Coloring, Covering and Partitioning

- Given a graph G,
  - How to paint its vertices such that no two adjacent vertices have the same color?
  - How to paint its edges such that no two adjacent edges have the same color?
  - How to paint its regions (only for planar graphs) such that no two adjacent regions have same color?
  - What is the minimum number of colors that you would require to color edges/vertices?
  - How to perform edge/vertex partitioning?

Chromatic number

- Painting all the vertices of a graph with colors such that no two adjacent vertices have the same color is called proper coloring or coloring of a graph.
- A graph in which every vertex has been assigned a color according to a proper coloring is called a properly colored graph.
- A graph can be colored in many different ways.

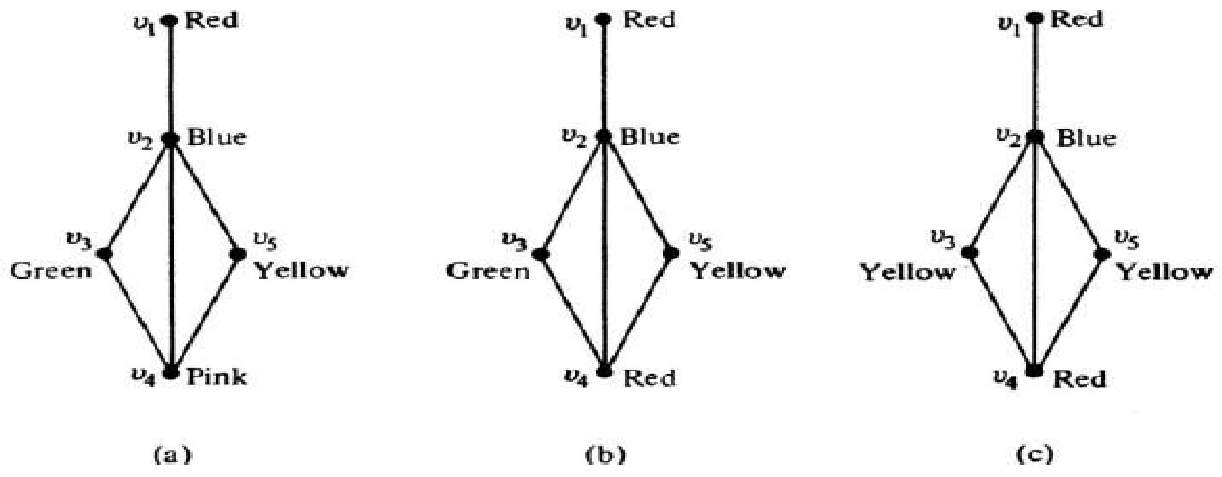


Fig. 8-1 Proper colorings of a graph.

- A simple connected graph G that requires κ different colors for its proper coloring and no less, is called κ-chromatic graph, and the number κ is called the chromatic number of G.

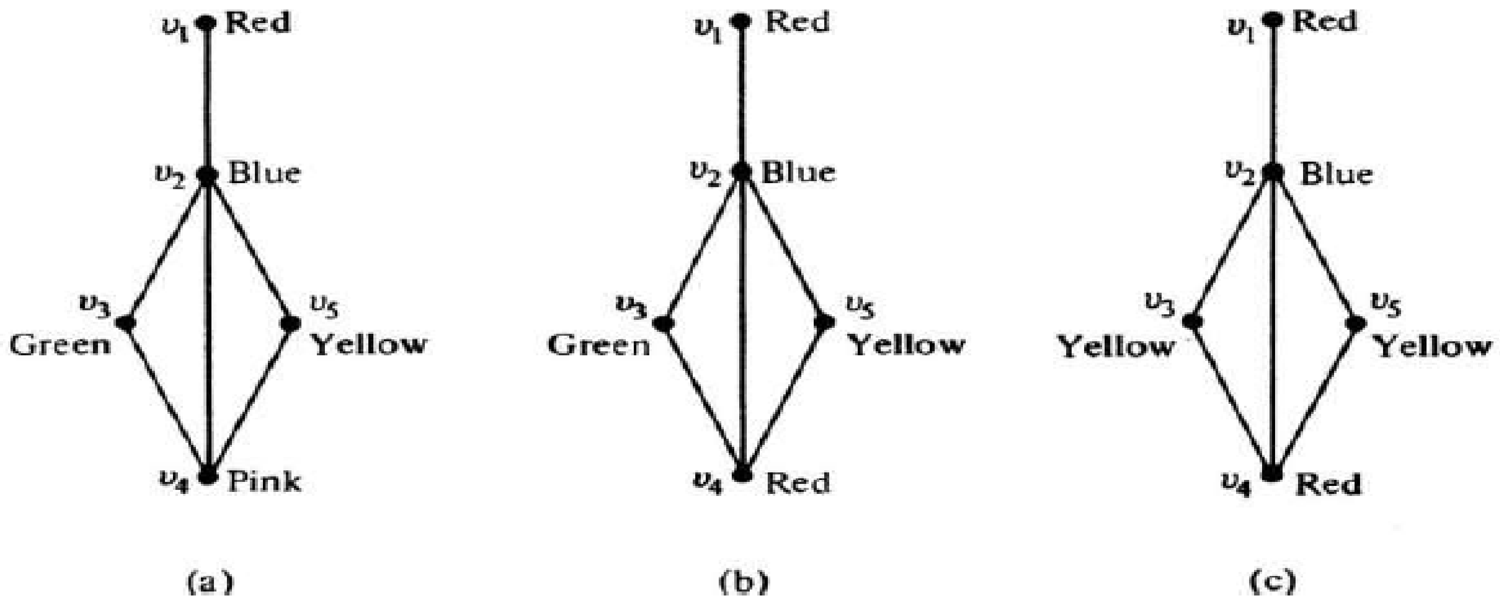


Fig. 8-1 Proper colorings of a graph.

- Some observations from the definition are:
  - A graph consisting of only isolated vertices is 1-chromatic.
  - A graph with one or more edges is atleast 2-chromatic.
  - A complete graph of n vertices is n-chromatic. Hence, a graph containing a sub-graph of r vertices is at least r-chromatic.
  - A graph consisting of simply one circuit with $n \geq 3$ vertices is 2-chromatic if n is even and 3-chromatic if n is odd.
- The identification of κ for any graph is quite difficult. Let’s focus on specific graphs now.

- **Theorem 8.1**: Every tree with two or more vertices is 2-chromatic.

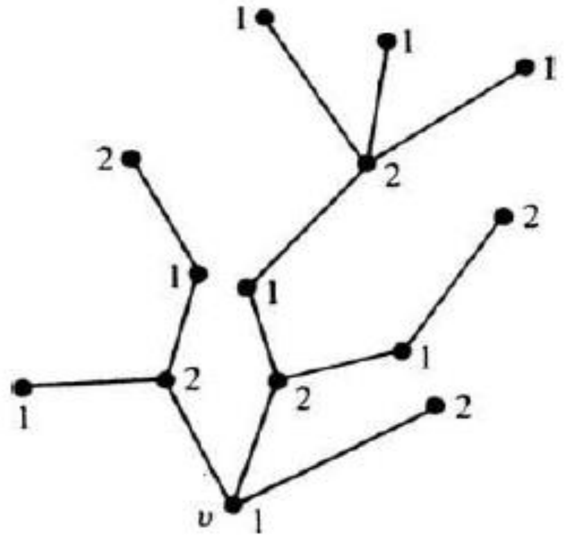


Fig. 8-2 Proper coloring of a tree.

- True/False: Though a tree is 2-chromatic, not very 2-chromatic graph is a tree.

- Utility graphs (bipartite graphs), and simple graph with cycle of even length are 2-chromatic but not tree.
- What is the characterization of a 2-chromatic graph?
- **Theorem 8.2**: A graph with at least one edge is 2-chromatic if and only if it has no circuits of odd length.
- The upper limit on the chromatic number of a graph is given by
- **Theorem 8.3**: If d_{max} is the maximum degree of the vertices in a graph G,

$$\boldsymbol{chromatic\ number\ of\ G\ \leq 1 +\ d_{max}}$$

The upper bound can be improved by 1 if G has no complete graph of d_{max} + 1 vertices .

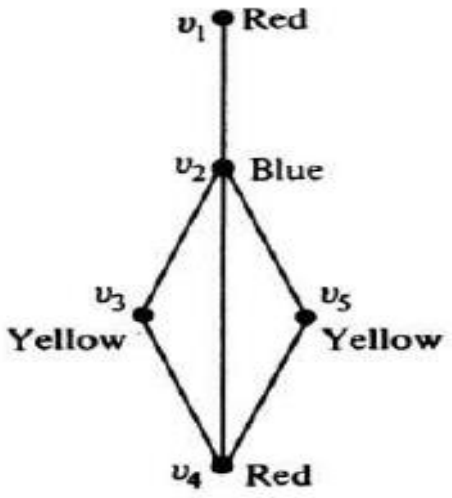
$$\boldsymbol{chromatic\ number\ of\ G\ \leq\ d_{max}}$$

- Every 2-chromatic graph is bipartite with an exception; a graph of two or more isolated vertices and with no edge is bipartite but is 1-chromatic.
- In general, a graph G is called *p*-partite if its vertex set can be decomposed into p disjoint subsets V_1, V_2, …, V_p, such that no edge in G joins the vertices in the same subset.
- A κ-chromatic graph is *p*-partite if and only if

$$\kappa \leq p$$

Chromatic Partitioning

- A proper coloring of a graph naturally induces a partitioning of the vertices into different subsets. For eg.



- Partition produces for the given figure are $\{v_1,v_4\},\{v_2\}.\{v_3,v_5\}$. No two vertices in any of these three subsets are adjacent.

- A set of vertices in a graph is said to be an independent set of vertices (internally stable set) if no two vertices in the set are adjacent.

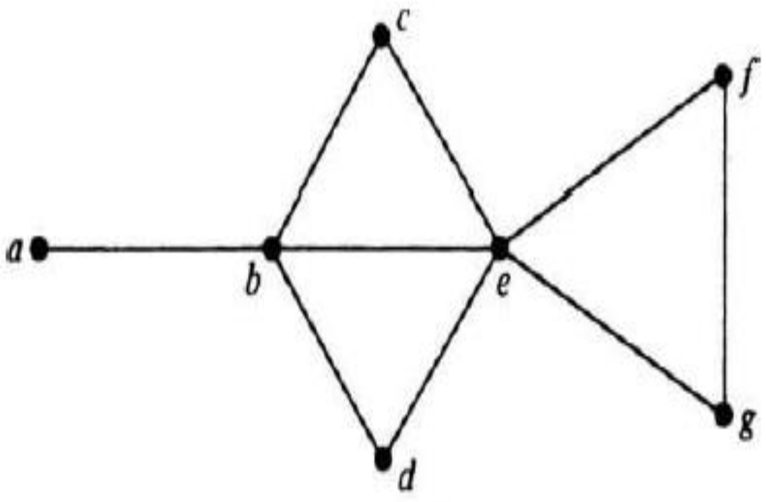


Fig. 8-3

- For the given figure, {a,c,d} is an independent set.
- A single vertex in any graph constitutes an independent set.

- A maximal independent set (maximal internally stable set) is an independent set to which no other vertex can be added without destroying its independence property.

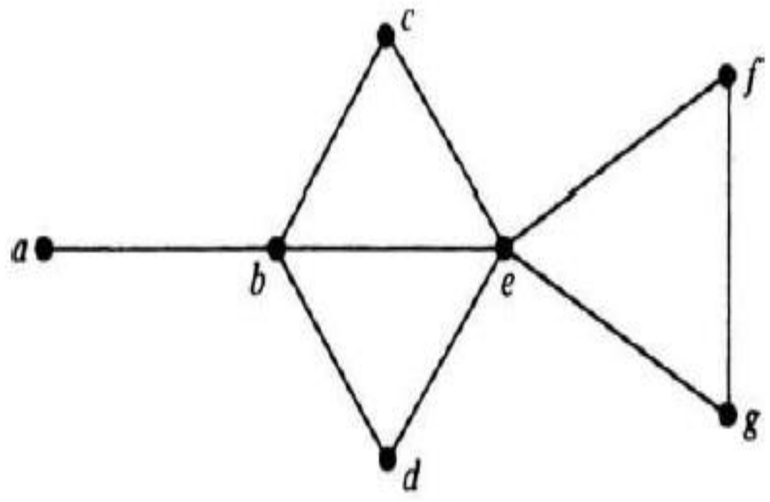


Fig. 8-3

- For the given figure, {a,c,d,g},{b,f} and {b,g} are maximal independent sets.
- A graph may have more than one maximal independent sets of variable size where the one with maximum number of vertices is of much interest.

Example

- Seven vertices of a graph represents possible code word to be used in some communication. Some words are so close to others that they might be confused for each other. Such pairs are joined with an edge.
- Find a largest set of code words for a reliable communication.
- The problem is to find maximal independent set with largest number of vertices.

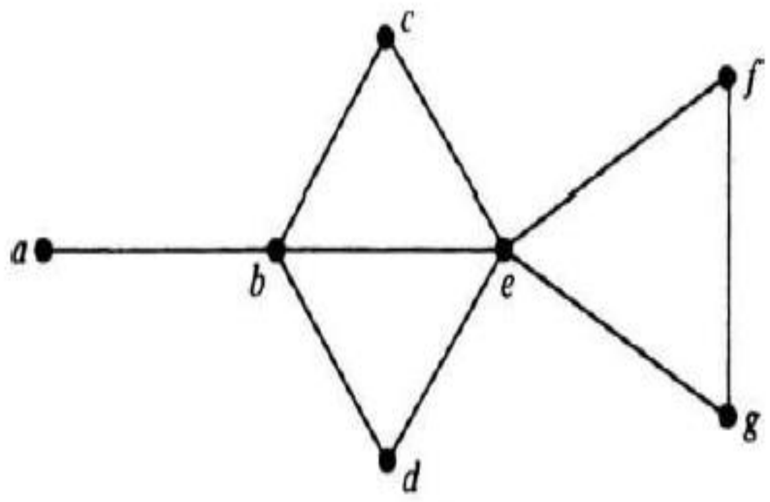


Fig. 8-3

- The number of vertices in the largest independent set of a graph G is called the independence number, $\beta(G)$.
- For a κ-chromatic graph, largest number of vertices in G with the same color cannot exceed the independence number , $\beta(G)$,

$$\beta(G) \geq \frac{n}{\kappa}$$

Finding a Maximal Independent Set

- To get the maximal independent set (not necessarily maximal independent set with largest number of vertices)
  - Start with any vertex v in G in the set.
  - Add more vertices to the set, selecting at each stage a vertex that is not adjacent to any of those already selected.

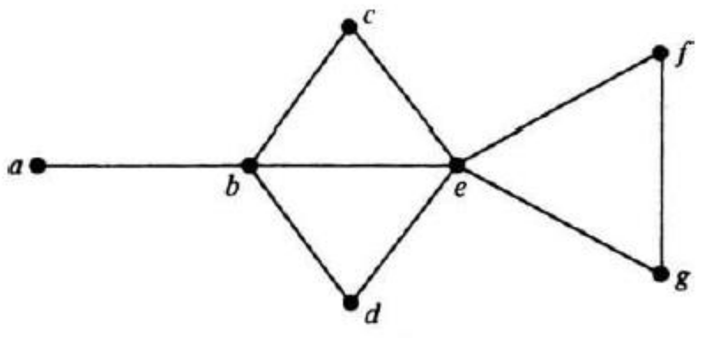


Fig. 8-3

Finding all maximal independent sets

- Use Boolean arithmetic on the vertices.
  - Let each vertex in the graph be treated as a Boolean variable.
  - Let
    - the logical sum $a+b$ denote the operation of including vertex a or b or both; and
    - logical multiplication ab denote the operation of including both vertices a and b; and
    - Boolean complement a' denote that vertex a is not included.
  - For a given graph G we must find a maximal subset of vertices that does not include the two end vertices of any edge in G. Let us express an edge (x,y) as a Boolean product, xy, of its end vertices x and y, and sum all such Boolean products in G to get a Boolean expression

$$\varphi = \sum xy \qquad \text{for all (x,y) in G.}$$

- Further take the Boolean complement φ' of this expression, and express it as a sum of Boolean products:

$$\varphi' = f_1 + f_2 + \ldots + f_n$$

- A vertex set is a maximal independent set if and only if $\varphi = 0$ (logically false), which is possible if and only if $\varphi' = 1$ (true), which is possible if and only if at least one $f_i = 1$, which is possible if and only if each vertex appearing in f_i (in complemented form) is excluded from the vertex set of G.
- The each f_i will yield a maximal independent set, and every maximal independent set will be produced by this method.

Example 1

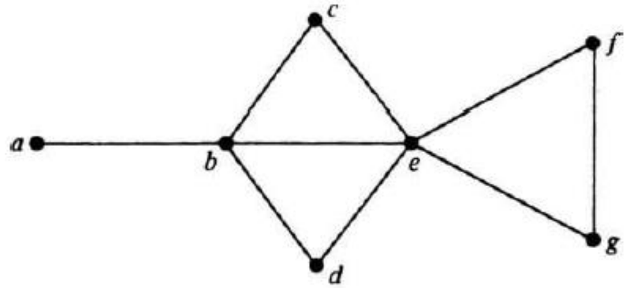


Fig. 8-3

- $\varphi = ab + bc + bd + be + ce + de + ef + eg + fg$
- $\varphi' = (a' + b')(b' + c')(b' + d')(b' + e')(c' + e')(d' + e')(e' + f')(e' + g')(f' + g')$

Use the usual identities:

1) **aa = a**
2) **a + a = a**
3) **a + ab = a**

$\varphi' = (a'+b')(b'+c')(b'+d')(b'+e')(c'+e')(d'+e')(e'+f')(e'+g')(f'+g')$

$\varphi' = (a'b' + a'c' + b'b' + b'c')(b'b' + b'e' + b'd' + d'e')(c'd' + c'e' + d'e' + e'e')$
$(e'e' + e'g' + e'f' + f'g')(f'+g')$

$\varphi' = (b' + a'c')(b' + d'e')(e' + c'd')(e' + f'g')(f'+g')$

$\varphi' = (b'b' + b'd'e' + a'b'c' + a'c'd'e')(e'e' + e'f'g' + c'd'e' + c'd'f'g')$
$(f'+g')$

$\varphi' = (b' + a'c'd'e')(e' + c'd'f'g')(f'+g')$

$\varphi' = (b'e' + b'c'd'f'g' + a'c'd'e' + a'c'd'e'f'g')(f'+g')$

$\varphi' = (b'e' + b'c'd'f'g' + a'c'd'e')(f'+g')$

$\varphi' = b'e'f' + b'e'g' + b'c'd'f'g' + \cancel{b'c'd'f'}g' + a'c'd'e'f' +$
$a'c'd'e'g'$

$\varphi' = b'e'f' + b'e'g' + b'c'd'f'g' + a'c'd'e'f' + a'c'd'e'g'$

$\therefore$ five maximal independent sets are
$\{a, c, d, g\}$, $\{a, c, d, f\}$, $\{g, e\}$, $\{b, g\}$ & $\{b, f\}$

Example 2

$$= (A'+B')(A'+D')(A'+E')(B'+C')(C'+E')(C'+F')(D'+E')(E'+F')$$

$$= (A'A'+A'D'+A'B'+B'D')(A'B'+A'C'+B'E'+C'E')(C'C'+C'F'+C'E'+E'F')(D'E'+D'F'+E'E'+E'F')$$

$$= (A'+B'D')(A'B'+A'C'+B'E'+C'E')(C'+E'F')(E'+D'F')$$

$$= (A'C'+A'E'F'+B'C'D'+B'D'E'F')(A'B'E'+A'B'D'F'+A'C'E'+A'C'D'F'+B'E'+B'D'E'F'+C'E'+C'D'F')$$

$$= A'B'C'E'+A'B'C'D'F'+A'C'E'+A'C'D'F'+A'B'C'E'+A'B'C'D'E'F'$$

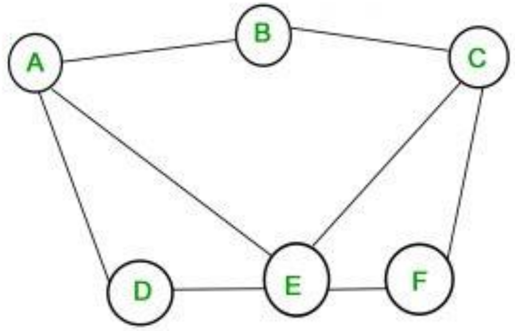
$$= (A'+B'D')(C'+E'F')(E'+D'F')(A'B'+A'C'+B'E'+C'E')$$

$$= (A'+B'D')(C'E'+C'D'F'+E'F'+D'E'F')(A'B'+A'C'+B'E'+C'E')$$

$$= (A'+B'D')(C'E'+C'D'F'+E'F')(A'B'+A'C'+B'E'+C'E')$$

$$= (A'C'E'+A'C'D'F'+A'E'F'+B'C'D'E'+B'C'D'F'+B'D'E'F')(A'B'+A'C'+B'E'+C'E')$$

$$= A'B'C'E'+A'C'E'+A'B'C'E'+A'C'E'+A'B'C'D'F'+A'C'D'F'$$
$$+A'B'C'D'E'F'+A'C'D'E'F'+A'B'E'F'+A'C'E'F'+A'B'E'F'+A'C'E'F'$$
$$+A'B'C'D'E'+A'B'C'D'E'+B'C'D'E'+B'C'D'E'+A'B'C'D'F'+A'B'C'D'E'F'$$
$$B'D'E'F'+B'C'D'E'F'$$



https://www.themathcitadel.com/using-boolean-algebra-to-find-all-maximal-independent-sets-in-a-graph/

Finding independence and chromatic numbers

- From the set of independence sets, find the set with the largest number of vertices to get the independence number, $\beta(G)$. For example 1, $\beta(G)$ = 4.
- To find the chromatic number of G, find the minimum number of these (maximal independence) sets, which collectively include all the vertices of G. For example 1, {a,c,d,f}, {b,g} and {a,e} sets satisfy this condition. Thus, the graph is 3-chromatic.

Chromatic partitioning

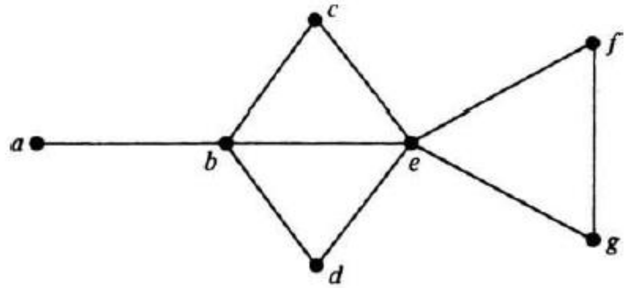


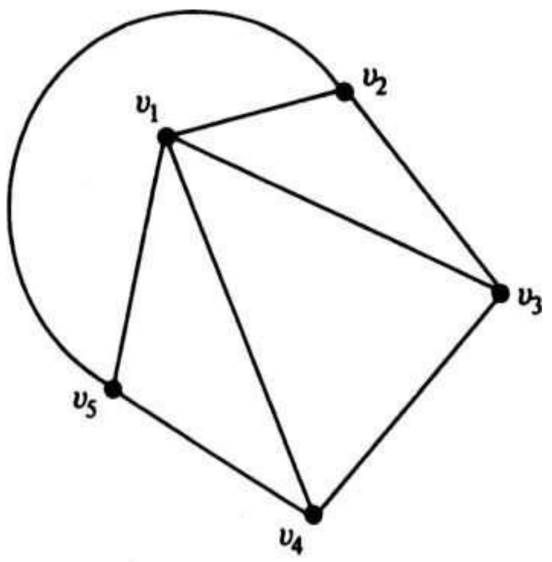
Fig. 8-3

- For a simple, connected graph G, chromatic partitioning can be obtained by partitioning all vertices of G into the smallest possible number of disjoint, independent sets.
- To get the chromatic partitioning, enumerate all maximal independent sets and select smallest number of sets that include all vertices of the graph.
- For example 1, maximal independent sets are {a,c,d,f}, {a,c,d,g}, {b,f}, {b,g} and {a,e} and its chromatic partitions are
  - {(a,c,d,f),(b,g),(e)}
  - {(a,c,d,g),(b,f),(e)}
  - {(c,d,f),(b,g),(a,e)}
  - {(c,d,g),(b,f),(a,e)}

This method of finding chromatic partitioning is inefficient and requires large amount of computer memory.

Uniquely colorable graphs

- A graph that has only one chromatic partition is called a uniquely colorable graph. The graph given here is uniquely colorable graph as it has only one chromatic partition, i.e. $\{(v_1),(v_2,v_4),(v_3,v_5)\}$.



Dominating sets

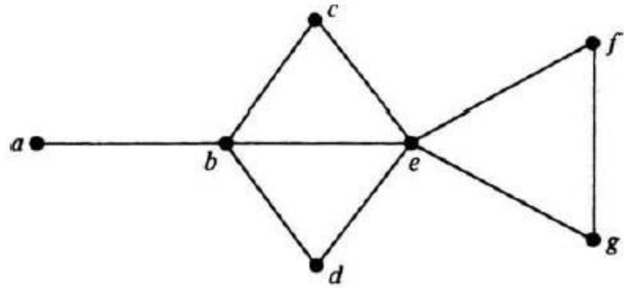


Fig. 8-3

- A dominating set (an externally stable set) in a graph G is a set of vertices that dominates every vertex *v* in G in the following sense: Either v is included in the dominating set or is adjacent to one or more vertices included in the dominating set.
- For eg. vertex set {b,g} and {a,b,e,f,g} are dominating sets.
- A dominating set need not be independent.
- For example, the set of all its vertices is trivially a dominating set in every graph.

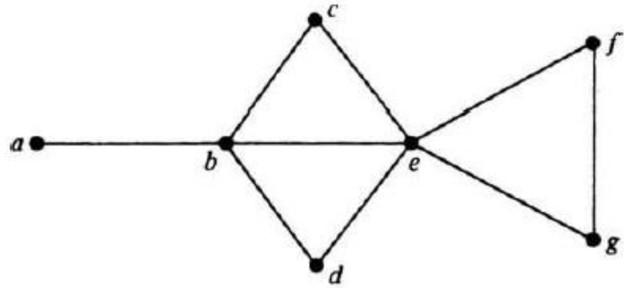


Fig. 8-3

- In many applications one is interested in finding minimal dominating sets defined as follows:
- A minimal dominating set is a dominating set from which no vertex can be removed without destroying its dominance property.
- For example, in Fig. 8-3, {b, e} is a minimal dominating set. And so is {a, c, d, f}.
- Observations that follow from these definitions are

1. Any one vertex in a complete graph constitutes a minimal dominating set.
2. Every dominating set contains at least one minimal dominating set.
3. A graph may have many minimal dominating sets, and of different sizes. [The number of vertices in the smallest minimal dominating set of a graph G is called the domination number, $\alpha(G)$.]
4. A minimal dominating set may or may not be independent.
5. Every maximal independent set is a dominating set. If an independent set does not dominate the graph, there is at least one vertex that is neither in the set nor adjacent to any vertex in the set. Such a vertex can be added to the independent set without destroying its independence. But then the independent set could not have been maximal.
6. An independent set has the dominance property only if it is a maximal independent set. Thus an independent dominating set is the same as a maximal independent set.
7. In any graph G, $\alpha(G) \leq ß(G)$.

Finding Minimal Dominating Sets

- Use Boolean arithmetic to find minimal dominating sets.
- To dominate a vertex v_i we must either include v_i or any of the vertices adjacent to v_i. A minimum set satisfying this condition for every vertex v_i is a desired set. Therefore, for every vertex v_i in G let us form a Boolean product of sums $(v_i + v_{i1} + v_{i2} + \ldots + v_{id})$, where $v_{i1}, v_{i2}, \ldots, v_{id}$ are the vertices adjacent to v_i , and d is the degree of v_i.

$$\theta = \Pi\ (v_i + v_{i1} + v_{i2} + \ldots + v_{id}) \text{ for all } v_i \text{ in } G$$

- When θ is expressed as a sum of products, each term in it will represent a minimal dominating set.

adj vertex

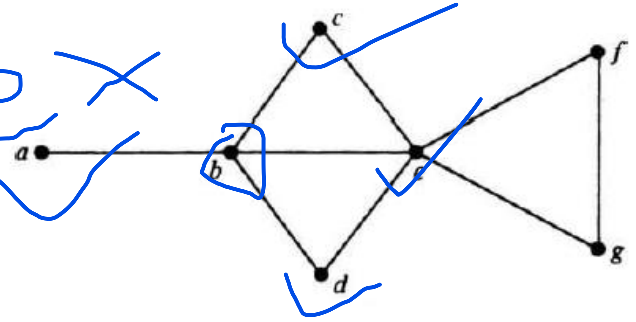


Fig. 8-3

$$\theta = (a + b)(b + c + d + e + a)(c + b + e)(d + b + e)(e + b + c + d + f + g)(f + e + g)(g + e + f)$$

- Since in Boolean arithmetic $(x + y)x = x$,

$$\theta = (a + b)(b + c + e)(b + d + e)(e + f + g)$$

$$\theta = ae + be + bf + bg + acdf + acdg.$$

- Each of the six terms in the preceding expression represents a minimal dominating set and $\alpha(G) = 2$.

Chromatic Polynomial

- A given graph G of n vertices can be properly colored in many different ways using a sufficiently large number of colors. This property of a graph is expressed elegantly by means of a polynomial. This polynomial is called the chromatic polynomial of G and is defined as follows :

  The value of the chromatic polynomial $P_n(\lambda)$ of a graph with n vertices gives the number of ways of properly coloring the graph using λ or fewer colors. Let c_i be the different ways of properly coloring G using exactly i different colors. Since i colors can be chosen out of λ colors in

$$\binom{\lambda}{i}\ different\ ways$$

- There are $c_i \binom{\lambda}{i}$ different ways of properly coloring G using exactly i colors out of λ colors.
- Since i can be any positive integer from 1 to n (it is not possible to use more than n colors on n vertices), the chromatic polynomial is a sum of these terms; that is,

$$
\begin{aligned}
P_n(\lambda) &= \sum_{i=1}^{n} c_i \binom{\lambda}{i} \\
&= c_1 \frac{\lambda}{1!} + c_2 \frac{\lambda(\lambda-1)}{2!} + c_3 \frac{\lambda(\lambda-1)(\lambda-2)}{3!} + \cdots \\
&\quad + c_n \frac{\lambda(\lambda-1)(\lambda-2)\cdots(\lambda-n+1)}{n!}.
\end{aligned}
$$

- Each c_i has to be evaluated individually for the given graph. For example, any graph with even one edge requires at least two colors for proper coloring, and therefore

$$c_1 = 0$$

- A graph with n vertices and using n different colors can be properly colored in n! ways; that is,

$$c_n = n!$$

Find the chromatic polynomial of the graph

$$P_5(\lambda) = c_1\lambda + c_2\frac{\lambda(\lambda-1)}{2} + c_3\frac{\lambda(\lambda-1)(\lambda-2)}{3!} + c_4\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!} + c_5\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{5!}.$$

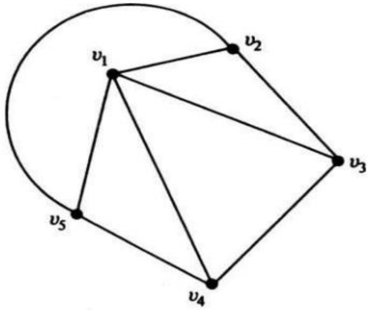


Fig. 8-4 A 3-chromatic graph.

- Since the graph has a triangle, it will require at least three different colors for proper coloring. Therefore, $c_1 = c_2 = 0$ and $c_5 = 5!$.
- Moreover, to evaluate c_3 , suppose that we have three colors x, y, and z. These three colors can be assigned properly to vertices v_1 , v_2 , and v_3 in 3! = 6 different ways. Having done that, we have no more choices left, because vertex v_5 must have the same color as v_3 , and v_4 must have the same color as v_2.
- Therefore, $c_3 = 6$.

- Similarly, with four colors, v_1 , v_2 , and v_3 can be properly colored in 4·6 = 24 different ways. The fourth color can be assigned to v_4 or v_5 , thus providing two choices. The fifth vertex provides no additional choice. Therefore, c_4 = 24·2 = 48.
- Substituting these coefficients in $P_5(\lambda)$, we get, for the graph in Fig. 8-4,
- $P_5(\lambda) = \lambda\ (\lambda - 1)(\lambda - 2) + 2\lambda(\lambda - 1)(\lambda - 2)(\lambda - 3) + \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)(\lambda - 4)$
- $P_5(\lambda) = \lambda(\lambda - 1)(\lambda - 2)(\lambda\ 2 - 5\lambda + 7)$.
- The presence of factors λ – 1 and λ – 2 indicates that G is at least 3-chromatic.

- **Theorem 8.4** - A graph of n vertices is a complete graph if and only if its chromatic polynomial is

$$P_n(\lambda) = \lambda(\lambda - 1)(\lambda - 2) \dots (\lambda - n + 1).$$

- **Theorem 8.5** - An n-vertex graph is a tree if and only if its chromatic polynomial

$$P_n(\lambda) = \lambda(\lambda - 1)^{n-1}$$

- **Theorem 8.6** - Let a and b be two nonadjacent vertices in a graph G. Let G′ be a graph obtained by adding an edge between a and b. Let G″ be a simple graph obtained from G by fusing the vertices a and b together and replacing sets of parallel edges with single edges. Then

$$P_n(\lambda) \text{ of } G = P_n(\lambda) \text{ of } G' + P_{n-1}(\lambda) \text{ of } G''.$$

Matching

- A matching in a graph is a subset of edges in which no two edges are adjacent.

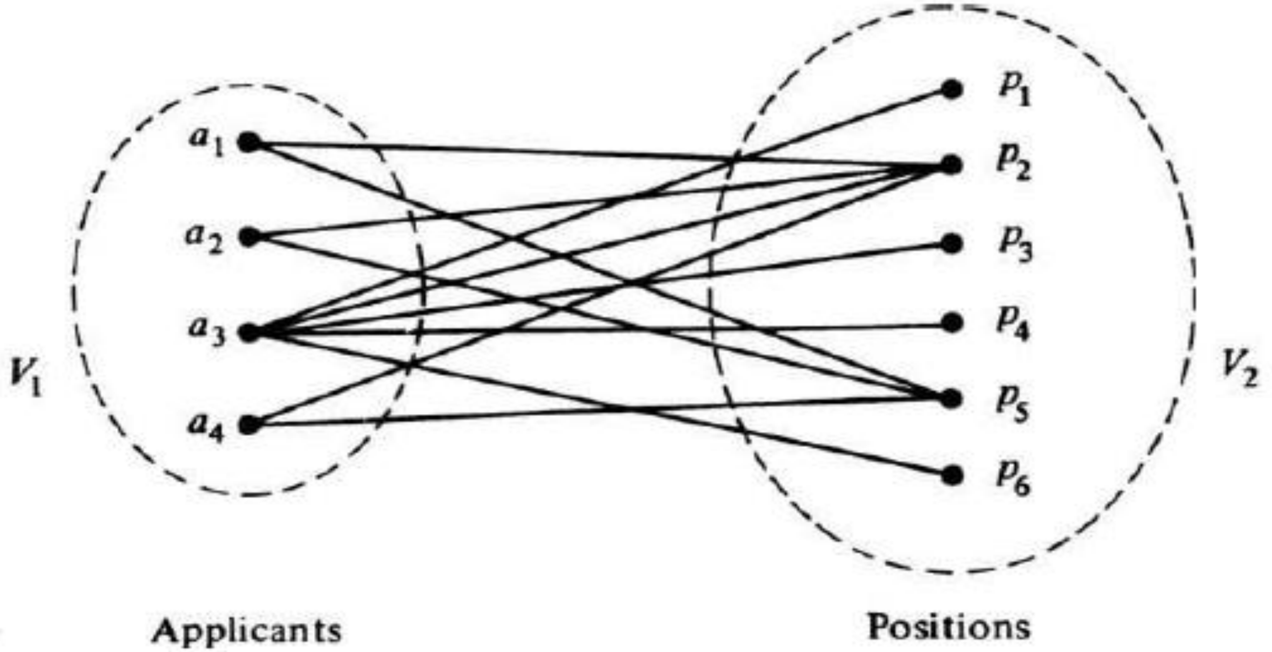


Fig. 8-6 Bipartite graph.

- A *maximal matching* is a matching to which no edge in the graph can be added. For example, in a complete graph of three vertices (i.e., a triangle) any single edge is a maximal matching.

Fig. 8-7 Graph and two of its maximal matchings.

- Clearly, a graph may have many different maximal matchings, and of different sizes. Among these, the maximal matchings with the largest number of edges are called the *largest maximal matchings*.
- The number of edges in a largest maximal matching is called the *matching number* of the graph.

- In a bipartite graph having a vertex partition V_1 and V_2, a *complete matching* of vertices in set V_1 into those in V_2 is a matching in which there is one edge incident with every vertex in V_1. In other words, every vertex in V_1 is matched against some vertex in V_2.
- Clearly, a complete matching (if it exists) is a largest maximal matching, whereas the converse is not necessarily true.
- For the existence of a complete matching of set V_1 into set V_2,
  - There must be at least as many vertices in V_2 as there are in V_1 (This condition is not sufficient, e.g. fig. 8.6).
  - Every subset of r vertices in V_1 must collectively be adjacent to at least r vertices in V_2 , for all values of r = 1, 2, . . . , | V_1|. (This condition is not satisfied for fig. 8.6, hence complete matching is not possible for it.)

- **Theorem 8.7**: A complete matching of V_1 into V_2 in a bipartite graph exists if and only if every subset of r vertices in V_1 is collectively adjacent to r or more vertices in V_2 for all values of r.
- Example:
- *Problem of Distinct Representatives*: Five senators $s1$, $s2$, $s3$, $s4$, and $s5$ are members of three committees, $c1$, $c2$, and $c3$. The membership is shown in Fig. 8-8. One member from each committee is to be represented in a super-committee. Is it possible to send one distinct representative from each of the committees?

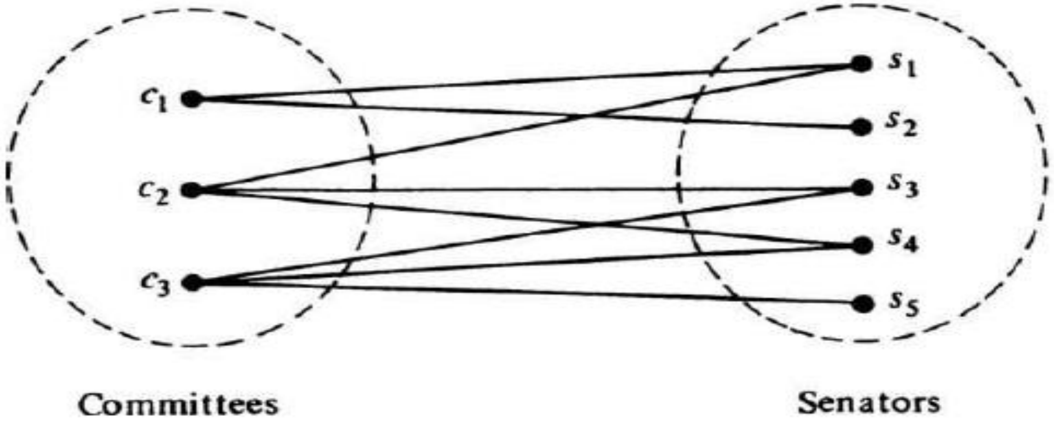


Fig. 8-8 Membership of committees.

	V_1	V_2	$r-q$
$r=1$	$\{c_1\}$	$\{s_1, s_2\}$	-1
	$\{c_2\}$	$\{s_1, s_3, s_4\}$	-2
	$\{c_3\}$	$\{s_3, s_4, s_5\}$	-2
$r=2$	$\{c_1, c_2\}$	$\{s_1, s_2, s_3, s_4\}$	-2
	$\{c_2, c_3\}$	$\{s_1, s_3, s_4, s_5\}$	-2
	$\{c_3, c_1\}$	$\{s_1, s_2, s_3, s_4, s_5\}$	-3
$r=3$	$\{c_1, c_2, c_3\}$	$\{s_1, s_2, s_3, s_4, s_5\}$	-2

Table 8-1

- The problem of distinct representatives just solved was a small one. A larger problem would have become unwieldy.
- If there are M vertices in V_1, Theorem 8-7 requires that we take all 2^M – 1 nonempty subsets of V_1 and find the number of vertices of V_2 adjacent collectively to each of these. In most cases, however, the following simplified version of Theorem 8-7 will suffice for detection of a complete matching in any large graph.
- **Theorem 8.8** - In a bipartite graph a complete matching of V_1 into V_2 exists if (but not only if) there is a positive integer m for which the following condition is satisfied :
- degree of every vertex in $V_1 \geq m \geq$ degree of every vertex in V_2.

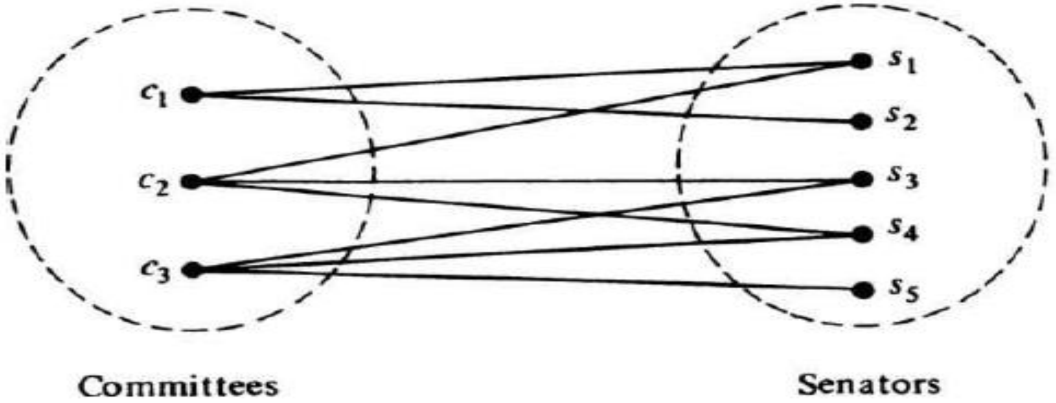


Fig. 8-8 Membership of committees.

- Consider a subset of r vertices in V_1.
- These r vertices have at least $m \cdot r$ edges incident on them.
- Each $m \cdot r$ edge is incident to some vertex in V_2.
- Since the degree of every vertex in set V_2 is no greater than m, these $m \cdot r$ edges are incident on at least $(m \cdot r)/m = r$ vertices in V_2. Thus any subset of r vertices in V_1 is collectively adjacent to r or more vertices in V_2. Therefore, according to Theorem 8-7, there exists a complete matching of V_1 into V_2.
- In the bipartite graph of Fig. 8-8,
- degree of every vertex in $V_1 \geq 2 \geq$ degree of every vertex in V_2.
- The condition of Theorem 8.8. is sufficient but not necessary condition.

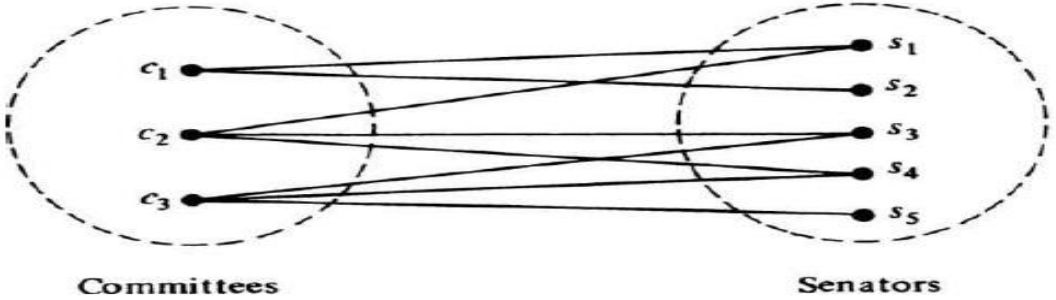


Fig. 8-8 Membership of committees.

- A set of r vertices in $V1$ is collectively incident on, say, q vertices of $V2$. Then the maximum value of the number $r - q$ taken over all values of r = 1, 2,. . . and all subsets of $V1$ is called the deficiency $\delta(G)$ of the bipartite graph G.
- Theorem 8-7, expressed in terms of the deficiency, states that a complete matching in a bipartite graph G exists if and only if, $\delta(G) \leq 0$.

	V_1	V_2	$r - q$
$r = 1$	$\{c_1\}$	$\{s_1, s_2\}$	-1
	$\{c_2\}$	$\{s_1, s_3, s_4\}$	-2
	$\{c_3\}$	$\{s_3, s_4, s_5\}$	-2
$r = 2$	$\{c_1, c_2\}$	$\{s_1, s_2, s_3, s_4\}$	-2
	$\{c_2, c_3\}$	$\{s_1, s_3, s_4, s_5\}$	-2
	$\{c_3, c_1\}$	$\{s_1, s_2, s_3, s_4, s_5\}$	-3
$r = 3$	$\{c_1, c_2, c_3\}$	$\{s_1, s_2, s_3, s_4, s_5\}$	-2

Table 8-1

- **Theorem 8.9** - The maximal number of vertices in set $V1$ that can be matched into $V2$ is equal to

$$\text{number of vertices in } V1 - \delta(G).$$

- For Fig 8.6, number of vertices in $V1 - \delta(G) = 4 - 1 = 3$.

- *Matching and Adjacency Matrix*:

COVERINGS

- In a graph G, a set g of edges is said to *cover* G if every vertex in G is incident on at least one edge in g. A set of edges that covers a graph G is said to be an *edge covering*, a *covering subgraph*, or simply a *covering* of G.
- For eg, a graph G is trivially its own covering. A spanning tree in a connected graph (or a spanning forest in an unconnected graph) is another covering. A Hamiltonian circuit (if it exists) in a graph is also a covering.
- the *minimal covering* is a covering from which no edge can be removed without destroying its ability to cover the graph.

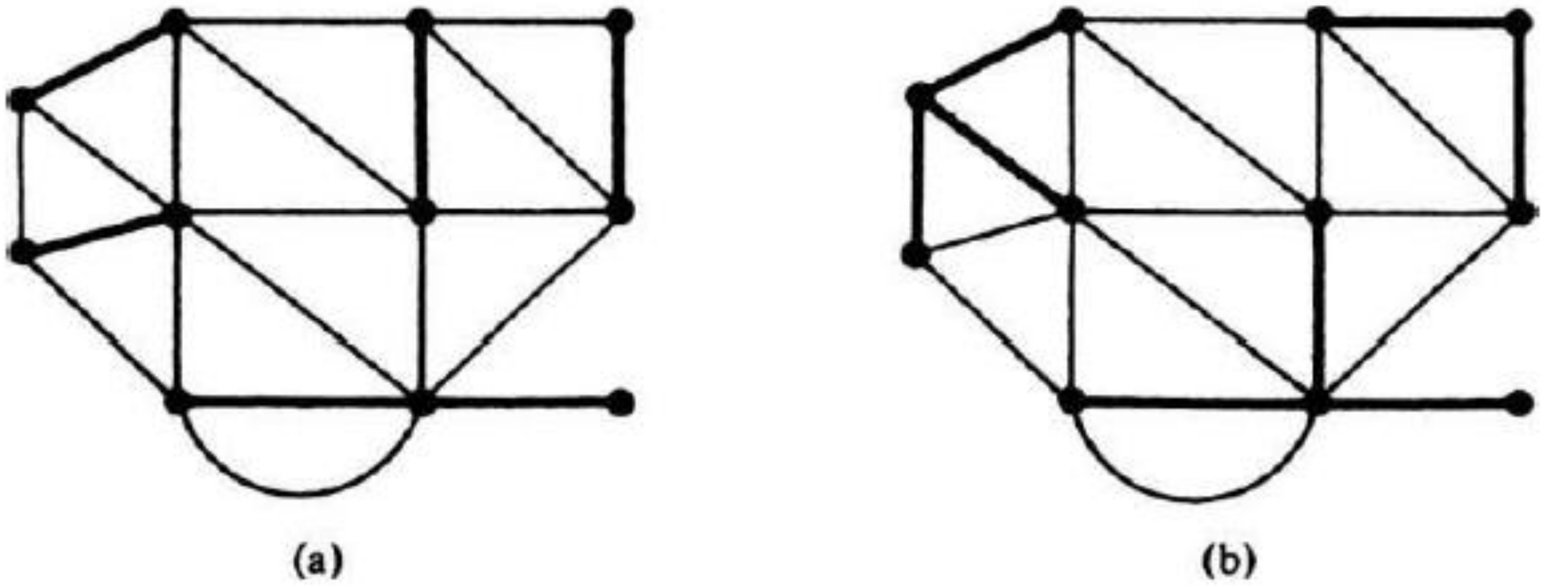


Fig. 8-9 Graph and two of its minimal coverings.

The following observations should be made:

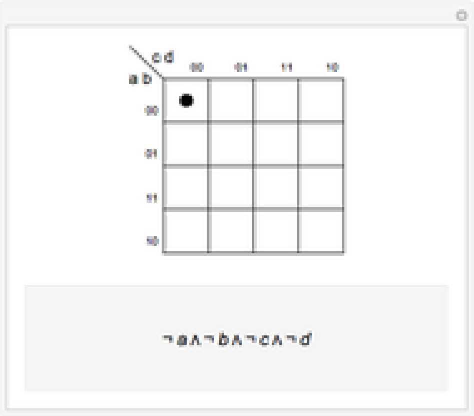
1. A covering exists for a graph if and only if the graph has no isolated vertex.
2. A covering of an n-vertex graph will have at least $\lceil n/2 \rceil$ edges. ($\lceil n \rceil$ denotes the smallest integer not less than x.)
3. Every pendant edge in a graph is included in every covering of the graph.
4. Every covering contains a minimal covering.
5. If we denote the remaining edges of a graph by $(G - g)$, the set of edges g is a covering if and only if, for every vertex V, the degree of vertex in $(G - g) \leq$ (degree of vertex v in G) – 1.

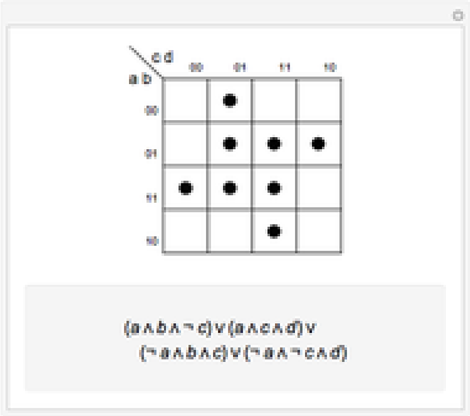
6. No minimal covering can contain a circuit, for we can always remove an edge from a circuit without leaving any of the vertices in the circuit uncovered. Therefore, a minimal covering of an n-vertex graph can contain no more than $n – 1$ edges.
7. A graph, in general, has many minimal coverings, and they may be of different sizes (i.e., consisting of different numbers of edges). The number of edges in a minimal covering of the smallest size is called the *covering number* of the graph.

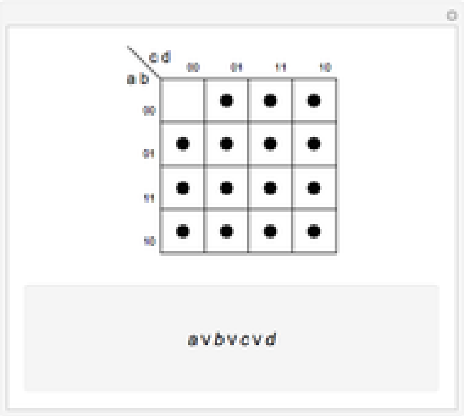
- Theorem 8.10 - A covering g of a graph is minimal if and only if g contains no paths of length three or more.
- Application of minimal covering
  - The street map of a part of a city where each of the vertices is a potential trouble spot and must be kept under the surveillance of a patrol car, assign a minimum number of patrol cars to keep every vertex covered.
  - Dimer Problem
  - The matching M is called perfect if for every $v \in V$, there is some $e \in M$ which is incident on v. The matching is also a covering if and only if it is perfect matching.

Minimization of Switching Functions to find covering of a graph

- The Karnaugh map is a useful graphical tool for minimizing switching functions of up to six variables. A four-variable map has 16 fields, each corresponding to a unique conjunction (AND) of inputs (a minterm).
- An important step in the logical design of a digital machine is to minimize Boolean functions before implementing them. Suppose we are interested in building a logical circuit that gives the following function F of four Boolean variables w, x, y, and z.







https://demonstrations.wolfram.com/FourVariableKarnaughMap/#:~:text=The%20Karnaugh%20map%20is%20a,of%20inputs%20(a%20minterm).

Five variable map

yz \ vwx	000	001	011	010	110	111	101	100
00	0	4	12	8	24	28	20	16
01	1	5	13	9	25	29	21	17
11	3	7	15	11	27	31	23	19
10	2	6	14	10	26	30	22	18

- For fig 8.11, the function is

$$F = \bar{w}\bar{x}\bar{y}\bar{z} + \bar{w}\bar{x}y\bar{z} + w\bar{x}\bar{y}\bar{z} + \bar{w}\bar{x}yz + \bar{w}xy\bar{z} + \bar{w}xyz + wxyz,$$

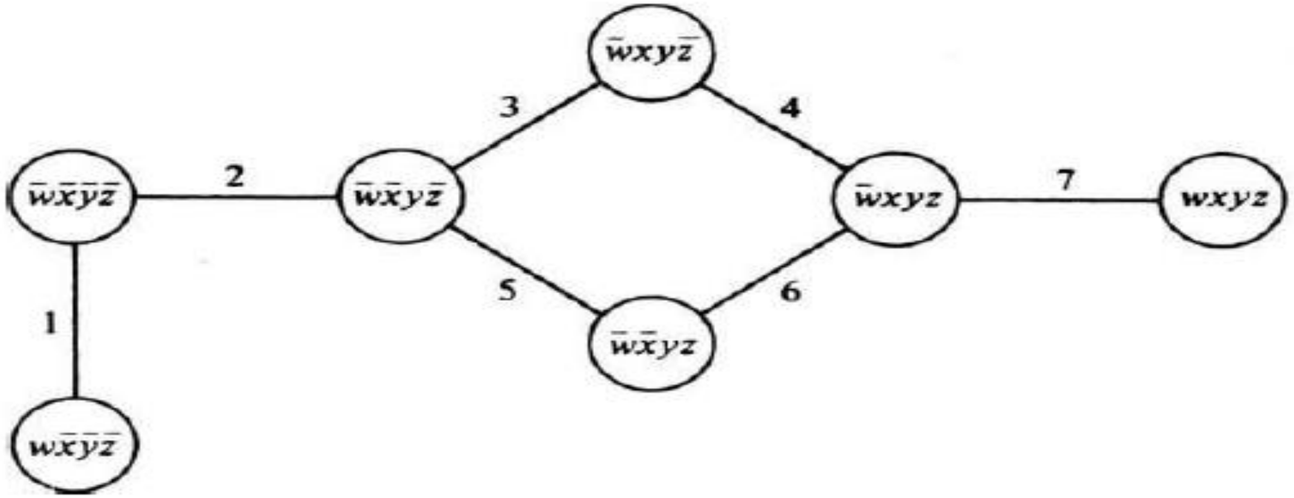


Fig. 8-11 Graph representation of a Boolean function.

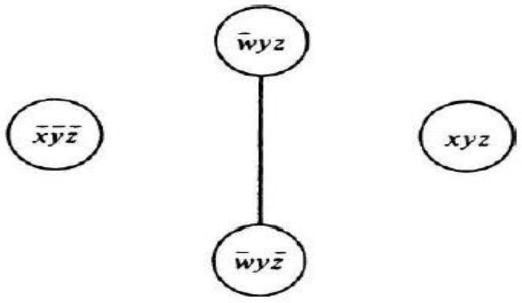


Fig. 8-12

- A minimal cover of this graph will represent a simplified form of *F*, performing the same function as *F*, but with less logic hardware.
- The pendant edges 1 and 7 must be included in every covering of the graph. Therefore, the terms, $\overline{xyz}$ and xyz are essential.
- Two additional edges 3 and 6 (or 4 and 5 or 3 and 5) will cover the remainder. Thus a simplified version of *F* is

$$F = \bar{x}\bar{y}\bar{z} + xyz + \bar{w}y\bar{z} + \bar{w}yz.$$

- This expression can again be represented by a graph of four vertices, as shown in Fig. 8-12.
- The essential terms the terms, $\overline{xyz}$ and xyz cannot be covered by any edge, and hence cannot be minimized further. One edge will cover the remaining two vertices in Fig. 8-12. Thus the minimized Boolean expression is

$$F = \bar{x}\bar{y}\bar{z} + xyz + \bar{w}y.$$

FOUR-COLOR PROBLEM

- The regions of a planar graph are said to be properly colored if no two *contiguous* or *adjacent regions* have the same color.
- Two regions are said to be adjacent if they have a common edge between them. Note that one or more vertices in common does not make two regions adjacent.)
- The proper coloring of regions is also called *map coloring*, referring to the fact that in an atlas different countries are colored such that countries with common boundaries are shown in different colors.

- The most famous *four-color conjecture* conjecture in graph theory is that every map (i.e., a planar graph) can be properly colored with four colors.
- Two remarks may be made here in passing.
    - Paradoxically, for surfaces more complicated than the plane (or sphere) corresponding theorems have been proved. For example, it has been proved that seven colors are necessary and sufficient for properly coloring maps on the surface of a torus.
    - Second, it has been proved that all maps containing less than 40 regions can be properly colored with four colors. Therefore, if in general the four-color conjecture is false, the counterexample has to be a very complicated and large one.

Vertex Coloring Versus Region Coloring

- A graph has a dual if and only if it is planar. Therefore, coloring the regions of a planar graph G is equivalent to coloring the vertices of its dual G^*, and vice versa.
- Thus the four-color conjecture can be restated as follows: *Every planar graph has a chromatic number of four or less*.

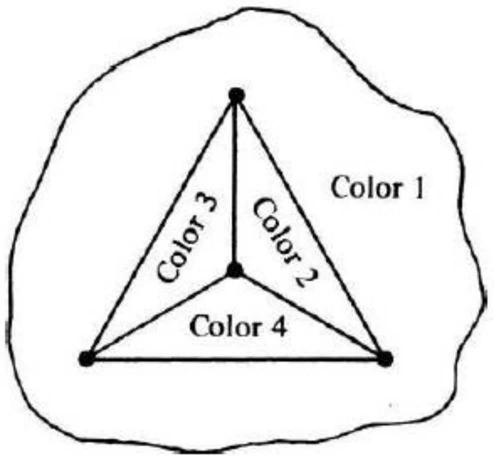


Fig. 8-14 Necessity of four colors.

- *Five-Color Theorem*: That every planar map can be properly colored with five colors.
- **Theorem 8.11** - The vertices of every planar graph can be properly colored with five colors. (Consider the planar graph G with n $(n \geq 4)$ vertices (except K_4). As G is planar, it must have at least one vertex with degree five or less).

- *Regularization of a Planar Graph*
  - Removing every vertex of degree one (together with the pendant edge) from the graph *G* does not affect the regions of a planar graph. Nor does the elimination of every vertex of degree two, by merging the two edges in series have any effect on the regions of a planar graph.
  - Now consider a typical vertex *v* of degree four or more in a planar graph. Let us replace vertex *v* by a small circle with as many vertices as there were incidences on *v*. This results in a number of vertices each of degree three (see Fig. 8-16).

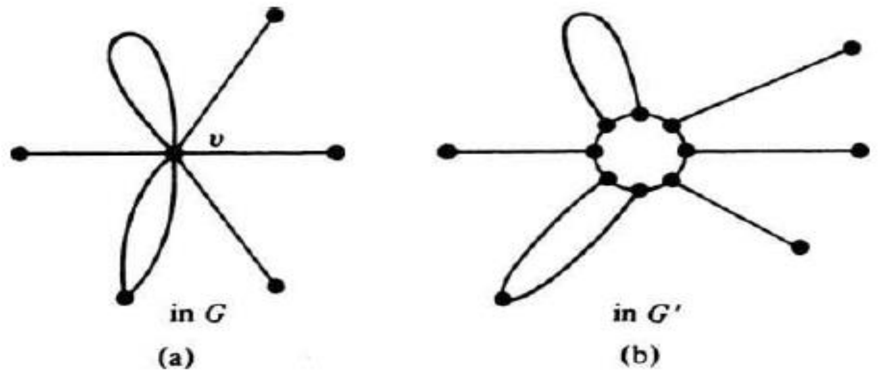


Fig. 8-16 Regularization of a graph.

- Performing this transformation on every vertex of degree four or more in a planar graph G will produce another planar graph H in which every vertex is of degree three.
- When the regions of H have been properly colored, a proper coloring of the regions of G can be obtained simply by shrinking each of the new regions back to the original vertex.
- Such a transformation may be called *regularization* of a planar graph, because it converts a planar graph G into a regular planar graph H of degree three. Clearly, if H can be colored with four colors, so can G.

- Thus, for map-coloring problems, it is sufficient to confine oneself to (connected) planar, regular graphs of degree three. And the four-color conjecture may be restated as follows:
- The regions of every planar, regular graph of degree three can be colored properly with four colors.
- If, in a planar graph G, every vertex is of degree three, its dual G^* is a planar graph in which every region is bounded by three edges; that is, G^* is a triangular graph. Thus the four-color conjecture may again be restated as follows:
- The chromatic number of every triangular, planar graph is four or less.