Classical Damped Coupled Oscillators & Analogy to Quantum Damped Two-level Systems

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Contents

1 Introduction

Quantum Mechanics is the basic foundation of almost all modern branches of physics. It describes the interaction between particles at a very microscopic level. Classical Mechanics on the other hand, fairly describes many aspects of physics which we encounter in our everyday life. Although quantum mechanics govern the processes, classical instruments in the classical world register the outcomes of these processes. Hence, it only makes sense that we draw some analogies between classical and quantum systems. Many of these analogies can be found in [?]. For this experiment, we used the the analogy between the semiclassical Rabi model of a two-level atom interacting with an oscillating electric field and the classical system of two coupled oscillating masses. The classical coupled oscillator has been discussed in Section 2 followed by the experimental setup and the preliminary results in Section 3. The quantum two level system has been touched upon in Section 4. We look at the density matrix elements for such a two level system and introduce decay in the population levels manually. The behaviour of our system is found to be analogous to that of the decaying solutions for the two level systems.

2 Classical Coupled Oscillator

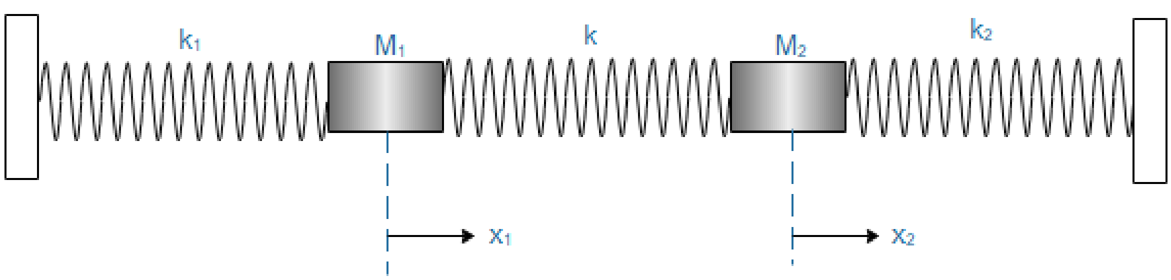

Figure 1: A system of classical coupled oscillator

We consider the system in Figure 1. There are two masses M_1 and M_2 connected by springs of spring constant k_1 and k_2 to the wall, and to each other by a spring of spring constant k.

For any general coupled oscillator system, the equations of motion are given by

$$\sum_j (A_{jk} - \omega^2 m_{jk}) a_j = 0 \tag{1}$$

where A_{jk} is derived from the potential energy part of the Lagrangian, given by U and is given by $A_{jk} = \frac{\partial^2 U}{\partial q_j \partial q_k}$. a_j is the real amplitude part of the *ansatz* solution $q_j(t) = a_j e^{i(\omega t - \delta)}$ and δ is an arbitrary constant and gives the phase. ω is the frequency of the oscillation.

However, for this system, straightforward application of Newton's laws suffice. The displacements of the two masses are given by x_1 and x_2 respectively. This gives us the following sets of equations

$$
\begin{aligned}
M_1 \ddot{x}_1 &= -k_1 x_1 + k_2(x_2 - x_1) \qquad (2) \\
M_2 \ddot{x}_2 &= k(x_1 - x_2) - k_2 x_2 \qquad (3)
\end{aligned}
$$

In matrix form, this can be written as

$$
-\frac{d^2 \boldsymbol{X}}{dt^2} = \boldsymbol{A}\boldsymbol{X} \qquad (4)
$$

$$
\boldsymbol{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \Bigg| \quad \boldsymbol{A} = \begin{bmatrix} \omega_1^2 & -\Omega_1^2 \\ -\Omega_2^2 & \omega_2^2 \end{bmatrix} \qquad (5)
$$

Here, $\omega_i^2 = \frac{k_i + k}{M_i}$ and $\Omega_i^2 = \frac{k}{M_i}$ and $i = 1, 2$.
The eigenvalues of $\boldsymbol{A}$ are the eigenfrequencies and are given by

$$
\omega_{\pm}^2 = \frac{1}{2}\left(\omega_1^2 + \omega_2^2 \pm \sqrt{4\Omega_1^2\Omega_2^2 + (\omega_1^2 - \omega_2^2)^2}\right) \qquad (6)
$$

The corresponding eigenvectors are given by

$$
\boldsymbol{X}_{\pm} = \begin{bmatrix} 2\Omega_1^2 \\ \omega_1^2 - \omega_2^2 \mp \sqrt{4\Omega_1^2\Omega_2^2 + (\omega_1^2 - \omega_2^2)^2} \end{bmatrix} \qquad (7)
$$

The two frequencies correspond to the two modes of vibration of the system. The general equation of motion is given by

$$
\boldsymbol{X} = C_+ \boldsymbol{X}_+ \cos(\omega_+ t + \phi_+) + C_- \boldsymbol{X}_- \cos(\omega_- t + \phi_-) \qquad (8)
$$

The four constants $C_{\pm}$ and $\phi_{\pm}$ are determined from four initial conditions, usually $x_i(0)$ and $\dot{x_i}(0)$ for $i = 1, 2$.

3 The Semiclassical two level Rabi Model

Consider an atom with two distinct eigenstates $|g\rangle$ and $|e\rangle$, having energies $\hbar\omega_g$ and $E_0 + \hbar\omega_e$. The system, initially in ground state, interacts with a time-varying electric field given by $\epsilon = \epsilon_0 e^{i\omega_L t}$. In the basis of the energy eigenstates, the Hamiltonian is

$$\boldsymbol{H} = \begin{bmatrix} \hbar\omega_g & -\mu\epsilon_0\cos(\omega_L t) \\ -\mu^*\epsilon_0\cos(\omega_L t) & \hbar\omega_e \end{bmatrix} \tag{9}$$

Instead of working with a time-dependent Hamiltonian, we can transform to a more convenient basis without explicit time dependence, through a unitary transformation. The transformation $\hat{U} = |g\rangle\langle g| + e^{-i\omega_L t}|e\rangle\langle e|$ is a common choice for this ordeal. Then, the transformed Hamiltonian is given by

$$\hat{\tilde{H}} = \hat{U}^\dagger \hat{H}\hat{U} - i\hbar\hat{U}^\dagger\frac{d\hat{U}}{dt} \tag{10}$$

Therefore, our Hamiltonian becomes

$$\tilde{H} = \begin{bmatrix} \hbar\omega_g & -\frac{\mu\epsilon_0}{2}(e^{-2i\omega_L t}+1) \\ -\frac{\mu^*\epsilon_0}{2}(e^{-2i\omega_L t}+1) & \hbar(\omega_e - \omega_L) \end{bmatrix} \approx -\hbar\begin{bmatrix} \omega_g & -\Omega/2 \\ -\Omega/2 & \omega_g - \Delta \end{bmatrix} \tag{11}$$

Here $\Delta = \omega_L - (\omega_e - \omega_g)$ and $\Omega = \frac{\mu\epsilon_0}{2}$.

The Schrödinger equation for time evolution of a state $|\psi\rangle$ is $|\dot{\psi}\rangle = \hat{H}|\psi\rangle$. For this two level system, the state becomes $|\psi\rangle = c_1(t)|\tilde{g}\rangle + c_2(t)|\tilde{e}\rangle$, where $|\tilde{g}\rangle$ and $|\tilde{e}\rangle$ are the new transformed basis states ($|\tilde{g}\rangle = |g\rangle$ & $|\tilde{e}\rangle = e^{-i\omega_L t}|e\rangle$).

Again, casting it in a matrix form, we have $i\hbar\frac{d\boldsymbol{X}}{dt} = \tilde{H}\boldsymbol{X}$ where $\boldsymbol{X} = (c_1, c_2)$. If initially, $|\psi(0)\rangle = |\tilde{g}\rangle$, then

$$\begin{aligned} c_1 &= \frac{\Delta + \sqrt{\Delta^2+\omega^2}e^{i(\Delta-\sqrt{\Delta^2+\Omega^2}t/2+\omega_g t)} - \Delta - \sqrt{\Delta^2+\omega^2}e^{i(\Delta+\sqrt{\Delta^2+\Omega^2}t/2+\omega_g t)}}{2\sqrt{\Delta^2+\Omega^2}} \\ c_2 &= \frac{\Omega e^{i(\Delta+\sqrt{\Delta^2+\Omega^2}t/2+\omega_g t)} - \Omega e^{i(\Delta-\sqrt{\Delta^2-\Omega^2}t/2+\omega_g t)}}{2\sqrt{\Delta^2+\Omega^2}} \end{aligned} \tag{12}$$

The solution is a linear combination of of two sinusoidal functions with frequency $\omega_\pm = \omega_0 + (\Delta \pm \sqrt{\Delta^2+\Omega^2})/2$, which is of a similar form as the expressions in coupled oscillator.

4 Comparing the coupled oscillator and the two level system

The analogy isn't obvious between the two systems. The coupled oscillator is governed by a second order differential equation while the two-level system is governed by the first order differential equation, the Schrodinger equation. To establish the similarity, we look at the Schrodinger equation

$$i\hbar\frac{d}{dt}|\tilde{\psi}\rangle = \tilde{H}|\tilde{\psi}\rangle \tag{13}$$

For this 2-D Hilbert space, we can express $|\tilde{\psi}\rangle$ and $\tilde{H}$ in the basis $(|\tilde{g}\rangle, |\tilde{e}\rangle)$. This allows us to write Eq. 13 as a matrix equation. Differentiating 13 with respect to time, we have

$$-\frac{d^2}{dt^2}|\tilde{\psi}\rangle = \frac{1}{\hbar^2}\tilde{H}^2|\tilde{\psi}\rangle \tag{14}$$

Now we have two vectors $\boldsymbol{X}$ and $\tilde{\boldsymbol{\psi}}$ on equal footing in Eqns. 2 and 14, both parts of two eigenvalue equations. If $\tilde{\boldsymbol{\psi}}$ is taken as analogous to $\boldsymbol{X}$, then we have

$$\boldsymbol{A} = \frac{1}{\hbar^2}\tilde{\boldsymbol{H}} \tag{15}$$

Here $\tilde{\boldsymbol{H}}$ is a Hermitian matrix, therefore we have to assure that $\boldsymbol{A}$ is Hermitian. One of the conditions for hermiticity is to have real eigenvalues which is obviously satisfied by $\boldsymbol{A}$ as all the entries are real. However, hermiticity also invokes $\tilde{\boldsymbol{H}}^\dagger = \tilde{\boldsymbol{H}}$, which for a real matrix translates to $\boldsymbol{A}^T = \boldsymbol{A}$. To uphold this condition, we need to assure that the $\Omega_1 = \Omega_2$ for the classical coupled oscillator. We will see the effects of deviation from this condition in later sections. Then, the displacements x_1 and x_2 observed from the experiment are then analogous to the real parts of the coefficients c_1 and c_2 in Eqn. 12 and differ only up to some normalisation constants. Although they are completely different physical quantities, they evolve in a similar fashion mathematically. For a classical coupled oscillator, the quantities Δ and Ω have no exact physical meaning, however comparing both sides of x_1 and $\mathbb{R}[c_1]$, we can roughly relate them to the coupling strength and the difference between the natural frequencies respectively.

5 Density Matrix Analysis

When there is a population of quantum particles in the ground and excited states, the equation of state is best expressed by the density of states. From Von-Neumann equation, we know that

$$\dot{\rho} = -\frac{i}{\hbar}[\hat{H}, \rho] \tag{16}$$

Here ρ has the form $\begin{bmatrix} \rho_{gg}(t) & \rho_{ge}(t) \\ \rho_{eg}(t) & \rho_{ee}(t) \end{bmatrix}$ where $\rho_{gg}(t)$ and $\rho_{ee}(t)$ represent, loosely speaking, the population of the ground and excited states at time t. In a closed system, for a normalised density matrix, $Tr(\rho) = \rho_{gg} + \rho_{ee} = 1$. Also, $\rho_{ge} = \rho_{eg}^*$.

For a statistical ensemble of the system discussed above, we use this equation to find the density matrix elements. They will be coupled differential equations which, in theory, can be integrated to give the elements. The off-diagonal terms in the density matrix determine *coherence* and as the system evolves in time, we expect the coherence terms to go to 0 i.e *decoherence* where we will have meaningful measurement.

$$\dot{\rho} = \begin{bmatrix} \dot{\rho_{gg}} & \dot{\rho_{ge}} \\ \dot{\rho_{eg}} & \dot{\rho_{ee}} \end{bmatrix} = \begin{bmatrix} -i\frac{\Omega\hbar}{2}(\rho_{ge} - \rho_{eg}) & -i\frac{\Omega\hbar}{2}(\rho_{gg} - \rho_{ee}) - i\Delta\rho_{ge} \\ -i\frac{\Omega\hbar}{2}(\rho_{ee} - \rho_{gg}) + i\Delta\rho_{eg} & -i\frac{\Omega\hbar}{2}(\rho_{eg} - \rho_{ge}) \end{bmatrix} \tag{17}$$

Solving these four coupled equations(one each for the diagonal elements as they are real and two for one of the off-diagonal elements as they can be complex), we can get the density matrix elements. For a closed system, the oscillations are shown in Fig. 2.

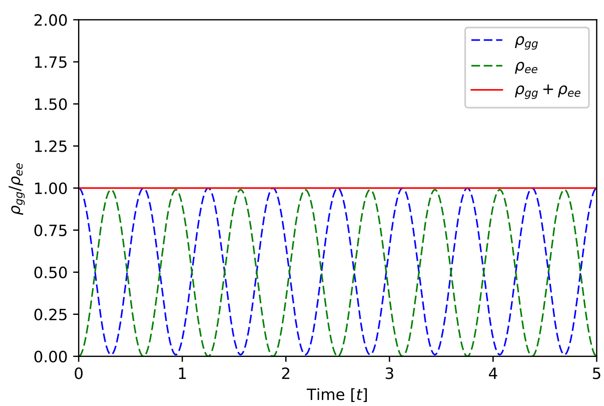

Figure 2: Diagonal elements of the density matrix. For a closed system, the trace is conserved. Here, $\Omega = 10$ and $\Delta = 1$

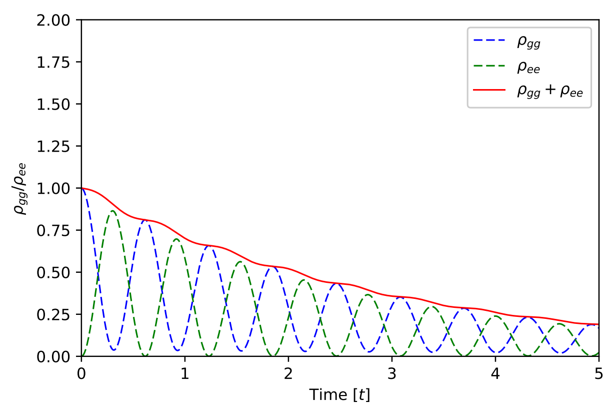

Figure 3: Diagonal elements of the density matrix where the system suffers loss. Here, $\Omega = 10$, $\Delta = 2$, $\Gamma_g = 0.1$ and $\Gamma_e = 0.6$.

For an open system sustaining loss in population of the states over time, the trace of the density matrix is not conserved. We add some loss terms by hand to our system to see the behaviour over time. The equations for the density matrix elements then become

$$\begin{aligned}
\dot{\rho_{gg}} &= \Omega\ \mathrm{Im}(\rho_{ge}) - \Gamma_g\rho_{gg} && (18)\\
\dot{\rho_{ee}} &= -\Omega\ \mathrm{Im}(\rho_{ge}) - \Gamma_e\rho_{ee} && (19)\\
\mathrm{Re}(\dot{\rho_{ge}}) &= \Delta\ \mathrm{Im}(\rho_{ge}) - \frac{\Gamma_g+\Gamma_e}{2}\ \mathrm{Re}(\rho_{ge}) && (20)\\
\mathrm{Im}(\dot{\rho_{ge}}) &= -\Delta\ \mathrm{Re}(\rho_{ge}) + \frac{\Omega}{2}(\rho_{ee}-\rho_{gg}) - \frac{\Gamma_g+\Gamma_e}{2}\ \mathrm{Im}(\rho_{ge}) && (21)
\end{aligned}$$

Analytically finding a closed form solution for these equations is a tedious job, however a numerical analysis gives us the general behaviour and how the states decay over time. One such result is shown in Fig. 3.

The solution for ρ_{ee} when there is no loss of population to the environment is given by

$$\rho_{ee} = \frac{\Omega^2}{\Omega^2+\Delta^2}\sin^2\left(\frac{\sqrt{\Omega^2+\Delta^2}}{2}t\right) \tag{22}$$

The ground state population is then given by

$$\rho_{gg} = 1 - \frac{\Omega^2}{\Omega^2+\Delta^2}\sin^2\left(\frac{\sqrt{\Omega^2+\Delta^2}}{2}t\right) \tag{23}$$

However, since the populations in both the states are undergoing decay, we add an exponentially decaying factor to this solution. Then, for the ground state, we have

$$\rho_{gg} = e^{-gt}\left(1 - \frac{\Omega^2}{\Omega^2+\Delta^2}\sin^2\left(\frac{\sqrt{\Omega^2+\Delta^2}}{2}t\right)\right) \tag{24}$$

6 Analysis of the long term decay with simulation

We looked at the long term decays of the solutions of the coupled oscillator system. We solved the coupled differential equations, with damping, numerically. We looked at various cases where the damping constant was time-independent. Further, we introduced some "random" time dependence in the value of the damping parameters in search for decoherence between the two modes. We also looked at a different method of introducing friction in the system, by allowing the a normalised random negative "kick" in the displacement at every time step in our integrator. Some of the results are attached below.

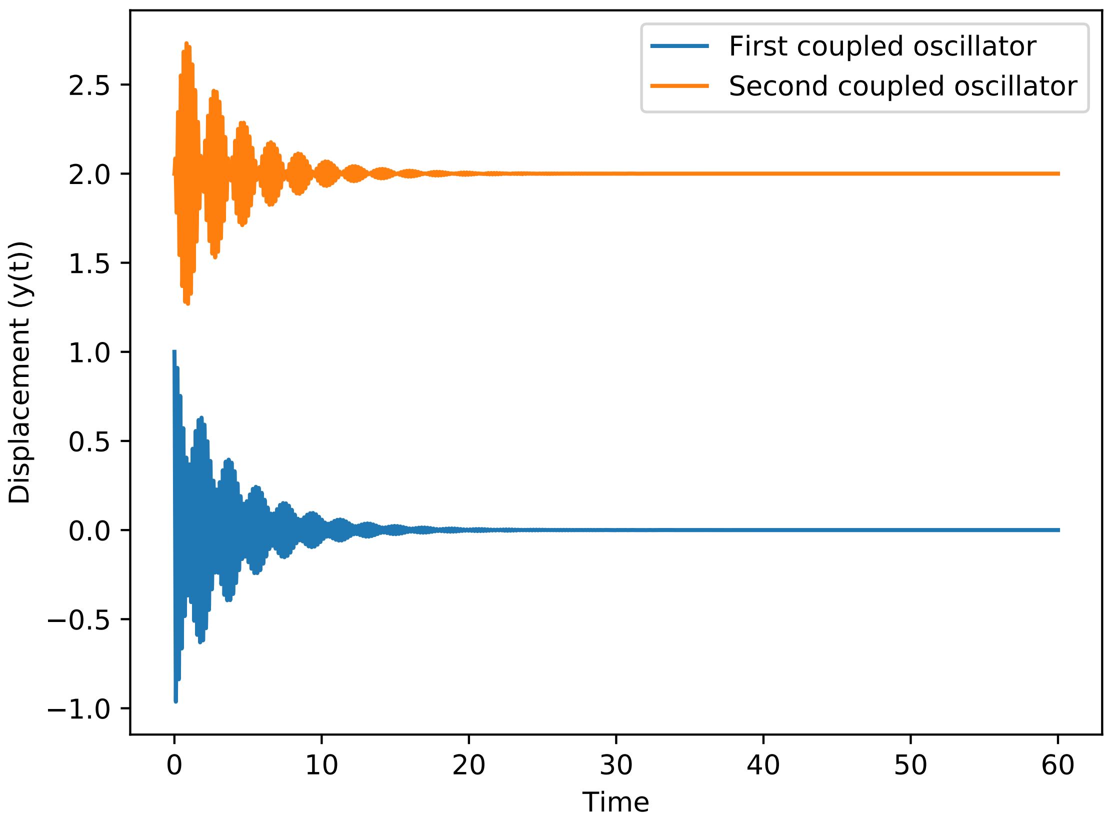

Figure 4: Damped coupled harmonic oscillator solution using a "random negative kick" in the displacement

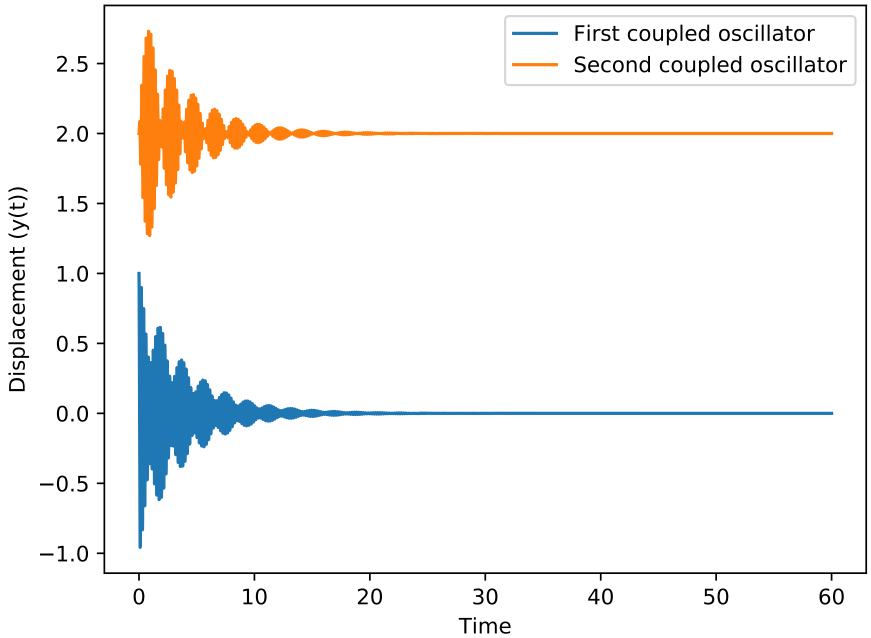

Figure 5: Damped coupled harmonic oscillator solution using a damping term with "random" time dependent fluctuations

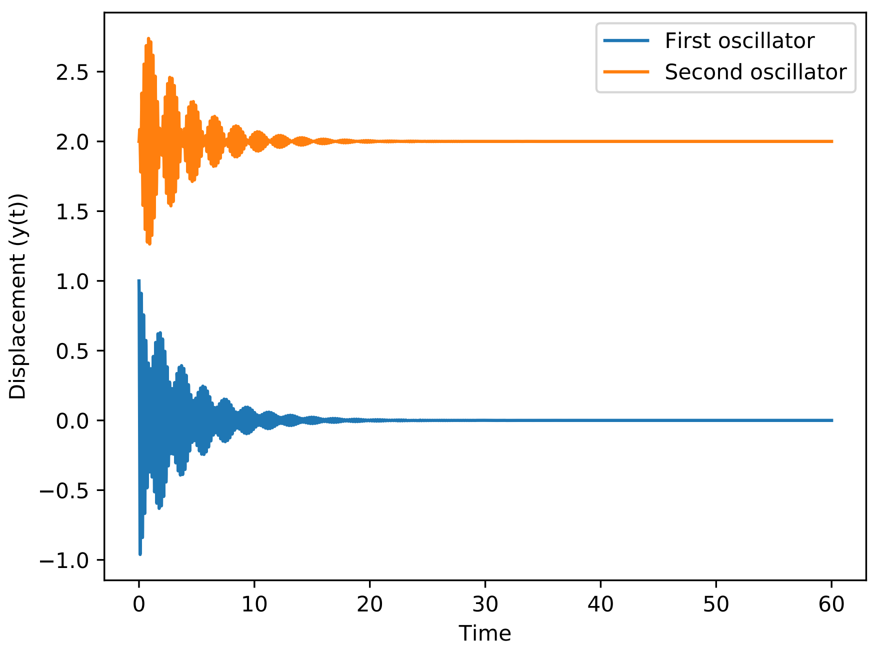

Figure 6: Damped coupled harmonic oscillator solution using a constant damping term

We normalised the displacements after getting a decay parameter from fitting and then plotted all of them simultaneously below. As can be seen, there is no phase decoherence developing in the system of equations, the one we had hoped to model.

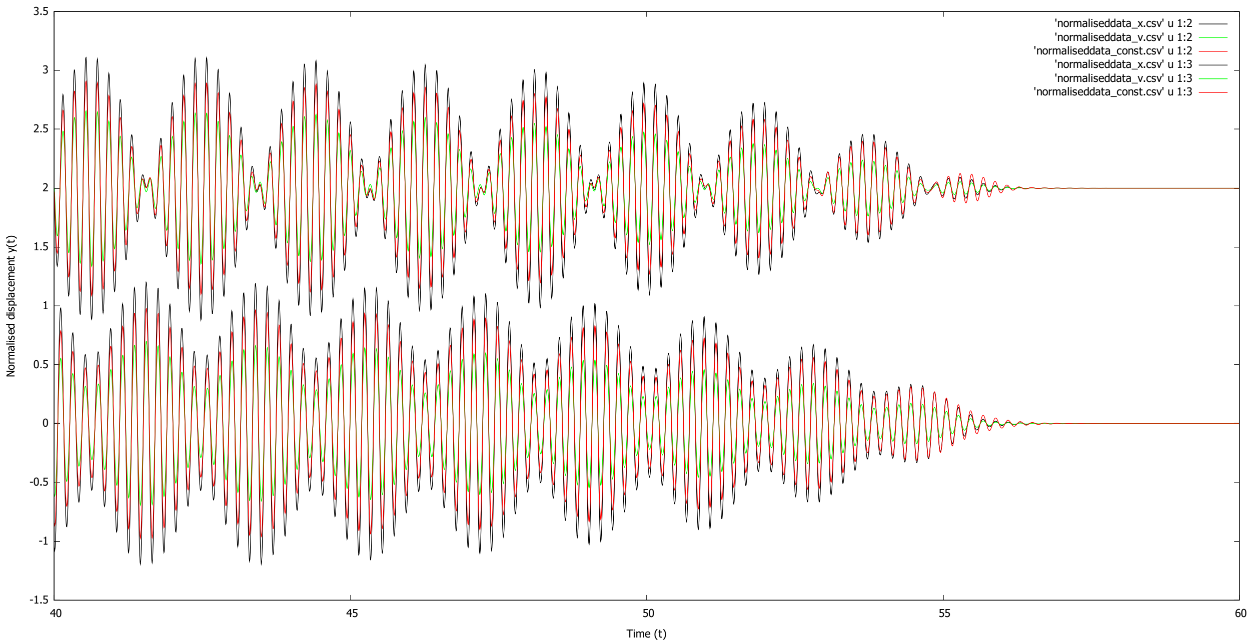

Figure 7: Normalised displacements for both the oscillators using the three different methods, described above.

7 Conclusion

Clearly, introducing a random fluctuation in the decay parameters do not change the phase correlation of the system. We now want to look at the effect of introducing fluctuations in the coupling constant k between the coupled oscillator. It seems that the decoherence might be a result of the coupling constant taking different values over the course of time. This is physically possible, as we are using rubber bands as couplers which when stretched, might have a slight dependence on orientation and shape. We will continue with this work in the next semester.