

Sample L^AT_EX Document

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1. [Convolution of probability measures] [540]

$$f(x) := \int_{-\infty}^{\infty} g(x-y)h(y) dy. \quad (4.19)$$

Proof. Let X and Y be independent **random variables** that are η and τ -distributed, with densities $g(\cdot)$ and $h(\cdot)$. Then, their joint density is

$$f_{X,Y}(x,y) = g(x) \cdot h(y).$$

We need the density of $X + Y$.

$$\begin{aligned} F_z &= P(X + Y \leq z) = \iint_{x+y \leq z} f_{X,Y}(x,y) dx dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} g(x)h(y) dx dy \\ &= \int_{-\infty}^{\infty} h(y) \int_{-\infty}^{z-y} g(x) dx dy \end{aligned}$$

$$\begin{aligned} \frac{d}{dz} F_z &= f_z = \frac{d}{dz} \int_{-\infty}^{\infty} h(y) \int_{-\infty}^{z-y} g(x) dx dy \\ &= \int_{-\infty}^{\infty} h(y) \frac{d}{dz} \int_{-\infty}^{z-y} g(x) dx dy \\ &= \int_{-\infty}^{\infty} h(y)g(z-y) dy \\ &= \int_{-\infty}^{\infty} g(z-y)h(y) dy \end{aligned}$$

Now, suppose that $0 < |x - 4| < \delta$. Then,

Scratch Work

$$\begin{aligned}
|f(x) - 15| &= |(x^2 + x - 5) - 15|, \text{ by the definition of } f, & |f(x) - 15| &< \epsilon \\
&= |x^2 + x - 15| & |(x^2 + x - 5) - (15)| &< \epsilon \\
&= |(x - 4)(x + 5)| & |x^2 + x - 20| &< \epsilon \\
&= |x - 4||x + 5|, \text{ by properties of absolute value,} & |(x + 5)(x - 4)| &< \epsilon \\
&< \delta \cdot |x + 5|, \text{ by the assumption } |x - 4| < \delta, & |(x - 4)| \cdot |(x + 5)| &< \epsilon \\
&\leq \frac{\epsilon}{10}|x + 5|, \text{ since } \delta \leq \frac{\epsilon}{10}, & |x - 4| &< \frac{\epsilon}{|x + 5|} \\
&= \frac{\epsilon}{10}|(x - 4) + 9| & \delta = 1 \implies |x - 4| &< 1 \\
&\leq \frac{\epsilon}{10}(|x - 4| + |9|), \text{ by properties of absolute value,} & -1 < x - 4 < 1 \\
&< \frac{\epsilon}{10}(\delta + 9), \text{ since } |x - 4| < \delta, & 8 < x + 5 < 10 \\
&\leq \frac{\epsilon}{10}(1 + 9), \text{ since } \delta \leq 1, & |x + 5| &< 10 \\
&= \left(\frac{\epsilon}{10}\right)(10) = \epsilon
\end{aligned}$$

All together, this shows that for any $\epsilon > 0$, if we choose $\delta = \min \left\{ 1, \frac{\epsilon}{10} \right\}$, then $0 \leq |x - 4| < \delta$ implies that $|f(x) - 15| < \epsilon$. Thus, $\lim_{x \rightarrow 4} x^2 + x - 5 = 15$.

□

1.5.15 Evaluate the given limits of the piecewise defined function f .

$$f(x) = \begin{cases} x^2 - 1 & \text{if } x < -1 \\ x^3 + 1 & \text{if } -1 \leq x \leq 1 \\ x^2 + 1 & \text{if } x > 1 \end{cases}$$

(a) $\lim_{x \rightarrow -1^-} f(x)$

Since we are evaluating the limit as x approaches -1 from the left, we need to consider the form of the function for values of x that are less than -1 , $x^2 - 1$.

$$\begin{aligned} \lim_{x \rightarrow -1^-} f(x) &= \lim_{x \rightarrow -1^-} x^2 - 1 \\ &= (-1)^2 - 1, \text{ by Theorem 2,} \\ &= 0. \end{aligned}$$

(b) $\lim_{x \rightarrow -1^+} f(x)$

Since we are evaluating the limit as x approaches -1 from the right, we need to consider the form of the function for values of x that are greater than -1 , $x^3 + 1$.

$$\begin{aligned} \lim_{x \rightarrow -1^+} f(x) &= \lim_{x \rightarrow -1^+} x^3 + 1 \\ &= (-1)^3 + 1, \text{ by Theorem 2,} \\ &= 0. \end{aligned}$$

(c) $\lim_{x \rightarrow -1} f(x)$

Since $\lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^+} f(x) = 0$, $\lim_{x \rightarrow -1} f(x) = 0$ by Theorem 7.

(d) $f(-1)$

When $x = -1$, $f(x) = x^3 + 1$. So, $f(-1) = (-1)^3 + 1 = 0$.

(e) $\lim_{x \rightarrow 1^-} f(x)$

Since we are evaluating the limit as x approaches 1 from the left, we need to consider the form of the function for values of x that are less than (but near) 1 , $x^3 + 1$.

$$\begin{aligned} \lim_{x \rightarrow 1^-} f(x) &= \lim_{x \rightarrow 1^-} x^3 + 1 \\ &= (1)^3 + 1, \text{ by Theorem 2,} \\ &= 2. \end{aligned}$$

$$(f) \lim_{x \rightarrow 1^+} f(x)$$

Since we are evaluating the limit as x approaches 1 from the right, we need to consider the form of the function for values of x that are greater than (but near) 1, $x^2 + 1$.

$$\begin{aligned}\lim_{x \rightarrow 1^-} f(x) &= \lim_{x \rightarrow 1^+} x^2 + 1 \\ &= (1)^2 + 1, \text{ by Theorem 2,} \\ &= 2.\end{aligned}$$

$$(g) \lim_{x \rightarrow 1} f(x)$$

Since $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = 2$, $\lim_{x \rightarrow 1} f(x) = 2$ by Theorem 7.

$$(h) f(1)$$

When $x = 1$, $f(x) = x^3 + 1$. So, $f(1) = (1)^3 + 1 = 2$.

To help us visualize all of these limits, a graph of $y = f(x)$ is provided below.