Graph Fundamentals

Graph

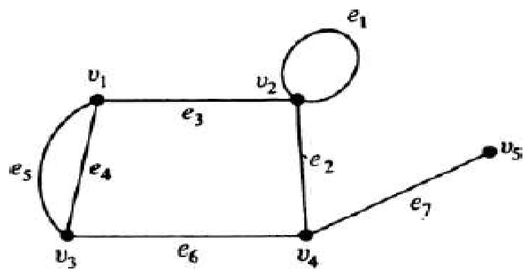


Fig. 1-1 Graph with five vertices and seven edges.

- A (linear) graph $G = (V, E)$ consists of a set of objects $V= \{v_1,v_2,\ldots..\}$ called vertices, and another set $E = \{e_1,e_2,\ldots..\}$, whose elements are called edges, such that each edges e_k is identified with an unordered pair (v_i,v_j) of vertices.
- The vertices v_i,v_j associated with edges e_k are called end vertices of e_k.

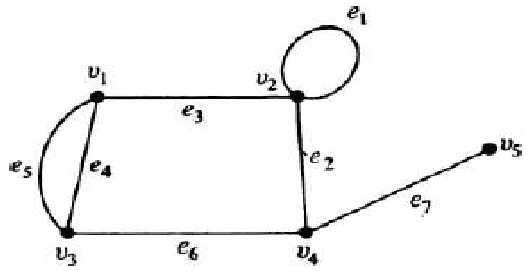


Fig. 1-1 Graph with five vertices and seven edges.

- The edge having the same vertex as both its end vertices is called a self-loop or simply a loop.
- More than one edges associated with same end vertices are called parallel edges.
- A graph that has neither self-loops nor parallel edges is called a simple graph.
- A graph that has self-loops and/or parallel edges is called a general graph.

- A graph is also called a linear complex, 1-complex or a one-dimensional complex.
- A vertex is also referred to as a node, a junction, a point, 0-cell, or an 0-simplex.
- An edge is also referred to as a branch, a line, an element, a 1-cell, an arc, and a 1-simplex.

Applications of graph

- Konigsberg Bridge Problem

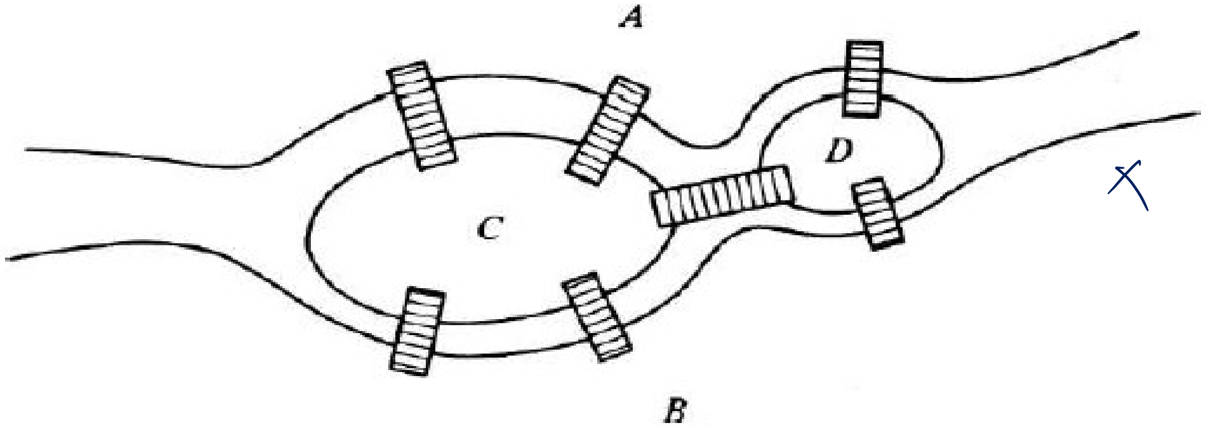
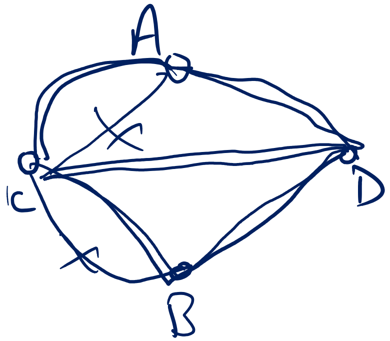


Fig. 1-4 Königsberg bridge problem.



- Utilities Problem

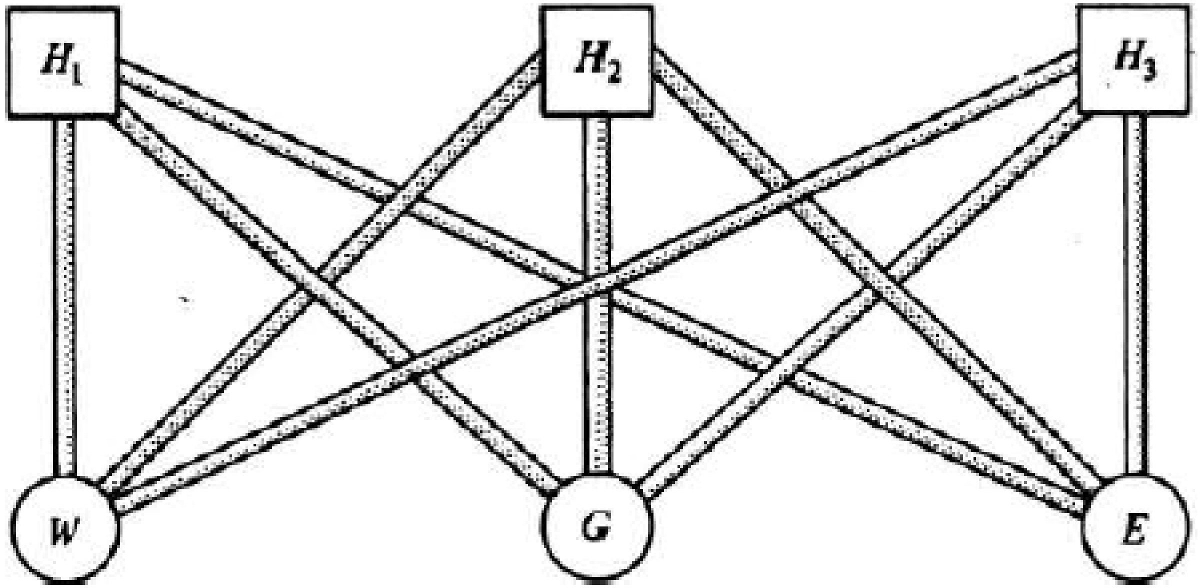


Fig. 1-6 Three-utilities problem.

- Electrical Network Problem
-

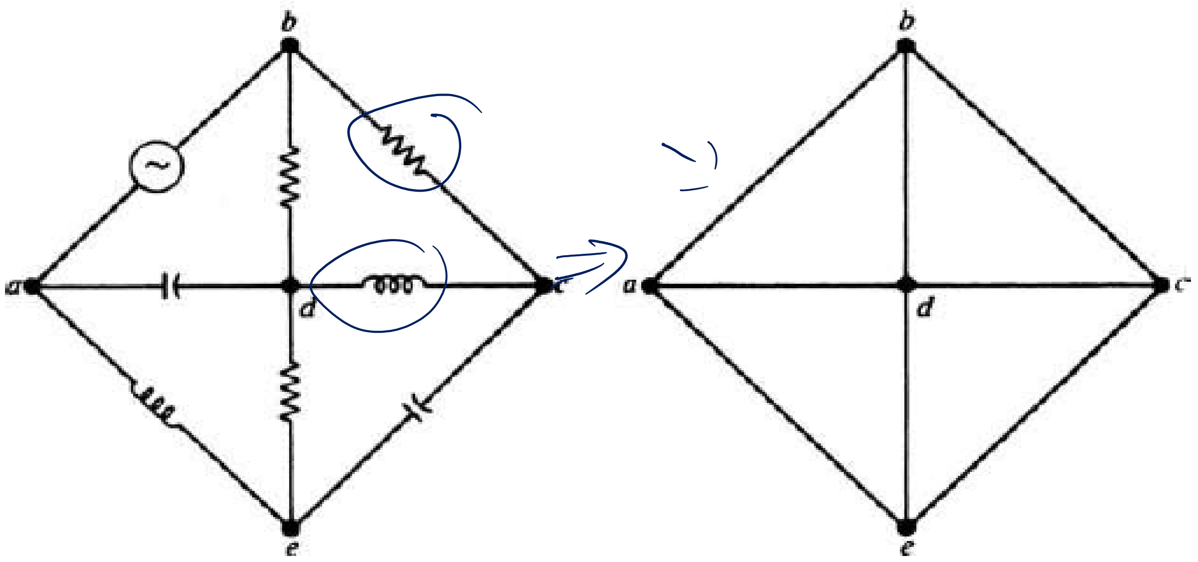


Fig. 1-8 Electrical network and its graph.

$$\frac{9-1}{2} = 4$$

$$\frac{16-2}{2} = 7$$

- Seating Problem: Nine members of a club meet each day for lunch at a round table. They decide to sit such that every member has different neighbours at each lunch. How many days can this arrangement last?
- Given n people, possible number of such arrangements is (n-1)/2 if n is odd, and (n-2)/2 if n is even.
- Solution: 1234567891, 1352749681, 1573928461, 1795836241

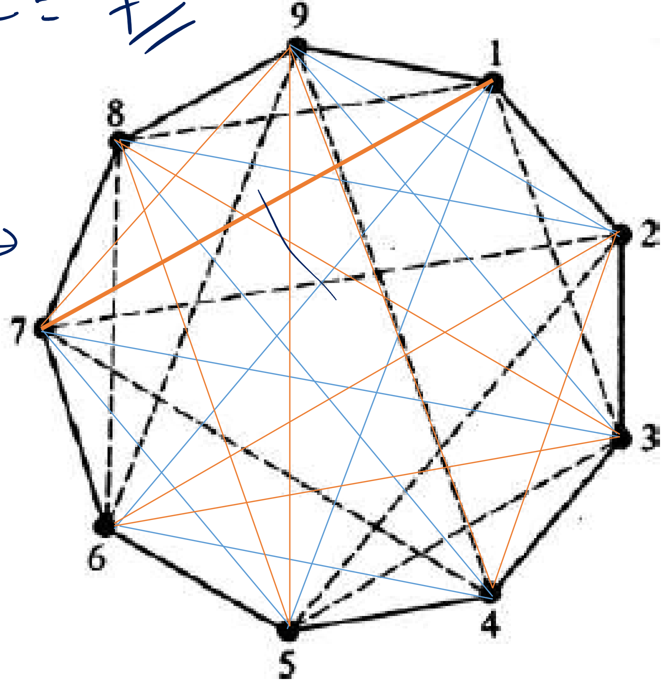


Fig. 1-9 Arrangements at a dinner table.

- A graph with a finite number of vertices as well as a finite number of edges is called a finite graph, otherwise, it is an infinite graph.

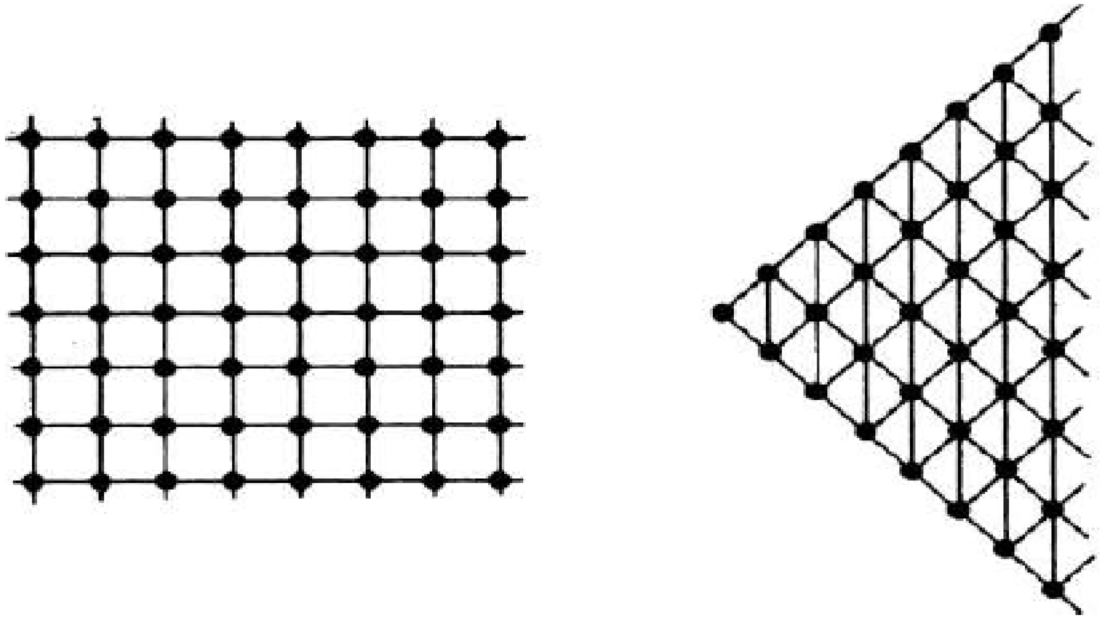


Fig. 1-10 Portions of two infinite graphs.

Incidence and Degree

- When a vertex v_i is an end vertex of some edge e_j, v_i and e_j are said to be incident with each other.
- Two nonparallel edges are said to be adjacent if they are incident on a common vertex.
- Two vertices are said to be adjacent if they are the end vertices of the same edge.

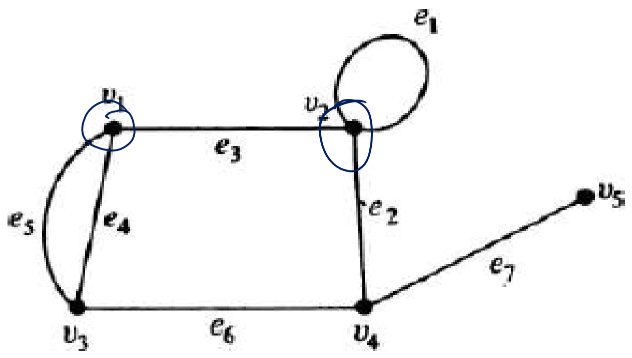


Fig. 1-1 Graph with five vertices and seven edges.

- The number of edges incident on a vertex v_i, with self-loop counted twice, is called the degree, $d(v_i)$, of vertex v_i. The degree of a vertex is sometimes also referred to as its valency.
- Handshaking Theorem: The sum of the degree of all vertices in G is twice the number of edges in G.

$$\sum_{i=1}^{n} d(v_i) = 2e.$$

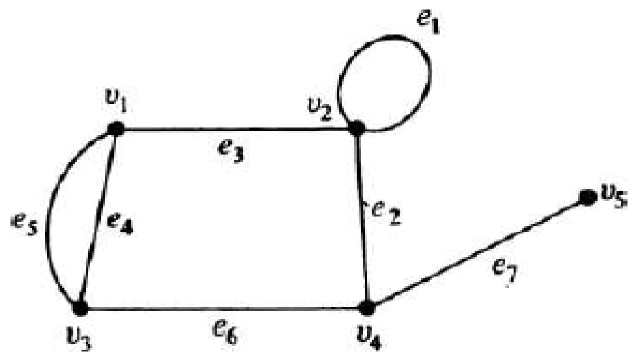


Fig. 1-1 Graph with five vertices and seven edges.

- THEOREM 1-1: The number of vertices of odd degree in a graph is always even.
- A graph in which all vertices are of equal degree is called a regular graph.

$$\sum_{i=1}^{n} d(v_i) = \sum_{\text{even}} d(v_j) + \sum_{\text{odd}} d(v_k)$$

$$\boxed{\text{even}} = \text{even} + \_\_\_$$

- A vertex having no incident edge is called a isolated vertex. Isolated vertices are vertices with zero degree.
- A vertex of degree one is called a pendant vertex or an end vertex.
- Two adjacent edges are said to be in series if their common vertex is of degree two.

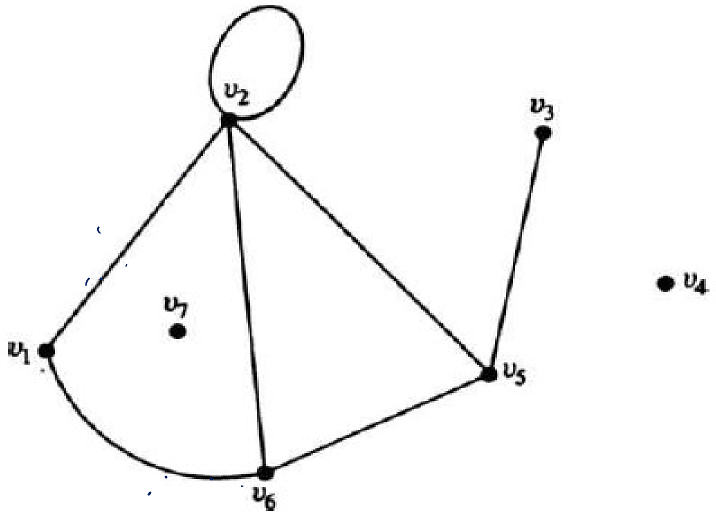


Fig. 1-11 Graph containing isolated vertices, series edges, and a pendant vertex.

- A graph without any edge is called a null graph. Every vertex in a null graph is an isolated vertex.
- For a graph, an edge set E can be empty, but vertex set V cannot be empty; otherwise there is no graph. A graph must have at least one vertex.

$G=(V,E)$ $E=\phi$

but! $V \neq \phi$

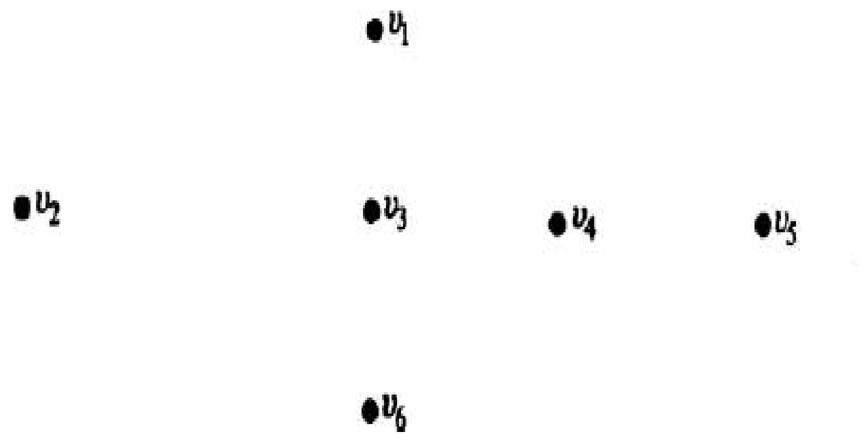
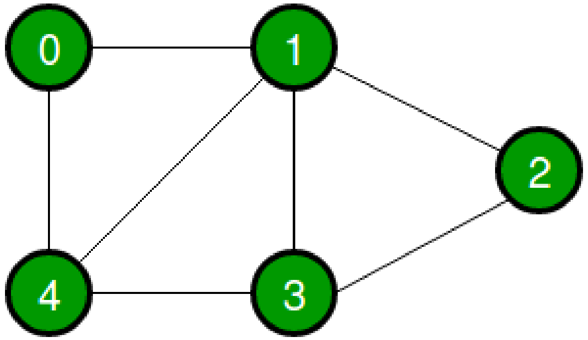


Fig. 1-12 Null graph of six vertices.

Graph Representation

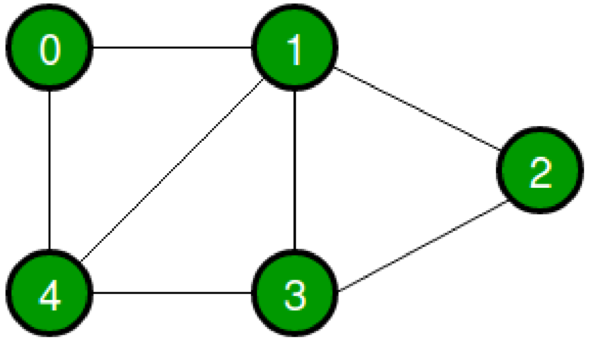
- Adjacency Matrix is a 2D array of size V x V where V is the number of vertices in a graph.
- Let the 2D array be adj[][], a slot adj[i][j] = 1 indicates that there is an edge from vertex i to vertex j.
- Adjacency matrix for undirected graph is always symmetric.
- Adjacency Matrix is also used to represent weighted graphs. If adj[i][j] = w, then there is an edge from vertex i to vertex j with weight w.

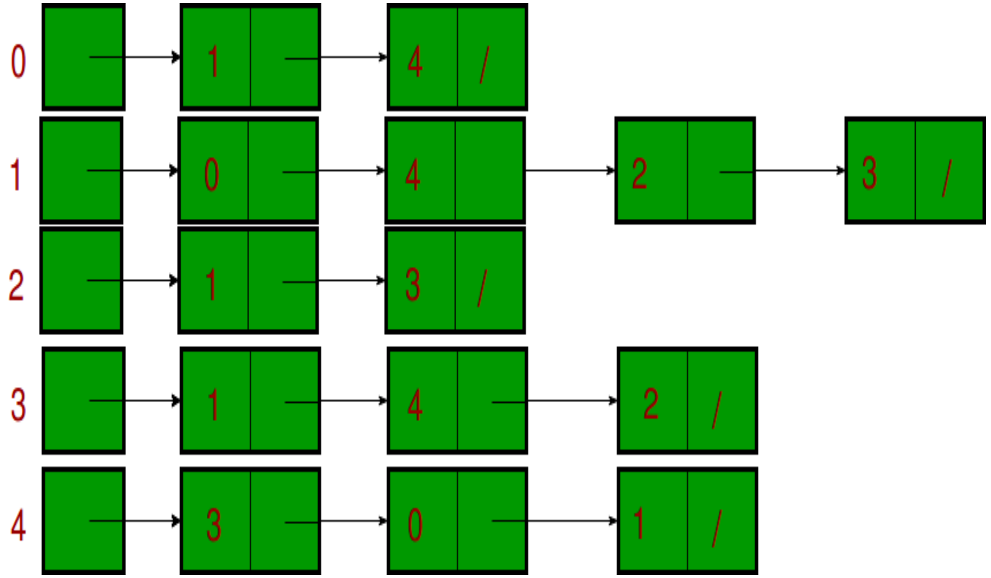


	0	1	2	3	4
0	0	1	0	0	1
1	1	0	1	1	1
2	0	1	0	1	0
3	0	1	1	0	1
4	1	1	0	1	0

https://www.geeksforgeeks.org/graph-and-its-representations/

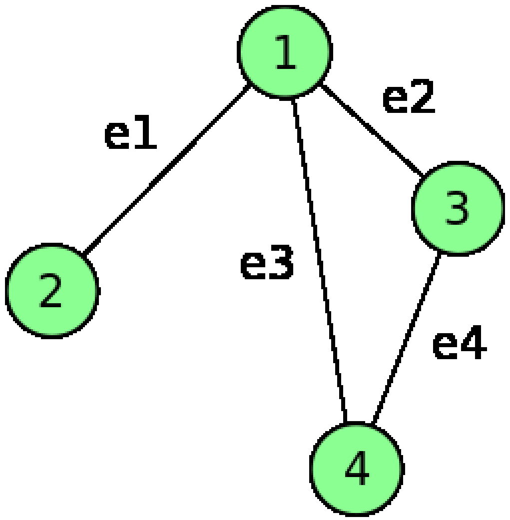
- **Adjacency List:** An array of lists is used.
- The size of the array is equal to the number of vertices.
- Let the array be an array[]. An entry array[i] represents the list of vertices adjacent to the ***i***th vertex.
- This representation can also be used to represent a weighted graph. The weights of edges can be represented as lists of pairs. Following is the adjacency list representation of the above graph.





https://www.geeksforgeeks.org/graph-and-its-representations/

- Incidence Matrix



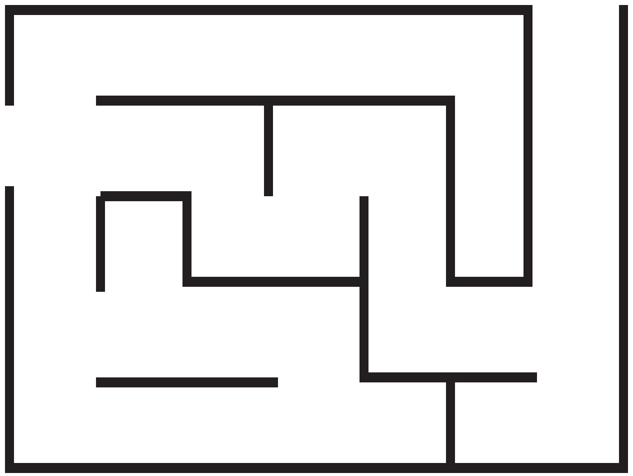
	e_1	e_2	e_3	e_4
1	1	1	1	0
2	1	0	0	0
3	0	1	0	1
4	0	0	1	1

$$= \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

https://en.wikipedia.org/wiki/Incidence_matrix

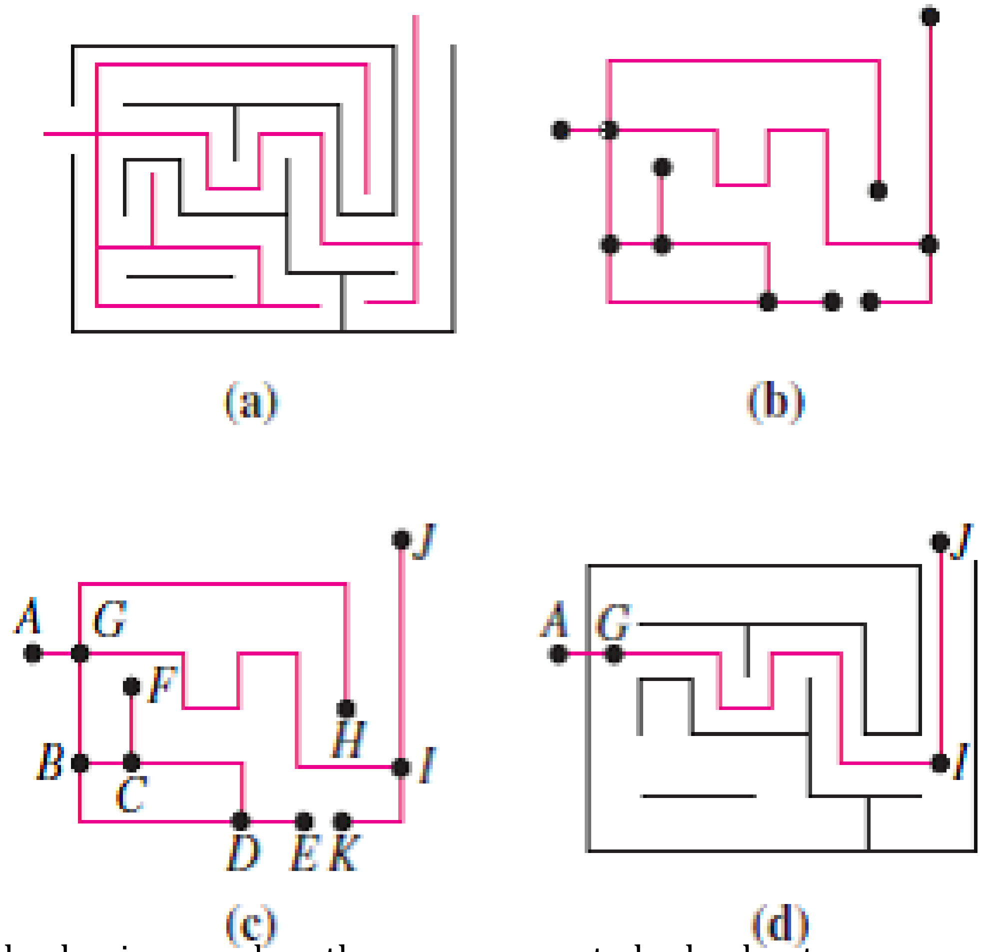
Maze representation using graph

- A maze can be represented and solved by using a graph.
- The pathways can be represented by edges, and the junctions and endpoints of the paths can be represented by vertices.
- One way to solve a maze is to construct a graph that represents the maze and find a path that leads from the starting vertex to the ending vertex of that graph.



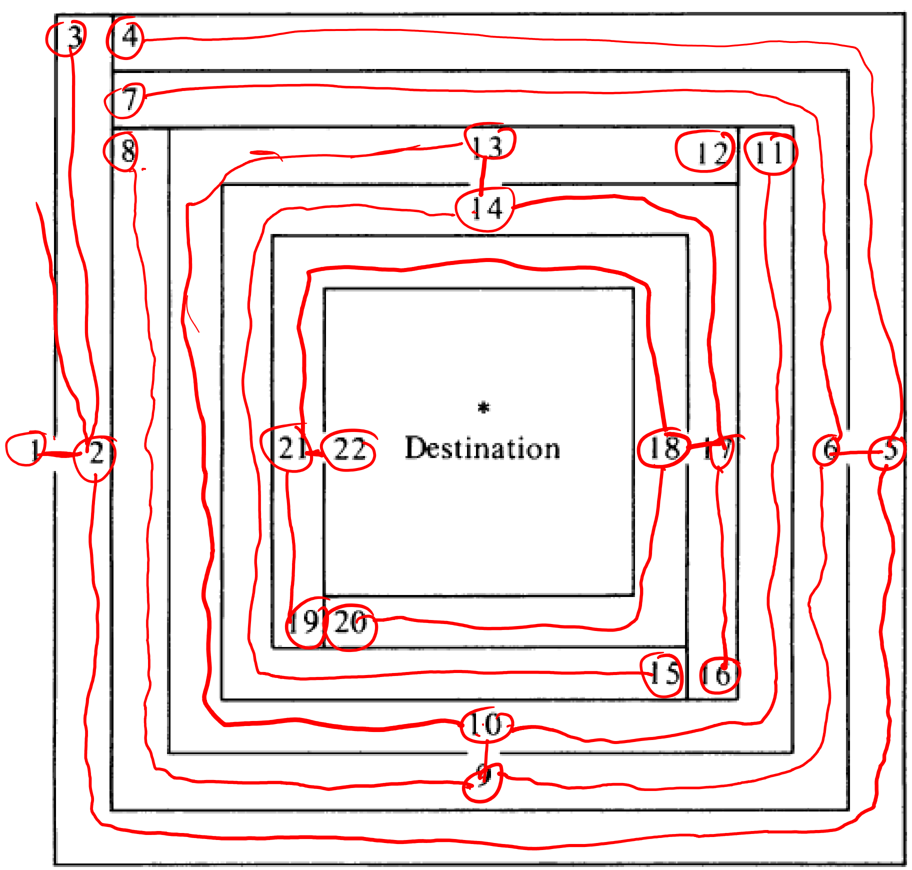
https://www.chegg.com/homework-help/maze-represented-solved-using-graph-pathways-represented-edg-chapter-6.2-problem-46p-solution-9780495388838-exc

- The first step in creating a graphical representation of the maze is to draw all pathways, as shown in (a).
- Next, identify all vertices at junctions (intersections) and endpoints as shown in (b).
- After labeling all vertices, as shown in (c), we see that the path that leads from vertex A, at the entrance, to vertex J, at the exit, is AGIJ.
- This path is the solution to the maze, as illustrated in (d).



https://www.chegg.com/homework-help/maze-represented-solved-using-graph-pathways-represented-edg-chapter-6.2-problem-46p-solution-9780495388838-exc

Exercise: Represent this maze by means of a graph such that a vertex denotes either a corridor or a dead end. An edge represents a possible path between two vertices. What is the length of a path from the entrance to the center of the maze.

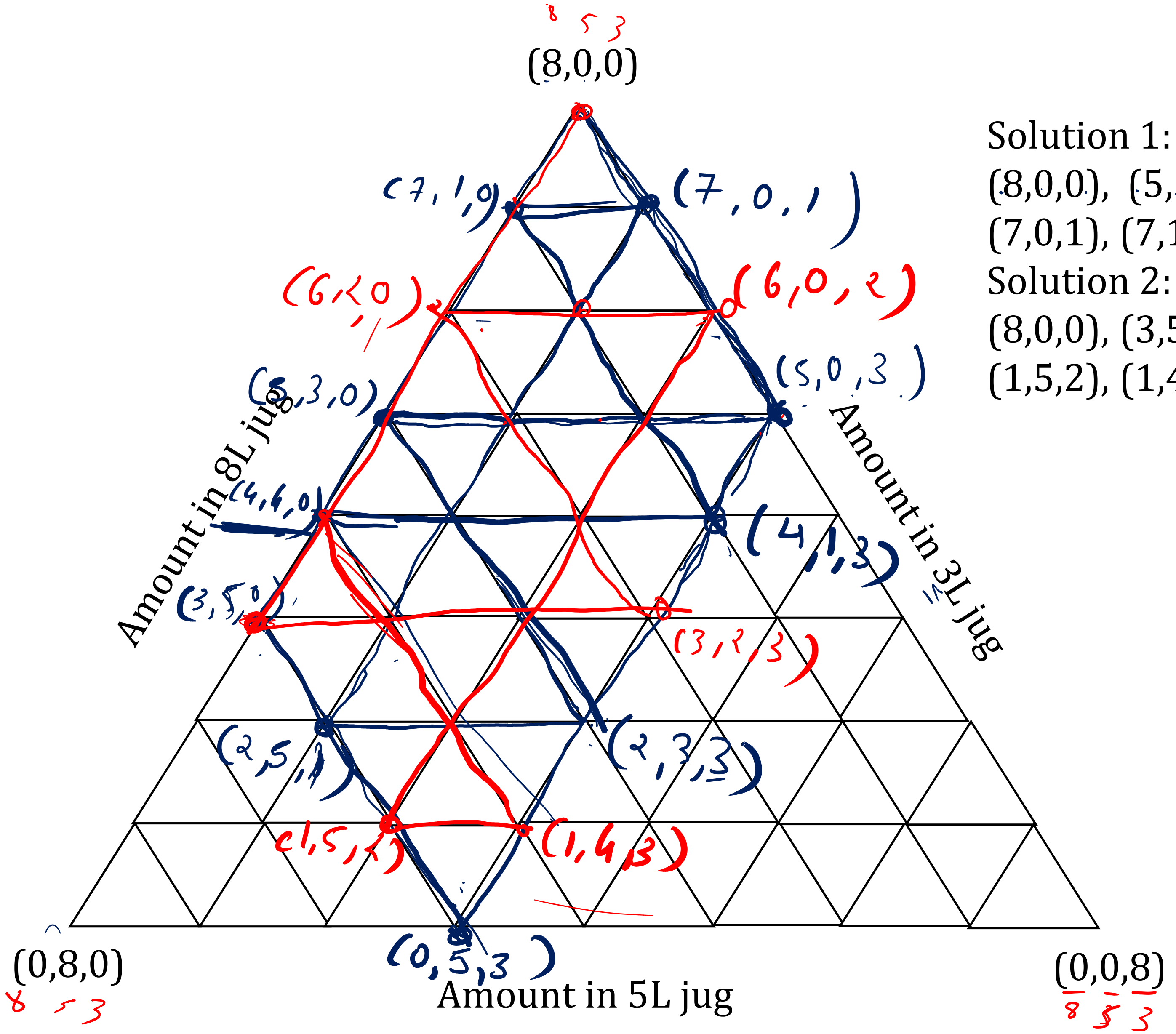


Solving Decanting Problem using graph

- You are given three vessels A, B, and C of capacities 8,5, and 3 gallons respectively. A is filled while B and C are empty. Divide the liquid in A into two equal quantities.
  - How many vertices are there in the graph which represents the possible states of jugs?
  - List out the possible states based on the conditions of the jugs?

	initial	end
A	8 L	4 L
B	0 L	4 L
C	0 L	0 L

Equilateral Triangle



Solution 1:
(8,0,0), (5,0,3), (5,3,0), (2,3,3), (2,5,1), (7,0,1), (7,1,0), (4,1,3), (4,4,0)

Solution 2:
(8,0,0), (3,5,0), (3,2,3), (6,2,0), (6,0,2), (1,5,2), (1,4,3), (4,4,0)

https://en.wikipedia.org/wiki/Water_pouring_puzzle

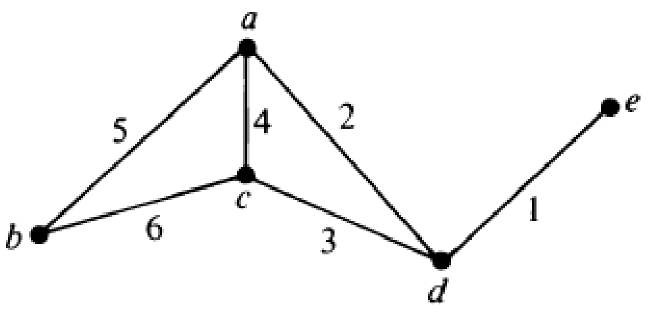
Isomorphic graphs

- Two graphs G and G' are isomorphic to each other if there is a one-to-one correspondence between their vertices and between their edges such that the incidence relationship is preserved.
- Such isomorphism shows identical behaviour in terms of geometric properties. Two graphs are thought of as equivalent if they have identical behaviour in terms of graph-theoretic properties.

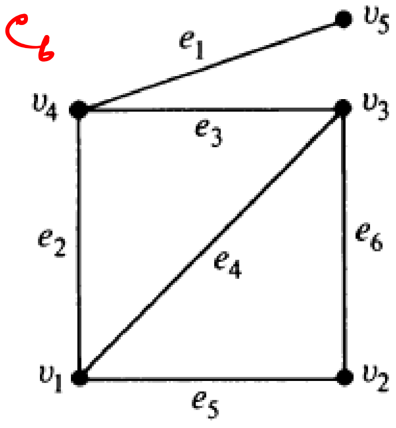
a, b, c, d, e
1, 2, 3, 4, 5, 6

$v_1, v_2, v_3\ v_4, v_5$
$e_1, e_2, e_3, e_4, e_5, e_6$

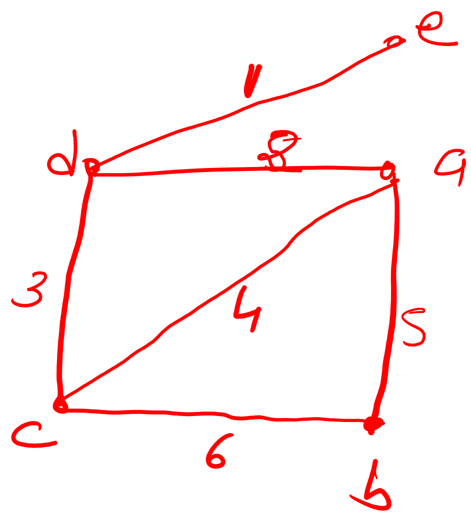
$e = v_5$
$d = v_4$
$b = v_2$
$a = v_3$
$c = v_1$



(a)



(b)



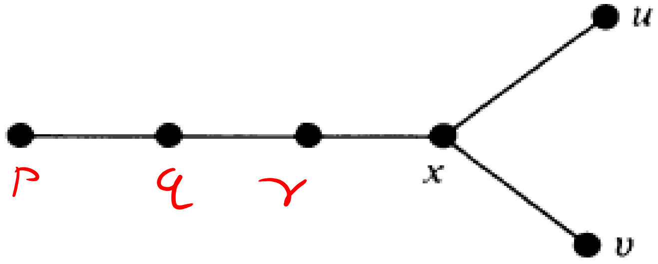
Term paper topic => check for isomorphism

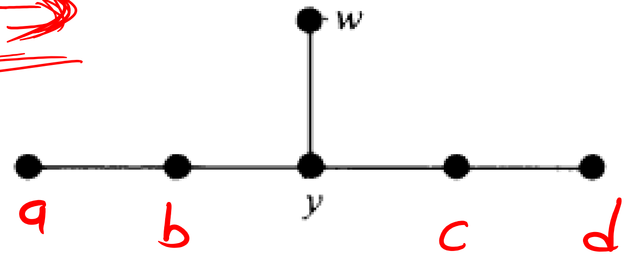
→ Keep record of adjacent vertices
→ check degree of adjacent vertices are same or not.

- Two isomorphic graphs must have:
  - The same number of vertices. →
  - The same number of edges.
  - An equal number of vertices with a given degree.
- However these conditions are necessary but not sufficient. For eg.

n vertices => n! →

d(p) = 1
d(q) = 2
d(r) = 2
d(x) = 3
d(u) = 1
d(v) = 1

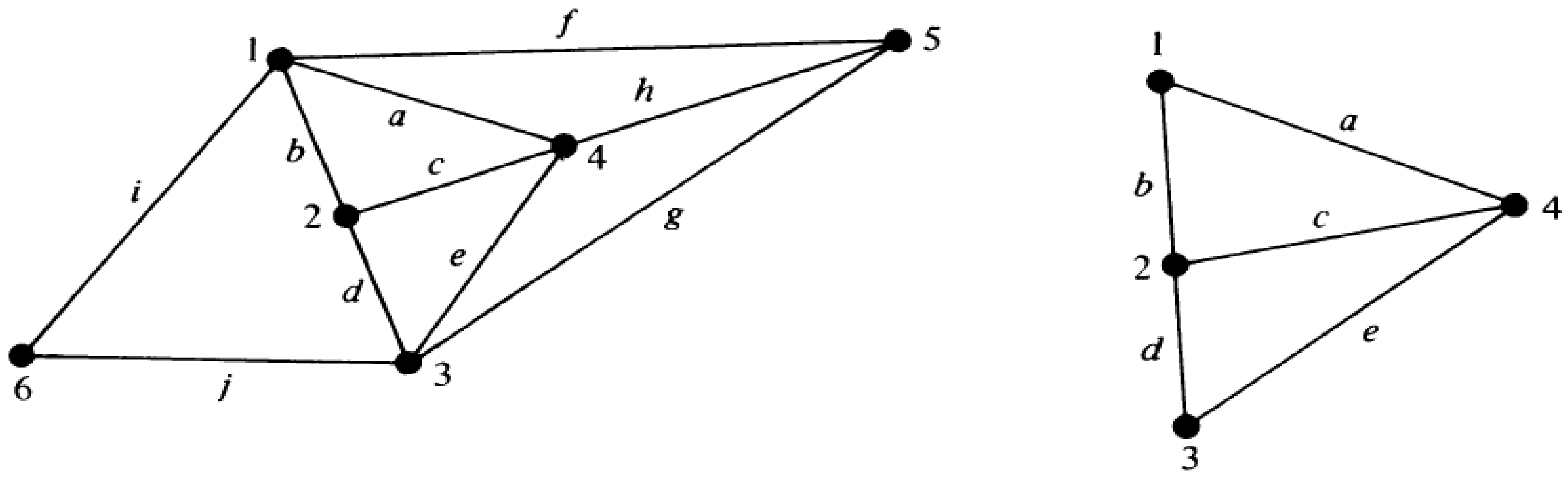




d(w) = 1
d(a) = 1
d(b) = 2
d(y) = 3
d(c) = 2
d(d) = 1

Subgraphs

- A graph g is said to be subgraph of a graph G if all the vertices and all the edges of g are in G.

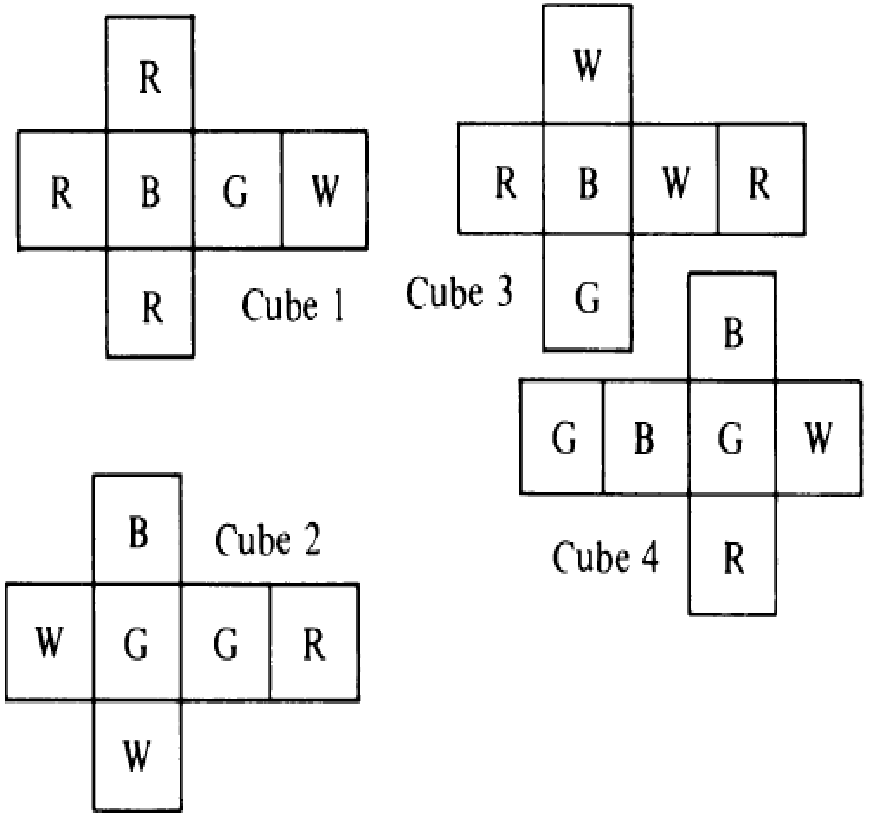


(a) (b)

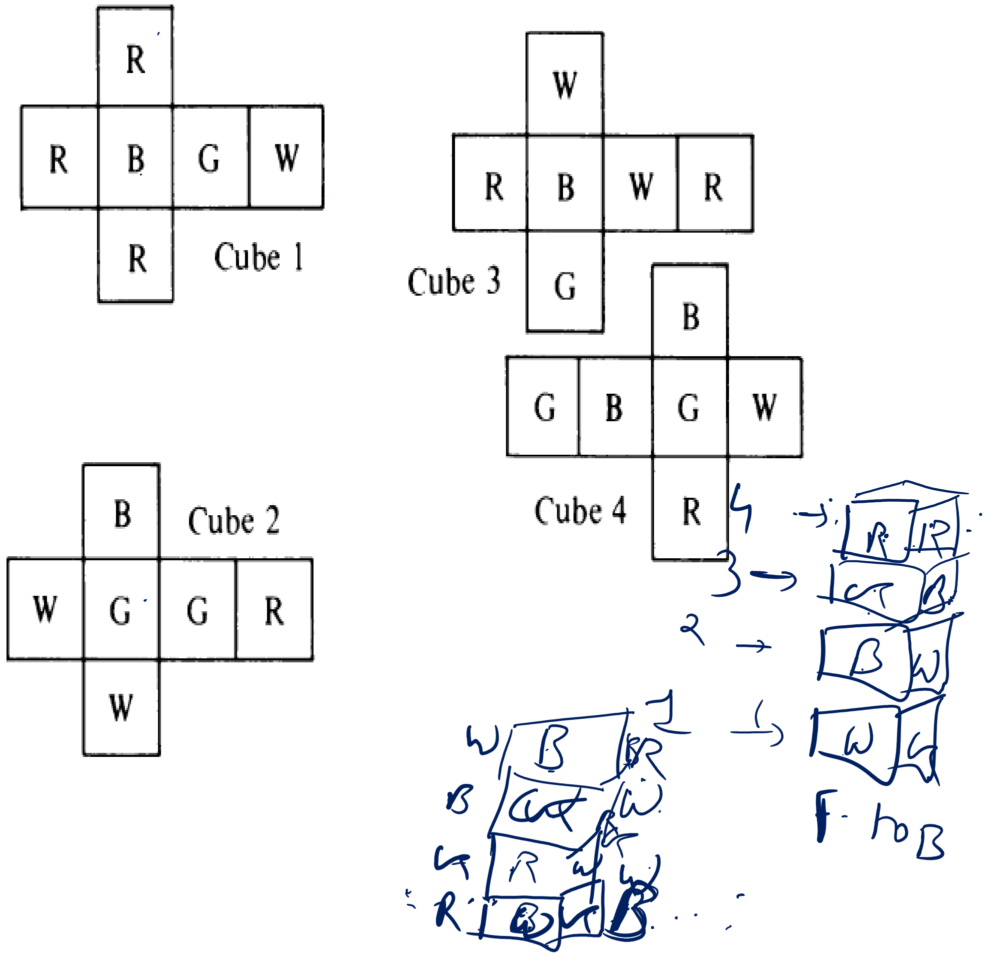
Observations

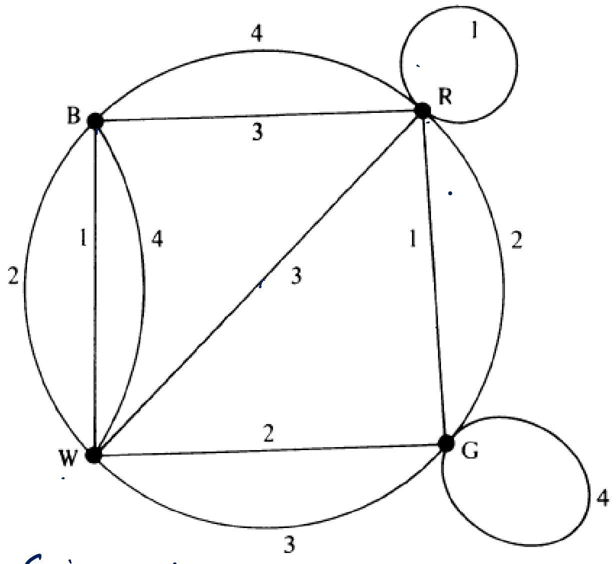
- Every graph is its own subgraph.
- A subgraph of a subgraph of G is a subgraph of G.
- A single vertex in a graph G is a subgraph of G.
- A single edge in G, together with its end vertices, is also a subgraph of G.
- Edge – disjoint subgraphs: Two subgraphs g1 and g2 of a graph G are said to be edge disjoint if g1 and g2 do not have any edge in common.
- Vertex – disjoint subgraphs: Two subgraphs g1 and g2 of a graph G are said to be vertex disjoint if g1 and g2 do not have any vertex in common.

A puzzle with multicolored Cubes

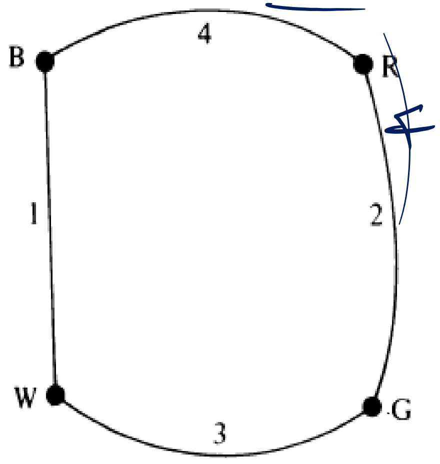


We are given four cubes. The six faces of cube are colored with blue, green, red or white. Is it possible to stack the cubes one on top of another to form a column such that no color appears twice on any of the four sides of the column?

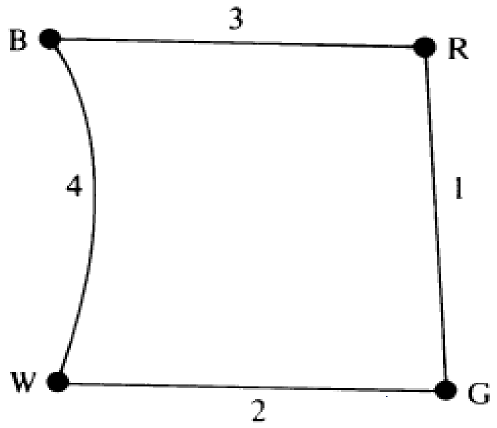




// circuit



(a) North-South Subgraph



(b) East-West Subgraph

Walks, Paths, and Circuits

A walk is defined as a finite alternating sequence of vertices and edges, beginning and ending with vertices, such that each edge is incident with the vertices preceding and following it. No edge appears (is covered or traversed) more than once in a walk. A vertex may appear more than once.

A walk is also referred to as an edge train or a chain.

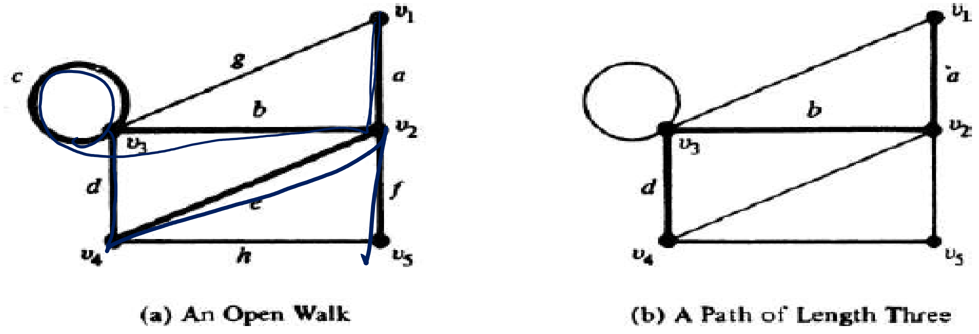


Fig. 2-8 A walk and a path.

$v_1\ a\ v_2\ b\ v_3\ c\ v_3\ d\ v_4\ e\ v_2\ f\ v_5$ is a walk.

True/False: The set of vertices and edges constituting a given walk in a graph G is a subgraph of G.

$V_1 a V_2 f V_5 \to$ path

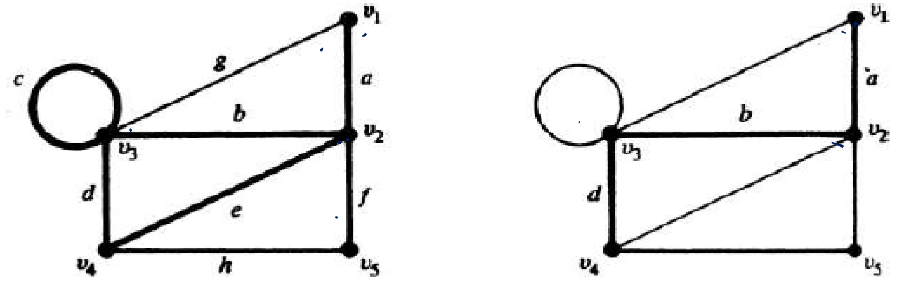
$\Rightarrow V_1 a V_2 b V_3 c V_3 d$

= path

$V_1 a V_2 b V_3 g V_1 a V_2$

$V_1 a V_2 b V_3 g V_1 \to$ closed walk

$V_1 a V_2 f V_3 \to$ open walk



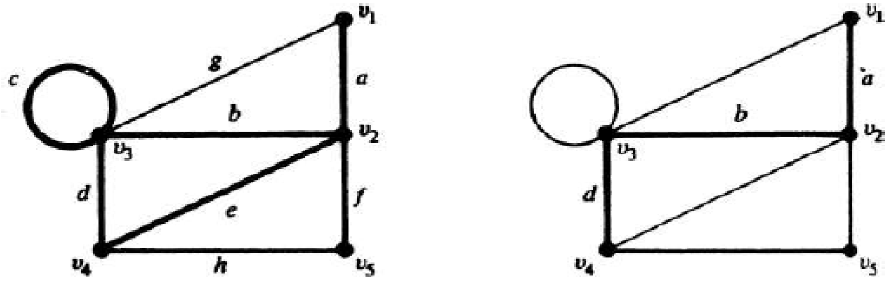
(a) An Open Walk (b) A Path of Length Three

Fig. 2-8 A walk and a path.

- Vertices with which a walk begins and ends are called its terminal vertices. A walk which begins and ends at the same vertex is called a closed walk. A walk which does not begin and end at the same vertex is called an open walk.
- An open walk in which no vertex appears more than once is called a path or simple path or an elementary path.
- A path does not interest itself.
- The number of edges in a path is called the length of a path.
- **True/False**: A self-loop can be included in a walk but not in path.

$v_1 a v_2 b v_3 g v_1$ $\underline{\underline{v_3 c v_3}}$

→ circuit

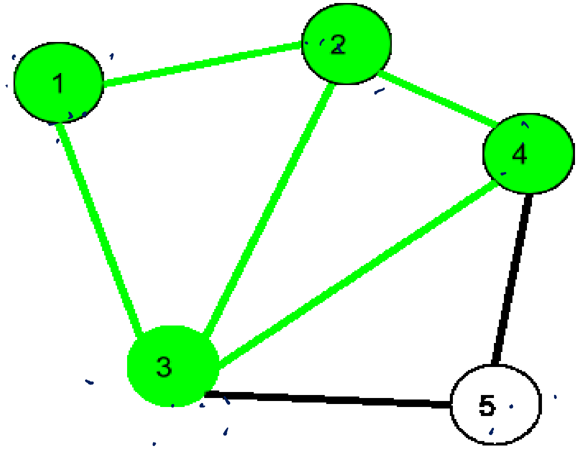


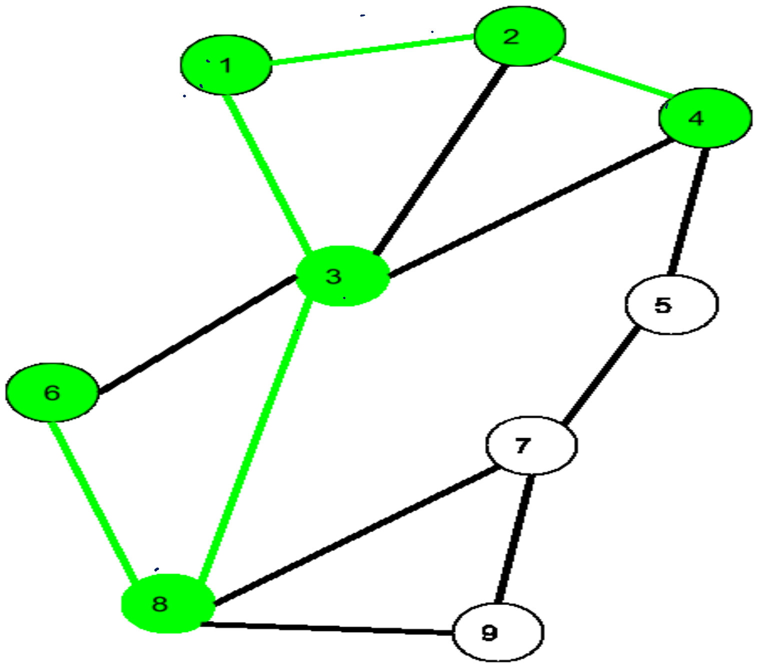
(a) An Open Walk (b) A Path of Length Three

Fig. 2-8 A walk and a path.

- The terminal vertices of a path are of degree one, and the rest of the vertices (intermediate vertices) are of degree two.
- A closed walk in which no vertex (except the initial and the final vertex) appears more than once is called a circuit. A circuit is a closed, non-intersecting walk.
- A circuit is also called a cycle, elementary cycle, circular path, and polygon.
- Every vertex in a circuit is of degree two.
- **True/False**: Every self-loop is a circuit, but not every circuit is a self-loop.

Example: Walk, path and circuit





Here 1->2->3->4->2->1->3 is a walk.

1->2->3->4->5->3-> is an open walk.

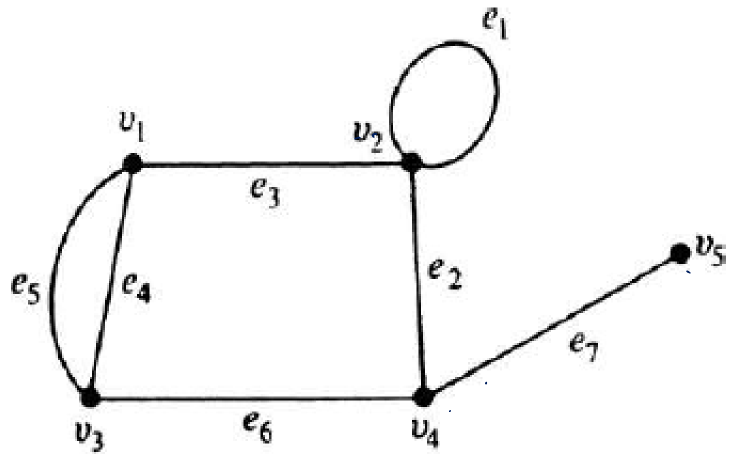
1->2->3->4->5->3->1-> is a closed walk.

Here 6->8->3->1->2->4 is a Path.

Here 1->2->4->3->1 is a circuit.

Connected graphs, Disconnected graphs and components

- A graph G is said to be connected if there is at least one path between every pair of vertices in G. Otherwise, G is disconnected.



Connected Graph

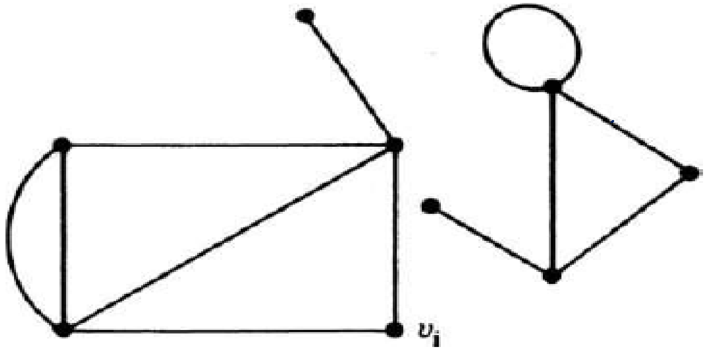


Fig. 2-11 A disconnected graph with two components.

- A null graph with more than one vertex is disconnected.
- A disconnected graph consists of two or more connected graphs. Each of these connected subgraphs is called a component.

- **Theorem 2-1:** A graph G is disconnected if and only if its vertex set V can be partitioned into two nonempty, disjoint sets V_1 and V_2 such that there exists no edge in G whose one end vertex is in subset V_1 and the other in subset V_2.
- **Theorem 2-2:** If a graph has exactly two vertices of odd degree, there must be a path joining these two vertices.
- **Theorem 2-3:** A simple graph with n vertices and k components can have at most $(n-k)(n-k+1)/2$ edges.

~~how many~~ maximum no. of edges in a simple graph of n vertices $= n(n-1)/2$

$\Rightarrow n_1, n_2 \ldots n_k \in N$ $\qquad n_1 + n_2 + \ldots + n_k = n$

$n_i \geq 1$

Then

$$\sum_{i=1}^{k} n_i^2 \leq \left(\sum_{i=1}^{k} n_i\right)^2 - (k-1)\left(2\sum_{i=1}^{k} n_i - k\right) \Rightarrow (n-k)(n-k+1)/2$$

$$\Rightarrow (n_1 - 1) + (n_2 - 1) + \ldots + (n_k - 1) = n_1 + n_2 + \ldots + n_k - k.$$

square on both sides

$$\Rightarrow [(n_1 - 1) + (n_2 - 1) + \ldots + (n_k - 1)]^2 = [(n_1 + n_2 + \ldots + n_k) - k]^2$$

$$\Rightarrow \underline{\sum_{i=1}^{k} (n_i - 1)^2} + \underbrace{\sum_{i=1}^{k} \sum_{j=1, j\neq i}^{k} (n_i - 1)(n_j - 1)}_{\text{positive}} = \underline{\sum_{i=1}^{k} (n_i)^2 - 2k\left(\sum_{i=1}^{k} n_i\right) + k^2}$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 - 2\sum_{i=1}^{k} n_i + k \leq \underline{\qquad\qquad}.$$

$$\sum_{i=1}^{k} n_i^2 \leq \left(\sum_{i=1}^{k} n_i\right)^2 - (k-1)\left(2\sum_{i=1}^{k} n_i - k\right)$$

Now,

$$\frac{1}{2}\sum_{i=1}^{k}(n_i-1)(n_i) = \frac{1}{2}\left(\sum_{i=1}^{k} n_i^2\right) - \frac{n}{2}$$

$$\leq \frac{1}{2}\left[n^2-(k-1)(2n-k)\right] - \frac{n}{2}$$

$$\leq$$

$$\vdots$$

$$\leq \frac{1}{2}\left[(n-k)(n-k+1)\right]$$

Euler Graph

- If some closed walk in a graph G contains all the edges of the graph, then the walk is called the Euler line and the graph is Euler graph.
- **True/False**: Euler graph (assumption: no isolated vertices in a graph) is always connected.
- **Theorem 2-4**: A given connected graph G is an Euler graph if and only if all vertices of G are of even degree.
- **Theorem 2-5**: In a connected graph G with exactly 2k odd vertices, there exist k edge-disjoint subgraphs such that they together contain all edges of G and that each is a unicursal graph.
- An open walk that includes (or traces) all edges of a **graph** without retracing any edge is called a **unicursal** line or open Euler line. A connected **graph** that has a **unicursal** line is called a **unicursal graph**.
- **Theorem 2-6**: A connected graph G in an Euler graph if and only if it can be decomposed into circuits.

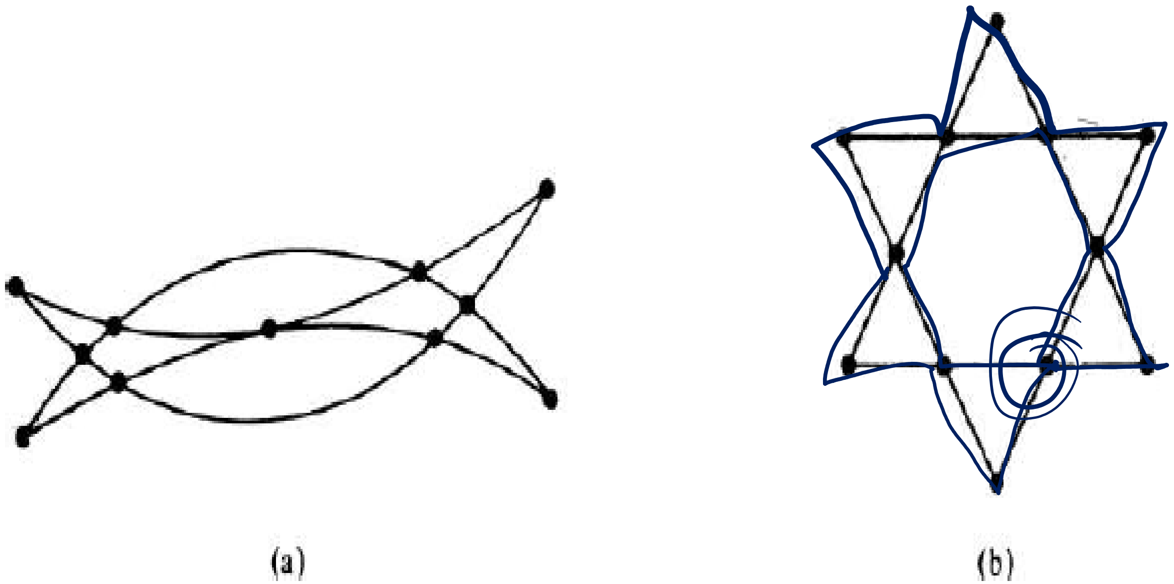


Fig. 2-12 Two Euler graphs.

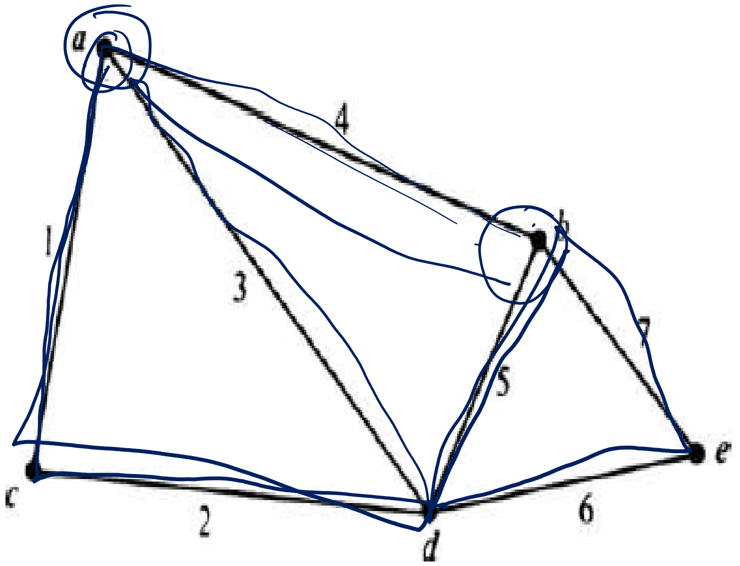


Fig. 2-13 Unicursal graph.

- What property must a vertex v in a Euler graph have such that an Euler line is always obtained when one follows any walk from vertex v according to the single rule that whenever one arrives at a vertex one shall select any edge (which has not been previously traversed)?
- **Theorem 2-7**: An Euler graph G is arbitrarily traceable from vertex v in G if and only if every circuit in G contain v.

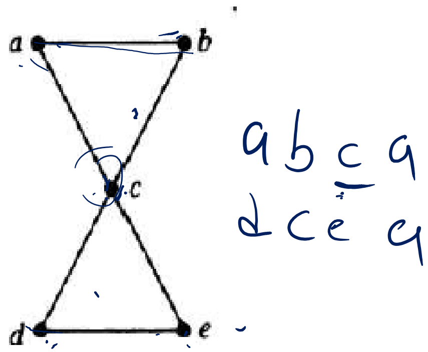


Fig. 2-17 Arbitrarily traceable graph from *c*.

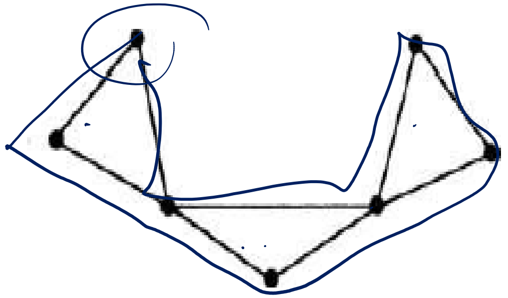


Fig. 2-18 Euler graph; not arbitrarily traceable.

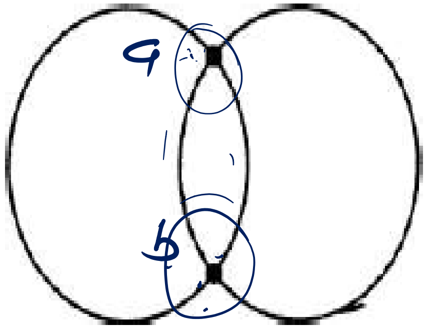


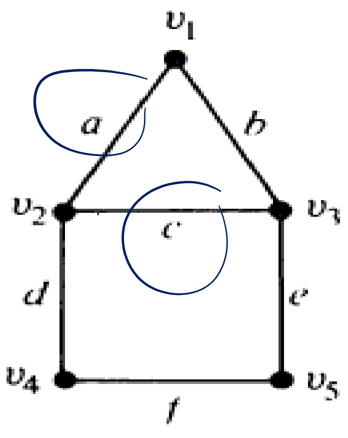
Fig. 2-19 Arbitrarily traceable graph from all vertices.

Operations on Graphs

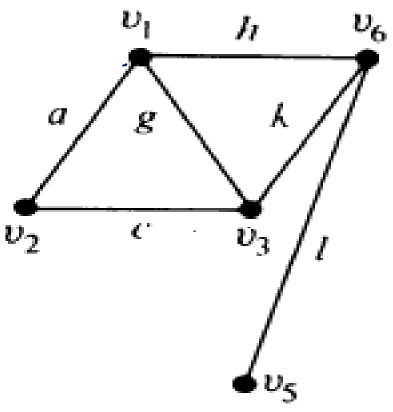
- Union: Union of two graphs $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ is another graph $G_3 = (V_3,E_3)$ whose vertex set $V_3 = V_1 \cup V_2$ and the edge set $E_3 = E_1 \cup E_2$.
- Intersection: Intersection of two graphs $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ is another graph $G_3 = (V_3,E_3)$ whose vertex set $V_3 = V_1 \cap V_2$ and the edge set $E_3 = E_1 \cap E_2$.
- Ring sum: The ring sum $(G_1 \oplus G_2)$ of two graphs $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ is a graph $G_3(V_3,E_3)$ whose vertex set $V_3 = V_1 \cup V_2$ and edges that are either in G_1 or G_2, but not both.

All the these operations are commutative.

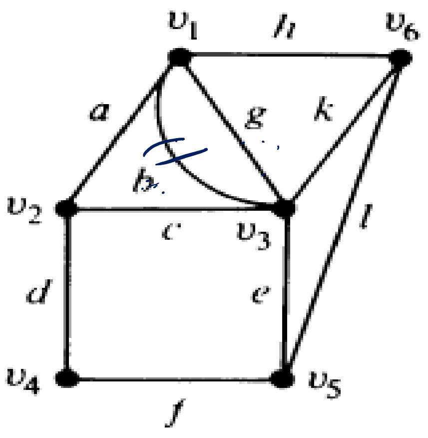
- If G_1 and G_2 are edge disjoint, then $G_1 \cap G_2$ is a null graph, and $G_1 \oplus G_2 = G_1 \cup G_2$.
- If G_1 and G_2 are vertex disjoint, then $G_1 \cap G_2$ is empty.
- For any graph G, $G_1 \cap G_2 = G_1 \cup G_2 = G$ and $G_1 \oplus G_2 =$ a null graph.
- If g is a subgraph if G, then $G \oplus g = G - g$, whenever $g \subseteq G$.



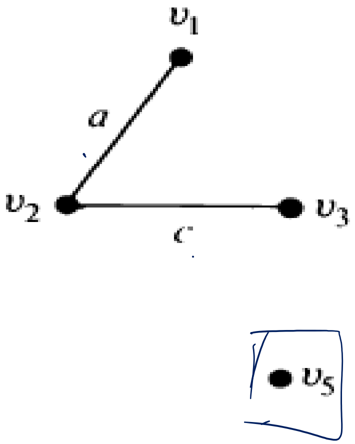
G_1



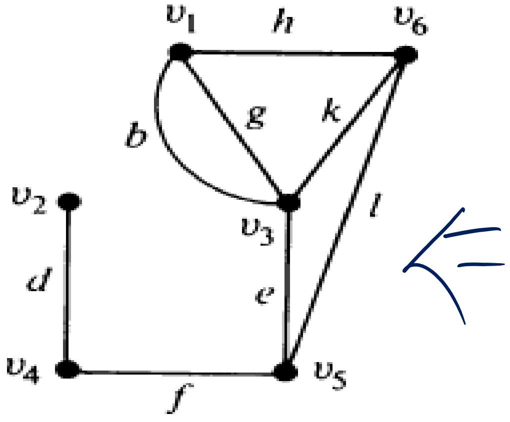
G_2



$G_1 \cup G_2$



$G_1 \cap G_2$



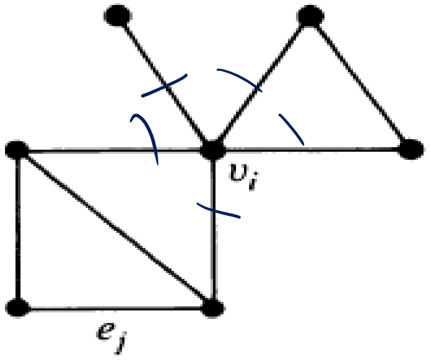
$G_1 \oplus G_2$

- Decomposition: A graph G is said to have been decomposed into two subgraphs, g_1 and g_2 if $g_1 \cup g_2 = G$ and $g_1 \cap g_2 =$ a null graph.
- In decomposition, isolated vertices are disregarded.
- A graph containing m edges $\{e_1,e_2,e_3,\ldots\ldots e_m\}$ can be decomposed in $\Rightarrow$ $2^{m-1} - 1$ different ways into pairs of subgraphs g_1,g_2.
- True/False: A graph can not be decomposed into more than two subgraphs?

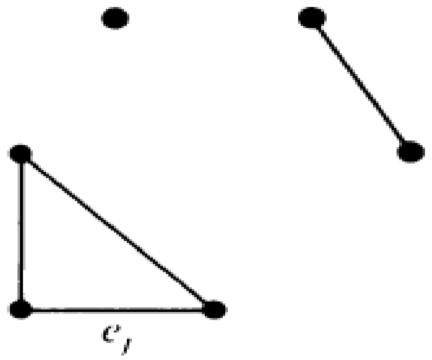
$\Rightarrow g_1 \cup g_2 \cup g_3 \ldots\ldots \cup g_k = G$

$\Rightarrow g_1 \cap g_2 \cap g_3 \ldots\ldots \cap g_k =$ null graph

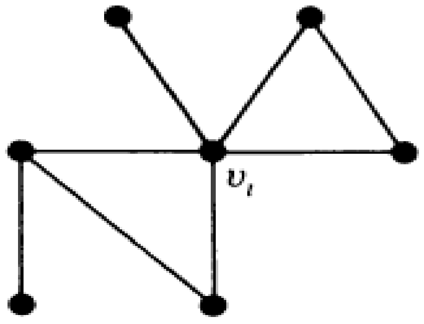
- Deletion: If v_i is a vertex in graph G, then G – v_i denotes a subgraph of G obtained by deleting v_i from G. Such vertex deletion always implies the deletion of all edges incident on that vertex. If e_j is an edge in G, then G-e_j is a subgraph of G obtained by deleting e_j from G. Such edge deletion does not imply deletion of its end vertices, i.e. $G - e_j = G \oplus e_j$.



G

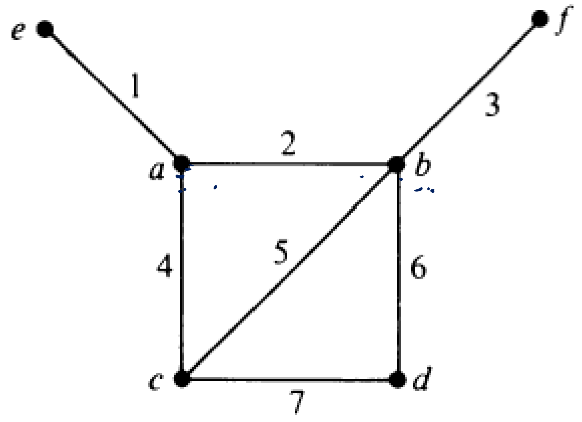


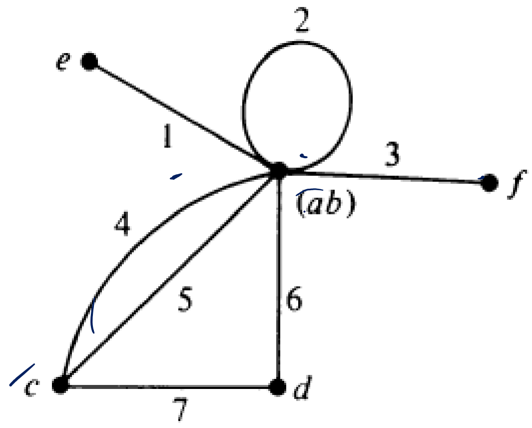
$(G - v_i)$



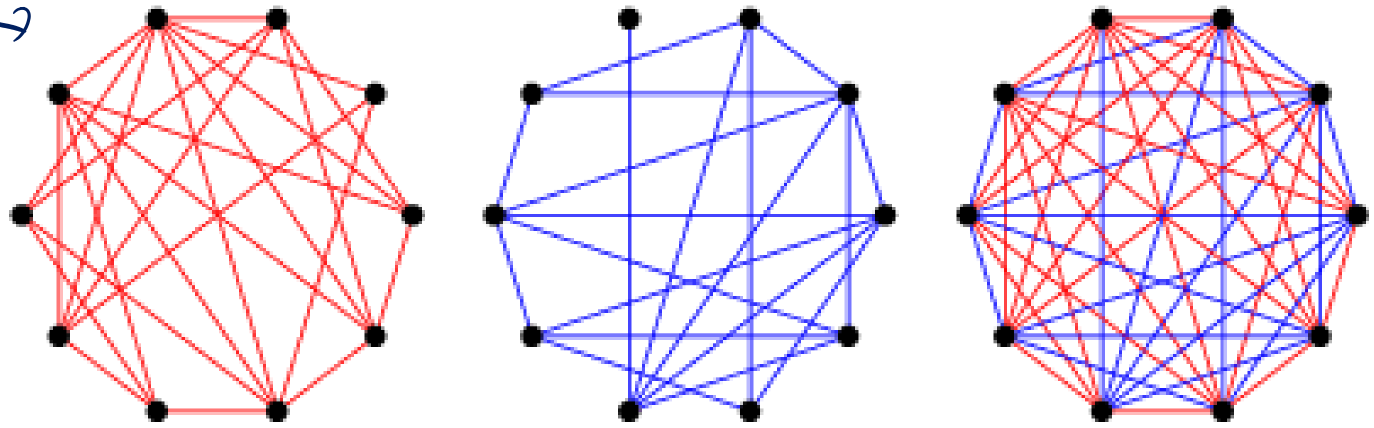
$(G - e_j)$

- Fusion: A pair of vertices a, b in a graph are said to be fused (merged or identified) if the two vertices are replaced by a single new vertex such that every edge that was incident on either a or b or both is incident on the new vertex.
- The fusion of two vertices does not alter the number of edges, but it reduces the number of vertices by one.

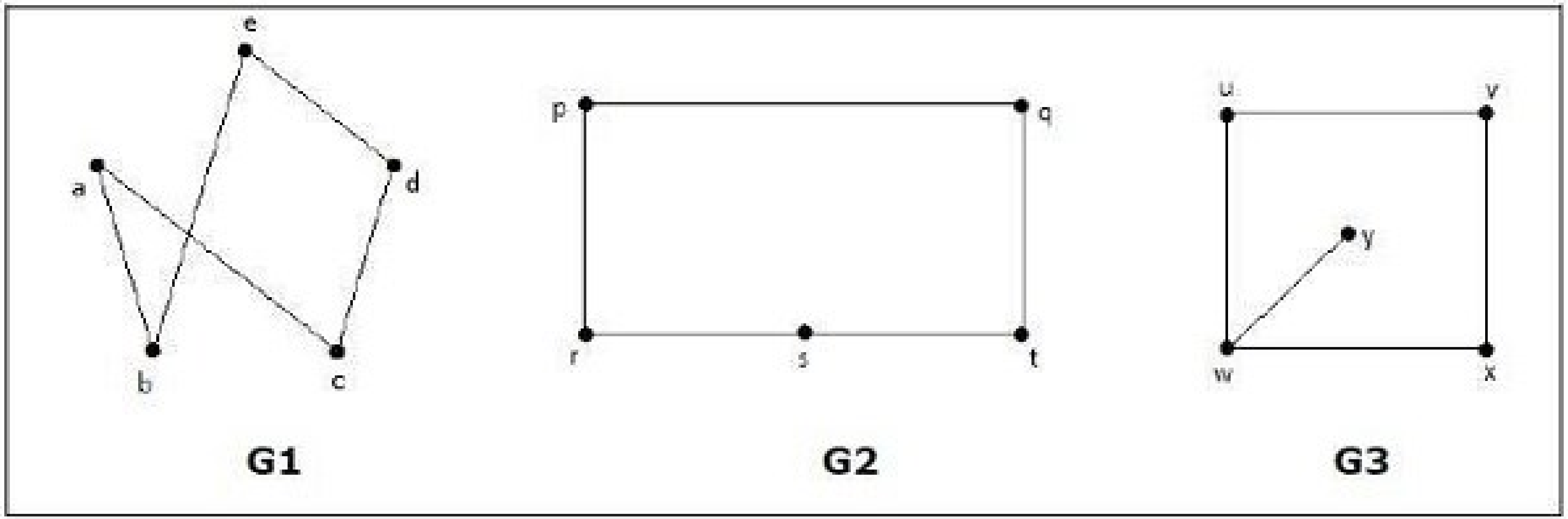




- Complement: For a graph $G = (V, E)$, the complement of G is the graph G' with vertex set V and edge set $E_n - E$, where E_n is the set of edges of the complete graph with V vertices.



Which of the following graphs are isomorphic?



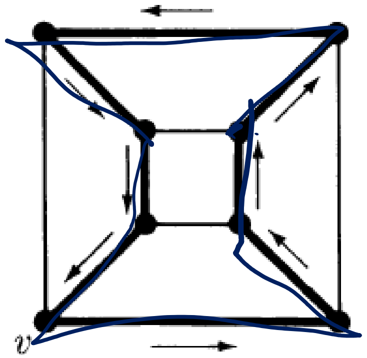
Spanning Subgraphs

$g = (v, e)$ $\quad$ $G = (V, E)$

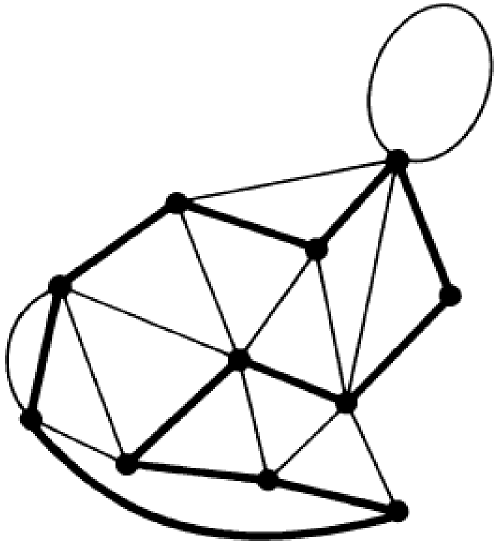
A subgraph g of a graph G is a spanning subgraph if $v = V$ and $e \subset E$.

Hamiltonian Paths and Circuits

- Hamiltonian circuit in a connected graph is defined as a closed walk that traverses every vertex of G exactly once except the starting vertex. A circuit in a connected graph *G* is said to be Hamiltonian if it includes every vertex of *G*. Hence a Hamiltonian circuit in a graph of n vertices consists of exactly n edges.
- Hamiltonian path in a graph G traverses every vertex of G. The length of a Hamiltonian path of a graph with n vertices is n-1.

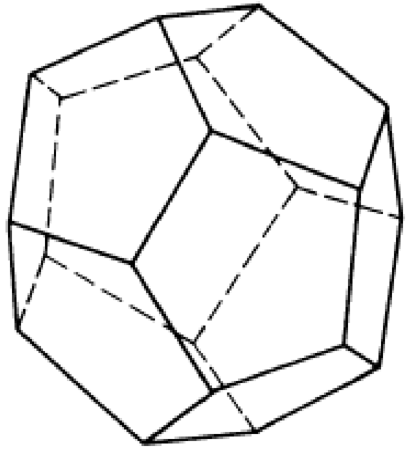


(a)

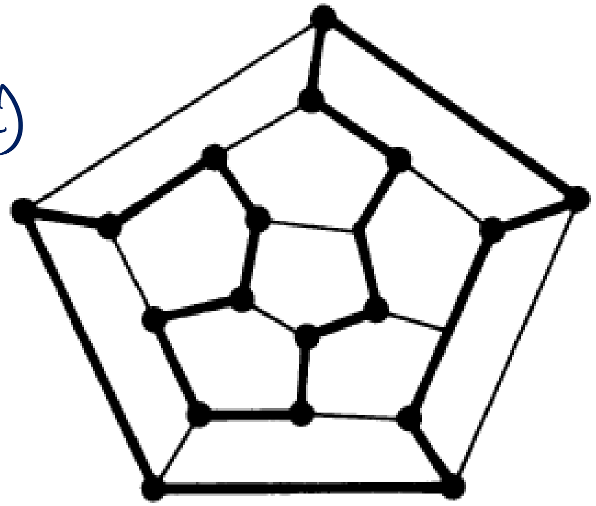


(b)

Dodecahedron



(a)



(b)

- What is a necessary and sufficient condition for a connected graph G to have a Hamiltonian circuit?

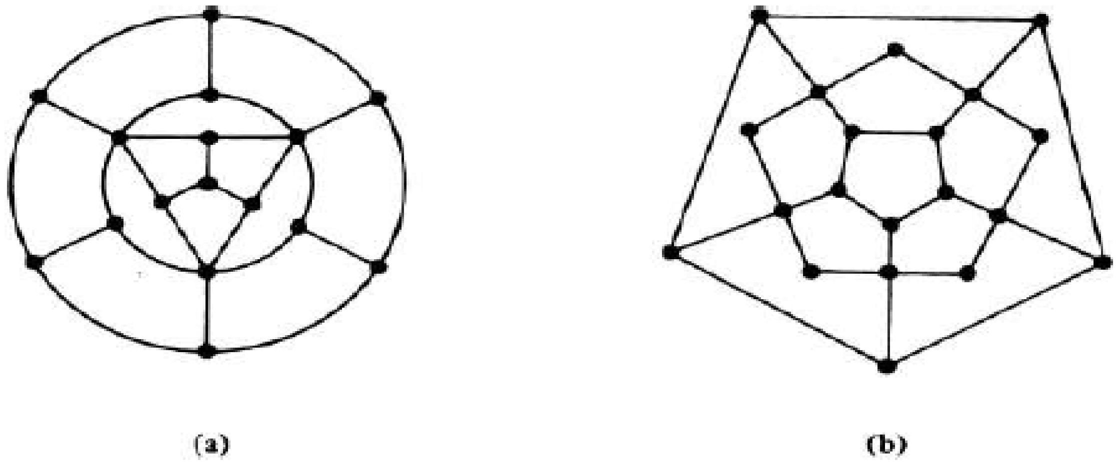


Fig. 2-22 Graphs without Hamiltonian circuits.

- Complete Graph is a simple graph in which there exists an edge between every pair of vertices. Complete graph is also known as universal graph or a clique.
- *Number of Hamiltonian Circuits in a Graph:* A given graph may contain more than one Hamiltonian circuit.
- True/False: If a graph contains more than one Hamiltonian circuits, than all such circuits are edge-disjoint.
- **Theorem 2-8**: In a complete graph with n vertices there are (n-1)/2 edge-disjoint Hamiltonian circuits, if n is an odd number $\geq$ 3.

=> Self study

- This theorem enables us to solve the problem of the seating arrangement at around table.

n is odd $\Rightarrow \frac{n-1}{2}$

n is even $\Rightarrow \frac{n-2}{2}$

$\Rightarrow n=9 \Rightarrow \frac{9-1}{2} = \underline{4}$

$\Rightarrow n=7 \Rightarrow$ 3 $\Rightarrow$ 8 8 8

$\Rightarrow n=5 \Rightarrow$

$n=11 \Rightarrow$

identify all possible arrangements ~~for~~ for seating

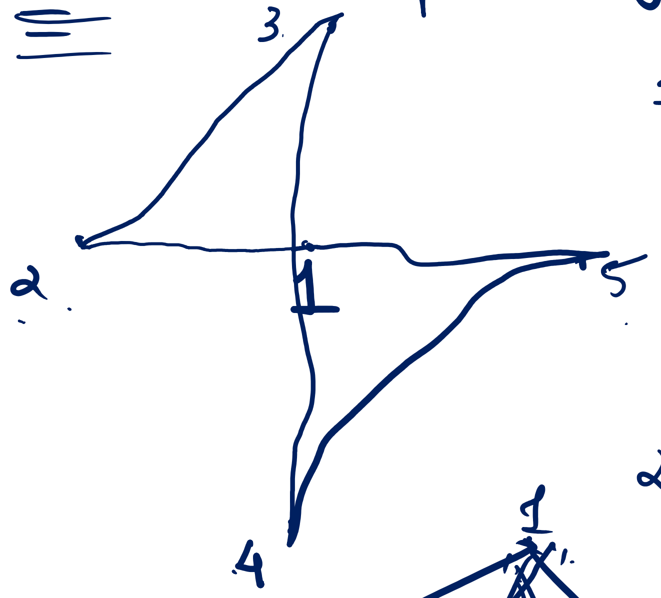
$\Rightarrow$ at round tables.

→ complete graph of n vertices $\frac{n(n-1)}{2}$ edges.

→ In a hamiltonian circuit you will get n edges.

→ such edge-disjoint hamiltonian circuits are $\frac{n-1}{2}$.

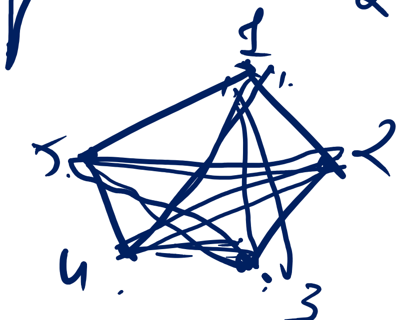
n=5 2 edge disjoint circuits are there.

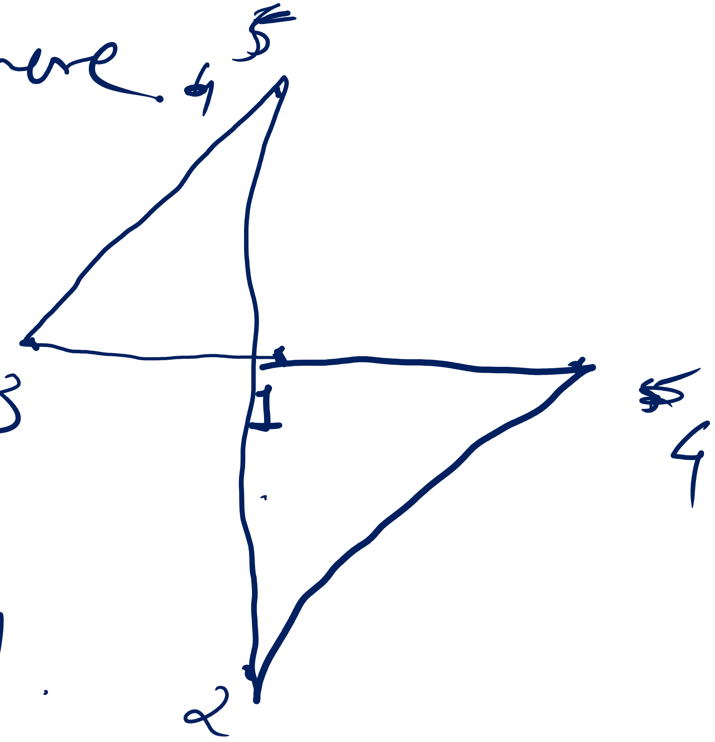


1) 123451

2) $\frac{360}{n-1} \Rightarrow \frac{360}{4} = 90.$

2) ~~13425~~ 1   135241





$n = 9 \Rightarrow \frac{n-1}{2} = \frac{9-1}{2} = 4$

$n=10$

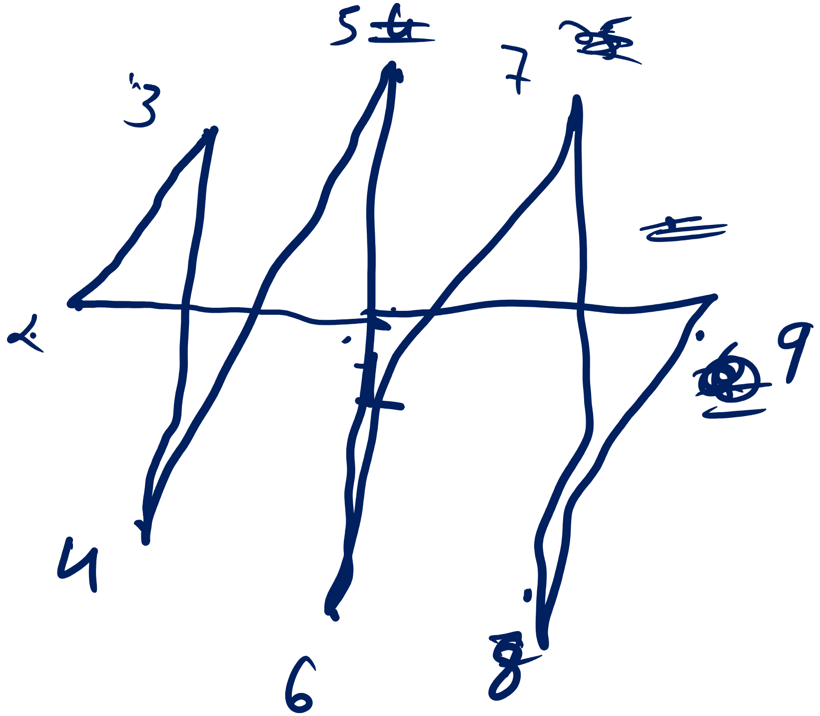
if n is even $\Rightarrow \frac{n-2}{2} = 4$

Which are these circuits?

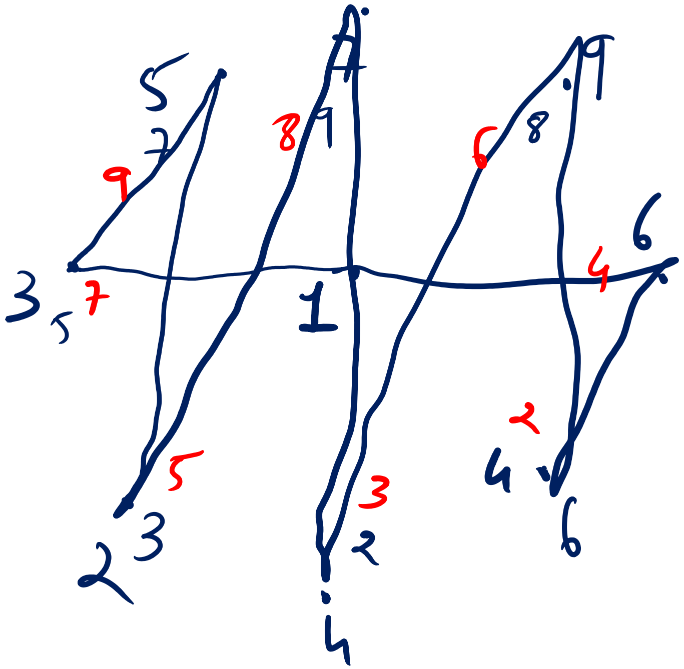
$\Rightarrow \frac{360}{n-1}, \frac{2 \cdot 360}{n-1} \cdots \frac{(n-3)}{2} \frac{360}{n-1}$

no. of rotation $\frac{6}{2} \cdot \frac{360}{8} = 45$

3 rotations of 45° each



V) 1 2 3 4 5 6 7 8 9 1



→ 1 3 5 2 7 4 9 6 8 1

→ 1 5 7 3 9 2 8 4 6 1

→ 1 7 9 5 8 3 6 2 4 1

Traveling-SalesMan Problem

- A salesman is required to visit a number of cities during a trip. Given the distances between the cities, in what order should he travel so as to visit every city precisely once and return home with the minimum mileage travelled?
  - Weighted graph
  - The number of non-edge disjoint Hamiltonian circuits in a complete graph of n vertices can be shown to be (n-1)! / 2. Why?

1 ⇒ _ _ n-1 _

2 ⇒ n-2

3 ⇒ n-3

⋮

1

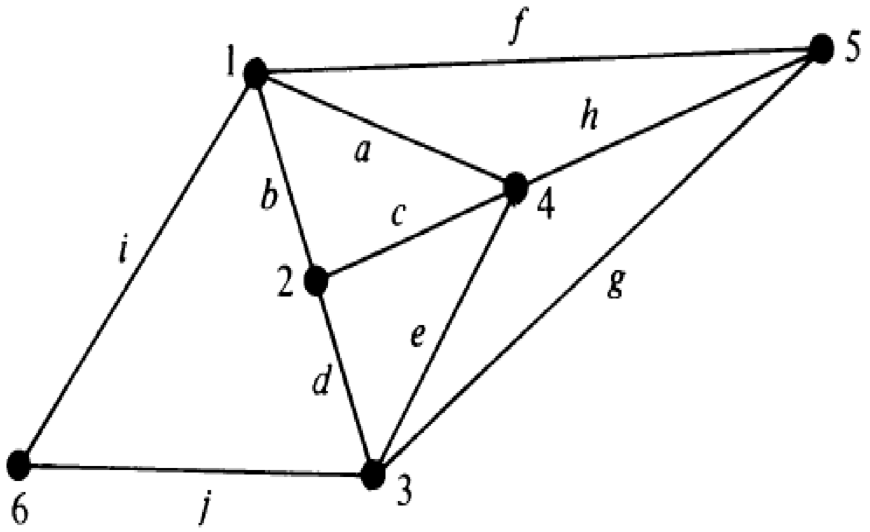
(n-1)

nearest neighbour algorithm

abcda

bcdab

List all the different paths between vertices 5 and 6.



- Group the listed paths into sets of edge disjoint paths. Check whether two edge disjoint paths form a circuit.

- You are given a 10-piece domino set whose tiles have the following set of dots: (1,2), (1,3),(1,4), (1,5), (2,3), (2,4), (2,5), (3,4), (3,5), (4,5). Discuss the possibilities of arranging the tiles in a connected series such that one number on a tile always touches the same number of its neighbour. (*Hint:* Use a five-vertex complete graph and see if it is an Euler graph.)
- Prove that complete graph with n vertices contains n(n-1)/2 edges.

Graph Theoretic Model for LAN
Graphic Sequence

Build a graph theoretical model of the LAN problems as mentioned below:

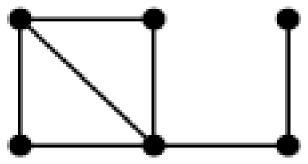
- In a college campus, there are seven blocks, Computer Center (C), Library (L), Academic (AC), Administrative (AD), Health Center (H), Guest House (G), Security (S).
- The problem is to design two LANs satisfying certain conditions:
  - LAN 1
    - Two of the blocks are connected to exactly five of the blocks.
    - Two of the blocks are connected to three of the blocks.
    - Three of the blocks are connected to two of the blocks.
  - LAN 2
    - Four of the blocks are connected to five of the blocks.
    - Three of the blocks are connected to two of the blocks.

5 5 3 3 2 2 2
degree seq nies

5 5 5 5 2 2 2
degree sequences

Degree Sequence

- Definition. If G is a graph on n vertices v1; v2; : : : ; vn with degrees d1; d2; : : : ; dn respectively, then the n-tuple (d1; d2; : : : ; dn) is called the degree sequence of G.
- Graph with its degree sequence is:



$S : (4, 3, 2, 2, 2, 1)$

- Every graph G gives rise to a sequence of integers (d1; d2; : : : ; dn).
- Given a sequence S of integers (d1; d2 : : : ; dn), does there exist a graph G with S as its degree sequence?

- Yes, by theorem: “summation of degree of vertices is even.”
- However, the question is more difficult if we ask for the existence of a simple graph G with degree sequence S.
- Definition. A sequence of non-negative integers S = (d1; d2; : : : ; dn) is said to be graphic, if there exists a simple graph G with n vertices v1; v2; : : : ; vn such that deg(vi) = di, for i = 1; 2; : : : ; n. When such a G exists, it is called a **realization** of S.
- This problem leads to the question: Design algorithms to construct a realization of S, if S is graphic.
- There are theorems which characterize graphic sequences.

Havel-Hakimi criterion

- A sequence S : (d1≥ d2 ≥ …. ≥ dn) of non-negative integers is graphic if and only if the reduced sequence
- S' : (*; d_2 - 1; d_3-1; : : : ; d_{d1+1}-1; d_{d1+2}; …. ; d_n)
- is graphic.
- (Here, S' is obtained from S by deleting d_1 and subtracting 1 from the next d_1 terms.
- Without loss of generality, assume that $d_1 \leq n - 1$.)

Realization of graphic sequence for LAN 1

S0 : 5 5 3 3 2 2 2 ⇐ graphical

⇐ will get simple graph

S1 : 0 4 2 2 1 1 2

rearrange

4 2 2 2 1 1 0

S2 : 0 1 1 1 0 1 0

rearrang

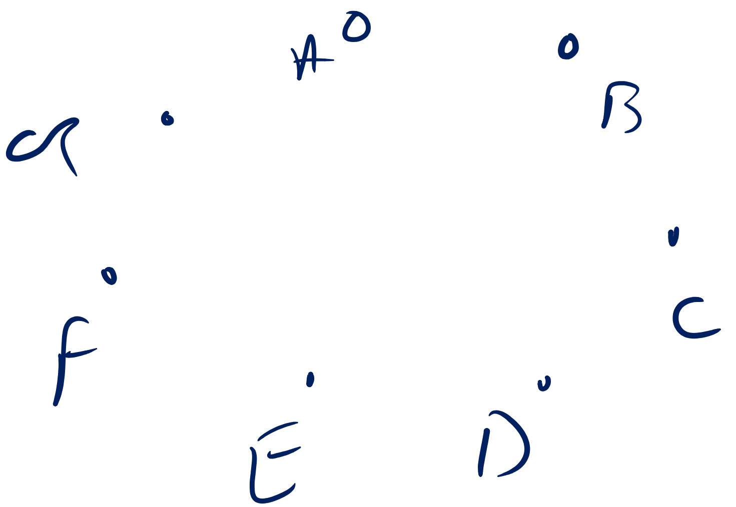
1 1 1 1 0 0 0

S3 : 0 0 1 1 0 0 0

rearr

1 1 0 0 0 0 0

S4 : 0 0 0 0 0 0 0 ⇐



Realization of graphic sequence for LAN 2

S0: 5 5 5 5 2 2 2 ⇐ not graphical

will not get simple graph

S1: 0 4 4 4 1 1 2

rearrangement

: 4 4 4 2 1 1 0

S2: 0 3 3 1 0 1 0

rearrang

3 3 1 1 0 0 0

S3: 0 2 0 0 0 0 0

rearrang

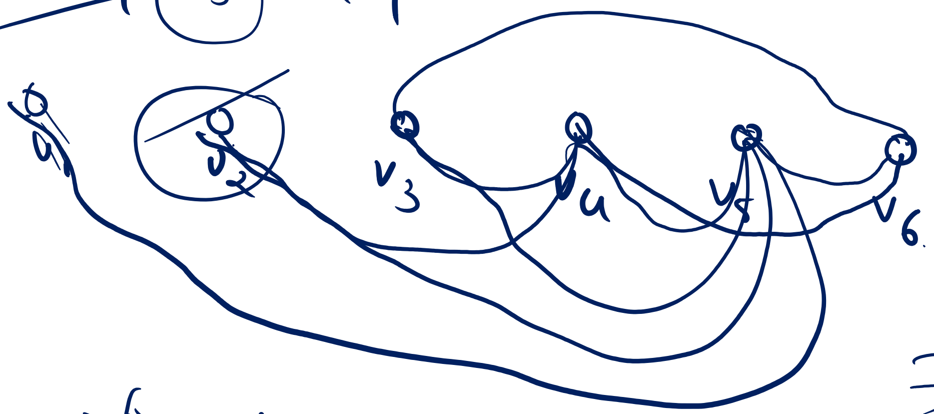
2 0 0 0 0 0 0

0 -1 -1 0 0 0 0 ⇐

- In a simple, connected graph on 6 vertices, the degrees of 5 vertices are 1, 2, 3, 4, 5 respectively. What may be the degree of the 6th vertex?

$v_5 \quad v_4 \quad v_3 \quad v_2 \quad v_1$

$\cancel{5} \quad \cancel{4} \quad 3 \quad 2 \quad 1$



$\Rightarrow 5\ 4\ 3\ 3\ 2\ 1$

degree of $v_6 = \underline{3}$

Solution

- Let us call the 5 vertices with known degree v1, v2, . . . v5, where d(vi) = i. The degree of v6 is unknown.
- Node v5 is connected by an edge to every node, so the only neighbor of v1 is v5.
- Node v4 is connected to very node except v1. Therefore the two neighbors of v2 are v5 and v4.
- For v6, we know that it is connected by an edge to v5 and v4 and not connected to v1 and v2. The same is true for v3. Since the degree of v3 is 3, v3 is connected by an edge to v6.
- Therefore the degree of v6 is 3.

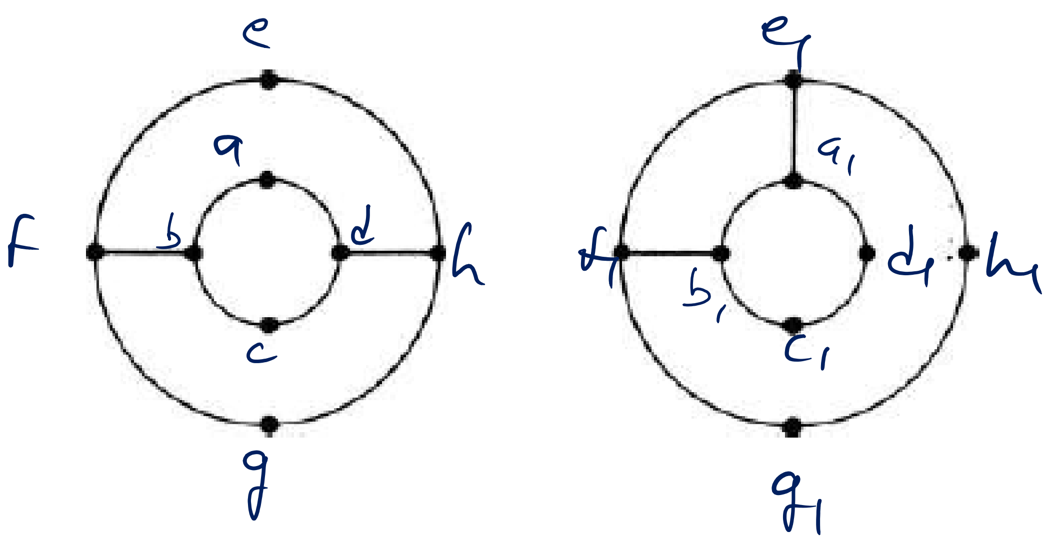
Are these two graphs isomorphic? Why?

→ same no. of vertices

→ same no. of edges

→ same no. of vertices with same degree

→ g



not isomorphic