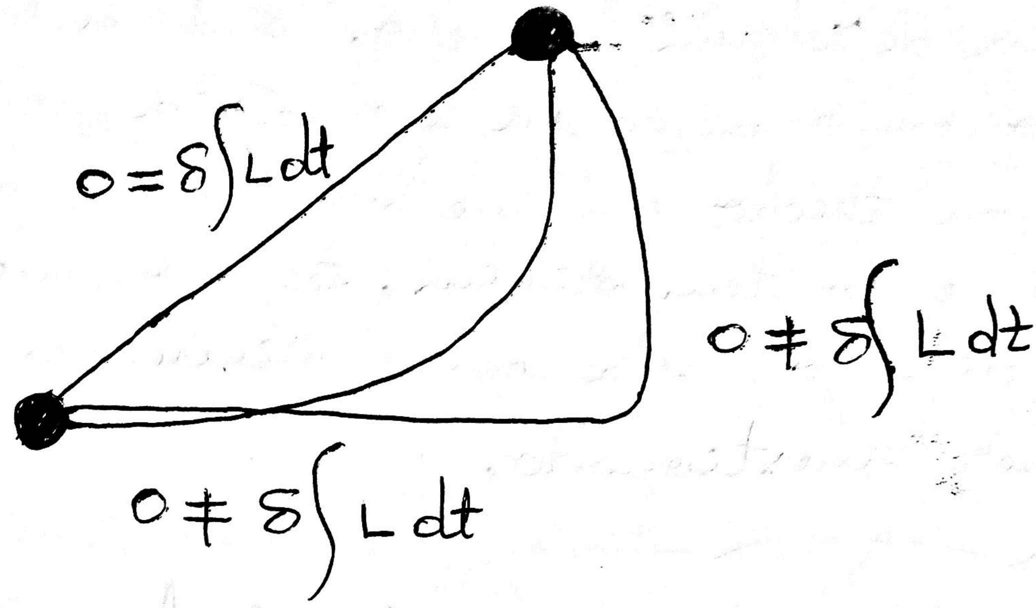
## Cap. 2 - Solution of the Eikonal equation 

**Princípio** significa o início, fundamento ou essência de algum fenômeno. Também pode ser definido como a causa primária, o momento, o local ou trecho em que algo, uma ação ou conhecimento tem origem. Sendo que o princípio de algo, seja como origem ou proposição pode ser questionado.

"A trajetória percorrida pela luz ao se propagar de um ponto a outro é tal que o tempo gasto em percorrê-la é um mínimo"

"A trajetória percorrida pela luz ao propagar-se de um ponto a outro é tal que o tempo gasto para percorrê-la é estacionário a respeito das possíveis variações de sua trajetória"

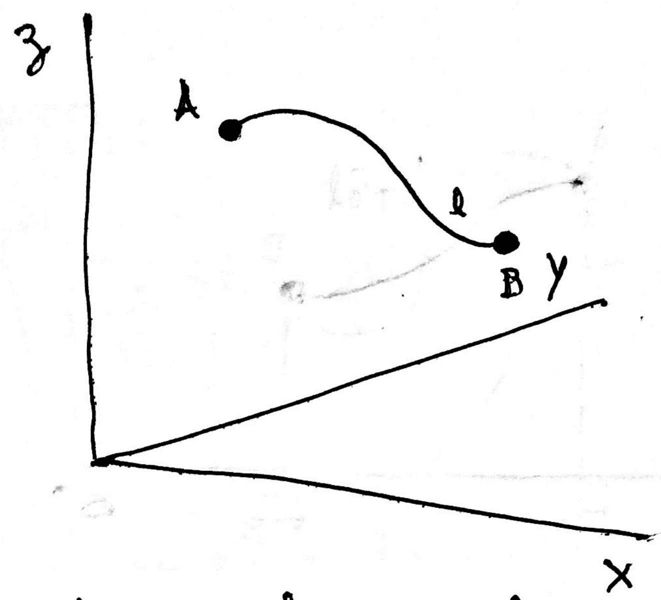
$$\delta L_{o_1 o_2}[n(\vec{r})] = \delta \int_{o_1}^{o_2} n(\vec{r})\, ds = 0$$




"Os trajetos próximos ao 'verdadeiro' requerem tempos aproximadamente iguais!"

## 2.1. Fermat's principle






$$l: \begin{cases} x(\sigma) = x \\ y(\sigma) = y \\ z(\sigma) = z \end{cases}$$

$$C = C(x, y, z)$$

O tempo de trânsito em $l$:

$$T(l) = \int_{(A)}^{(B)} dt = \int_{(A)}^{(B)} \frac{ds}{C} \quad \text{(Funcional de Fermat)}$$

O funcional de Fermat é uma integral em coordenadas curvilíneas e pode ser reduzido para uma integral ordinária:

$$\boxed{\dot{x} = \frac{dx}{d\sigma}}$$

$$ds = \sqrt{dx^2 + dy^2 + dz^2}$$

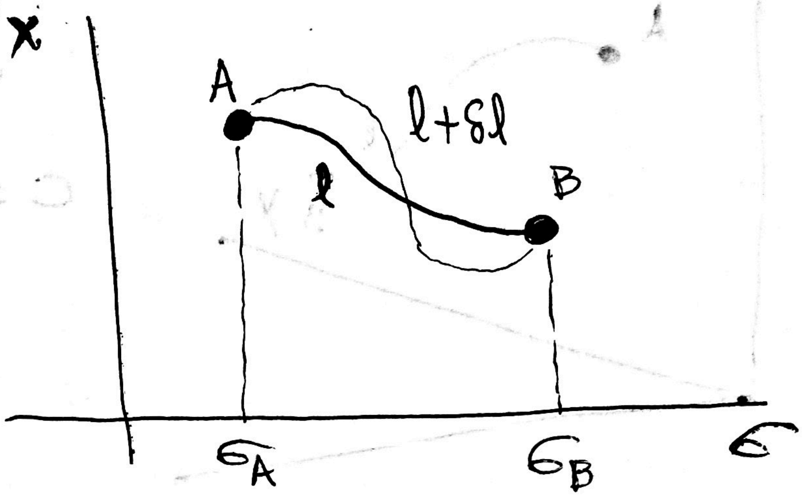
$$= \sqrt{\dot{x}^2(\sigma) + \dot{y}^2(\sigma) + \dot{z}^2(\sigma)}\, d\sigma$$

$$T(l) = \int_{\sigma_A}^{\sigma_B} \frac{\sqrt{\dot{x}^2(\sigma) + \dot{y}^2(\sigma) + \dot{z}^2(\sigma)}}{C(x(\sigma), y(\sigma), z(\sigma))}\, d\sigma = \int_{\sigma_A}^{\sigma_B} L(\dot{x}, \dot{y}, \dot{z}, x, y, z)\, d\sigma$$

Onde $L$ é a função lagrangiana!

# 2.2. Variation of a functional; Euler's equation

**Problema 1D**




$$T(l) = \int_{(A)}^{(B)} L(\dot{x}(\sigma), x(\sigma))\, d\sigma$$

$$\boxed{\begin{aligned} x &= x(\sigma) + \delta x(\sigma) \\ \dot{x} &= \dot{x}(\sigma) + \delta\dot{x}(\sigma) \end{aligned}}$$

$$T(l+\delta l) = \int_{(A)}^{(B)} L(\dot{x}+\delta\dot{x}, x+\delta x)\, d\sigma$$

Expandindo L em série de Taylor:

Em relação a $\dot{x}$ e $x$, série de Taylor 2D

$$L(x+\delta x) \approx L(\dot{x}, x) + \frac{\partial L}{\partial \dot{x}}\delta\dot{x} + \frac{\partial L}{\partial x}\delta x + O(\delta x)^2$$

$$\boxed{\text{série de Taylor: } f(x) \approx \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} x^n}$$

Deste modo:

$$\Delta T(\ell) = T(\ell+\delta\ell) - T(\ell)$$

$$= \int_A^B \left( \frac{\partial L}{\partial \dot{x}} \delta\dot{x} + \frac{\partial L}{\partial x} \delta x \right) d\sigma + \int_A^B \mathcal{O}(\delta x)^2 \, d\sigma$$

$\Delta T$ ou $\delta T$ é a primeira variação do funcional.

$$\delta T = \int_A^B \left( \frac{\partial L}{\partial \dot{x}} \delta\dot{x} + \frac{\partial L}{\partial x} \delta x \right) d\sigma$$

$\delta\dot{x}$ e $\delta x$ não são independentes. Integrando por partes:

Integração por partes

$$\int u\,dv = uv - \int v\,du$$

$$u = \frac{\partial L}{\partial \dot{x}} \qquad v = \delta x$$

$$du = \frac{d}{d\sigma}\frac{\partial L}{\partial \dot{x}}\, d\sigma \qquad dv = \frac{d}{d\sigma}\delta x = \delta\dot{x}$$

$$\boxed{\delta T = \frac{\partial L}{\partial \dot{x}} \delta x \Big|_A^B + \int_A^B \left( \frac{\partial L}{\partial x} - \frac{d}{d\sigma}\frac{\partial L}{\partial \dot{x}} \right) \delta x \, d\sigma}$$

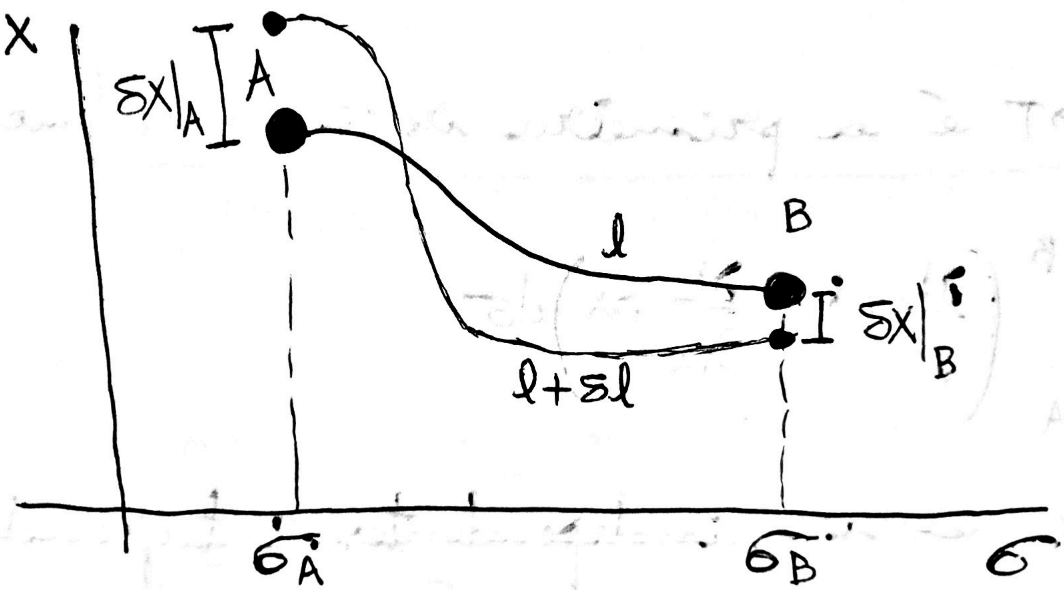
Esta é a fórmula para a primeira variação do funcional com pontos móveis ($\delta x|_A$ e $\delta x_B$ arbitrários). No caso de pontos A e B fixos:

$$\delta X\big|_A = \delta X\big|_B = 0$$

e,

$$\delta T = \int_A^B \left( \frac{\partial L}{\partial X} - \frac{d}{d\sigma} \frac{\partial L}{\partial \dot{X}} \right) \delta X \, d\sigma$$

Observe:




Se A e B são fixos, $\delta X|_A$ e $\delta X|_B$ são iguais a zero.

$$\delta T = \int_A^B \left( \frac{\partial L}{\partial X} - \frac{d}{d\sigma} \frac{\partial L}{\partial \dot{X}} \right) \delta X \, d\sigma = 0$$

Princípio de Fermat

Assim,

$$\boxed{\frac{d}{d\sigma} \frac{\partial L}{\partial \dot{X}} - \frac{\partial L}{\partial X} = 0}$$ (Equação de Euler)

$\boxed{\text{Problema em 3D}}$

$$T(l) = \int_A^B L(\dot{x},\dot{y},\dot{z},x,y,z)\,d\sigma$$

$$\boxed{\begin{array}{l} x \to x+\delta x \\ \dot{x} \to \dot{x}+\delta\dot{x} \\ y \to y+\delta y \\ \dot{y} \to \dot{y}+\delta\dot{y} \\ z \to z+\delta z \\ \dot{z} \to \dot{z}+\delta\dot{z} \end{array}}$$

Repetindo os cálculos do problema 1D para todos os coordenados:

$$T(l+\delta l) = \int_A^B L(\dot{x}+\delta\dot{x}, x+\delta x, \dot{y}+\delta\dot{y}, y+\delta y, \dot{z}+\delta\dot{z}, z+\delta z)\,d\sigma$$

Expandindo L em série de Taylor em relação a $\dot{x}, x$:

$$L(x+\delta x) \approx L(\dot{x},x) + \frac{\partial L}{\partial \dot{x}}\delta\dot{x} + \frac{\partial L}{\partial x}\delta x + O(\delta x)^2$$

Em relação a $\dot{y}$ e $y$:

$$L(y+\delta y) \approx L(\dot{y},y) + \frac{\partial L}{\partial \dot{y}}\delta\dot{y} + \frac{\partial L}{\partial y}\delta y + O(\delta y)^2$$

Em relação a $\dot{z}$ e $z$:

$$L(z+\delta z) \approx L(\dot{z},z) + \frac{\partial L}{\partial \dot{z}}\delta\dot{z} + \frac{\partial L}{\partial z}\delta z + O(\delta z)^2$$

Superpondo as soluções:

$$L(x+\delta x, y+\delta y, z+\delta z) \approx L(x,y,z) + \frac{\partial L}{\partial \dot{x}}\delta\dot{x} + \frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{y}}\delta\dot{y} + \cdots$$

Assim,

$$\delta T(\ell) = T(\ell + \delta \ell) - T(\ell)$$

$$= \int_A^B \left\{ \left( \frac{\partial L}{\partial \dot{x}} \delta \dot{x} + \frac{\partial L}{\partial x} \delta x \right) + \left( \frac{\partial L}{\partial \dot{y}} \delta \dot{y} + \frac{\partial L}{\partial y} \delta y \right) + \left( \frac{\partial L}{\partial \dot{z}} \delta \dot{z} + \frac{\partial L}{\partial z} \delta z \right) \right\} d\sigma$$

Integrando por partes o primeiro termo de cada um dos parênteses:

$$\boxed{\begin{array}{ll} u = \frac{\partial L}{\partial \dot{x}} & v = \delta x \\ du = \frac{d}{d\sigma} \frac{\partial L}{\partial \dot{x}} d\sigma & dv = \delta \dot{x} \end{array}}$$

$$\int_A^B \left( \frac{\partial L}{\partial \dot{x}} \delta \dot{x} + \frac{\partial L}{\partial x} \delta x \right) d\sigma = \frac{\partial L}{\partial \dot{x}} \delta x \Big|_A^B + \int_A^B \left( \frac{\partial L}{\partial x} - \frac{d}{d\sigma} \frac{\partial L}{\partial \dot{x}} \right) \delta x \, d\sigma$$

$$\boxed{\begin{array}{ll} u = \frac{\partial L}{\partial \dot{y}} & v = \delta y \\ du = \frac{d}{d\sigma} \frac{\partial L}{\partial \dot{y}} d\sigma & dv = \delta \dot{y} \end{array}}$$

$$\int_A^B \left( \frac{\partial L}{\partial \dot{y}} \delta \dot{y} + \frac{\partial L}{\partial y} \delta y \right) d\sigma = \frac{\partial L}{\partial \dot{y}} \delta y \Big|_A^B + \int_A^B \left( \frac{\partial L}{\partial y} - \frac{d}{d\sigma} \frac{\partial L}{\partial \dot{y}} \right) \delta y \, d\sigma$$

(2.2) (cont.)

$$\boxed{\begin{array}{ll} u = \dfrac{\partial L}{\partial \dot{z}} & v = \delta z \\[2ex] du = \dfrac{d}{d\sigma}\dfrac{\partial L}{\partial \dot{z}}\, d\sigma & dv = \delta \dot{z} \end{array}}$$

$$\int_A^B \left( \frac{\partial L}{\partial \dot{z}} \delta \dot{z} + \frac{\partial L}{\partial z} \delta z \right) d\sigma = \frac{\partial L}{\partial \dot{z}} \delta z \Big|_A^B + \int_A^B \left( \frac{\partial L}{\partial z} - \frac{d}{d\sigma} \frac{\partial L}{\partial \dot{z}} \right) \delta z \, d\sigma$$

Assim,

$$\delta T = \left( \frac{\partial L}{\partial \dot{x}} \delta x + \frac{\partial L}{\partial \dot{y}} \delta y + \frac{\partial L}{\partial \dot{z}} \delta z \right) \Big|_A^B$$

$$+ \int_A^B \left\{ \left( \frac{\partial L}{\partial x} - \frac{d}{d\sigma} \frac{\partial L}{\partial \dot{x}} \right) \delta \dot{x} + \left( \frac{\partial L}{\partial y} - \frac{d}{d\sigma} \frac{\partial L}{\partial \dot{y}} \right) \delta \dot{y} \right.$$

$$\left. + \left( \frac{\partial L}{\partial z} - \frac{d}{d\sigma} \frac{\partial L}{\partial \dot{z}} \right) \delta \dot{z} \right\} d\sigma$$

Da mesma forma (A e B fixos):

$$\delta x|_A = \delta x|_B = \delta y|_A = \delta y|_B = \delta z|_A = \delta z|_B = 0$$

Daí resulta:

$$\frac{d}{d\sigma}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$$

$$\frac{d}{d\sigma}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = 0$$

$$\frac{d}{d\sigma}\frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} = 0$$

Qualquer solução da equação de Euler é um extremal do cálculo variacional e um raio na geofísica

$$\frac{\partial y(u(t))}{\partial t} = \frac{\partial y}{\partial u} \cdot \frac{\partial u}{\partial t} \quad \text{(Regra da Cadeia)}$$

$$\frac{\partial L}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}}\left(\frac{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}{C(x,y,z)}\right) = \frac{2\dot{x}}{2}\,\frac{1}{C\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}$$

$$\frac{\partial L}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}{C}\right)$$

Aplicando a regra da cadeia para resolver estas derivadas

$$= \cancelto{0}{\frac{\partial}{\partial x}\left(\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}\right)}\frac{1}{C} + \sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}\,\frac{\partial}{\partial x}\cdot\frac{1}{C}$$

Resultados semelhantes serão obtidos para $\frac{\partial L}{\partial y}$, $\frac{\partial L}{\partial \dot{y}}$, $\frac{\partial L}{\partial z}$ e $\frac{\partial L}{\partial \dot{z}}$. Assim,

$$\frac{d}{d\sigma}\left(\frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}\cdot\frac{1}{C}\right) - \sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}\,\frac{\partial}{\partial x}\cdot\frac{1}{C} = 0$$

$$\frac{d}{d\sigma}\left(\frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}\,\frac{1}{C}\right) - \sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}\,\frac{\partial}{\partial y}\,\frac{1}{C} = 0$$

$$\frac{d}{d\sigma}\left(\frac{\dot{z}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}\,\frac{1}{C}\right) - \sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}\,\frac{\partial}{\partial z}\,\frac{1}{C} = 0$$

Exemplo 1: Tomando o parâmetro s (comprimento do arco ao longo do raio) ao invés de $\sigma$.

$$ds = \sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}\, d\sigma$$

$$\rightarrow \frac{d}{ds} = \frac{1}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}} \cdot \frac{d}{d\sigma}$$

Assim,

$$\frac{d}{ds}\left(\frac{1}{c}\frac{dx}{ds}\right) - \frac{\partial}{\partial x}\cdot\frac{1}{c} = 0$$

$$\frac{d}{ds}\left(\frac{1}{c}\frac{dy}{ds}\right) - \frac{\partial}{\partial y}\frac{1}{c} = 0$$

$$\frac{d}{ds}\left(\frac{1}{c}\cdot\frac{dz}{ds}\right) - \frac{\partial}{\partial z}\frac{1}{c} = 0$$

Na forma vetorial:

$$\frac{d}{ds}\left(\frac{\vec{t}}{c}\right) - \nabla(1/c) = 0$$

Onde:

$$\vec{t} = \frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j} + \frac{dz}{ds}\hat{k}$$

$\vec{t}$ é o vetor unitário tangente ao raio.

Pois, (o mesmo vale para y e z)

$$\frac{d}{d\sigma}\left(\frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}\cdot\frac{1}{c}\right) - \sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}\,\frac{\partial}{\partial x}\frac{1}{c} = \sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}\,\frac{d}{ds}\left(\frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}\,\frac{1}{c}\right) - \sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}\,\frac{\partial}{\partial x}\cdot\frac{1}{c}$$

$$= \frac{d}{ds}\left(\frac{1}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}\,\frac{dx}{d\sigma}\,\frac{1}{c}\right) - \frac{\partial}{\partial x}\frac{1}{c} = \frac{d}{ds}\left(\frac{1}{c}\frac{dx}{ds}\right) - \frac{\partial}{\partial x}\cdot\frac{1}{c}$$

Exemplo 2: Meio homogêneo.

$$c = \text{constante}$$

$$\frac{\partial}{\partial x}\frac{1}{c} = \frac{\partial}{\partial y}\frac{1}{c} = \frac{\partial}{\partial z}\frac{1}{c} \equiv 0$$

Isto significa que o lado direito da equação de Euler:

$$\frac{d}{ds}\left(\frac{1}{c}\frac{dx}{ds}\right) = 0$$

$$\frac{1}{c}\cdot\frac{dx}{ds} = a_1 = \text{constante}$$

$$\frac{dx}{ds} = a_1 c \rightarrow \int \frac{dx}{ds}\, ds = \int a_1 c\, ds$$

$$x(s) = a_1 c s + x(0)$$

$$\boxed{\text{Perceba: } x(s)\big|_{s=0} = x(0)}$$

Isto Posto:

- $\dfrac{1}{c}\cdot\dfrac{dx}{ds} = a_1, \quad x = a_1 c s + x(0);$

$\dfrac{1}{c}\cdot\dfrac{dy}{ds} = a_2, \quad y = a_2 c s + y(0);$

$\dfrac{1}{c}\cdot\dfrac{dz}{ds} = a_3, \quad z = a_3 c s + z(0);$

(2.2) (cont.)

$$s = \sqrt{(x - x(0))^2 + (y - y(0))^2 + (z - z(0))^2}$$

$$s = Cs\sqrt{a_1^2 + a_2^2 + a_3^2}$$

$$\boxed{\frac{1}{C} = \sqrt{a_1^2 + a_2^2 + a_3^2}}$$

$x - x(0) = a_1 Cs + x(0) - x(0)$

$x - x(0) = a_1 Cs$

Da mesma forma,

$y - y(0) = a_2 Cs$

$z - z(0) = a_3 Cs$

$$\boxed{s = \sqrt{a_1^2 C^2 s^2 + a_2^2 C^2 s^2 + a_3^2 C^2 s^2} = Cs\sqrt{a_1^2 + a_2^2 + a_3^2}}$$

## 2.3. Hamilton form of the functional and Euler's equation

No caso 1D

$$T = \int_A^B L(\dot{x}, x)\, d\sigma$$

Introduzindo o slowness generalizado $p = \partial L / \partial \dot{x}$ e a partir de agora devemos encontrar $\dot{x}$ como uma função de $p$, $\dot{x} = \dot{x}(p)$

$$H = (p\dot{x} - L)\Big|_{\dot{x} = \dot{x}(p)}$$

$$H = H(p, x)$$

$$L = p\dot{x} - H$$

Em mecânica, H significa a energia mecânica do sistema.

$$T = \int_A^B L(\dot{x},x)\,d\sigma = \int_A^B (p\dot{x} - H(p,x))\,d\sigma$$

$$T = \int_A^B p\,dx - H(p,x)\,d\sigma$$

Primeira variação de $T$: $x \to x + \delta x$;

$p \to p + \delta p$;

Expandindo em série de Taylor ao redor de $\delta x$ e $\delta p$:

$$L \approx p\dot{x} - H + \dot{x}\,\delta p + p\frac{\partial \dot{x}}{\partial p}\delta p - \frac{\partial H}{\partial p}\delta p + \cancelto{0}{\frac{\partial p}{\partial x}\dot{x}}\,\delta x$$

$$+ \cancelto{0}{p\frac{\partial \dot{x}(p)}{\partial x}}\,\delta x - \frac{\partial H}{\partial x}\delta x$$

$$L \approx p\dot{x} - H + \delta p\,\dot{x} + p\,\delta\dot{x} - \frac{\partial H}{\partial p}\delta p - \frac{\partial H}{\partial x}\delta x$$

$$\delta T = \int_A^B \left(\delta p\,\dot{x} + p\,\delta\dot{x} - \frac{\partial H}{\partial p}\delta p - \frac{\partial H}{\partial x}\delta x\right) d\sigma$$

$$\boxed{\begin{array}{ll} u = p & dv = \delta\dot{x}\,d\sigma \\ du = \dfrac{\partial p}{\partial \sigma}\,d\sigma & v = \delta x \end{array}}$$

Regra da integração por partes:

$$\int u\,dv = uv - \int v\,du$$

$$\delta T = p\,\delta x\Big|_A^B + \int_A^B \left\{\left(\dot{x} - \frac{\partial H}{\partial p}\delta p\right) - \left(\dot{p} + \frac{\partial H}{\partial x}\right)\delta x\right\} d\sigma$$

Então,

$$\dot{x} = \frac{\partial H}{\partial p} \quad e \quad \dot{p} = -\frac{\partial H}{\partial x}$$

(Equações de Hamilton)

Onde:
$x = x(\sigma)$
$p = p(\sigma)$

## Extensão para 3D

$$p_1 = \frac{\partial L}{\partial \dot{x}} \quad e \quad p_2 = \frac{\partial L}{\partial \dot{y}} \quad p_3 = \frac{\partial L}{\partial \dot{z}}$$

$$H = p_1 \dot{x} + p_2 \dot{y} + p_3 \dot{z} - L$$

$$T = \int_A^B L \, d\sigma = \int_A^B p_1 \dot{x} + p_2 \dot{y} + p_3 \dot{z} - H \, d\sigma$$

Assim,

$$\delta T = \left(p_1 \delta x + p_2 \delta y + p_3 \delta z\right)\Big|_A^B$$

$$+ \int_A^B \left\{ \left(\dot{x} - \frac{\partial H}{\partial p_1}\right)\delta p_1 + \left(\dot{y} - \frac{\partial H}{\partial p_2}\right)\delta p_2 \right.$$

$$+ \left(\dot{z} - \frac{\partial H}{\partial p_3}\right)\delta p_3 - \left(\dot{p}_1 + \frac{\partial H}{\partial x}\right)\delta x$$

$$\left. - \left(\dot{p}_2 + \frac{\partial H}{\partial y}\right)\delta y - \left(\dot{p}_3 + \frac{\partial H}{\partial z}\right)\delta z \right\} d\sigma$$

As equações de Euler na forma Hamiltoniana derivadas a partir de $\delta T$:

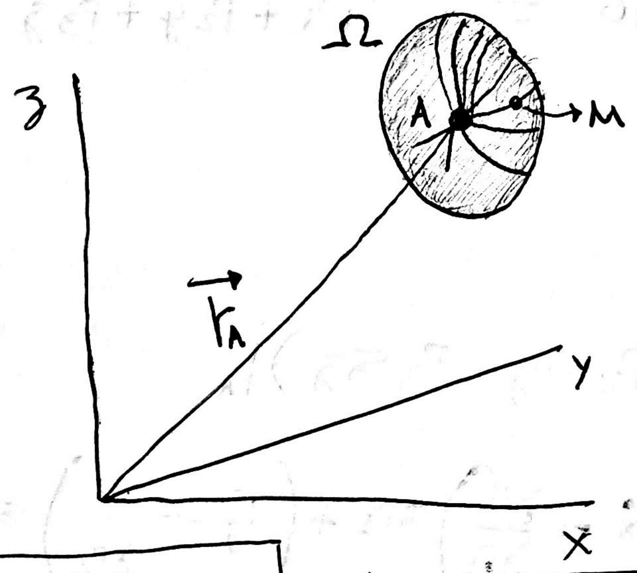
$$\dot{x} = \frac{\partial H}{\partial P_1}, \quad \dot{P}_1 = -\frac{\partial H}{\partial x};$$

$$\dot{y} = \frac{\partial H}{\partial P_2}, \quad \dot{P}_2 = -\frac{\partial H}{\partial y};$$

$$\dot{z} = \frac{\partial H}{\partial P_3}, \quad \dot{P}_3 = -\frac{\partial H}{\partial z};$$

## 2.4. Solution of the eikonal equation in the case of a point source

$$\boxed{\vec{r}(\sigma) = x(\sigma)\hat{x} + y(\sigma)\hat{y} + z(\sigma)\hat{k}}$$




$$\boxed{\vec{r}(\sigma)\Big|_{\sigma=0} = \vec{r}_A;} \qquad \boxed{\frac{d\vec{r}}{d\sigma}\Big|_{\sigma=0} = \vec{t}}$$

<u>Definition:</u> we say that this family of rays form a regular field of rays in a domain $\Omega$ if for each point $M \in \Omega$ there is one and only one ray which starts at A and reachs M.

(i) For each $M \in \Omega$ we must find the ray which reachs M.

(ii) Then we must compute Fermat's integral $T = \int_A^M ds/C$ along this ray between A and M. We thus obtain a function of M which we denote $\tau(M) = \tau(x,y,z)$,

$$\boxed{\begin{array}{l} M, \text{ móvel} \\ A, \text{ fixo} \end{array}}$$

$$\tau(M) = \int_A^M \frac{ds}{C} = \int_A^M \frac{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}{C(x,y,z)}\, d\sigma$$

This function $\tau(x,y,z)$ will satisfy the eikonal equation

$$\delta T = \left(\frac{\partial L}{\partial \dot{x}}\delta x + \frac{\partial L}{\partial \dot{y}}\delta y + \frac{\partial L}{\partial \dot{z}}\delta z\right)\Bigg|_M + \int_A^M \left\{\left(\frac{\partial L}{\partial x} - \frac{d}{d\sigma}\frac{\partial L}{\partial \dot{x}}\right)\delta x + \right.$$

$$\left.\left(\frac{\partial L}{\partial y} - \frac{d}{d\sigma}\frac{\partial L}{\partial \dot{y}}\right)\delta y + \left(\frac{\partial L}{\partial z} - \frac{d}{d\sigma}\frac{\partial L}{\partial \dot{z}}\right)\delta z\right\} d\sigma$$

As each time we compute T on a ray, Euler's equation are satisfied and because of that integral in equation vanishes, so

$$\delta T \equiv d\mathcal{Y} = \left( \frac{\partial L}{\partial \dot{x}} \delta x + \frac{\partial L}{\partial \dot{y}} \delta y + \frac{\partial L}{\partial \dot{z}} \delta z \right) \Big|_M$$

$$= \frac{\partial \mathcal{Y}}{\partial x} dx + \frac{\partial \mathcal{Y}}{\partial y} dy + \frac{\partial \mathcal{Y}}{\partial z} dz$$

e,

$$dx = \delta x \big|_M \, , \; dy = \delta y \big|_M \, , \; dz = \delta z \big|_M$$

assim,

$$\frac{\partial \mathcal{Y}}{\partial x} = \frac{\partial L}{\partial \dot{x}} = \frac{\dot{x}}{C\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} = \frac{1}{C\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} \frac{dx}{d\sigma}$$

$$\frac{\partial \mathcal{Y}}{\partial y} = \frac{\partial L}{\partial \dot{y}} = \frac{\dot{y}}{C\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} = \frac{1}{C\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} \frac{dy}{d\sigma}$$

$$\frac{\partial \mathcal{Y}}{\partial z} = \frac{\partial L}{\partial \dot{z}} = \frac{\dot{z}}{C\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} = \frac{1}{C\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} \frac{dz}{d\sigma}$$

$$\boxed{(\nabla \mathcal{Y})^2 = \frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{C^2(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)} = \frac{1}{C^2}}$$

A função $\mathcal{Y}(M)$ satisfaz a equação Eikonal.

Definition: Surfaces in 3D, defined by $\mathcal{S} = \text{const}$, are called wavefronts.

$$0 = \frac{\partial \mathcal{S}}{\partial x} dx + \frac{\partial \mathcal{S}}{\partial y} dy + \frac{\partial \mathcal{S}}{\partial z} dz$$

$$0 = \nabla \mathcal{S} \cdot d\vec{r}$$

$d\vec{r}$ pertence ao plano tangente à superfície $\mathcal{S} = \text{const}$.

$$\frac{\partial \mathcal{S}}{\partial x} = \frac{1}{c} \frac{dx}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} d\sigma$$

$$\boxed{\begin{gathered} d\sigma \longrightarrow ds \\ \frac{d}{ds} = \frac{1}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} \frac{d}{d\sigma} \end{gathered}}$$

$$\frac{\partial \mathcal{S}}{\partial x} = \frac{1}{c} \cdot \frac{dx}{ds}$$

E o mesmo para $\frac{\partial \mathcal{S}}{\partial y}$ e $\frac{\partial \mathcal{S}}{\partial z}$. Deste modo,

$$\nabla \mathcal{S} \equiv \frac{\partial \mathcal{S}}{\partial x} \hat{\imath} + \frac{\partial \mathcal{S}}{\partial y} \hat{\jmath} + \frac{\partial \mathcal{S}}{\partial z} \hat{k}$$

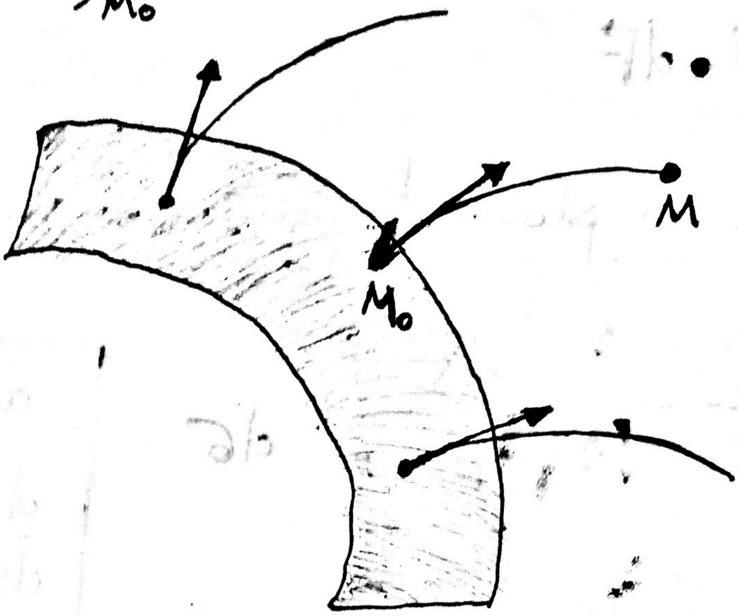
$$\nabla \mathcal{S} = \frac{1}{c} \left( \frac{dx}{ds} \hat{\imath} + \frac{dy}{ds} \hat{\jmath} + \frac{dz}{ds} \hat{k} \right) = \frac{1}{c} \vec{t}$$

Onde $\vec{t}$ é o vetor unitário tangente ao raio.

## 2.5. Solution of the eikonal equation when an initial wave is given

$$\tau = \tau_0 = \text{const}$$

$$\tau = \tau(M) = \int_{M_0}^{M} \frac{ds}{c} + \tau_0$$




$\tau = \tau_0 = \text{Const}$

Where the integral is a curvilinear integral along that unique ray which starts at $M_0$ reaches M. In a homogenious medium this procedure coincides with

## Huygen's principle.

"Each point on a wavefront $t = t_0$ is considered as a secondary source, which irradiates a spherical wavefront that propogates during a time interval $\Delta t$. An envelope of these spherical waves describes the wavefront at $t = t_0 + \Delta t$."