Hybrid Computers and Partial Differential Equations

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Universiteit Leiden, supervised by Henning Basold and Arjen Doelman

July 5, 2023 - Final presentation

Today

1. Introduction to analog computers

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2. Hybrid solutions for PDEs

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3. Problems with hybrid solutions

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2. Hybrid solutions for PDEs
3. Problems with hybrid solutions
4. **PTOC:** PDE to ODE compiler

Analog computers

- Continuous calculations
- Reconfigurable components: addition, multiplication, etc.
- Integration-component
- Compiler: LEGNO

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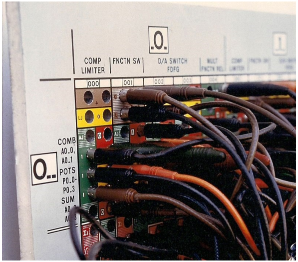

(a) EAI 8800

(b) Anadigm chips

Example: $\ddot{x} = C + x + \dot{x}^2$

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Rewrite with integrals:

$$\begin{cases} x & = & \int y \; dt \\ y & = & \int (C + x + y^2) \; dt \end{cases}$$

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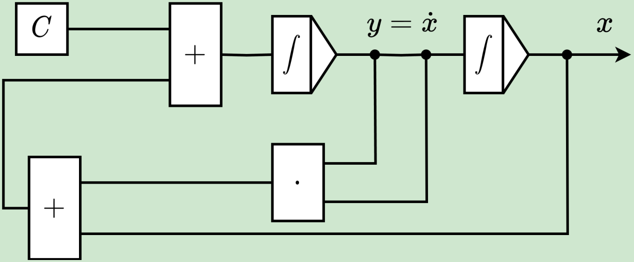

Partial differential equations

Note

An analog computer cannot solve the heat equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

Two solutions:

Partial differential equations

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Two solutions:

- "Extended analog computer"
  - Theoretical model that can solve PDEs
  - Not physically realized

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Two solutions:

- "Extended analog computer"
  - Theoretical model that can solve PDEs
  - Not physically realized
- Approximations with hybrid implementation
  - Discrete space, continuous time

Hybrid solution

- Create grid over spatial dimensions
- Replace spatial derivatives with approximations
- Solve the resulting system of ODEs

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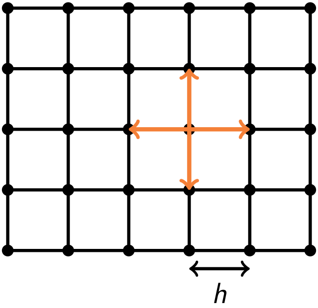

Hybrid solution

Multiple problems:

- Form of the PDE determines if it can be simulated
  - ODEs of the form $\dot{x} = f(t, x)$ are useful
  - Example: $\partial_{xx} u + \partial_{yy} u = 0$ is not converted to this form
  - Example: $\partial_t u = \partial_{xx} u$ is converted to this form

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  - Split up into groups

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- The system of ODEs can become very large
  - Split up into groups
- After splitting, values from different groups are required
  - Introduce iterative method
  - After an initial guess, values from the previous iteration are taken for values from different groups

Form of the PDE

- We want an ODE of the form $\dot{x} = f(t, x)$
- The PDE should have the form

$$\frac{\partial u}{\partial t} = F\left(t, x, \frac{\partial^{\alpha_1} u}{\partial x^{\alpha_1}}, \ldots, \frac{\partial^{\alpha_1} u}{\partial x^{\alpha_m}}\right)$$

- Initial conditions and boundary conditions should be given

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- Initial conditions and boundary conditions should be given
- Some PDEs can be rewritten to the correct form
- All PDEs of the following form can be rewritten:

$$\frac{\partial^{k} u}{\partial t^{k}} = F\left(t, x, \frac{\partial^{\alpha_1} u}{\partial x^{\alpha_1}}, \ldots, \frac{\partial^{\alpha_m} u}{\partial x^{\alpha_m}}\right)$$

Form of the PDE

Example

Wave equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

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Rewriting to a system of PDEs gives

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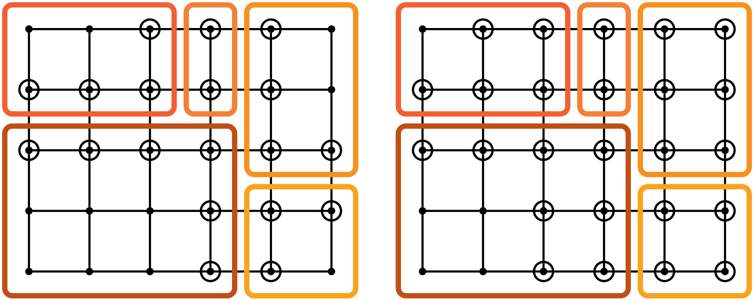
- Extra initial conditions needed for $\partial_t u|_{t=0}$
- New boundary conditions follow from given boundary conditions

Division of groups

- Values from the previous iteration should be stored at the boundary of groups
- Boundary with a certain width



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- Several heuristics to divide grid groups:

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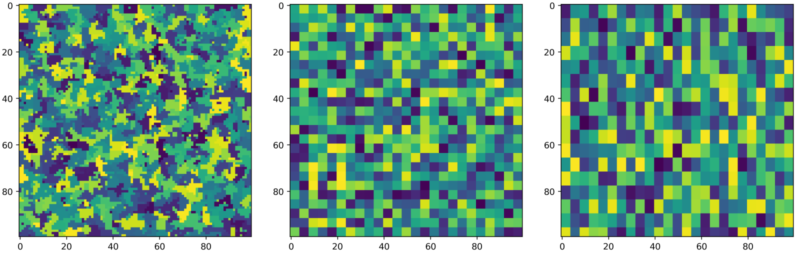

Random: 6.740% Squares: 27.040% Rectangles: 35.360%

Figure: The proportion of internal cells for different heuristics

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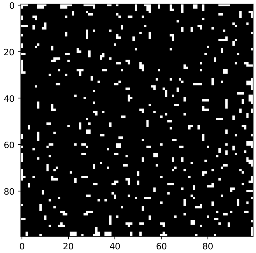

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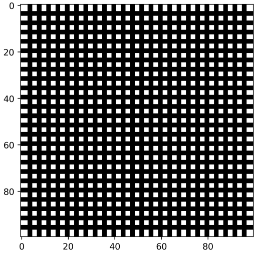

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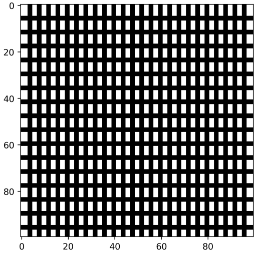

Rectangles: 35.360%

Figure: The proportion of internal cells for different heuristics

Iterative method

- By using values from previous iterations, we get an iterative system:

$$\begin{aligned} &\text{Original:} && \dot{x} = f(t, x, x), \\ &\text{Iterations:} && \dot{x}_{k+1} = f(t, x_{k+1}, x_k) \end{aligned}$$

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- Questions:
  - Can we use the iterative ODEs to find a solution of the original ODE?
  - Are we guaranteed to find a solution?
  - How can we find a solution?

Convergence of the iterative method

- If the iterative method converges, it converges to a solution

Lemma

Let $\Omega \subset \mathbb{R}^n$ and $f : [0, \tau] \times \Omega^2 \to \mathbb{R}^n$ be continuously differentiable. Let $x : [0, \tau] \to \Omega$ and $x_k : [0, \tau] \to \Omega$ be continuously differentiable, such that

$$\dot{x}_{k+1} = f(t, x_{k+1}, x_k),$$
$$x_k(0) = x(0),$$
$$\lim_{k \to \infty} \sup_{t \in [0, \tau]} |x_k(t) - x(t)| = 0.$$

Then x is a solution of the system $\dot{x} = f(t, x, x)$.

Convergence of the iterative method

- Local convergence can be proven in a similar way to Picard-Lindelöf

Theorem

Let $\Omega \subset \mathbb{R}^n$ and $f : [0, \tau] \times \Omega^2 \to \mathbb{R}^n$ continuously differentiable. Let $x : [0, \tau] \to \Omega$ be continuously differentiable with $\dot{x} = f(t, x, x)$. There exists a $T > 0$ and $x_k : [0, T] \to \Omega$ continuously differentiable, such that

$$\begin{aligned} \dot{x}_{k+1} &= f(t, x_{k+1}, x_k), \\ x_k(0) &= x(0), \end{aligned}$$

and such that $x_k \to x$ uniformly on $[0, T]$.

Convergence of the iterative method

Theorem

If f is globally Lipschitz and $\Omega = \mathbb{R}^n$, the sequence $\{x_k\}$ exists and converges uniformly on $[0, \tau]$ (not just $[0, T]$).

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Corollary

If f is linear and $\Omega = \mathbb{R}^n$, the sequence $\{x_k\}$ exists and converges uniformly on $[0, \tau]$.

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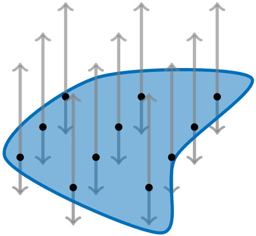

$$\Omega = \mathbb{R}^{11}$$

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The interval $[0, T]$ on which the iterative method converges, can be made independent of the initial condition.

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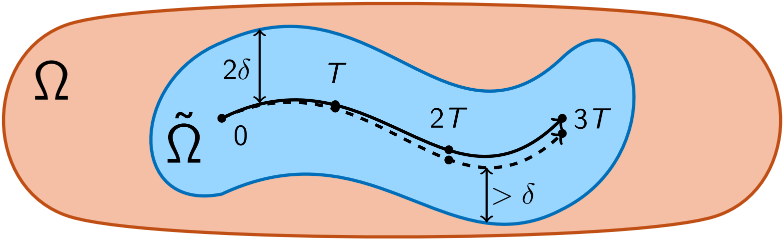
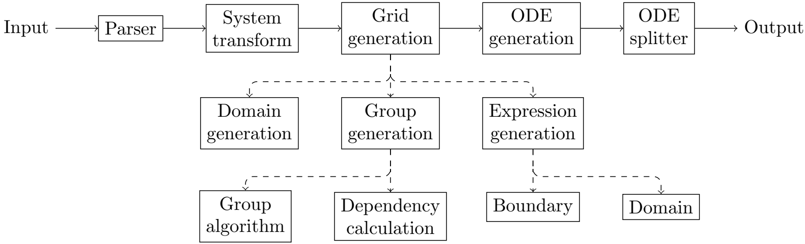
Corollary

Repeating the iterative procedure on smaller intervals, the entire solution can be found.



PTOC

- Automating steps:
  - Converts PDE to the correct form
  - Divides grid into groups
  - Generates system of ODEs from the PDE
  - Generates iterative system



PTOC syntax

```
pde {
    dims [x]; domain x * x < 100; pivot [0]; scale 0.08;
    equation dt(dt(u)) = dx(dx(u));
    init u = sin(x * 3.1415 / 10); init dt(u) = 0;
    boundary u = 0;
    interval u = [-2, 2]; interval dt(u) = [-2, 2];
    time 20; iterations 1; emit u as u;
}
```

Figure: PTOC PDE syntax for the wave equation $\partial_{tt} u = \partial_{xx} u$

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Figure: PTOC PDE syntax for the wave equation $\partial_{tt} u = \partial_{xx} u$

```
system {
    var x = integ(5 * x - 3, 1);
    emit x as x;
    interval x = [-5, 5];
    time 1;
}
```

Figure: PTOC ODE syntax for the IVP $\dot{x} = 5x - 3, x(0) = 1$

PTOC simulation

- PTOC has a tool to simulate the generated PDEs, for graphical output

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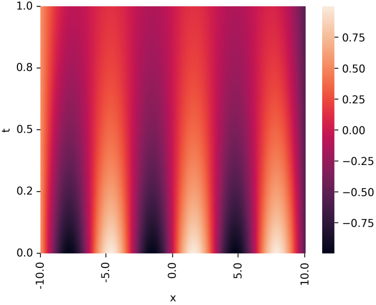
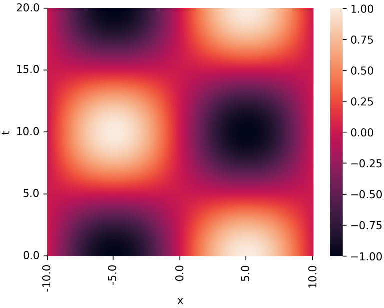


(a) Heat equation



(b) Wave equation

Figure: Simulation without iterations

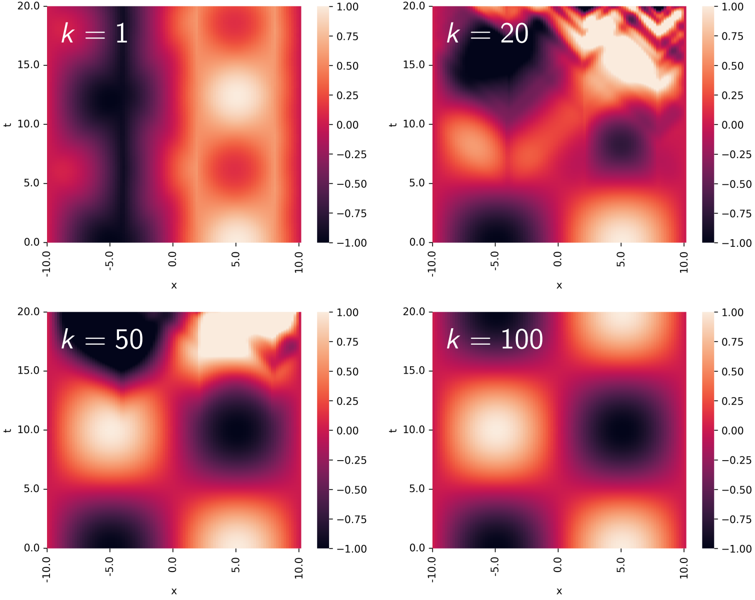


Figure: Simulations with k iterations and group size 30

Further research

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- Is the interval of convergence for the iterative method the same as the interval of existence of the solution?
- PTOC can be connected to a compiler for analog computers
- Automatically determine bounds on the values the grid points can attain
  - Analog computers: operational range
  - Type system
  - Analysis for specific types of PDEs