Isomorphism of Graphs
Graph Traversal

Graph Algorithms

CS3104

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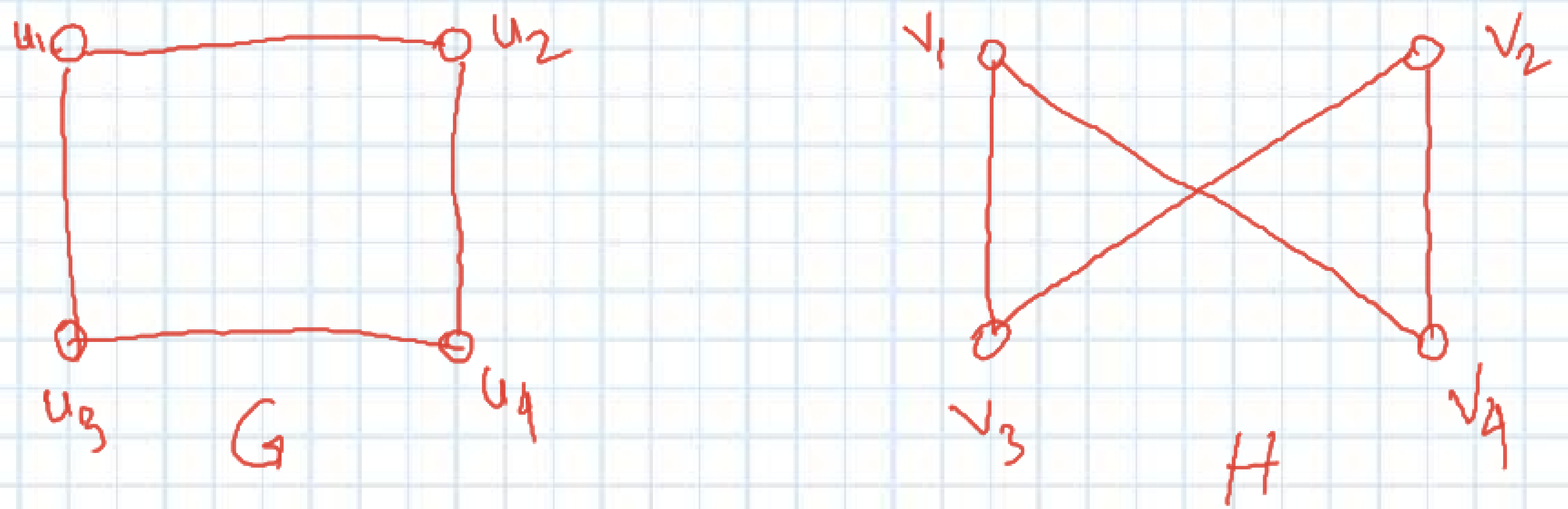
- **Two simple graphs are isomorphic if**:
  - there is a one-to one correspondence between the vertices of the two graphs
  - the adjacency relationship is preserved

- The simple graphs $G_1=(V_1, E_1)$ and $G_2=(V_2, E_2)$ are *isomorphic* if there is a one-to-one and onto function $\boldsymbol{f}$ from $\boldsymbol{V_1}$ to $\boldsymbol{V_2}$ with the property that $\boldsymbol{a}$ and $\boldsymbol{b}$ are adjacent in $\boldsymbol{G_1}$ iff $\boldsymbol{f(a)}$ and $\boldsymbol{f(b)}$ are adjacent in $\boldsymbol{G_2}$, for all a and $\boldsymbol{b}$ in $\boldsymbol{V_1}$.

Example: Isomorphism

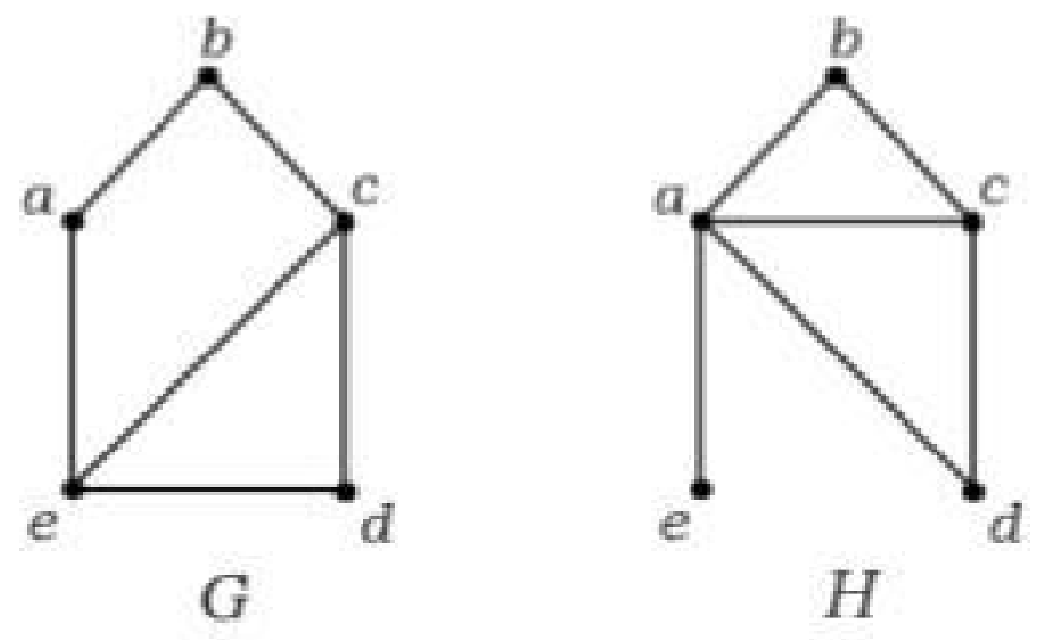


Are G & H are Isomorphic ?

$f(u_1) = v_1$, $f(u_2) = v_4$, $f(u_3) = v_3$, $f(u_4) = v_2$

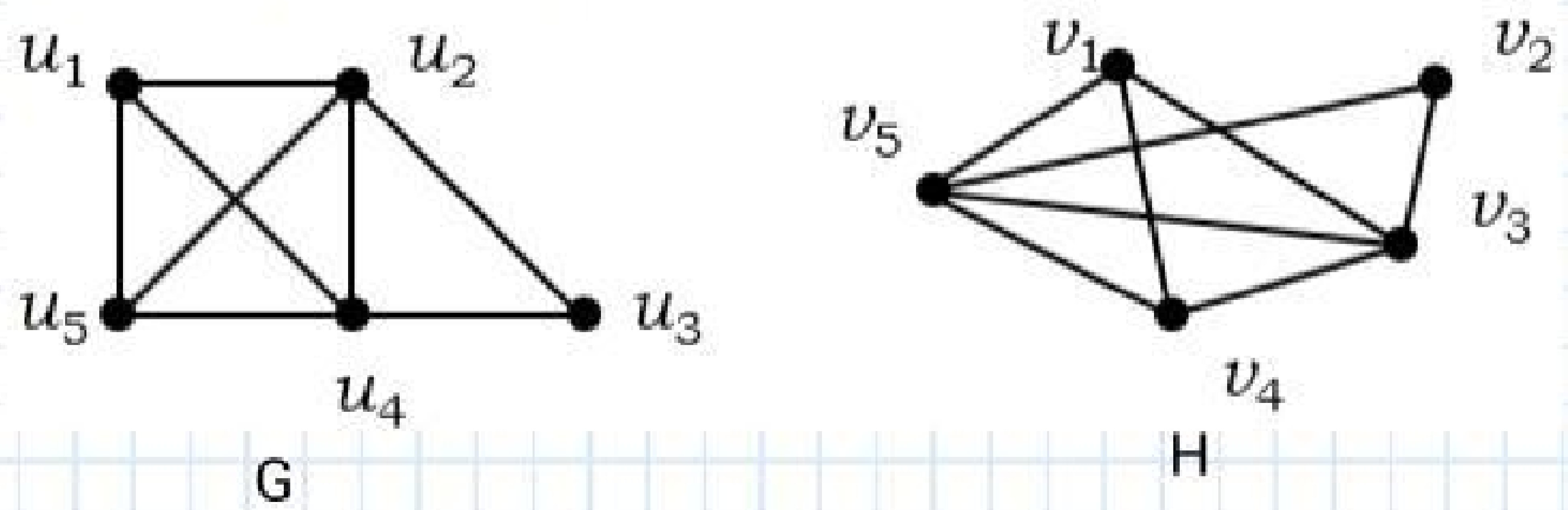
Example: Isomorphism

- **Invariants:** properties that two simple graphs must have in common to be isomorphic
  - Same number of vertices
  - Same number of edges
  - Degrees of corresponding vertices are the same
  - If one is bipartite, the other must be; if one is complete, the other must be; and others …



Are G and H isomorphic?

- **Are these two graphs isomorphic?**



- They both have 5 vertices
- They both have 8 edges
- They have the same number of vertices with the same degrees: 2, 3, 3, 4, 4.

Isomorphism Example

$$\begin{array}{c} \text{G} \\ \begin{array}{c|ccccc} & u_1 & u_2 & u_3 & u_4 & u_5 \\ \hline u_1 & 0 & 1 & 0 & 1 & 1 \\ u_2 & 1 & 0 & 1 & 1 & 1 \\ u_3 & 0 & 1 & 0 & 1 & 0 \\ u_4 & 1 & 1 & 1 & 0 & 1 \\ u_5 & 1 & 1 & 0 & 1 & 0 \end{array} \end{array} \quad \begin{array}{c} \text{H} \\ \begin{array}{c|ccccc} & v_1 & v_2 & v_3 & v_4 & v_5 \\ \hline v_1 & 0 & 0 & 1 & 1 & 1 \\ v_2 & 0 & 0 & 1 & 0 & 1 \\ v_3 & 1 & 1 & 0 & 1 & 1 \\ v_4 & 1 & 0 & 1 & 0 & 1 \\ v_5 & 1 & 1 & 1 & 1 & 0 \end{array} \end{array} \quad \Bigg| \quad \begin{array}{c} \text{H} \rightarrow \text{G?} \\ \begin{array}{c|ccccc} & v_1 & v_3 & v_2 & v_5 & v_4 \\ \hline v_1 & & & & & \\ v_3 & & & & & \\ v_2 & & & & & \\ v_5 & & & & & \\ v_4 & & & & & \end{array} \end{array}$$

- G and H don't appear to be isomorphic.
- However, we haven't tried mapping vertices from G onto H yet.

Isomorphism Example

- Start with the vertices of degree 2 since each graph only has one:

  $\deg(u_3) = \deg(v_2) = 2$ therefore $f(u_3) = v_2$

- Now consider vertices of degree 3

  $\deg(u_1) = \deg(u_5) = \deg(v_1) = \deg(v_4) = 3$

  therefore we must have either one of

  $f(u_1) = v_1$ and $f(u_5) = v_4$ or $f(u_1) = v_4$ and $f(u_5) = v_1$

- Now try vertices of degree 4:

  $\deg(u_2) = \deg(u_4) = \deg(v_3) = \deg(v_5) = 4$

  therefore we must have one of:

  $f(u_2) = v_3$ and $f(u_4) = v_5$ or $f(u_2) = v_5$ and $f(u_4) = v_3$

Isomorphism Example

- There are four possibilities
  - $f(u_1) = v_1, f(u_2) = v_3, f(u_3) = v_2, f(u_4) = v_5, f(u_5) = v_4$
  - $f(u_1) = v_4, f(u_2) = v_3, f(u_3) = v_2, f(u_4) = v_5, f(u_5) = v_1$
  - $f(u_1) = v_1, f(u_2) = v_5, f(u_3) = v_2, f(u_4) = v_3, f(u_5) = v_4$
  - $f(u_1) = v_4, f(u_2) = v_5, f(u_3) = v_2, f(u_4) = v_3, f(u_5) = v_1$

Isomorphism Example

$$
\begin{array}{c}
G \\
\begin{array}{cccccc}
 & u_1 & u_2 & u_3 & u_4 & u_5 \\
u_1 & 0 & 1 & 0 & 1 & 1 \\
u_2 & 1 & 0 & 1 & 1 & 1 \\
u_3 & 0 & 1 & 0 & 1 & 0 \\
u_4 & 1 & 1 & 1 & 0 & 1 \\
u_5 & 1 & 1 & 0 & 1 & 0
\end{array}
\end{array}
\quad
\begin{array}{c}
H \\
\begin{array}{cccccc}
 & v_1 & v_2 & v_3 & v_4 & v_5 \\
v_1 & 0 & 0 & 1 & 1 & 1 \\
v_2 & 0 & 0 & 1 & 0 & 1 \\
v_3 & 1 & 1 & 0 & 1 & 1 \\
v_4 & 1 & 0 & 1 & 0 & 1 \\
v_5 & 1 & 1 & 1 & 1 & 0
\end{array}
\end{array}
\quad\Bigg|\quad
\begin{array}{c}
H' \\
\begin{array}{cccccc}
 & v_1 & v_3 & v_2 & v_5 & v_4 \\
v_1 & 0 & 1 & 0 & 1 & 1 \\
v_3 & 1 & 0 & 1 & 1 & 1 \\
v_2 & 0 & 1 & 0 & 1 & 0 \\
v_5 & 1 & 1 & 1 & 0 & 1 \\
v_4 & 1 & 1 & 0 & 1 & 0
\end{array}
\end{array}
$$

- We permute the adjacency matrix of *H* (per function choices above) to see if we get the adjacency of *G*. Let's try:

$$f(u_1) = v_1, f(u_2) = v_3, f(u_3) = v_2, f(u_4) = v_5, f(u_5) = v_4$$

- Does G = H'? Yes!

Isomorphism - Application

- Chemists use molecular graphs to model chemical compounds. Vertices represent atoms and edges represent chemical bonds. When a new compound is synthesized, a database of molecular graphs is checked to determine whether the graph representing the new compound is isomorphic to the graph of a compound that this already known.
- Electronic circuits are modeled as graphs in which the vertices represent components and the edges represent connections between them.
    - Graph isomorphism is the basis for the verification that a particular layout of a circuit corresponds to the design's original schematics.
    - determining whether a chip from one vendor includes the intellectual property of another vendor.