HW1

Saturday, February 9, 2019 9:55 AM

Problems : Grinstead Snell (7, 10, 16, 18, 25, 27, 28, 38)

7). Probability of six showing $= \frac{1}{6}$.

It follows geometric distribution.

a) $p(T) = \left(\frac{5}{6}\right)^{T-1}\left(\frac{1}{6}\right)$ {T-1 failures followed by 1 success}.

b) $p(T>3) = 1 - \sum_{i=1}^{3} p(T=i)$

$$= 1 - \left(\frac{1}{6}\right) - \left(\frac{5}{6}\right)\left(\frac{1}{6}\right) - \left(\frac{25}{36}\right)\left(\frac{1}{6}\right)$$

$$= 1 - \frac{1}{6}\left(1 + \frac{5}{6} + \frac{25}{36}\right) = 1 - \frac{91}{216} = \frac{125}{216}$$

c) $p(T>6 \mid T>3) \Rightarrow$ We are sure of first 3 outcomes.

So this is similar to:

$$1 - \underbrace{p(T=1)}_{4} + \underbrace{p(T=2)}_{5} + \underbrace{p(T+3)}_{6.} = \left(\frac{125}{216}\right)$$ from previous

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10) $N \leftarrow$ Total population

$n_1 \leftarrow$ people counted 1^{st} time.

$n_2 \leftarrow$ people counted 2^{nd} time.

$n_{12} \leftarrow$ people counted both time.

	Census 1	Census 2	
N	n_1	n_2	total counted
	0	n_{12}	repeated.
	$N - n_1$	$N - n_1 - n_2 + n_{12}$	Not counted

a) Probability of $n_{12} = k$.

$$= \frac{\begin{pmatrix}\text{Probability of selecting} \\ k \text{ out of } n_1\end{pmatrix} \times \begin{pmatrix}\text{Probability of selecting} \\ n_2 - k \text{ out of } N - n_1\end{pmatrix}}{\begin{pmatrix}\text{Total probability of} \\ \text{selecting } n_2 \text{ out of } N\end{pmatrix}}$$

$$= \frac{\binom{n_1}{k}\binom{N-n_1}{n_2-k}}{\binom{N}{n_2}}$$ — Answer.

b) Let $X = n_{12}$

N which maximizes $X = n_{12}$

Probability of hypergeometric is highest at mean.

so. $n_{12} = \left(\frac{n_1}{N}\right)$ gives $N = \frac{n_1}{n_{12}}$ — Ans

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16) $\lambda = \text{mean} = (0.01) \times (60 \times 5) = 3$.

$P(\text{miss at most } 1) = P(0) + P(1)$.

Approximating to poison.

$$P(X = c) = \frac{\lambda^c}{c!} e^{-\lambda}$$

$$P(0) = \frac{3^0}{0!} e^{-3} = \frac{1}{e^3} = 0.0497 = 4.97\%$$

$$P(1) = \frac{3^1}{1!} e^{-3} = \frac{3}{e^3} = 0.1493 = 14.93\%$$

$$P(\text{at most } 1 \text{ call}) = \underline{19.9\%} \text{ Ans}$$

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18) 600 raisins 400 chocolate chips

500 cookies.

$$\lambda_{\text{raisin}} = \frac{600}{500} = 1.2 \qquad \lambda_{\text{choc.}} = \frac{400}{500} = 0.8.$$

a) $$P_{\text{no raisin}} = \frac{(1.2)^0}{0!} e^{-1.2} = \frac{1}{e^{1.2}} = 0.3011$$

$$= \underline{30.11\%} \text{ Ans}$$

b) $$P_{2\text{ chocs}} = \frac{(0.8)^2}{2!} e^{-0.8} = \frac{0.64}{2} e^{-0.8} = 0.1437$$

$$= \underline{14.37\%} \text{ Ans}$$

c) $P(\text{at least 2 bits}) = 1 - P(\text{0 bit} \vee \text{1 bit})$

$P(\text{0 bit}) = P_{\text{no choc}} \times P_{\text{no raisin}}.$

$= \dfrac{1}{e^{0.8}} \times \dfrac{1}{e^{1.2}} = 0.1352$

$= 13.52\%$

$P(\text{1 bit only}) = P_{\underline{1}\text{choc}} \times P_{\underline{0}\text{raisin}} + P_{\underline{0}\text{ choc}} \times P_{\underline{1}\text{ raisin}}$

$= \dfrac{(0.8)^1}{e^{0.8}} \times \dfrac{(1)}{e^{1.2}} + \dfrac{1.2}{e^{1.2}} \times \dfrac{1}{e^{0.8}} = 0.108 + 0.1624$

$= 0.2706 = \underline{27.06\%}$ Ans

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25) $\text{Money})_{\text{parking meter}} = 0.1 \times 100 = 10\$$

$p)_{\text{get caught}} = 0.05$

$\text{mean})_{\text{caught every 100 times}} = 0.05 \times 100 = 5.$

$P(0) = \dfrac{5^0 e^{-5}}{0!} = \dfrac{1}{e^5} = 0.0067 = 0.66\%$

$P(1) = \dfrac{5}{1} e^{-5} = \dfrac{5}{e^5} = 0.0336 = 3.36\%$

$P(2) = \dfrac{25}{2} e^{-5} = 12.5/e^5 = 0.084 = 8.4\%$

$$P(>2) = 1 - \left(\frac{18.5}{e^5}\right) = 87.6\%$$

(Expected) money when caught $\Rightarrow$

$$= 0 \times (P(0) + P(1)) + 2 \times P(2) + \sum_{X=3}^{100} [5(x-2)+2]\, P(X)$$

$$= 0 \times 4\% + 2 \times 8.4\% - 8(87.6\%) + 5\sum_{X=3}^{100} x P(x)$$

$$= 0.168 - 7.008 + 5\left(\sum_{x=0}^{100} x P(x)\right) - 5\left(\sum_{X=0}^{2} x P(x)\right)$$

$$= 0.168 - 7.008 + 5 \times \underbrace{(0.05 \times 100)}_{\lambda N} - 5\left(0 \times P(0) + 1 \times P(1) + 2 \times P(2)\right)$$

$$= 18.16 - 5(0.033 + 2 \times 0.084)$$

$$= 18.16 - 5(0.201) = \underline{17.16\,\$}.$$

Thus, Reese Prosser is expected to pay 17.6\$ in the long run. It's better for him to follow the law.

27) Solving with Poisson.

Mean $= \lambda = 100 \times 0.001 = 0.01$.

P(at least 1 accident)

$= 1 - [P(0 \text{ accident})]$.

$$P(0 \text{ accident}) = \frac{1}{e^{(0.1)}} = \frac{1}{1.105} = 0.904$$

P(at least 1 such accident) $= 1 - 0.904 = 0.096 = 9.6\%$

(b) Solving using binomial

$P(0 \text{ accident}) = (0.999)^{100}$

$P(\text{at least } 1) = 1 - (0.999)^{100}$

$= 1 - 0.9047 = 0.0953 = 9.53\%$

Ans

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28) P(all find seat) $= 1 - P(100 \vee 99 \text{ show up})$.

$P(100 \text{ show up}) = (1 - p_{\text{not show}})^{100}$

$= (0.96)^{100} = 0.01687$

$= 1.68\%$

$$P(99 \text{ show up}) = \binom{100}{1}(0.04)^1(0.96)^{99}$$

$$= 100 \times 0.04 \times (0.96)^{99}$$

$$= 0.0702 \Rightarrow 7.02\%$$

So $P(\text{all find seat}) = 1 - 0.01687 - 0.0702 = 0.9133$

$$= \underline{\underline{91.33\%}} \text{ Ans}$$

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38) Total population $= 20$

Defective $= 5$

number selected $= 5$

$P(1 \text{ item defective})$?

a) Without replacement $\Rightarrow$.

Hypergeometric distribution can be used.

$$P(1 \text{ defective}) = \underbrace{\binom{15}{4}}_{\substack{\text{4 good items} \\ \text{selected from} \\ \text{15 good items}}} \underbrace{\binom{5}{1}}_{\substack{\text{1 bad item} \\ \text{selected from 5} \\ \text{bad item}}} \Big/ \underbrace{\binom{20}{5}}_{\substack{\text{5 items} \\ \text{selected from} \\ \text{20.}}}$$

P(1 defective w/o replacement)

$$= \frac{15 \times \cancel{14}^{7} \times 13 \times \cancel{12}}{\cancel{4 \times 3 \times 2} \times 1} \times \frac{5}{\frac{\cancel{20} \times 19 \times \cancel{18}^{3} \times 17 \times 16}{\cancel{5 \times 4 \times 3 \times 2} \times 1}}$$

$$^{15}C_4 = \frac{15 \times 7 \times 13 \times 5}{19 \times 3 \times 17 \times 16} = 0.4402 = \underline{\underline{44.02\%}}$$

Ans

b) With replacement.

We can use binomial distribution to model.

P(1 defective in 5 trials).

$$P(1 \text{ defective}) = 5/20 = 1/4.$$

$$P(1 \text{ def. in } 5) = \binom{5}{1}\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^4$$

$$= 5 \times \frac{1}{4} \times \frac{3^4}{4^4} = \frac{5 \times 3^4}{2^{10}} = \frac{5 \times 81}{1024}$$

$$= 0.3955 = \underline{\underline{39.55\%}}$$

Ans

Section 5.2

Problem 1, 7, 21, 37

1. $U \sim \text{Unif}[0,1]$

a) $Y = U + 2.$

$= \text{Unif}[0,1] + 2. \quad = \text{Unif}[2,3]$

$f(Y) = 1 \qquad 2 \leq Y \leq 3$

$F(Y) = \int_{-\infty}^{Y} f(Y)$

$= \int_{2}^{Y} f(Y) = Y\Big]_2^Y = \underline{\underline{Y-2}}$

b) $Y = U^3$

$F_Y(y) = P(Y \leq y)$

$= P(U^3 \leq y) = P(U \leq y^{\frac{1}{3}}) = F_{U}(y^{\frac{1}{3}}).$

$p_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_U(y^{\frac{1}{3}})$

$= \frac{d}{du} F_U(y^{\frac{1}{3}}) \cdot \frac{du}{dy}.$

$= \cancel{P_{U}(y^{\frac{1}{3}})} \cdot \frac{d(y^{\frac{1}{3}})}{dy} = \frac{1}{3} y^{-\frac{2}{3}}.$

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$$f_Y(y) = \frac{1}{3} y^{-2/3}.$$

(can also be derived from $f_Y(y) = f_X(\phi^{-1}(y)) \frac{d(\phi^{-1}(y))}{dy}$.)

$$F_Y(y) = \int_0^y \frac{1}{3} y^{-2/3} . dy = \frac{1}{3(\frac{-1}{3})} y^{\frac{1}{3}} \Big]_0^y = y^{1/3}$$

— Answer

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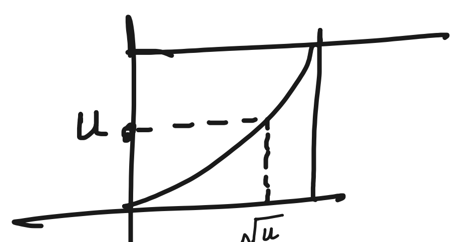
7. $$F_x \begin{cases} 0 & x < 0 \\ x^2 & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

$$x = F_x^{-1}(rnd)$$



$$u = F_x(x) = x^2$$

$$\sqrt{u} = x.$$

To simulate this density:

1) Draw from uniform density between 0,1
2) Take the root of drawn rnd variable.

— Answer

17.

$$F_x = \begin{cases} 0 & x < 0 \\ \sin^2\left(\frac{\pi x}{2}\right) & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

a) $P_x. = \frac{d}{dx}\left(F_x\right).$

$$\begin{cases} 0 & x < 0 \\ 2\sin\left[\frac{\pi x}{2}\right] \cos\left[\frac{\pi x}{2}\right]\left[\frac{\pi}{2}\right] & 0 \le x \le 1 \\ 0 & x > 1 \end{cases}$$

$$= \frac{\pi}{2}\sin(\pi x)$$

b) $P\left(x \le 1/4\right)$

$$F_x\left(1/4\right) = \sin^2\left[\frac{\pi}{2}\left[\frac{1}{4}\right]\right] = \sin^2\left[\frac{\pi}{8}\right]$$

$$= 0.1464 \qquad \sim 14.6\,\%$$

—— Answer

21 To proove:

$Y = F(x)$ where X is cummulative density, is uniformly distributed in 0,1.

Range of $Y = [0,1]$: from range of cumm. density.

To proove. $P(Y) = y$.

$P(F(X) \leq y)$ since F is monotically increasing.

$= P(X \leq F^{-1}(y))$ [From theorem 5.1].

$= F(F^{-1}(y))$ [From def. of cumm. density].

$= y$. ——— Answer

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37. Find. log normal density of variabe Y : $Y = e^X$ where X is normal density.

$$f_Y(y) = f_X(\phi^{-1}(y)) \cdot \frac{d(\phi^{-1}(y))}{dy}.$$

$y = e^x$

$$x = \log(y)$$

$$\phi^{-1}(y) = \log(y)$$

$$f_Y(y) = f_X(\log(y)) \cdot \frac{d(\log(y))}{dy}.$$

$$= f_X(\log(y)) \times \frac{1}{y}.$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\left[\frac{\log(y)^2}{2}\right]} \frac{1}{y}$$

$$= \frac{1}{y\sqrt{2\pi}} e^{-\frac{(\log y)^2}{2}}$$

Answer.