Writing Mathematics

How I Write Mathematics

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Abstract. *Writing mathematics clearly, carefully, and accurately is a meaningful skill for people of all ages to learn and understand. It requires considerable focus, such as understanding concepts with specialized language and notation. Writing mathematics can be difficult, but it is worth pursuing because it helps people comprehend complex ideas and prepares them for advancing their own.*

Introduction

One of the greatest challenges for students is learning to express mathematics fluently and accurately. Trying to decipher complex formulas or explaining mathematical ideas in writing can be a daunting task, even for those with an advanced understanding of mathematics. However, mastering this skill can result in tremendous rewards. Expressing mathematical ideas effectively through written language provides us with a greater understanding and appreciation of the subject, not only from an academic perspective but from its relevance to our daily lives as well. It's important to understand that although making sense of mathematics on paper may seem difficult at first, it's worth pursuing because developing this proficiency can help tremendously in class discussions and exams while also providing insights into real-world problems we might face on a regular basis.

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What Makes Writing Mathematics Different?

Writing mathematics is an artful combination of two distinct languages, natural and mathematics. Natural language offers the writer numerous expression possibilities but can be ambiguous. In contrast, writing with mathematical symbols requires clarity yet allows succinctness when conveying complex concepts. Accurate language is a necessity when it comes to mathematics. To properly communicate your thoughts and opinions, ensure that each word carries the correct connotation you intend for them to have.

> The precision of words is essential in math-based conversations.

To comprehend mathematically expressed ideas, readers must take their time and read content several times while allowing themselves ample pauses for contemplation along the way. Further, mathematics writing often serves as reference material, which necessitates that its contents should ideally be accessed piece by piece upon demand rather than require deep immersion into text volumes at once.

Writing mathematics has rules – some narrow, others broad. Small conventions relate to sentence structure (including punctuation) and are easily verifiable. Broader ones involve general style and strategies that depend on the author's discretion. For a full primer on mathematical writing see Krantz (2017).

Crafting an effective document takes more than just following a few simple rules; it requires mastering composition. This skillfulness is where the art of writing comes in, with broad and deep strategies for your entire work. It's all about connection: linking individual sentences together for clarity and flow while adhering to specific sentence structure regulations like commas or mathematical terminologies. At its most complex level, these interrelated requirements can create intricate webs that are both precise yet engaging – and (if followed) can result in a masterpiece.

Now it's time to dive into each of **the five levels for writing mathematics**: sentence, paragraph, section, document, and beyond.

Sentence Level

Breaking complex mathematical ideas into more accessible parts is essential for effective communication. Using the right voice and symbols in your writing is crucial for success.

> These small details can make comprehension natural – but mastering them is fundamental when creating potent written works.

Here are ten rules that I use when writing mathematics.

1. Do not start a sentence with a mathematical symbol.
2. Do not end a sentence with a mathematical symbol, instead use the correct punctuation.
3. Replace logical symbols with words, unless you are writting on logic.
4. Do not use a colon at the end of a sentence. Do not use sentence fragments, just use complete statements only.
5. Do spell out small numbers, when they are not being used mathematically.
6. Use "we" ("you" and "me" together) for formal exposition, use "I" ("you" and "I") for informal discussions.
7. Use "that" to indicate a specific object (a restrictive, essential clause) and not for making a point of inference.
8. Use "which" to add information to objects (a nonrestrictive, nonessential clause) and not as a conjunction.
9. Separate formulas with words. Do not list formulas, but rather communicate with a reader.
10. Write a sentence that flows logically from left to right to eliminate confusion.

There are two obvious observations to make about these rules. First, it is impossible to delineate a complete list of rules for natural language. So think of these rules given here not as a complete list, but only as a guide that you should (strongly) consider following when writing mathematics.

The second observation is that these rules do not take into account mathematical accuracy. Of course, your mathematics should be accurate. This is not up for debate because mathematics that is not accurate is simply not mathematics. Even a slight exchange of words can lead to impotence or even nonsense. For example, consider the following variation of mathematical induction.

> If A is a subset of the natural numbers $\mathbb{N}$ with the following two properties: *i*) 0 is in A and *ii*) for all k in $\mathbb{N}$, $k \in A$ implies $k+1 \in A$, then A is the set of natural numbers.

Now let's switch two words and see what you think.

> If A is **the** subset of the natural numbers $\mathbb{N}$ with the following two properties: *i*) 0 is in A and *ii*) for all k in $\mathbb{N}$, $k \in A$ implies $k+1 \in A$, then A is **a** set of natural numbers.

In mathematics, precision is everything. For more on these rules see Knuth et al. (1989).

Paragraph Level

Crafting the perfect opening paragraph is essential to grabbing a reader's attention and inviting them on an engaging journey through your prose. To do so, divide your work into linear segments to be presented in a hierarchical development for ease of comprehension. Retain an even flow by keeping notation and terms familiar while avoiding lengthy explanations or eloquent phrasing that could confuse readers. Furthermore, provide previews of upcoming topics before diving deeper into the content so they can prepare themselves mentally as you write. Adding visuals where appropriate will also aid understanding – use examples and counterexamples along with suggestive references when necessary to maximize clarity without sacrificing creativity.

1. Begin a paragraph with your best sentence to ensure that readers stay hooked.
2. Each sentence needs its own distinct beat. Peruse your writing and fine-tune it until everything flows smoothly.
3. A theorem (definition, lemma, corollary, etc.) should stand firmly on its own two feet and not rely upon what came before.
4. Pay particular attention to terms like "therefore", as they are essential for achieving the right cadence and has many variations such as "whence","hence", "and so", etc.
5. Break up paragraphs by displaying important formulas on a line by themselves.
6. Don't over complicate things with unnecessary subscripts, superscripts, or other vertically spaced symbols.
7. Break up consecutive long sentences into simpler ones, and break up mathematics (but not important formulas) into readable text.
8. Don't be tempted to resort to the use of technical jargon without good reason.
9. Every paragraph should have a (mathematical) point, make it clear.
10. Write a paragraph that flows logically from sentence to sentence to eliminate confusion.

Mathematics writing requires you to effectively communicate your thought process and convince the reader that your conclusions are valid. A first-rate mathematical exposition should provide precise explanations while also being persuasive enough to satisfy a skeptical audience.

Section Level

Crafting a clear and concise section requires consistent information, deliberate structure, and reader-minded readability.

1. **Commmunicate**. While writing, the writer must envision what areas of confusion could arise for their reader and strive to ensure that these are addressed. This foresight is crucial in order to effectively communicate with readers; otherwise it risks creating misunderstandings instead of a meaningful exchange. To ensure accuracy and clarity, having an audience in mind while writing is not simply beneficial but necessary.

2. **Examples**. Mastering the art of example and counterexample can be a powerful tool. However, it is important to not overuse this power – examples should have some "spark" or element that will engage readers with thought-provoking insight. Additionally, don't forget the importance of providing context by using both examples and counterexamples which illustrate definitions/results while simultaneously clarifying any underlying essential assumptions.

3. **Figures**. Whenever possible, facilitate understanding by utilizing visual elements that have the power to communicate complex ideas quickly and accurately. Design figures with captions as reinforcement for concepts discussed in textual form; this will help bring key points into clear focus without being bogged down in minutiae. Work diligently to utilize graphs over tables when illustrating main topics.

4. **Efficiency**. When writing proofs, strive for efficiency by taking advantage of earlier results and avoiding unnecessary repetition. If a current proof seems to be significantly similar to one already constructed in the past, think beyond simple alterations; try finding an interesting overarching generalization that ties both together.

5. **Strategize**. When crafting especially intricate arguments, provide strategic comments throughout your construction – noting the purpose behind each step

taken as well as providing overviews can make lengthy demonstrations easier to follow. When tackling a complex proof, take the time to strategize before you write. Start by outlining how you plan to approach it and give periodic updates on your progress. At the appropiate time and place, demonstrate why particular steps are necessary and look for similarities between results in order to avoid reiterating similar arguments.

> Be strategically efficient.

Crafting a perfect section for your chapter is no easy feat. It requires the careful placement of each element – from preliminary discussions to lemmas, proofs and more – into an order that both makes sense logically while also being effectively digested by readers. The ideal outcome presents detail in such a way so as to capture attention, yet surprise them with unexpected connections between elements. All parts fit together like pieces of a puzzle, forming an explanation or argument far greater than its individual components.

1. While writing, the writer must envision what areas of confusion could arise for their reader and strive to ensure that these are addressed. This foresight is crucial in order to effectively communicate with readers; otherwise it risks creating misunderstandings instead of a meaningful exchange. To ensure accuracy and clarity, having an audience in mind while writing is not simply beneficial but necessary.
2. Mastering the art of example and counterexample can be a powerful tool. However, it is important to not overuse this power – examples should have some "spark" or element that will engage readers with thought-provoking insight. Additionally, don't forget the importance of providing context by using both examples and counterexamples which illustrate definitions/results while simultaneously clarifying any underlying essential assumptions.
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similar to one already constructed in the past, think beyond simple alterations; try finding an interesting overarching generalization that ties both together.

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Document Level

Establishing an introduction that introduces the important theorems, provides insight into what matters, and previews what's to come can "hook" readers in for further exploration. Make sure your document has these components.

Research. To effectively communicate your mathematics, it's essential to take into account the background knowledge of your intended audience. Do the research necessary to know your audience. This may mean teaching classes, going to workshops, collaborating on documents, refereeing journals, or even reading education journals that specialize in learning about your audience. While you are accomplishing this research, keep in mind the following three points.

Firstly, don't let unfamiliar terminology confuse them; provide additional explanation as needed for clarity and comprehension. Secondly, consider adding appendices when discussing more complex or specialized topics that may require deeper context in order to be fully understood by less experienced readers. Thirdly, presenting too much detail can be overwhelming for those without an expert's understanding; however, avoiding explanations altogether risks leaving readers feeling lost or confused.

> Speak to someone and know that someone as well as you can.

Organize. A well-crafted document requires a sound strategy. To achieve this, start by outlining the expected sections in your paper or chapter so you know exactly what to include in each step of its development. Establish necessary definitions that will inform and structure every result and proof before testing them for accuracy with lemmas. Next, dabbling in some examples along the way allows readers to

apply concepts practically. Repeat these steps until there is an obvious logical flow from beginning to end, then add interstitial comments as needed. To a reader, these comments are often the linchpin needed. They should provide further context on certain arguments, compare opposing points within the literature or simply warn against any potential missteps.

Make sure to keep your reader informed throughout; start each segment with a brief introduction and perhaps an outline of what's ahead. Avoid stretches without any clear direction. Instead, provide hints as you progress by stating outcomes such as "It follows that ...". Finally, be sure to reiterate essential points in conclusion for easy recall later on.

When composing equations and propositions, don't make it difficult for readers to find the references; clearly number each item along with providing a content/name descriptor. To ensure concentration isn't broken by excessive page flipping, repeat any simple math expressions used in addition to reminding the reader of earlier analysis or unusual notations they may have encountered. Don't be afraid to reiterate information if necessary – however remain mindful that too much repetition can become exhausting.

> Say something in the right place, at the right time, in the right manner.

Of course, the researching of your audience and the organizing of your document go hand-in-hand. Or as I like to say:

> Research your audience and organize for them.

And Beyond

By reflecting my own personal tastes, this article offers unique insights into the art of mathematical writing. Although some may not agree or share similar opinions, I believe many people will find it valuable and meaningful. While it may not satisfy all readers looking to learn more about this topic, I trust that many have come here seeking a connection between their passions and mine. If you are looking for more on this topic see Halmos (1970), Steenrod (1973), and Higham (1998).

> While it might be hard to get started, writing mathematics is an invaluable skill that requires patience and dedication. **But if you put in the work your mastery of mathematical exposition can be**

> **a force of change that improves lives**.

Finally, I'd like to mention that this is where many beginners believe math ends – with books and articles (documents). But writing your research program, (turning your suspicions, conjectures, unproven ideas, and theories into written words) should be started as soon as possible. Turning your ideas into written mathematics can enable you to harness this force of change.

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