ash

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[Stirling's approximation]

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{r_n}$$

여기서 r_n은 다음 부등식을 만족한다.

$$\frac{1}{12n+1} < r_n < \frac{1}{12n}$$

Proof. Let

$$S_n = \log(n!) = \sum_{p=1}^{n-1} \log(p+1)$$

and write

$$log(p+1) = A_p + b_p - \epsilon_p$$

where

$$\begin{aligned} A_p &= \int_p^{p+1} \log x \, dx, \\ b_p &= \frac{1}{2}\big(\log(p+1) - \log p\big), \\ \epsilon_p &= \int_p^{p+1} \log x \, dx - \frac{1}{2}\big[\log(p+1) + \log p\big]. \end{aligned}$$

then

$$\begin{aligned} S_n &= \sum_{p=1}^{n-1}(A_p + b_p - \epsilon_p) = \int_1^n \log(x)dx + \frac{1}{2}\log(n) - \sum_{p=1}^{n-1}\epsilon_p \\ &= (n + \frac{1}{2}\log(n) - n + 1) - \sum_{p=1}^{n-1}\epsilon_p, \\ \epsilon_p &= \frac{2p+1}{2}\log(\frac{p+1}{p}) - 1. \end{aligned}$$

using Taylor expansion of $\log(1+x)$, we have

$$\log(\frac{1+x}{1-x}) = 2(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots)$$

valid for $|x| < 1$, let $x = (2p+1)^{-1}$, so that $\frac{1+x}{1-x} = \frac{p+1}{p}$.

$$\epsilon_p = \frac{1}{3(2p+1)^2} + \frac{1}{5(2p+1)^4} + \cdots$$

Upper bound of ϵ_p:
Since the coefficients decrease ($\frac{1}{3} > \frac{1}{5} > \frac{1}{7} > \cdots$),

$$\begin{aligned}\epsilon_p <& \frac{1}{3}x^2(1 + x^2 + x^4 + \cdots) \\ =& \frac{x^2}{3} \cdot \frac{1}{1-x^2} = \frac{1}{12}(\frac{1}{p} - \frac{1}{p+1})\end{aligned}$$

lower bound of ϵ_p:

$$\epsilon_p > \frac{x^2}{3}(1 + \frac{x^2}{3} + \frac{x^4}{3^2}) = \frac{x^2}{3} \cdot \frac{1}{1 - \frac{x^2}{3}} = \frac{1}{12p^2 + 12p + 2} > \frac{1}{12p^2 + 14p + \frac{13}{12}}$$

rewrite:

$$\sum_{p=1}^{n-1} \epsilon_p = \underbrace{\sum_{p=1}^{\infty} \epsilon_p}_{B} - \underbrace{\sum_{p=n}^{\infty} \epsilon_p}_{r_n}$$

$$\frac{1}{13} < B < \frac{1}{12}$$

$$\frac{1}{12n+1} < r_n < \frac{1}{12n}$$

rewrite S_n:

$$S_n = \left(n + \frac{1}{2}\right) \log n - n + 1 - B + r_n$$

so,

$$\begin{aligned}n! &= e^{S_n} \\ &= e^{1-B} \cdot n^{n+\frac{1}{2}} e^{-n} \cdot e^{r_n}\end{aligned}$$

and $e^{1-B} = \sqrt{2\pi}$.

$\square$

2 CLT

Theorem 1 (Central Limit Theorem)**.** *assume that the random variables $X_1, X_2, \ldots$ are i.i.d with expectation μ and variance σ^2. Then:*

$$P\left(\frac{X_1 + X_2 + \cdots + X_n - \mu n}{\sigma\sqrt{n}} \leq x\right) \to P(Z \leq x)$$

Proof. let

$$T_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}$$

we will show $\phi_{T_n}(t) \to \phi_Z(t)$.
let $Y_i = \frac{X_i - \mu}{\sigma}$, then $E[Y_i] = 0$, $Var(Y_i) = 1$

$$\begin{aligned}
\phi_{T_n}(t) &= E\left[e^{tT_n}\right] \\
&= E\left[e^{\frac{t}{\sqrt{n}}Y_1 + \cdots + Y_n}\right] \\
&= E\left[e^{\frac{t}{\sqrt{n}}Y}\right]^n \\
&= E\left[1 + \frac{t}{\sqrt{n}}Y + \frac{t^2}{2n}Y^2 + \cdots\right]^n \\
&= E\left[1 + 0 + \frac{t^2}{2n}Y^2\right]^n \\
&= \left(1 + \frac{t^2}{2n}E[Y^2]\right)^n \\
&= (1 + \frac{t^2}{2n})^n && \text{since } E[Y^2] = Var[Y] + (E[Y])^2 \\
&= e^{\frac{t^2}{2}} \quad \text{as } n \to \infty \\
&= \phi_Z(t)
\end{aligned}$$

$\square$

3 Markov Chain