Homework 2 – part a Posted 10 Feb 2019 // Due 1535 18 Feb 2019

1. Suppose that one letter is to be selected at random from the 42 letters in the sentence, "The shortest distance between two points is a taxi." If Y denotes the number of letters in the word in which the selected letter appears, what is the value of E(Y)?

2. Suppose that $X$and Y have a continuous joint distribution for which the joint ppf is:

   $f(x, y) = 12y^2$ for $0 \leq y \leq x \leq 1$

   Find the value of $E(XY)$.

3. Suppose that three random variables X_1, X_2, X_3 form a random sample from the uniform distribution on the interval $[0, 1]$. Find $E[(X_1 - 2X_2 + X_3)^2]$.

4. X has pdf

   $f(x) = e^{-x}, \quad x > 0$

   $Y = e^{\frac{3X}{4}}$

   Find $E(Y)$

5. X is the outcome of rolling a fair die.

   $Y = g(X) = 2X^2 + 1$

   Find $E(Y)$

6. X has pdf

   $f(x) = 2(1 - x), \quad 0 < x < 1$

   $Y = (2X + 1)$

   Find $E(Y^2)$.

7. Remember the binomial theorem: $(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \quad forn \in \mathbb{Z}^+$

   Show that $E[(ax + b)^n] = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i E(X^{n-i})$

8. The proportion of defective parts in a large shipment is p. A random sample of n parts is selected from the shipment. Let X denote the number of defective parts in the sample, and Y denote the number of good parts in the sample. Find $E(X - Y)$.

   If the sample size is 20 and p is 5%, what is $E(X - Y)$? Write out your answer as a complete sentence that expresses the meaning of your result.

① $f(x)$: Discrete variable.

$= $ num of letters / total letter $= \frac{n\_letter}{42}$

$y =$ num of letters.

$$E(Y) = \sum_{word\ 1}^{9} (y)\, f(x).\, dx$$

[By rule of lazy statisticians.]

$$= \frac{1}{42}\left(3^2 + 8^2 + 8^2 + 7^2 + 3^2 + 6^2 + 2^2 + 1 + 4^2\right)$$

$$= \frac{252}{42} = 6.$$

② $f(x,y) = 12y^2$ for $0 \leq y \leq x \leq 1$

$$\int_0^1 \int_0^x 12y^2 .\, dy .\, dx$$

checking if pdf

$$\int_0^1 \left[12 \frac{y^3}{3} \right]_0^x .\, dx = \int_0^1 4x^3 .\, dx = \left.\frac{4x^4}{4}\right]_0^1$$

$$= 1.$$

$$E(XY) = \int_0^1 \int_0^x xy\, 12y^2 .\, dy .\, dx$$

$$= \int_0^1 \int_0^x x(12y^3)\, dy\, dx$$

$$= \int_0^1 x \left(\frac{\cancel{12}^{3} y^4}{\cancel{4}} \right) \Bigg]_0^x . dx$$

$$= \int_0^1 3x^4 . dx = \frac{3}{5} x^5 \Bigg]_0^1 = \frac{3}{5}$$

③

$$E\left[(X_1 - 2X_2 + X_3)^2 \right] = K$$

$$E(X_1) = E(X_2) = E(X_3) = 0.5.$$

Using law of lazy statistician

$$K = \left(E(X_1) - 2E(X_2) + E(X_3) \right)^2$$

$$= (0.5 - 1 + 0.5)^2 = 0$$

④

$$f(x) = e^{-x} , \quad x > 0$$

$$Y = e^{3x/4}.$$

$$E(Y) = \int_0^\infty e^{\frac{3x}{4}} . e^{-x} . dx$$

$$= \int_0^{\infty} e^{-\frac{x}{4}} . dx$$

$$-\frac{x}{4} = t$$

$$x = -4t$$

$$dx = -4dt$$

$$= \int_{-\frac{0}{4}}^{-\infty} e^{t} . (-4dt)$$

$$= \int_{-\infty}^{0} 4e^{t} . dt = 4e^{t}\Big]_{-\infty}^{0} = 4\left[e^{0} - e^{-\infty}\right]$$

$$= 4[1-0] = \underline{\underline{4}}$$

(5)

$$Y = g(x) = 2x^2 + 1$$

How many sided die? Assume $n = 6$.

$$\sum_{x=1}^{n} (2x^2+1)\left(\frac{1}{n}\right)$$

$$\sum_{x=1}^{6} (2x^2+1)\left(\frac{1}{n}\right) = \frac{1}{n}\left[2\left(\frac{n(n+1)(2n+1)}{6}\right) + n\right]_6$$

$$= \left(\frac{\cancel{2}(n+1)(2n+1)}{\cancel{6}3.} + 1\right)_6 = \frac{7(13)}{3} + 1 = 31.\bar{3}$$

⑥ $Y = (2x+1) \qquad f(x) = 2(1-x) \qquad 0 < x < 1.$

$$\int_0^1 2(1-x)$$
$$= 2 - \frac{2x^2}{2}\Big]_0^1 = 2-1$$
$$= 1 \checkmark$$

pdf ✓

$$E(Y) = \int_0^1 2(2x+1).(1-x).dx.$$

$$= \int_0^1 4x + 2 - 2x - 4x^2 . dx$$

$$= \int_0^1 2x + 2 - 4x^2 . dx = x^2 + 2x - \frac{4x^3}{3}\Big]_0^1$$

$$= 1 + 2 - \frac{4}{3} = 3 - \frac{4}{3} = \frac{5}{3}.$$ Ans.

⑦

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$E\left((ax+b)^n\right] = E\left[\sum_{k=0}^{n} \binom{n}{k} (ax)^{n-k} \times b^k\right].$$

$\because$ a, b are constants.

$$= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} . b^k E(X^{n-k})$$

Hence prooved.

(8) X = number of defective item in sample

p = proportion of defective items

Y = no. of good item. $= n - X$.

$$E(x - Y) = E(x - (n - x))$$

$$= E(2x - n) = E(2x) - n = 2pn - n$$

$$= n(2p - 1)$$

$$E(X - Y) = 20(2(0.05) - 1). = -18.$$

It is expected that 1 part $(5\% \times 20)$ will be defective and 19 parts will be good. So

E(defective part − good part) = −18.