Cap.3 – Solution of the transport equation

3.1. Ray coordinates

$$d\tau = \frac{ds}{c} = \frac{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}\; d\sigma}{c}$$

$$\frac{d}{ds} = \frac{1}{c}\frac{d}{d\tau} = \frac{1}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}\frac{d}{d\sigma}$$

$$\vec{r} = \vec{r}(\tau,\alpha,\beta)$$

$$\vec{r} = x(\tau,\alpha,\beta)\,\hat{i} + y(\tau,\alpha,\beta)\,\hat{j} + z(\tau,\alpha,\beta)\,\hat{k}$$

Definition: We say that τ, α, β form the ray coordinates in a domain Ω, where the central ray is regular.

3.2. Auxiliary formulas

Determinantes funcionais

$$\begin{cases} x = x(\xi,\nu,\zeta) \\ y = y(\xi,\nu,\zeta) \\ z = z(\xi,\nu,\zeta) \end{cases}$$

É possível encontrar? $\begin{cases} \xi = \xi(x,y,z) \\ \nu = \nu(x,y,z) \\ \zeta = \zeta(x,y,z) \end{cases}$

Admitindo $x = y = z = 0$ em $\xi = \nu = \zeta = 0$

$$x \simeq \frac{\partial x}{\partial \xi}\xi + \frac{\partial x}{\partial \nu}\nu + \frac{\partial x}{\partial \zeta}\zeta$$

$$y \simeq \frac{\partial y}{\partial \xi}\xi + \frac{\partial y}{\partial \nu}\nu + \frac{\partial y}{\partial \zeta}\zeta$$

$$z \simeq \frac{\partial z}{\partial \xi}\xi + \frac{\partial z}{\partial \nu}\nu + \frac{\partial z}{\partial \zeta}\zeta$$

O determinante do sistema acima é

$$\Delta = \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \nu} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \nu} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \nu} & \frac{\partial z}{\partial \zeta} \end{vmatrix}$$

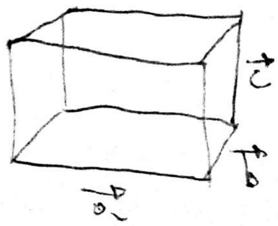
Este determinante é chamado de determinante funcional e é denotado por

$$\Delta = \frac{D(x,y,z)}{D(\xi,\nu,\zeta)}$$

Este determinante é diferente de zero se o sistema possuir solução

Assim, o determinante funcional Δ regula a possibilidade do sistema ter solução com respeito a ξ, ν, ζ.

- Interpretação geométrica do determinante funcional



$$\text{Volume} = |\vec{a} \times (\vec{b} \cdot \vec{c})|$$

$$d_\xi \vec{r} = \frac{\partial \vec{r}}{\partial \xi} d\xi = \left(\frac{\partial x}{\partial \xi} \hat{i} + \frac{\partial y}{\partial \xi} \hat{j} + \frac{\partial z}{\partial \xi} \hat{k} \right) d\xi$$

$$d_\nu \vec{r} = \frac{\partial \vec{r}}{\partial \nu} d\nu = \left(\frac{\partial x}{\partial \nu} \hat{i} + \frac{\partial y}{\partial \nu} \hat{j} + \frac{\partial z}{\partial \nu} \hat{k} \right) d\nu$$

$$d_\zeta \vec{r} = \frac{\partial \vec{r}}{\partial \zeta} d\zeta = \left(\frac{\partial x}{\partial \zeta} \hat{i} + \frac{\partial y}{\partial \zeta} \hat{j} + \frac{\partial z}{\partial \zeta} \hat{k} \right) d\zeta$$

Onde $\vec{r} = \vec{r}(\xi, \nu, \zeta)$. Assim, o elemento dV:

$$dV = \left| (d_\xi \vec{r} \times (d_\nu \vec{r} \cdot d_\zeta \vec{r}) \right| = \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \nu} & \frac{\partial y}{\partial \nu} & \frac{\partial z}{\partial \nu} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{vmatrix} d\xi \, d\nu \, d\zeta$$

$$dV = \left| \frac{D(x,y,z)}{D(\xi,\nu,\zeta)} \right| d\xi \, d\nu \, d\zeta = \Delta \, d\xi \, d\nu \, d\zeta$$

O determinante funcional Δ aparece como um fator escalar para o elemento dV no novo sistema de coordenadas ξ, ν, ζ. Esta é a interpretação geométrica do determinante funcional. Este fator escalar é chamado Jacobiano.

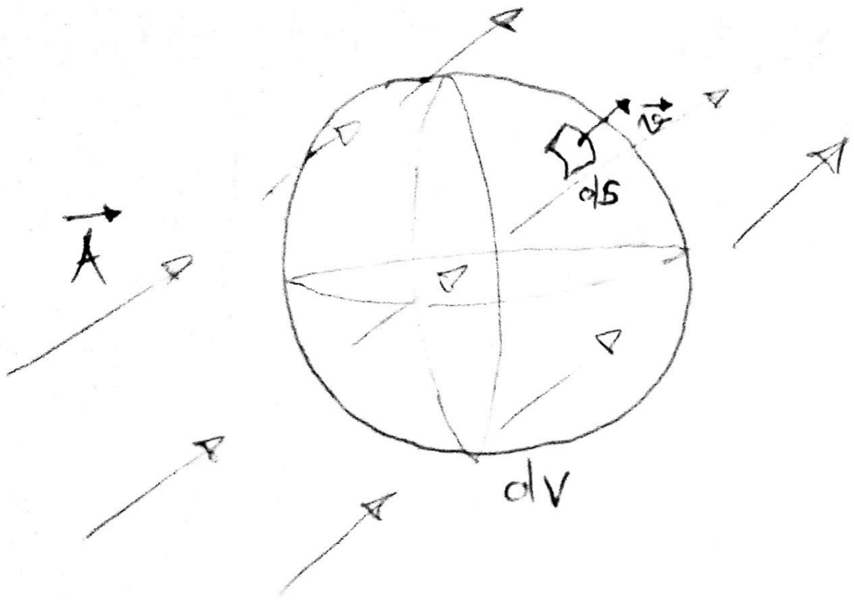
• <u>Definição de divergência</u>

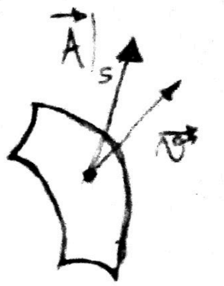
$$\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \qquad \vec{A} = \vec{A}(x,y,z)$$

Definição: Considere um pequeno volume circundado pela superfície S. Denote o volume por ΔV e o vetor $\vec{v}$ seja o vetor unitário normal a S. Então $(\vec{A}\cdot\vec{v})|_S$ é o produto escalar dos vetores $\vec{A}$ e $\vec{v}$ em S. Assim,

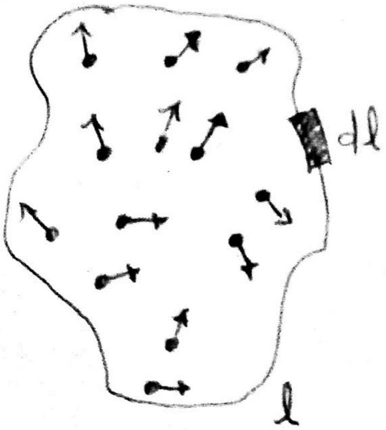
$$\nabla\cdot\vec{A} = \lim_{\Delta V\to 0} \frac{\iint_{(S)} (\vec{A}\cdot\vec{v})|_S \, dS}{\Delta V}$$

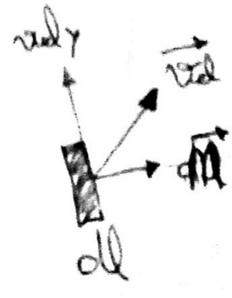
→ Fluxo do campo vetorial $\vec{A}$ através de S





$\vec{A}\cdot\vec{v}|_s$



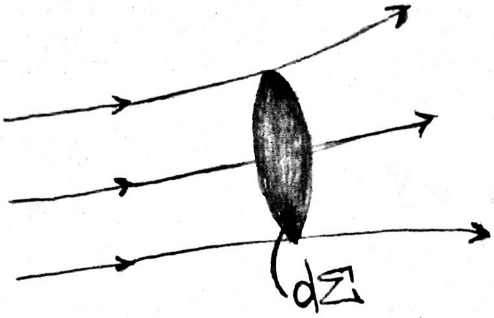


$vel_x = \vec{vel}\cdot\vec{n}$

$$\Phi_l = \oint_l \vec{vel}\cdot\vec{n}\, dl$$

Fluxo de partículas para fora de l.

3.3. Geometrical spreading



$\vec{r} = \vec{r}(\gamma, \alpha, \beta)$

γ é o eikonal

α e β são parâmetros que especificam um raio

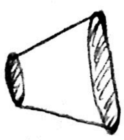
Cross section Ray tube by Wave front $\gamma = \gamma_0$

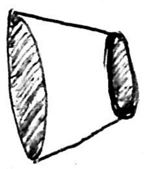
$$\alpha_0 \leq \alpha \leq \alpha_0 + d\alpha \ , \quad \beta_0 \leq \beta \leq \beta_0 + d\beta$$

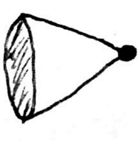
Formam um tubo de raios

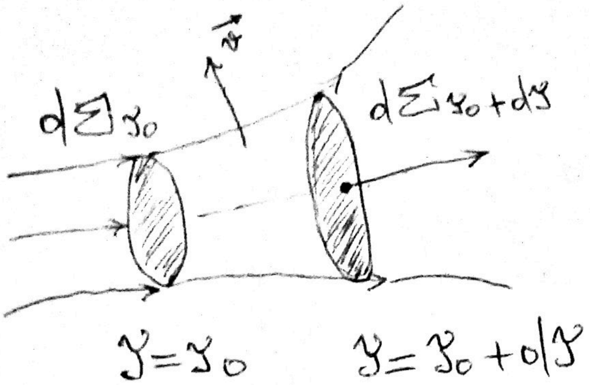
$$d\Sigma = \left| d_\beta \vec{r} \times d_\alpha \vec{r} \right| = \left| \frac{\partial \vec{r}}{\partial \beta} \times \frac{\partial \vec{r}}{\partial \alpha} \right| d\alpha \, d\beta = J \, d\alpha \, d\beta$$

Onde: $J = \left| \dfrac{\partial \vec{r}}{\partial \beta} \times \dfrac{\partial \vec{r}}{\partial \alpha} \right|$ Espalhamento Geométrico

 J cresce

 J decresce

 $J = 0$ Ponto de Foco



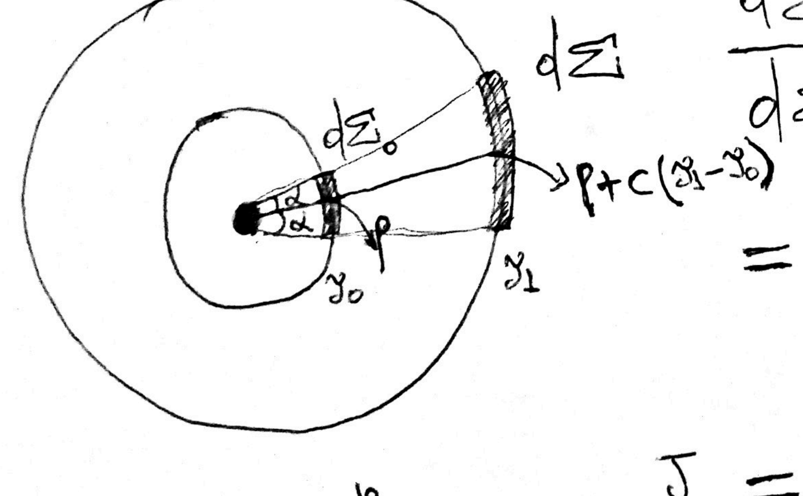
$$d\gamma = \frac{ds}{c}$$

$$dV = \left|\frac{D(x,y,z)}{D(\gamma,\alpha,\beta)}\right| d\gamma\, d\alpha\, d\beta = \left|\frac{D(x,y,z)}{D(\gamma,\alpha,\beta)}\right| \frac{1}{c}\, ds\, d\alpha\, d\beta$$

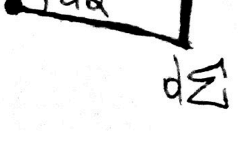
$$dV = d\Sigma\, ds \qquad d\Sigma = J\, d\alpha\, d\beta$$

$$dV = J\, d\alpha\, d\beta\, ds$$ Onde: $$\boxed{J = \frac{1}{c}\left|\frac{D(x,y,z)}{D(\gamma,\alpha,\beta)}\right|}$$ //

Exemplo (meio homogêneo):



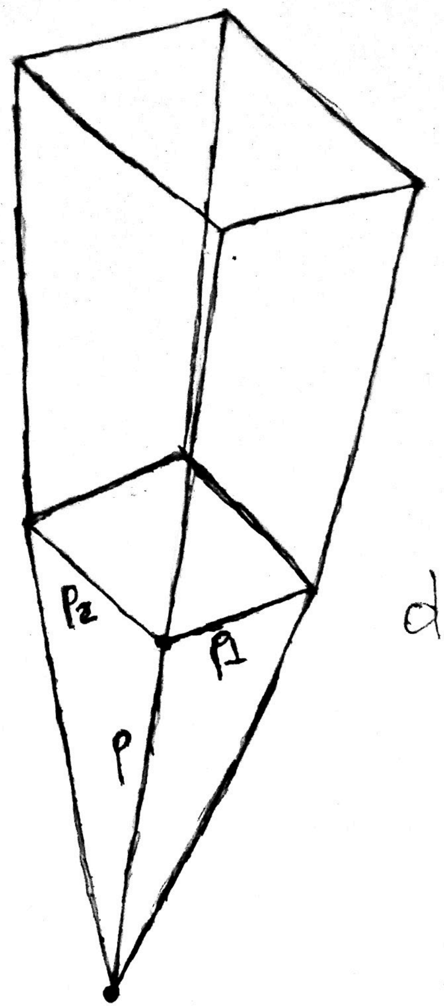
$$\frac{d\Sigma}{d\Sigma_0} \equiv \frac{J\, d\alpha}{J_0\, d\alpha} = \frac{[\rho + c(\gamma_1-\gamma_0)]\, d\alpha}{\rho\, d\alpha}$$



$$\frac{J}{J_0} = \frac{\rho + c(\gamma-\gamma_0)}{\rho}$$

$$\boxed{J(\gamma) = J(\gamma_0)\frac{\rho + c(\gamma-\gamma_0)}{\rho}}$$

(3D)



$$d\Sigma = (\rho_1 + d\rho_1)(\rho_2 + d\rho_2)$$

$$d\Sigma_0 = \rho_1 \rho_2$$

$$\frac{d\Sigma}{d\Sigma_0} = \frac{J}{J_0} = \frac{(\rho_1 + c(\tau-\tau_0))(\rho_2 + c(\tau-\tau_0))}{\rho_1 \rho_2}$$

$$J(\tau) = J_0(\tau_0)\,\frac{[\rho_1 + c(\tau-\tau_0)][\rho_2 + c(\tau-\tau_0)]}{\rho_1 \rho_2}$$

3.4. Solução das equações de transporte

$$2\,\nabla\tau\cdot\nabla A_0 + A_0\nabla^2\tau = 0$$

Esta é uma equação diferencial parcial, mas pode ser apresentada como uma equação diferencial ordinária ao longo do raio. Relembrando a definição da derivada ao longo da curva, para,

$$x = x(s),\ y = y(s),\ z = z(s)$$

uma curva em 3D (raio) e considerando $f = f(x,y,z)$. A derivada da função ao longo da curva

$$\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial z}\frac{dz}{ds} = \nabla f\cdot\vec{t}$$

Onde $\vec{t}$ é um vetor unitário tangente a curva.

Se,

$$\nabla\tau = \vec{t}/c$$

Então,

$$\nabla\tau\cdot\nabla A_0 = \nabla A_0\cdot\frac{\vec{t}}{c} = \frac{1}{c}\nabla A_0\cdot\vec{t} = \frac{1}{c}\frac{dA_0}{ds}$$

Ao invés de s, nós podemos utilizar o eikonal τ como parâmetro ao longo do raio. Neste caso,

$$ds = c\,d\tau \longrightarrow \frac{d}{ds} = \frac{1}{c}\frac{d}{d\tau}$$

Assim,

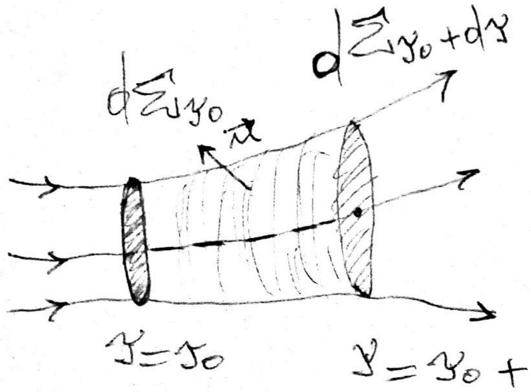
$$\nabla\gamma\cdot\nabla A_0 = \frac{1}{c}\frac{dA_0}{ds} = \frac{1}{c^2}\frac{dA_0}{d\gamma}$$

Finalmente,

$$2\nabla\gamma\cdot\nabla A_0 + A_0\nabla^2\gamma = \frac{2}{c^2}\frac{dA_0}{d\gamma} + A_0\nabla^2\gamma = 0$$

Esta é ~~(a pr)~~ uma equação diferencial ordinária de primeira ordem ao longo do raio. Consequentemente, teremos de calcular o $\nabla^2\gamma$ ao longo do raio.

Sabemos que $\nabla^2\gamma = \nabla\cdot\nabla\gamma$. Assim, o campo vetorial é formado por $\nabla\gamma = \vec{t}/c$. Então, tomando um volume ΔV formado por um pedaço do tubo de raios e $\vec{n}$ é o vetor unitário normal a superfície S que envolve ΔV.



Na lateral da superfície S,

$$\vec{A}\cdot\vec{n} = \frac{1}{c}\vec{t}\cdot\vec{n} = 0$$

Na base,

$$\vec{A}\cdot\vec{n} = \frac{1}{c}\vec{E}\cdot\vec{n} = \frac{-1}{c}$$

No topo,

$$\vec{A}\cdot\vec{n} = \frac{1}{c}\vec{E}\cdot\vec{n} = \frac{1}{c}$$

Então,

$$\iint_{(S)} \vec{A}\cdot\hat{n}\, dS \cong -\frac{1}{c} d\Sigma_{\gamma_0} + \frac{1}{c} d\Sigma_{\gamma_0+d\gamma}$$

$$\cong \left(\frac{1}{c} J \Big|_{\gamma+d\gamma} - \frac{1}{c} J \Big|_{\gamma} \right) d\alpha\, d\beta$$

lembrando, $\quad d\Sigma = J\, d\alpha\, d\beta$

Então, para o elemento de volume dV,

$$dV = \left| \frac{D(x,y,z)}{D(\gamma,\alpha,\beta)} \right| d\gamma\, d\alpha\, d\beta = c\, J\, d\gamma\, d\alpha\, d\beta$$

Pois,

$$J = \frac{1}{c} \left| \frac{D(x,y,z)}{D(\gamma,\alpha,\beta)} \right|$$

Então,

$$\nabla^2\gamma = \lim_{\Delta V \to 0} \frac{\iint_{(S)} \vec{A}\cdot\hat{n}\, dS}{\Delta V}$$

$$\nabla^2\gamma = \lim_{\Delta V \to 0} \frac{\left(\frac{1}{C} J|_{\gamma+d\gamma} - \frac{1}{C} J|_{\gamma}\right) d\alpha\, d\beta}{C J\, d\gamma\, d\alpha\, d\beta} = \lim_{\Delta V \to 0} \frac{\left(\frac{1}{C} J|_{\gamma+d\gamma} - \frac{1}{C} J|_{\gamma}\right)}{C J\, d\gamma}$$

Finalmente,

$$2\nabla\gamma \cdot \nabla A_0 + A_0 \nabla^2\gamma = 0 \Rightarrow \frac{2}{C^2}\frac{dA_0}{d\gamma} + \frac{A_0}{CJ}\frac{d}{d\gamma}\left(\frac{J}{C}\right) = 0$$

Pois,

$$\frac{d}{d\gamma}\left(\frac{J}{C}\right) = \lim_{d\gamma \to 0} \frac{\left(\frac{1}{C} J|_{\gamma+d\gamma} - \frac{1}{C} \gamma|_{\gamma}\right)}{d\gamma}$$

Assim,

$$\frac{2}{A_0 C^2}\frac{dA_0}{d\gamma} = -\frac{1}{CJ}\frac{d}{d\gamma}\left(\frac{J}{C}\right) \Rightarrow \frac{1}{A_0}\frac{dA_0}{d\gamma} = -\frac{1}{2}\frac{C^2}{C\gamma}\frac{d}{d\gamma}\left(\frac{J}{C}\right)$$

$$\frac{1}{A_0}\frac{dA_0}{d\gamma} = -\frac{1}{2}\frac{C}{\gamma}\frac{d}{d\gamma}\left(\frac{J}{C}\right)$$

Se,

$$\frac{d}{dx}(\ln u(x)) = \frac{u'(x)}{u(x)}$$

Então,

(3.4.)(cont.)

$$\frac{1}{A_0}\frac{d}{d\gamma}A_0 = -\frac{1}{2}\frac{c}{J}\frac{d}{d\gamma}\left(\frac{J}{c}\right)$$

$$\ln A_0 = -\frac{1}{2}\ln\left(\frac{J}{c}\right) + \ln C_1$$

Pois,

$$\frac{d}{d\gamma}(\ln A_0) = \frac{d}{d\gamma}\ln(J/c)\cdot -\frac{1}{2}$$

$$\int \frac{d}{d\gamma}(\ln A_0)\,d\gamma = \int \frac{d}{d\gamma}\ln(J/c)\cdot -\frac{1}{2}\,d\gamma$$

$$\ln A_0 = -\frac{1}{2}\ln(J/c) + \ln C_1$$

$$\ln A_0 = -\frac{1}{2}\ln\left[(J/c)\cdot C_1\right] = \ln\left[(J/c)\cdot C_1\right]^{-1/2}$$

$$A_0 = \left(\frac{J}{c}\right)^{-1/2}\cdot C_1^{-1/2} = \frac{C_2}{\sqrt{J/c}}$$

Por fim,

$$A_0 = \frac{\psi_0(\alpha,\beta)}{\sqrt{J/c}}$$

Onde $\psi_0(\alpha,\beta)$ é uma constante de integração em relação a γ, assim depende apenas de α, β!

A equação de transporte para ordens maiores,

$$2(\nabla\gamma \cdot \nabla A_{n+1}) + A_{n+1}\nabla^2\gamma = \nabla^2 A_n \qquad n = -1, 0, 1, \dots$$

$$A_{-1} \equiv 0$$

Da mesma forma,

$$\frac{2}{C^2}\frac{dA_{n+1}}{d\gamma} + \frac{A_{n+1}}{CJ}\frac{d}{d\gamma}\left(\frac{J}{C}\right) = \nabla^2 A_n$$

logo,

$$\frac{d}{d\gamma}A_{n+1} + \frac{A_{n+1}}{2}\frac{C}{J}\frac{d}{d\gamma}\left(\frac{J}{C}\right) = \frac{C^2}{2}\nabla^2 A_n$$

A solução da equação homogênea, $(\nabla^2 A_n = 0)$

$$A_{n+1} = \frac{Const}{\sqrt{J/C}} = \frac{\psi_{n+1}(\alpha,\beta)}{\sqrt{J/C}}$$

Agora precisamos encontrar a solução da equação inomogênea. Propomos a solução na forma:

$$\bar{A}_{n+1} = U(\gamma)A_{n+1}(\gamma)$$

Onde $U(\gamma)$ é uma função desconhecida. Inserindo a solução na equação de transporte,

$$\frac{d}{d\gamma}\bar{A}_{n+1} + \frac{\bar{A}_{n+1}}{2}\left(\frac{J}{C}\right)^{-1}\frac{d}{d\gamma}\left(\frac{J}{C}\right) = \frac{C^2}{2}\nabla^2 A_n$$

$$A_{n+1}\frac{dU}{dy} + U\frac{dA_{n+1}}{dy} + U A_{n+1}\frac{1}{2}\left(\frac{J}{c}\right)^{-1}\frac{d}{dy}\left(\frac{J}{c}\right) = \frac{c^2}{2}\nabla^2 A_n$$

O termo,

$$\frac{dA_{n+1}}{dy} = \frac{d}{dy}\; c_1\left(\frac{J}{c}\right)^{-1/2} = -\frac{1}{2}\,c_1\left(\frac{J}{c}\right)^{-3/2}\frac{d}{dy}\left(\frac{J}{c}\right)$$

$$\frac{dA_{n+1}}{dy} = -\frac{1}{2}\,c_1\left(\frac{J}{c}\right)^{-1/2}\left(\frac{J}{c}\right)^{-1}\frac{d}{dy}\left(\frac{J}{c}\right)$$

$$\frac{dA_{n+1}}{dy} = -\frac{1}{2}\,A_{n+1}\left(\frac{J}{c}\right)^{-1}\frac{d}{dy}\left(\frac{J}{c}\right)$$

Substituindo,

$$A_{n+1}\frac{dU}{dy} + U A_{n+1}\cdot -\frac{1}{2}\left(\frac{J}{c}\right)^{-1}\frac{d}{dy}\left(\frac{J}{c}\right) + U A_{n+1}\frac{1}{2}\left(\frac{J}{c}\right)^{-1}\frac{d}{dy}\left(\frac{J}{c}\right) = \frac{c^2}{2}\nabla^2 A_n$$

Eliminando as duas últimas parcelas do lado esquerdo pois são simétricas,

$$A_{n+1}\frac{dU}{dy} = \frac{c^2}{2}\nabla^2 A_n$$

Passando A_{n+1} dividindo e depois integrando em y,

$$\int\frac{dU}{dy}\,dy = \int A_{n+1}^{-1}\,\frac{c^2}{2}\nabla^2 A_n\,dy$$

Então,

$$U = \int_{\gamma_0}^{\gamma} A_{n+1}^{-1} \frac{c^2}{2} \nabla^2 A_n \, d\gamma$$

Esta equação satisfaz a condição inicial $U(\gamma_0) = 0$.

Fazendo $\psi_{n+1}(\alpha, \beta) = 1$, a solução geral,

$$A_{n+1} = \frac{\psi_{n+1}(\alpha,\beta)}{\sqrt{J/c}} + \int_{\gamma_0}^{\gamma} \frac{c^2}{2} \nabla^2 A_n \, d\gamma$$

Pois,

$$\bar{A}_{n+1} = U(\gamma) A_{n+1}(\gamma) = \int_{\gamma_0}^{\gamma} A_{n+1} A_{n+1}^{-1} \frac{c^2}{2} \nabla^2 A_n \, d\gamma$$

$$\bar{A}_{n+1} = \int_{\gamma_0}^{\gamma} \frac{c^2}{2} \nabla^2 A_n \, d\gamma$$

Colocando $\sqrt{c/J}$ em evidência na solução geral,

$$A_{n+1} = \sqrt{\frac{c}{J}} \left(\psi_{n+1}(\alpha,\beta) + \int_{\gamma_0}^{\gamma} \frac{c^2}{2} \sqrt{\frac{J}{c}} \nabla^2 A_n \, d\gamma \right)$$

O principal problema do cálculo de A_1 é causado pelo termo,

$$\nabla^2 A_0 = \nabla^2 \psi_0(\alpha,\beta) / \sqrt{J/c}$$