Planar and Dual Graphs

Prof. Rupal Kapdi, Assistant Professor
Computer Science and Engineering Department,
Institute of Technology, Nirma University.

Planarity of a graph

Applications:

Design of a circuit without any extra layer of insulation.
Supply of the utilities without crossing the lines.

Combinatorial(abstract) and Geometric Graph

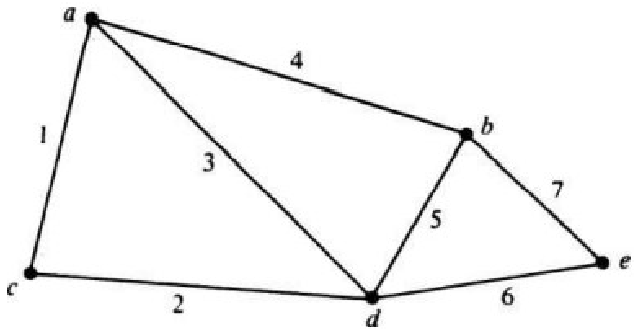


Fig. 2-13 Unicursal graph.

An abstract graph can be defined as:

$$G1 = (V, E, \Psi)$$

where the set V consists of the five objects named a, b, c, d, and e, that is, V = {a, b, c, d, e}, and
the set E consists of seven objects (none of which is in set V) named 1, 2, 3, 4, 5, 6, and 7, that is, E = {1, 2, 3, 4, 5, 6, 7}, and
the relationship between the two sets is defined by the mapping Ψ, which consists of

$$\Psi = \begin{bmatrix} 1 \longrightarrow (a, c) \\ 2 \longrightarrow (c, d) \\ 3 \longrightarrow (a, d) \\ 4 \longrightarrow (a, b). \\ 5 \longrightarrow (b, d) \\ 6 \longrightarrow (d, e) \\ 7 \longrightarrow (b, e) \end{bmatrix}$$

Here, the symbol $1 \to (a, c)$ says that object 1 from set E is mapped onto the (unordered) pair (a, c) of objects from set V.

Now it so happens that this combinatorial abstract object G1 can also be represented by means of a geometric figure.

In fact, the sketch in Fig. 2-13 is one such geometric representation of this graph.

Moreover, it is also true that any graph can be represented by means of such a configuration in three dimensional Euclidean space.

It will often be necessary to make a distinction between the abstract (or combinatorial) graph and a geometric representation of a graph.

Planar Graph

A graph G is said to be planar if there exists some geometric representation of G which can be drawn on a plane such that no two of its edges intersect.

A graph that cannot be drawn on a plane without a crossover between its edges is called nonplanar.

A drawing of a geometric representation of a graph on any surface such that no edges intersect is called embedding.

Thus, to declare that a graph G is nonplanar, we have to show that of all possible geometric representations of G none can be embedded in a plane.

Equivalently, a geometric graph G is planar if there exists a graph isomorphic to G that is embedded in a plane. Otherwise, G is nonplanar.

An embedding of a planar graph G on a plane is called a plane representation of G.

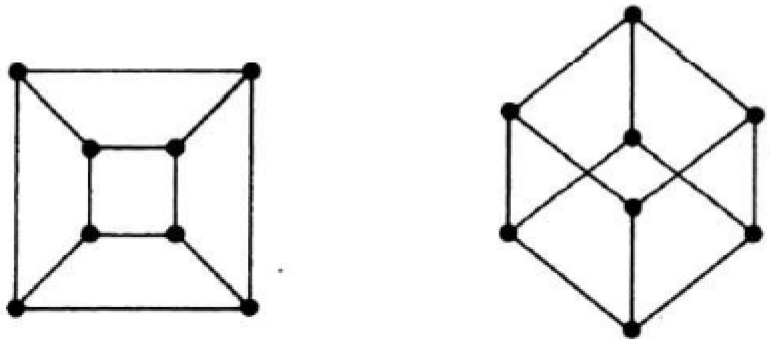
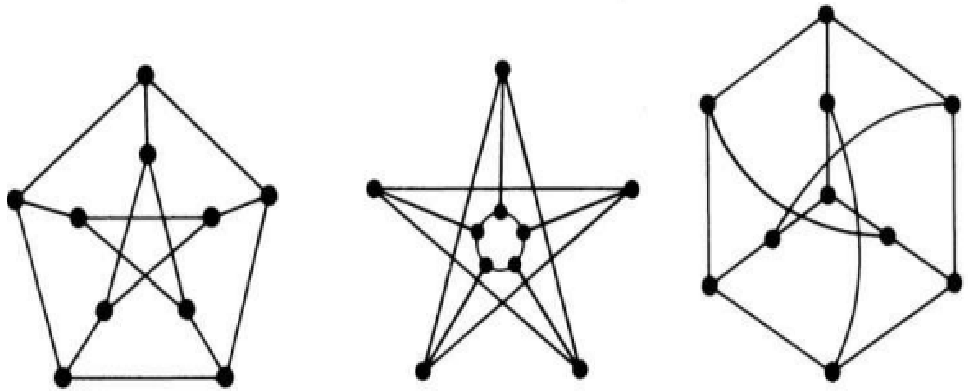


Fig. 2-2 Isomorphic graphs.



(a) (b) (c)

Fig. 2-3 Isomorphic graphs.

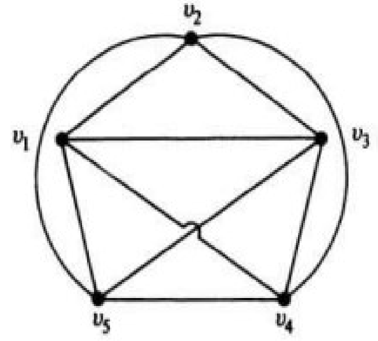
Planar graphs

non-planar graphs

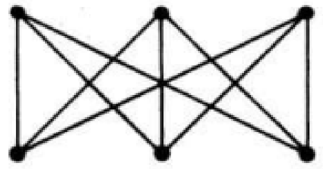
How to check whether the given graph G (abstract or geometric) is planar?

Kuratowski's Two Graphs

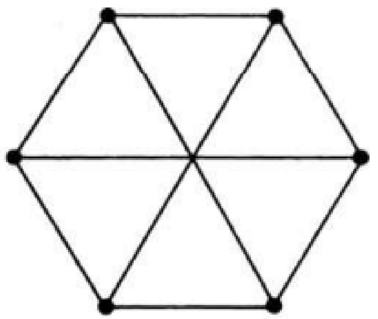
THEOREM 5.1 - The complete graph of five vertices is nonplanar.



THEOREM 5.2 - Kuratowski's second graph is also nonplanar.



(a)



(b)

Several properties common to the two graphs of Kuratowski are

1. Both are regular graphs.
2. Both are nonplanar.
3. Removal of one edge or a vertex makes each a planar graph.
4. Kuratowski's first graph is the nonplanar graph with the smallest number of vertices, and Kuratowski's second graph is the nonplanar graph with the smallest number of edges.

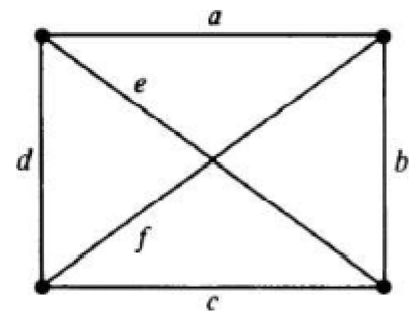
Thus both are the simplest nonplanar graphs.

Different representations of a planar graphs.

THEOREM 5.3 - Any simple planar graph can be embedded in a plane such that every edge is drawn as a straight line segment.

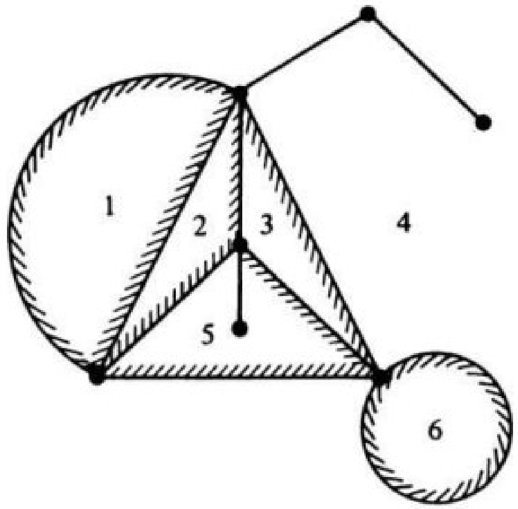
Region: A plane representation of a graph divides the plane into regions. A region is characterized by the set of edges (or the set of vertices) forming its boundary.

Note that a region is not defined in a nonplanar graph or even in a planar graph not embedded in a plane.



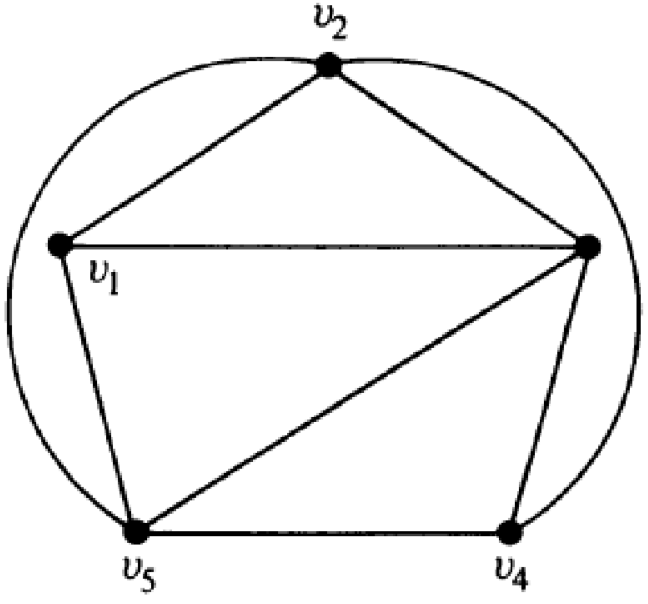
Thus a region is a property of the specific plane representation of a graph and not of an abstract graph.

Infinite Region: The portion of the plane lying outside a graph embedded in a plane, such as region 4 in Fig. 5-4, is infinite in its extent. Such a region is called the infinite, unbounded, outer, or exterior region for that particular plane representatic

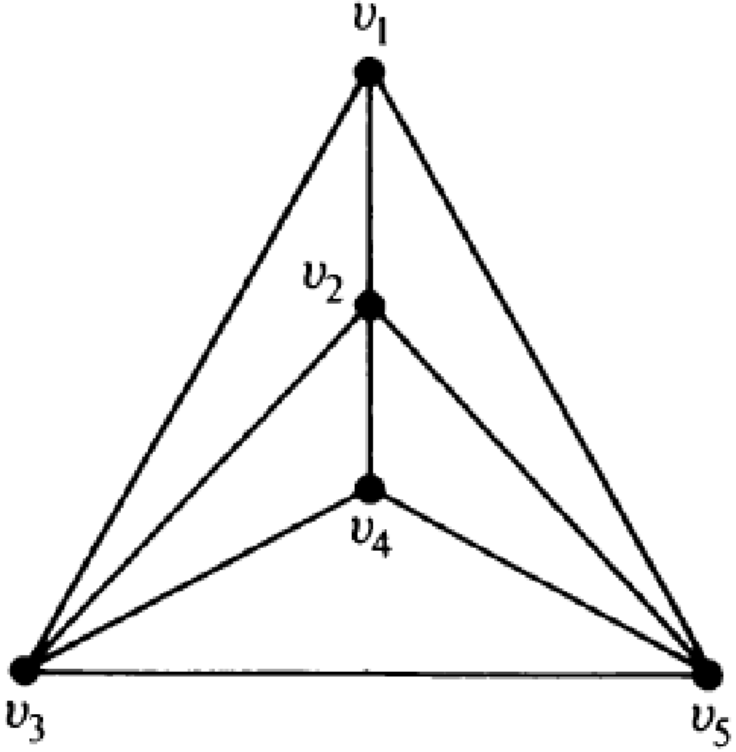


Different representations of a Planar Graph

Changing the embedding of a given planar graph, changes the infinite region.



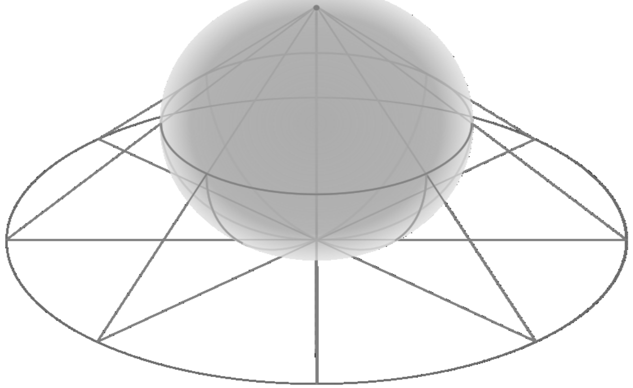
(d)



Embedding on a sphere

Embedding on a Sphere: To eliminate the distinction between finite and infinite regions, a planar graph is often embedded in the surface of a sphere.

It is accomplished by stereographic projection of a sphere on a plane. Put the sphere on the plane and call the point of contact SP (south pole). At point SP, draw a straight line perpendicular to the plane, and let the point where this line intersects the surface of the sphere be called NP (north pole).



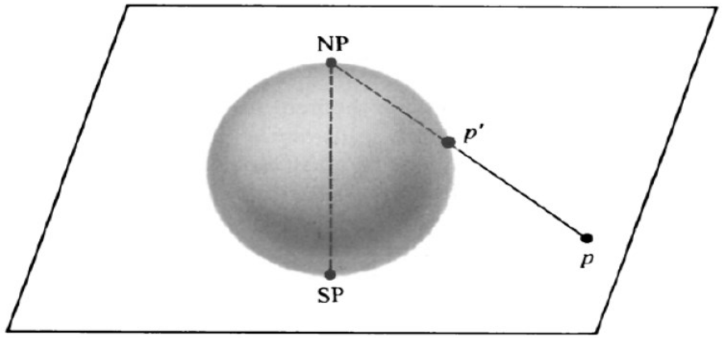


Fig. 5-5 Stereographic projection.

- Corresponding to any point p on the plane, there exists a unique point p’ on the sphere and vice versa, where p’ is the point at which the straight line from point p to point NP intersects the surface of the sphere.
- Thus there is a one-to-one correspondence between the points of the sphere and the finite points on the plane, and points at infinity in the plane correspond to the points NP on the sphere.
- A planar graph embedded in the surface of the sphere divides the surface into different regions. Each region on the sphere is finite, and the infinite region on the plane is mapped onto the region containing the point NP.
- By suitably rotating the sphere we can make any specified region map onto the infinite region on the plane.
- **THEOREM 5.4 -** A graph can be embedded in the surface of a sphere if and only if it can be embedded in a plane.

- Which leads to the theorem:
- **Theorem 5.5 -** A planar graph may be embedded in a plane such that any specified region (i.e. specified by the edge forming it) can be made the infinite region.
- There is no real difference between the infinite region and the finite region on the plane, the infinite region is also included in the planar representation of a graph.
- There is no difference between an embedding of a planar graph on a plane or on a sphere, the term “plane representation” of a graph is often used to include spherical as well as planar embedding.

- Since a planar graph may have different plane representations, the number of regions resulting from each embedding is always same.
- Euler's formula:
- **Theorem 5.6** - A connected planar graph with n vertices and e edges has $\boldsymbol{e-n+2}$ regions.
- Prove with mathematical induction.

Proof with direct method:

- any planar graph can be drawn such that each region is a polygon (a polygonal net). Let the polygonal net representing the given graph consist of f regions or faces, and let k_p be the number of p-sided regions. Since each edge is on the boundary of exactly two regions,

$$3\,k_3 + 4k_4 + 5k_5 + \ldots + rk_r = 2e$$

- where k_r is the number of polygons, with maximum r edges, and

$$k_3 + k_4 + k_5 + \ldots + k_r = f$$

- The sum of all angles subtended at each vertex in the polygonal net is = $2\pi n$.

- Recal that the sum of all interior angles of a p-sided polygon is $\boldsymbol{\pi(p-2)}$**,** and the sum of the exterior angles is $\boldsymbol{\pi(p+2)}$.
- Let us compute the summation of all the angles as the grand sum of all interior angles of f – 1 finite regions plus the sum of the exterior angles of the polygon defining the infinite region as,
- $\pi(3-2)k_3 \ + \pi(4-2)k_4 \ + \pi(5-2)k_5 \ + \ \ldots + \pi(r-2)k_r \ + 4\pi = 2\pi n$
- $\pi\ [3k_3 \ - 2k_3 \ + 4k_4 \ - 2k_4 \ + 5k_5 \ - 2k_5 \ ++ \ \ldots + 5k_r \ - 2k_r\,] + 4\pi = 2\pi n$
- $\pi\ [3k_3 \ + 4k_4 \ + 5k_5 \ + \ \ldots + 5k_r \ - 2k_3 \ - 2k_4 \ - 2k_5 \ - \cdots - 2k_r\,] + 4\pi = 2\pi n$
- $\pi\ [2e - 2f] + 4\pi = 2\pi n$
- $2\pi\ [e - f] + 4\pi = 2\pi n$
- $e - f + 2 \ = n$
- $e - n + 2 \ = f$

- **Corollary:** In any simple, connected planar graph with f regions, n vertices, and e edges ($e > 2$), the following inequalities must hold :

  For the graphs with the regions made up of at least 3 edges:

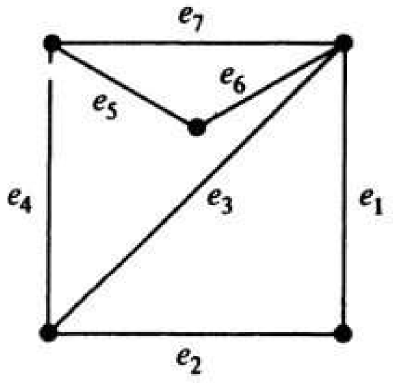
$$e \geq \frac{3}{2} f \Rightarrow e \leq 3n - 6$$

  For the graphs with the regions made up of at least 4 edges:

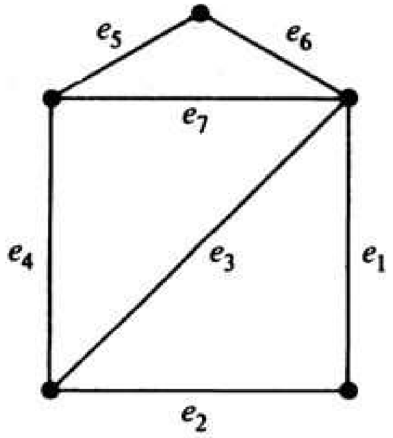
$$2e \geq 4f \Rightarrow e \leq 2n - 4$$

Plane Representation and Connectivity

- In a disconnected graph the embedding of each component can be considered independently. Therefore, a disconnected graph is planar if and only if each of its components is planar.
- In a separable graph the embedding of each block can be considered independently. Therefore a separable graph is planar if and only if each of its blocks is planar.
- Unique embedding: Two embeddings of a planar graph on sphere are not distinct if the embeddings can be made to coincide by suitably rotating one sphere with respect to the other and possibly distorting regions (without letting a vertex cross an edge). If of all possible embeddings on a sphere no two are distinct, the graph is said to have a unique embedding on a sphere.
- Does a nonseparable planar graph G have a unique embedding on a sphere?



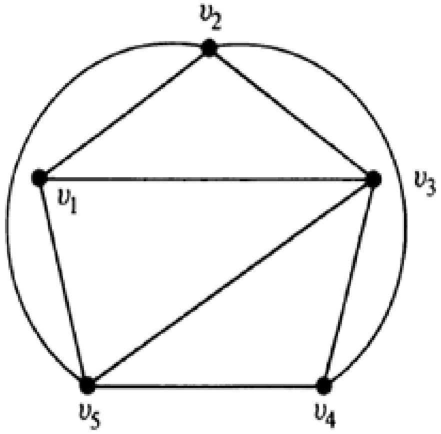
(a)



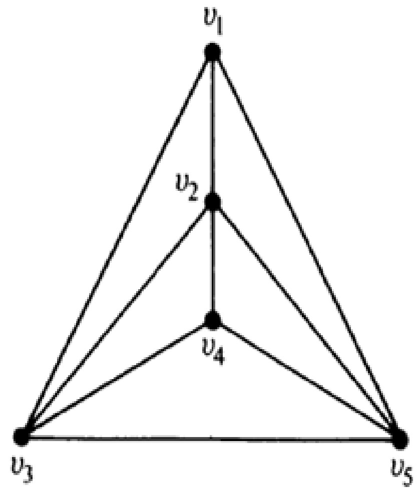
(b)

Two distinct representations of the same graph do not have unique embedding on sphere as (a) does not have any region with five edges whereas (b) have.

Two same graphs as shown in (c) and (d) can be made to coincide by rotating one sphere with respect to other and possibly distorting regions (without letting a vertex cross an edge).



(c)



(d)

- **Theorem 5.7** - The spherical embedding of every planar 3-connected graph is unique.

Detection of Planarity

- **Elementary Reduction**

- ***Step 1***: Since a disconnected graph is planar if and only if each of its components is planar, we need consider only one component at a time. Also, a separable graph is planar if and only if each of its blocks is planar. Therefore, for the given arbitrary graph G, determine the set

$$G = \{G_1, G_2, \dots, G_k\}$$

  where each G_i is a nonseparable block of G. Then we have to test each G_i for planarity.

- ***Step 2***: Since addition or removal of self-loops does not affect planarity, remove all self-loops.

- ***Step 3***: Since parallel edges also do not affect planarity, eliminate edges in parallel by removing all but one edge between every pair of vertices.

- ***Step 4***: Elimination of a vertex of degree two by merging two edges in series does not affect planarity. Therefore, eliminate all edges in series.

- Let the nonseparable connected graph G_i be reduced to a new graph H_i after the repeated application of steps 3 and 4. What will graph H_i look like?
- **Theorem 5.8** -

Graph H_i is

1. A single edge, or

2. A complete graph of four vertices, or

3. A nonseparable, simple graph with $n \geq 5$ and $e \geq 7$.

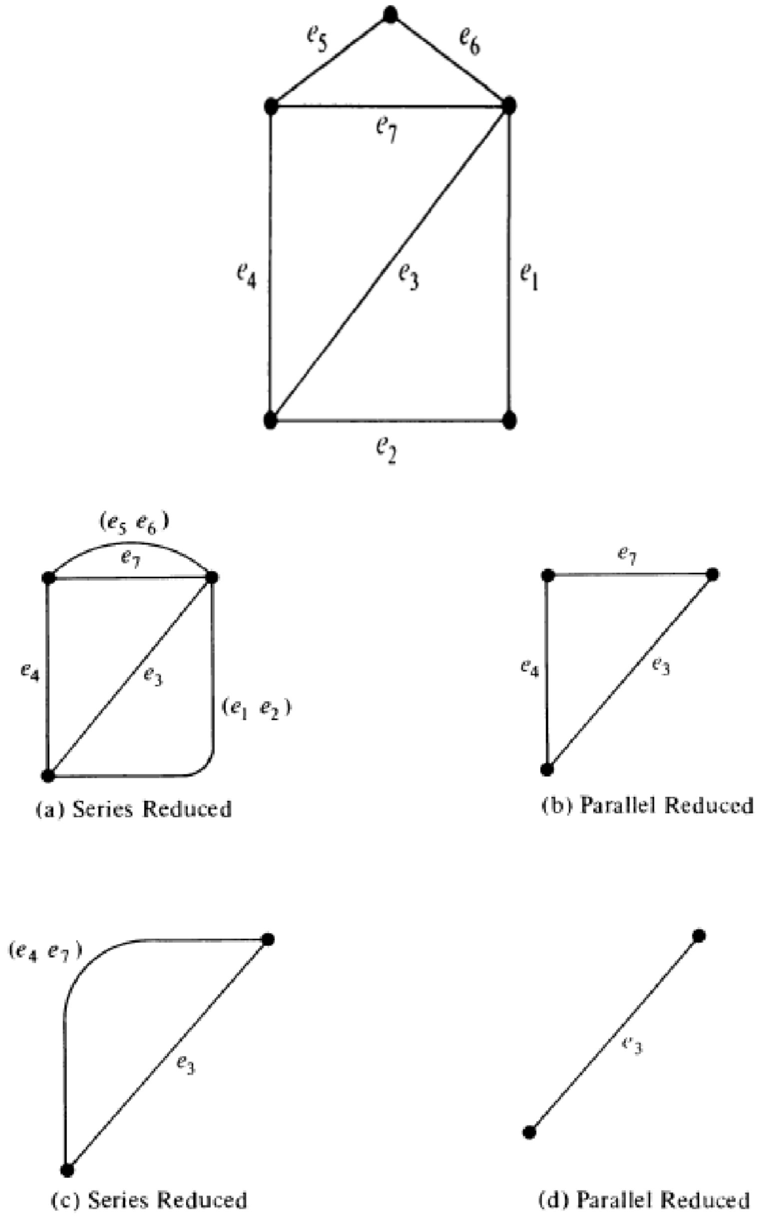


Fig. 5-7 Series-parallel reduction of the graph in Fig. 5-6(b).

- In Theorem 5-8, all H_i falling in categories 1 or 2 are planar and need not be checked further.
- For category 3, investigate only *simple, connected, nonseparable graphs of at least five vertices and with* ***every vertex of degree three or more***.
- Check to see if $e \leq 3n - 6$. If this inequality is not satisfied, the graph H_i is nonplanar. If the inequality is satisfied, test the graph further for the graphs homomorphic to Kuratowski's $K_5 or\ K_{3,3}$ *graphs.*
- ***Homeomorphic Graphs*****:** Two graphs are said to be *homeomorphic* if one graph can be obtained from the other by the creation of edges in series (i.e., by insertion of vertices of degree two) or by the merger of edges in series.

Below given three graphs are homeomorphic to each other, for instance. A graph G is planar if and only if every graph that is homeomorphic to G is planar. (This is a restatement of series reduction, step 4 in this section.)

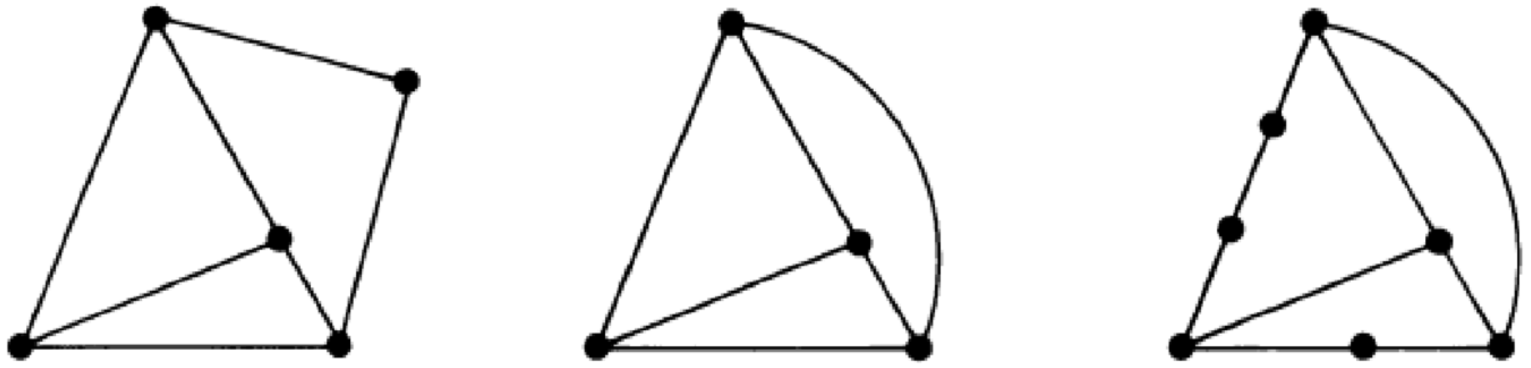


Fig. 5-8 Three graphs homeomorphic to each other.

- **Theorem 5.9 -** A necessary and sufficient condition for a graph G to be planar is that G does not contain either of Kuratowski's two graphs or any graph homeomorphic to either of them.

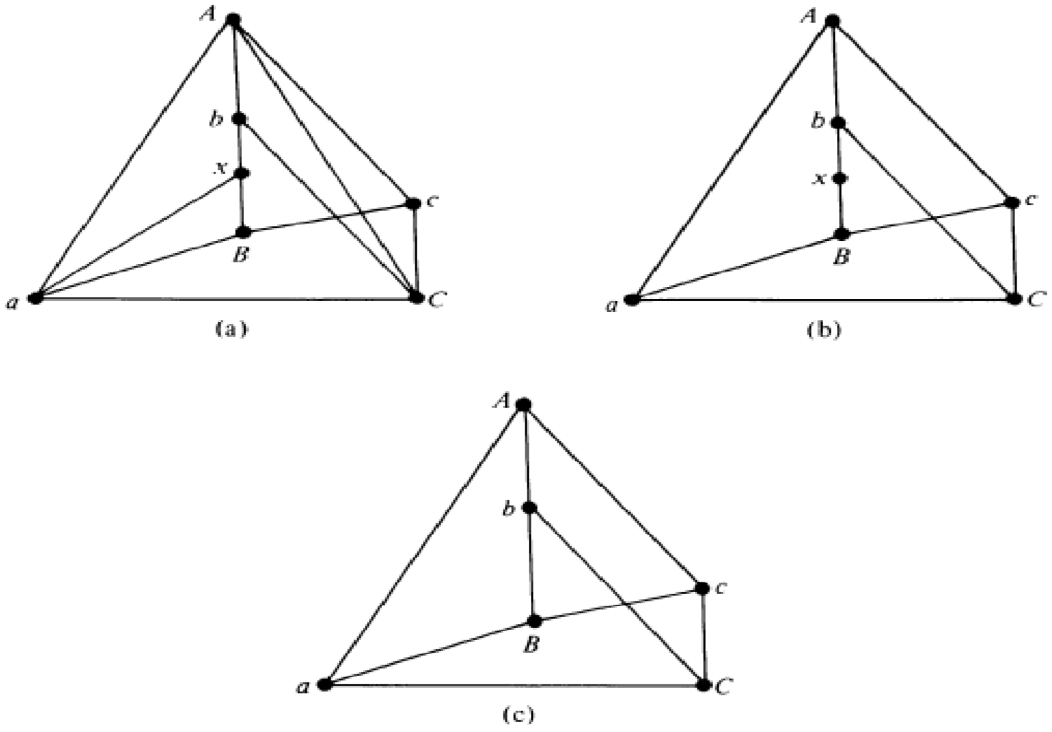
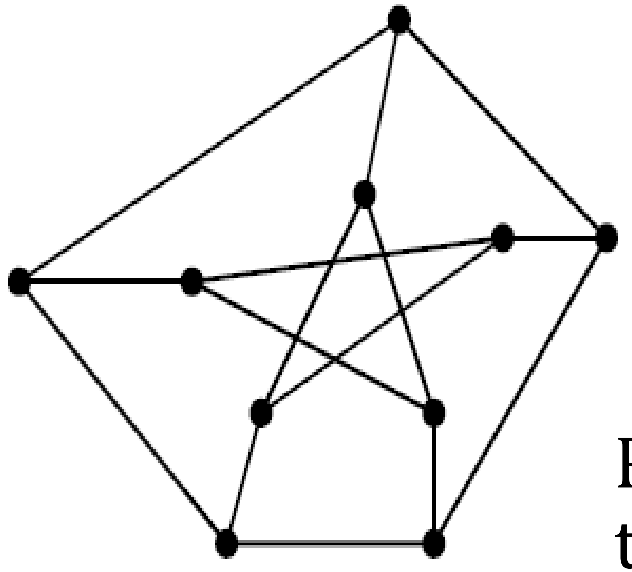


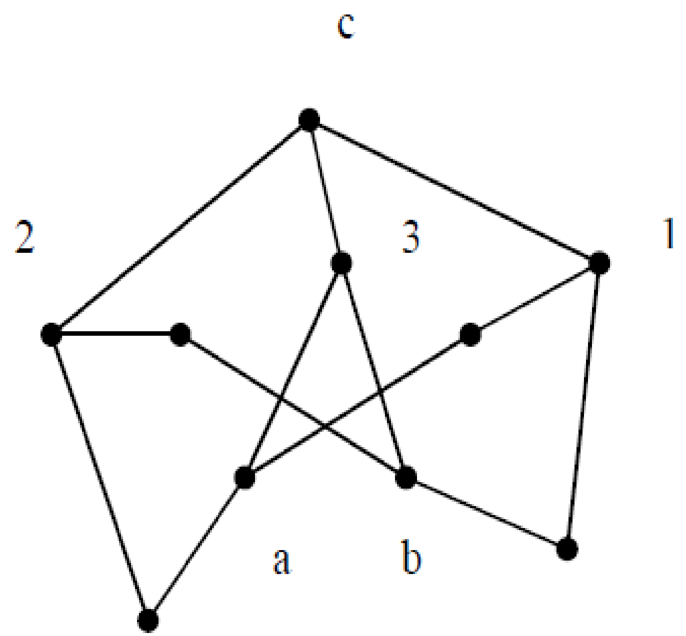
Fig. 5-9 Nonplanar graph with a subgraph homeomorphic to $K_{3,3}$.

Pertersen Graph



Petersen Graph

Petersen graph satisfies both the conditions $e \leq 3n - 6$ and $e \leq 2n - 4$) but still the graph is non planar as it has the subgraph homeomorphic to $K_{3,3}$).

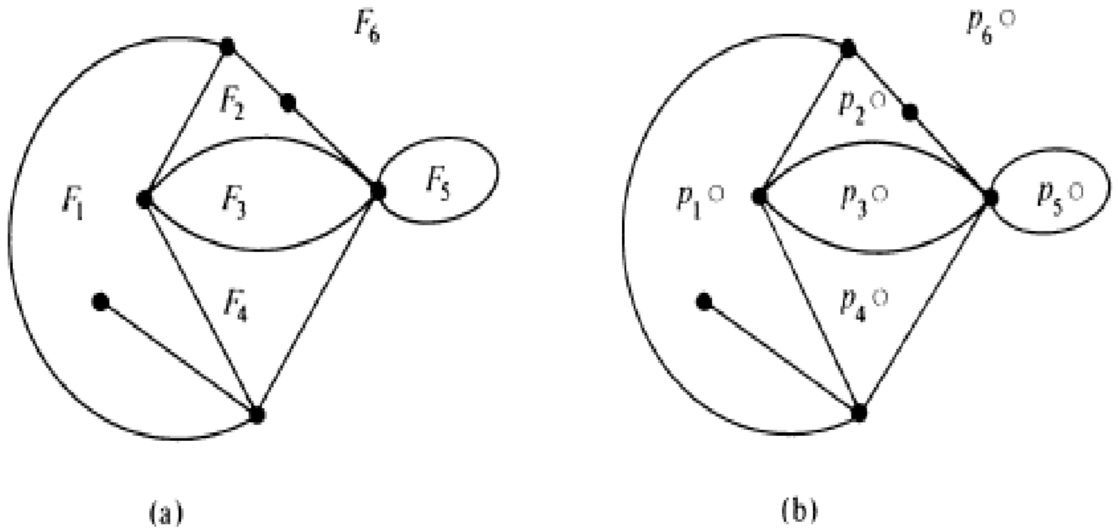


Petersen subgraph homeomorphic to $K_{3,3}$

- An elegant and simple-looking criterion (Theorem 5.9) for planarity of a graph, the theorem is difficult to apply in the actual testing of a large graph (say, a simple, nonseparable graph of 25 vertices, each of degree three or more).
- There have been several alternative characterizations of a planar graph.
- One of these characterizations, the existence of a **dual graph**.

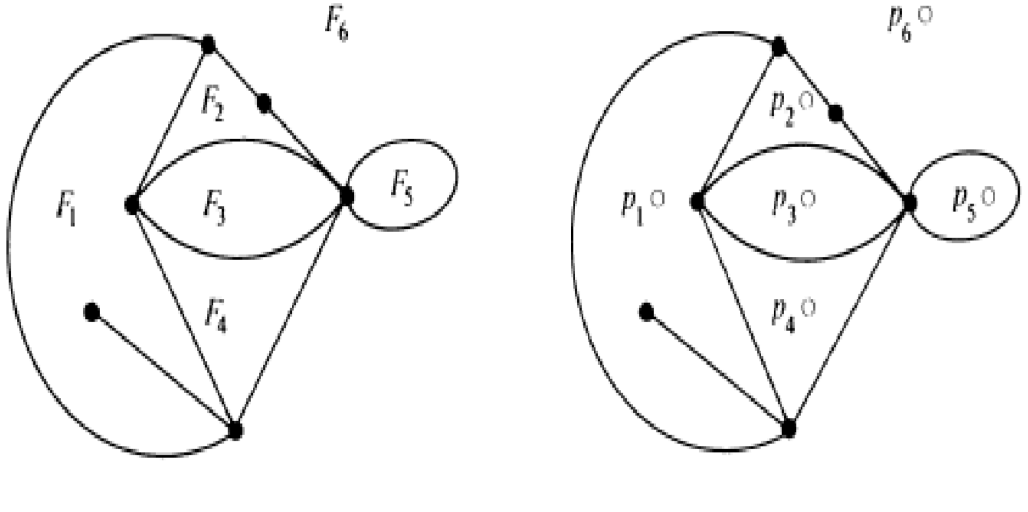
Geometric Dual

- Consider the plane representation of a graph as shown in Fig (a), with six regions or faces F_1, F_2, F_3, F_4, F_5, and F_6.
- Let us place six points p_1, p_2, p_3, p_4, p_5, and p_6 one in each of the regions, as shown in Fig (b).
- Next let us join these six points according to the procedure:



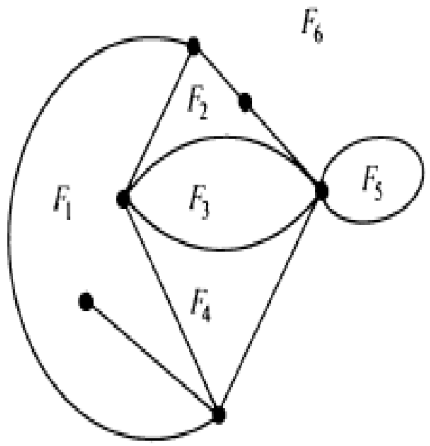
If two regions F_i and F_j are adjacent (i.e., have a common edge), draw a line joining points p_i and p_j that intersects the common edge between F_i and F_j exactly once. If there is more than one edge common between F_i and F_j, draw one line between points p_i and p_j for each of the common edges.

For an edge e lying entirely in one region, say F_k, draw a self-loop at point p_k intersecting e exactly once.

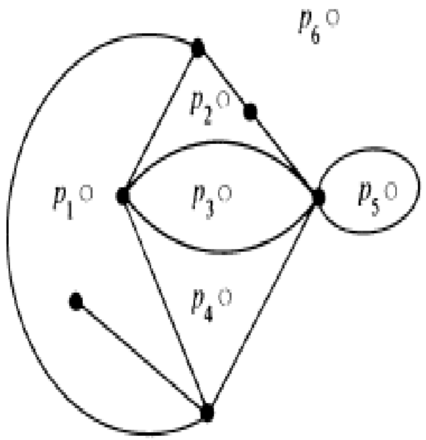


(a) (b)

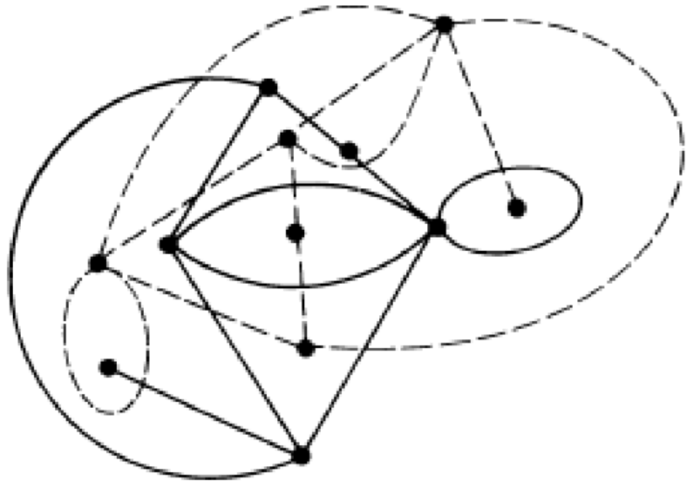
By this procedure we obtain a new graph G^* [in broken lines in Fig. (c)] consisting of six vertices, p_1, p_2, p_3, p_4, p_5, p_6 and edges joining these vertices. Such a graph G^* is called a ***geometric dual*** of G.



(a)

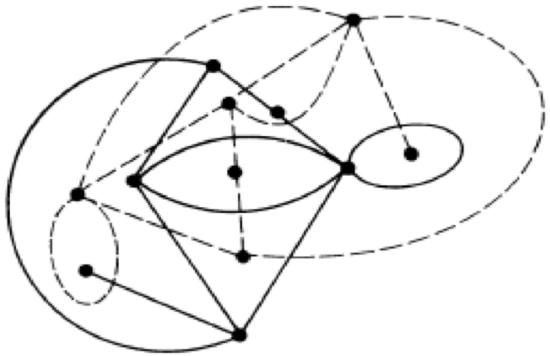


(b)



(c)

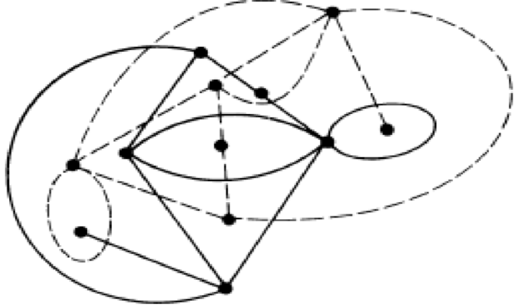
- Clearly, there is a one-to-one correspondence between the edges of graph G and its dual G^*–one edge of G^* intersecting one edge of G. Some simple observations that can be made about the relationship between a planar graph G and its dual G^* are
    1. An edge forming a self-loop in G yields a pendant edge in G^*.
    2. A pendant edge in G yields a self-loop in G^*.
    3. Edges that are in series in G produce parallel edges in G^*.
    4. Parallel edges in G produce edges in series in G^*.



(c)

5. Remarks 1-4 are the result of the general observation that the number of edges constituting the boundary of a region F_i in G is equal to the degree of the corresponding vertex p_i in G^*, and vice versa.
6. Graph G^* is also embedded in the plane and is therefore planar.
7. Considering the process of drawing a dual G^* from G, it is evident that G is a dual of G^*. Therefore, instead of calling G^* a dual of G, we usually say that G and G^* are dual graphs (**symmetric property**).
8. If n, e, f, r, and μ denote as usual the numbers of vertices, edges, regions, rank, and nullity of a connected planar graph G, and if n^*, e^*, f^*, r^*, and μ^* are the corresponding numbers in dual graph G^*. then

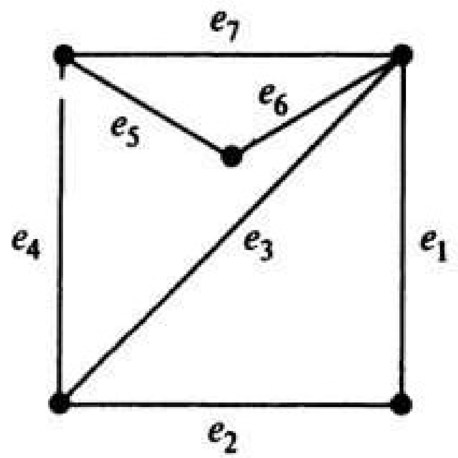
$$n^* = f, e^* = e, f^* = n, \; r^* = \mu \,, \;\; \mu^* = r$$



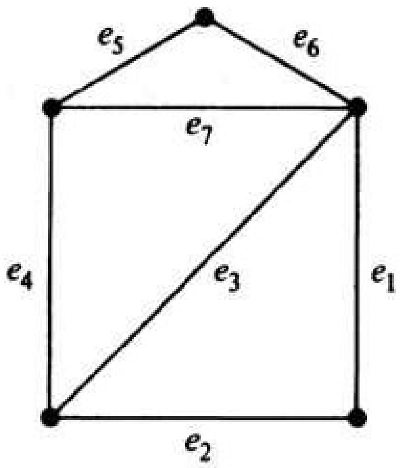
(c)

Is a (geometric) dual of a graph unique? OR Are all duals of a given graph isomorphic?

- Find the duals of the following graphs.

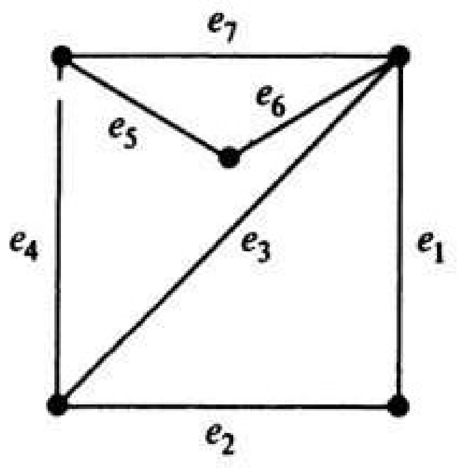


(a)

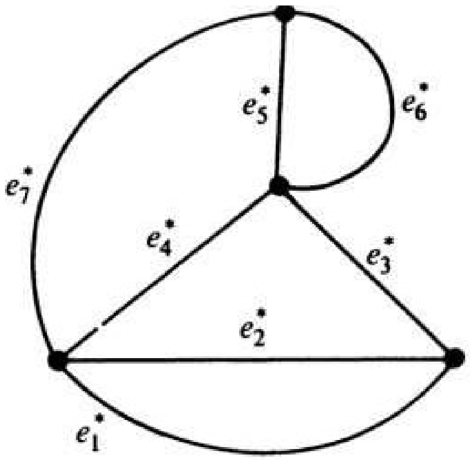


(b)

- Duals of the graphs are:

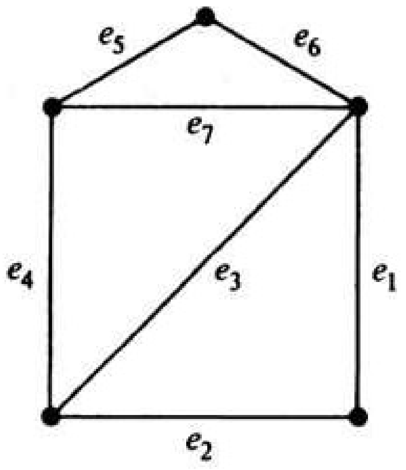


(a)

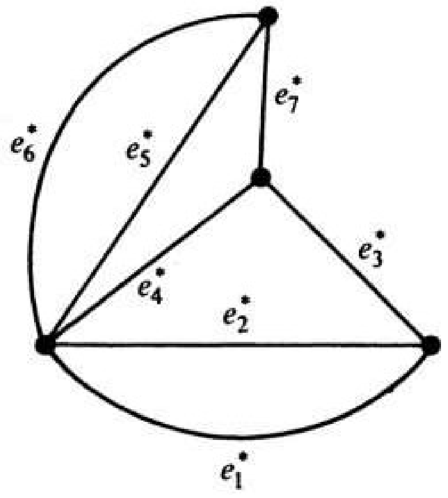


Dual of (a)

The same graph (isomorphic) which has two distinct embeddings, (a) and (b) leads to the duals of these isomorphic graphs which are nonisomorphic. However these duals are 2-isomorphic.



(b)



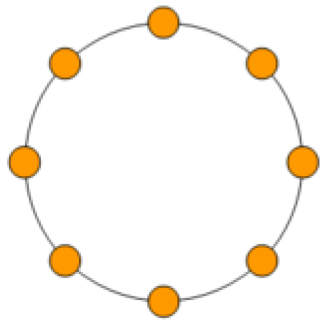
Dual of (b)

Uniqueness of dual graphs

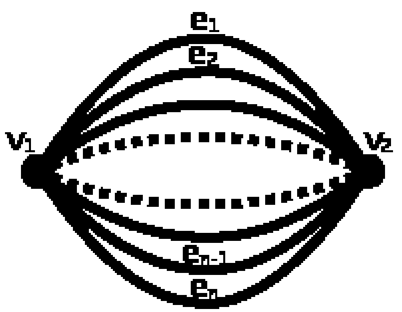
- Is a (geometric) dual of a graph unique? OR Are all duals of a given graph isomorphic? **ANSWER:** A planar graph G will have a unique dual if and only if it has a unique plane representation or unique embedding on a sphere.
- **THEOREM 5.10 -** All duals of a planar graph G are 2-isomorphic; and every graph 2-isomorphic to a dual of G is also a dual of G.
- Since a 3-connected planar graph has a unique embedding on a sphere, its dual must also be unique. In other words, all duals of a 3-connected graph are isomorphic.
- It is quite appropriate to refer to *a dual* as *the dual* of a planar graph.

The unique planar embedding of a cycle graph divides the plane into only two regions, the inside and outside of the cycle (**Jordan curve theorem**). However, in an *n*-cycle, these two regions are separated from each other by *n* different edges. Therefore, the dual graph of the *n*-cycle is a **multigraph** with two vertices (dual to the regions), connected to each other by *n* dual edges. Such a graph is called a **dipole graph**.

Conversely, the dual to an *n*-edge dipole graph is an *n*-cycle.



Cycle graph

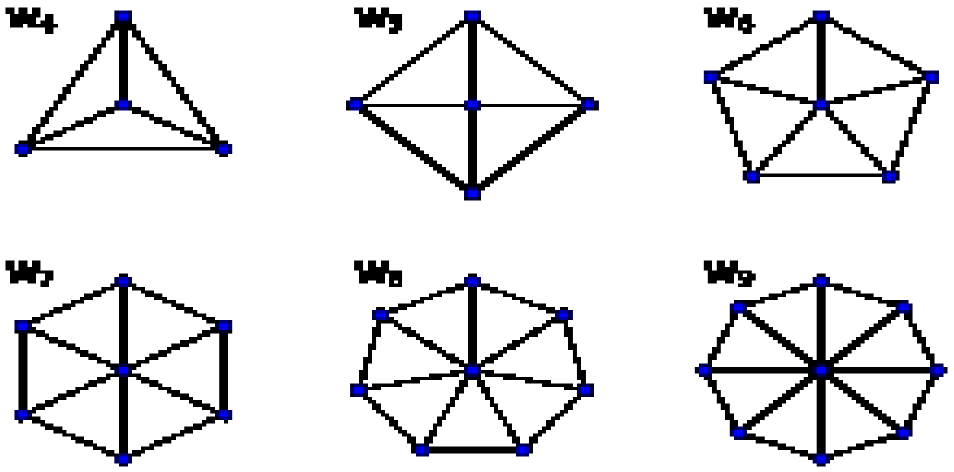


Dipole graph

https://en.wikipedia.org/wiki/Dual_graph

Self-dual graphs

- A plane graph is said to be **self-dual** if it is **isomorphic** to its dual graph. The **wheel** graphs provide an infinite family of self-dual graphs. Some of the wheel graphs are:

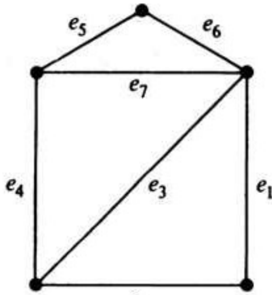


- The four-vertex complete graph is a self-dual graph.
- Which wheel graph is same as four vertex complete graph?

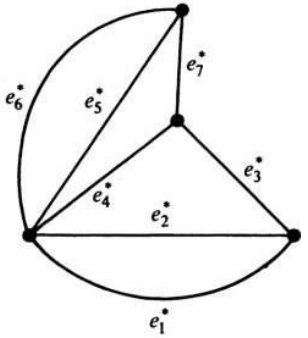
https://en.wikipedia.org/wiki/Dual_graph

Combinatorial Dual

- Independent definition of duality independent of the geometric representation is :
- **THEOREM 5.11 -** A necessary and sufficient condition for two planar graphs G_1 and G_2 to be duals of each other is as follows: There is a one-to-one correspondence between the edges in G_1 and the edges in G_2 such that a set of edges in G_1 forms a circuit if and only if the corresponding set in G_2 forms a cut-set.



(b)

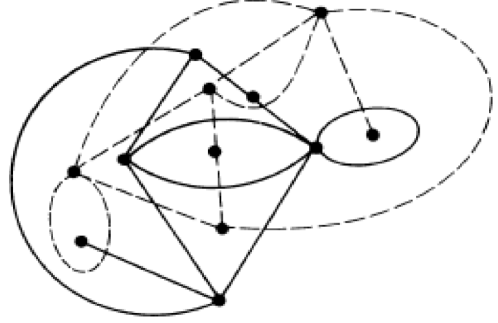


Dual of (b)

- Proof: Let us consider a plane representation of a planar graph G. Let us also draw (geometrically) a dual G^* of G. Then consider an arbitrary circuit Γ in G.
- Clearly, Γ will form some closed simple curve in the plane representation of G dividing the plane into two areas. (Jordan Curve Theorem). Thus the vertices of G^* are partitioned into two nonempty, mutually exclusive subsets−one inside Γ and the other outside.
- In other words, the set of edges Γ^* in G^* corresponding to the set Γ in G is a cut-set in G^*. Likewise it is apparent that corresponding to a cut-set S^* in G^* there is a unique circuit consisting of the corresponding edge-set S in G such that S is a circuit. This proves the necessity portion of Theorem 5.11.

- To prove the sufficiency, let G be a planar graph and let G' be a graph for which there is a one-to-one correspondence between the cut-sets of G and circuits of G', and vice versa.
- Let G^* be a dual graph of G.
- There is a one-to-one correspondence between the circuits of G' and cut-sets of G, and also between the cut-sets of G and circuits of G^*. Therefore there is a one-to-one correspondence between the circuits of G' and G^*, implying that G' and G^* are 2-isomorphic (Theorem 4-15).
- According to Theorem 5-10, G' must be a dual of G.

How to obtain ***Dual of a Subgraph?***



(c)

- Let G be a planar graph and G^* be its dual. Let a be an edge in G, and the corresponding edge in G^* be a^*. Suppose that we delete edge a from G and then try to find the dual of $G - a$.
- If edge a was on the boundary of two regions, removal of a would merge these two regions into one. Thus the dual $(\mathrm{G} - a)^*$ can be obtained from G^* by deleting the corresponding edge a^* and then fusing the two end vertices of a^* in $G^* - a^*$. On the other hand, if edge a is not on the boundary, a^* forms a self-loop. In that case $G^* - a^*$ is the same as $(\mathrm{G} - a)^*$.
- Thus if a graph G has a dual G^*, the dual of any subgraph of G can be obtained by successive application of this procedure.

How to obtain *Dual of a Homeomorphic Graph*?

- Let G be a planar graph and G^* be its dual.
- Let a be an edge in G, and the corresponding edge in G^* be a^*.
- Suppose that we create an additional vertex in G by introducing a vertex of degree two in edge a (i.e., a now becomes two edges in series). How will this addition affect the dual?
- It will simply add an edge parallel to a^* in G^*. Likewise, the reverse process of merging two edges in series (step 4 in Section 5-5) will simply eliminate one of the corresponding parallel edges in G^*.
- Thus if a graph G has a dual G^*, the dual of any graph homeomorphic to G can be obtained from G^* by the specified procedure.

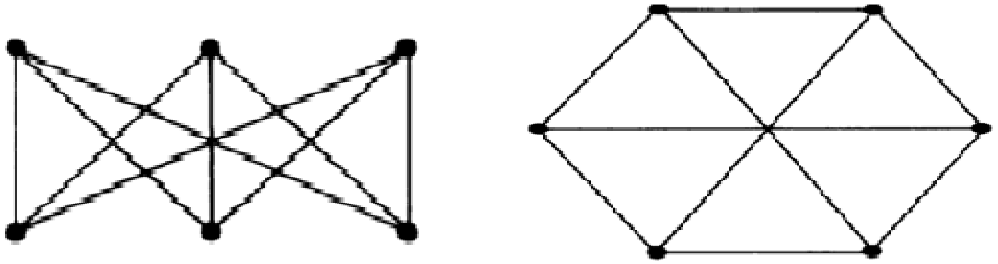
- The duality depends on the graph being embedded in a plane. However, now that Theorem 5-11 provides us with an equivalent abstract definition of duality (namely, the correspondence between circuits and cut-sets), which does not depend on a plane representation of a graph.
- Can the concept of duality be extended to nonplanar graphs also. In other words, given a nonplanar graph G, can we find another graph G' with one-to-one correspondence between their edges such that every circuit in G corresponds to a unique cut-set in G', and vice versa? **ANSWER: No**

- **Theorem 5.12: (Whitney's theorem)** A graph has a dual if and only if it is planar.

We only need to prove that a nonplanar graph does not have a dual. Let G be a nonplanar graph. Then according to Kuratowski's theorem, G contains K_5 or $K_{3,3}$ or a graph homeomorphic to either of these.

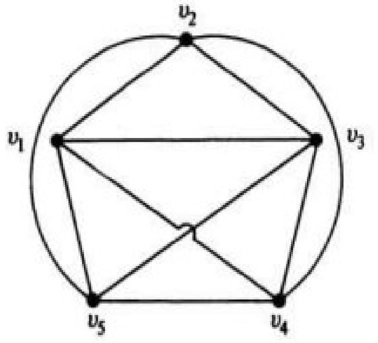
We have already seen that a graph G can have a dual only if every subgraph g *of* G and every graph homeomorphic to g has a dual. Thus if we can show that neither K_5 nor $K_{3,3}$ has a dual, we have proved the theorem.

Prove by contradiction that neither K_5 nor $K_{3,3}$ has a dual.



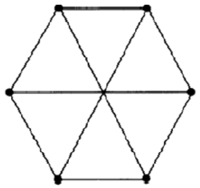
$K_{3,3}$

- Suppose that $K_{3,3}$ has a dual D. Observe that the cut-sets in $K_{3,3}$ correspond to circuits in D and vice versa (Theorem 5.10).
- Since $K_{3,3}$ has no cut-set consisting of two edges, D has no circuit consisting of two edges. That is, D contains no pair of parallel edges.
- Since every circuit in $K_{3,3}$ is of length four or six, D has no cut-set with less than four edges. Therefore, the degree of every vertex in D is at least four.
- As D has no parallel edges and the degree of every vertex is at least four, D must have at least five vertices each of degree four or more.
- That is, D must have at least (5 x 4)/2 = 10 edges. This is a contradiction, because $K_{3,3}$ has nine edges and so must its dual. Thus $K_{3,3}$ cannot have a dual.



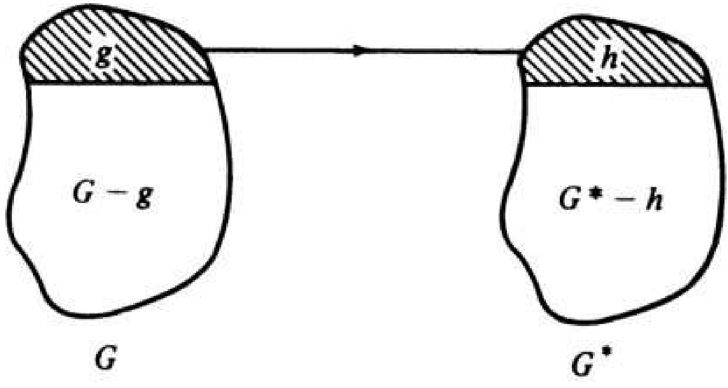
K_5

- Suppose that the graph K_5 has a dual H.
- Note that K_5 has: (1) 10 edges, (2) no pair of parallel edges, (3) no cut-set with two edges, and (4) cut-sets with only four or six edges.
- Consequently, graph H must have (1) 10 edges, (2) no vertex with degree less than three, (3) no pair of parallel edges, and (4) circuits of length four and six only.
- Now graph H contains a hexagon (a circuit of length six), and no more than three edges can be added to a hexagon without creating a circuit of length three or a pair of parallel edges.



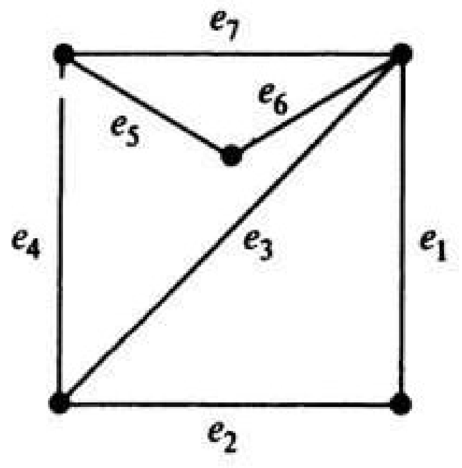
- Since both of these are forbidden in H and H has 10 edges, there must be at least seven vertices in H.
- The degree of each of these vertices is at least three. This leads to H having at least 11 edges. A contradiction.

- There is yet another equivalent combinatorial definition of duality, also given by Whitney and proved equivalent to the earlier two definitions [5-10].
- Two planar graphs G and G^* are said to be duals (or *combinatorial duals*) of each other if there is a one-to-one correspondence between the edges of G and G^* such that if g is any subgraph of G and h is the corresponding subgraph of G^*, then
- $rank\ of\ (G^* - h) = rank\ of\ G^*\ - nullity\ of\ g$.

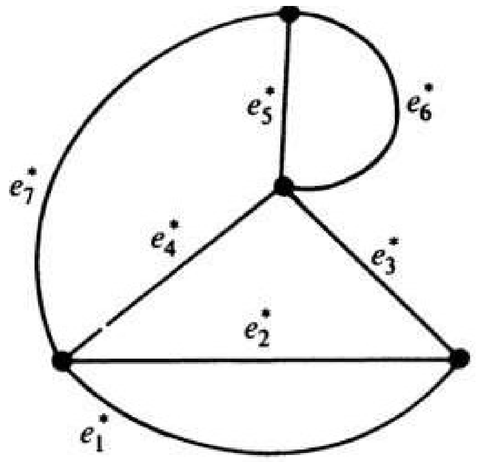


Rank of (G^* − h) = Rank of G^* − Nullity of g

Example



(a)



Dual of (a)

Consider a subgraph $\{e_4, e_5, e_6, e_7\}$ from G and $\{e_4{}^*, e_5{}^*, e_6{}^*, e_7{}^*\}$ from G^*.

$$rank\ of\ (G^* - \{e_4{}^*, e_5{}^*, e_6{}^*, e_7{}^*\}) = rank\ of \{e_1{}^*, e_2{}^*, e_3{}^*\} = 2$$

$$rank\ of\, G^* = 3$$

$$nullity\ of \{e_4, e_5, e_6, e_7\} = 1$$

$$\text{And } 2 = 3 - 1$$

Clearly, this definition is also independent of the geometric connotation. It is therefore often preferred for proving results in purely algebraic fashion.

The third classic planarity criterion by MacLane

- ***Set of Basic Circuits*:** A set C of circuits in a graph is said to be *a complete set of basic circuits* if

(i) every circuit in the graph can be expressed as a ring sum of some or all circuits in C, and

(ii) no circuit in C can be expressed as a ring sum of others in C.

It may, however, be mentioned here that whereas a set of fundamental circuits (as defined in Chapter 3 with respect to a spanning tree) always constitutes a complete set of basic circuits, the converse does not hold for all graphs (Problem 5-15).

In a planar graph a complete set of basic circuits has an additional property.

- In a plane representation of a planar, connected graph G the set of circuits forming the interior regions constitutes a complete set of basic circuits.
- For any circuit Γ in G can be expressed as the ring sum of the circuits defining the regions contained in Γ.
- Observe that every edge appears in at most two of these basic circuits. Thus for every planar graph G we can find a complete set of basic circuits such that no edge appears in more than two of these basic circuits.
- This result and its converse lead to another well-known characterization of planar graphs.
- **Theorem 5.13** - A graph G is planar if and only if there exists a complete set of basic circuits (i.e., all μ of them, μ being the nullity of G) such that no edge appears in more than two of these circuits.

- All three of these classic characterizations suffer from two shortcomings. **First**, they are extremely difficult to implement for a large graph. **Second**, in case the graph is planar they do not give a plane representation of the graph.
- These drawbacks have prompted recent discoveries of several *mapconstruction* methods, where the testing of planarity itself is based on an attempt to produce a plane representation of the graph.
- One such method is given by Tutte [5-9]. Several other construction methods, some of them quite similar, have been implemented on digital computers [5-2, 5-8].
- In most of these methods, the given graph is first reduced to one or more *simple, nonseparable graphs with every vertex of degree three or more* and with $e \leq 3n - 6$. Then the construction algorithm is applied such that either one succeeds in obtaining a planar realization of the graph or the graph is nonplanar.

Thickness

- Having found that a given graph G is nonplanar, it is natural to ask, what is the minimum number of planes necessary for embedding G?
- The least number of planar subgraphs whose union is the given graph G is called the *thickness* of G.
- In a printed-circuit board, for instance, the number of insulation layers necessary is the thickness of the corresponding graph.
- By definition, then, **the thickness of a planar graph is one**.
- The thickness of each of Kuratowski's graphs is clearly two.
- The thickness of the complete graph of eight vertices is two, while the thickness of the complete graph of nine vertices is three.
- Although there are several results available on the thickness of special types of graphs, the thickness of an arbitrary graph is in general, difficult to determine.

- Another question one might ask about a nonplanar graph is: What is the fewest number of crossings (or intersections) necessary in order to “draw” the graph in a plane?
- The crossing number of a planar graph is, by definition, zero, and of either of Kuratowski’s graphs, it is one.
- The crossing numbers of only a few graphs have been determined. No formula exists to give the crossing number of an arbitrary graph.