Chapter 6: Circular motion:

Circular motion:

The motion of a body moving in a circle is called the circular motion. When a body moves in a circle at constant speed, the motion is called uniform circular motion.

It is the case of two dimensional motions in which a particle moves in circular path in such a way that the magnitude of both velocity and acceleration remains constant but their directions change continuously.

e.g. a satellite moving in a circular orbit, motion of the tip of second in a clock.

Angular displacement(θ):

It is the angle swept by the line joining the position of particle and centre of circular path.

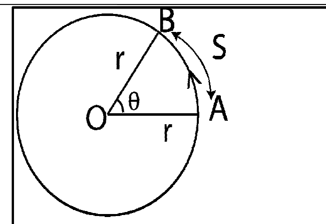


Fig (1) motion in a cirlce

$$\text{Angular displacement} = \frac{\text{Arc length}}{\text{Radius}} = \frac{AB}{OA} = \frac{s}{r} \quad \therefore \theta = \frac{s}{r}$$

Its unit: radian (rad) → S.I. unit

It is scalar quantity but small displacement (dθ) is vector quantity. It is dimensionless quantity.

Angular Velocity (ω) :

The time rate of change of angular displacement is called angular velocity. It is also known as angular frequency.

$$\text{Angular velocity} = \frac{\text{Angular displacement}}{\text{time interval}}$$

$$\boldsymbol{\omega} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}$$

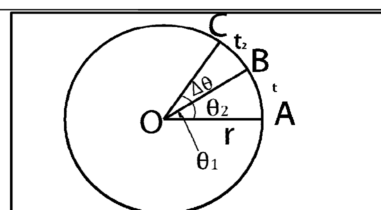


Fig (1) Angular velocity in a circularotation

Brij Kumar Singh M.Sc Physics , CDP, TU, Kirtipur
Physics lecturer at Shree Secondary Technical &Vocational School Dharapani Dhanusha Nepal

$\therefore \overrightarrow{\overline{\boldsymbol{\omega}}} = \frac{\Delta\theta}{\Delta t} = \overline{\boldsymbol{\omega}}$ (average angular velocity)

units: radian per second → S. I. unit

Dimensional formula = $[M^o L^o T^{-1}]$ and it is vector quantity.

Instantaneous angular velocity:

Velocity of particle at any instant is known as instantaneous angular velocity.

i.e. $\boldsymbol{\omega}_{ins} = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$

Angular Acceleration(α):

The rate of change of angular velocity with respect to time if the motion is not uniform, then it is called angular acceleration.

i.e. angular acceleration $(\propto) = \frac{\text{change in angualr velocity}}{\text{time}} = \frac{\omega_2 - \omega_1}{t}$

where ω_1 = initial angular velocity , ω_2 = final angular velocity

Note:

(a) Average angular acceleration

$$\overline{\propto} = \frac{\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1}{t_2 - t_1} = \frac{\triangle\boldsymbol{\omega}}{\triangle t}$$

(b) Instantaneous angular acceleration

$$\propto_{ins} = \lim_{\Delta t \to 0} \frac{\triangle\boldsymbol{\omega}}{\triangle t} = \frac{d\boldsymbol{\omega}}{dt} = \frac{d^2\theta}{dt^2}$$

It is the limiting value of average angular acceleration.

Unit: $\frac{rad}{s^2}$

And dimension: $[M^0 L^o T^{-2}]$

Relation between Linear velocity and Angular velocity (Also Relation between linear acceleration and angular acceleration)

Suppose a particle of mass 'm' moving in a circular path of radius r with constant speed v. Let θ be the angular displacement when particles goes from point A to B in time t as shown in fig (1). If s be the arc length AB then

Angular displacement (θ) = $\frac{\text{s(arc length)}}{\text{r (radius)}}$

i.e. $\theta = \frac{s}{r}$,

or, $s = r\theta$ ……. (1)

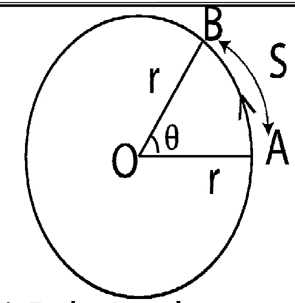


Fig (1) Relation between linear velocity and angular velocity

Differentiating eq. (1) w.r.t time on both sides,

$\frac{ds}{dt} = r\frac{d\theta}{dt}$ Where, r is constant

or, $v = r\omega$ ….. (2)

where, $v = \frac{ds}{dt}$ is the linear velocity

and $\omega = \frac{d\theta}{dt}$ is the angular velocity

Hence, Equation (2) is the required relation between linear velocity and angular velocity.

Again, differentiating equation (2) w.r.t time on both sides, then we get

$\frac{dv}{dt} = r\frac{d\omega}{dt}$ Where r is constant

or, $a = r \propto$ …. (3)

where, $a = \frac{dv}{dt}$ is linear acceleration

and $\propto = \frac{dw}{dt}$ is angular acceleration.

Brij Kumar Singh M.Sc Physics , CDP, TU, Kirtipur
Physics lecturer at Shree Secondary Technical &Vocational School Dharapani Dhanusha Nepal

Hence, equation (3) is the required relation between linear acceleration and angular acceleration.

Note

(1) Relation between linear and angular displacement :

$$\theta = \frac{s}{r} \ldots\ldots\ldots. (1)$$

(2) Relation between linear and angular velocity:

$$v = r\omega \ldots\ldots\ldots. (2)$$

(3) Relation between linear and angular acceleration:

$$a = r\propto \ldots\ldots\ldots. (3)$$

(4) Relation between time period, frequency & angular velocity: $f = \frac{1}{T}$

$$\omega = r\pi f = \frac{2\pi}{T}$$

Linear (1D)	Angular (2D)
1) Linear displacement = s	1) Angular displacement = θ Also $\theta = \frac{s\ (\text{arc length})}{r(\text{radius})}$
2) Linear velocity $v = \frac{ds}{dt}\left(\text{or } v = \frac{s}{t}\right)$	2) Angular velocity $\omega = \frac{d\theta}{dt}$
3) Linear acceleration $a = \frac{dv}{dt} = \frac{d^2S}{dt^2}$	3) acceleration $= \frac{d\,\omega}{dt} = \frac{d^2\theta}{dt^2}$
4) Centripetal force $F_c = \frac{mv^2}{r}$	4) $F_c = \frac{m}{r}(r\,\omega)^2 = mr\,\omega^2$ $\therefore \quad F_c = mr\,\omega^2$
5) Centripetal acceleration $a = \frac{v^2}{r}$	5) Centripetal acceleration $\propto = \frac{v^2}{r} = \omega^2 r$

Note :

(1) Time period → time taken to complete one revolution.

(2) Frequency → the number of revolution completed per second.

Centripetal acceleration$\left(a = \frac{v^2}{r} = m\omega^2\right)$ & centripetal force$\left(F_c = \frac{mv^2}{r} = mr\omega^2\right)$ (2051,2053,2056,2058,2061,2066,2069, 2070)

Q. Why a force is necessary to keep a body moving with uniform speed in a circular motion? Deduce its expression.

Or,

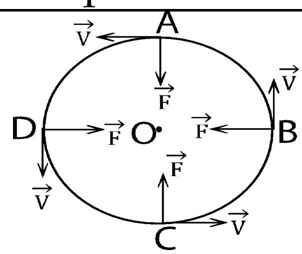
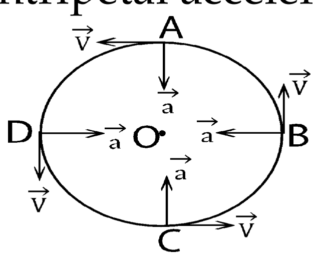
Q. Derive an expression for the force required to make a particle of mass 'm' move in a circle of radius 'r' with uniform angular velocity 'ω'.

Or

Q. Define centripetal force. Calculate the force acting on a body moving with a uniform speed along a circular path.

Q. Define centripetal acceleration. Derive an expression for it.

Q. Show that the acceleration of body moving in a circular path of radius r with uniform speed v is $\frac{v^2}{r}$. Also show direction of acceleration.

Centripetal force:  Fig (I) Direction of centripetal force	The force that acts on body moving in circle towards the centre of the circle is called centripetal force.
Centripetal acceleration  Fig (I) Centripetal acceleration directed towards centre	Acceleration acting on a body moving in circle towards its centre is called centripetal acceleration.

Brij Kumar Singh M.Sc Physics , CDP, TU, Kirtipur
Physics lecturer at Shree Secondary Technical &Vocational School Dharapani Dhanusha Nepal

Expression: $F = m\frac{v^2}{r} = mr\omega^2$

Let us consider a body of mass m is moving in a circle of radius r with uniform linear velocity v and angular velocity w, as shown in fig (I).

Then, we have v=r ω ………… (1)

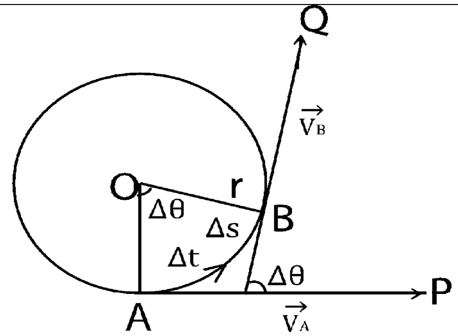


Fig (1) Determination of centriptel acceleration

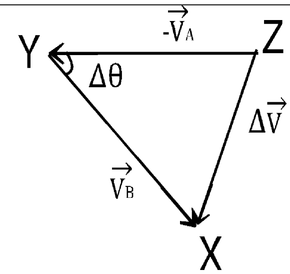


Fig (II) vectors $\vec{V_A}$ and $\vec{V_B}$ are redraus

Suppose at any instant, the body is at point A and after a small time Δt, it is at point B so that it revolves through an angle Δθ about the centre '0' of the circle and moves Δs distance. Then, velocity $\overrightarrow{V_A}$ of the body at point A has magnitude v and direction along tangent of AP; and its velocity $\overrightarrow{V_B}$ at point B has magnitude v and direction along new tangent BQ.

Thus, the change in velocity $\Delta\vec{v}$ of the body in time Δt is given by;

$\Delta\vec{v} = \overrightarrow{V_B} - \overrightarrow{V_A} = \overrightarrow{V_B} + (-\overrightarrow{V_A})$ ………… (2)

Since, Δt is very small, so Δθ is also very small and

$|\overrightarrow{V_B}| = |-\overrightarrow{V_A}| = v(\text{say})$ … …(3)

(∴ magnitude of velocity must remains constant)

Now, two triangles OAB and XYZ are similar, both being isosceles triangle and having the same angle Δθ, so

$$\frac{\Delta v}{v} = \frac{\Delta s}{r}$$

(since, Corresponding sides of similar triangles are proportional)

or, $\Delta v = \frac{v}{r}\Delta s$ ………… (4)

Dividingequation (4)by the time Δt,

$\frac{\Delta v}{\Delta t} = \frac{v}{r}\frac{\Delta s}{\Delta t}$, ... (5)

Taking limit $\Delta t \to 0$ on both sides of equation(5)

$$\lim_{\Delta t\to 0} \frac{\Delta v}{\Delta t} = \lim_{\Delta t\to 0} \frac{v}{r}\frac{\Delta s}{\Delta t}$$

$\lim_{\Delta t\to 0} \frac{\Delta v}{\Delta t} = \frac{v}{r} \lim_{\Delta t\to 0} \frac{\Delta s}{\Delta t}$ ......(6)

Where, a= $\lim_{\Delta t\to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$ is the instantaneous acceleration of the body at point A

Also, v= $\lim_{\Delta t\to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$ is the instantaneous velocity of the body.

Now, from (6) $a = \frac{v}{r}.v = \frac{v^2}{r}$ $\quad \therefore a = \frac{v^2}{r}$ .... (7)

Using equation (1) $a = \frac{(rw)^2}{r} = r\omega^2$ $\quad \therefore a = r\omega^2$ .... (8)

This gives the magnitude of centripetal acceleration.

The centripetal force $F_c = ma = m\frac{v^2}{r} = mr\omega^2$

$\therefore F_C = mr\omega^2$ .... (9)

Note: **Centrifugal Force:** A body revolving in a circular path when suddenly released from centripetal force, it would leave the circular path, and this force is called centrifugal force. Its magnitude is equal to the centripetal force i.e. $F_c = \frac{mv^2}{r}$.

The outward force experienced on a body when it changes its direction of motion is called centrifugal force. It is not a real force; it is due to inertial property of the body.

Conical pendulum (Horizontal pendulum): (2066,2068,2073,2074,2076)

Q. What is conical pendulum? Show that the period of oscillation of this pendulum is given by $T = 2\pi\sqrt{\frac{l\cos\theta}{g}}$ where symbols have their usual meanings.

Q. Describe motion of a conical pendulum and derive an expression for its time period.

Q. Write difference between conceptual designs of a conical pendulum w.r.t. that of a simple pendulum? Derive relation of time period and frequency of a conical pendulum?

Conical Pendulum	Simple pendulum
(I) It's motion is circular	(I) It's motion is to and fro
(II) It is the simple pendulum whirled in a horizontal circle.	(II) It is a massive bob suspended by an inextensible, weightless,& inflexible string
(III) $T = 2\pi\sqrt{\frac{l\cos\theta}{g}}$	(III) $T = 2\pi\sqrt{\frac{l}{g}}$
(IV) Considering the motion of string, it is 3D	(IV) Considering the motion of string, it is of 2D.

The conical pendulum is a heavy metal bob B of mass m tired to a string AB and whirled in a horizontal circle of radius r with uniform speed v so that the string AB revolves in a cone making an angle θ with the vertical AO. The weight 'mg' of the bob acts vertically downward.

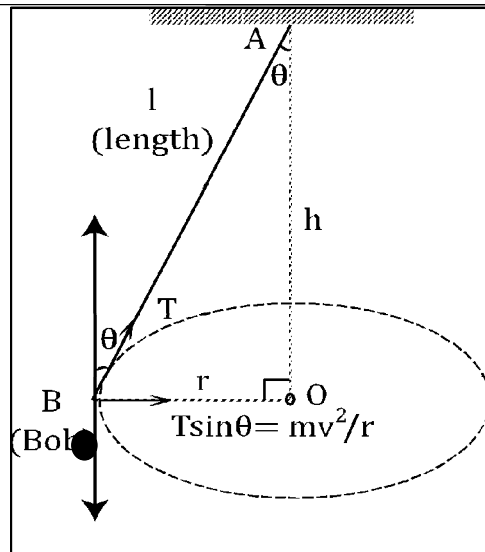


Fig (I) conical pendulum

The tension T in the string can be resolved into two components:

(I) Tcosθ vertically upward which balances the weight 'mg', and

(II) Tsinθ Horizontally towards the centre O, which provides the necessary centripetal force $\frac{mv^2}{r}$.

Thus we have,

$$T\cos\theta = mg \text{ , ...(1)}$$

$$T\sin\theta = \frac{mv^2}{r} \quad ....(2) \quad \text{'Where r is radius'}$$

Dividing equation (2) by equation (1), we have

$$\frac{T\sin\theta}{T\cos\theta} = \frac{\frac{mv^2}{r}}{mg}$$

Or, $\tan\theta = \frac{v^2}{rg}$ .... (3)

If ω be angular speed of the bob, then v=rω

So, from (3) $\tan\theta = \frac{r\omega^2}{g}$, ... (4)

Or, $\omega = \sqrt{\frac{g\tan\theta}{r}}$ .... (5)

∴ Time period(T) of the Conical pendulum is given by,

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{g\tan\theta}{r}}} = 2\pi\sqrt{\frac{r}{g\tan\theta}} \qquad \therefore T = 2\pi\sqrt{\frac{r}{g\tan\theta}} \quad .... (6)$$

From fig (I), in ΔAOB, $\sin\theta = \frac{BO}{AB} = \frac{r}{l}$

Or, $r = l\sin\theta$

$$T = 2\pi\sqrt{\frac{l\sin\theta}{g\tan\theta}} = 2\pi\sqrt{\frac{l\cos\theta}{g}} \qquad \therefore T = 2\pi\sqrt{\frac{l\cos\theta}{g}} \quad .... (7)$$

This is the required expression for Time period of conical pendulum.

Also, $f = \frac{1}{T}$ $\quad \therefore f = \frac{1}{2\pi}\sqrt{\frac{g}{l\cos\theta}}$ ....(8)

This is req. expression for frequency of conical pendulum.

MOTION IN A VERTICAL CIRCLE:

Q. A solid tied at the end of a string is resolved in vertical. At what point the tension in the string will be greater? (2052) (ans: up to eq. 2)

It is an example of non-uniform circular motion.

When a bob of mass 'm' tied to a string in whirled in a vertical circle of radius 'r' with uniform speed v, then one can feel a minimum tension T_{min} at the top of circle A, a maximum tension T_{max} at its bottom B and an intermediate tension T at point C when the string OC is horizontal, as shown in fig (I).

The weight 'mg' of the bob acts vertically downward at all points A, B and C. Since the centripetal force needed towards the centre O is $\frac{mv^2}{r}$,

So at the top A, we have

$T_{min} + mg = \frac{mv^2}{r}$ (here, v =v_A)

or, $T_{min} = \frac{mv^2}{r} - mg$ ………………… (1)

Similarly at the bottom B, we have

or, $T_{max} - mg = \frac{mv^2}{r}$ (here, v =v_A)

or, $T_{max} = \frac{mv^2}{r} + mg$ ………………… (2)

At point C, since the horizontal components of the weight 'mg' of the bob is zero, so we have,

$T = \frac{mv^2}{r}$ ……………… (3)

(since, mg = 0) (here (v=Vc)

[simillar for pointD ,$T = \frac{mv^2}{r}$ ……….. (4)]

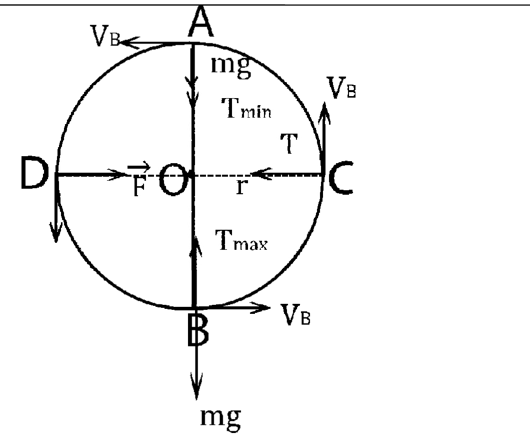


Fig (I) motion of body in a verticle circle

Note: According to conservation of energy

Total energy at point B = Total energy at A

But, Total energy at point B = $\frac{1}{2}mv_B^2$ ………… (5)

Total energy at point A = K.E. + P.E. = $\frac{1}{2}mv_A^2$+ mg(AB)

Brij Kumar Singh M.Sc Physics , CDP, TU, Kirtipur
Physics lecturer at Shree Secondary Technical &Vocational School Dharapani Dhanusha Nepal

$$= \frac{1}{2}mv_A^2 + \quad mg(2r) \quad \ldots\ldots\ldots\ldots \quad (6)$$

From equation (5)and (6):

$$\frac{1}{2}mv_B^2 = \frac{1}{2}mv_A^2 + mg(2r)$$

$$\therefore \frac{1}{2}mv_B^2 = \frac{1}{2}mrg + mg(2r)$$

$(\therefore \frac{v_A^2}{rg} = 1 \rightarrow v_A^2 = rg$, is critical velocity)

$$\frac{1}{2}v_B^2 = \frac{1}{2}rg + 2rg = \frac{5rg}{2}$$

$$\therefore v_B = \sqrt{5rg} = \sqrt{5v_A^2} \quad \ldots\ldots\ldots\ldots\ldots \text{(7)}$$

Therefore, for completing the loop, the velocity at lowest point must be $\sqrt{5}$ times critical velocity at highest point.
This shows that, the motion in a vertical plane is non-uniform.

Applications of circular motion:

(I) Bending of cyclist or Motion of a cyclist around a circular.

Q.1 Why a cyclist leans/bends while turning a corner? Derive an expression for the inclination of the bicycle from the vertical.

Q. 2 Derive centripetal force in the case of motion of a bicycle on a curved road. (2054,2055,2067,2070)

Suppose a cyclist moves around a circular path of radius r with uniform speed v, as shown in fig (1). In order to provide necessary centripetal force, he should incline his bicycle through an angle θ from the vertical towards the centre O of the circular path.
The total weight mg of the

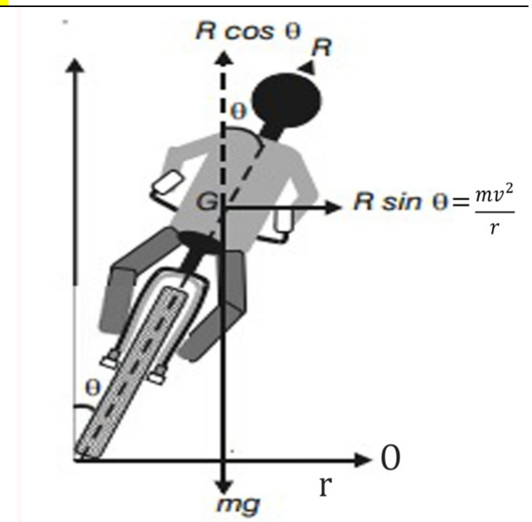


Fig (I) Bending of a cyclist at circulas path

bicycle with rider acts vertically downward through the centre of gravity. The reaction R of the ground on bicycle can be resolved into two components are:	

(I) $R\cos\theta$ vertically upward with balances the weight 'mg'

i.e. $R\cos\theta = mg$ ……… (1) [m=mass of cycle)

(II) $R\sin\theta$ Horizontally towards centre O which provides the necessary centripetal force $\frac{mv^2}{r}$ i.e. $R\sin\theta = \frac{mv^2}{r}$ ……… (2)

Dividing equation (2) by (1), then we have,

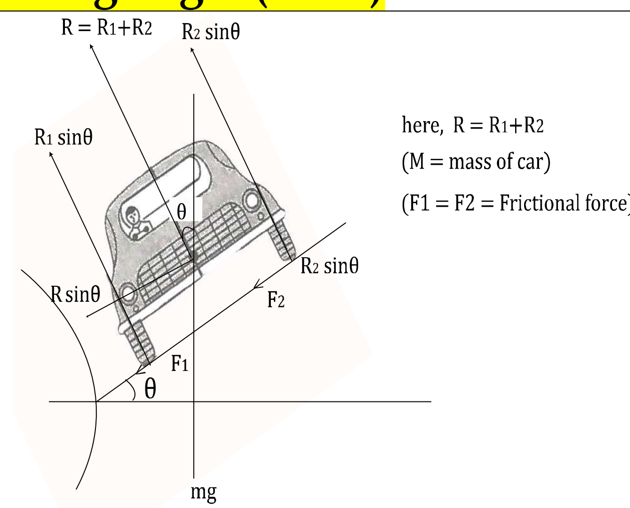
$$\tan\theta = \frac{v^2}{rg}, \text{ …………… (3)}$$

This is the required expression for the inclination of bicycle from vertical.

(II) Banking of track: Motion of a car around a banked circular track

Q. Discuss the motion of a car moving round in a circular banked track. (4 marks) (2068)

Q. What do you mean by the banking of a curved path? Derive an expression for the banking angle.(2071)

The main purpose of banking is to tilt the normal reaction of ground inward so that its horizontal component provides necessary centripetal force. The large amount of friction between tyre and road produces	

considerable wear and tear of the tyre. To avoid this the curved road is given an inclination.	

The system of raising the outer edge of a curved road above its inner edge is called banking of curved (path.)

Suppose a car moves on a banked circular track of radius r and banking angle θ with uniform speed v, as shown in fig (1). The weight mg of the car with its passengers acts vertically downward.

The normal reactions R_1 and R_2 of the ground on the inner and outer wheels can be resolved into two components:

(I) $R_1\cos\theta, R_2\cos\theta$vertically upward which together balance the weight mg,

i.e. $(R_1 + R_2)\cos\theta = mg$ ……….. (1)

(II) $R_1\sin\theta, R_2\sin\theta$ Horizontally towards centre O of the circular track which together provide the necessary centripetal force$\frac{mv^2}{r}$,

i.e. $(R_1 + R_2)\sin\theta = \frac{mv^2}{r}$……….. (2)

Dividing equation (2) by equation (1), we have $\tan\theta = \frac{v^2}{rg}$ …………. (3)

This is the required banking angle of the circular track of radius r for a uniform speed v.

Motion of a car on a level curved path

Let a car of mass 'm' is going round a circular turn of radius 'r' with uniform velocity 'v' as shown in fig (1).

The weight 'mg' of car acts vertically downwards and normal reaction R_1 and R_2 of ground in inner and outer tyre act vertically upward as shown in fig (I).

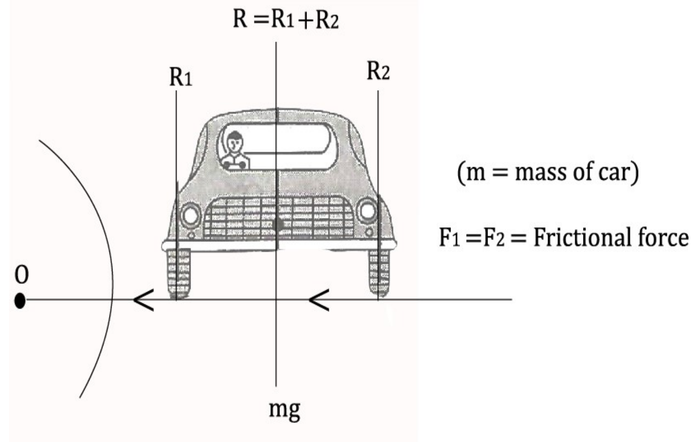


Fig (I) motion of a car in a level curved path

As there can be no motion of car along vertically, the total normal reaction $R = R_1 + R_2$ balances weight mg.

i.e. $R_1 + R_2 = mg$ ....(1)

The frictional force F_1 and F_2 are acting inward provide the necessary centripetal force i.e. $F_1 + F_2 = \frac{mv^2}{r}$, ... (2)

If μ be the coefficient of friction between road and tyres then

$F_1 = \mu R_1$ and $F_2 = \mu R_2$

$\therefore \mu R_1 + \mu R_2 = \frac{mv^2}{r}$

$\mu(R_1 + R_2) = \frac{mv^2}{r}$ .... (3)

Dividing equation (3) by (1)

$$\frac{\mu(R_1+R_2)}{R_1+R_2} = \frac{\frac{mv^2}{r}}{mg}$$

$$\mu = \frac{v^2}{rg}$$

$\therefore v = \sqrt{\mu rg}$ .... (4)

This is maximum velocity with which a vehicle can take a safe circular turn of radius r.

Thus, if μ is large, vehicle can take safe turn over with greater velocity.

If μ is constant, vehicle has to take a turn of greater radius 'r' when v is large.

Brij Kumar Singh M.Sc Physics , CDP, TU, Kirtipur
Physics lecturer at Shree Secondary Technical &Vocational School Dharapani Dhanusha Nepal
Former Physics Lecturer at Nepal Adarsha School/Campus Waling-8, Syangja
Former secondary Science teacher at Everest Academy Lalbandi, Sarlahi Nepal