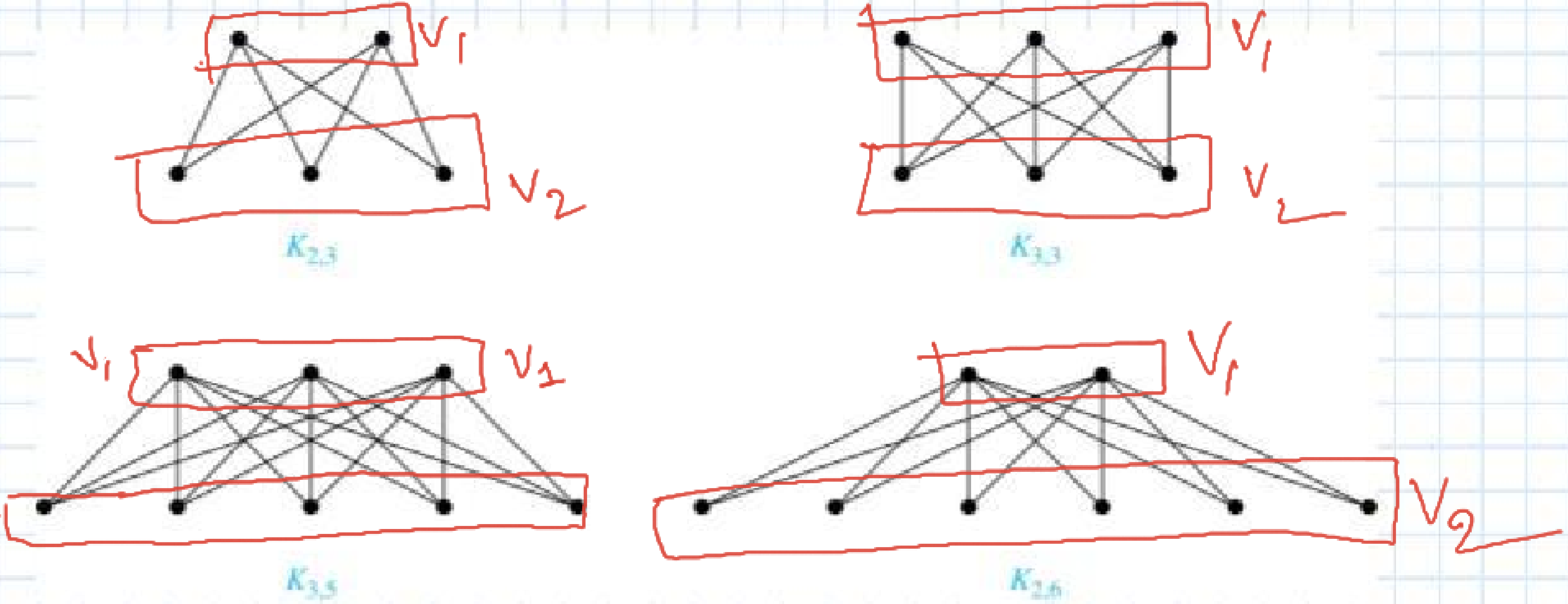
- **Example:** Show that C_6 is bipartite.
- **Solution:** We can partition the vertex set into $V_1 = \{v_1, v_3, v_5\}$ and $V_2 = \{v_2, v_4, v_6\}$ so that every edge of C_6 connects a vertex in V_1 and V_2.





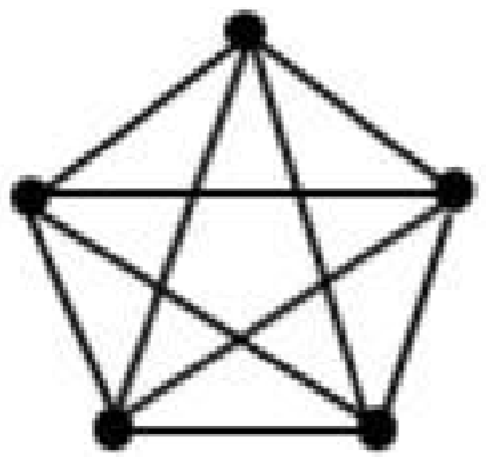
Complete Bipartite Graphs

- A complete bipartite graph $K_{m,n}$ is a graph that has its vertex set partitioned into two subsets V_1 of size m and V_2 of size n such that there is an edge from every vertex in V_1 to every vertex in V_2.
- **Example:** We display four complete bipartite graphs here:

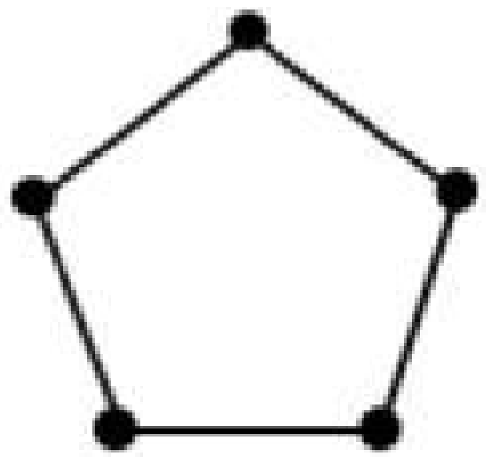


Subgraph

- A *subgraph* of a graph ***G = (V, E)*** is a graph ***H = (W, F)*** where ***W ⊆ V*** and ***F ⊆ E***.



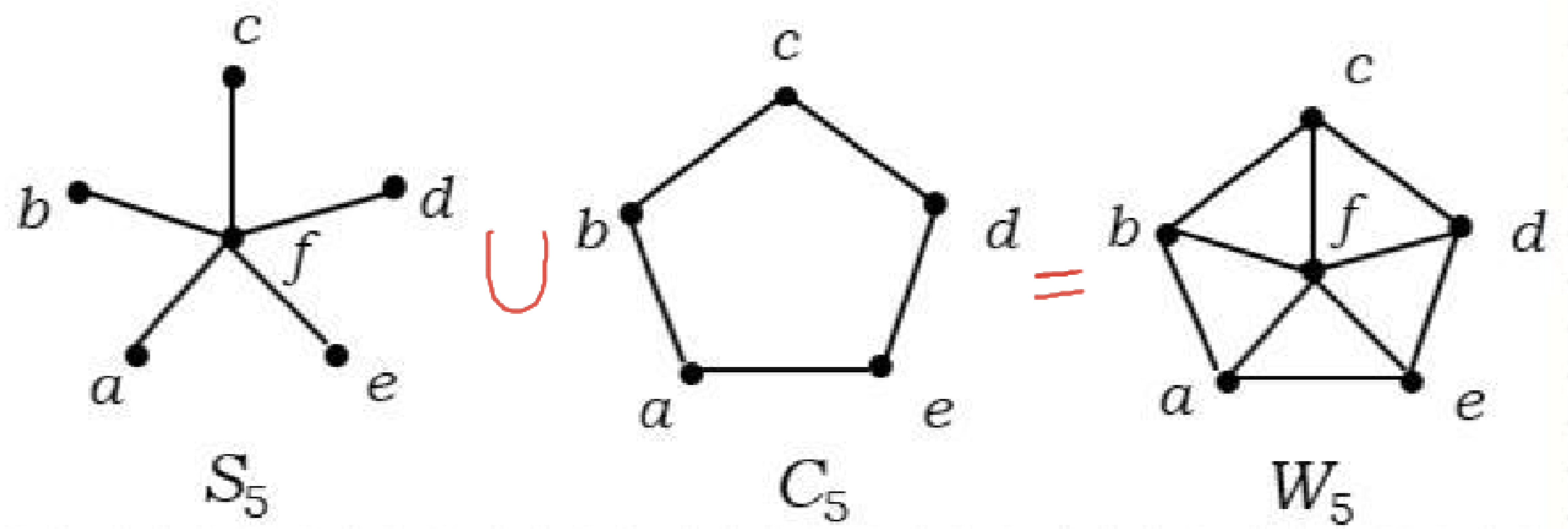
K_5



C_5

Union

- The *union* of 2 simple graphs $\boldsymbol{G_1 = (V_1, E_1)}$ and $\boldsymbol{G_2 = (V_2, E_2)}$ is the simple graph with vertex set $\boldsymbol{V = V_1 \cup V_2}$ and edge set $\boldsymbol{E = E_1 \cup E_2}$. The union is denoted by $\boxed{\boldsymbol{G_1 \cup G_2}}$.

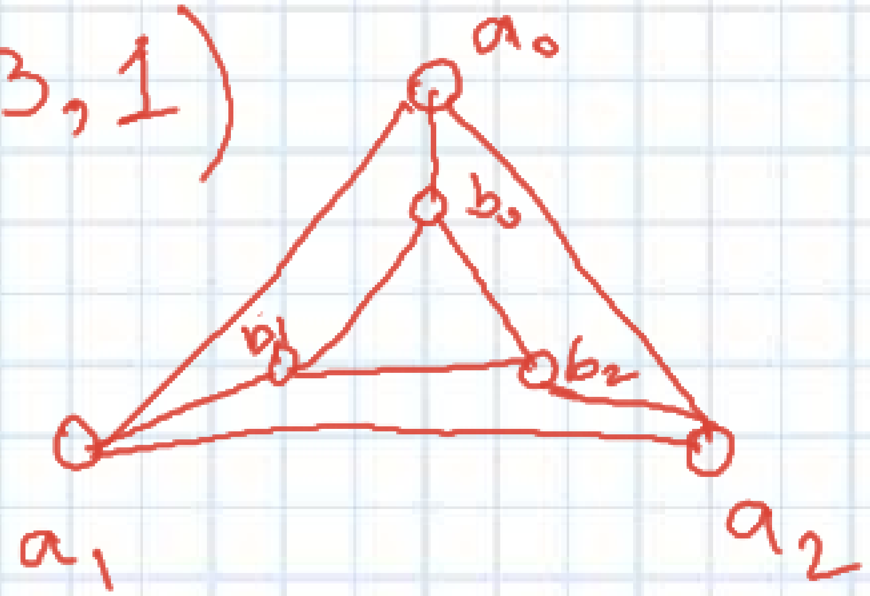


$\boldsymbol{S_5 \cup C_5 = W_5}$

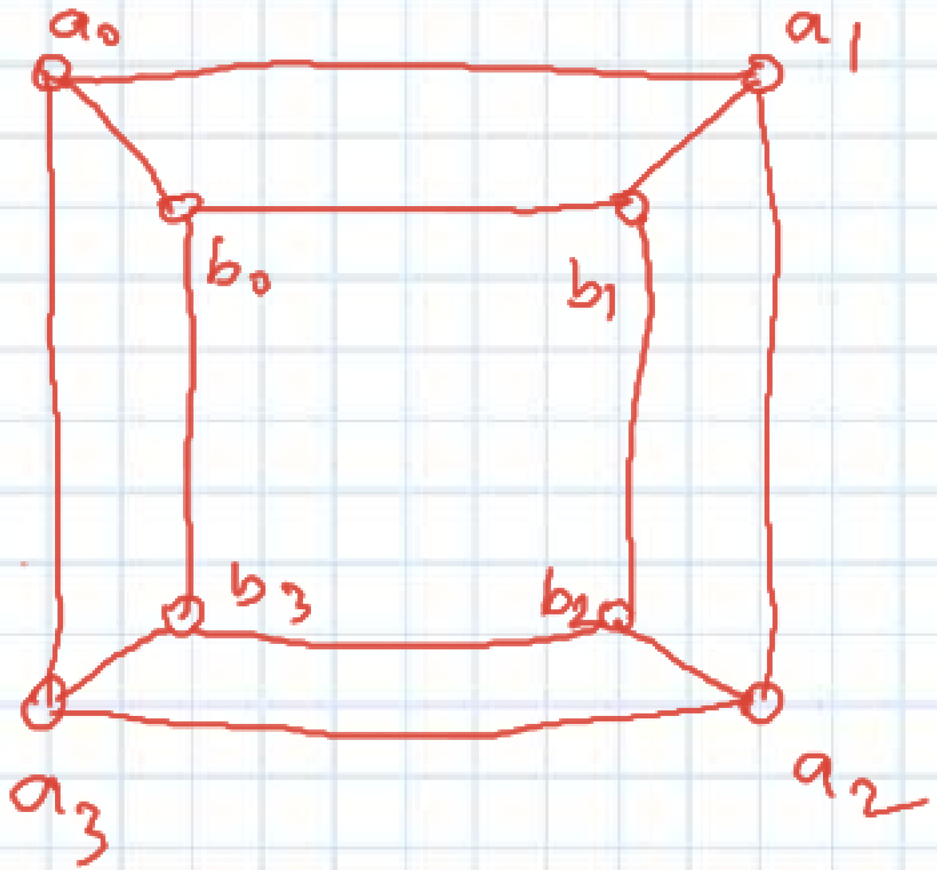
- A Generalized petersen Graph ***P(n, k)*** where ***$3 \leq n$*** and ***$k < n/2$***, is a simple graph consisting of an n-Cycle ***$C_n = a_0a_1 \ldots a_{n-1}a_0$*** and vertices ***$b_0, b_1, b_{n-1}$*** all are not in ***$C_n$*** and having edges ***$a_ib_i$*** and ***$b_ib_{i+k}$*** where **i+k** is read modulo **n**.
- The classical Petersen graph is ***P(5, 2)***.

$P(3,1)$

C_3



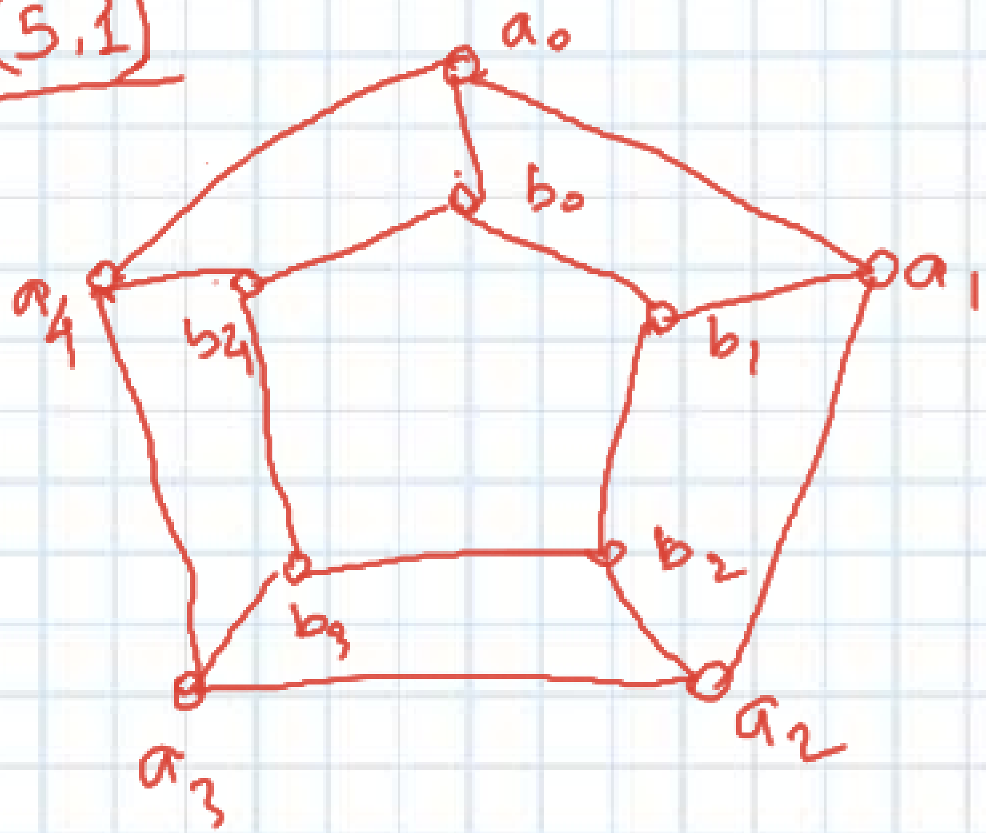
$P(4,1)$



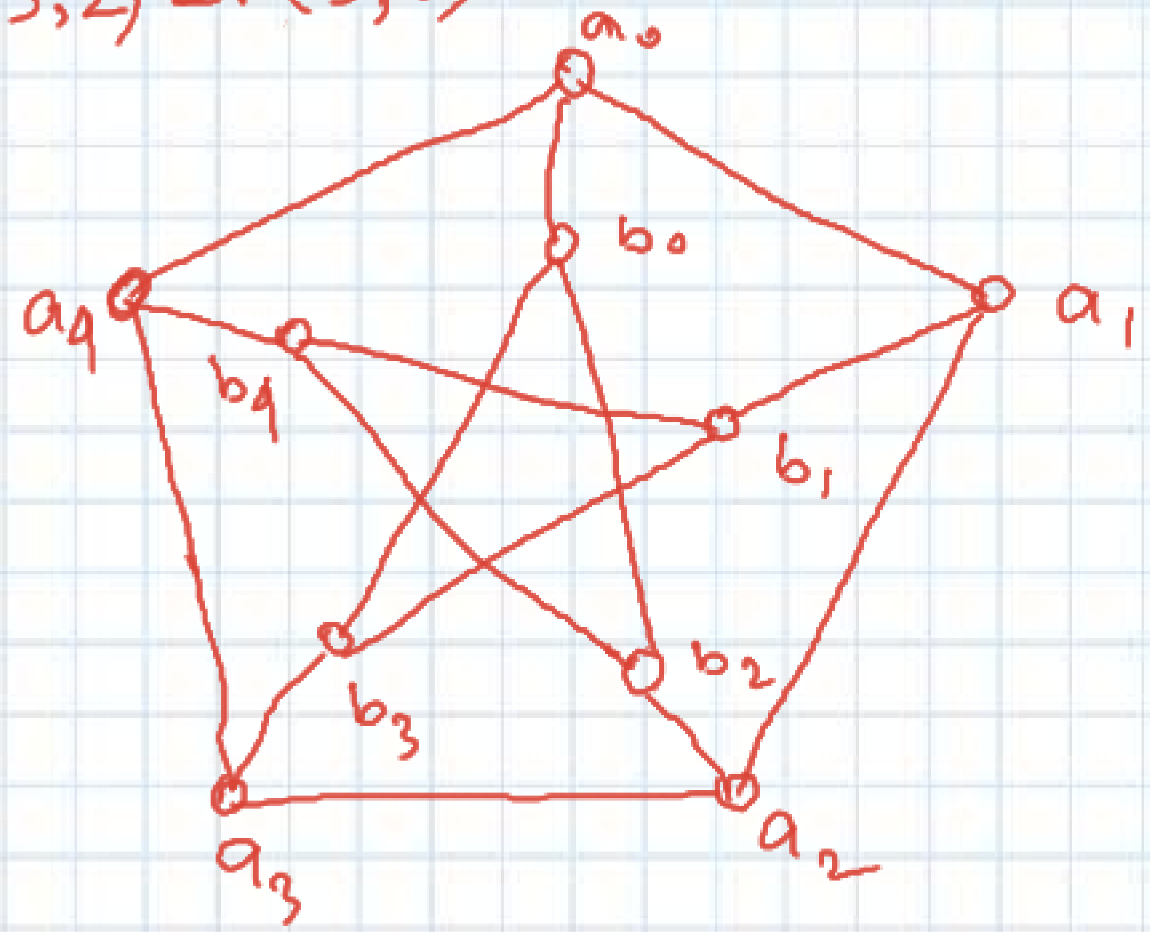
Example Generalized Petersen Graph P(n, k)

P(5,1)

C_5



$P(5,2) = P(5,3)$



$\boxed{P(5,3)}$