PHYS 414 Final Project

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This document is the report of my term project in PHYS414: Computational Physics at Koç University. In this project, the structures of various types of stars in Newtonian gravity, general relativity (GR) are calculated. For numerical parts Python 3.0 and Wolfram Mathematica 12.0 are used. The project files can be found in this link.

I. INTRODUCTION

This project calculates the structures of various types of stars in Newtonian gravity, general relativity (GR) using the problem set titled "Stars from Newton to Einstein (and maybe beyond)" as a guide. In this project, there are two main parts as follows: Newton and Einstein. In the Newton part, we focused on the White dwarfs (WDs) which are the end stages of relatively low-mass stars. And in the Einstein part, we looked at neutron stars (NSs).

II. NEWTON

Part a

We consider the hydrostatic equilibrium of stars in Newtonian gravity. In this system, we have the following ODEs:

$$\frac{dm(r)}{dr} = 4\pi r^2 \rho(r) \tag{1}$$

$$\frac{dP(r)}{dr} = -\frac{Gm(r)\rho(r)}{r^2} \tag{2}$$

Mutliplying both sides of the Eq. (2) with $\frac{r^2}{\rho(r)}$ and differentiating w.r.t. r we get:

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -G\frac{dm}{dr} \tag{3}$$

Using Eq. (1) and (3):

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho \tag{4}$$

We assume a polytropic EOS such that $P = K\rho^{1+\frac{1}{n}}$ where $\rho = \rho_c\theta^n$. Using these and chain rule in Eq. (4):

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 K\rho_c^{1/n}(n+1)\frac{d\theta}{dr}\right) = -4\pi G\rho_c\theta^n \tag{5}$$

By defining $\alpha^2 = \frac{(n+1)K\rho_c^{1/n-1}}{4\pi G}$ and substituting $r = \alpha\xi$ in Eq. (5):

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) + \theta^n = 0 \tag{6}$$

which is the Lane-Emden equation.

Observe that $\theta(0) = 1$ Using Mathematica, for $n = 0$ we get the $\theta(\xi) = 1 - \frac{\xi^2}{6}$. When we look for $n = 1$, this time, we get $\theta(\xi) = 1 - \frac{\xi^2}{6} + \frac{\xi^4}{120} + ...$ For $n > 1$, Mathematica returned no analytic solutions. Also, from the solution for $n = 0$ and $n = 1$, we see that $\theta'(0) = 0$.

To find the mass, let $R = \alpha\xi_n$:

$$M = \int_0^R 4\pi r^2\rho dr = 4\pi\rho_c\alpha^3\int_0^{\xi_n}\xi^2\theta^n d\xi \tag{7}$$

and using Lande-Emden equation, we know that $\xi^2\theta^n = -\frac{d}{d\xi}(\xi^2\frac{d\theta}{d\xi})$. Hence:

$$M = -4\pi\rho_c\alpha^3\int_0^{\xi_n}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)d\xi \tag{8}$$

$$= -4\pi\rho_c\alpha^3\xi_n^2\frac{d\theta}{d\xi}\Big|_{\xi_n} \tag{9}$$

$$= 4\pi\rho_c R^3\left(\frac{-\theta'(\xi_n)}{\xi_n}\right) \tag{10}$$

gives us M.

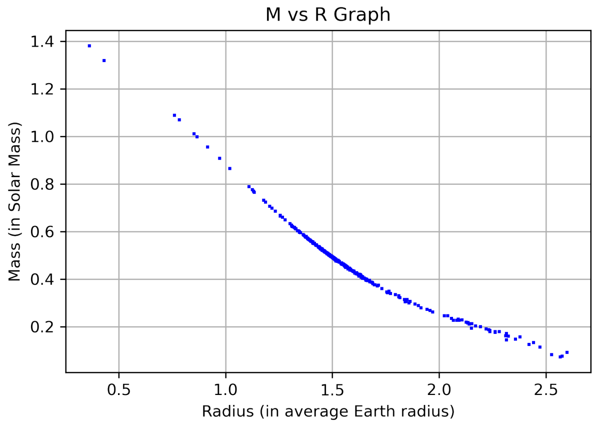
Further, solving Eq. (9) for ρ_c and usin α expression, we can get:

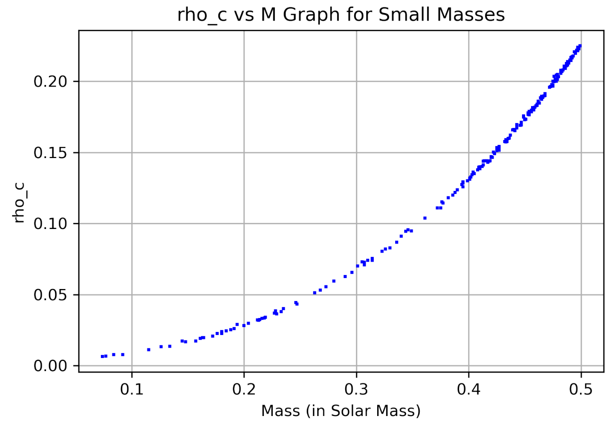
$$M = \beta R^{\frac{3-n}{1-n}} \tag{11}$$

where $\beta = -4\pi(\frac{n+1}{4\pi}\frac{K}{G})^{\frac{n}{n+1}}\xi_n^{\frac{n+1}{n-1}}\theta'(\xi_n)$ is the proportionality constant.

Part b

We wrote a function to read the white dwarf data file. The M vs R plot below is plotted using solar masses and average Earth radii as units.





Part c

Using Mathematica we obtained the series expansion for P s.t. $P = \frac{8Cx^5}{5} + ...$ where $x = (\frac{\rho}{D})^{1/q}$. Keeping only the leading term, defining $K_* = \frac{8C}{5D^{5/q}}$ and $n_* = q/(5-q)$ we see that the equation becomes $P = K_* \rho^{1+\frac{1}{n_*}}$.

To find the unknowns we made a fit for Eq. (11) using the data points with mass < 0.5 solar mass. We found the following:

$$n_* = 1.544164172849014,$$
$$\beta = 1.5077564207770797$$
$$q = 3.03471802120345$$

Note that we know q is an integer from the theory; hence, we will take $q = 3$ and $n_* = 1.5$ to satisfy that condition.

As, now, we know the index n, we solved the Lane-Emden equation for $n_* = 1.5$ and found:

$$\xi_n = 3.6500999999999664$$
$$\theta'(\xi_n) = -0.1946172255527264$$

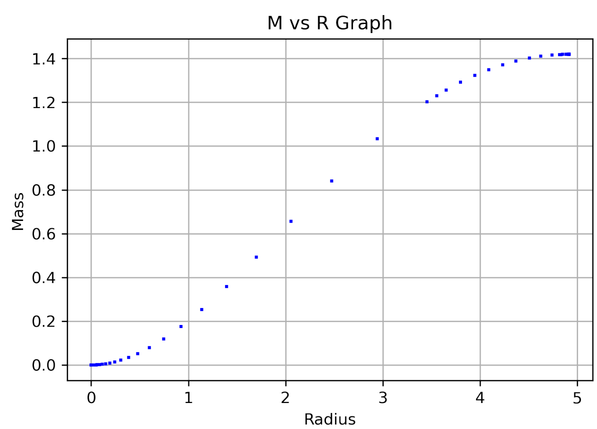
Then, using Eq. (10) we get the central density ρ_c of the WDs we used in the fit, and plotted them with respect to M. The ρ_c vs M plot can be found below. Also, as we, now, know ξ_n and $\theta'(\xi_n)$ we can find K_* using the relation for β above.

III. EINSTEIN

Part a

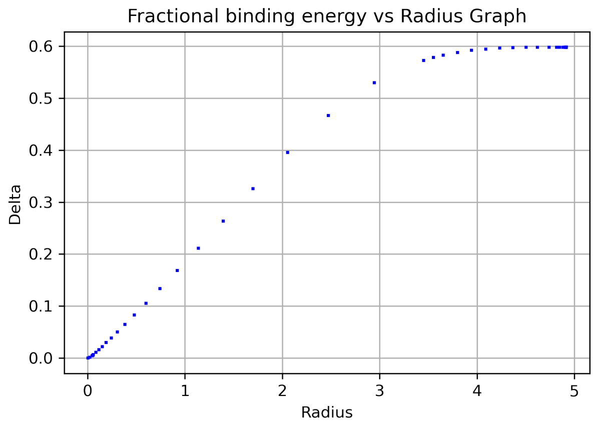
To solve the Tolman-Oppenheimer-Volko (TOV) equations, we wrote a function. To obtain the M vs R curve for NSs, we integrated the TOV equations from the center out and stopped at where ρ vahishes, i.e. $\rho = 0$. The initial conditions were $m(0) = 0$, and $P(0) = P_c$.

The M vs R curve for $\rho_c = 1$ (chosen for simplicity) is below:



Part b

In this part, we looked at the baryonic mass. Using the same function as part a, with a small adjustment, we found M_P values. Defining $\Delta = \frac{M_P - M}{M}$, i.e. the fractional binding energy, and knowing the M values from part a, we get the graph below:



Part e

To find $\nu(r > R)$, we simply integrated ν' from R to r using Mathematica. The result is following:

$$\nu(r > R) = \nu(R) + ln(1 - \frac{2M}{r}) - ln(1 - \frac{2M}{R}) \quad (12)$$