# Graph Algorithms

CS3104

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# All Pairs Shortest Path

- *The problem:* find the shortest path between every pair of vertices of a graph
- ***The graph:*** may contain negative edges but no negative cycles
- ***A representation:*** a weight matrix where
  - W(i,j)=0 if i=j
  - W(i,j)=$\infty$ if there is no edge between i and j
  - W(i,j)="weight of edge"

# All Pairs Shortest Path

- Run **BELLMAN-FORD** once from each vertex:
  - $O(V^2E)$, which is $O(V^4)$ if the graph is dense ($E = \Theta(V^2)$)
- If no negative-weight edges, could run **Dijkstra's** algorithm once from each vertex:
  - $O(VE\lg V)$ with binary heap, $O(V^3\lg V)$ if the graph is dense
- We can solve the problem in $O(V^3)$, with no elaborate data structures

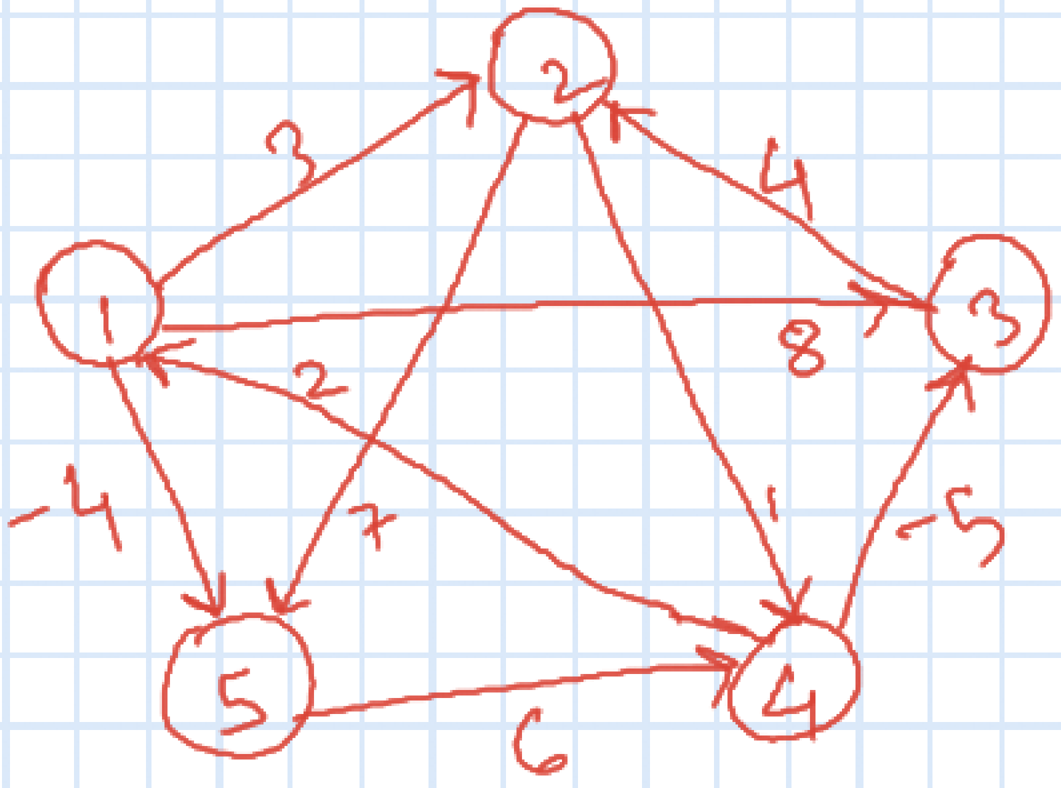
# The Floyd-Warshall algorithm

$$d_{ij}^{(k)} = \begin{cases} w_{ij} & \text{if } k = 0 \\ \min\left(d_{ij}^{(k-1)},\ d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right) & \text{if } k \geq 1 \end{cases}$$

Constructing the shortest path.

$$\pi_{ij}^{(0)} = \begin{cases} \text{NIL} & \text{if } i == j \text{ or } w_{ij} = \infty \\ i & \text{if } i \neq j \text{ and } w_{ij} < \infty \end{cases}$$

$$\pi_{ij}^{(k)} = \begin{cases} \pi_{ij}^{(k-1)} & \text{if } d_{ij}^{k-1} \leq d_{ik}^{k-1} + d_{kj}^{k-1} \\ \pi_{kj}^{k-1} & \text{if } d_{ij}^{k-1} > d_{ik}^{k-1} + d_{kj}^{k-1} \end{cases}$$




$$D^{(0)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix} \end{array}$$

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$$\min\Big(\underbrace{d^0_{23}}_{\infty}, \underbrace{d^0_{21} + d^0_{13}}_{\infty + 8}\Big)$$

$$\min\Big(\underbrace{d^0_{24}}_{1}, \underbrace{d^0_{21} + d_{14}}_{\infty + (-\infty)}\Big)$$

$$\min\Big(\underbrace{d^0_{25}}_{7}, \underbrace{d^0_{21} + d^0_{15}}_{\infty + (-4)}\Big)$$

$$\min\Big(\underline{d^0_{32}}, \underline{d^0_{31} + d^0_{12}}\Big)$$

$$\min\Big(\underbrace{d^0_{42}}_{\infty}, \underbrace{d^0_{41} + d^0_{12}}_{2+3}\Big) \quad \pi^{(0)}_{12} = 1$$

$$D^{(1)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix} \end{array}$$

$$\Pi^{(1)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} NIL & 1 & 1 & NIL & 1 \\ NIL & NIL & NIL & 2 & 2 \\ NIL & 3 & NIL & NIL & NIL \\ 4 & 1 & 4 & NIL & 1 \\ NIL & NIL & NIL & 5 & NIL \end{bmatrix} \end{array}$$

$$D^{(1)} = \begin{array}{c|ccccc|} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 3 & 8 & \infty & -4 \\ 2 & \infty & 0 & \infty & 1 & 7 \\ 3 & \infty & 4 & 0 & \infty & \infty \\ 4 & 2 & 5 & -5 & 0 & -2 \\ 5 & \infty & \infty & \infty & 6 & 0 \\ \hline \end{array}$$

$$\Pi^{(1)} = \begin{array}{c|ccccc|} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & NIL & 1 & 1 & NIL & 1 \\ 2 & NIL & NIL & NIL & 2 & 2 \\ 3 & NIL & 3 & NIL & NIL & NIL \\ 4 & 4 & 1 & 4 & NIL & 1 \\ 5 & NIL & NIL & NIL & 5 & NIL \\ \hline \end{array}$$

$$D^{(2)} = \begin{array}{c|ccccc|} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & & & & & \\ 2 & & & & & \\ 3 & & & & & \\ 4 & & & & & \\ 5 & & & & & \\ \hline \end{array}$$

$$\Pi^{(2)} = \begin{array}{c|ccccc|} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & & & & & \\ 2 & & & & & \\ 3 & & & & & \\ 4 & & & & & \\ 5 & & & & & \\ \hline \end{array}$$

$\vdots$

$D^{(5)} = ?$

$\Pi^{(5)} = ?$

$D^{(0)} = W$

for $k = 1$ to $n$

  for $i = 1$ to $n$

    for $j = 1$ to $n$

      $d_{ij}^{(k)} = \min\left(d_{ij}^{(k-1)},\ d_{ik}^{(k-1)} + d_{kj}^{k-1}\right) \rightarrow O(1)$

return $D^{(n)}$

Total running time $\rightarrow O(n^3)$

$n$ = total number of vertices