Variação linear da velocidade com a profundidade ①

$$v(z) = v_0 + az \qquad a \geq 0$$

No turning point,

$$p\,v(z) = p(v_0 + az) = 1$$

Então,

$$z_{TP} = \frac{1 - p v_0}{p a}$$

Partindo da equação do tempo de trânsito T,

$$T = 2\int_0^H \frac{dz}{v(z)\sqrt{1 - p^2 v^2(z)}}$$

$$T = \frac{2}{a}\int_0^H \frac{a p\, dz}{p v\sqrt{1 - p^2 v^2}} = \frac{2}{a}\int_{p v_0}^{p V_H} \frac{dy}{y\sqrt{1 - y^2}}$$

$$y = p(v_0 + az) \qquad dy = pa\, dz \qquad y(0) = p v_0$$

$$y(H) = p v_0 + paH = p V_H$$

A integral, na tabela de integrais

$$\int \frac{dy}{y\sqrt{a^2 - y^2}} = -\frac{1}{a}\ln\left|\frac{a + \sqrt{a^2 - y^2}}{y}\right|$$

Para $a^2 = 1$,

$$T = -\frac{2}{a} \ln\left[\frac{1+\sqrt{1-y^2}}{y}\right]_{y=pv_0}^{y=pV_H}$$

$$\left(\cancel{= -\frac{2}{a}\left\{\frac{1+\sqrt{1-p^2V_H^2}}{pV_H} - \right.}\right)$$

$$T = -\frac{2}{a}\left\{\ln\left[\frac{1+\sqrt{1-p^2V_H^2}}{pV_H}\right] - \ln\left[\frac{1+\sqrt{1-p^2v_0^2}}{pv_0}\right]\right\}$$

$$T = \frac{2}{a}\left\{\ln\left[\frac{1+\sqrt{1-p^2v_0^2}}{pv_0} \cdot \frac{pv_H}{1+\sqrt{1-p^2v_H^2}}\right]\right\}$$

$$T = \frac{2}{a}\ln\left[\frac{v_H}{v_0} \cdot \frac{1+\sqrt{1-p^2v_0^2}}{1+\sqrt{1-p^2v_H^2}}\right]$$

Para a distância horizontal percorrida pelo raio,

$$X = \int_0^H \frac{p\,v(z)}{\sqrt{1-p^2v^2(z)}}\,dz$$

corresponde a metade da distância. Se,

$$X = \int_0^H \frac{pv}{\sqrt{1-p^2v^2}}\,dz = \frac{1}{ap}\int_0^H \frac{ap^2v\,dz}{\sqrt{1-p^2v^2}} = \frac{1}{ap}\int_0^H \frac{ap^2(v_0+az)}{\sqrt{1-p^2v^2}}\,dz$$

$$X = \frac{-1}{ap} \sqrt{1 - p^2(v_0 + az)^2} \Big|_0^H$$

Pois, para,

$$X = \frac{1}{ap} \int_0^H \frac{a p^2 v}{\sqrt{1 - p^2 v^2}} dz = \frac{1}{ap} \int_0^H \frac{a p^2 (v_0 + az)}{\sqrt{1 - p^2(v_0 + az)^2}} dz$$

Se, $u = \sqrt{1 - p^2(v_0 + az)^2}$

$$du = \frac{d}{dz} \sqrt{1 - p^2(v_0 + az)^2} \, dz$$

$$du = \frac{1}{2} \frac{1}{\sqrt{1 - p^2(v_0 + az)^2}} \cdot -p^2 \cdot 2(v_0 + az) \cdot a \, dz$$

Assim,

$$du = \frac{-p^2 a (v_0 + az) dz}{\sqrt{1 - p^2(v_0 + az)^2}}$$

Deste modo,

$$X = \frac{-1}{ap} \int du = -\frac{1}{ap} u \Big|_0^H = \frac{-1}{ap} \sqrt{1 - p^2(v_0 + az)^2} \Big|_0^H$$

Retomando,

$$X = -\frac{1}{ap}\sqrt{1-p^2(v_0+az)^2}\Bigg|_0^H$$

$$X = -\frac{1}{ap}\sqrt{1-p^2(v_0+aH)^2} + \frac{1}{ap}\sqrt{1-p^2v_0^2}$$

$$X - \frac{1}{ap}\sqrt{1-p^2v_0^2} = -\frac{1}{ap}\sqrt{1-p^2(v_0+aH)^2}$$

Elevando os dois lados da expressão ao quadrado,

$$\left[X - \frac{1}{ap}\sqrt{1-p^2v_0^2}\right]^2 = \frac{1}{a^2p^2}\left(1-p^2(v_0+aH)^2\right)$$

O lado direito da expressão,

$$\frac{1}{a^2p^2}\left(1-p^2(v_0+aH)^2\right) = \frac{1}{a^2p^2} - \frac{1}{a^2}(v_0+aH)^2$$

$$= \frac{1}{a^2p^2} - \frac{a^2}{a^2}\left(\frac{v_0}{a}+H\right)^2 = \frac{1}{a^2p^2} - \left(\frac{v_0}{a}+H\right)^2$$

Pois,

$$(v_0+aH)^2 = \left[a\left(\frac{v_0}{a}+H\right)\right]^2 = a^2\left(\frac{v_0}{a}+H\right)^2$$

Deste modo,

Variação linear da velocidade com a profundidade ③

$$\left[x - \frac{1}{ap}\sqrt{1-p^2v_0^2}\right]^2 = \frac{1}{a^2p^2} - \left[\frac{v_0}{a} + H\right]^2$$

$$\left[x - \frac{1}{ap}\sqrt{1-p^2v_0^2}\right]^2 + \left[\frac{v_0}{a} + H\right]^2 = \frac{1}{a^2p^2}$$

Esta é a equação de um arco circular de raio $1/ap$, onde,

$$(x_c, z_c) = \left(\frac{\sqrt{1-p^2v_0^2}}{ap}, -\frac{v_0}{a}\right)$$

Para o raio ascendente,

$$x(z) = x(H) + \int_H^z \frac{dx}{dz'}\,dz'$$

Se,

$$\int_H^0 \frac{dx}{dz'}\,dz' = \int_H^z \frac{dx}{dz'}\,dz' + \int_z^0 \frac{dx}{dz'}\,dz'$$

$$\int_H^z \frac{dx}{dz'}\,dz' = \int_H^0 \frac{dx}{dz'}\,dz' - \int_z^0 \frac{dx}{dz'}\,dz'$$

Então,

$$X(z) = X(H) + \int \frac{dx}{dz'}\, dz' = X(H) + \int_H^0 \frac{dx}{dz'}\, dz' - \int_z^0 \frac{dx}{dz'}\, dz'$$

$$X(z) = 2X(H) + \int_0^z \frac{1}{ap} \cdot \frac{ap^2(v_0 + az')\, dz'}{\sqrt{1 - p^2(v_0+az')^2}}$$

Assim,

$$X(z) = 2X(H) + \frac{1}{ap}\sqrt{1 - p^2(v_0+aH)^2} - \frac{1}{ap}\sqrt{1 - p^2 v_0^2}$$

$$\left[X(z) - 2X(H) + \frac{1}{ap}\sqrt{1-p^2v_0^2}\right]^2 = \frac{1}{a^2p^2}\left(1 - p^2(v_0+aH)^2\right)$$

$$\left[X(z) - 2X(H) + \frac{1}{ap}\sqrt{1-p^2v_0^2}\right] = \frac{1}{a^2p^2} - \left[\frac{v_0}{a} + H\right]^2$$

$$\left[X(z) - 2X(H) + \frac{1}{ap}\sqrt{1-p^2v_0^2}\right]^2 + \left[\frac{v_0}{a} + H\right]^2 = \frac{1}{a^2p^2}$$

O raio deste arco circular continua $1/ap$, o centro,

$$(X_c, Z_c) = \left(2X(H) - \frac{1}{ap}\sqrt{1-p^2v_0^2},\ -\frac{v_0}{a}\right)$$