Galois Connections

A Complete Introduction

David A. Smith, Direct Knowledge, USA [*†‡]

Thursday, February 9, 2023

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Example 0.1. Show that the statement

$$(p \wedge \neg q) \to [(\neg p \vee \neg q) \to (p \wedge \neg q)] \tag{1}$$

is a tautology.

1 Introduction

2 What are Galois Connections?

Let $(X, \preceq)$ and $(Y, \leqslant)$ be partially ordered sets.

Definition 2.1. If $f_* : X \to Y$ and $f^* : Y \to X$ are functions such that

$$f_*(x) \leqslant y \Longleftrightarrow x \preceq f^*(y) \tag{2}$$

[*]email: david@directknowledge.com

for all $x \in X$ and all $y \in Y$, then (f_*, f^*) is called a **Galois connection** between $(X, \preceq)$ and $(Y, \leqslant)$.

There are several definitions of Galois connections in the literature; however they are all order-isomorphic to the definition above.

Proposition 2.1. *Let $f_* : X \to Y$ and $f^* : Y \to X$ be functions. Then (f_*, f^*) is a Galois connection if and only if*

1. *both f_* and f^* are monotone,*
2. *$x \preceq (f^* \circ f_*)(x)$ for all $x \in X$, and*
3. *$(f_* \circ f^*)(y) \leqslant y$ for all $y \in Y$.*

Proof. Suppose (f_*, f^*) is a Galois connection. By (2) it follows

$$f_*(x) \leqslant f_*(x) \iff x \preceq (f^* \circ f_*)(x).$$

Since $\leqslant$ is reflexive, $x \preceq (f^* \circ f_*)(x)$ follows immediately. Similarly, by (2) it follows

$$(f_* \circ f^*)(y) \leqslant y \iff f^*(y) \preceq f^*(y)$$

proving that (**??**) also holds. Assume $x_1 \preceq x_2$. By (**??**) we have $x_2 \preceq (f^* \circ f_*)(x_2)$. By (2) it follows $f_*(x_1) \leqslant f_*(x_2)$ and so f_* is monotone. Assume $y_1 \leqslant y_2$. By (**??**) we have $(f_* \circ f^*)(y_1) \leqslant y_1$. By (2) it follows $f^*(y_1) \preceq f^*(y_2)$ and so f^* is also monotone.

Conversely, assume (**??**), (**??**), and (**??**) all hold. Assume $f_*(x) \leqslant y$. By (**??**) and (**??**), it follows $(f^* \circ f_*)(x) \preceq f^*(y)$ and $x \preceq (f^* \circ f_*)(x)$, respectively. By transitivity of $\preceq$, we have $x \preceq f^*(y)$ as needed. Assume $x \preceq f^*(y)$. By (**??**) and (**??**), it follows $f_*(x) \leqslant (f_* \circ f^*)(y)$ and $(f_* \circ f^*)(y) \leqslant y$. By transitivity of $\leqslant$, we have $f_*(x) \leqslant y$ as needed. Therefore, (2) holds and so (f_*, f^*) is a Galois connection. □

Proposition 2.2. *If (f_*, g) and (f_*, h) are Galois connections between $(X, \preceq)$ and $(Y, \leqslant)$, then $g = h$. Likewise, if (g, f^*) and (h, f^*) are Galois connections between $(X, \preceq)$ and $(Y, \leqslant)$, then $g = h$.*

Proof. Let $f_* : X \to Y$ and $g : Y \to X$ be a Galois connection. Also let f_* and $h : Y \to X$ be a Galois connection. By (2) we have the following

$$f_*(x) \leqslant y \iff x \preceq g(y) \tag{3}$$

$$f_*(x) \leqslant y \iff x \preceq h(y) \tag{4}$$

By (3) we have $(f_* \circ h)(y) \leqslant y \iff h(y) \preceq g(y)$. Notice $(f_* \circ h)(y) \leqslant y$ holds by (**??**).(**??**); and thus $h(y) \preceq g(y)$. By (4) we have $(f_* \circ g)(y) \leqslant y \iff g(y) \preceq h(y)$. Notice $(f_* \circ g)(y) \leqslant y$ holds by (**??**).(**??**); and thus $h(y) \preceq g(y)$. Since $\preceq$ is antisymmetric, it follows $g(y) = h(y)$ for arbitrary y. The second statement is the dual of the first and follows just as easily using (**??**).(**??**). $\square$

If (f, g) and (f, h) are Galois connections between $(P, \leq_P)$ and $(Q, \leq_Q)$, then $g = h$. To see this, observe that $p \leq_P g(q)$ iff $f(p) \leq_Q q$ iff $p \leq_P h(q)$, for any $p \in P$ and $q \in Q$. In particular, setting $p = g(q)$, we get $g(q) \leq_P h(q)$ since $g(q) \leq_P g(q)$. Similarly, $h(q) \leq_P g(q)$, and therefore $g = h$. By a similarly argument, if (g, f) and (h, f) are Galois connections between $(P, \leq_P)$ and $(Q, \leq_Q)$, then $g = h$. Because of this uniqueness property, in a Galois connection $f = (f^*, f_*)$, f^* is called **the upper adjoint** of f_* and f_* **the lower adjoint** of f^*.

Proposition 2.3. *If (f_*, f^*) is a Galois connection between $(X, \preceq)$ and $(Y, \leqslant)$, then*

1. *$f^*(y) =$ the maximum of $\{x \in X : f_*(x) \leqslant y\}$ and*
2. *$f_*(x) =$ the minimum of $\{y \in Y : x \preceq f^*(y)\}$.*

Proof. Let $M = \{x \in X : f_*(x) \leqslant y\}$. By (**??**).(**??**) we have $f^*(y) \in M$. Let $x \in M$. Then $f_*(x) \leqslant y$ and since f^* is monotone, it follows $(f^* \circ f_*)(x) \preceq f^*(y)$. By (**??**).(**??**), we have $x \preceq (f^* \circ f_*)(x)$. By transitivity of $\preceq$, we have $x \preceq f^*(y)$ and thus $f^*(y)$ is the maximum of M. For the second statement, let $N = \{y \in Y : x \preceq f^*(y)\}$. By (**??**).(**??**) we have $f_*(x) \in N$. Let $y \in N$. Then $x \preceq f^*(y)$ and since f_* is monotone, it follows $f_*(x) \leqslant (f_* \circ f^*)(y)$. By (**??**).(**??**), we have $(f_* \circ f^*)(y) \leqslant y$ and so by transitivity, it follows $f_*(x) \leqslant y$. Thus $f_*(x)$ is the minimum of N. $\square$

Proposition 2.4. *If (f_*, f^*) is a Galois connection between $(X, \preceq)$ and $(Y, \leqslant)$, then*

1. *$f_* \circ f^* \circ f_* = f_*$, $f^* \circ f_* \circ f^* = f^*$,*
2. *$x \in f^*(Y)$ if and only if x is a fixed point of $f^* \circ f_*$,*
3. *$y \in f_*(X)$ if and only if y is a fixed point of $f_* \circ f^*$,*
4. *$f^*(Y) = (f^* \circ f_*)(X)$, and $f_*(X) = (f_* \circ f^*)(Y)$.*

Proof. (**??**): Using (**??**), we have $f_*(x) \leqslant (f_* \circ f^* \circ f_*)(x)$. By (2) with $x := (f^* \circ f_*)(x)$ and $y := f_*(x)$ it follows $(f_* \circ f^* \circ f_*)(x) \leqslant f_*(x)$ using that $\preceq$ is reflexive.
Since $\leqslant$ is antisymmetric, it follows $f_*(x) = (f_* \circ f^* \circ f_*)(x)$ for arbitrary x, thus proving (**??**) holds. (**??**): By definition, $x \in f^*(Y)$ is equivalent to $f^*(y) = x$ for some $y \in Y$. Then

$$(f^* \circ f_*)(x) = (f^* \circ f_* \circ f^*)(y) = f^*(y) = x$$

follows by (**??**). (**??**): It follows by (**??**) that $f^*(Y) \subseteq (f^* \circ f_*)(X)$. Conversely, let $x \in (f^* \circ f_*)(X)$. Then $x = f^*(y)$ for some $y \in f_*(X) \subseteq Y$. By definition, $x \in f^*(Y)$ and so $(f^* \circ f_*)(X) \subseteq f^*(Y)$. □

Proposition 2.5. *If (f_*, f^*) is a Galois connection between $(X, \preceq)$ and $(Y, \leqslant)$, then*

1. $x \preceq f^*(y) \iff f_*(x) \leqslant (f_* \circ f^*)(y) \iff f_*(x) \leqslant y \iff (f^* \circ f_*)(x) \preceq f^*(y)$,
2. $f^*(x) \preceq f^*(y) \iff (f_* \circ f^*)(x) \leqslant (f_* \circ f^*)(y) \iff (f_* \circ f^*)(x) \leqslant y$, *and*
3. $f_*(x) \leqslant f_*(y) \iff (f^* \circ f_*)(x) \preceq (f^* \circ f_*)(y) \iff x \preceq (f^* \circ f_*)(x)$,
4. $f^*(x) = f^*(y) \iff (f_* \circ f^*)(x) = (f_* \circ f^*)(y)$, *and*
5. $f_*(x) = f_*(y) \iff (f^* \circ f_*)(x) = (f^* \circ f_*)(y)$.

Proof. For the first statement we have

$$\begin{aligned} x \preceq f^*(y) &\implies f_*(x) \leqslant (f_* \circ f^*)(y) \implies f_*(x) \leqslant y \\ &\implies (f^* \circ f_*)(x) \preceq f^*(y) \implies x \preceq f^*(y) \end{aligned}$$

For the third statement we have

$$f^*(x) \preceq f^*(y) \implies (f_* \circ f_*)(x) \leqslant (f_* \circ f^*)(y) \implies (f_* \circ f_*)(x) \leqslant y$$

For the fourth statement we have

$$\begin{aligned} f^*(x) = f^*(y) &\iff f^*(x) \preceq f^*(y) \wedge f^*(y) \preceq f^*(x) \\ &\iff (f_* \circ f^*)(x) \leqslant (f_* \circ f^*)(y) \wedge (f_* \circ f^*)(y) \leqslant (f_* \circ f^*)(x) \\ &\iff (f_* \circ f^*)(x) = (f_* \circ f^*)(y) \end{aligned}$$

The remaining statements are the dual and easily proved. □

By an **order isomorphism** from an partially ordered set X to another partially ordered set Y we shall mean an isotone bijection $f : X \to Y$ whose inverse $f^{-1} : Y \to X$ is also an isotone.

Proposition 2.6. *Partially ordered sets $(X, \preceq)$ and $(Y, \leqslant)$ are isomorphic if and only if there is a surjective mapping $f : X \to Y$ such that*

$$x \preceq y \iff f(x) \leqslant f(y).$$

Proof. The necessity is clear. Suppose conversely that such a surjective mapping f exists. Then f is also injective; for if $f(x) = f(y)$ then from $f(x) \leqslant f(y)$ we obtain $x \preceq y$ and from $f(y) \leqslant f(x)$ we obtain $y \preceq x$, so that $x = y$. Hence f is a bijection. Clearly, f is isotone; and so also is f^{-1}, since $x \preceq y$ can be written $f(f^{-1}))(x) \leqslant f(f^{-1})(y)$ which gives $f^{-1}(x) \preceq f^{-1}(y)$. □

Proposition 2.7. *If (f_*, f^*) is a Galois connection between $(X, \preceq)$ and $(Y, \leqslant)$, then $f^*(Y)$ and $f_*(X)$ are order-isomorphic.*

Proof. This follows immediately from (**??**) and (**??**). □

Definition 2.2. A function f on X is called a **(co-)closure function** if

1. f is extensive, $\forall x \in X, x \preceq f(x)$, $(\forall x \in X, f(x) \preceq x)$,
2. f is monotone, $x_1 \preceq x_2 \implies f(x_1) \preceq f(x_2)$, for all $x_1, x_2 \in X$, and
3. f is idempotent, $f(f(x)) = f(x)$ for all $x \in X$.

Proposition 2.8. *If (f_*, f^*) is a Galois connection between $(X, \preceq)$ and $(Y, \leqslant)$, then $f^* \circ f_*$ is a closure function for X and $f_* \circ f^*$ is a co-closure function for Y.*

Proof. Let $x \in X$. Then $x \preceq (f^* \circ f_*)(x)$ follows by (**??**).(**??**). Since both f^* and f_* are monotone we have, $x_1 \preceq x_2 \implies (f^* \circ f_*)(x_1) \preceq (f^* \circ f_*)(x_2)$. Thus $f^* \circ f_*$ is also monotone. By associativity of functions and (**??**).(**??**) we have

$$(f^* \circ f_*) \circ (f^* \circ f_*) = f^* \circ (f_* \circ f^* \circ f_*) = f^* \circ f_*$$

as needed. The dual statement is proved just as easily. □

By (**??**), the closed elements of $f^* \circ f_*$ and $f_* \circ f^*$ are precisely the elements that are an image of some element under f^*, respectively f_*.

There is a one-to-one correspondence between Galois connections and (co-)closure functions.

Proposition 2.9. *If f is a closure (respectively co-closure) function, then there is a Galois connection (f_*, f^*) such that $f = f^* \circ f_*$ (respectively $f = f_* \circ f^*$).*

Proof. Let $f : X \to X$ be a closure over $(X, \preceq)$. Let $\overline{X}$ be the set of closed elements of f that is $f(X) = \overline{X}$. We will construct a Galois connection between $\overline{X}$ and X using (**??**). Let $f_* = f$, that is $f_* : X \to \overline{X}$ defined by $f_*(x) = f(x)$ for all $x \in X$. Let $f^* : \overline{X} \to X$ be the inclusion mapping, that is $f^*(x) = x$ for all $x \in \overline{X}$. Notice $f_* \circ f^*$ is the identity on $\overline{X}$ and $f^* \circ f_* = f$.

1. Notice f_* is monotone since f is monotone and that f^* is monotone since the identity is monotone.
2. Let $x \in X$. Since f is extensive, we have $x \preceq f(x)$. Thus it follows,
$$x \preceq f(x) = (f^* \circ f)(x) = (f^* \circ f_*)(x).$$
3. Let $y \in \overline{X}$. There exists $x \in X$ such that $y = f(x)$. Since f is idempotent we have
$$f(y) = (f \circ f)(y) \preceq y$$
for all $y \in \overline{X}$ as needed.

Therefore, $(f_*, f^*) = (f, f^*)$ where $f^* : \overline{X} \to X$ is the inclusion mapping is a Galois connection between $(X, \prec 0$ and $(\overline{X}, \preceq)$. □

Proposition 2.10. *Let R be a relation between X and Y. Let*

$$f_R(A) = \{b \in Y : \forall a(a \in A \implies (a, b) \in R)\} \text{ and } \tag{5}$$

$$f^R(B) = \{a \in X : \forall b(b \in B \implies (a, b) \in R)\}. \tag{6}$$

Then (f_R, f^R) *is a Galois connection between* $(P(X), \subseteq)$ *and* $(P(Y), \supseteq)$

Proof. Clearly, $f_R : P(X) \to P(Y)$ and $f^R : P(Y) \to P(X)$ are functions. By (2) we must show

$$f_R(A) \supseteq B \Longleftrightarrow A \subseteq f^R(B) \tag{7}$$

for all $A \in P(X)$ and all $B \in P(Y)$. Assume $B \subseteq f_R(A)$. We will show $A \subseteq f^R(B)$. Let $x \in A$. If $y \in B$, then $y \in f_R(A)$. Then, by (5), it follows $(x, y) \in R$. So we have shown, $y \in B \implies (x, y) \in R$ as needed to show $x \in f^R(B)$. Conversely, assume $A \subseteq f^R(B)$. We will show $B \subseteq f_R(A)$. Let $y \in B$. If $x \in A$, then $x \in f^R(B)$. Then, by (6), it follows $(x, y) \in R$. So we have shown, $x \in A \implies (x, y) \in R$ as needed to show $y \in f_R(A)$. Therefore, (7) holds. $\square$