KT 10.7 We give a greedy algorithm to color a graph with $w+1$ colors, and show that it always succeeds. Let T be a tree decomposition of G with width w. For the root node $t \in T$, assign $w+1$ colors to the $w+1$ vertices in V_t arbitrarily. Then traverse the nodes in T in a top-down fashion. For each node $t \in T$, let t_p be the parent of t. First, for vertices in $V_t \cap V_{t_p}$, assign the same colors as in V_{t_p}. Then assign the remaining colors to vertices in $V_t - V_{t_p}$ arbitrarily. Due to the coherence property, the same vertex must receive the same color. Also, two adjacent vertices must have different colors because every edge must be covered by at least one node in T, and vertices in one node have received different colors.

To compute the chromatic number, we check for every $k = 1, 2, \ldots, w+1$ if k colors are sufficient. For each k, we run the following dynamic programming algorithm in a bottom-up fashion. For any node $t \in T$, and let $\pi_t : V_t \to [k]$ be a coloring of the at most $w+1$ vertices in V_t. Note that there are at most $(w+1)^k$ colorings for each t. We say that π_t is valid if it assigns different colors to adjacent vertices in V_t. Set a $z(t, \pi_t) = True$ if there is a valid k-coloring for all vertices in the subtree of t, subject to the condition that vertices in V_t are colored according to π_t, and $False$ otherwise.

Base case: For a leaf $t \in T$, $z(t, \pi_t) = True$ iff π_t is valid on V_t.

Recurrence: For an internal node t, let $t_1, \ldots, t_d$ be its children. For any π_t, we set $z(t, \pi_t) = True$ iff (1) π_t is valid on V_t, and (2) for every i, there is at least one π_{t_i} such that $z(t_i, \pi_{t_i}) = True$ and π_t and π_{t_i} assign the same colors to the same vertices in $V_t \cap V_{t_i}$.

Finally, we return "yes" if for the root t, there is one π_t such that $z(t, \pi_t) = True$.

For each $z(t, \pi_t)$, we spend $O((w+1)^k d)$ time to compute. There are $O((w+1)^k n)$ subproblems. So the total running time is $O((w+1)^{2k} n) = O((w+1)^{2w} n)$.