# Problem set: March 21

1. To estimate the proportion $p$ of butterflies that have a special marking on their wings. Consider two approaches:

   (a) Capture butterflies one at a time until five with the special marking have been collected. A total of 43 butterflies are required to collect the five. what is the M.L.E. of p?

   (b) Collect butterflies all day and count those with the special mark. 58 are captured. Three have the Mark. What is the M.L.E. of p?

2. Consider a random sample $X_1, \cdots, X_n \sim Uniform(0, \theta), \theta$ unknown. Show that the sequence of MLE's of $\theta$ is a consistent sequence.

3. Consider a random sample $X_1, \cdots, X_n \sim N(\mu, \sigma^2), \mu, \sigma$ unknown. Find the MLE of the 0.95 quantile.

4. Consider again a random sample $X_1, \cdots, X_n \sim N(\mu, \sigma^2), \mu, \sigma$ unknown. Find the MLE of $v = P(X > 2)$.

5. Find the MLE estimator for $\theta$ in the Cauchy distribution given the sample X = (-22.33, -10.29, -1.35, -1.73, 6.91, -0.52, 0.43, -0.00, -8.66, -7.16, 1.15, 1.15, -3.75, 2.54, 7.31, 0.65, 6.66, 5.52, 2.02, -1.48).

6. Consider a random sample of 21 observations from exponential($\lambda$). Mean ($\mu > 0$). 20 of the observations are collected without incident and have a mean of 6. The 21st observation was not measured exactly except that it is greater than 15.

   Find the MLE of $\mu$.

7. $X_1, \cdots, X_n$ form a random sample from a Poisson distribution for which the mean is unknown. Determine the MLE of the standard deviation of the distribution.

8. Consider a random sample $X_1, \cdots, X_n \sim exp(\beta), \beta$ is unknown. Determine the MLE of the median of the distribution.

## Problem set: 22 March Due on 29 March

For each of these distributions show that the specified statistic T is suficient for the parameter.

1. The Bernoulli distribution with parameter $p$.
   $(0 < p < 1)$, $T = \sum_{i=i} X_i$.

2. The geometric distribution with parameter $p$.
   $(0 < p < 1)$, $T = \sum_{i=i} X_i$.

3. The negative binomial distribution with parameters $r$ and $p$.
   r is known. $(0 < p < 1)$, $T = \sum_{i=i} X_i$.

4. The gamma distribution with parameters $\alpha$ and $\beta$. $\alpha$ is known.
   $(\beta > 0)$, $T = \sum_{i=i} X_i$.

5. The gamma distribution with parameters $\alpha$ and $\beta$. $\beta$ is known.
   $(\alpha > 0)$, $T = \prod_{i=i} X_i$.

Assignment - 4

Problem set 1

I Is Specified statistics T sufficient for parameter?

1. Bernouli dist. with parameter 'p'

$$T = \sum_{i=1}^{n} X_i$$

$$f(x_i, p) = p^{x_i} (1-p)^{1-x_i}$$

$$\prod_{i=1}^{n} f(x_i, p) = \prod_{i=1}^{n} p^{x_i} (1-p)^{1-x_i} = p^{\Sigma x_i} \frac{(1-p)^n}{(1-p)^{\Sigma x_i}}$$

Using

$$T = \sum_{i=1}^{n} X_i \Rightarrow \text{ we get}$$

$$f(x_1 - x_n, p) = \left[\frac{p^T (1-p)^n}{(1-p)^T}\right] \times 1$$

$$= V(p, T) \times h(x)$$

$h(x) = 1$ doesn't depend on $p$

and $V(p, T) = \dfrac{p^T (1-p)^n}{(1-p)^T}$, hence $T$ is sufficient statistics.

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2. Geometric $\rightarrow f(x,p) = (1-p)^{x} p$

$$f(x_1 - x_n, p) = \prod_{i=1}^{n} (1-p)^{x_i} p = \frac{(1-p)^{n} p}{(1-p)^{\Sigma x_i}}$$

Given $T = \Sigma x_i$ hence. $f(x_1 - x_n, p) = \underbrace{\left(\frac{(1-p)^{n} p}{(1-p)^{T}}\right)}_{v(T,p)} \times \underbrace{1}_{h(x)}$.

Hence $T$ is sufficient statistic.

3. Negative Binomial. $r = \#$ success $\quad p =$ probability of success

$$f(x_1 - x_n \mid p, r) = \prod_{i=1}^{n} \binom{x_i - 1}{r-1} p^{r} (1-p)^{x_i - r}$$

$$f(x_1 - x_n \mid p, r) = p^{nr} \cdot \prod_{i=1}^{n} \binom{x_i - 1}{r-1} \frac{(1-p)^{\Sigma x_i}}{(1-p)^{nr}}$$

Since $\Sigma x_i = T$

$$= \underbrace{\left(\frac{p^{nr}}{(1-p)^{nr}} (1-p)^{T}\right)}_{v(T,p,r)} \times \underbrace{\left(\prod_{i=1}^{n} \binom{x_i - 1}{r-1}\right)}_{h(x,p,r)}$$

Hence $T$ is sufficient statistic.

4. Gamma dist. $f(x_1 - x_n, \alpha, \beta) = \prod_{i=1}^{n} \frac{\beta^{\alpha}}{\Gamma(\alpha)} x_i^{\alpha-1} e^{-\beta x_i}$

$$= \underbrace{\left(\prod_{i=1}^{n} \frac{\beta^{\alpha}}{\Gamma(\alpha)} x_i^{\alpha-1}\right)}_{h(x,\alpha,\beta)} \times \underbrace{\left(e^{-\beta \Sigma x_i}\right)}_{v(t,\beta,\alpha)}$$

Problem set 2.

1 (a) Estimate $p$ through MLE.

Using geometric distribution (r) # success = 5 $p = ?$

\# trials = $x$

$$L(p, r \mid x) \Rightarrow \prod_{i=1}^{n} \binom{x_i - 1}{r - 1} p^{r} (1-p)^{x_i - r}$$

$$l = \log(L(p, r \mid x)) = \log\left(\prod_{i=1}^{n}\left[\binom{x_i - 1}{r - 1}\right]\right) + r \log p + (\Sigma x_i - r)\log(1-p)$$

$$\frac{dl}{dp} = 0 + \frac{r}{p} + \frac{(\Sigma_i x_i - r)(-1)}{(1-p)} = 0$$

$$\Rightarrow \frac{r}{p} = \frac{\Sigma x_i - nr}{(1-p)} \Rightarrow n = 1 \; ; \; x_i = \{43\} \; ; \; r = 5$$

$$\frac{5}{\hat{p}} = \frac{43 - 5}{1 - \hat{p}} = \frac{38}{1-p}$$

$$\frac{1}{\hat{p}} = \frac{38+1}{5} = \frac{43}{5} \rightarrow \boxed{\hat{p} = 5/43}$$

(b) Using binomial dist.

$$L(p, n, x) = \binom{n}{x} p^{x} (1-p)^{n-x} \qquad l = \log\binom{n}{x} + x \log p + n - x \log(1-p)$$

$$\frac{dl}{dp} = \frac{x}{p} + \frac{n - x(-1)}{(1-p)} = 0 \rightarrow \frac{x}{p} = \frac{n-x}{1-p} \Rightarrow \frac{58-3}{1-\hat{p}} = \frac{3}{\hat{p}}$$

$$\text{So} \quad \hat{p} = \frac{3}{58}$$

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2. $X_1 - X_n \sim$ Uniform $(0,\theta)$. Show that sequence of MLE of $\theta$ is a consistent sequence.

$$f(x_i,\theta) = \frac{1}{\theta}, \quad 0<x<\theta$$

$$L(\theta, x_1 - x_n) = \prod_{i=1}^{n} \frac{1}{\theta} = \left(\frac{1}{\theta}\right)^n = \theta^{-n}$$

as $n \to \infty$ lim $\Sigma_i X_n = \frac{n\theta}{2}$ To proove $E\left[(\hat{\theta_n} - \theta)^2\right] \to 0$ as $n \to \infty$

$L(\theta, X_1 - X_N) = \theta^n \, I(X_1 \ldots X_n < \theta) = \theta^n I(\max(X_1 - X_n) < \theta)$.

for maximium L we need min$^m$ $\theta$ s.t. $\theta \geq \max(X_1 - X_N)$. Hence its ~~convergent~~ consistent

3. $X_1 - X_n \sim N(\mu, \sigma^2)$

$$f(x, \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\left(\frac{(x-\mu)^2}{2\sigma^2}\right)}$$

$$l(\mu,\sigma^2) = -\frac{n}{2}\log(2\pi\sigma^2) - \sum_{i=1}^{n}\left(\frac{(x-\mu)^2}{2\sigma^2}\right)$$

$$\frac{dl}{d\mu} = 0 + \Sigma_i x_i - n\mu = 0 \rightarrow \mu = \frac{\Sigma_i x_i}{n}$$

$$\frac{dl}{d\sigma^2} = -\frac{n}{2}\frac{1}{\sigma^2} - \frac{1(-1)}{2\sigma^4}\left(2(x-\mu)^2\right).$$

$$\Rightarrow \sigma^2 = \frac{(x-\mu)^2}{n} \qquad \sigma = \sqrt{\frac{(x-\mu)^2}{n}}$$

MLE for 0.95 quantile $= \mu + $ Z-score $(0.95) \times \sigma$

$$= \mu + (1.64)\left(\sqrt{\frac{(x-\mu)^2}{n}}\right)$$

$\gamma$

4 MLE of $P(x>2)$.

From Question 3 :

$\hat{\mu} = \frac{\sum_i x_i}{N}$ $\hat{\sigma} = \sqrt{\frac{(x - \sum_i x_i/n)^2}{n}}$

$P(x>2) \Rightarrow 1 - P(x \leq 2) = 1 - CDF(2$ [redacted] $;\hat{\mu}$ [redacted] $, \hat{\sigma})$

where $CDF(2) = \phi\left( \frac{2 - \bar{x}}{\sqrt{(x - \frac{\sum x_i}{n})^2 / n}} \right)$

5. MLE of $\theta$ for Cauchy distribu$^n$ $x_1 - x_n$. with $\gamma = 1$

$\hat{l}(x_0, \gamma \mid x_1 - x_n) = -n \log(\gamma\pi) - \sum \log\left(1 + \left(\frac{x - x_0}{\gamma}\right)^2\right)$

$\frac{dl}{dx_0} = 0$ & $\frac{dl}{d\gamma} = 0$ gives :

① $\frac{\sum 2(x_i - x_0)}{\gamma^2 + [x_i - x_0]^2} = 0.$ $\{\gamma = 1\}$.

② $\cancel{\frac{x_i - x_0}{1 + (x_i - x_0)^2}}$ $\cancel{\sum \frac{\gamma^2}{\gamma^2 + [x_i - x_0]^2}}$ $\cancel{- \frac{n}{2} = 0}$

We will use 'R' to find the root through 'uniroot' function.

$x_0 = 0.407$.

6. Exponential. $n = 21$

20 observation $\rightarrow \mu = 16$

1 item $(>15)$.

find mle of $\mu$.

$f(x) = \lambda e^{-\lambda x}$

$$L(\mu \mid x_1 - x_{21})$$

$$= \prod_{i=1}^{20} \lambda e^{-\lambda x_i} \times P(x_{21} > 15 \mid \mu)$$

$$= \lambda^{20} \left(e^{-\lambda \sum_i x_i}\right) \times 1 - P(x_{21} < 15)$$

$$\times\ 1 - (1 - e^{-\lambda(15)})$$

$$= \lambda^{20} e^{-\lambda \sum_i x_i} \times e^{-\lambda(15)}$$

$$l = \log L = 20 \log \lambda - \lambda \sum_i x_i + (-15\lambda)$$

$$\frac{dl}{d\lambda} = \frac{20}{\lambda} - \sum_i x_i - 15 = 0.$$

$$\Rightarrow \frac{20}{\lambda} = \sum_i x_i + 15 = 20 \times 6 + 15 = 135$$

$$\hat{\lambda} = \frac{20}{135}$$

$$\hat{\mu} = \frac{1}{\hat{\lambda}} = \frac{135}{20} = 6.75.$$

7. $x_1 - x_N \sim$ Poisson  for poission: $\lambda = \cancel{\frac{1}{4}}\ 4$

MLE for $\sigma$.

$$f(x,\lambda) = \frac{\lambda^x e^{-\lambda}}{x!} = \left(\frac{\lambda^x}{x!}\right) e^{-\lambda}$$

$$\sigma = \sqrt{\frac{(x-\mu)^2}{n}} \qquad \mu ?$$

$$L = \prod_{i=1}^{n} \frac{\lambda^x}{x!} e^{-\lambda} = \frac{\lambda^{\sum x_i} e^{-\lambda n}}{\prod x!}$$

$$\log L. \Rightarrow \sum_i x_i \log\lambda - \lambda n - \log(\prod x!) = 0$$

$$\frac{dl}{d\lambda} = \frac{\sum x_i}{\lambda} - n = 0 \rightarrow \lambda = \frac{\sum_i x_i}{n} = 4.$$

$$\sigma^2 = \int (x-\lambda)^2 f(x)\,dx = \int (x)^2 (\lambda)^2 \frac{\lambda^x e^{-\lambda}}{x!}.dx$$

$$= \lambda \rightarrow \sigma = \sqrt{\lambda}.$$

$$\text{so} \quad \sigma = \sqrt{\frac{\sum x_i}{N}}.$$

8. $x_1 - x_N \sim exp(\beta)$

$$f(x|\beta) = \beta e^{-\beta x}$$

mle of median. ie. CDF (0.5)

$$\phi(x) = \int \beta e^{-\beta x} = 1 - e^{-\beta x_0} = 0.5$$

where $\hat{x_0}$ is median

$$1 - e^{-\beta x_0} = 0.5$$

$$e^{-\beta x_0} = 0.5 \longrightarrow \left[ -\frac{\log(0.5)}{\beta} = x_0 \right]$$

$$= \frac{\log(2)}{\beta} = x_0 .$$

and from question 6 : $\beta = \frac{\Sigma x_i}{N}$

Hence $\boxed{x_0 = \frac{N \ln 2}{\Sigma x_i}}$ Ans.