Graph Algorithms

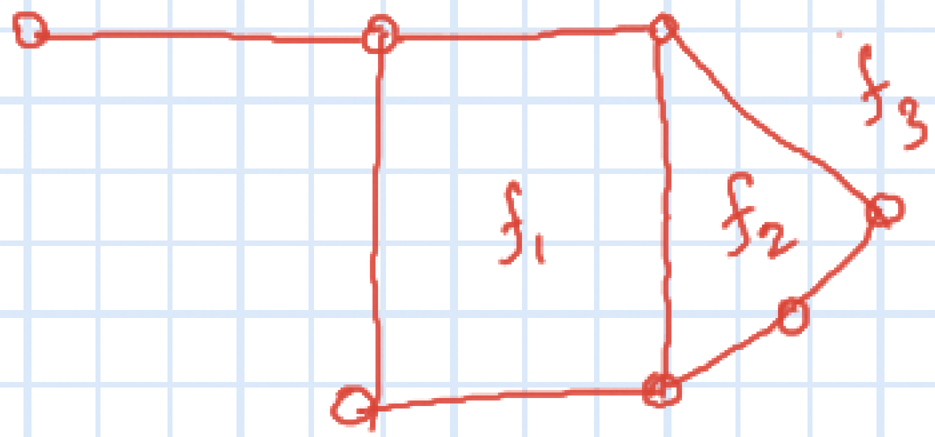
CS3104

Dr. Samit Biswas, *Assistant Professor,*
Department of Computer Sc. and Technology,
Indian Institute of Engineering Science and Technology, Shibpur

Email: samit@cs.iiests.ac.in

Characterization of Planar Graphs:

If f is any face then degree on f is the number of edges encountered in a walk around the boundary on the face.



$d(f_1) = 4$

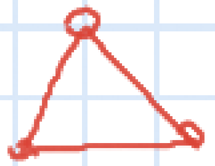
$d(f_2) = 4$

$d(f_3) = 8$

$\boxed{\text{Sum of face degree equal to } 2e.}$

Lemma 1: Let G be a connected planar simple graph with $v \geq 3$ vertices and e edges then we can say $\boxed{e \leq 3v - 6}$

Proof: Each face must have degree ≥ 3.



$$2e = \sum_{f} \deg(f) \geq 3f$$

So $2e \geq 3f$

$$\Rightarrow f \leq \frac{2e}{3}$$

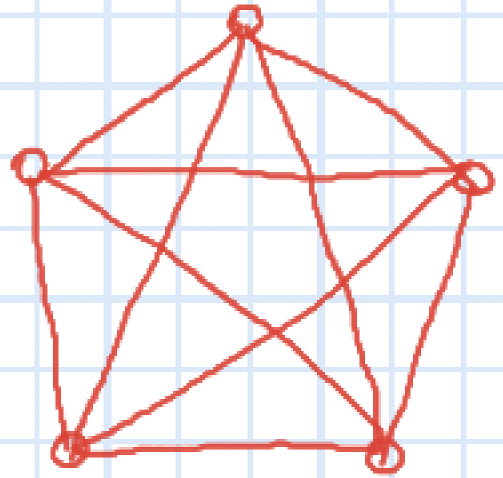
Euler formula, $v - e + f = 2$

$$\Rightarrow v - e + \frac{2e}{3} \geq 2$$

$$\Rightarrow v - 2 \geq e/3$$

$$\Rightarrow e \leq 3v - 6$$

K_5 is non Planar.



K_5

$v = 5$

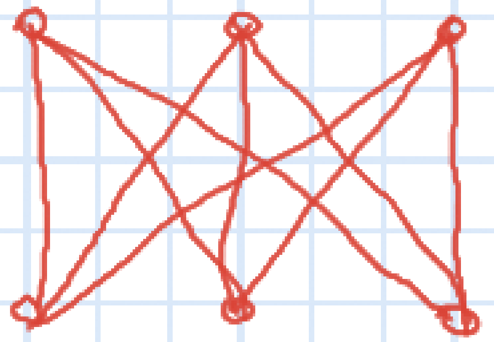
$e = 10$

$3v - 6 = 3*5 - 6 = 9$

$\boxed{e \leq 3v - 6}$ - Contradiction

K_5 is non - Planar.

$K_{3,3}$ is non-Planar



$v = 6$

$e = 9$

$\boxed{e \leq 3v - 6}$

$e = 9$ $\leq$ $3v - 6$

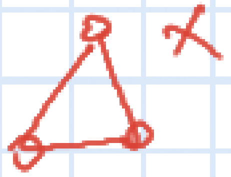
$= 3 \times 6 - 6$

$= 12$

we can not use lemma-1 to prove $K_{3,3}$ as non-planar

Lemma 2: Let G be a connected planar simple graph with v vertices, e edges and no triangles then

$$\boxed{e \le 2v - 4}$$



Proof: G is a triangle free graph

So each face has degree $\geqslant 4$

$$2e = \sum_f \deg(f) \geqslant 4f$$

$$\Rightarrow \frac{e}{2} \geqslant f$$

Use Euler formula $\quad v - e + f = 2$

$$\Rightarrow v - e + \frac{e}{2} \geqslant 2$$

$$\Rightarrow v - e \geqslant \frac{e}{2}$$

$$\Rightarrow \boxed{e \le 2v - 4}$$

$K_{3,3}$ is not planar

$$\boxed{e \leq 2v-4}$$

Proof.

Suppose $K_{3,3}$ is planar

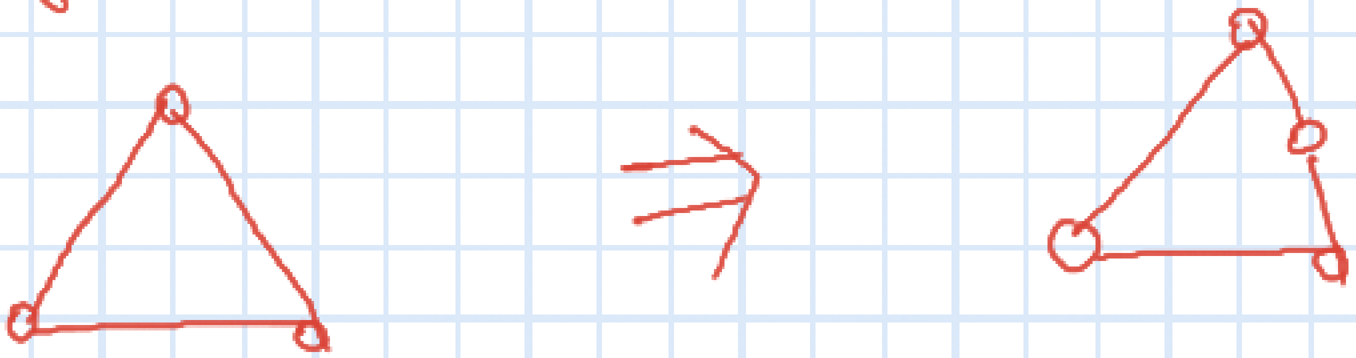
So, $K_{3,3}$ has $v=6$, $e=9$, triangle free, so it follows the lemma:

$$e = 9 \leq 2v-4$$
$$= 2*6 - 4$$
$$= 8$$

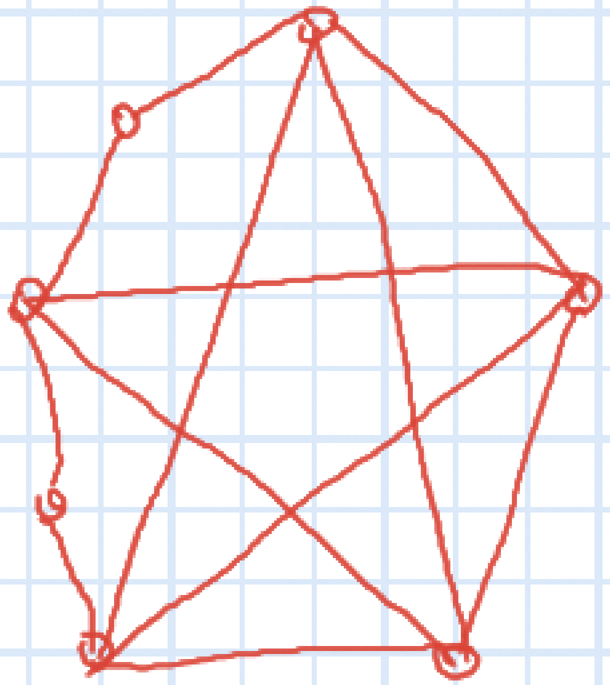
This is a contradiction, it shows that $K_{3,3}$ is non-planar.

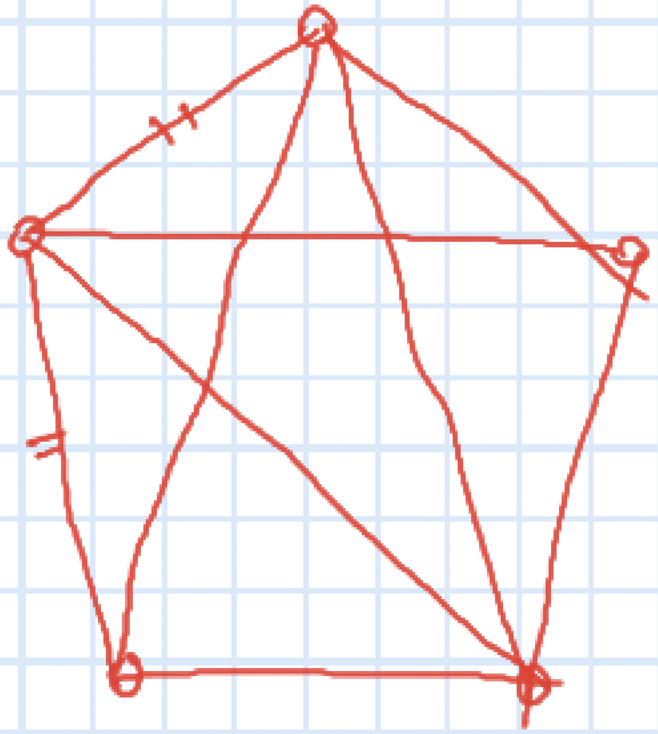
Kuratowski's theorem (1930)

Two graphs are homeomorphic if one can make one graph into the other by edge subdivision



Theorem A finite graph G is planar iff it has no subgraph that is homeomorphic to K_5 or $K_{3,3}$





G

G is homeomorphic to K_5

So G is non-planar

$\boxed{K_5, K_{3,3}}$ — non-planar.