Circular motion

Angular displacement

The angular displacement of the object moving in a circular direction is defined as the angle traced out by the radius vector at the center of the circular path in a given time. The SI-unit of angular displacement is radian.

Angular velocity

The time rate of angular displacement is called angular velocity and is denoted by ω.

Mathematically, ω_{av} = angular displacement / time interval

$$= (\theta_2 - \theta_1)/(t_2 - t_1)$$

$$= \Delta\theta/\Delta t$$

- Time period
- The period of an object in circular motion is defined as the time taken to complete one revolution. It is denoted by T. since in time T, the object completes one revolution i.e. an angle of 2π radians, angular velocity of an object is
- $\omega = 2\pi/T$
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- Frequency
- The number of revolutions completed per second by an object in circular motion is called frequency and is denoted by f. the relation between the frequency, f and period, T is
- $F = 1/T$
- Therefore, we have
- The unit of frequency is hertz, Hz.

- Angular acceleration
- The rate of change of angular velocity with respect to time is called angular acceleration. It is denoted by α.
- .: Angular acceleration, α = Change in angular velocity/time taken
- If ω_0 and ω are the initial and final angular velocities and t is the time taken to change the angular velocity, then,
- $\alpha = (\omega - \omega_0)/ t$
- $\omega = \omega_0 + \alpha. T$
- Angular acceleration is measured in radian per second square (rad. s^{-2}) and its dimensional formula is $[M^0L^0T^{-2}]$.
- Angular acceleration is also given by, $d\omega/dt$

Relation between Linear Velocity and Angular Velocity

- **Suppose a particle of mass m moving in a circular path of radius r with constant speed v. let θ be the angular displacement when particle goes from points P to Q in time t as shown in the figure. If s is the arc of length PQ, then**
- **$\theta = s/r$**
- **or, $s = r.\ \theta$**
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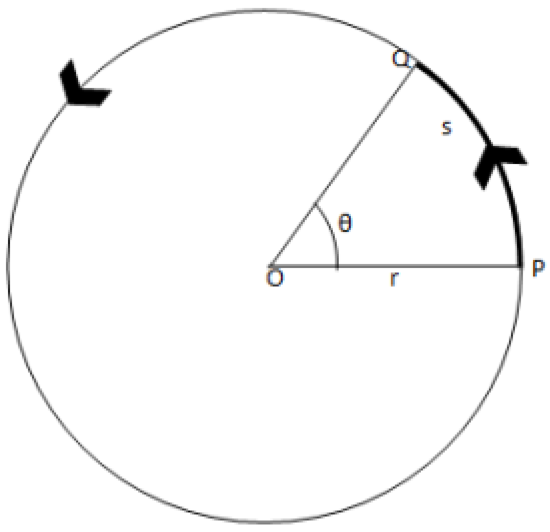
- **Differentiating both sides with respect to time, we have**
- **$ds/dt = r.\ d\theta/dt$**



- As r is constant, $v = ds/dt$, and angular velocity, $\omega = d\theta/dt$
- Then, above equation can be written as
- $V = r.\ \omega$
- This is the relation between linear velocity and angular velocity.
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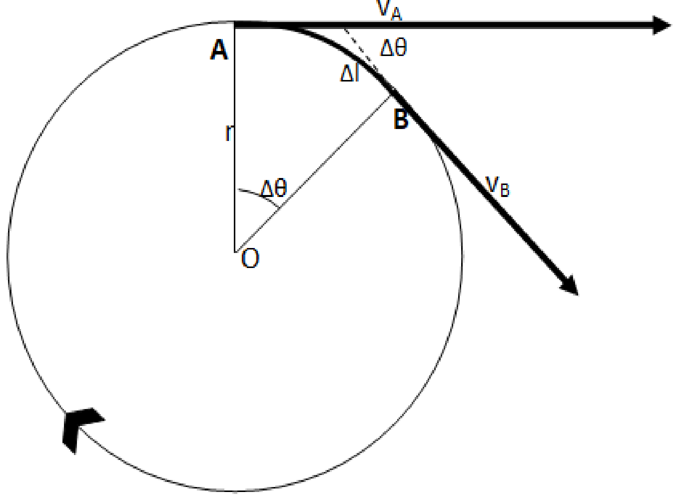
- Since linear acceleration is given by $a = dv/dt$
- And angular acceleration is given by $\alpha = d\omega/dt$
- Differentiating both sides w.r.t time in $V = r.\ \omega$, we get
- $a = r.\ \alpha$

Expression for Centripetal Acceleration

- **Consider a particle moving with a constant speed v in a circular path of radius r with center O. At a point A on its path, its velocity v_A is along the tangent at A and at B, its velocity v_B is along the tangent at B. since the direction of the velocity at A and B are different, an acceleration occurs from A to B.**
- **Let Δt be the time taken to move from A to B covering a small displacement Δl and angular displacement Δθ.**



- **As shown in figure v_A and v_B are redrawn where Δv represents the change in velocity, $\Delta v = v_B - v_A$, which form three sides of a triangle PQR. Two triangles OAB and PQR are similar since both are isosceles triangles and angles labelled $\Delta\theta$ are the same. So, the ratios of the corresponding sides of such triangles are equal.**

- **Thus,**
- **$\Delta v/v = \Delta l/r$**
- **or, $\Delta v = \Delta l/r \cdot \Delta l$**
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- **The rate of change of velocity is**
- **$\Delta v/\Delta t = v/t \cdot \Delta l/\Delta t$**
-
- **For $\Delta t \to 0$,**
- **Limit $\Delta t \to 0$ $\Delta v/\Delta t = a$ and,**
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- **Limit $\Delta t \to 0$ $\Delta l/\Delta t = v$**
- **Then, the above equation is written as**
- **$a = v/r \cdot v$**
- **or, $a = v^2/ r$**
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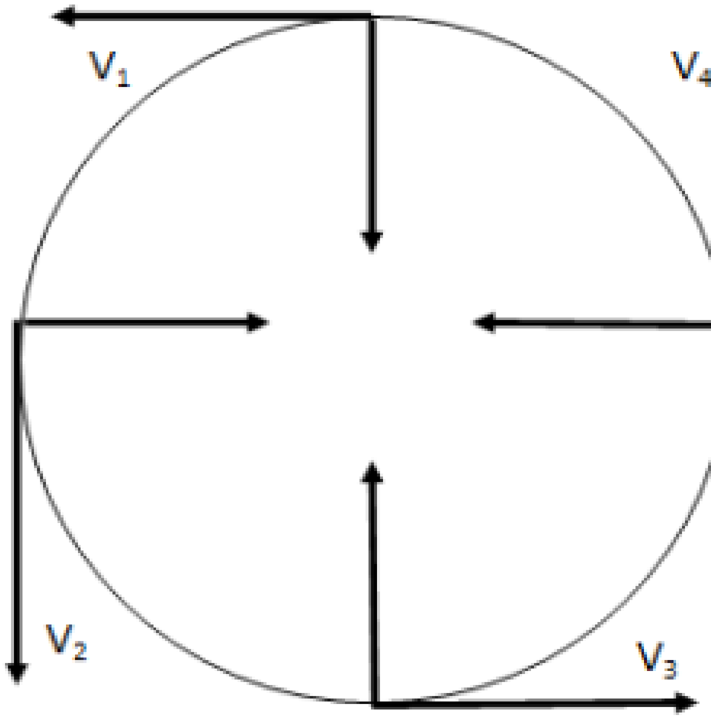
- **Since $v = r. \omega$, the centripetal acceleration can be written as**
- **$a = r. \omega^2$**

Centripetal Force

Centripetal force is the force required to move a body uniformly in a circle. This force acts along the radius and is directed towards the center of the circle



- When a body moves in a circle, its direction of motion at any instant is along the tangent to the circle at that instant. As shown in figure, the direction of velocity of the body goes in changing continuously and due to this change, there is an acceleration called the centripetal acceleration given as,
- $a = v^2/ r = r. \omega^2$
- Where v is linear velocity, ω is angular velocity of the body, and r is radius of the circular path.
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- As F = m. a, centripetal force is
- F = mass * acceleration
- $= mv^2/r = m. r. \omega^2$
- And this force acts along the radius and is directed towards the center of the circle.
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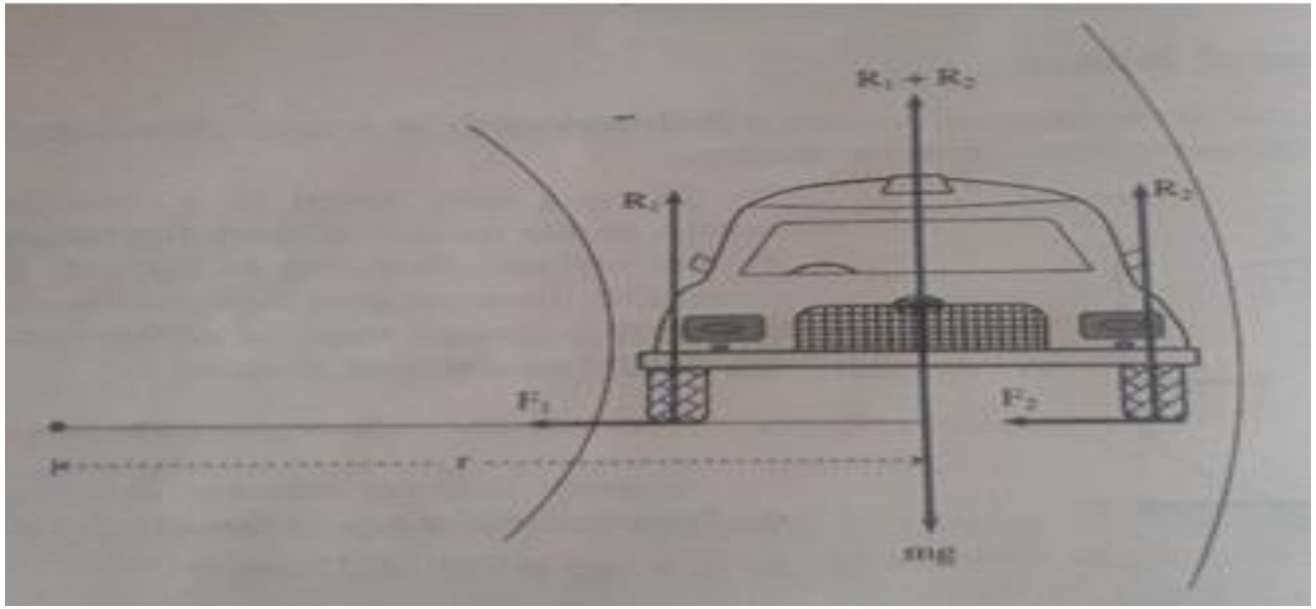
Application of Centripetal force

- **1. Motion of a car on a Level Curved Path**
- **Let F_1 and F_2 be the forces of friction between the wheels and the road, directed towards the center of the horizontal curved track of radius r as shown in the figure. From the law of Friction,**
- $$F_1 + F_2 = \mu (R_1 + R_2) \;\ldots\ldots\; \text{(i)}$$
- **Where μ is the coefficient of sliding friction between tires and the road and R_1 and R_2 are the forces of normal reaction of the road on the wheels.**



This equation gives the maximum velocity with which the car can take a circular turn of radius r and coefficient of friction μ. This shows that much more friction is required as the turning speed is to be increased and if the velocity is greater than determined by equation (iv), the skidding of the car occurs and moves out of the road. This equation holds for a vehicle of any mass from a bicycle to a heavy truck.

As there is no motion of the car in the vertical direction, we have

$R_1 + R_2 = m.\ g$ ……. (ii)

As the force of friction provides the necessarily centripetal force, then

$F_1 + F_2 = m.\ v^2/\ r$ ……… (iii)

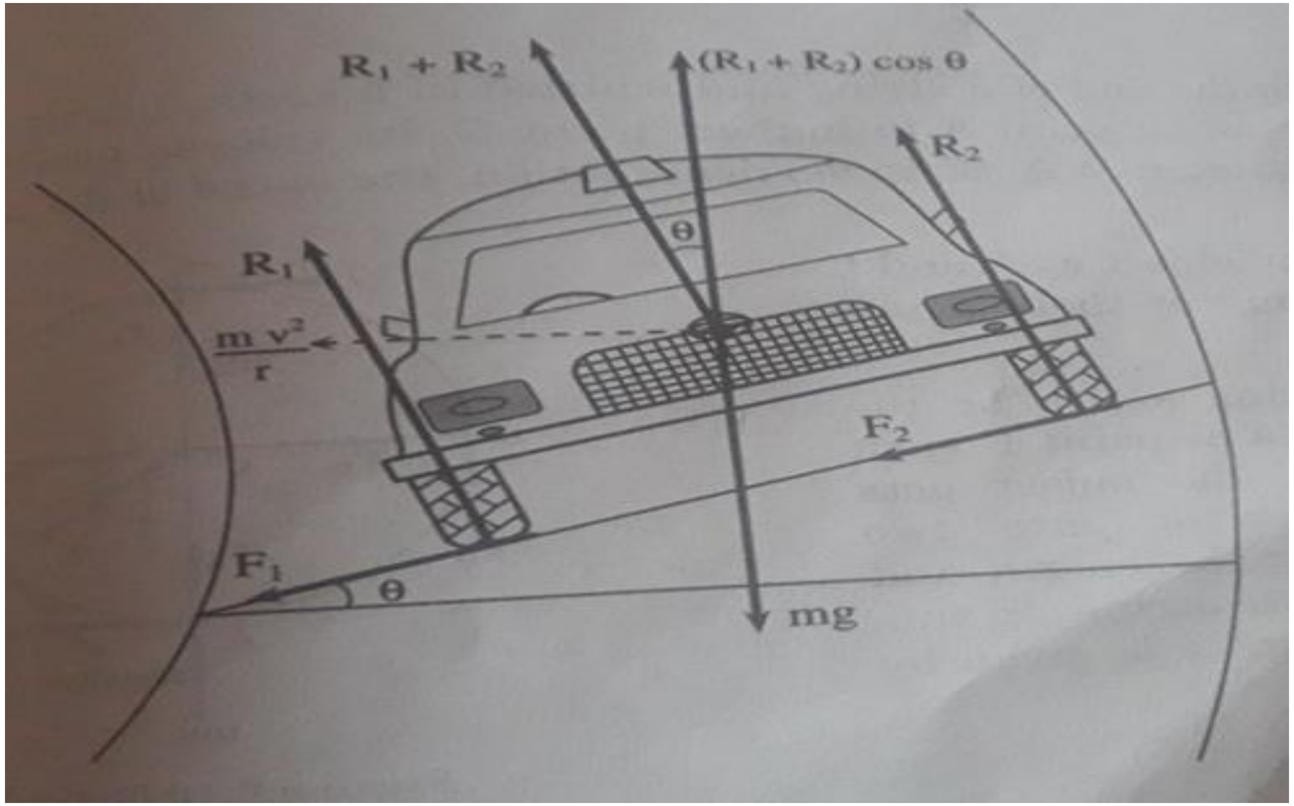
So, From equation (ii) and (iii), we have

$\mu mg = mv^2/\ r$

or, $\mu = v^2/\ rg$

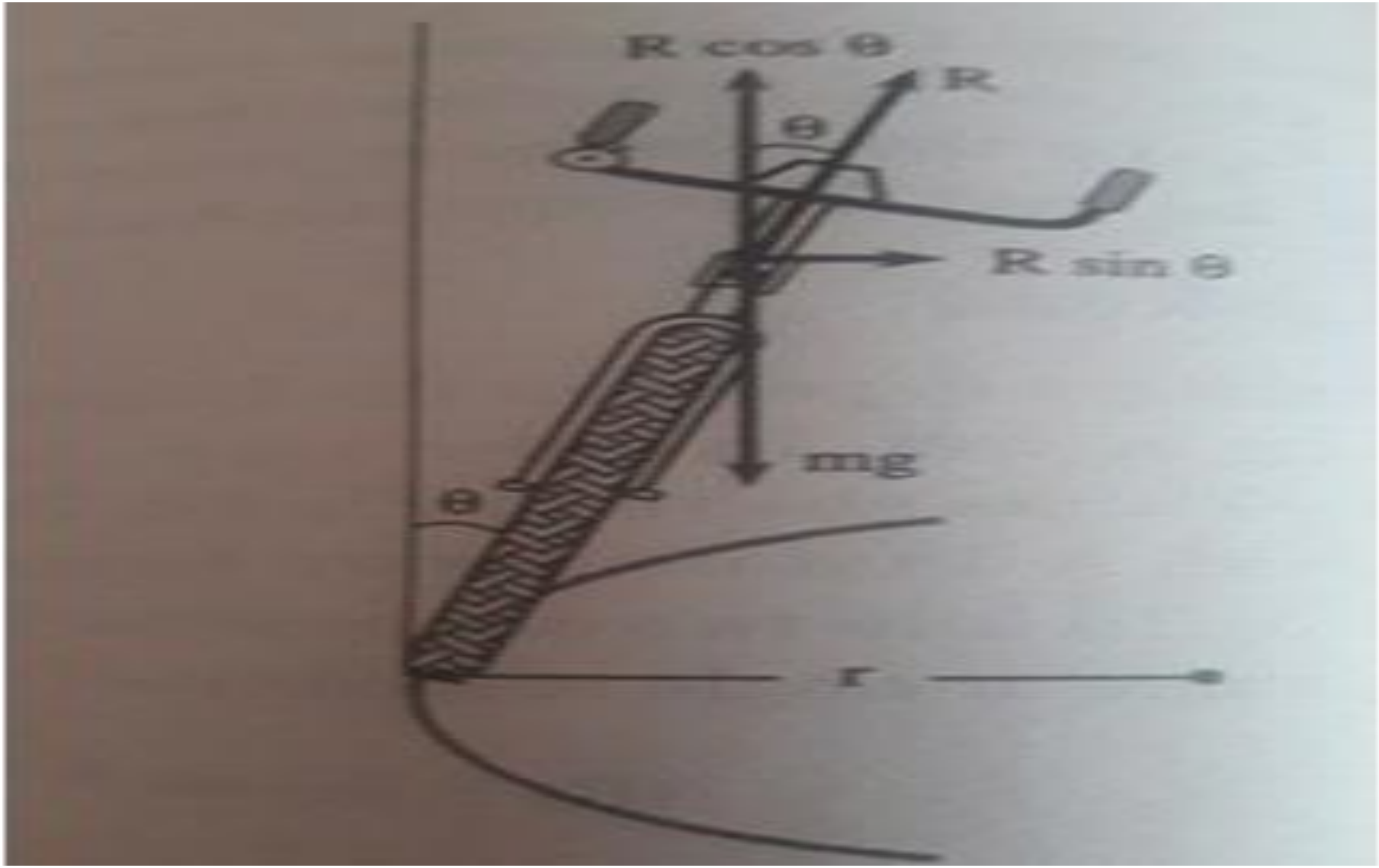
.: $v = \sqrt{\mu\ rg}$ ……. (iv)

- **2. Banking of road**
- **We can not always depend on the friction to move a car or other vehicles around a curve, especially if the road is icy or wet. In order not to depend on friction or to reduce wear on the tires on road, the roads are banked as shown in the figure.**



- The effect of banking is to tilt the normal toward the center of the curved road, so that it's inward horizontal component can supply the required centripetal force. So, resolving $(R_1 + R_2)$ into two components $(R_1 + R_2)\sin\theta$ towards the center and $(R_1 + R_2)\cos\theta$ where θ is the banking angle or angle of inclination of the plane with the horizontal, we have
- $(R_1 + R_2) \sin\theta = mv^2/r$
-
- And vertically
- $(R_1 + R_2) \cos\theta = mg$
-
- From these two equations, we have
- $(R_1 + R_2) \sin\theta/(R_1 + R_2) \cos\theta = mv^2/ r * 1/ mg$
- or, $\tan\theta = v^2/ rg$
-
- The above equation gives the angle for banking for a speed v and a curved track of horizontal radius r. so the tangent of the banking angle is directly proportional to the sped and inversely proportional to the radius of the path.

- **3. Bending of a Cyclist**
- **When a cyclist takes a turn, he bends toward the center of the circular path and the horizontal component of normal reaction directed towards the center provides the necessary centripetal force for the motion. Suppose a cyclist of mass m going in a circular path of radius r with constant speed v tilting at an angle of θ with the vertical. Then R can be resolved into two rectangular components: R cosθ in the vertically upward direction and R sinθ, along the horizontal, towards the center of the circular track as shown in the figure. In equilibrium, R cosθ balances the weight mg of the cyclist and R sinθ provides the necessary centripetal force, mv^2/r.**



R cos θ
R
θ
R sin θ
mg
θ
r

- So,
- $R \cos\theta = mg$ …….. (i)
-
- And,
- $R \sin\theta = mv^2/ r$ ……... (ii)
-
- Dividing equation (ii) by equation (i), we get
- $\tan\theta = v^2/ rg$
- This relation gives the angle through which a cyclist tilts with the vertical while going in a circular road of radius r at a speed of v.
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