Multivariable Calculus

To my wife, Sally, and our children,

Courtney, Rebecca, and Benjamin,

whose support made this book possible

David A. Smith

Multivariable Calculus

With Python

First Edition

Direct Knowledge

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Direct Knowledge

Preface

Writing Philosophy

I recognize the importance of writing confidently, crafting original ideas, and thoughtfully-argued positions. I take my proficiency in language seriously, seeking to hone good grammar usage and proper punctuation for captivating mathematical exposition. To learn more about how I approach writing mathematics please visit https://directknowledge.com/writing https://directknowledge.com/writing

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David A. Smith \ Fort Worth, Texas

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Chapter 1

Vector Functions

Vector-valued functions are functions that assign a vector to every point in a given space. In this book, we will explore the properties of these functions and how they can be used to model real-world situations. We'll start with the basics and work our way up to more complex concepts. Along the way, you'll learn about space curves and how to calculate their derivatives and integrals. So whether you're a beginner or a pro, this book has something for you!

A vector function (or a, vector-valued function) is a function that takes one or more inputs and outputs a vector. Vector functions have a wide range of applications in science and technology.

Vector functions are important in mathematics, physics, and engineering. In physics, vector functions are used to describe the motion of particles and the behavior of waves. In engineering, vector functions are used to design bridges, buildings, and other structures. And in mathematics, vector functions are used to study geometry and topology.

Vector functions are quite interesting once you wrap your head around them. So let's take a few minutes to do just that.

A vector function is simply a function that takes one or more scalars as input and outputs a vector. In other words, it's a function that gives you a vector when you plug in a number(s). For example, the function $f(x) = (2x, 3x - 1)$ is a vector function. You can think of it as a machine that takes in a number (x) and spits out a vector (2x, 3x-1). Pretty simple, right?

Now, vector functions can be used to model all sorts of things in the real world, from the trajectory of a baseball to the movement of particles in a fluid. So the next time you're struggling with a vector function problem, just remember that you're really just trying to understand the movement of something in the world around us. And that's not so tough after all.

Derivatives and integrals of vector functions may sound like a mouthful, but they're really just mathematical tools that allow us to use vector functions. In other words, they let us find out how vector functions change over time or space. And that can be really useful information, whether we're trying to figure out the trajectory of a projectile or the motion of a planet in orbit.

So what exactly are derivatives and integrals of vector functions? Well, derivatives tell us how vector functions change with respect to changes in their input variables, while integrals give us information about how vector functions change over time or space. Basically, they're two sides of the same coin: one tells us how a vector function changes in response to changes in its input variables, while the other tells us how a vector function changes over time or space. But together, they give us a pretty complete picture of how vector functions behave.

Vector functions are used in a variety of fields, such as physics, engineering, and economics. One example of their use is in the analysis of electrical circuits. Vector-valued functions can be used to model the voltage and current in an electrical circuit. This information can then be used to design products that are more efficient and have fewer problems.

In addition, vector-valued functions can be used to model the movement of objects in space. This information can be used to optimize the trajectory of a spacecraft or to predict the path of a hurricane. As you can see, vector-valued functions have a wide range of applications in the real world.

There's more to a curve than meets the eye. In fact, one of the most important properties of a curve is its arc length. Arc length is simply the length of the curve between two points, and it can be calculated using vector functions.

Of course, this assumes that you have a vector function for your curve. If you don't, you could always try finding one. In any case, arc length is a vital property of curves, and it's something that everyone should know about.

It's a well-known fact that the shortest distance between two points is a straight line. But what if those points are moving? This is where the concept of arc length comes in. Arc length is simply the distance traveled by a point moving along a curve. In other terms, it's the length of a vector function. And just like with regular old straight-line distances, we can use arc length to calculate things like speed and acceleration.

So the next time you find yourself on a winding road, take comfort in knowing that you're traveling the mathematically shortest distance possible!

Curvature is a measure of how vector functions change as you move along them. In other words, it's a way to quantify how "curved" a function is. For example, the function $f(x) = x^2$ is more curved than the function $g(x) = x$. In this chapter, you'll learn how to find the "curvature vector". This vector tells you how quickly the direction of the vector function is changing at any given point. It also tells you which way the vector function is curving - whether it's curving to the left or right. Curvature can be positive or negative, depending on which way the vector function is curving.

And finally, the curvature can be calculated for any (smooth) vector function - not just those that represent curves in two-dimensional space.

This book is for anyone who wants to learn about vector-valued functions and their applications. Whether you're a student, a professional, or just someone curious about the topic, this book will give you a thorough understanding of vector-valued functions. We'll start with the basics and work our way up to more advanced material:

- What are vector-valued functions and what do they do?
- How are vector-valued functions used in the real world?
- What is arc length and how can it be calculated?
- What is curvature and how can it be calculated?

In this book, you'll find short, concise chapters that get straight to the point. I'll also be using a lot of pictures and diagrams to help you understand the material. And finally, I'll be giving you plenty of opportunities to practice what you've learned with lots of exercises.

In general, the best way I teach vector functions is to start with simple examples and then gradually increase the complexity of the examples. This gives you a chance to learn the basics and solidify your understanding of the material before being overwhelmed by too much information at once.

So if you're ready to learn about vector-valued functions, then let's get started!

1.1 Space Curves

Before we prove any theorems and develop any definitions let's see how natural it is to use vector functions to model the world around us.

In the professor's classroom, there is little to no air movement so we assume the only force acting on the arrow is gravity.

Example 1.1. To start a calculus semester a professor wants to light a torch with a flaming arrow that sits on top of a podium on the other side of a large classroom.

Solution. Suppose the professor lit the arrow and shot it at a height of 6 ft above the ground level 90 ft from the 30 ft high podium and he wanted the arrow to reach a maximum height exactly 4 ft above the center of the podium. What will the initial firing angle of the arrow be?

Assume that the arrow is fired from the initial point $(x_0, y_0) = (0, 6)$ corresponding to $t = 0$. Using the definition of sine and cosine the initial velocity is found to be

$$\begin{aligned}\vec{v}_0 &= (x_0 + ||\vec{v}_0|| \cos\theta)\,\vec{i} + (y_0 + ||\vec{v}_0|| \sin\theta)\,\vec{j} \\ &= ||\vec{v}_0|| \cos\theta\,\vec{i} + (6 + ||\vec{v}_0|| \sin\theta)\,\vec{j}\end{aligned}$$

where θ is the angle that $\vec{v}_0$ makes with the horizontal. Recall Newton's second law of motion which says the the force acting on the projectile is equal to the projectile's mass m times its acceleration, or

$$\vec{F} = m\frac{d^2\,\vec{r}}{dt^2}$$

when $\vec{r}$ is the projectile's position vector and t is time. In the
We assume the only force acting on the flaming arrow is the gravitational force $-mg\,\vec{j}$, then

$$\frac{d^2\,\vec{r}}{dt^2} = -g\,\vec{j}, \quad \vec{r}_0 = 0\,\vec{i} + 6\,\vec{j}, \quad \text{and} \quad \left.\frac{d\vec{r}}{dt}\right|_{t=0} = \vec{v}_0. \tag{1.1}$$

The first integration gives $\frac{d\vec{r}}{dt} = -gt\,\vec{j} + \vec{v}_0$. The second integration yields

$$\vec{r}(t) = \frac{-gt^2}{2}\,\vec{j} + \vec{v}_0 t + \vec{r}_0. \tag{1.2}$$

Now using substitution of the initial conditions in (1.1) into equation (1.2) we find an expression for the position function of the arrow

$$\vec{r}(t) = (||\vec{v}_0|| \cos\theta)\,t\,\vec{i} + \left(6 + (||\vec{v}_0|| \sin\theta)t - \frac{1}{2}gt^2\right)\,\vec{j}. \tag{1.3}$$

The arrow reaches its highest point when $\frac{dy}{dt} = 0$, and solving for t we obtain

$$t = \frac{||\vec{v}_0|| \sin\theta}{g}.$$

For this value of t, the value of y is

$$6 + \frac{(||\vec{v}_0|| \sin\theta)^2}{2g}.$$

Using $y_{\max} = 34$ and $g = 32$, we see that

$$34 = 6 + \frac{(||\vec{v}_0|| \sin\theta)^2}{2(32)}$$

or $||\vec{v}_0|| \sin\theta = \sqrt{(28)(64)}$.

In order to find θ we wish to find a similar expression for $||\vec{v}_0|| \cos\theta$. When the arrow reaches $y_{\text{max}} = 34$, the horizontal distance is $x = 90$ ft and by substitution into equation (1.3) we obtain

$$90 = (||\vec{v}_0|| \cos\theta) \left(\frac{||\vec{v}_0|| \sin\theta}{g} \right).$$

Therefore we find

$$\tan\theta = \frac{||\vec{v}_0|| \sin\theta}{||\vec{v}_0|| \cos\theta} = \frac{\left(\sqrt{(28)(64)}\right)^2}{(90)(32)} = \frac{28}{45}.$$

So the approximate firing angle will be

$$\theta = \tan^{-1}(28/45) \approx 31.8908^\circ. \tag{1.4}$$

1.2 Definition of a Vector-Valued Function

Vector functions can be used to model the motion of an object. For example, suppose a fly takes off from the top of a coffee cup at the front of a classroom. The path the fly traces as it travels through the classroom is a one-dimensional path which can be described as a vector-valued function. You may want to study the path (geometry only), but you may also want to know its speed, direction of motion, and acceleration at each point in time, in that case you will want more than its path – you will want a vector-valued function that describes this fly's motion in three dimensions. Given a vector-valued function defined at each point in time, you will not only have position but also its speed, direction of motion, and acceleration at each point in time.

A vector-valued function can be defined with more than one variable and with more components. In general, a vector-valued function is a function that takes an n-tuple of variables and outputs a unique vector with m components. We start of by defining a vector function of one variable and then give several examples.

Definition 1.1. A **vector-valued function** $\vec{F}$ of a real variable t with D assigns to each number t in the set D a unique vector $\vec{F}(t)$. The set of all vectors $\vec{v}$ of the form $\vec{v} = \vec{F}(t)$ for t in D is the **range** of $\vec{F}$.

In three dimensions vector functions can be expressed in the form

$$\vec{F}(t) = f_1(t)\vec{i} + f_2(t)\vec{j} + f_3(t)\vec{k}$$

where f_1, f_2, and f_3 are real-valued functions of the real variable t defined on the domain set D. A vector function may also be denoted by

$$\vec{F}(t) = \langle f_1(t), f_2(t), f_3(t) \rangle.$$

Unless stated otherwise, the **domain** of a vector function $\vec{F}$ is the intersection of the domains of the scalar component functions f_1, f_2, and f_3.

Next we take a given vector-valued function and explicitly say what the scalar component functions are and ascertain the domain.

Example 1.2. Given the vector function

$$\vec{F}(t) = t^3\,\vec{\imath} + \ln(3-t)\,\vec{\jmath} + \sqrt{t}\,\vec{k}$$

find the component functions and the domain of $\vec{F}$, and then evaluate $\vec{F}$ at $t = 1$.

Solution. The component functions are $f(t) = t^3$, $g(t) = \ln(3-t)$, and $h(t) = \sqrt{t}$
where $\vec{F}(t) = f(t)\,\vec{\imath} + g(t)\,\vec{\jmath} + h(t)\,\vec{k}$. The domain of $\vec{F}(t)$ is the interval $[0, 3)$ since $\sqrt{t}$ requires $t \geq 0$ and $\ln(3-t)$ requires $t < 3$. We can evaluate the vector function $\vec{F}$ at $t = 1$ as follows

$$\vec{F}(1) = f(1)\,\vec{\imath} + g(1)\,\vec{\jmath} + h(1)\,\vec{k} = \vec{\imath} + \ln 2\,\vec{\jmath} + \vec{k}.$$

1.3 Graphs of Vector Functions

Vector functions are useful for tracing out graphs of curves and for describing motion along a path. Often the variable t represents time and since each $\vec{F}(t)$ represents a vector, we have a position (x, y, z) at time t. That is to say, given a time value of t we have a vector

$$\vec{F}(t) = \langle f_1(t), f_2(t), f_3(t) \rangle$$

which represents a point (x, y, z) where $x = f_1(t)$, $y = f_2(t)$, and $z = f_3(t)$. In this manner, we use arrowheads on the curve to indicate the curve's orientation by pointing in the direction of increasing values of t.

Sketching the graph a given vector function can be time-consuming, especially if we are given an unfamiliar function. In the next example we notice that each scalar component function is linear and so claim that the vector function is linear in three dimensions. Thus graphing this vector function is easy; just pick any two points in the domain you wish, plot them and then draw a straight line.

Example 1.3. Sketch the graph of the vector-valued function defined by

$$\vec{F}(t) = -3\sin t\vec{\imath} + 3\cos t\vec{\jmath} + 0.1t\vec{k}.$$

Solution. The graph is the set of all points (x, y, z) with

$$x = -3\sin t, \quad y = 3\cos t, \quad \text{and} \quad z = 0.1t.$$

The graph is a circular helix that lies on the surface of the cylinder with equation

$$x^2 + y^2 = (-3\sin t)^2 + (3\cos t)^2 = 9.$$

The cylinder is centered at (0,0) in the xy-plane as shown in **??**.

Example 1.4. Sketch the graph of the vector-valued function

$$\vec{F}(t) = (5 - 2t)\vec{\imath} + (3 + 2t)\vec{\jmath} + 2t\vec{k}.$$

Solution. The graph is the set of all points (x, y, z) with $x = 5 - 2t$, $y = 3 + 2t$, and $z = 5t$. The graph is a line that passes through the point $(5, 3, 0)$ (when $t = 0$) and the point $(3, 5, 5)$ (when $t = 1$) as shown in **??**.

1.4 Spaces Curves and Parameterizations

It is important to realize that there is not a one-to-one correspondence between a one-dimensional graph in three dimensions and a vector function. That is to say a vector-valued function has a graph, and only one set of points in three dimensions is its graph. But a graph (set of points in three dimensions) can be represented by more than one functional rule. The classic two dimensional example is that of the unit circle, which can be parametrized as a vector-valued function by $\vec{F}(t) = \sin t\vec{\imath} + \cos t\vec{\jmath}$ with \$0 t<2 \$; and can also be parametrized as a vector-valued function by $\vec{F}(t) = \sin(2t)\vec{\imath} + \cos(2t)\vec{\jmath}$ with \$0 t< \$. With both of these vector-valued functions we have the same graph: the unit circle. It is this relationship between algebra and geometry that make vector-valued functions extremely useful and important to study.

Finding a functional rule for a vector function, from given geometrical information, can also be a challenging endeavor. In fact, given a geometrical object in three dimensions, there can be quite a few ways to represent that object using algebra. In short, parametrizing a geometrical object, even one as simple as the unit circle, can lead to many parametric representations and thus different vector functions.

In the next few examples we illustrate two ways in which this process might be carried out. We are given a geometric object and asked to find a vector function, (find a parametrization), that represents the given object.

There are many other ways to accomplish the task of introducing the variable t, e.g. try $x = 2t$. What's interesting is that no matter how you introduce t the vector function obtained will have the same graph.

Example 1.5. Find a vector-valued function $\vec{F}$ whose graph is the curve of intersection of the hemisphere $z = \sqrt{1 - x^2 - y^2}$ and the parabolic cylinder $y = x^2$.

Solution. One way to accomplish the task is by letting $x = t$. Then $y = t^2$ and

$$z = \sqrt{1 - x^2 - y^2} = \sqrt{1 - t^2 - t^4}.$$

Therefore

$$\vec{F}(t) = t\vec{i} + t^2\vec{j} + \sqrt{1 - t^2 - t^4}\vec{k}.$$

is a value-valued function for this intersection.

Try reworking this example with $x = 2t$. and graph the resulting vector function.

Example 1.6. Find a vector-valued function $\vec{F}$ whose graph is the curve of intersection of the plane $2x + y + 3z = 6$ and the plane $x - y - z = 1$.

Solution. One way to accomplish the task is by letting $x = t$. Then to find relations for y and z we will solve the system

$$\begin{cases} t - y - z = 1 \\ 2t + y + 3z = 6. \end{cases}$$

Eliminating y we have, $3t + 2z = 7$ and so $z = (7 - 3t)/2$. Solving the first for y we find

$$y = t - 1 - z = t - 1 - \left(\frac{7 - 3t}{2}\right) = \frac{2t - 2 - 7 + 3t}{2} = \frac{5t - 9}{2}$$

Therefore

$$\vec{F}(t) = t\,\vec{i} + \frac{5t - 9}{2}\vec{j} + \frac{7 - 3t}{2}\vec{k}. \tag{1.5}$$

is a vector-valued function for this intersection.

1.5 Operations with Vector Functions

Next we show how to perform basic operations like addition, subtraction, dot product and the cross product with vector functions.

Definition 1.2. Let $\vec{F}$ and $\vec{G}$ be vector-valued functions of the real variable t, and let $f(t)$ be a real-valued function. Then $\vec{F}+\vec{G}$, $\vec{F}-\vec{G}$, $f\vec{F}$, $\vec{F}\times\vec{G}$, and $\vec{F}\cdot\vec{G}$ are vector functions defined as follows

- $(\vec{F}+\vec{G})(t)=\vec{F}(t)+\vec{G}(t)$
- $(f\vec{F})(t)=f(t)\vec{F}(t)$
- $(\vec{F}-\vec{G})(t)=\vec{F}(t)-\vec{G}(t)$
- $(\vec{F}\cdot\vec{G})(t)=\vec{F}(t)\cdot\vec{G}(t)$
- $(\vec{F}\times\vec{G})(t)=\vec{F}(t)\times\vec{G}(t)$ These operations are defined on the intersection of the domain of the vector-valued and real-valued functions that occur in the definitions, respectively.

Example 1.7. Let $\vec{F}$, $\vec{G}$, and $\vec{H}$ be the vector functions defined by

$$\vec{F}(t)=(2t)\vec{\imath}-5\vec{\jmath}+t^2\vec{k}, \quad \vec{G}(t)=(1-t)\vec{\imath}+\frac{1}{t}\vec{k}, \quad \vec{H}(t)=(\sin t)\vec{\imath}+e^t\vec{\jmath}.$$

Find the vector functions

- $\vec{F}(t)\times\vec{G}(t)$
- $2e^t\vec{F}(t)+t\vec{G}(t)+10\vec{H}(t)$
- $\vec{G}(t)\cdot[\vec{H}(t)\times\vec{F}(t)]$

Solution. • Using 1.2, we find

$$\begin{aligned} &2e^t\vec{F}(t)+t\vec{G}(t)+10\vec{H}(t)\\ &\qquad=\left(4te^t+t-t^2+10\sin t\right)\vec{\imath}+\left(2t^2e^t+1\right)\vec{k}. \end{aligned}$$

- Using the determinant formula for a cross product of vectors, we find

$$\begin{aligned} \vec{F}(t)\times\vec{G}(t) &= \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 2t & -5 & t^2 \\ 1-t & 0 & \frac{1}{t} \end{vmatrix} \\ &= \left(\frac{-5}{t}\right)\vec{\imath}+\left(-t^3+t^2-2\right)\vec{\jmath}+(-5t+5)\vec{k}. \end{aligned}$$

- We find

$$\vec{G}(t)\cdot[\vec{H}(t)\times\vec{F}(t)]=(1-t)t^2e^t-\frac{2te^t+5\sin t}{t}$$

by using

$$\begin{aligned} &\vec{G}(t)\cdot[\vec{H}(t)\times\vec{F}(t)]\\ &\qquad=\left[(1-t)\vec{\imath}+\frac{1}{t}\vec{k}\right]\cdot\begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \sin t & e^t & 0 \\ 2t & -5 & t^2 \end{vmatrix}. \end{aligned}$$

1.6 Limits of Vector Functions

The idea is now to extend the notion of a limit of a one-variable function and continuity of a one-variable function to vector-valued functions by capitalizing on the properties of vector functions.

Definition 1.3. Suppose the components function f_1, f_2, and f_3 of the vector-valued function

$$\vec{F}(t) = f_1(t)\vec{\imath} + f_2(t)\vec{\jmath} + f_3(t)\vec{k} \tag{1.6}$$

all have finite limits as $t \to t_0$, where t_0 is any real number or $\pm\infty$$. Then the limit of $\vec{F}(t)$ as $t \to t_0$ is defined as the vector

$$\lim_{t\to t_0} \vec{F}(t) = \left(\lim_{t\to t_0} f_1(t)\right)\vec{\imath} + \left(\lim_{t\to t_0} f_2(t)\right)\vec{\jmath} + \left(\lim_{t\to t_0} f_3(t)\right)\vec{k}.$$

::: {#lem- } [Limits of Vector-Functions] If the vector-valued functions $\vec{F}$ and $\vec{G}$ are functions of a real variable t and $h(t)$ is a real-valued function such that all three functions have finite limits as $t \to t_0$, then

- $\lim_{t\to t_0}[\vec{F}(t) + \vec{G}(t)] = \lim_{t\to t_0} \vec{F}(t) + \lim_{t\to t_0} \vec{G}(t)$
- $\lim_{t\to t_0}[\vec{F}(t) - \vec{G}(t)] = \lim_{t\to t_0} \vec{F}(t) - \lim_{t\to t_0} \vec{G}(t)$
- $\lim_{t\to t_0}[h(t)\vec{F}(t)] = \left[\lim_{t\to t_0} h(t)\right]\left[\lim_{t\to t_0} \vec{F}(t)\right]$
- $\lim_{t\to t_0}[\vec{F}(t) \cdot \vec{G}(t)] = \left[\lim_{t\to t_0} \vec{F}(t)\right] \cdot \left[\lim_{t\to t_0} \vec{G}(t)\right]$
- $\lim_{t\to t_0}[\vec{F}(t) \times \vec{G}(t)] = \left[\lim_{t\to t_0} \vec{F}(t)\right] \times \left[\lim_{t\to t_0} \vec{G}(t)\right]$

These limit formulas are also valid when $t \to \pm\infty$ provided all limits are finite. :::

Proof. Let $\vec{F}$ and $\vec{G}$ have (respectively) standard forms

$$\vec{F}(t) = f_1(t)\vec{\imath} + f_2(t)\vec{\jmath} + f_3(t)\vec{k} \qquad \vec{G}(t) = g_1(t)\vec{\imath} + g_2(t)\vec{\jmath} + g_3(t)\vec{k}. \tag{1.7}$$

(i) Using (1.7) we find,

$$\begin{aligned}
&\lim_{t\to t_0}[\vec{F}(t) + \vec{G}(t)] \\
&= \left(\lim_{t\to t_0}(f_1 + g_1)(t)\right)\vec{\imath} + \left(\lim_{t\to t_0}(f_2 + g_2)(t)\right)\vec{\jmath} + \left(\lim_{t\to t_0}(f_3 + g_3)(t)\right)\vec{k} \\
&= \lim_{t\to t_0}[f_1(t)\vec{\imath} + f_2(t)\vec{\jmath} + f_3(t)\vec{k}] + \lim_{t\to t_0}[g_1(t)\vec{\imath} + g_2(t)\vec{\jmath} + g_3(t)\vec{k}] \\
&= \lim_{t\to t_0} \vec{F}(t) + \lim_{t\to t_0} \vec{G}(t)
\end{aligned}$$

(iv) Again using (1.7) we find,

$$\begin{aligned}\lim_{t\to t_0}[\vec{F}(t)\cdot\vec{G}(t)] &= \lim_{t\to t_0}[f_1(t)g_1(t)+f_2(t)g_2(t)+f_3(t)g_3(t)]\\ &= [f_1(t_0)g_1(t_0)+f_2(t_0)g_2(t_0)+f_3(t_0)g_3(t_0)]\\ &= \vec{F}(t_0)\cdot\vec{G}(t_0)\\ &= \left[\lim_{t\to t_0}\vec{F}(t)\right]\cdot\left[\lim_{t\to t_0}\vec{G}(t)\right]\end{aligned}$$

The remainder of the proof is left as Exercise **??**.

$\square$

Example 1.8. Find $\lim_{t\to 0} h(t)\vec{F}(t)$ given

$$\vec{F}(t) = (\sin t)\vec{i} - t\vec{k}$$

and $h(t) = 1/(t^2+t-1)$.

Solution. By 1.6, we find

$$\begin{aligned}&\lim_{t\to 0}\left(\frac{\sin t}{t^2+t-1}\right)\vec{i} - \frac{t}{t^2+t-1}\vec{k}\\ &\quad= \lim_{t\to 0}\left(\frac{\sin t}{t^2+t-1}\right)\vec{i} - \lim_{t\to 0}\left(\frac{t}{t^2+t-1}\right)\vec{k} = (0)\vec{i}-(0)\vec{k} = \vec{0}.\end{aligned}$$

Example 1.9. Given $\vec{F}(t) = 2\vec{i} - t\vec{j} + e^t\vec{k}$ and $\vec{G}(t) = t^2\vec{i} + 4\sin t\vec{j}$ find

$$\lim_{t\to 2}\vec{F}\times\vec{G}.$$

Solution. By 1.6, we find

$$\begin{aligned}\lim_{t\to 2}\vec{F}\times\vec{G} &= \left(2\vec{i}-2\vec{j}+e^2\vec{k}\right)\times(4\vec{i}+4\sin 2\vec{j})\\ &= \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2 & -2 & e^2\\ 4 & 4\sin 2 & 0\end{vmatrix}\\ &= \left(-4e^2\sin 2\right)\vec{i} - \left(-4e^2\right)\vec{j} + (8\sin 2+8)\vec{k}.\end{aligned}$$

1.7 Continuous Vector Functions

One of the reasons why vector-valued functions are so favorable is that it is easy to extend known results from functions of one-variable and results in two-dimensional space to results concerning vector functions of one-variable, or **space curves**, that have graphs in 3 (or more) dimensions. Continuity of vector functions is one such example.

Definition 1.4. A vector-valued function $\vec{F}$ is **continuous** at t_0 means t_0 is in the domain of $\vec{F}$ and $\lim_{t\to t_0}\vec{F}(t)=\vec{F}(t_0)$. Further, a vector function is **continuous on an interval** I if it is continuous at every point in the interval.

::: {#thm- } Continuous Vector Functions A vector function is continuous at a if and only if each of its component functions is continuous at a. :::

Proof. Suppose that $\vec{F}$ is a vector function that is continuous at a. Let

$$\vec{F}(t)=f_1(t)\vec{i}+f_1(2)\vec{j}+f_3(t)\vec{k}.$$

Then we have

$$\begin{aligned}\vec{F}(a)=\lim_{t\to a}\vec{F}(t)&=\lim_{t\to a}\left[f_1(t)\vec{i}+f_1(2)\vec{j}+f_3(t)\vec{k}\right]\\&=\left(\lim_{t\to a}f_1(t)\right)\vec{i}+\left(\lim_{t\to a}f_2(t)\right)\vec{j}+\left(\lim_{t\to a}f_3(t)\right)\vec{k}\end{aligned}$$

Hence

$$\lim_{t\to a}f_1(t)=f_1(a),\quad \lim_{t\to a}f_2(t)=f_2(a),\quad \lim_{t\to a}f_3(t)=f_3(a).$$

Therefore, each of the component functions of $\vec{F}$ is continuous at a. The reminder of the proof is Exercise **??**.

□

Example 1.10. Determine where the vector-valued function

$$\vec{F}(t)=te^t\vec{i}+\frac{e^t}{t}\vec{j}+3e^t\vec{k}$$

is continuous.

Solution. The component functions te^t and $3e^t$ are continuous for all real numbers. The component function e^t/t is continuous on its domain. By 1.7, the function $\vec{F}$ is continuous for all real numbers in its domain which is $\{t\in\mathbb{R}\mid t\neq 0\}$.

Example 1.11. Determine where the vector-valued function $\vec{F}$ defined by

$$\vec{F}(t) = \frac{\vec{u}}{\|\vec{u}\|}, \qquad \vec{u} = t\vec{i} + \frac{1}{\sqrt{t}}\vec{j} + e^t\vec{k}$$

is continuous.

Solution. By 1.7, the function $\vec{F}$ is continuous for all real numbers in its domain which is $\{t \in \mathbb{R} : t > 0\}$ because

$$\|\vec{u}\| = \sqrt{t^2 + \left(\frac{1}{\sqrt{t}}\right)^2 + e^{2t}}$$

and therefore

$$\begin{aligned}\lim_{t\to t_0} \vec{F}(t) &= \lim_{t\to t_0} \left(\frac{t\vec{i} + \frac{1}{\sqrt{t}}\vec{j} + e^t\vec{k}}{\sqrt{t^2 + \left(\frac{1}{\sqrt{t}}\right)^2 + e^{2t}}} \right) = \frac{t_0\vec{i} + \frac{1}{\sqrt{t_0}}\vec{j} + e^{t_0}\vec{k}}{\sqrt{(t_0)^2 + \left(\frac{1}{\sqrt{t_0}}\right)^2 + e^{2t_0}}} \\ &= \left(\frac{t_0}{\sqrt{t_0^2 + \frac{1}{t_0} + e^{2t_0}}} \right)\vec{i} + \left(\frac{\frac{1}{\sqrt{t_0}}}{\sqrt{t_0^2 + \frac{1}{t_0} + e^{2t_0}}} \right)\vec{j} + \left(\frac{e^{t_0}}{\sqrt{t_0^2 + \frac{1}{t_0} + e^{2t_0}}} \right)\vec{k}\end{aligned}$$

for all real numbers such that $t_0 > 0$.

1.8 Exercises

Exercise 1.1. Find the domain of the vector function

$$\vec{F}(t) = (1-t)\vec{i} + \left(\sqrt{t}\right)\vec{j} - \frac{1}{t-2}\vec{k}.$$

Exercise 1.2. Find the domain of the vector function

$$\vec{F}(t) = (\cos t)\vec{i} - (\cot t)\vec{j} + (\csc t)\vec{k}.$$

Exercise 1.3. Find the domain of the vector function $\vec{F}(t) + \vec{G}(t)$ where $\vec{F}(t) = 3t\vec{j} + t^{-1}\vec{k}$ and $\vec{G}(t) = 5t\vec{i} + \sqrt{10-t}\vec{j}$.

Exercise 1.4. Find the domain of the vector function $\vec{F}(t) \times \vec{G}(t)$ where $\vec{F}(t) = t^2\vec{i} - t\vec{j} + 2t\vec{k}$ and $\vec{G}(t) = \frac{1}{t+2}\vec{i} + (t+4)\vec{j} - \sqrt{-t}\vec{k}$.

Exercise 1.5. Describe the graph in words and sketch a graph by hand for the vector function $\vec{F}$ defined by

$$\vec{F}(t) = (a\cos t)\vec{i} + (a\sin t)\vec{j} + t\vec{k}.$$

Exercise 1.6. Show that the vector function $\vec{F}$ defined by

$$\vec{F}(t) = t\vec{i} + 2t\cos t\vec{j} + 2t\sin t\vec{k}$$

lies on the cone $4x^2 = y^2 + z^2$. Sketch the curve.

Exercise 1.7. Describe the graph in words and sketch a graph by hand for the vector function $\vec{F}$ defined by

$$\vec{F}(t) = (e^{at})\,\vec{i} + (e^{at})\,\vec{j} + (e^{-t})\,\vec{k}.$$

Exercise 1.8. How many revolutions are made by the circular helix

$$\vec{F}(t) = (4\sin t)\vec{i} + (4\cos t)\vec{j} + \frac{7}{12}t\,\vec{k}$$

in a vertical distance of 12 units?

Exercise 1.9. Find the domain of the vector function $\vec{F}$ defined by $(\vec{F} \times \vec{G}) \times \vec{H}$ where

- $\vec{F}(t) = t^2\vec{i} - t\vec{j} + 2t\vec{k}$,
- $\vec{G}(t) = \frac{1}{t+2}\vec{i} + (t+4)\vec{j} - \sqrt{-t}\vec{k}$, and
- $\vec{H}(t) = \frac{1}{t+3}\vec{i} + t^2\vec{j} - \sqrt{t}\vec{k}$.

Exercise 1.10. Find a vector function whose graph is the curve of intersection of the hemisphere $z = \sqrt{9 - x^2 - y^2}$ and the parabolic cylinder $x = y^2$.

Exercise 1.11. Find a vector function whose graph is the line of intersection of the planes $2x + y + 3z = 6$ and $x - y - z = 1$.

Exercise 1.12. Determine the component functions for the vector function defined by
$\vec{D}(t) = 2e^t\vec{F}(t) + t\vec{G}(t) + 10\vec{H}(t) \times \vec{G}(t)$
where $\vec{F}(t) = 2t\vec{\imath} - 5\vec{\jmath} + t^2\vec{k}$, $\vec{G}(t) = (1-t)\vec{\imath} + \frac{1}{t}\vec{k}$, and $\vec{H}(t) = (\sin t)\vec{\imath} + e^t\vec{\jmath}$.

Exercise 1.13. Determine the function given by

$$\vec{E}(t) = \vec{F}(t) \cdot [\vec{G}(t) \times \vec{H}(t)]$$

where $\vec{F}(t) = 2t\vec{\imath} - 5\vec{\jmath} + t^2\vec{k}$, $\vec{G}(t) = (t+1)\vec{\imath} + \frac{1}{t-1}\vec{k}$, and $\vec{H}(t) = (\cos t)\vec{\imath} + e^{-t}\vec{\jmath}$.

- Determine a function A that satisfies

$$A(t)e^t + \frac{5}{t}\sin t = \vec{H}(t) \cdot [\vec{G}(t) \times \vec{F}(t)]$$

  where $\vec{F}(t) = 2t\vec{\imath} - 5\vec{\jmath} + t^2\vec{k}$, $\vec{G}(t) = (1-t)\vec{\imath} + \frac{1}{t}\vec{k}$, and $\vec{H}(t) = (\sin t)\vec{\imath} + e^t\vec{\jmath}$.

Exercise 1.14. Find the limit of each of the following vector-valued function.

- $\lim_{t\to 1}\left[\frac{t^3-1}{t-1}\vec{\imath} + \frac{t^2-3t+2}{t^2+t-2}\vec{\jmath} + \left(t^2+1\right)e^{t-1}\vec{k}\right]$
- $\lim_{t\to\infty}\left[\left(e^{-t}\right)\vec{\imath} + \left(\frac{t-1}{t+1}\right)\vec{\jmath} + \left(\tan^{-1} t\right)\vec{k}\right]$
- $\lim_{t\to 0^+}\left[\frac{\sin 3t}{\sin 2t}\vec{\imath} + \frac{\ln(\sin t)}{\ln(\tan t)}\vec{\jmath} + (t\ln t)\vec{k}\right]$

Exercise 1.15. Determine all real numbers a that satisfies

$$\lim_{t\to 0}\left[\frac{t}{\sin at}\vec{\imath} + \frac{a}{a-\cos t}\vec{\jmath} + \left(e^{a-t}\right)\vec{k}\right] = \frac{3}{2}\vec{\imath} - 2\vec{\jmath} + e^{2/3}\vec{k}.$$

Exercise 1.16. Find a vector function $\vec{F}(t) = \langle f_1(t), f_2(t), f_3(t)\rangle$ such that $\|\vec{F}(t)\|$ is continuous at $t = 0$ but $\vec{F}(t)$ is not continuous at $t = 0$.

Exercise 1.17. Determine the intervals for which both vector functions $\vec{F}(t) = \langle e^{-t}, t^2, \tan t\rangle$ and $\vec{G}(t) = \langle 8, \sqrt{t}, \sqrt[3]{t}\rangle$ are continuous.

Exercise 1.18. Show that if $\vec{F}$ is a vector function that is continuous at c, then $\|\vec{F}\|$ is continuous at c.

Exercise 1.19. Find the interval(s) on which the vector function $\vec{r}(t) = e^{-2t}\vec{i} + \cos\sqrt{9-t}\,\vec{j} + \frac{1}{t^2-1}\vec{k}$ is continuous.

1.9 Derivatives of Vector Functions

As with functions of one variable we define the derivative as the limit of a difference quotient; and then we develop a theorem which allows us to compute derivatives based on previously known differentiation rules.
Recall the definition of difference quotient from single variable calculus, namely, the difference quotient of $y = f(x)$ with respect to a change Δx in x is defined by

$$\frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

We now generalize this idea to vector functions.

Definition 1.5. The **difference quotient** of a vector function $\vec{F}$ is the vector function

$$\frac{\Delta \vec{F}}{\Delta t} = \frac{\vec{F}(t+\Delta t) - \vec{F}(t)}{\Delta t}$$

where Δt is an increment of the variable t.

Notice that if the component functions of a vector function $\vec{F}$ are f_1, f_2, and f_3, then

$$\frac{\Delta \vec{F}}{\Delta x} = \frac{\Delta f_1}{\Delta x}\vec{i} + \frac{\Delta f_2}{\Delta x}\vec{j} + \frac{\Delta f_3}{\Delta x}\vec{k} \tag{1.8}$$

In light of 1.8, the definition of the derivative of vector function seems natural.

Definition 1.6. The **derivative** of a vector function $\vec{F}$ is the vector function $\vec{F}\,'$ defined as the limit

$$\vec{F}\,'(t) = \lim_{\Delta t \to 0} \frac{\Delta \vec{F}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\vec{F}(t+\Delta t) - \vec{F}(t)}{\Delta t}$$

provided this limit exists. If $\vec{F}\,'(t)$ exists for a given value of t, then we say $\vec{F}$ is **differentiable** at t.

::: {#lem- } [Differentiable Vector Functions] Any vector function

$$\vec{F}(t) = f_1(t)\vec{i} + f_2(t)\vec{j} + f_3(t)\vec{k}$$

is differentiable whenever the component functions f_1, f_2, and f_3 are each differentiable and in this case

$$\vec{F}\,'(t) = f_1'(t)\vec{\imath} + f_2'(t)\vec{\jmath} + f_3'(t)\vec{k}.$$

:::

Proof. By the definition of the derivative of a vector function and 1.8 we find that

$$\begin{aligned}
\vec{F}\,'(t) &= \lim_{\Delta t \to 0} \frac{\vec{F}(t+\Delta t) - \vec{F}(t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \left[\frac{\Delta f_1}{\Delta x}\vec{\imath} + \frac{\Delta f_2}{\Delta x}\vec{\jmath} + \frac{\Delta f_3}{\Delta x}\vec{k} \right] \\
&= \left(\lim_{\Delta t \to 0} \frac{\Delta f_1}{\Delta x} \right) \vec{\imath} + \left(\lim_{\Delta t \to 0} \frac{\Delta f_2}{\Delta x} \right) \vec{\jmath} + \left(\lim_{\Delta t \to 0} \frac{\Delta f_3}{\Delta x} \right) \vec{k}
\end{aligned}$$

By the hypothesis that the component functions f_1, f_2, and f_3 are each differentiable we have

$$\vec{F}\,'(t) = f_1'(t)\vec{\imath} + f_2'(t)\vec{\jmath} + f_3'(t)\vec{k}.$$

as needed.

□

Example 1.12. Find the derivative of the vector function

$$\vec{F}(t) = (\ln t)\vec{\imath} + \frac{1}{2}t^3\vec{\jmath} - t\vec{k}.$$

Solution. By 1.9, we find that

$$\vec{F}\,'(t) = \frac{1}{t}\vec{\imath} + \frac{3}{2}t^2\vec{\jmath} - \vec{k}$$

is the derivative is the vector function $\vec{F}$.

::: {#lem- } [Derivative Rules for Vector Functions] If the vector functions $\vec{F}$, $\vec{G}$ and the scalar function h are differentiable at t, and if a and b are constants, then $a\,\vec{F} + b\,\vec{G}$, $\vec{F} \cdot \vec{G}$, and $\vec{F} \times \vec{G}$ are differentiable at t and,

- $(a\,\vec{F} + b\,\vec{G})'(t) = a\,\vec{F}\,'(t) + b\,\vec{G}'(t)$
- $(h\,\vec{F})'(t) = h'(t)\vec{F}(t) + h(t)\vec{F}\,'(t)$

- $(\vec{F} \cdot \vec{G})'(t) = \vec{F}\,'(t) \cdot \vec{G}(t) + \vec{F}(t) \cdot \vec{G}'(t)$
- $(\vec{F} \times \vec{G})'(t) = \vec{F}\,'(t) \times \vec{G}(t) + \vec{F}(t) \times \vec{G}'(t)$

- $\vec{F}(h(t))' = h'(t)\vec{F}\,'(h(t))$.

:::

Proof. The proof is left for the reader.

□

Example 1.13. Compute the derivative of the vector function given by $\vec{F}(t) \times \vec{G}(t)$ where $\vec{F}(t) = t^2\vec{\imath} + t\vec{\jmath} + \vec{k}$ and $\vec{G}(t) = \vec{\imath} + t\vec{\jmath} + t^2\vec{k}$.

Solution. By 1.9, we find

$$
\begin{aligned}
[\vec{F}(t) \times \vec{G}(t)]' &= \vec{F}\,'(t) \times \vec{G}(t) + \vec{F}(t) \times \vec{G}'(t) \\
&= \left[(2t\vec{\imath} + \vec{\jmath}) \times \left(\vec{\imath} + t\vec{\jmath} + t^2\vec{k}\right)\right] + \left[\left(t^2\vec{\imath} + t\vec{\jmath} + \vec{k}\right) \times (\vec{\jmath} + 2t\vec{k})\right] \\
&= \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 2t & 1 & 0 \\ 1 & t & t^2 \end{vmatrix} + \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ t^2 & t & 1 \\ 0 & 1 & 2t \end{vmatrix} \\
&= \left(3t^2 - 1\right)\vec{\imath} - 4t^3\vec{\jmath} + \left(3t^2 - 1\right)\vec{k}.
\end{aligned}
$$

1.10 Tangent Vectors

::: {#thm- } Tangent Vectors Suppose $\vec{F}(t)$ is differentiable at t_0 and that $\vec{F}'\left(t_0\right) \neq 0$. Then $\vec{F}'\left(t_0\right)$ is a tangent vector to the graph of $\vec{F}(t)$ at the point where $t = t_0$ and points in the direction of increasing t.
:::

Proof. Let t_0 be a number in the domain of the vector function $\vec{F}$, and let P be the point on the graph of $\vec{F}$ that corresponds to t_0. Then for any positive number Δt, the difference quotient

$$\frac{\Delta \vec{F}}{\Delta t} = \frac{\vec{F}(t + \Delta t) - \vec{F}(t)}{\Delta t}$$

is a vector that points in the same direction as the secant vector

$$PQ = \vec{F}\left(t_0 + \Delta t\right) - \vec{F}\left(t_0\right)$$

where Q is the point on the graph of $\vec{F}$ that corresponds to $t = t_0 + \Delta t$. Suppose the difference quotient $\Delta \vec{F}/\Delta t$ has a limit as $\Delta t \to 0$ and that

$$\lim_{\Delta \to 0} \frac{\Delta \vec{F}}{\Delta t} \neq 0.$$

Then, as $\Delta t \to 0$, the direction of the **secant vector**, PQ, and hence that of the difference quotient $\Delta \vec{F} \Delta t$, will approach the direction of the tangent vector of P. Thus we expect the tangent vector at P to be the limit vector

$$\lim_{\Delta\to 0}\frac{\Delta\vec{F}}{\Delta t}$$

which is the vector derivative $\vec{F}'(t_0)$.

$\square$

Example 1.14. Find a tangent vector at the point where $t = 2$ for

$$\vec{F}(t) = (t^2 + t)\,\vec{\imath} - e^t\vec{\jmath} + \sqrt{t}\vec{k}.$$

Also find parametric equations for the tangent line to the graph of $\vec{F}$ that passes through the point corresponding to $t = 2$.

Solution. Using 1.9, we find a tangent vector

$$\vec{F}\,'(2) = \left.(2t+1)\vec{\imath} - e^t\vec{\jmath} + \frac{1}{2\sqrt{t}}\vec{k}\right|_{t=2} = 5\vec{\imath} - e^2\vec{\jmath} + \frac{1}{2\sqrt{2}}\vec{k};$$

and the tangent line to the graph of $\vec{F}(t)$ for $t = 2$ is the line that passes through the point $\left(6, -e^2, \sqrt{2}\right)$ and is determined by the parametric equations

$$x(t) = 6 + 5t, \quad y(t) = -e^2 - e^2 t, \quad \text{and} \quad z(t) = \sqrt{2} + \frac{1}{2\sqrt{2}}t$$

because this line passes through $\vec{F}(2) = 6\vec{\imath} - e^2\vec{\jmath} + \sqrt{2}\vec{k}$ and is parallel to the tangent vector at $t = 2$.

Example 1.15. Find parametric equations for the tangent line to the graph of

$$\vec{R}(t) = te^{-2t}\vec{\imath} + t^2\vec{\jmath} + te^{-2t}\vec{k}$$

at the highest point on the graph.

Solution. We want $\frac{dz}{dt} = 0$ where $z = te^{-2t}$ and since
$\frac{dz}{dt} = e^{-2t} - 2te^{-2t}$
we find that $t = 1/2$. Further,

$$\vec{R}'(t) = \left(e^{-2t} - 2te^{-2t}\right)\vec{\imath} + 2t\vec{\jmath} + \left(e^{-2t} - 2te^{-2t}\right)\vec{k}$$

and so

$$\vec{R}'\left(\frac{1}{2}\right) = \left(e^{-1} - e^{-1}\right)\vec{\imath} + \vec{\jmath} + \left(e^{-1} - e^{-1}\right)\vec{k} = 0\vec{\imath} + \vec{\jmath} + 0\vec{k}.$$

Also,

$$\vec{R}\left(\frac{1}{2}\right) = \frac{1}{2e^{1/2}}\vec{i} + \frac{1}{4}\vec{j} + \frac{1}{2e}\vec{k}$$

and therefore,

$$x(s) = \frac{1}{2e}, \quad y(s) = s + \frac{1}{4}, \quad z(s) = \frac{1}{2e}.$$

are the parametric equations for the tangent line at the highest point.

1.11 Unit Tangent and Unit Normal Vectors

Definition 1.7. If the graph of the vector function $\vec{R}(t)$ is smooth, then at each point t a **unit tangent vector** is defined by

$$\vec{T}(t) = \frac{\vec{R}'(t)}{\left|\left|\vec{R}'(t)\right|\right|}$$

and the principal **unit normal vector function** is defined by

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\left|\left|\vec{T}'(t)\right|\right|}.$$

The unit normal vector is orthogonal (or normal, or perpendicular) to the unit tangent vector and hence to the curve as well.

Lemma 1.1. *Suppose that $\vec{R}$ is a vector such that $||R(t)|| = c$ for all t. Then $\vec{R}(t)$ is orthogonal to $\vec{R}'(t)$.*

Proof. Notice $\vec{R}(t) \cdot \vec{R}(t) = \left|\left|\vec{R}(t)\right|\right|^2 = c^2$, for all t. We differentiate with respect to t to find,

$$\vec{R}'(t) \cdot \vec{R}(t) + \vec{R}'(t) \cdot \vec{R}(t) = 0$$

which yields $\vec{R}'(t) \cdot \vec{R}(t) = 0$ and so $\vec{R}'(t)$ is orthogonal to $\vec{R}(t)$.

□

Example 1.16. Given the vector function defined by

$$\vec{R}(t) = \langle -4t, \sin 2t, -\cos 2t\rangle$$

find the unit tangent vector $\vec{T}(t)$ and unit normal vector $\vec{N}(t)$.

Solution. We find

$$\begin{aligned}
\vec{T}(t) &= \frac{\vec{R}'(t)}{\left\|\vec{R}'(t)\right\|} \\
&= \frac{\langle -4, 2\cos 2t, 2\sin 2t\rangle}{\sqrt{(-4)^2 + 4\cos^2 2t + 4\sin^2 2t}} \\
&= \frac{\langle -4, 2\cos 2t, 2\sin 2t\rangle}{2\sqrt{5}} \\
&= \left\langle -\frac{4}{2\sqrt{5}}, \frac{2}{2\sqrt{5}}\cos 2t, \frac{2}{2\sqrt{5}}\sin 2t \right\rangle \\
&= \left\langle -\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\cos 2t, \frac{1}{\sqrt{5}}\sin 2t \right\rangle
\end{aligned}$$

and

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\left\|\vec{T}'(t)\right\|} = \frac{\left\langle 0, \frac{-2}{\sqrt{5}}\sin 2t, \frac{2}{\sqrt{5}}\cos 2t \right\rangle}{\sqrt{\frac{4}{5}}} = \langle 0, -\sin 2t, \cos 2t\rangle.$$

as the unit normal vector.

1.12 Integrals of Vector Functions

Next we study **vector integration** . Since integration is a linear process, studying vector functions and integration together is natural. Indeed recall that with functions of one variable, definite integration is the process of taking a limit of a Riemann sum. Basically, since taking limits and Riemann sums both are linear processes, so is integration.

Definition 1.8. Let
$\vec{F}(t) = f_1(t)\vec{i} + f_2(t)\vec{j} + f_3(t)\vec{k}$,
where $f_1(t)$, $f_2(t)$, and $f_3(t)$ are continuous on the closed interval $a \le t \le b$. Then the indefinite integral of $\vec{F}(t)$ is

$$\int \vec{F}(t)\,dt = \left[\int f_1(t)\,dt\right]\vec{i} + \left[\int f_2(t)\,dt\right]\vec{j} + \left[\int f_3(t)\,dt\right]\vec{k}.$$

Recall integration by parts:

$$\int u\,dv = uv - \int v du$$

and so use $u = \ln t$ and $dv = t\,dt$.

Example 1.17. Evaluate $\int \langle t\ln t, -\sin(1-t), t\rangle\, dt$.

Solution. We find that the given integral is equal to

$$\begin{aligned} &= \left(\int t\ln t\,dt\right)\vec{i} + \left(\int -\sin(1-t)\,dt\right)\vec{j} + \left(\int t\,dt\right)\vec{k} \\ &= \left(\frac{-t^2}{4} + \frac{t^2}{2}\ln t\right)\vec{i} - \cos(1-t)\vec{j} + \left(\frac{t^2}{2}\right)\vec{k} + \vec{C} \end{aligned}$$

where $\vec{C}$ is a constant vector.

Definition 1.9. Let $\vec{F}(t) = f_1(t)\vec{i} + f_2(t)\vec{j} + f_3(t)\vec{k}$,
where $f_1(t)$, $f_2(t)$, and $f_3(t)$ are continuous on the closed interval $a \le t \le b$. Then the definite integral of $\vec{F}(t)$ is the vector

$$\int_b^a \vec{F}(t)\,dt = \left[\int_b^a f_1(t)\,dt\right]\vec{i} + \left[\int_b^a f_2(t)\,dt\right]\vec{j} + \left[\int_b^a f_3(t)\,dt\right]\vec{k}.$$

Example 1.18. Given the vector function

$$\vec{F}(t) = \left(t\sqrt{1+t^2}\right)\vec{i} + \left(\frac{1}{1+t^2}\right)\vec{j}.$$

Find a value of a for which $\int_0^a \vec{F}(t)\,dt = \frac{2\sqrt{2}}{3}\vec{i} + \frac{\pi}{4}\vec{j}$.

Solution. By definition, we find

$$\begin{aligned} \int_0^a \vec{F}(t)\,dt &= \left(\int_0^a t\sqrt{1+t^2}\,dt\right)\vec{i} + \left(\int_0^a \frac{1}{1+t^2}\,dt\right)\vec{j} \\ &= \left(\frac{1}{3}\sqrt{\left(1+a^2\right)^3}\right)\vec{i} + \left(\tan^{-1} a\right)\vec{j}. \end{aligned}$$

Thus we require $\left(1+a^2\right)^3 = 8$ and $\tan^{-1} a = \frac{\pi}{4}$. Therefore we find $a = 1$.

vMotion of an Object

If the graph of a vector function $\vec{R}(t)$ is a smooth curve C then the nonzero derivative $\vec{R}'(t)$ is tangent to C at the point P that corresponds to t; and in this case we can make the following definition.

Definition 1.10. If an object moves in such a way that its position at time t is given by the vector function $\vec{R}(t)$ whose graph is a smooth curve C, then

- $\vec{V}(t) = \frac{d\vec{R}}{dt}$ is the object's velocity vector at time t,

- $\|\vec{V}(t)\|$ is the speed of the object at time t,
- $\frac{\vec{V}(t)}{\|\vec{V}(t)\|}$ is the direction of the object's motion at time t, and
- $\frac{d\vec{V}(t)}{dt}$ is the object's acceleration at time t.

Example 1.19. Suppose the position vector for a particle in space at time t is given by

$$\vec{R}(t) = e^t\vec{i} + e^{-t}\vec{j} + e^{2t}\vec{k}.$$

Find the particle's velocity vector, acceleration vector, speed, and direction of motion vector at time $t = \ln 2$.

Solution. At time $t = \ln 2$, the particle's velocity vector is

$$\vec{R}'(t)|_{t=\ln 2} = e^t\vec{i} - e^{-t}\vec{j} + 2e^{2t}\vec{k}|_{t=\ln 2} = 2\vec{i} - \frac{1}{2}\vec{j} + 8\vec{k},$$

the acceleration vector is

$$\vec{R}''(t)|_{t=\ln 2} = e^t\vec{i} + e^{-t}\vec{j} + 4e^{2t}\vec{k}|_{t=\ln 2} = 2\vec{i} + \frac{1}{2}\vec{j} + 16\vec{k},$$

the speed is

$$\vec{R}'(t)|_{t=\ln 2} = \sqrt{2^2 + \left(\frac{-1}{2}\right)^2 + 8^2} = \sqrt{68.25},$$

and the direction of motion vector is

$$\frac{2}{\sqrt{68.25}}\vec{i} - \frac{1}{2\sqrt{68.25}}\vec{j} + \frac{8}{\sqrt{68.25}}\vec{k}.$$

Example 1.20. Find the position vector $\vec{R}(t)$ and velocity vector $\vec{V}(t)$, given the acceleration vector function

$$\vec{A}(t) = t^2\vec{i} - 2\sqrt{t}\vec{j} + e^{3t}\vec{k},$$

initial position vector $\vec{R}(0) = 2\vec{i} + \vec{j} - \vec{k}$, and initial velocity vector $\vec{V}(0) = \vec{i} - \vec{j} - 2\vec{k}$.

Solution. Given $\vec{A}(t) = t^2\vec{i} - 2\sqrt{t}\vec{j} + e^{3t}\vec{k}$ the velocity vector function is

$$\vec{V}(t) = \left(\int t^2 dt\right)\vec{i} - \left(\int 2\sqrt{t}dt\right)\vec{j} + \left(\int e^{3t}dt\right)\vec{k}$$

$$\vec{V}(t) = \left(\frac{t^3}{3} + C_1\right)\vec{i} - \left(\frac{4t^{3/2}}{3} + C_2\right)\vec{j} + \left(\frac{e^{3t}}{3} + C_3\right)\vec{k}$$

where C_1, C_2, and C_3 are constants to be determined. By using

$$\vec{V}(0) = \vec{i} - \vec{j} - 2\vec{k} = (0 + C_1)\,\vec{i} + (-0 + C_2)\,\vec{j} + \left(\frac{1}{3} + C_3\right)\vec{k}$$

we find $C_1 = 1$, $C_2 = 1$, $C_3 = -7/3$. Therefore,

$$\vec{V}(t) = \left(\frac{t^3}{3} + 1\right)\vec{i} + \left(-\frac{4t^{3/2}}{3} - 1\right)\vec{j} + \left(\frac{e^{3t} - 7}{3}\right)\vec{k}.$$

So the position vector function is

$$\vec{R}(t) = \left(\int \left(\frac{t^3}{3} + 1\right)\,dt\right)\vec{i} + \left(\int \left(-\frac{4t^{3/2}}{3} - 1\right)\,dt\right)\vec{j} + \left(\int \frac{e^{3t} - 7}{3}\,dt\right)\vec{k}$$

which is

$$\vec{R}(t) = \left(\frac{t^4}{12} + t + K_1\right)\vec{i} + \left(-\frac{8t^{5/2}}{15} - t + K_2\right)\vec{j} + \left(\frac{1}{3}\left(\frac{e^{3t}}{3} - 7t\right) + k_3\right)\vec{k}$$

where K_1, K_2, and K_3 are constants to be determined. By using

$$\vec{R}(0) = 2\vec{i} + \vec{j} - \vec{k} = (0 + K_1)\,\vec{i} + (-0 + K_2)\,\vec{j} + \left(\frac{1}{9} + K_3\right)\vec{k}$$

we find $K_1 = 2$, $K_2 = 1$, $K_3 = -10/9$. Therefore,

$$\vec{R}(t) = \left(\frac{t^4}{12} + t + 2\right)\vec{i} + \left(-\frac{8t^{5/2}}{15} - t + 1\right)\vec{j} + \left(\frac{1}{3}\left(\frac{e^{3t}}{3} - 7t\right) - \frac{10}{9}\right)\vec{k}$$

is the required vector function.

Theorem 1.1. *Neglecting air resistance, the path of a projectile launched from an initial height h with initial speed v_0 and an angle of elevation θ is described by the vector function*

$$\vec{R}(t) = (v_0 \cos\theta)t\vec{i} + \left[h + (v_0 \sin\theta)t - \frac{1}{2}gt^2)\right]\vec{j} \tag{1.9}$$

where g is he gravitional constant.

Proof. Suppose a projectile of mass m is launched from an initial position $\vec{R}_0$ with an initial velocity $\vec{V}_0$. First we will find its position vector as a function of time. We begin with acceleration $\vec{A}(t) = -g\vec{j}$ and we integrate twice, namely,

$$\vec{V}(t) = \int \vec{A}(t)\,dt = \int -g\vec{j}\,dt = -gt\vec{j} + \vec{C}_1$$
$$\vec{R}(t) = \int \vec{V}(t)\,dt = \int (gt\vec{j} + \vec{C}_1)\,dt = -\frac{1}{2}gt^2\vec{i} + \vec{C}_1 t + \vec{C}_2$$

where $\vec{C_1}$ and $\vec{C_2}$ can be determined from the initial conditions. Using $\vec{V}(0) = \vec{V_0}$, $\vec{R}(0) = \vec{R_0}$, produces $\vec{C_1} = \vec{V_0}$ and $\vec{C_2} = \vec{R_0}$. Therefore the position vector function is

$$\vec{R}(t) = -\frac{1}{2}gt^2\vec{j} + t\vec{V_0} + \vec{R_0}. \tag{1.10}$$

Now from the given height h we realize that

$$\vec{R_0} = h\vec{j}$$

and because speed is the magnitude of the velocity, that is $V_0 = ||\vec{V_0}||$, we have

$$\vec{V_0} = x\vec{i} + y\vec{j} = \left(||\vec{V_0}|| \cos\theta\right)\vec{i} + \left(||\vec{V_0}|| \sin\theta\right)\vec{j} = V_0\cos\theta\vec{i} + V_0\sin\theta\vec{j}.$$

Now we can derive 1.9 as follows

$$\begin{aligned} R(t) &= -\frac{1}{2}gt^2\vec{j} + t\vec{V_0} + \vec{R_0} \\ &= -\frac{1}{2}gt^2\vec{j} + tV_0\cos\theta\vec{i} + t(V_0\cos\theta\vec{i} + V_0\sin\theta\vec{j}) + h\vec{j} \\ &= (v_0\cos\theta)t\vec{i} + \left[h + (v_0\sin\theta)t - \frac{1}{2}gt^2)\right]\vec{j} \end{aligned}$$

as desired.

□

1.13 Exercises

Exercise 1.20. Evaluate $\int_0^6 \langle t^3 - t, t^2 - 2\rangle\, dt$.

Exercise 1.21. Evaluate $\int \left(2t\vec{i} + 9t^2\vec{j} + 7\vec{k}\right) dt$.

Exercise 1.22. Find the vector derivative $\vec{F}\,'$ for each of the following functions.

- $\vec{F}(s) = \left(s\vec{i} + s^2\vec{j} + s^2\vec{k}\right) + \left(2s^2\vec{i} - s\vec{j} + 3\vec{k}\right)$.
- $\vec{F}(s) = (1 - 2s^2)\,\vec{i} + (s\cos s)\vec{j} - s\vec{k}$.
- $\vec{F}(\theta) = \left(\sin^2\theta\right)\vec{i} + (\cos 2\theta)\vec{j} + \theta^2\vec{k}$.

Exercise 1.23. Find the indefinite vector integral for each of the following.

- $\int \langle \cos t, \sin t, -2t \rangle \, dt$.
- $\int \langle 3e, t^2, t \sin t \rangle \, dt$.
- $\int e^{-t} \langle 3, t, \sin t \rangle \, dt$.
- $\int \langle \sinh t, -3, \cosh t \rangle \, dt$.

Exercise 1.24. Given $\vec{F}(t) = t^2\vec{\imath} + 2t\vec{\jmath} + \left(t^3 + t^2\right)\vec{k}$ determine all real numbers a such that $\vec{F}\,'(0) + \vec{F}\,'(1) + \vec{F}\,'(-1) = a\vec{\jmath} + a\vec{k}$.

Exercise 1.25. Given $\vec{F}(t) = t^2\vec{\imath} + \cos t\vec{\jmath} + t^2 \cos t\vec{k}$ determine all real numbers a such that $\vec{F}\,'(0) + \vec{F}\,'(a) = \pi\vec{\imath} - \vec{\jmath} - a^2\vec{k}$.

Exercise 1.26. Determine all real numbers a such that the parametric equations for the tangent line to the graph of the vector function $\vec{F}(t) = t^{-3}\vec{\imath} + t^{-2}\vec{\jmath} + t^{-1}\vec{k}$ at the point corresponding to $t = -1$ are

$$x = \frac{-3}{a} - at, \quad y = \frac{a}{3} + \frac{6}{a}t, \quad z = \frac{-3}{a} - \frac{3}{a}t.$$

Exercise 1.27. Find the first and second derivatives for each of the following functions.

- $\vec{F}(t) = (\ln t)\left[t\vec{\imath} + 5\vec{\jmath} - e^t\vec{k}\right]$
- $\vec{F}(t) = (\sin t)\vec{\imath} + (\cos t)\vec{\jmath} + t^2\vec{k}$
- $f(x) = \left\|\left(x\vec{\imath} + x^2\vec{\jmath} - 20\vec{k}\right) + \left(x^3\vec{\imath} + x\vec{\jmath} - x\vec{k}\right)\right\|$.

Exercise 1.28. Determine all real numbers a such that

$$f'(x) = -ax^2 - \frac{18}{a}x$$

given

$$f(x) = [x\vec{\imath} + (x+1)\vec{\jmath}] \cdot \left[2x\vec{\imath} - 3x^2\vec{\jmath}\right].$$

Exercise 1.29. Determine all real numbers a such that

$$F''(t) = a\vec{i} + at^{-3}\vec{j} + 2ae^{at}\vec{k}$$

given $\vec{F}(t) = t^2\vec{i} + t^{-1}\vec{j} + e^{2t}\vec{k}$.

Exercise 1.30. Determine all real numbers a such that $\vec{F}\,'(t)$ and $\vec{F}\,'''(t)$ are parallel for all t given $\vec{F}(t) = e^{at}\vec{i} + e^{-at}\vec{j}$.

Exercise 1.31. Prove that if $\vec{F}$ is a differentiable vector function such that $\vec{F}(t) \neq 0$, then

$$\frac{d}{dt}\left(\frac{\vec{F}(t)}{\|\vec{F}(t)\|}\right) = \frac{\vec{F}\,'(t)}{\|\vec{F}(t)\|} - \frac{[\vec{F}(t)\cdot\vec{F}'(t)]\vec{F}(t)}{\left\|\vec{F}(t)\right\|^3}.$$

Exercise 1.32. Prove that if $\vec{F}, \vec{G}$, and $\vec{H}$ are differentiable vector functions, then

$$[\vec{F}\cdot(\vec{G}\times\vec{H})]' = \vec{F}\,'\cdot(\vec{G}\times\vec{H}) + \vec{F}\cdot(\vec{G}\,'\times\vec{H}) + \vec{F}\cdot(\vec{G}\times\vec{H}\,').$$

Exercise 1.33. Prove that if $\vec{F}, \vec{G}$, and $\vec{H}$ are differentiable vector functions, then

$$[\vec{F}\times(\vec{G}\times\vec{H})]' = [(\vec{H}\cdot\vec{F})\vec{G}]' - [(\vec{G}\cdot\vec{F})\vec{H}]'.$$

Exercise 1.34. Find a value of a and b such that

$$\int_0^a \left[t\sqrt{1+t^2}\vec{i} + \left(\frac{1}{1+t^2}\right)\vec{j}\right] dt = \frac{2\sqrt{2}-1}{3}\vec{i} + \frac{\pi}{4}\vec{j}$$

and

$$\int_0^b [\cos t\vec{i} + \sin t\vec{j} + \sin t\cos t\vec{k}]\,dt = \vec{i} + \vec{j} + \frac{1}{2}\vec{k}.$$

Exercise 1.35. Show that if $\vec{R}(t) = f(t)\,\vec{i} + g(t)\,\vec{j} + h(t)\,\vec{k}$ is differentiable at $t = t_0$, then it is continuous at t_0 as well.

Exercise 1.36. Suppose that the scalar function $u(t)$ and the vector function $\vec{R}(t)$ are both defined for $a \leq t \leq b$. Show that $u\,\vec{R}$ is continuous on $[a,b]$ if u and $\vec{R}$ are continuous on $[a,b]$. If u and $\vec{R}$ are both differentiable on $[a,b]$, show that $u\,\vec{R}$ is differentiable on $[a,b]$ and that

$$\frac{d}{dt}(u\,\vec{R}) = u\,\frac{d\vec{R}}{dt} + \vec{R}\frac{du}{dt}.$$

Exercise 1.37. Use the Mean Value Theorem to show that if $\vec{R}_1(t)$ and $\vec{R}_2(t)$ have identical derivatives on an interval I, then the vector functions differ by a constant vector value throughout I.

Exercise 1.38. Suppose $\vec{r}$ is a continuous vector function on $[a,b]$. Show that if $\vec{R}$ is any antiderivative of a $\vec{r}$ on $[a,b]$ then if

$$\vec{r}(t) = \frac{d}{dt}\int_a^t \vec{r}(s)\,ds \quad \text{and} \quad \int_a^b \vec{r}(t)\,dt = \vec{R}(a) - \vec{R}(b).$$

Exercise 1.39. Find a vector function describing the curve of intersection of the plane $x+y+2z=2$ and the paraboloid $z=x^2+y^2$. Also find the point(s) on the curve that are closest to and farthest from the origin.

Exercise 1.40. Prove that

$$\frac{d}{dt}\left[\vec{R}(t)\times\vec{R}\,'(t)\right] = \vec{R}(t)\times\vec{R}\,''(t).$$

Exercise 1.41. Let $\vec{R}(t) = \sin t\,\vec{\imath} + \cos t\,\vec{\jmath} + t\,\vec{k}$ and $c = 2\,\vec{\imath} + 3\,\vec{\jmath} - \vec{k}$ show that

$$\int_a^b \vec{c}\cdot\vec{R}(t) = \vec{c}\cdot\int_a^b \vec{R}(t);$$

also show that this formula holds true for any vector function $\vec{R}(t)$ that is integrable on $[a,b]$ and for any vector constant $\vec{c}$.

Exercise 1.42. Find the unit tangent vector $\vec{T}(t)$ for $\vec{r}(t) = 8t\vec{\imath} + 8t\vec{\jmath} + 4t\vec{k}$ at $t=2$.

Exercise 1.43. Find the unit tangent vector $\vec{T}(t)$ and the unit normal vector $\vec{N}(t)$.

- $\vec{R}(t) = t^2\vec{\imath} + \sqrt{t}\vec{\jmath}$, with $t > 0$
- $\vec{R}(t) = (t\cos t)\vec{\imath} + (t\sin t)\vec{\jmath}$
- $\vec{R}(t) = t\vec{\imath} + (\ln\cos t)\vec{\jmath}, \ -\pi/2 < t < \pi/2$
- $\vec{R}(t) = (2t+3)\vec{\imath} + (4-t^2)\vec{\jmath}$
- $\vec{R}(t) = (\ln\sec t)\vec{\imath} + t\vec{\jmath} \ -\pi/2 < t < \pi/2$
- $\vec{R}(t) = \sin t\vec{\imath} - \cos t\vec{\jmath} + t\vec{k}$
- $\vec{R}(t) = (3\sin t)\vec{\imath} + (3\cos t)\vec{\jmath} + 4t\vec{k}$
- $\vec{R}(t) = (e^t\cos t)\vec{\imath} + (e^t\sin t)\vec{\jmath} + 4\vec{k}$
- $\vec{R}(t) = (t^3/3)\vec{\imath} + \vec{\jmath} + (t^2/2)\vec{k}, \ t > 0$
- $\vec{R}(t) = (\cos^3 t)\vec{\imath} + (\sin^3 t)\vec{\jmath} + \vec{k}, \ 0 < t < \pi/2$

Exercise 1.44. Given $\vec{R}(t) = (\ln t)\vec{\imath} + t^2\vec{\jmath}$ find a so that

$$\vec{N}(t) = \frac{-2}{\sqrt{1+at^a}}\vec{\imath} + \frac{1}{\sqrt{1+at^a}}\vec{\jmath}.$$

Exercise 1.45. Given $\vec{R}(t) = (\cos t)\vec{\imath} + (\sin t)\vec{\jmath} + t\vec{k}$ find a so that $\vec{N}(t) = (-\cos at)\vec{\imath} + (-\sin at)\vec{\jmath}$.

Exercise 1.46. Determine values for a and b so that the following vector functions are smooth over $[a, b]$.

- $\vec{F}(t) = at^3\vec{\imath} + bt^2\vec{\jmath} + abt\vec{k}$
- $\vec{G}(t) = (a\sin bt)\vec{\imath} + (a\cos bt)\vec{\jmath} + ab\vec{k}$
- $\vec{H}(t) = (e^{at} - bt)\,\vec{\imath} + t^{a/b}\vec{\jmath} + (b\cos at)\vec{k}$

Exercise 1.47. The velocity of a particle moving in space is $\vec{V}(t) = e^t\vec{\imath} + t^2\vec{\jmath}$. Find the vector $\vec{R}(0)$ so that the particle's position as a function of t is

$$\vec{R}(t) = e^t\vec{\imath} + \left(\frac{1}{3}t^3 - 1\right)\vec{\jmath}.$$

Exercise 1.48. The acceleration of a moving particle is $\vec{A}(t) = 24t^2\vec{\imath} + 4\vec{\jmath}$. Find the vectors $\vec{R}(0)$ and $\vec{V}(0) = 0$ so that the particle's position as a function of t is

$$\vec{R}(t) = (2t^4 + 1)\,\vec{\imath} + (2t^2 + 2)\,\vec{\jmath}.$$

Exercise 1.49. Show that $\vec{N}(t) = -g'(t)\vec{i} + f'(t)\vec{j}$ and $-\vec{N}(t) = g'(t)\vec{i} - f'(t)\vec{j}$ are both normal to the curve $\vec{R}(t) + f(t)\vec{i} + g(t)\vec{j}$ at the point $(f(t), g(t))$.

Exercise 1.50. A baseball is hit above ground at 100 feet per second at an angle of $\pi/4$ with respect to the ground. Find the maximum height reached by the baseball. Will it clear a 10-foot high fence located 300 feet from home plate?

Exercise 1.51. Find the vector function for the path pf a projectile launched at a height of 10 feet above the ground with an initial velocity of 88 feet per second and at an angle of 30° above the horizontal. Sketch the path of the projectile.

Exercise 1.52. Find the angle at which an object must be thrown to obtain (a) the maximum range, and (b) he maximum m height.

Exercise 1.53. A projectile is fired from ground level at an angle of 10° with the horizontal. Find the minimum initial velocity necessary of the projectile is to have a range of 100 feet.

Exercise 1.54. Use linear approximation by finding a set of parametric equations for the tangent line to the graph at $t = t_0$ and use the equations for the line to approximate $\vec{R}(t_0 + .01)$.

- $\vec{R}(t) = t\vec{i} - t^2\vec{j} + \frac{t^3}{4}$, $t_0 = 1$
- $\vec{R}(t) = t\vec{i} + \sqrt{25 - t^2}\vec{j} + \sqrt{25 - t}$, $t_0 = 3$

1.14 Smooth Curves

The one variable function $f(x) = |x|$ is not differentiable at $x = 0$ because it has a `corner" at $x=0.$ In three dimensions we have the concept of a vector function representing a smooth curve; that is, where there are no so called`corners". With an extra degree of freedom we require the derivative to not only exist but also to be continuous and nonzero.

Definition 1.11. The graph of the vector function defined by $\vec{F}(t)$ is **smooth** on any interval of t where $\vec{F}\,'$ is continuous and $\vec{F}'(t) \neq \vec{0}$.

Example 1.21. Find the intervals on which the epicycloid C given by

$$\vec{R}(t) = (5\cos t - \cos 5t)\vec{i} + (5\sin t - \sin 5t)\vec{j}, \qquad 0 \le t \le 2\pi$$

is smooth.

Solution. The derivative of $\vec{R}(t)$ is

$$\vec{R}(t) = (-5\sin t + 5\sin 5t)\vec{i} + (5\cos t - 5\cos 5t)\vec{j}.$$

Notice that $\vec{R}'$ is continuous on $[0, 2\pi)$. In the interval $[0, 2\pi)$, the only values of t for which $\vec{R}'(t) = 0\vec{i}+0\vec{j}$ are found by solving the trigonometric equations:

$$-5\sin t + 5\sin 5t = 0 \qquad 5\cos t - 5\cos 5t = 0, \qquad \text{on } [0, 2\pi).$$

The first equation yields:

$$t = 0, \quad t = \frac{7\pi}{6}, \quad t = \frac{3\pi}{2}, \quad t = \frac{11\pi}{6}, \quad t = \frac{\pi}{6}, \quad t = \frac{\pi}{2}, \quad t = \frac{5\pi}{6}$$

The second equation yields:

$$t = 0, \quad t = \pi, \quad t = \frac{4\pi}{3}, \quad t = \frac{3\pi}{2}, \quad t = \frac{5\pi}{3}, \quad t = \frac{\pi}{3}, \quad t = \frac{\pi}{2}, \quad t = \frac{2\pi}{3}$$

Therefore, we can conclude that C is smooth on the intervals

$$\left(0, \frac{\pi}{2}\right), \qquad \left(\frac{\pi}{2}, \pi\right), \qquad \left(\pi, \frac{3\pi}{2}\right), \qquad \left(\frac{3\pi}{2}, 2\pi\right),$$

as shown in **??**

The graph of a vector function $\vec{F}$ is **piecewise smooth** on any interval that can be subdivided into a finite number of subintervals on which $\vec{F}$ is smooth. For instances, the graph in **??** is piecewise smooth on $[0, 2\pi)$ since the interval $[0, 2\pi)$ can be subdivided into a finite number of subintervals on which $\vec{F}$ is smooth.

Example 1.22. Determine where the graph of the vector function

$$\vec{F}(t) = \left(2t^3 + \left(4 - \frac{3\pi}{2}\right)t^2 - 4\pi t + 2\pi\right)\vec{i} + \left(-\frac{1}{2}\cos 2t\right)\vec{j} + (\sin t)\vec{k}.$$

is piecewise smooth.

Solution. The graph of the vector function $\vec{F}$ is smooth over any interval not containing $t = \pi/2$ because

$$\vec{F}'(t) = (6t^2 + (8 - 3\pi)t - 4\pi)\,\vec{i} + (\sin 2t)\vec{j} + (\cos t)\vec{k},$$

$\vec{F}'(t) \neq \vec{0}$ for any t except $t = \frac{\pi}{2}$, and $\vec{F}'$ is continuous everywhere.

Example 1.23. Determine where the graph of the vector function

$$\vec{F}(t) = (2t^2 - \pi t + 2\pi)\,\vec{\imath} + \left(\frac{1}{2}\sin 2t\right)\vec{\jmath} + \left(\frac{-1}{4}\cos 4t\right)\vec{k}.$$

is piecewise smooth.

Solution. The graph of the vector function $\vec{F}$ is piecewise smooth everywhere because

$$\vec{F}'(t) = (4t - \pi)\vec{\imath} + (\cos 2t)\vec{\jmath} + (\sin 4t)\vec{k},$$

$\vec{F}'(t) \neq \vec{0}$ except for $t = \frac{\pi}{4}$, and $\vec{F}'$ is continuous everywhere.

1.15 Arc Length

In this section, we first generalize the arc length formula from two to three dimensions. Recall that the arc length of the graph of a differentiable function $y = f(x)$ on the interval $[a, b]$ is given by

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Now if the graph is parametrized by equations $x = x(t)$ and $y = y(t)$ then by the chain rule

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$$

whenever $dx/dt \neq 0$. After algebraic manipulation we find that

$$s = \int_{t_1}^{t_2} \sqrt{\frac{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}{\left(\frac{dx}{dt}\right)^2}} \left(\frac{dx}{dt}\right) dt = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

represents the arc length of the curve $y = f(x)$ from the points corresponding to $x = a$ to $x = b$.

Suppose that C is a plane curve described by the vector function

$$\vec{R}(t) = x(t)\vec{\imath} + y(t)\vec{\jmath}$$

instead. Then $\vec{R}'(t) = x'(t)\vec{\imath} + y'(t)\vec{\jmath}$ where $y = f(x)$ and

$$|\vec{R}'(t)| = \sqrt{R'(t) \cdot R'(t)} = \sqrt{[x'(t)]^2 + [y'(t)]^2}$$

form which we see that s can also be written in the form

$$s = \int_{t_1}^{t_2} \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt = \int_{t_1}^{t_2} |R'(t)|\, dt$$

In $\mathbb{R}^3$ we have

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

But in fact the arc length of a graph is independent of its parametrization and thus,

$$s(t) = \int_{t_0}^{t} \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du$$

where P_0 is the base point corresponding to t_0.
This discussion motivates the following theorem.

::: {#thm- } [Arc Length Function] Let C be a piecewise smooth curve that is the graph of the vector function described parametrically by

$$\vec{R}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}, \qquad a \leq t \leq b$$

and let $P_0 = P(a)$ be a particular point on C (**base point**). If C is transversed exactly once as t increases from a to t, then the length of C from the base point P_0 to the variable $P(t)$ is given by the **arc length function**

$$s(t) = \int_{a}^{t} \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du = \int_{a}^{t} ||\vec{R}'(t)||\, dt$$

:::

Example 1.24. Find the arc length of the curve

$$\vec{R}(t) = (1 - 2\cos t)\vec{i} + 2\sin t\vec{j} + 0.3t\vec{k}$$

from $t = -5$ to $t = 5$.

Solution. We find the arc length to be $\sqrt{409}$ units because

$$\frac{dx}{dt} = 2\sin t, \quad \frac{dy}{dt} = 2\cos t, \quad \frac{dz}{dt} = 0.3$$

and so the arc length is

$$\int_{-5}^{5} \sqrt{4\sin^2 t + 4\cos^2 t + (0.3)^2}\, dt = \int_{-5}^{5} \frac{\sqrt{409}}{10}\, dt = \sqrt{409}.$$

as claimed.

::: {#cor- } [Speed Along an Arc] Suppose an object moves along a smooth curve C that is the graph of the position function

$$\vec{R}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k},$$

where $\vec{R}'(t)$ is continuous on the interval $[t_1, t_2]$. Then the object has speed ds/dt for $t_1 \leq t \leq t_2$ where

$$s(t) = \int_{t_1}^{t} \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du.$$

:::

Proof. Given that $\vec{R}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ is the position vector function for an object which moves along the graph of $\vec{R}$ and given that $\vec{R}'(t)$ is continuous on $[t_1, t_2]$ we can apply the Fundamental Theorem of Calculus to

$$s(t) = \int_{t_1}^{t} \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du$$

and obtain

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \left\|\vec{R}'(t)\right\| = \left\|\vec{V}(t)\right\|.$$

□

Example 1.25. If a moving object has a position vector function of

$$\vec{R}(t) = \langle e^{3t}, \sqrt{18}t - 5, -e^{-3t} \rangle$$

then find the speed of the object at time t and the distance traveled by the object between times $t = 0$ and $t = 1$.

Solution. The speed of the object at time t is $3e^{3t} + 3e^{-3t}$ because

$$\vec{V}(t) = \langle 3e^{3t}, \sqrt{18}, 3e^{-3t} \rangle,$$

$$\|\vec{V}(t)\| = \sqrt{9e^{6t} + 18 + 9e^{-6t}} = 3\sqrt{\left(e^{3t} + e^{-3t}\right)^2} = 3\left(e^{3t} + e^{-3t}\right)$$

and the distance traveled by the object between times $t = 0$ and $t = 1$ is

$$s = \int_0^1 \left(3e^{3t} + 3e^{-3t}\right)\, dt = e^3 - \frac{1}{e^3}.$$

Example 1.26. Express the vector function $\vec{R}(t) = \langle e^{-t}, 1 - e^{-t} \rangle$ in terms of arc length measured from the point corresponding to $t = 0$, in the direction of increasing t.

Solution. We have

$$s = s(t) = \int_0^t \sqrt{e^{-2u} + e^{-2u}}\, du = \sqrt{2} \int_0^t e^{-u}\, du = \sqrt{2} - \sqrt{2}e^{-t}.$$

Solving for e^{-t} we have $e^{-t} = \frac{\sqrt{2}-s}{\sqrt{2}}$. Thus

$$\vec{R}(s) = \left\langle \frac{\sqrt{2}-s}{\sqrt{2}}, \frac{s}{\sqrt{2}} \right\rangle.$$

which is expressed in terms of arc length s.

::: {#thm- } [Normal Vectors]

If $\vec{R}(t)$ has a piecewise smooth graph and is represented as $\vec{R}(s)$ in terms of the arc length parameter s, then the unit tangent vector $\vec{T}$ and the principal unit **normal vector** satisfies

$$\vec{T} = \frac{d\vec{R}}{ds} \qquad \text{and} \qquad \vec{N} = \frac{1}{\kappa}\frac{d\vec{T}}{ds}$$

where $\kappa = \left|\left|d\vec{T}/ds\right|\right|$ is a scalar function of s. :::

Geometrically, the curvature κ measures how fast the unit tangent vector to the curve rotates. If a curve keeps close to the same direction, the unit tangent vector changes very little and the curvature is small; where the curve undergoes a tight turn, the curvature is large.

Proof. Given a piecewise smooth graph represented by $\vec{R}(t)$ and in terms of arc length by $\vec{R}(s)$, then by the chain rule,

$$\frac{d\vec{R}}{ds} = \frac{\frac{d\vec{R}}{dt}}{\frac{ds}{dt}} = \frac{\vec{R}'(t)}{\left|\left|\vec{R}'(t)\right|\right|} = \vec{T}.$$

Also

$$\vec{N} = \frac{\vec{T}'(t)}{\left|\left|\vec{T}'(t)\right|\right|} = \frac{d\vec{T}/dt}{\left|\left|\vec{T}'(t)\right|\right|} = \frac{d\vec{T}}{ds} \cdot \frac{ds/dt}{\left|\left|\vec{T}'(t)\right|\right|}$$

and since $ds/dt > 0$ and $\left|\left|\vec{T}'(t)\right|\right| > 0$, $\vec{N}$ points in the same direction as $d\vec{T}/ds$ and since $\vec{N}$ is a unit vector

$$\vec{N} = \frac{\frac{d\vec{T}}{ds}}{||ds/dt||} = \frac{1}{\kappa}\frac{d\vec{T}}{ds}$$

where $\kappa = ||ds/dt||$.

□

1.16 Curvature

Suppose the smooth curve C is the graph of the vector function $\vec{R}(s)$, parametrized in terms of the arc length s. Then the **curvature** of C is the function

$$\kappa(s) = \left\| \frac{d\vec{T}}{ds} \right\|$$

where $\vec{T}(s)$ is the unit tangent vector.

Example 1.27. Find the curvature of a circle.

Solution. A circle can be parametrized by $\vec{R}(t) = \langle r\cos t, r\sin t\rangle$ where r is the radius. We determine the arc length function as

$$s = \int_0^t \sqrt{r^2\cos^2 u + r^2\sin^2 u}\, du = rt.$$

Solving for t we find the component functions to be

$$x(s) = r\cos\left(\frac{s}{r}\right) \qquad \text{and} \qquad y(s) = r\sin\left(\frac{s}{r}\right).$$

Thus the unit tangent and unit normal vectors are

$$\vec{T}(s) = \left\langle -\sin\left(\frac{s}{r}\right), \cos\left(\frac{s}{r}\right)\right\rangle$$

and

$$\vec{N}(s) = \left\langle -\frac{1}{r}\cos\left(\frac{s}{r}\right), -\frac{1}{r}\sin\left(\frac{s}{r}\right)\right\rangle \left(\frac{1}{\kappa}\right)$$

where

$$\kappa = \sqrt{\frac{1}{r^2}\cos^2\left(\frac{s}{r}\right) + \frac{1}{r^2}\sin^2\left(\frac{s}{r}\right)} = \frac{1}{r}.$$

Therefore the curvature of a circle is $\frac{1}{r}$.

Example 1.28. Let C be the curve given as the graph of the vector function

$$\vec{R}(t) = (t - \sin t)\vec{i} + (1 - \cos t)\vec{j} + \left(4\sin\frac{t}{2}\right)\vec{k}.$$

Find the unit tangent vector $\vec{T}(t)$ to C, $\frac{d\vec{T}}{ds}$, and the curvature $\kappa(t)$.

Solution. The derivative is

$$\vec{R}'(t) = (1 - \cos t)\vec{\imath} + (\sin t)\vec{\jmath} + \left(2\cos\frac{t}{2}\right)\vec{k}$$

and the magnitude of the derivative is

$$\begin{aligned}\left\|\vec{R}'(t)\right\| &= \sqrt{(1-\cos t)^2 + \sin^2 t + 4\cos^2\frac{t}{2}} \\ &= \sqrt{2 - 2\cos t + 4\cos^2\frac{t}{2}} = \sqrt{4\sin^2\frac{t}{2} + 4\cos^2\frac{t}{2}} = 2\end{aligned}$$

thus the unit tangent vector is

$$\vec{T}(t) = \frac{\vec{R}'(t)}{\left\|\vec{R}'(t)\right\|} = \frac{1}{2}\left[(1-\cos t)\vec{\imath} + (\sin t)\vec{\jmath} + \left(2\cos\frac{t}{2}\right)\vec{k}\right].$$

Then,

$$\vec{T}'(t) = \frac{1}{2}\left[(\sin t)\vec{\imath} + (\cos t)\vec{\jmath} - \left(\sin\frac{t}{2}\right)\vec{k}\right]$$

because $\frac{d\vec{T}}{ds} = \frac{\vec{T}(t)}{ds/dt}$ we have

$$\frac{d\vec{T}}{ds} = \frac{1}{4}\left[(\sin t)\vec{\imath} + (\cos t)\vec{\jmath} - \left(\sin^2\frac{t}{2}\right)\vec{k}\right]$$

Therefore the curvature is

$$\kappa(t) = \left\|\frac{d\vec{T}}{ds}\right\| = \frac{1}{4}\sqrt{\sin^2 t + \cos^2 t + \sin^2\frac{t}{2}} = \frac{1}{4}\sqrt{1 + \sin^2\frac{t}{2}}.$$

as a function of t.

::: {#cor- } [Curvature in Vector Form] Suppose the smooth curve C is the graph of the vector function $\vec{R}(t)$. Then the curvature is given by

$$\kappa(t) = \frac{\left\|\vec{R}'(t) \times \vec{R}''(t)\right\|}{\left\|\vec{R}'(t)\right\|^3}. \tag{1.11}$$

:::

Proof. Given two vector functions $\vec{R}(t)$ and $\vec{R}(s)$ with the same smooth graph C and where $\vec{R}(s)$ is a parametrization vector function in terms of the arc length of C, then according to the rule

$$\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds}\frac{ds}{dt}$$

we have

$$\kappa(t) = \left\| \frac{d\vec{T}}{ds} \right\| = \frac{\left\| \frac{d\vec{T}}{dt} \right\|}{\left\| \vec{R}'(t) \right\|}.$$

Therefore,

$$\begin{aligned}
\kappa(t) &= \frac{\left\|\vec{T}'(t)\right\|}{\left\|\vec{R}'(t)\right\|} = \frac{\left\|\vec{T}'(t)\right\| \left\|\vec{R}'(t)\right\|^2}{\left\|\vec{R}'(t)\right\|^3} = \frac{\left\|\vec{T}'(t)\right\| \left(\frac{ds}{dt}\right)^2}{\left\|\vec{R}'(t)\right\|^3} \\
&= \frac{\left\|\vec{T}'(t)\right\| \left\|\vec{T}(t)\right\| \left(\frac{ds}{dt}\right)^2}{\left\|\vec{R}'(t)\right\|^3} = \frac{\left\|\vec{T}'(t)\right\| \left\|\vec{T}(t)\right\| \sin\left(\frac{\pi}{2}\right) \left(\frac{ds}{dt}\right)^2}{\left\|\vec{R}'(t)\right\|^3} \\
&= \frac{\left\|\vec{T}(t) \times \vec{T}'(t)\right\| \ \left(\frac{ds}{dt}\right)^2}{\left\|\vec{R}'(t)\right\|^3}.
\end{aligned}$$

Since

$$\vec{R}'(t) = \left\|\vec{R}'(t)\right\| \vec{T}(t) = \left(\frac{ds}{dt}\right) \vec{T}(t)$$

and

$$\vec{R}''(t) = \frac{d}{dt}\left(\frac{ds}{dt}\right) \vec{T}(t) + \frac{ds}{dt}\vec{T}'(t) = \frac{d^2 s}{dt^2}\vec{T}(t) + \frac{ds}{dt}\vec{T}'(t)$$

we have,

$$\begin{aligned}
\vec{R}'(t) \times \vec{R}''(t) &= \left(\frac{ds}{dt}\right) \vec{T}(t) \times \left(\frac{d^2 s}{dt^2}\vec{T}(t) + \frac{ds}{dt}\vec{T}'(t)\right) \\
&= \left(\frac{ds}{dt}\right)\left(\frac{d^2 s}{dt^2}\right) (\vec{T}(t) \times \vec{T}(t)) + \left(\frac{ds}{dt}\right)^2 (\vec{T}(t) \times \vec{T}'(t)) \\
&= \left(\frac{ds}{dt}\right)^2 (\vec{T}(t) \times \vec{T}'(t)).
\end{aligned}$$

Therefore,

$$\left\|\vec{R}'(t) \times \vec{R}''(t)\right\| = \left(\frac{ds}{dt}\right)^2 \left\|\vec{T}(t) \times \vec{T}'(t)\right\|.$$

Finally

$$\kappa(t) = \frac{\left\|\vec{R}'(t) \times \vec{R}''(t)\right\|}{\left\|\vec{R}'(t)\right\|^3}$$

as desired.

□

Example 1.29. Given the curve defined by

$$\vec{R}(t) = \sin t\vec{\imath} + \cos t\vec{\jmath} + t\vec{k}$$

find a unit tangent vector $\vec{T}$ at the point on the curve where $t = \pi$, the curvature at $t = \pi$, and find the length of the curve from $t = 0$ to $t = \pi$.

Solution. We have $\vec{R}'(t) = \cos t\vec{\imath} - \sin t\vec{\jmath} + \vec{k}$, $\vec{R}''(t) = -\sin t\vec{\imath} - \cos t\vec{\jmath}$, and

$$\begin{aligned} \left\|\vec{R}'(t) \times \vec{R}''(t)\right\| &= \left\| \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \cos t & -\sin t & 1 \\ -\sin t & -\cos t & 0 \end{vmatrix} \right\| \\ &= \left\|(\cos t)\vec{\imath} + (-\text{sint})\vec{\jmath} + \left(-\cos^2 t - \sin^2 t\right)\vec{k}\right\| \\ &= \sqrt{2}. \end{aligned}$$

Therefore the curvature is $\kappa = \frac{\sqrt{2}}{\left(\sqrt{2}\right)^3} = \frac{1}{2}$, and also

$$\vec{T} = \left(\frac{1}{\sqrt{2}}\cos t\right)\vec{\imath} + \left(-\frac{1}{\sqrt{2}}\sin t\right)\vec{\jmath} + \left(\frac{1}{\sqrt{2}}\right)\vec{k}$$

and

$$\int_0^\pi \left\|\vec{R}'\right\| dt = \int_0^\pi \sqrt{2}\, dt = \sqrt{2}\pi.$$

1.17 Maximum Curvature

Now that we know that the curvature formula in (1.11) is a one variable function we can apply one variable calculus to find maximum curvature. We will derive two more curvature formulas, one for planar functional form 1.12 and another for parametric form 1.14.

There is a fourth formula (for polar form) which we leave for the reader to explore as Exercise **??**. For example, consider the cardioid given by equation $r = 2(1 + \cos\theta)$ with derivative $dr/d\theta = -2\sin\theta$. We can use this information to determine where the curvature is maximum. From the graph (**??**) can you tell at which points will the curvature be maximum?

::: {#cor- } [Curvature in Planar Form] The graph C of the function $y = f(x)$ has curvature

$$\kappa(x) = \frac{|f''(x)|}{\left(1 + [f'(x)]^2\right)^{3/2}} \tag{1.12}$$

where $f(x)$, $f'(x)$, and $f''(x)$ all exist. :::

Proof. Given the vector function $x(t) = t$ and $y(t) = f(t)$ defined by $y = f(x)$ we have $\vec{R}(t) = t\vec{i} + f(t)\vec{j}$. Using the formula

$$\kappa(t) = \frac{\left\|\vec{R}'(t) \times \vec{R}''(t)\right\|}{\left\|\vec{R}'(t)\right\|^3},$$

we have

$$\kappa(t) = \frac{\left\|\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & f'(t) & 0 \\ 0 & f''(t) & 0 \end{vmatrix}\right\|}{\left\|\vec{R}'(t)\right\|^3} = \frac{|f''(t)|}{\left(\sqrt{1 + [f''(t)]^2}\right)^3}.$$

□

Example 1.30. Find the maximum curvature for the graph of $y = \ln x$.

Solution. We have $y' = \frac{1}{x}$, $y'' = \frac{-1}{x^2}$, and

$$\kappa(x) = \frac{\frac{1}{x^2}}{\left(1 + \frac{1}{x^2}\right)^{3/2}} = \frac{1}{\left(1 + \frac{1}{x^2}\right)^{3/2} x^2}$$

To maximize the curvature we find the first derivative

$$\frac{d\kappa}{dx} = \frac{3}{\left(1 + \frac{1}{x^2}\right)^{5/2} x^5} - \frac{2}{\left(1 + \frac{1}{x^2}\right)^{3/2} x^3} = \frac{1 - 2x^2}{\sqrt{1 + \frac{1}{x^2}} x \left(x^2 + 1\right)^2}$$

Applying the first derivative test we have the maximum curvature at $x = \frac{1}{\sqrt{2}}$ with curvature of $\kappa\left(\frac{1}{\sqrt{2}}\right) = \frac{2}{3\sqrt{3}}$.

Example 1.31. Find the maximum curvature for the graph of $y = e^{2x}$.

Solution. We have $y' = 2e^{2x}$, $y" = 4e^{2x}$, and

$$\kappa = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{4e^{2x}}{(1 + 4e^{4x})^{3/2}}$$

To maximize the curvature we find the first derivative

$$\frac{d\kappa}{dx} = -\frac{96e^{6x}}{(1 + 4e^{4x})^{5/2}} + \frac{8e^{2x}}{(1 + 4e^{4x})^{3/2}} = -\frac{8e^{2x}\left(-1 + 8e^{4x}\right)}{(1 + 4e^{4x})^{5/2}}$$

Applying the first derivative test we have the maximum curvature at $x = \frac{-3}{4}\ln 2$ with curvature of

$$\kappa\left(\frac{-3}{4}\ln 2\right) = \frac{4\sqrt{3}}{9}. \tag{1.13}$$

::: {#cor- } [Curvature in Parametric Form] If C is a smooth curve in $\mathbb{R}^2$ described by the parametric equations $x(t)$ and $y(t)$, then the curvature is given by

$$\kappa = \frac{|x'y'' - y'x''|}{\left[(x')^2 + (y')^2\right]^{3/2}}. \tag{1.14}$$

:::

Example 1.32. Find the point(s) where the ellipse $9x^2 + 4y^2 = 36$ has maximum curvature.

Solution. Write the ellipse as

$$\frac{x^2}{2^2} + \frac{y^2}{3^2} = 1$$

and so we parametrize using $x(t) = 2\cos t$ and $y(t) = 3\sin t$. Then we find

$$x'(t) = -2\sin t, \quad y'(t) = 3\cos t, \quad x''(t) = -2\cos t, \quad y''(t) = -3\sin t.$$

Now using 1.17, we find

$$\begin{aligned}
\kappa &= \frac{|x'y'' - y'x''|}{\left[(x')^2 + (y')^2\right]^{3/2}} \\
&= \frac{|(-2\sin t)(-3\sin t) - (3\cos t)(-2\cos t)|}{\left[(-2\sin t)^2 + (3\cos t)^2\right]^{3/2}} \\
&= \frac{6}{\left(4\sin^2 t + 9\cos^2 t\right)^{3/2}} \\
&= 6\left(4 + 5\cos^2 t\right)^{-3/2}
\end{aligned}$$

See **??** for a sketch of the curvature function. Then we apply the first derivative test, using

$$\frac{d\kappa}{dt} = -9\left(4 + 5\cos^2 t\right)^{-5/2}(-10\cos t\sin t) = 0$$

when $t = 0, \pi/2, \pi, 3\pi/2, 2\pi$. The point(s) where the ellipse $9x^2+4y^2 = 36$ has maximum curvature are $(0, \pm 3)$ from when $t = \pi/2$ and $t = 3\pi/2$ with curvature $\kappa = 3/4$.

1.18 Exercises

Exercise 1.55. Find the length of the given curve over the given interval.

- $\vec{R}(t) = t\vec{i} + 3t\vec{j}$ over the interval $[0, 4]$
- $\vec{R}(t) = t\vec{i} + 2t\vec{j} + 3t\vec{k}$ over the interval $[0, 2]$
- $\vec{R}(t) = \cos^3 t\vec{i} + \cos^2 t\vec{k}$ over the interval $[0, \frac{\pi}{2}]$

Exercise 1.56. Express the following vector functions in terms of the arc length parameter s measured from the point where $t = 0$ in the direction of increasing t.

- $\vec{R}(t) = \langle \sin t, \cos t \rangle$ at the point where $t = 0$
- $\vec{R}(t) = \langle 3\cos t + 3t\sin t, 2t^2, 3\sin t - 3t\cos t \rangle$ at the point where $t = 0$

Exercise 1.57. $\vec{R}(t) = [\ln(\sin t)]\vec{i} + [\ln(\cos t)]\vec{j}$ at the point where $t = \pi/3$

Exercise 1.58. Find the length of the given curve described by the vector function $\vec{R}(t) = 3t\vec{i} + (3\cos t)\vec{j} + (3\sin t)\vec{k}$ over the interval $[0, \frac{\pi}{2}]$.

Exercise 1.59. Determine a so that the length of the given curve described by the vector function $\vec{R}(t) = a\cos t\vec{i} + (a\sin t)\vec{j} + (5t)\vec{k}$ has length $\sqrt{41}\pi$ over the interval $[0, \pi]$.

Exercise 1.60. Express the vector function $\vec{R}(t) = \langle 2 + 3t, 1 - t, -4t - 9 \rangle$ in terms of the arc length parameter s measured from the point where $t = 0$ in the direction of increasing t.

Exercise 1.61. Determine a so that the vector function $\vec{R}(t) = \langle 2 - 3t, 1 + t, -4t \rangle$ written in terms of the arc length parameter s measured from the point where $t = 0$ in the direction of increasing t is

$$\vec{R}(s) = \left\langle 2 - \frac{3s}{a}, 1 + \frac{s}{a}, \frac{-4s}{a} \right\rangle.$$

Exercise 1.62. Find the curvature at the given point.

- $\vec{R}(t) = (2t + 3)\vec{i} + (5 - t^2)\vec{j}$
- $\vec{R}(t) = t\vec{i} + \cos t\vec{j}$

- $\vec{R}(t) = t\vec{i} + e^t\vec{j}$
- $\vec{R}(t) = 3\cos t\vec{i} + 2\sin t\vec{j}$
- $\vec{R}(t) = (2t-1)\vec{i} + (t-3)\vec{j} + (2-3t)\vec{k}$, $(1,-2,-1)$
- $\vec{R}(t) = t\vec{i} + \ln t\vec{j} + 0.5t\vec{k}$, $(2, \ln 2, 1)$
- $\vec{R}(t) = \sin t\vec{i} + \cos t\vec{j} + 2t\vec{k}$, $(1, 0, \pi)$
- $\vec{R}(t) = (\cos t + t\sin t)\vec{i} + (\sin t - t\cos t)\vec{j} + 3\vec{k}$
- $\vec{R}(t) = (e^t \sin t)\vec{i} + (e^t \cos t)\vec{j} + 2\vec{k}$
- $\vec{R}(t) = 6\sin 2t\vec{i} + 6\cos 2t\vec{j} + 5t\vec{k}$

Exercise 1.63. Find the curvature of the plane curve at the given location.

- $y = x - \frac{1}{9}x^2$, $x = 3$
- $y = \sin x$, $x = \frac{\pi}{2}$
- $y = \ln x$, $x = 1$
- $y = 1/\ln x$, $x = 2$
- $y = x^3 - 3x^2 + 2$, $x = 2$
- $y = e^{x^2}$, $x = 0$
- $y = \sec x$, $\pi/3$
- $1/x^2$, $x = 2$
- $y = \cos 2x$, 0
- $y = x^4 - x^2 + 3$, $x = -1$

Exercise 1.64. Find the curvature of the plane curve at the given location.

- $y = \ln(\cos x)$, $-\pi/2 < x < \pi/2$
- $x = t - 1, y = \sqrt{t}$, $(x, y) = (3, 2)$
- $x = t - t^2, y = 1 - t^3$, $(x, y) = (0, 1)$
- $x = 1 - \sin t, y = 2 + \cos t$, $(x, y) = (1, 3)$
- $x = \cos^3 t, y = \sin^3 t$, $(x, y) = (\sqrt{2}/4, \sqrt{2}/4)$

Exercise 1.65. Given the vector function

$$\vec{R}(t) = \sin t\vec{i} + \cos t\vec{j} + t\vec{k}$$

and $t_0 = \pi$, find the unit tangent vector and the curvature at t_0.

Exercise 1.66. Find all points on the curve at which the curvature is zero.

- $y = \cot x$
- $y = e^{-x^2}$

- $y = x^4 - 12x^2$
- $\vec{R}(t) = t^2\vec{i} - t^2\vec{j}$
- $\vec{R}(t) = e^x\vec{i} + \ln x\vec{j}$
- $\vec{R}(t) = e^{-x}\vec{i} + e^x\vec{j}$

Exercise 1.67. Find the point(s) where the ellipse $36x^2 + 9y^2 = 121$ has maximum curvature.

Exercise 1.68. Find the points on the curve in the xy-plane described by $y = 3x - x^3$ at which the curvature is maximum.

Exercise 1.69. Show that the curvature of the parabola $y = x^2 + 3$ approaches 0 as $x \to \infty$.

Exercise 1.70. Show that the curvature at every point on a line is 0.

Exercise 1.71. Show that the maximum curvature of a parabola occurs at the vertex.

Exercise 1.72. Show that for an ellipse, the maximum and minim curvature occurs at ends of the major and minor axes, respectively.

- Show that for a hyperbola, the maximum curvature occurs at the ends of the traverse axis.

Exercise 1.73. Show that the curvature of the helix

$$\vec{R}(t) = a\cos t\vec{i} + a\sin t\vec{j} + bt$$

is $\kappa = a/(a^2 + b^2)$. What is the largest value that κ can have for a given value of b?

Exercise 1.74. Suppose a curve C in the xy-plane described by a polar equation $r = f(\theta)$. Show that the curvature of C is

$$\kappa(\theta) = \frac{\mid [f(\theta)]^2 + 2[f'(\theta)]^2 - f(\theta)f''(\theta) \mid}{\{[f(\theta)]^2 + [f'(\theta)]^2\}^{3/2}}. \tag{1.15}$$

Exercise 1.75. Find the curvature of the polar curve at $P(r, \theta)$.

- the cardoid $r = a(1 - \cos\theta)$, where $0 < \theta < 2\pi$.
- the four-leafed rose $r = \sin 2\theta$, where $0 < \theta < \pi/2$
- the spiral $r = e^{a\theta}$
- $r = 1 - \cos\theta$, $\theta = 3\pi/2$
- $r = 1 - \sin\theta - \cos\theta$, $\theta = 2\pi$

Exercise 1.76. Show that lines and circles are the only planes curves that have constant curvature. Is this true for space curves?

Chapter 2

Limits and Continuity

2.1 Multivariable Functions

Partial differentiation is a powerful mathematical tool that can be used to solve a variety of problems. In this book, we will discuss the fundamental concepts and principles of partial differentiation. We will also provide some examples to illustrate how partial differentiation can be used to find maxima and minima, optimize functions, and more.

Partial differentiation is the process of taking the partial derivative of a function with respect to one or more of its variables. The partial derivative of a function is the rate of change of that function with respect to one of its variables, holding all other variables constant. Partial differentiation is used in calculus and other mathematical disciplines to find maxima and minima, as well as to study the behavior of functions near points of discontinuity.

Partial differentiation has two key concepts: the derivative and the limit. The derivative is a measure of how a function changes as one of its inputs changes. The limit is the value that a function approaches as one of its inputs approaches a particular value. Partial differentiation relies on both of these concepts to quantify how a function changes near a point in space. In order to take the partial derivative of a function, we must first be able to take the derivative with respect to one variable while holding all other variables constant.

Partial differentiation is a powerful tool that can be used to solve problems in many different fields, including physics, engineering, and economics.

A multivariable function is a mathematical function of two or more variables. Examples of multivariable functions include the position of a moving particle in space (which depends on its velocity and acceleration), the electric and magnetic fields in electromagnetism, and the pressure, temperature, and density of a fluid in fluid mechanics. Multivariable functions can be graphed in three-dimensional space, and their behavior can be studied using calculus. In general, multivariable functions are more complicated than their single-variable counterparts. However, they can often be decomposed into simpler functions, making them easier to understand and work with.

Multivariable functions can be partial differentiated with respect to each of their variables. This partial differentiation is a measure of how the function changes when one of its variables is changed, while the other variables are held constant. In other words, it helps us to understand how a function behaves when we change one of its inputs. Partial differentiation can be used to optimize a function, by finding the input that will produce the desired output. It can also be used to find maxima and minima, and to solve problems in physics and engineering. So, if you're ever stuck trying to figure out a multivariable function, remember that partial differentiation is your friend.

Multivariable limits and continuity are often some of the most challenging concepts for students to grasp in mathematics. In partial differentiation, we take the limit of a function as one or more of its variables approach infinity while the others remain finite. This can be a difficult concept to wrap one's head around, but it is crucial for understanding many advanced mathematical concepts.

Continuity, on the other hand, is a relatively straightforward concept: it simply means that a function is uninterrupted and smooth. However, proving continuity can often be quite tricky. In general, multivariable limits and continuity can be quite challenging concepts, but they are essential for understanding advanced mathematics.

When it comes to math, there are a lot of big words that can be intimidating. Differentials and differentiability may sound like they belong in a horror movie, but they're actually just two concepts that are closely related.

In calculus, a differential is a small change in a variable, while differentiability refers to the ability to calculate these small changes. These concepts are important in calculus and other branches of mathematics, as they allow for the determination of things like rates of change and slopes of curves. So the next time you see a differential or differentiability, don't be scared off - they're just math terms!

The chain rule is one of the most important concepts in calculus, and it can be a little tricky to wrap your head around at first. Basically, the chain rule says that if you have a function that is the composition of two other functions, you can take the derivative of that function by taking the derivative of each of the individual functions and then multiplying them together.

The chain rule can be generalized to functions with more than two components, but the basic idea is always the same: to take the derivative of a composite function, you need to take the derivative of each individual function and then multiply them all together.

Partial differentiation can also be used when taking derivatives with respect to more than one variable. However, it's important to remember that this rule is only useful with functions that are composed of other functions - if a function is not composed of other functions, then the chain rule is not that useful at all. So, if you ever come across a function that you're not sure how to differentiate, make sure to check whether or not it's composed of other functions before trying using the chain rule.

Partial differentiation simply means taking the derivative with respect to one variable while holding all other variables constant. For example, consider the function $f(x, y) = x^2 + y^2$. The partial derivative of this function with respect to x is 2x, and the partial derivative with respect to y is 2y. This means that as we move across the graph of this function, it is increasing at a rate of 2x in the direction of x and at a rate of 2y in the direction of y.

The gradient vector is simply a vector that contains all partial derivatives of a given function. So, for our example function $f(x, y) = x^2 + y^2$, the gradient vector would be $\langle 2x, 2y\rangle$. This vector can be thought of as a "directional derivative" because it tells us which way the function is increasing and by how much. The magnitude of the gradient vector tells us how steeply the function is increasing (i.e., how steeply it is "pointing" in a particular direction), and the direction tells us which way it is pointing.

There are many applications for partial derivatives and gradient vectors in physics and engineering. One common application is finding local maxima and minima (i.e., points where the derivative is zero).

Another common application is optimization; for example, finding the shortest path between two points or minimizing fuel consumption in a car engine. Partial derivatives and gradient vectors also play a central role in Newton's Method for solving differential equations numerically.

In short, partial derivatives and gradient vectors are essential tools for anyone who wants to understand how functions change as their inputs vary.

Differentiation is all about finding rates of change. And in the world of partial differential equations, we're usually interested in rates of change with respect to more than one variable. Enter the tangent plane and the normal line.

The tangent plane to a surface at a point is the plane that just barely touches the surface at that point. It's like if you took a really thin piece of paper and tried to flatten it out on the surface. The normal line is the line that is perpendicular to the tangent plane at the point of contact.

Partial differentiation is all about taking derivatives with respect to more than one variable, so tangent planes and normal lines are going to be our best friends. We can use them to find rates of change in multiple directions, which will come in handy when we're trying to solve partial differential equations.

Partial differentiation is a way of finding the extreme values of a function of two variables. In other words, it helps us find the points where the function is at its highest or lowest.

To do this, we take partial derivatives with respect to each of the variables. The partial derivative with respect to x tells us how the function changes as x changes, and the partial derivative with respect to y tells us how the function changes as y changes. If both partial derivatives are positive, then the function is increasing in both variables, and so the point is not an extreme value.

However, if both partial derivatives are negative, then the function is decreasing in both variables, and so the point is an extreme value. If one partial derivative is positive and the other is negative, then the function is stylized and we need to use the second partial derivative test to determine whether the point is an extreme value.

So, partial differentiation can be used to find extreme values of functions of two variables.

What are Lagrange multipliers? In short, they're a mathematical way of solving optimization problems. But what does that mean, exactly?

Partial differentiation is a powerful tool that allows us to optimize functions subject to constraints. In many cases, the constrained optimization problem can be simplified by introducing Lagrange multipliers. Lagrange multiplier methods are named after Joseph-Louis Lagrange, who first developed the technique in the 18th century.

The key idea is to introduce new variables, called Lagrange multipliers, which represent the constraints of the optimization problem. These multipliers can then be used to simplify the optimization problem by eliminating the need for explicit constraints. As a result, the Lagrange multiplier method is a powerful tool for solving constrained optimization problems.

If you're interested in learning partial differentiation, this book is for you. It covers all the basics, from the definition of partial derivatives to the chain rule and beyond. Plus, it includes plenty of worked examples to help you understand and master every concept.

How does one go about teaching partial differentiation? It's simply a matter of conveying an understanding clearly and concisely. Of course, that's often easier said than done. partial differentiation can be a tricky topic to wrap your head around, let alone explain to someone else. But with a little patience and perseverance, you should be able to do it. After all, if you can partially differentiate, anyone can!

2.2 Functions of Several Variables

A **polynomial function** of two variables is a sum of terms of the form $cx^m y^n$, where c is a real number and both m and n are nonnegative integers. For example the function

$$f(x,y) = 3x^3y^5 - 3x^2y^4 - xy + 7$$

is a polynomial function with terms $3x^3y^5$, $-3x^2y^4$, $-xy$, and 7. A **rational function** of two variables is a ratio of polynomial functions. For example the function

$$f(x,y) = \frac{3x^3y^5 - 3x^2y^4 - xy + 7}{x^3 - y^2}$$

is a rational function with numerator $3x^3y^5 - 3x^2y^4 - xy + 7$ and denominator $x^3 - y^2$.

A function of three variables is a rule that assigns to each ordered pair (x, y, z) in a set D a unique number $f(x, y, z)$. The set D is called the **domain** of the function and is a subset of $\mathbb{R}^3$. The collection of corresponding values $f(x, y, z)$ constitutes the **range** which is a subset of $\mathbb{R}$.

Example 2.1. Let

$$f(x,y,z) = x^2 - 2xyz + 2yz - x.$$

Find the domain of f and evaluate $f(1,2,3)$, $f(t,2t,3t)$, $f(x+y,x-y,0)$, and

$$\frac{f(x+h,y,z)-f(x,y,z)}{h}, \quad \frac{f(x,y+h,z)-f(x,y,z)}{h}, \quad \frac{f(x,y,z+h)-f(x,y,z)}{h}.$$

where h is an unknown quantity.

Solution. Substitution into the function yields the following: $f(1,2,3) = 0$,

$$f(t,2t,3t) = -t + 13t^2 - 12t^3 \qquad f(x+y,x-y,0) = -x - y + (x+y)^2$$

and

$$\frac{f(x+h,y,z)-f(x,y,z)}{h} = \frac{-h-x^2+(h+x)^2+2xyz-2(h+x)yz}{h} \tag{2.1}$$

$$\frac{f(x,y+h,z)-f(x,y,z)}{h} = \frac{-h-x^2+(h+x)^2+2xyz-2(h+x)yz}{h} \tag{2.2}$$

$$\frac{f(x,y,z+h)-f(x,y,z)}{h} = \frac{2yz-2xyz-2y(h+z)+2xy(h+z)}{h} \tag{2.3}$$

The domain of f is **not** all real numbers, but rather $\mathbb{R}^3$.

Definition 2.1. A **function** of n variables is a rule that assigns to each ordered pair $(x_1, x_2, \dots, x_n)$ in a set D a unique number $f(x_1, x_2, \dots, x_n)$. The set D is called the **domain** of the function and the set of corresponding values $f(x_1, x_2, \dots, x_n)$ is called the **range**.

The **graph** of a function of several variables $f(x_1, \dots, x_n)$ is the collection of all ordered $(n+1)$-tuples $(x_1, \dots, x_n, x_{n+1})$ such that $(x_1, \dots, x_n)$ is in the domain of f and $x_{n+1} = f(x_1, \dots, x_n)$. Sketching by hand the graph of a function with several variables can be challenging. Let's look at a few particular functions.

For example the function f defined by

$$f(x,y,z) = 2x^3 + 4y^4 + 9z^6 \tag{2.4}$$

is a three variable polynomial function with domain $\mathbb{R}^3$ and range $(-\infty, +\infty)$.
As another example consider the function g defined by

$$g(w,x,y,z) = z^2 + \sin wx + \cos wy. \tag{2.5}$$

The function g is a four variable function with domain $\mathbb{R}^4$ and range $[-2, +\infty)$. Let h be the function defined by

$$h(x,y,z) = e^{xy} + z^4 \sqrt[6]{x^2-4}. \tag{2.6}$$

Then h is a three variable function with domain all ordered triples (x, y, z) with the requirement $x^2 - 4 \geq 0$. The range of h is the set of all real numbers greater than 0.

Definition 2.2. Let f and g be functions of the variables $x_1, x_2, \dots, x_n$. Then defined point-wise, the following functions

- $(f+g)(x_1, x_2, \dots, x_n) := f(x_1, x_2, \dots, x_n) + g(x_1, x_2, \dots, x_n)$

- $(f-g)(x_1, x_2, \dots, x_n) := f(x_1, x_2, \dots, x_n) - g(x_1, x_2, \dots, x_n)$
- $(fg)(x_1, x_2, \dots, x_n) := f(x_1, x_2, \dots, x_n)\, g(x_1, x_2, \dots, x_n)$
- $(f/g)(x_1, x_2, \dots, x_n) := \dfrac{f(x_1, x_2, \dots, x_n)}{g(x_1, x_2, \dots, x_n)}$ where $g(x_1, x_2, \dots, x_n) \neq 0$.

are also functions of the variables $x_1, x_2, \dots, x_n$.

For example, considering the functions f and h defined in 2.4 and 2.6, respectively. Is $f+h$ defined as a function? Yes, by 2.2

$$(f+h)(x,y,z) = 2x^3 + 4y^4 + 9z^6 + e^{xy} + z^4\sqrt[6]{x^2-4}$$

and notice the domain of $f+h$ is all ordered triples (x,y,z) with the requirement $x^2 - 4 \geq 0$ and the range is $(-\infty, +\infty)$.

Is $g+h$ defined as a function? If we use the functional rule defining h, namely $e^{xy} + z^4\sqrt[6]{x^2-4}$ then the function defined by

$$h_1(w,x,y,z) = e^{xy} + z^4\sqrt[6]{x^2-4}$$

(defined as a four variable function), then the function $(g+h_1)$ is defined as a function by

$$(g+h_1)(w,x,y,z) = z^2 + \sin wx + \cos wy + e^{xy} + z^4\sqrt[6]{x^2-4}.$$

using 2.2. Notice the domain of $g+h_1$ is $\mathbb{R}^4$ and the range is $[-2, +\infty)$.

2.3 Functions of Two Variables

When working with functions f of two variables x and y, we write $z = f(x,y)$ where x and y are the **independent variables** and z is the **dependent variable**. The **domain** is defined to be the largest set of points for which the functional formula is defined and real-valued.

Example 2.2. Find the domain and range for the function

$$f(x,y) = \frac{1}{\sqrt{x-y}}.$$

Solution. The domain is

$$\{(x,y) \in \mathbb{R}^2 \mid y < x\}$$

because of the square root in the denominator of f and the range is

$$\{z \in \mathbb{R} \mid z > 0\}.$$

Notice the domain is a subset of $\mathbb{R}^2$ and the range is a subset of $\mathbb{R}$.

Example 2.3. Find the domain and range for the function

$$f(x, y) = \sqrt{\frac{x}{y}}.$$

Solution. The domain is

$$\{(x, y) \in \mathbb{R}^2 \mid xy \geq 0 \text{ and } y \neq 0\}$$

because of the square root and the range is

$$\{z \in \mathbb{R} \mid z \geq 0\}.$$

Notice the domain is a subset of $\mathbb{R}^2$ and the range is a subset of $\mathbb{R}$.

In three dimensions, the graph of $z = f(x, y)$ is a surface in $\mathbb{R}^3$ whose **projection** onto the xy-plane is the domain D. When the plane $z = C$ intersects the surface

$$z = f(x, y),$$

the result is the curve with the equation $f(x, y) = C$ and such an intersection is called the **trace** of the graph of f in the plane $z = C$. The set of points (x, y) in the xy-plane that satisfies

$$f(x, y) = C$$

is called the **level curve** of f at C and an entire family of level curves is generated as C varies over the range of f. Level curves are obtained by projecting a trace onto the xy-plane. Because level curves are used to show the shape of a surface, they are sometimes called **contour curves** .

Definition 2.3. The curves $f(x, y) = C$ in the xy-plane are called the **level curves** of the function f of two variables x and y, where C is a constant in the range of f.

Example 2.4. Sketch a few level curves for the function

$$f(x, y) = 2x - 3y = C$$

with $C \geq 0$.

Solution. Graphing the lines

$$y = \frac{2}{3}x - \frac{C}{3}$$

we have the family of level curves corresponding to $C = 1, \ldots, 10$. The level curves show that the graph of the function f is a plane in $\mathbb{R}^3$.

Example 2.5. Sketch a few level curves for the function

$$f(x,y) = x^2 + \frac{y^2}{4} = C$$

with $C \geq 0$.

Solution. Graphing the ellipses

$$\frac{x^2}{1^2} + \frac{y^2}{2^2} = C$$

in $\mathbb{R}^2$, we have the family of level curves corresponding to $C = 1, \ldots, 10$. The level curves show that the graph of the function f is a elliptic paraboloid in $\mathbb{R}^3$.

Example 2.6. Find the domain and range for the function

$$f(x,y) = \sqrt{\frac{y}{x-2}}$$

and sketch some level curves for $f(x,y) = C$ with $C = 0, 1, 2, 3, 4$.

Solution. The domain of f is

$$\left\{(x,y) \in \mathbb{R}^2 \mid \frac{y}{x-2} > 0\right\}.$$

To sketch some level curves

$$\sqrt{\frac{y}{x-2}} = C$$

let's square both sides $\frac{y}{x-2} = C^2$ and so

$$y = C^2(x-2).$$

The range of f is $\{z \in \mathbb{R} \mid z \geq 0\}$.

2.4 Exercises

Exercise 2.1. Let $f(x,y,z) = x^2ye^{2x} + (x+y-z)^2$. Find each of the following.

- $f(0,0,0)$
- $f(1,-1,1)$
- $f(-1,1,-1)$

- $\frac{d}{dx} f(x, x, x)$,
- $\frac{d}{dy} f(1, y, 1)$
- $\frac{d}{dz} f\left(1, 1, z^2\right)$

Exercise 2.2. Find the domain and range for the multivariate function.

- $f(x, y) = \frac{1}{\sqrt{x-y}}$
- $f(x, y) = \sqrt{\frac{y}{x}}$.
- $f(u, v) = \sqrt{u \sin v}$.
- $f(x, y) = e^{(x+1)/(y-2)}$.
- $f(x, y) = \frac{1}{\sqrt{9-x^2-y^2}}$.

Exercise 2.3. Sketch some level curves of the function.

- $f(x, y) = x^2 - y^2 = C$
- $f(x, y) = \frac{x}{y} = C$
- $f(x, y) = x^2 - y = C$
- $f(x, y) = x^2 + \frac{y^2}{4} = C$
- $f(x, y) = x^3 - y = C$
- $f(x, y) = 1/\sqrt{x^2 - y^2}$
- $g(x, y) = \sqrt{x \sin y}$
- $h(x, y) = \ln(y - x)$

Exercise 2.4. Describe the trace of the quadratic surface in each coordinate plane, then sketch the surface.

- $\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 1$
- $\frac{x^2}{9} - y^2 - z^2 = 1$

Exercise 2.5. Sketch the graph of the multivariate function.

- $f(x, y) = x$
- $f(x, y) = x^3 - 1$
- $f(x, y) = x^2 - y$
- $f(x, y) = x^2 - y^2$
- $f(x, y) = \sqrt{x + y}$
- $f(x, y) = 1/\sqrt{x^2 - y^2}$
- $g(x, y) = \sqrt{x \sin y}$
- $h(x, y) = \ln(y - x)$

2.5 Multivariable Limits

Recall when considering

$$\lim_{x \to c} f(x) = L$$

we need to examine the approach of x to c from two directions –namely the left-hand limit and the right-hand limit. However for functions of two variables, we write $(x, y) \to (a, b)$ to mean that the point (x, y) is allowed to approach (a, b) along any path in the domain of f that passes through (a, b).

Open and closed disks are analogous to open and closed intervals on a coordinate line. An **open disk** (or **open ball** centered at the point (a, b) is the set of all points (x, y) such that

$$\sqrt{(x-a)^2 + (y-b)^2} < r$$

for $r > 0$. If the **boundary** of the disk is included, the disk is called a **closed disk**.

A point (a, b) is called an **interior point** of a set S in $\mathbb{R}^2$ if some open disk centered at (a, b) is contained entirely within S. If S is the empty set, or if every point of S is an interior point, then S is called an **open set** . The point (a, b) is called a **boundary point** of S if every open disk centered at (a, b) contains both points that belong to S and points that do not. The collection of all boundary points of S is called the **boundary** of S, and S is a closed disk if it contains its boundary.

Definition 2.4. Let f be a function that is defined for all points (x, y) in an open disk around (a, b) with the possible exception of (a, b) itself. Then the **limit** of $f(x, y)$ as (x, y) approaches (a, b) is L, written by

$$\lim_{(x,y) \to (a,b)} f(x, y) = L$$

if $f(x, y)$ can be made as close to L as we please by restricting (x, y) to be sufficiently close to (a, b).

2.6 Limit Properties

If a, b, and c are real numbers then

$$\lim_{(x,y) \to (a,b)} c = c, \qquad \lim_{(x,y) \to (a,b)} x = a, \qquad \lim_{(x,y) \to (a,b)} y = b. \tag{2.7}$$

The three limits in 2.7 can be used with 2.6 to show that

$$\lim_{(x,y) \to (a,b)} p(x, y) = p(a, b) \qquad \text{and} \qquad \lim_{(x,y) \to (a,b)} r(x, y) = r(a, b) \tag{2.8}$$

where p is a polynomial function of x and y and r is a rational function of x and y, provided (a, b) is in the domain of r.

::: {#thm- } [Limit Law] Suppose

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = M$$

and a is a real number. Then

- $\lim_{(x,y)\to(x_0,y_0)} (af)(x,y) = aL$
- $\lim_{(x,y)\to(x_0,y_0)} (f+g)(x,y) = L + M$
- $\lim_{(x,y)\to(x_0,y_0)} (fg)(x,y) = LM$
- $\lim_{(x,y)\to(x_0,y_0)} (f/g)(x,y) = L/M$ whenever $M \neq 0$. :::

Proof. The proof is left for the reader. □

Example 2.7. Evaluate $\lim_{(x,y)\to(1,1)} \dfrac{x^4+y^4}{x^2+y^2}$.

Solution. Since

$$r(x,y) = (x^4+y^4)/(x^2+y^2)$$

is a rational function and $(1,1)$ is in the domain of r, we use 2.6 to find

$$\lim_{(x,y)\to(1,1)} \frac{x^4+y^4}{x^2+y^2} = r(2,2) = 1.$$

Example 2.8. Evaluate $\lim_{(x,y)\to(0,0)} \dfrac{x^4-y^4}{x^2+y^2}$.

Solution. By canceling

$$\lim_{(x,y)\to(0,0)} \frac{(x^2-y^2)(x^2+y^2)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} (x^2-y^2) = 0. \tag{2.9}$$

Notice the functions f and g defined by

$$f(x,y) = x^2 - y^2 \qquad \text{and} \qquad g(x,y) = \frac{x^4-y^4}{x^2+y^2}$$

are not the same functions, because they have different domains. However f and g have the same limit because whenever $(x,y) \neq (0,0)$ we do indeed have $(x^4-y^4)/(x^2+y^2) = x^2-y^2$. Therefore, the "canceling" step in 2.9 is valid.

Example 2.9. Evaluate $\lim_{(x,y)\to(1,2)} \dfrac{\left(x^2-1\right)\left(y^2-4\right)}{(x-1)(y-2)}$.

Solution. By factoring and then canceling we have

$$\begin{aligned}
\lim_{(x,y)\to(1,2)} \frac{\left(x^2-1\right)\left(y^2-4\right)}{(x-1)(y-2)} &= \lim_{(x,y)\to(1,2)} \frac{(x-1)(x+1)(y-2)(y+2)}{(x-1)(y-2)} \\
&= \lim_{(x,y)\to(1,2)} (x+1)(y+2) = (1+1)(2+2) = 8.
\end{aligned}$$

Example 2.10. Evaluate $\lim_{(x,y)\to(a,a)} \dfrac{x^4-y^4}{x^2-y^2}$.

Solution. By factoring and then canceling we have

$$\begin{aligned}
\lim_{(x,y)\to(a,a)} \frac{x^4-y^4}{x^2-y^2} &= \lim_{(x,y)\to(a,a)} \frac{\left(x^2-y^2\right)\left(x^2+y^2\right)}{x^2-y^2} \\
&= \lim_{(x,y)\to(a,a)} \left(x^2+y^2\right) = 2a^2.
\end{aligned}$$

2.7 Limits that Do Not Exist

It is sometimes possible to show that the limit of a function does not exist by showing that the limit has different values depending on which path in the domain is used.

::: {#thm- } [Limits Along Paths]

If $f(x,y)$ approaches two different numbers as (x,y) approaches (a,b) along two different paths, then the limit

$$\lim_{(x,y)\to(a,b)} f(x,y)$$

does not exist. :::

Proof. The proof is left for the reader.

□

Example 2.11. Evaluate $\lim_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$.

Solution. Along the x-axis we have

$$\lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{xy}{x^2+y^2} = \lim_{(x,0)\to(0,0)} \frac{0}{x^2+0^2} = \lim_{x\to0} 0 = 0.$$

We have along the y-axis,

$$\lim_{\substack{(x,y)\to(0,0)\\ x=0}} \frac{xy}{x^2+y^2} = \lim_{(0,y)\to(0,0)} \frac{0}{0^2+y^2} = \lim_{y\to 0} 0 = 0.$$

Along the line $y = x$ we have

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x}} \frac{xy}{x^2+y^2} = \lim_{(x,x)\to(0,0)} \frac{x^2}{x^2+x^2} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}.$$

Therefore, by 2.7, the given limit does not exist (see **??**).

Example 2.12. Evaluate $\lim_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^4}$.

Solution. Along the curves $y = mx$ we have

$$\lim_{\substack{(x,y)\to(0,0)\\ y=mx}} \frac{xy^2}{x^2+y^4} = \lim_{(x,mx)\to(0,0)} \frac{m^2x^3}{x^2+m^4x^4} = \lim_{x\to 0} \frac{m^2x}{1+m^4x^2} = 0.$$

But along the curve $x = y^2$ we have

$$\lim_{\substack{(x,y)\to(0,0)\\ x=y^2}} \frac{xy^2}{x^2+y^4} = \lim_{(y^2,y)\to(0,0)} \frac{y^4}{y^4+y^4} = \lim_{y\to 0} \frac{y^4}{y^4+y^4} = \frac{1}{2}.$$

Therefore, by 2.7, the given limit does not exist.

Example 2.13. Evaluate $\lim_{(x,y)\to(0,0)} \dfrac{x^2y^2}{x^4+y^4}$.

Solution. Let m be any real number, then along the paths $y = mx$ we have

$$\lim_{\substack{(x,y)\to(0,0)\\ y=mx}} \frac{x^2y^2}{x^4+y^4} = \lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+y^4} = \lim_{(x,mx)\to(0,0)} \frac{x^2(mx)^2}{x^4+(mx)^4} = \frac{m^2}{1+m^4}.$$

Notice as the path $y = mx$ changes (as m changes) so does the value $m^2/(1+m^4)$. Therefore, by 2.7, the given limit does not exist.

Example 2.14. Evaluate $\lim_{(x,y)\to(0,0)} \dfrac{x^4y^4}{(x^2+y^4)^3}$.

Solution. Along the line $y = 0$ we have

$$\lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{x^4y^4}{\left(x^2+y^4\right)^3} = \lim_{x\to 0} \frac{0}{\left(x^2+0\right)^3} = 0$$

and along the path $y = \sqrt{x}$ we have

$$\lim_{\substack{(x,y)\to(0,0)\\ y=\sqrt{x}}} \frac{x^4y^4}{\left(x^2+y^4\right)^3} = \lim_{x\to 0} \frac{x^4x^2}{\left(2x^2\right)^3} = \frac{1}{8}.$$

Therefore, by 2.7, the given limit does not exist.

Example 2.15. Find a function $f(x, y)$ and a point (x_0, y_0) such that

$$\lim_{x\to x_0}\left(\lim_{y\to y_0} f(x,y)\right) \neq \lim_{y\to y_0}\left(\lim_{x\to x_0} f(x,y)\right).$$

Solution. Consider the function $f(x, y) = \frac{ax+by}{cx+dy}$ and the point $(0, 0)$. Then

$$\lim_{x\to 0}\left(\lim_{y\to 0} \frac{ax+by}{cx+dy}\right) = \frac{a}{c} \neq \lim_{y\to 0}\left(\lim_{x\to 0} \frac{ax+by}{cx+dy}\right) = \frac{b}{d}$$

for some values of $a, b, c,$ and d. What does this say about

$$\lim_{(x,y)\to(0,0)}\left(\frac{ax+by}{cx+dy}\right) \ ?$$

Example 2.16. Consider the function

$$f(x,y) = \frac{xy}{x^2y^2+(x-y)^2}.$$

Notice that

$$\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right) = \lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right). \tag{2.10}$$

Is it true, then, that

$$\lim_{(x,y)\to(0,0)}\left(\frac{xy}{x^2y^2+(x-y)^2}\right) \tag{2.11}$$

exists?

Solution. First we notice the (iterated) limits in 2.10 are both $-1/2$ as the following shows

$$\lim_{x\to 0}\left[\lim_{y\to 0}\left(\frac{xy}{x^2y^2+(x-y)^2}\right)\right]=\lim_{x\to 0}\left[\lim_{y\to 0}\left(\frac{x}{2x^2y+2(x-y)(-1)}\right)\right]=-\frac{1}{2}$$

$$\lim_{y\to 0}\left[\lim_{x\to 0}\left(\frac{xy}{x^2y^2+(x-y)^2}\right)\right]=\lim_{y\to 0}\left[\lim_{x\to 0}\left(\frac{y}{2y^2x+2(x-y)(1)}\right)\right]=-\frac{1}{2}$$

using L'Hopitals rule. Secondly, we find that the limit in 2.11 does not exist because along the paths $y=mx$ we have

$$\lim_{\substack{(x,y)\to(0,0)\\ y=mx}}\frac{xy}{x^2y^2+(x-y)^2}=\lim_{x\to 0}\left(\frac{mx^2}{m^2x^4+(x-mx)^2}\right)=\frac{m}{(1-m)^2}$$

which varies as the path $y=mx$ does.

2.8 Continuity of a Function of Two Variables

Definition 2.5. Let f be a function that is defined on any open disk around (a,b). Then f is **continuous** at the point (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y)=f(a,b).$$

::: {#thm- } [Continuous Multivariable Functions] If k is a real number and f and both g are continuous functions at (a,b), then the following functions are also continuous at (a,b), provided (a,b) is in the domain of the function.

- kf
- $f+g$
- $f-g$
- f/g
- $f\circ g$
- $\sqrt[n]{f}$ :::

Proof. The basic properties of limits can be used to proof this theorem. The details are left for the reader.

□

Since any polynomial and rational function can be built out of the continuous functions $f(x)=x$, $g(x,y)=y$, and $h(x,y)=k$ we can use the limits in 2.8 to realize that every polynomial is continuous on the entire plane and that rational functions are continuous on their domains.

Example 2.17. What is the largest set on which the function

$$f(x,y) = \begin{cases} \dfrac{3x^2y}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$$

is continuous?

Solution. We know f is continuous for $(x,y) \neq (0,0)$ since f is a rational function. Since

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$$

we have that f is continuous on $\mathbb{R}^2$.

Example 2.18. What is the largest set on which the function

$$h(x,y) = \ln\left(x^2+y^2-1\right)$$

is continuous?

Solution. Let $f(x,y) = x^2+y^2-1$ and $g(t) = \ln t$. Then

$$h(x,y) = \ln\left(x^2+y^2-1\right) = (g \circ f)(x,y).$$

Now f is continuous everywhere since it is a polynomial and g is continuous on its domain $\{t \mid t \geq 0\}$. Thus, h is continuous on its domain

$$D = \left\{(x,y) \mid x^2+y^2 > 1\right\}$$

which consists of all points outside the circle $x^2+y^2=1$.

2.9 Exercises

Exercise 2.6. Where is $f(x,y) = \sqrt{\frac{x^2+y^2}{(x-y)^2+1}}$ is continuous?

Exercise 2.7. Given $\lim_{(x,y)\to(0,0)} \frac{x^4y^4}{(x^2y^2)^{3/2}}$, find the limit.

Exercise 2.8. Find $\lim_{(x,y)\to(\pi/2,1)} \frac{1+\cos 2x}{y-e^y}$. Explain why.

Exercise 2.9. Evaluate the limit.

- $\lim_{(x,y)\to(-1,0)} \left(xy^2 + x^3y + 5\right)$
- $\lim_{(x,y)\to(0,0)} \left(5x^2 - 2xy + y^2 + 3\right)$
- $\lim_{(x,y)\to(0,1)} e^{x^2+x} \ln\left(ey^2\right)$
- $\lim_{(x,y)\to(a,a)} \frac{x^4-y^4}{x^2-y^2}$
- $\lim_{(x,y)\to(1,2)} \frac{(x^2-1)(y^2-4)}{(x-1)(y-2)}$
- $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+y^4}$
- $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2}$
- $\lim_{(x,y)\to(0,0)} \sin x + \sin y$
- $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+4}-2}$
- $\lim_{(x,y)\to(0,0)} \frac{1-\cos(x^2+y^2)}{x^2+y^2}$
- $\lim_{(x,y)\to(0,0)} \frac{\sin(x^3+y^3)}{\sqrt{x^6+y^6}}$
- $\lim_{(x,y)\to(-1,0)} \left(xy^2 + x^3y + 5\right)$
- $\lim_{(x,y)\to(0,0)} \left(5x^2 - 2xy + y^2 + 3\right)$
- $\lim_{(x,y)\to(0,1)} e^{x^2+x} \ln\left(ey^2\right)$
- $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2}$
- $\lim_{(x,y)\to(0,0)} \sin x + \sin y$
- $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+4}-2}$
- $\lim_{(x,y)\to(0,0)} \frac{1-\cos(x^2+y^2)}{x^2+y^2}$
- $\lim_{(x,y)\to(0,0)} \frac{\sin(x^3+y^3)}{\sqrt{x^6+y^6}}$

Exercise 2.10. Is the function f defined by

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

continuous at $(0,0)$? Explain.

Exercise 2.11. Is the function f defined by

$$g(x,y) = \begin{cases} \dfrac{xy^3}{x^2+y^6} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

continuous at $(0,0)$? Explain.

Exercise 2.12. Given that the function f defined by

$$f(x,y) = \begin{cases} \dfrac{3x^3 - 3y^3}{x^2 - y^2} & x^2 \neq y^2 \\ B & \text{otherwise} \end{cases}$$

is continuous at the point $(0,0)$ what is the value of B? Explain.

Exercise 2.13. Given that the function f defined by

$$f(x,y) = \begin{cases} \dfrac{x^3 + y^3}{x^2 + y^2} & (x,y) \neq (0,0) \\ A & (x,y) = (0,0) \end{cases}$$

is continuous at the point $(0,0)$ what is the value of A? Explain.

Exercise 2.14. Let $f(x,y) = \begin{cases} \frac{x^2y^2}{x^4+y^4+1} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$ Explain why f is continuous at $(0,0)$.

Chapter 3

Differentiation

3.1 Partial Derivatives

For the **partial differentiation** of a function of two variables, $z = f(x, y)$, we find the partial derivative with respect to x by regarding y as a constant while differentiating the function with respect to x. Similarly, the partial derivative with respect to y is found by regarding x as a constant while differentiating with respect to y.

If $z = f(x, y)$ then the partial derivatives of f with respect to x and y are the functions f_x and f_y, respectively, defined by

$$f_x(x, y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

and

$$f_y(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta \mathrm{y}) - f(x, y)}{\Delta y}.$$

The partial derivatives f_x and f_y are denoted by

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = \frac{\partial}{\partial x} f(x, y) = D_x(f)$$

and

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial z}{\partial y} = \frac{\partial}{\partial y} f(x, y) = D_y(f).$$

Example 3.1. Find the partial derivatives of the function

$$f(x, y) = x^3 + x^2 y^3 - 2y^2$$

at $P = (2, 1)$.

Solution. Holding y constant and differentiating with respect to x, we get

$$f_x(x, y) = 3x^2 + 2xy^3$$

and so

$$f_x(2, 1) = 3\left(2^2\right) + 2(2)\left(1^3\right) = 16.$$

Holding x constant and differentiating with respect to y, we get

$$f_y(x, y) = 3x^2y^2 - 4y$$

and so $f_y(2, 1) = 3(2)^2(1)^2 - 4(1) = 8$.

Example 3.2. Find the partial derivatives of the function f defined by

$$f(x, y) = 4 - x^2 - 2y^2$$

at $P = (1, 1)$. Explain graphical.

Solution. We have

$$f_x(x, y) = -2x \quad \text{and} \quad f_y(x, y) = -4y,$$

and so $f_x(1, 1) = -2$ and $f_y(1, 1) = -4$. The graph of f is the paraboloid

$$z = 4 - x^2 - 2y^2$$

and the vertical plane $y = 1$ intersects it in the parabola $z = 2 - x^2, y = 1$. The slope of the tangent line to this parabola at the point $(1, 1, 1)$ is $f_x(1, 1) = -2$. Similarly the curve in which the plane $x = 1$ intersects the paraboloid is the parabola $z = 3 - 2y^2, x = 1$ and the slope of the tangent line at $(1, 1, 1)$ is $f_y(1, 1) = -4$.

Definition 3.1. If f is a function of the variables $x_1, \dots, x_n$, then the **partial derivative** of f with respect to x_k is the function f_{x_k} defined by

$$f_{x_k}(x_1, \dots, x_n) = \lim_{\Delta x_k \to 0} \frac{f\left(x_1, \dots, x_k + \Delta x_k, x_{k+1}, \dots, x_n\right) - f\left(x_1, \dots, x_n\right)}{\Delta x_k}$$

provided this limits exists.

Example 3.3. Find the partial derivatives f_x and f_y given

$$f(x, y) = \sin\left(\frac{x}{1 + y}\right).$$

Solution. Using the **chain rule** for functions of one variable, we have

$$f_x = \cos\left(\frac{x}{1+y}\right)\frac{\partial}{\partial x}\left(\frac{x}{1+y}\right) = \cos\left(\frac{x}{1+y}\right)\cdot\frac{1}{1+y}$$

and

$$f_y = \cos\left(\frac{x}{1+y}\right)\frac{\partial}{\partial y}\left(\frac{x}{1+y}\right) = -\sin\left(\frac{x}{1+y}\right)\cdot\frac{x}{(1+y)^2}.$$

Example 3.4. Find the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if z is defined **implicitly** as a function of x and y by the equation

$$x^3 + y^3 + z^3 + 6xyz = 1.$$

Solution. To find $\frac{\partial z}{\partial x}$ we differentiate implicitly with respect to x, being careful to treat y as a constant:

$$3x^2 + 3z^2\frac{\partial z}{\partial x} + 6yz + 6xy\frac{\partial z}{\partial x} = 0.$$

Solving this equation for $\frac{\partial z}{\partial x}$ we obtain

$$\frac{\partial z}{\partial x} = -\frac{x^2 + 2yz}{z^2 + 2xy}.$$

Similarly, **implicit differentiation** with respect to y gives

$$\frac{\partial z}{\partial y} = -\frac{y^2 + 2xz}{z^2 + 2xy}.$$

Example 3.5. Suppose the system

$$\begin{cases} xu + yv - uv &= 0 \\ yu - xv + uv &= 0 \end{cases}$$

can be solved for u and v in terms of x and y, so that $u = u(x, y)$ and $v = v(x, y)$. Use implicit differentiation to find the partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$.

Solution. We use implicit differentiation on $xu + yv - uv = 0$ to find $\frac{\partial u}{\partial x}$. We have,

$$u + x\frac{\partial u}{\partial x} + y\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}v - \frac{\partial v}{\partial x}u = 0$$

and so

$$(x-v)\frac{\partial u}{\partial x}+(y-u)\frac{\partial v}{\partial x}=-u$$

Also,

$$y\frac{\partial u}{\partial x}-x\frac{\partial v}{\partial x}-v+u\frac{\partial v}{\partial x}+v\frac{\partial u}{\partial x}=0$$

and so

$$(y+v)\frac{\partial u}{\partial x}+(-x+u)\frac{\partial v}{\partial x}=v$$

So we can solve the system

$$\begin{cases}(x-v)\frac{\partial u}{\partial x}+(y-u)\frac{\partial v}{\partial x} & =-u\\ (y+v)\frac{\partial u}{\partial x}+(-x+u)\frac{\partial v}{\partial x} & =v\end{cases}$$

to obtain

$$\frac{\partial u}{\partial x}=\frac{u^2-uv-ux+vy}{x^2+y^2-ux-uy-vx+vy}$$

$$\frac{\partial v}{\partial x}=\frac{v^2-uv-ux+vy}{x^2+y^2-ux-uy-vx+vy}$$

3.2 Second-Order Partial Derivatives

The partial derivative is a function, so it is possible to take the partial derivative of a partial derivative. This is very much like taking the second derivative of a function of one variable if we take two consecutive partial derivatives with respect to the same variable, and the resulting derivative is called the second-order partial derivative with respect to that variable. However, we can also take the partial derivative with respect to one variable and then take a second partial derivative with respect to a different variable, producing what is called a **second-order partial derivative**.

The higher-order partial derivatives for a function of two variables $f(x, y)$ are denoted as

$$\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=(f_x)_x=f_{xx}$$

and

$$\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=(f_y)_y=f_{yy}$$

and the mixed second partial derivatives are denoted as

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=(f_y)_x=f_{yx}$$

and

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = (f_x)_y = f_{xy}.$$

Example 3.6. Find the second order partial derivatives of

$$f(x,y) = x^3 + x^2y^3 - 2y^2.$$

Solution. The first partial derivatives are $f_x(x,y) = 3x^2 + 2xy^3$ and $f_y(x,y) = 3x^2y^2 - 4y$. Therefore the second derivatives are

$$\begin{aligned} f_{xx} &= \frac{\partial}{\partial x}\left(3x^2 + 2xy^3\right) = 6x + 2y^3 & f_{xy} &= \frac{\partial}{\partial y}\left(3x^2 + 2xy^3\right) = 6xy^2 \\ f_{yx} &= \frac{\partial}{\partial x}\left(3x^2y^2 - 4y\right) = 6xy^2 & f_{yy} &= \frac{\partial}{\partial y}\left(3x^2y^2 - 4y\right) = 6x^2y - 4 \end{aligned}$$

::: {#thm- } [Mixed Second-Order Partial Derivatives]

If the function $f(x,y)$ has mixed second-order partial derivatives f_{xy} and f_{yx}, that are continuous on an open disk containing (a,b), then

$$f_{yx}(a,b) = f_{xy}(a,b).$$

:::

Proof. For small values of h with $h \neq 0$, consider the difference

$$\Delta(h) = [f(a+h, b+h) - f(a+h, b)] - [f(a, b+h) - f(a,b)].$$

Notice that if we let

$$g(x) = f(x, b+h) - f(x,b),$$

then

$$\Delta(h) = g(a+h) - g(a).$$

By the Mean Value Theorem, there is a number c between a and $a+h$ such that

$$g(a+h) - g(a) = g'(c)h = h\left[f_x(c, b+h) - f_x(c,b)\right].$$

Applying the Mean Value Theorem **mean value theorem** again, this time to f_x we get a number d between b and $b+h$ such that

$$f_x(c, b+h) - f_x(c,b) = f_{xy}(c,d)h.$$

Combining these equations, we obtain $\Delta(h) = h^2 f_{xy}(c,d)$. If $h \to 0$, then $(c,d) \to (a,b)$, so the continuity of f_{xy} at (a,b) gives

$$\lim_{h\to 0} \frac{\Delta(h)}{h^2} = \lim_{(c,d)\to(a,b)} f_{xy}(c,d) = f_{xy}(a,b).$$

Similarly, by writing

$$\Delta(h) = [f(a+h,b+h) - f(a,b+h)] - [f(a+h,b) - f(a,b)]$$

and using the Mean Value Theorem twice and the continuity of f_{yx} at (a,b), we obtain

$$\lim_{h\to 0} \frac{\Delta(h)}{h^2} = f_{yx}(a,b).$$

It follows that $f_{xy}(a,b) = f_{yx}(a,b)$ as desired.

$\square$

Example 3.7. Use implicit differentiation to find $\frac{\partial^2 z}{\partial x \partial y}$ given $z^2 + \sin x = \tan y$.

Solution. We use implicit differentiation with respect to y first to find $\frac{\partial z}{\partial y}$ as follows,

$$2z\frac{\partial z}{\partial y} + 0 = \sec^2 y$$

(recall $\frac{d}{du}(\tan u) = \sec^2 u$) and so

$$\frac{\partial z}{\partial y} = \frac{\sec^2 y}{2z}$$

Now to find $\frac{\partial^2 z}{\partial x \partial y}$ we use implicit differentiation with respect to x but first we write

$$\frac{\partial z}{\partial y} = (\sec^2 y)\,(2z)^{-1}$$

Then,

$$\frac{\partial^2 z}{\partial x \partial y} = (\sec^2 y)\,(-2z)^{-2}\frac{\partial z}{\partial x} \tag{3.1}$$

so in order to find $\frac{\partial^2 z}{\partial x \partial y}$ we now need to find $\frac{\partial z}{\partial x}$. We find,

$$2z\frac{\partial z}{\partial x} + \cos x = 0$$

and solving for $\frac{\partial z}{\partial x}$ we find,

$$\frac{\partial z}{\partial x} = -\frac{\cos x}{2z}.$$

Now substituting back into (3.1), we find

$$\frac{\partial^2 z}{\partial x \partial y} = \left(\sec^2 y\right)\left(-2z^{-2}\right)\left(-\frac{\cos x}{2z}\right).$$

Therefore, after simplifying

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\cos x \sec^2 y}{4z^3}$$

Example 3.8. Use implicit differentiation to find $\frac{\partial^2 z}{\partial x \partial y}$ given

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 3.$$

Solution. Using implicit differentiation with respect to y we find $-y^{-2} - z^{-2} z_y = 0$
and solving for z_y yields

$$z_y = \frac{y^{-2}}{-z^{-2}} = -\frac{z^2}{y^2}.$$

Using implicit differentiation with respect to x we find,

$$z_{yx} = \frac{-1}{y^2} 2z \; z_x \tag{3.2}$$

and so we need to find z_x in order to finish with z_{yx}. So using implicit differentiation with respect to x we find, $-x^{-2} - z^{-2} z_x = 0$ and solving for z_x, yields

$$z_x = -\frac{z^2}{x^2}$$

which is easily seen from the symmetry of the given equation. Now then we substitute into (3.2) and find

$$z_{yx} = \frac{-1}{y^2} 2z \left(-\frac{z^2}{x^2}\right) = \frac{2z^3}{x^2 y^2}$$

as desired.

3.3 Verifying Partial Differential Equations

Example 3.9. Verify that the function $u(x,t) = \sin(x - at)$ is a solution of the **wave equation**

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}.$$

where a is a constant.

Solution. We find that $u_x = \cos(x - at)$,
$u_{xx} = -\sin(x - at)$, $u_t = -a\cos(x - at)$, and $u_{tt} = -a^2\sin(x - at) = a^2 u_{xx}$.
We verify as follows:

$$u_{tt} = -a^2 \sin(x - at) = a^2 u_{xx}.$$

Example 3.10. Verify that the function $u(x,y) = e^x \sin y$ is a solution of **Laplace's equation**

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Solution. We find that $u_x = e^x \sin y$, $u_{xx} = e^x \sin y$, $u_y = e^x \cos y$, and $u_{yy} = -e^x \sin y$. We verify as follows

$$u_{xx} + u_{yy} = e^x \sin y - e^x \sin y = 0.$$

Example 3.11. Verify that the functions

$$u(x,y) = \ln\left(x^2 + y^2\right) \quad \text{and} \quad v(x,y) = 2\tan^{-1}\left(\frac{y}{x}\right)$$

satisfy the **Cauchy-Riemann** equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Solution. We find that

$$\frac{\partial u}{\partial x} = \frac{2x}{x^2 + y^2} = \frac{\partial v}{\partial y}$$

and

$$\frac{\partial u}{\partial y} = \frac{2y}{x^2 + y^2} = -\left(-\frac{2y}{x^2 + y^2}\right) = -\frac{\partial v}{\partial y}.$$

Example 3.12. Verify that the function

$$u = 1 \Big/ \sqrt{x^2 + y^2 + z^2}$$

is a solution of the three-dimensional **Laplace equation** $u_{xx} + u_{yy} + u_{zz} = 0$.

Solution. We compute

$$\begin{aligned} u_x &= -\tfrac{x}{(x^2+y^2+z^2)^{3/2}} & \qquad u_{xx} &= \tfrac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}} \\ u_y &= -\tfrac{y}{(x^2+y^2+z^2)^{3/2}} & u_{yy} &= -\tfrac{x^2-2y^2+z^2}{(x^2+y^2+z^2)^{5/2}} \\ u_z &= -\tfrac{z}{(x^2+y^2+z^2)^{3/2}} & u_{zz} &= -\tfrac{x^2+y^2-2z^2}{(x^2+y^2+z^2)^{5/2}}. \end{aligned}$$

We verify as follows

$$\frac{2x^2 - y^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}} - \left[\frac{x^2 - 2y^2 + z^2}{(x^2 + y^2 + z^2)^{5/2}}\right] - \left[\frac{x^2 + y^2 - 2z^2}{(x^2 + y^2 + z^2)^{5/2}}\right] = 0.$$

Example 3.13. Show that the function $z = xe^y + ye^x$ is a solution of the partial differential equation

$$z_{xxx} + z_{yyy} = xz_{xyy} + yz_{xxy}.$$

Solution. The needed partial derivatives are

$$\begin{aligned} z_x &= e^y + e^x y & \qquad z_{yy} &= e^y x & \qquad z_{yyy} &= e^y x \\ z_y &= e^x + e^y x & z_{xy} &= e^x + e^y & z_{xyy} &= e^y \\ z_{xx} &= e^x y & z_{xxx} &= e^x y & z_{xxy} &= e^x. \end{aligned}$$

We verify as follows

$$z_{xxx} + z_{yyy} = e^x y + e^y x = x\left(e^y\right) + y\left(e^x\right) = xz_{xyy} + yz_{xxy}$$

as desired.

3.4 Exercises

Exercise 3.1. Find $f_{xz} + f_{yz}$ given $f(x, y, z) = 16e^{-(x^2+y^2+z^2)}$.

Exercise 3.2. Find $\frac{\partial x}{\partial y}$ when $x^2y^2 = 2z^2$.

Exercise 3.3. Let $f(x, y, z) = x \cos yz$. Find f_{xyz}.

Exercise 3.4. Find z_y if $y^2 z^2 + x \sin yz = 3$.

Exercise 3.5. Let $z = f(x, y)$ be a differentiable function where x and y are both differentiable functions of s, t, and v. Find z_v

Exercise 3.6. The function $z = f(x, y)$ is implicitly defined by the equation $xy^2 - x^2 e^z + yz^2 = 0$. What is $\frac{\partial z}{\partial x}$?

Exercise 3.7. Find the first and second order partial derivatives.

- $f(x, y) = (x^2 - 2xy + y)^5$
- $f(x, y) = \ln(\sin xy)$.
- $f(x, y) = (x + xy + y)^3$
- $f(x, y) = \ln(2x + 3y)$.
- $f(x, y) = (x + xy + 10y)^{-2} - xy^2$
- $f(x, y) = \ln(2x + 3y)$.
- $f(x, y) = (\sin \sqrt{x}) \ln y^2$
- $f(x, y) = \tan\left(\sqrt{x} \ln y^2\right)$
- $f(x, y) = x^2 e^{x+y} \cos y$.
- $f(x, y) = \cos^{-1}(xy)$.
- $f(x, y, z) = \frac{x+y^2}{z}$.
- $f(x, y, z) = \sin(xy + z)$.

Exercise 3.8. Determine the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ by differentiating implicitly.

- $3x^2 y + y^3 z - z^2 = 1$
- $\ln(xy + yz + xz) = 5$

Exercise 3.9. Compute the slope of the tangent line to the graph of the function $f(x, y) = x^2 \sin(x + y)$ at the point $\left(\frac{\pi}{2}, \frac{\pi}{2}, 0\right)$ in the direction of the xz-plane and of the yz-plane.

Exercise 3.10. Determine the partial derivatives f_x and f_y given

$$f(x, y) = \int_x^y \left(t^2 + 2t + 1\right) dt.$$

Exercise 3.11. Show that the function $f(x, y) = \ln(x^2 + y^2)$ is harmonic on the xy-plane with the point $(0, 0)$ removed.

Exercise 3.12. Show that the mixed partial derivatives are identical.

- $f(x, y) = \cos xy^2$
- $f(x, y) = \left(\sin^2 x\right)(\sin y)$
- Let f be the function defined by

  $$f(x, y, z) = x^2 + y^2 - 2xy\cos z.$$

  Determine $f_{xzy} - f_{yzz}$.

Exercise 3.13. Determine f_x, f_y, and f_z.

- $f(x, y, z) = \ln(x + y^2 + z^3)$
- $f(x, y, z) = \frac{xy+yz}{xz}$

Exercise 3.14. Assume $z = f(x, y)$, determine $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ by differentiating implicitly.

- $x^3 - xy^2 + yz^2 - z^3 = 0$
- $\sqrt{x} + y^2 + \sin xz = 2.$

Exercise 3.15. Show that $f_x(0, 0) = 0$ but $f_y(0, 0)$ does not exist, given the following function.

$$f(x, y) = \begin{cases} (x^2 + y)\sin\left(\frac{1}{x^2+y^2}\right) & \text{if}(x, y) \neq (0, 0) \\ 0 & \text{if}(x, y) = (0, 0). \end{cases}$$

Exercise 3.16. The partial differential equation

$$\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}.$$

is often called the *heat equation.* Determine whether the function satisfies the heat equation.

- $z = e^{-t}\left(\sin\frac{x}{c} + \cos\frac{x}{c}\right)$
- $z = \sin(3ct)\sin(3x)$
- $z = \sin(5ct)\cos(5x)$

- $z = \tan(5ct)\cot(5x)$

Exercise 3.17. The *Cauchy-Riemann* equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

where u and v are functions of x and y. Determine which pair of functions u and v satisfies the Cauchy-Riemann equations.

- $u = e^{-x}\cos y, v = e^{-x}\sin y$
- $u = x^2 + y^2, v = 2xy$

3.5 Differentials

Definition 3.2. Let $z = f(x, y)$ and let Δx and Δy be increments of x and y, respectively. The **differentials** dx and dy of the independent variables x and y are defined by $dx = \Delta x$ and $dy = \Delta y$. The **differential** dz, also called the **total differential** , is defined by

$$dz = f_x(x, y)dx + f_y(x, y)dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

Theorem 3.1. *If $f(x, y)$ and its partial derivatives f_x and f_y are defined on an open region R containing the point $P(x_0, y_0)$ and both f_x and f_y are continuous at P, then*

$$\Delta f = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) \approx f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$$

so that

$$f(x_0 + \Delta x, y_0 + \Delta y) \approx f(x_0, y_0) + f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y. \quad (3.3)$$

Example 3.14. If

$$z = f(x, y) = x^2 + 3xy - y^2,$$

find the differential dz. Further, if x changes from 2 to 2.05 and y changes from 3 to 2.96, compare the values of Δz and dz. Which is easier to compute Δz or dz?

Solution. By definition

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = (2x+3y)dx + (3x-2y)dy.$$

Putting $x = 2$, $dx = \Delta x = 0.05$, $y = 3$, and $dy = \Delta y = -0.04$, we get

$$dz = [2(2) + 3(3)]0.05 + [3(2) - 2(3)](-0.04) = 0.65.$$

The increment of z is

$$\begin{aligned}\Delta z &= f(2.05, 2.96) - f(2,3)\\ &= \left[(2.05)^2 + 3(2.05)(2.96) - (2.96)^2\right] - \left[2^2 + 3(2) - 3^2\right] = 0.6449.\end{aligned}$$

Notice that $\Delta z \approx dz$ but dz is easier to compute.

Example 3.15. Use differentials to approximate the real number

$$\sqrt{9(1.95)^2 + (8.1)^2}.$$

Solution. Consider the function

$$z = f(x,y) = \sqrt{9x^2 + y^2}$$

and observe that we can easily calculate $f(2,8) = 10$. Therefore, we take $a = 2, b = 8, dx = \Delta x = -0.05$, and $dy = \Delta y = 0.1$ in our linear approximation formula with the multivariate function $f(x,y)$, so

$$f(a + \Delta x, b + \Delta y) \approx f(a,b) + dz.$$

Next we compute the partial derivatives

$$f_x(x,y) = \frac{9x}{\sqrt{9x^2+y^2}} \qquad \text{and} \qquad f_y(x,y) = \frac{y}{\sqrt{9x^2+y^2}}$$

Now using 3.3

$$\begin{aligned}&\sqrt{9(1.95)^2 + (8.1)^2} = f(1.95, 8.1) \approx f(2,8) + dz\\ &= f(2,8) + f_x(2,8)dx + f_y(2,8)dy = 10 + \frac{18}{10}(-0.05) + \frac{8}{10}(0.1) = 9.99\end{aligned}$$

This approximation is accurate to two decimal places.

Example 3.16. Use differentials to approximate the real number

$$8.94\sqrt{9.99 - (1.01)^3}.$$

Solution. Consider the multivariate function $z = f(w, x, y) = w\sqrt{x - y^3}$ and observe that we can easily calculate

$$f(9, 10, 1) = 9\sqrt{10 - 1^3} = 27.$$

Therefore, we take $a = 9$, $b = 10$, $c = 1$, $dw = \Delta w = -0.06$, $dx = \Delta x = -0.01$ and $dy = \Delta y = 0.01$ in

$$f(a + \Delta x, b + \Delta y, c + \Delta z) \approx f(a, b, c) + dz.$$

We find the partial derivatives

$$f_w(w, x, y) = \sqrt{x - y^3}, \quad f_x(w, x, y) = \frac{w}{2\sqrt{x - y^3}}, \quad f_y(w, x, y) = -\frac{3wy^2}{2\sqrt{x - y^3}}$$

Now using 3.3

$$\begin{aligned} 8.94\sqrt{9.99 - (1.01)^3} &= f(8.94, 9.99, 1.01) \approx f(9, 10, 1) + dz \\ &= f(9, 10, 1) + f_w(9, 10, 1)dw + f_x(9, 10, 1)dx + f_y(9, 10, 1)dy \\ &= 27 + 3(-0.06) + \frac{3}{2}(-0.01) + -\frac{9}{2}(0.01) = 26.75\overline{9}. \end{aligned}$$

This approximation is accurate to two decimal places.

3.6 Differentiability

For a function of two variables $z = f(x, y)$, if x and y are given increments Δx and Δy , then the corresponding increment of z is

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y).$$

Definition 3.3. If $z = f(x, y)$, then f is **differentiable** at (a, b) provided Δz can be expressed in the form

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$. Additionally, $f(x, y)$ is said to be differentiable in the region R of the plane if f is differentiable at each point in R.

::: {#thm- } [Differentiability Implies Continuity] Let f be a function of two variables with (a, b) in the domain of f.

- If f is differentiable at (a, b) it is also continuous at (a, b).
- If f is a function of x and y, and f, f_x, f_y are continuous in a disk D centered at (a, b), then f is differentiable at (a, b).

:::

Proof. The proof is left to the reader. ☐

Example 3.17. Let f be the function defined by

$$f(x,y) = \begin{cases} 1 & \text{if } x > 0 \text{ and } y > 0 \\ 0 & \text{otherwise} \end{cases} \tag{3.4}$$

Show that the partial derivatives f_x and f_y exist at the origin, but f is not differentiable there.

Solution. Since $f(0,0) = 0$, we have

$$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(0+\Delta x, 0) - f(0,0)}{\Delta x} = 0$$

and similarly $f_y(0,0) = 0$. Thus, the partial derivatives both exist at the origin. If $f(x,y)$ were differentiable at the origin, it would have to be continuous there. Thus, we can show that f is not differentiable by showing that it is not continuous at $(0,0)$. Toward this end, note that

$$\lim_{(x,y)\to(0,0)} f(x,y) \tag{3.5}$$

is 1 along the line $y = x$ in the first quadrant but it is 0 if the approach is along the x-axis. This means that the limit in 3.5 does not exist. Thus $f(x,y)$ is not differentiable at $(0,0)$.

Therefore $f(x,y)$ is an example of a **non-differentiable** function for which f_x and f_y exist, or in other words, the word differentiable means more than just the partial derivatives exist because the existence of partial derivatives does not guarantees that a function is differentiable.

Example 3.18. Show that $f(x,y) = x^2y + xy^3$ is differentiable for all (x,y).

Solution. We find the partial derivatives

$$f_x(x,y) = \frac{\partial}{\partial x}\left(x^2y + xy^3\right) = 2xy + y^3$$

and

$$f_y(x,y) = \frac{\partial}{\partial y}\left(x^2y + xy^3\right) = x^2 + 3xy^2.$$

Because f, f_x and f_y are all polynomials in x and y they are continuous throughout the plane. Therefore by 3.6, f must be differentiable for all x and y.

Example 3.19. Determine whether the function

$$f(x,y)=\frac{x^2y}{x^4+y^2}$$

is differentiable at either $(0,0)$ or $(1,1)$. Explain why.

Solution. Since $(0,0)$ is not in the domain of f, f is not continuous at $(0,0)$, and thus not differentiable at $(0,0)$ by 3.6. To show that f is differentiable at $(1,1)$ we compute the partial derivatives

$$f_x=\frac{-2x^5y+2xy^3}{\left(x^4+y^2\right)^2}\quad\text{and}\quad f_y=\frac{x^6-x^2y^2}{\left(x^4+y^2\right)^2}.$$

Since these partial derivatives and f are continuous on any open disk not containing $(0,0)$ we conclude that f is differentiable at any point except $(0,0)$; and in particular at $(1,1)$.

Example 3.20. Determine whether the function defined by

$$f(x,y)=\begin{cases}\frac{2(x-1)(y-1)}{(x-1)^2+(y-1)^2} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$$

is differentiable at either $(0,0)$ or $(1,1)$. Explain why.

Solution. The limit,

$$\lim_{(x,y)\to(1,1)}\frac{2(x-1)(y-1)}{(x-1)^2+(y-1)^2}$$

does not exist because along $y=1$

$$\lim_{\substack{(x,y)\to(1,1)\\ y=1}}\frac{2(x-1)(y-1)}{(x-1)^2+(y-1)^2}=\lim_{x\to1}\frac{0}{(x-1)^2}=0$$

and along $y=x$

$$\lim_{\substack{(x,y)\to(1,1)\\ y=x}}\frac{2(x-1)(y-1)}{(x-1)^2+(y-1)^2}=\lim_{x\to1}\frac{2(x-1)^2}{(x-1)^2+(x-1)^2}=1.$$

Therefore f is not continuous at $(1,1)$. By 3.6, f is not differentiable at $(1,1)$. To show that f is differentiable at $(0,0)$ we compute the partial derivatives,

$$f_x(x,y)=\frac{[(x-1)^2+(y-1)^2][2(y-1)]-[2(x-1)(y-1)][2(x-1)]}{(x-1)^2+(y-1)^2}\quad\text{for } (x,y)\neq(0,0)$$

and

$$f_y(x,y) = \frac{[(x-1)^2 + (y-1)^2][2(x-1)] - [2(x-1)(y-1)][2(y-1)]}{(x-1)^2 + (y-1)^2} \quad \text{for } (x,y)$$

Since f, f_x and f_y are rational functions they are continuous on any open disk not containing $(1,1)$. We conclude that f is differentiable at any point except $(1,1)$; and in particular at $(0,0)$.

3.7 Exercises

Exercise 3.18. Find the maximum rate of change of the function at the given point and the direction in which it occurs $f(x,y) = \sqrt{x^2 + 2y}$, $(4,10)$

Exercise 3.19. Show that if x and y are sufficiently close to zero and f is differentiable at $(0,0)$, then

$$f(x,y) \approx f(0,0) + xf_x(0,0) + yf_y(0,0)$$

Use this approximation for the expressions

$$\frac{1}{1+x-y} \qquad \text{and} \qquad \frac{1}{(x+1)^2 + (y+1)^2}$$

around $(0,0)$.

- Find the total differential of the function $z = x^2 - 2xy + 3y^2$.

Exercise 3.20. Find the tangent plane to the surface $z = x\sin y - x\cos x$ at $(\pi, 0)$.

Exercise 3.21. Find an equation for each horizontal tangent plane to the surface $z = 5 - x^2 - y^2 + 4y$.

Exercise 3.22. Determine the total differential.

- $z = 5x^2y^3$
- $z = \cos x^2 y$
- $z = ye^x$
- $w = \sin x + \sin y + \cos z$

- $w = z^2 \sin(2x - 3y)$
- $w = 3y^2 z \cos x$

Exercise 3.23. Show that each functions is differentiable on $\mathbb{R}^2$.

- $f(x, y) = xy^3 + 3xy^2$
- $f(x, y) = \sin\left(x^2 + 3y\right)$
- $f(x, y) = x^2 + 4x - y^2$
- $f(x, y) = e^{2x+y^2}$

Exercise 3.24. Determine the standard-form equation for the tangent plane to the surface at the specified point.

- $f(x, y) = \sqrt{x^2 + y^2}$ at $P_0\left(3, 1, \sqrt{10}\right)$
- $f(x, y) = x^2 + y^2 + \sin xy$ at $P_0(0, 2, 4)$
- $f(x, y) = e^{-x} \sin y$ at $P_0\left(0, \frac{\pi}{2}, 1\right)$
- $z = 10 - x^2 - y^2$ at $P(2, 2, 2)$
- $z = \ln\left|x + y^2\right|$ at $P(-3, -2, 0)$

Exercise 3.25. Use linear approximation (differentials) to find approximate value.

- $\left(\sqrt{\frac{\pi}{2}} - 0.01\right)$
- $\sin\left(\sqrt{\frac{\pi}{2}} + 0.01\right)$
- $e^{1.01^2 0.98^2}$

Exercise 3.26. Show that if x and y are sufficiently close to zero and f is differentiable at $(0, 0)$, then

$$f(x, y) \approx f(0, 0) + x f_x(0, 0) + y f_y(0, 0)$$

Use this approximation for the expressions

$$\frac{1}{1 + x - y} \qquad \text{and} \qquad \frac{1}{(x + 1)^2 + (y + 1)^2}$$

around $(0, 0)$.

Exercise 3.27. Find a unit vector that is normal to the given graph at the point $P_0\left(x_0, y_0\right)$ on the graph.

- the circle $x^2 + y^2 = a^2$
- the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Exercise 3.28. Find a unit vector that is normal to each surface given at the prescribed point, and the standard form of the equation of the tangent plane at that point.

- $\ln\left(\frac{x}{y-z}\right) = 0$ at $(2, 5, 3)$
- $ze^{x^2-y^2} = 3$ at $(1, 1, 3)$

3.8 The Chain Rule

Recall that the chain rule for functions of a single variable gives the rule for differentiating a composite function: if $y = f(x)$ and $x = g(t)$, where f and g are differentiable functions, then y is a a differentiable function of t and

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}.$$

3.9 Chain Rule Involving One Independent Variable

There are several versions of the chain rule for functions of more than one variable, each of them giving a rule for differentiating a composite function.

::: {#lem- } Chain Rule Involving One Independent Variable Let $f(x, y)$ be a differentiable function of x and y, and let $x = x(t)$ and $y = y(t)$ be differentiable functions of t. Then $z = f(x, y)$ is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}. \tag{3.6}$$

:::

Proof. Because $z = f(x, y)$ is differentiable, we can write the increment Δz in the following form:

$$\Delta z = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as both $\Delta x \to 0$ and $\Delta y \to 0$. Dividing by $\Delta t \neq 0$, we obtain

$$\frac{\Delta z}{\Delta t} = \frac{\partial z}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial z}{\partial y}\frac{\Delta y}{\Delta t} + \epsilon_1 \frac{\Delta x}{\Delta t} + \epsilon_2 \frac{\Delta y}{\Delta t}.$$

Because x and y are function of t, we can write their increments as

$$\Delta x = x(t+\Delta t) - x(t) \qquad \text{and} \qquad \Delta y = y(t+\Delta t) - y(t).$$

We know that x and y vary continuously with t, because x and y are differentiable, and it follows that $\Delta x \to 0$ and $\Delta y \to 0$ as $\Delta t \to 0$ so that $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as $\Delta t \to 0$. Therefore, we have

$$\begin{aligned}
\frac{dz}{dt} &= \lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t} \\
&= \lim_{\Delta t \to 0} \left(\frac{\partial z}{\partial x} \frac{\Delta x}{\Delta t} + \frac{\partial z}{\partial y} \frac{\Delta y}{\Delta t} + \epsilon_1 \frac{\Delta x}{\Delta t} + \epsilon_2 \frac{\Delta y}{\Delta t} \right) \\
&= \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} + (0) \frac{\Delta x}{\Delta t} + (0) \frac{\Delta y}{\Delta t}
\end{aligned}$$

as desired.

$\square$

Example 3.21. If $z = x^2 y + 3xy^4$, where $x = e^t$ and $y = \sin t$, find $\frac{dz}{dt}$. The chain rule gives,

$$\begin{aligned}
\frac{dz}{dt} &= \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \\
&= \left(2e^t \sin t + 3\mathrm{sint}^4 t\right) e^t + \left(e^{2t} + 12e^t \sin^3 t\right) \cos t.
\end{aligned}$$

Example 3.22. Two objects are traveling in elliptical paths given by the following parametric equations

$$x_1(t) = 2\cos t, \quad y_1(t) = 3\sin t \quad x_2(t) = 4\sin 2t, \quad y_2(t) = 3\cos 2t.$$

At what rate is the distance between the two objects changing when $t = \pi$?

Solution. The distance s between the two objects is given by

$$s = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

and that when $t = \pi$, we have $x_1 = -2$, $y_1 = 0$, $x_2 = 0$, and $y_2 = 3$. So

$$s = \sqrt{(0+2)^2 + (3-0)^2} = \sqrt{13}.$$

When $t=\pi$, the partial derivatives of s are as follows.

$$
\begin{aligned}
\left.\frac{\partial s}{\partial x_1}\right|_{t=\pi} &= \left.\frac{-(x_2-x_1)}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}\right|_{t=\pi} = \frac{-2}{\sqrt{13}} \\
\left.\frac{\partial s}{\partial y_1}\right|_{t=\pi} &= \left.\frac{-(y_2-y_1)}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}\right|_{t=\pi} = \frac{-3}{\sqrt{13}} \\
\left.\frac{\partial s}{\partial x_2}\right|_{t=\pi} &= \left.\frac{(x_2-x_1)}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}\right|_{t=\pi} = \frac{2}{\sqrt{13}} \\
\left.\frac{\partial s}{\partial y_2}\right|_{t=\pi} &= \left.\frac{(y_2-y_1)}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}\right|_{t=\pi} = \frac{3}{\sqrt{13}}
\end{aligned}
$$

When $t=\pi$, the derivatives of x_1, y_1, x_2, and y_2 are

$$
\begin{aligned}
\left.\frac{dx_1}{dt}\right|_{t=\pi} &= -2\sin t|_{t=\pi} = 0 & \qquad \left.\frac{dy_1}{dt}\right|_{t=\pi} &= 3\cos t|_{t=\pi} = -3 \\
\left.\frac{dx_2}{dt}\right|_{t=\pi} &= 8\cos 2t|_{t=\pi} = 8 & \qquad \left.\frac{dy_2}{dt}\right|_{t=\pi} &= -6\sin 2t|_{t=\pi} = 0
\end{aligned}
$$

So using the chain rule

$$\frac{ds}{dt} = \frac{\partial s}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial s}{\partial y_1}\frac{dy_1}{dt} + \frac{\partial s}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial s}{\partial y_2}\frac{dy_2}{dt}$$

When $t=\pi$, we find that the distance is changing at a rate of

$$\left.\frac{ds}{dt}\right|_{t=\pi} = \left(\frac{-2}{\sqrt{13}}\right)(0) + \left(\frac{-3}{\sqrt{13}}\right)(-3) + \left(\frac{2}{\sqrt{13}}\right)(8) + \left(\frac{3}{\sqrt{13}}\right)(0) = \frac{25}{\sqrt{13}}.$$

3.10 Chain Rule Involving Two Independent Variables

Next we work through an example which illustrates how to find partial derivatives of two variable functions whose variables are also two variable functions. The proof of this chain rule is motivated by appealing to a previously proven chain rule with one independent variable.

::: {#lem- } Chain Rule Involving Two Independent Variables Suppose $z=f(x,y)$ is a differentiable function at (x,y) and that the partial derivatives of $x=x(u,v)$ and $y=y(u,v)$ exist at (u,v). Then the composite function $z=f(x(u,v),y(u,v))$ is differentiable at (u,v) with

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \qquad \text{and} \qquad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}.$$

:::

Example 3.23. If $z = e^x \sin y$ where $x = st^2$ and $y = s^2 t$, find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

Solution. Applying the chain rule we obtain

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \\ &= \left(e^x \sin y\right)\left(t^2\right) + \left(e^x \cos y\right)(st) = t^2 e^{st^2} \sin\left(s^2 t\right) + 2ste^{st^2} \cos\left(s^2 t\right)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \\ &= \left(e^x \sin y\right)(2st) + \left(e^x \cos y\right)\left(2s^2\right) = 2ste^{st^2} \sin\left(s^2 t\right) + s^2 e^{st^2} \cos\left(s^2 t\right).\end{aligned}$$

Example 3.24. The **Cauchy-Riemann** equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

where $u = u(x, y)$ and $v = v(x, y)$. Show that if x and y are expressed in terms of polar coordinates, the Cauchy-Riemann equations become

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \text{and} \qquad \frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}.$$

Solution. Using $x = r\cos\theta$ and $y = r\sin\theta$ we can state the chain rule to be used:

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \qquad \text{and} \qquad \frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta}.$$

By the chain rule

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta \qquad \text{and} \qquad \frac{\partial v}{\partial \theta} = -\frac{\partial v}{\partial x}(r\sin\theta) + \frac{\partial v}{\partial y}(r\cos\theta).$$

Substituting,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

we obtain

$$\frac{\partial u}{\partial r} = \frac{\partial v}{\partial y}\cos\theta - \frac{\partial v}{\partial x}\sin\theta$$

and so

$$\frac{\partial u}{\partial r} = \frac{1}{r}\left[\frac{\partial v}{\partial y}(r\cos\theta) - \frac{\partial v}{\partial x}(r\sin\theta)\right] = \frac{1}{r}\frac{\partial v}{\partial \theta}.$$

Similarly the chain rule is to be used

$$\frac{\partial v}{\partial r}=\frac{\partial v}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial r} \qquad \text{and} \qquad \frac{\partial u}{\partial \theta}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}.$$

By the chain rule

$$\frac{\partial v}{\partial r}=\frac{\partial v}{\partial x}\cos\theta+\frac{\partial v}{\partial y}\sin\theta$$

and

$$\frac{\partial u}{\partial \theta}=-\frac{\partial u}{\partial x}(r\sin\theta)+\frac{\partial u}{\partial y}(r\cos\theta).$$

Substituting

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x},$$

we obtain

$$\frac{\partial v}{\partial r}=-\frac{\partial u}{\partial y}\cos\theta-\frac{\partial u}{\partial x}\sin\theta$$

and also

$$\frac{\partial u}{\partial r}=-\frac{1}{r}\left[\frac{\partial u}{\partial y}(r\cos\theta)-\frac{\partial u}{\partial x}(r\sin\theta)\right]=-\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

3.11 Chain Rule Involving Several Independent Variable

::: {#thm- } Chain Rule Involving Several Independent Variable If $w=f(x_1,\dots,x_n)$ is a differentiable function of the n variables $x_1,\dots,x_n$ which in turn are differentiable functions of m parameters $t_1,\dots,t_m$ then the composite function is differentiable and

$$\frac{\partial w}{\partial t_1}=\sum_{k=1}^{n}\frac{\partial w}{\partial x_k}\frac{\partial x_k}{\partial t_1}, \quad \dots \quad, \frac{\partial w}{\partial t_m}=\sum_{k=1}^{n}\frac{\partial w}{\partial x_k}\frac{\partial x_k}{\partial t_m}.$$

:::

Example 3.25. Write out the chain rule for the case for the case when $n=4$ and $m=2$ where $w=f(x,y,z,t)$, $x=x(u,v)$, $y=y(u,v)$, $z=z(u,v)$, and $t(u,v)$.

Solution. The chain rule for the case when $n=4$ and $m=2$ yields the following the partial derivatives:

$$\frac{\partial w}{\partial u}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial u}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial u}+\frac{\partial w}{\partial t}\frac{\partial t}{\partial u}$$

and

$$\frac{\partial w}{\partial v}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial v}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial v}+\frac{\partial w}{\partial t}\frac{\partial t}{\partial v}.$$

Example 3.26. If $u = x^4y + y^2z^3$ where $x = rse^t$, $y = rs^2e^{-t}$, and $z = r^2s\sin t$, find the value of $\frac{\partial u}{\partial s}$ when $r = 2$, $s = 1$, and $t = 0$.

Solution. By the chain rule,

$$\begin{aligned}\frac{\partial u}{\partial s} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s} \\ &= \left(4x^3y\right)\left(re^t\right) + \left(x^4 + 2yz^3\right)\left(2rse^{-t}\right) + \left(3y^2z^2\right)\left(r^2\sin t\right).\end{aligned}$$

When $r = 2$, $s = 1$, and $t = 0$, we have $x = 2$, $y = 2$, and $z = 0$, so

$$\frac{\partial u}{\partial s} = (64)(2) + (16)(4) + (0)(0) = 192.$$

Example 3.27. If $F(u, v, w)$ is differentiable where $u = x - y$, $v = y - z$, and $w = z - x$, then find

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}.$$

Solution. We compute,

$$\begin{aligned}\frac{\partial F}{\partial x} &= \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial F}{\partial w}\frac{\partial w}{\partial x} \\ &= \frac{\partial F}{\partial u}(1) + \frac{\partial F}{\partial v}(0) + \frac{\partial F}{\partial w}(-1) \\ &= \frac{\partial F}{\partial u} - \frac{\partial F}{\partial w}.\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{\partial F}{\partial y} &= \frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial F}{\partial w}\frac{\partial w}{\partial y} \\ &= \frac{\partial F}{\partial u}(-1) + \frac{\partial F}{\partial v}(1) + \frac{\partial F}{\partial w}(0) \\ &= -\frac{\partial F}{\partial u} + \frac{\partial F}{\partial v}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial F}{\partial z} &= \frac{\partial F}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial F}{\partial w}\frac{\partial w}{\partial z} \\ &= \frac{\partial F}{\partial u}(0) + \frac{\partial F}{\partial v}(-1) + \frac{\partial F}{\partial w}(1) \\ &= -\frac{\partial F}{\partial v} + \frac{\partial F}{\partial v}\end{aligned}$$

Therefore the required expression is

$$\left[\frac{\partial F}{\partial u}-\frac{\partial F}{\partial w}\right]+\left[-\frac{\partial F}{\partial u}+\frac{\partial F}{\partial v}\right]+\left[-\frac{\partial F}{\partial v}+\frac{\partial F}{\partial v}\right]=0.$$

Example 3.28. If f is differentiable and $z=u+f\left(u^2v^2\right)$, show that

$$u\frac{\partial z}{\partial u}-v\frac{\partial z}{\partial v}=u.$$

Solution. Let $w=u^2v^2$, so $z=u+f(w)$. Then according to the chain rule,

$$\frac{\partial z}{\partial u}=1+\frac{df}{dw}\frac{\partial w}{\partial u}=1+f'(w)\left(2uv^2\right)$$

and

$$\frac{\partial z}{\partial v}=1+\frac{df}{dw}\frac{\partial w}{\partial v}=f'(w)\left(2u^2v\right)$$

so that

$$\begin{aligned}u\frac{\partial z}{\partial u}-v\frac{\partial z}{\partial v}&=u\left[1+f'(w)\left(2uv^2\right)\right]-v\left[f'(w)\left(2u^2v\right)\right]\\&=u+f'(w)\left[u\left(2uv^2\right)-v\left(2u^2v\right)\right]=u.\end{aligned}$$

Example 3.29. Find $\frac{\partial w}{\partial s}$ if $w=4x+y^2+z^3$, where $x=e^{rs^2}$, $y=\ln\left(\frac{r+s}{t}\right)$, and $z=rst^2$.

Solution. We have

$$\begin{aligned}\frac{\partial w}{\partial s}&=\frac{\partial w}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial s}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial s}\\&=\left[\frac{\partial}{\partial x}\left(4x+y^2+z^3\right)\right]\left[\frac{\partial}{\partial s}\left(e^{rs^2}\right)\right]\\&\qquad+\left[\frac{\partial}{\partial y}\left(4x+y^2+z^3\right)\right]\left[\frac{\partial}{\partial s}\left(\ln\frac{r+s}{t}\right)\right]\\&\qquad\qquad+\left[\frac{\partial}{\partial z}\left(4x+y^2+z^3\right)\right]\left[\frac{\partial}{\partial s}\left(rst^2\right)\right]\\&=4\left[e^{rs^2}(2rs)\right]+2y\left(\frac{1}{\frac{r+s}{t}}\right)\left(\frac{1}{t}\right)+3z^2\left(rt^2\right)\\&=8rse^{rs^2}+2\frac{y}{r+s}+3rt^2z^2.\end{aligned}$$

Example 3.30. If $u = f(x, y)$, where $x = e^s \cos t$ and $y = e^s \sin t$, show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{-2s}\left[\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2}\right].$$

Solution. By the chain rule we have

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} = \frac{\partial u}{\partial x} e^s \cos t + \frac{\partial u}{\partial y} e^s \sin t$$

and

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial u}{\partial x}\left(-e^s \sin t\right) + \frac{\partial u}{\partial y} e^s \cos t.$$

Therefore

$$\frac{\partial^2 u}{\partial s^2} = \frac{\partial u}{\partial x} e^s \cos t + \frac{\partial}{\partial s}\left(\frac{\partial u}{\partial x}\right) e^s \cos t + \frac{\partial u}{\partial y} e^s \sin t + \frac{\partial}{\partial s}\left(\frac{\partial u}{\partial y}\right) e^s \sin t$$

and

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial u}{\partial x}\left(-e^s \cos t\right) + \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)\left(-e^s \sin t\right) + \frac{\partial u}{\partial y}\left(-e^s \sin t\right) + \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right) e^s \cos t.$$

Also

$$\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial s}\right) &= \frac{\partial^2 u}{\partial x^2} e^s \cos t + \frac{\partial^2 u}{\partial x \partial y}\left(e^s \sin t\right), \\
\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial s}\right) &= \frac{\partial^2 u}{\partial x \partial y}\left(e^s \cos t\right) + \frac{\partial^2 u}{\partial y^2} e^s \sin t, \\
\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) &= \frac{\partial^2 u}{\partial x^2}\left(-e^s \sin t\right) + \frac{\partial^2 u}{\partial x \partial y} e^s \cos t, \\
\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t}\right) &= \frac{\partial^2 u}{\partial x \partial y}\left(-e^s \sin t\right) + \frac{\partial^2 u}{\partial y^2} e^s \cos t.
\end{aligned}$$

Finally

$$
\begin{aligned}
&e^{-2s}\left[\frac{\partial^2 u}{\partial s^2}+\frac{\partial^2 u}{\partial t^2}\right] \\
&= e^{-2s}\left[\frac{\partial u}{\partial x}e^s\cos t+\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial x}\right)e^s\cos t+\frac{\partial u}{\partial y}e^s\sin t\right. \\
&\qquad +\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial y}\right)e^s\sin t+\frac{\partial u}{\partial x}(-e^s\cos t)+\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)(-e^s\sin t) \\
&\qquad \left.+\frac{\partial u}{\partial y}(-e^s\sin t)+\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right)e^s\cos t\right] \\
&= e^{-2s}\left[\frac{\partial u}{\partial x}e^s\cos t+\left[\frac{\partial^2 u}{\partial x^2}e^s\cos t+\frac{\partial^2 u}{\partial x\partial y}(e^s\sin t)\right]e^s\cos t\right. \\
&\qquad +\frac{\partial u}{\partial y}e^s\sin t+\left[\frac{\partial^2 u}{\partial x\partial y}(e^s\cos t)+\frac{\partial^2 u}{\partial y^2}e^s\sin t\right]e^s\sin t \\
&\qquad +\frac{\partial u}{\partial x}(-e^s\cos t)+\left[\frac{\partial^2 u}{\partial x^2}(-e^s\sin t)+\frac{\partial^2 u}{\partial x\partial y}e^s\cos t\right](-e^s\sin t) \\
&\qquad \left.+\frac{\partial u}{\partial y}(-e^s\sin t)+\left[\frac{\partial^2 u}{\partial x\partial y}(-e^s\sin t)+\frac{\partial^2 u}{\partial y^2}e^s\cos t\right]e^s\cos t\right] \\
&= e^{-2s}\left[\frac{\partial u}{\partial x}e^s\cos t+\frac{\partial^2 u}{\partial x^2}e^{2s}\cos^2 t+\frac{\partial^2 u}{\partial x\partial y}\left(e^{2s}\cos t\sin t\right)\right. \\
&\qquad +\frac{\partial u}{\partial y}e^s\sin t+\frac{\partial^2 u}{\partial x\partial y}\left(e^{2s}\sin t\cos t\right)+\frac{\partial^2 u}{\partial y^2}e^{2s}\sin^2 t+\frac{\partial u}{\partial x}(-e^s\cos t) \\
&\qquad +\frac{\partial^2 u}{\partial x^2}\left(e^{2s}\sin^2 t\right)+\frac{\partial^2 u}{\partial x\partial y}\left(-e^{2s}\cos t\sin t\right) \\
&\qquad \left.+\frac{\partial u}{\partial y}(-e^s\sin t)+\frac{\partial^2 u}{\partial x\partial y}\left(-e^{2s}\cos t\sin t\right)+\frac{\partial^2 u}{\partial y^2}e^{2s}\cos^2 t\right] \\
&= e^{-2s}\left[\frac{\partial^2 u}{\partial x^2}e^{2s}\cos^2 t+\frac{\partial^2 u}{\partial y^2}e^{2s}\sin^2 t+\frac{\partial^2 u}{\partial x^2}\left(e^{2s}\sin^2 t\right)+\frac{\partial^2 u}{\partial y^2}e^{2s}\cos^2 t\right] \\
&= \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}.
\end{aligned}
$$

3.12 Exercises

Exercise 3.29. Let $w=\ln(x+y)$, $x=uv$, $y=\frac{u}{v}$. What is $\frac{\partial w}{\partial v}$?

Exercise 3.30. Write out the chain rule for the function $t=f(u,v)$ where $u=u(x,y,z,w)$ and $v=v(x,y,z,w)$.

Exercise 3.31. Write out the chain rule for the function $w = f(x, y, z)$ where $x = x(s, t, u)$, $y = y(s, t, u)$, and $z = z(s, t, u)$.

Exercise 3.32. Use the chain rule to find $\frac{dw}{dt}$. Leave your answer in mixed form (x, y, z, t).

- $w = \ln\left(x + 2y - z^2\right)$, $x = 2t - 1$, $y = \frac{1}{t}$, and $z = \sqrt{t}$.
- $w = \sin xyz$, $x = 1 - 3t$, $y = e^{1-t}$, and $z = 4t$.
- $w = ze^{xy^2}$, $x = \sin t$, $y = \cos t$, and $z = \tan 2t$.
- $w = e^{x^3+yz}$, $x = \frac{2}{t}$, $y = \ln(2t - 3)$, and $z = t^2$.
- $w = \frac{x+y}{2-z}$, $x = 2rs$, $y = \sin rt$, and $z = st^2$.

Exercise 3.33. Find dy/dx, assuming each of the following the equations defines y as a differentiable function of x.

- $\left(x^2 - y\right)^{3/2} + x^2 y = 2$
- $\tan^{-1}\left(\frac{x}{y}\right) = \tan^{-1}\left(\frac{y}{x}\right)$

Exercise 3.34. Find the following higher order partial derivatives.

- $\frac{\partial^2 z}{\partial x \partial y}$
- $\frac{\partial^2 z}{\partial x^2}$
- $\frac{\partial^2 z}{\partial y^2}$
- $\ln(x + y) = y^2 + z$.
- $x^{-1} + y^{-1} + z^{-1} = 3$.

- $z^2 + \sin x = \tan y$
- $x^2 + \sin z = \cot y$

Exercise 3.35. Use the chain rule for one parameter to find the first order partial derivatives.

- $f(x, y) = \left(1 + x^2 + y^2\right)^{1/2}$ where $x(t) = \cos 5t$ and $y(t) = \sin 5t$
- $g(x, y) = xy^2$ where $x(t) = \cos 3t$ and $y(t) = \tan 3t$.

Exercise 3.36. Use the chain rule for two parameters with each of the following.

- $F(x, y) = x^2 + y^2$ where $x(u, v) = u \sin v$ and $y(u, v) = u - 2v$
- $F(x, y) = \ln xy$ where $x(u, v) = e^{uv^2}$ and $y(u, v) = e^{uv}$.

Exercise 3.37. Let (x, y, z) lie on the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

without solving for z, find $\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial x \partial y}$.

Exercise 3.38. If $w = f\left(\frac{r-s}{s}\right)$, show that

$$r\frac{\partial w}{\partial r} + s\frac{\partial w}{\partial s} = 0.$$

Exercise 3.39. If $z = xy + f\left(x^2 + y^2\right)$, show that

$$y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = y^2 - x^2.$$

Exercise 3.40. Let $w = f(t)$ be a differentiable function of t, where $t = \left(x^2 + y^2 + z^2\right)^{1/2}$. Show that

$$\left(\frac{dw}{dt}\right)^2 = \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2.$$

Exercise 3.41. If $z = f(x, y)$, where $x = r\cos\theta$, $y = r\sin\theta$, show that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r}\frac{\partial z}{\partial r}.$$

Chapter 4

Applications of Derivatives

4.1 Definition of Directional Derivative

Partial derivatives find the rate of change of $z = f(x, y)$ in the directions of the x and y axis; that is in the direction of the unit vectors $\vec{i}$ and $\vec{j}$, respectively.

Definition 4.1. Let f be a function of two variables, and let $\vec{u} = u_1\vec{i}+u_2\vec{j}$ be a unit vector. The **directional derivative** of f at $P(a, b)$ in the direction of $\vec{u}$ is defined by

$$D_{\vec{u}}f(a, b) = \lim_{h\to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h}$$

provided the limit exists.

::: {#thm- } [Directional Deriviative]

Let $f(x, y)$ be a function that is differentiable at $P(a, b)$. Then f has a directional derivative in the direction of the unit vector $\vec{u} = u_1\vec{i} + u_2\vec{j}$ which is given by

$$D_{\vec{u}}f(a, b) = f_x(a, b)u_1 + f_y(a, b)u_2. \tag{4.1}$$

:::

::: {.proof } We define a function F of a single variable h by $F(h) = f(a + hu_1, b + hu_2)$ so that

$$D_{\vec{u}}f(a, b) = \lim_{h\to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h} = \lim_{h\to 0} \frac{F(h) - F(0)}{h} = F'(0).$$

Applying the chain rule with $x = a + hu_1$ and $y = b + hu_2$

$$F'(h) = \frac{dF}{dh} = \frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh} = f_x(x,y)u_1 + f_y(x,y)u_2.$$

When $h = 0$, we have $x = a$ and $y = b$ so that

$$D_{\vec{u}}f(a,b) = F'(0) = \frac{\partial f}{\partial x}u_1 + \frac{\partial f}{\partial y}u_2 = f_x(a,b)u_1 + f_y(a,b)u_2.$$

:::

Example 4.1. Find the directional derivative $D_{\vec{u}}f(x,y)$ where f is the function defined by

$$f(x,y) = x^3 - 3xy + 4y^2$$

and $\vec{u}$ is the unit vector given by the angle $\theta = \pi/6$. What is $D_{\vec{u}}f(1,2)$?

Solution. We let $\vec{u} = \cos\left(\frac{\pi}{6}\right)\vec{\imath} + \sin\left(\frac{\pi}{6}\right)\vec{\jmath}$ and use 4.1 to find

$$\begin{aligned} D_{\vec{u}}f(x,y) &= f_x(x,y)\cos\frac{\pi}{6} + f_y(x,y)\sin\frac{\pi}{6} \\ &= \left(3x^2 - 3y\right)\frac{\sqrt{3}}{2} + (-3x + 8y)\frac{1}{2} = \frac{1}{2}\left[3\sqrt{3}x^2 - 3x + \left(8 - 3\sqrt{3}\right)y\right]. \end{aligned}$$

Therefore, by 4.1, the directional derivative is

$$D_{\vec{u}}f(1,2) = \frac{1}{2}\left[3\sqrt{3}(1)^2 - 3(1) + 8 - 3\sqrt{3}\right](2) = \frac{13 - 3\sqrt{3}}{2}.$$

Example 4.2. Find the directional derivative of the function f defined by

$$f(x,y) = y^2 + 3yx^2$$

at $P = (-1,-2)$ in the direction towards the origin.

Solution. A vector in the direction from $(-1,-2)$ to $(0,0)$ is $\vec{\imath} + 2\vec{\jmath}$, so a unit vector in this direction is therefore

$$\vec{u} = \frac{1}{\sqrt{5}}\vec{\imath} + \frac{2}{\sqrt{5}}\vec{\jmath}.$$

We find that

$$f_y(-1,-2) = \left(2y + 3x^2\right)\Big|_{(-1,-2)} = -1 \qquad \text{and} \qquad f_x(-1,-2) = 6xy\big|_{(-1,-2)} = 12.$$

Therefore, by 4.1, the directional derivative is

$$D_{\vec{u}}f(-1,-2) = \frac{1}{\sqrt{5}}(12) + \frac{2}{\sqrt{5}}(-1) = 2\sqrt{5}.$$

Example 4.3. Find the directional derivative of the function f defined by

$$f(x,y) = x^2 + 3xy^2$$

at $P = (1,2)$ in the direction towards the origin.

Solution. A vector in the direction from $(1,2)$ to $(0,0)$ is $-\vec{\imath} - 2\vec{\jmath}$, so a unit vector in this direction is therefore

$$\vec{u} = \frac{-1}{\sqrt{5}}\vec{\imath} - \frac{2}{\sqrt{5}}\vec{\jmath}.$$

We find that

$$f_x(1,2) = \left.(2x + 3y^2)\right|_{(1,2)} = 14 \qquad \text{and} \qquad f_y(1,2) = \left.6xy\right|_{(1,2)} = 12.$$

Therefore, the directional derivative is given by

$$D_{\vec{u}}f(1,2) = \frac{-1}{\sqrt{5}}(14) - \frac{2}{\sqrt{5}}(12) = \frac{-38}{\sqrt{5}}.$$

4.2 The Gradient of a Function

Definition 4.2. Let f be a differentiable function at (a,b). Then the **gradient** of f is denoted by $\nabla f(x,y) = f_x(x,y)\vec{\imath} + f_y(x,y)\vec{\jmath}$.
The value of the gradient at the point $P(a,b)$ is denoted by

$$\nabla f_P = f_x(a,b)\vec{\imath} + f_y(a,b)\vec{\jmath}.$$

Example 4.4. Find the gradient ∇f of the function f defined by

$$f(x,y) = \sin x + e^{xy}$$

and evaluate the gradient at $(0,1)$.

Solution. If $f(x,y) = \sin x + e^{xy}$, then

$$\nabla f(x,y) = \langle \cos x + ye^{xy}, xe^{xy} \rangle$$

and $\nabla f(0,1) = \langle 2, 0 \rangle$.

::: {#thm- } [Directional Derivative and Gradient] If f is a differentiable function of x and y, then the directional derivative of f at the point $P(a,b)$ in the direction of the unit vector $\vec{u}$ is

$$D_{\vec{u}}f(a,b) = \nabla f_P \cdot \vec{u}.$$

:::

Proof. Since $\nabla f_P = f_x(a,b)\vec{i} + f_y(a,b)\vec{j}$ and $\vec{u} = u_1\vec{i} + u_2\vec{j}$, we have

$$\nabla f_P \cdot \vec{u} = f_x(a,b)u_1 + f_y(a,b)u_2 = D_{\vec{u}} f(a,b).$$

□

Example 4.5. Find the directional derivative of the function f defined by

$$f(x,y) = x^2 + xy + y^2$$

at $P(1,-1)$ in the direction towards the origin.

Solution. We use the unit vector

$$\vec{u} = \left(-\frac{1}{\sqrt{2}}\right)\vec{i} + \left(\frac{1}{\sqrt{2}}\right)\vec{j}$$

and find
$\nabla f = (2x+y)\vec{i} + (2y+x)\vec{j}$
to determine

$$D_{\vec{u}} f(1,-1) = \nabla f(1,-1) \cdot \vec{u} = (\vec{i} - \vec{j}) \cdot \left(-\frac{1}{\sqrt{2}}\right)\vec{i} + \left(\frac{1}{\sqrt{2}}\right)\vec{j} = -\sqrt{2}.$$

Example 4.6. Find the directional derivative of the function f defined by

$$f(x,y) = x^3y^4$$

at the point $(6,-1)$ in the direction of the vector $\vec{v} = 2\vec{i} + 5\vec{j}$.

Solution. We first compute the gradient vector at $(6,-1)$

$$\nabla f(x,y) = 3x^2y^4\vec{i} + 4x^3y^3\vec{j} \qquad \text{and} \qquad \nabla f(6,-1) = 108\vec{i} - 864\vec{j}.$$

Note that $\vec{v}$ is not a unit vector, but since $|\vec{v}| = \sqrt{29}$, the unit vector in the direction of $\vec{v}$ is

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{2}{\sqrt{29}}\vec{i} + \frac{5}{\sqrt{29}}\vec{j}.$$

Therefore, by 4.2, we have

$$D_{\vec{u}} f(6,-1) = \nabla f(6,-1) \cdot \vec{u} = (108\vec{i} - 864\vec{j}) \cdot \left(\frac{2}{\sqrt{29}}\vec{i} + \frac{5}{\sqrt{29}}\vec{j}\right) = \frac{-4104}{\sqrt{29}}.$$

Example 4.7. Let $f(x,y,z)=xyz$, and let $\vec{u}$ be a unit vector perpendicular to both $\vec{v}=\vec{i}-2\vec{j}+3\vec{k}$ and $\vec{w}=2\vec{i}+\vec{j}-\vec{k}$. Find the directional derivative of f at $P(1,-1,2)$ in the direction $\vec{u}$.

Solution. The gradient is

$$\nabla f=\nabla(xyz)=yz\vec{i}+xz\vec{j}+xy\vec{k}$$

and $\nabla f_P=-2\vec{i}+2\vec{j}-\vec{k}$. Now since we are looking for a unit vector perpendicular to both $\vec{v}=\vec{i}-2\vec{j}+3\vec{k}$ and $\vec{w}=2\vec{i}+\vec{j}-\vec{k}$ we find, $\vec{v}\times\vec{w}=-\vec{i}+7\vec{j}+5\vec{k}$ and so $\vec{u}=\frac{1}{\sqrt{75}}(-\vec{i}+7\vec{j}+5\vec{k})$. Therefore, by 4.2, we have

$$D_{\vec{u}}f=(-2\vec{i}+2\vec{j}-\vec{k})\cdot\left(\frac{-1}{\sqrt{75}}\vec{i}+\frac{7}{\sqrt{75}}\vec{j}+\frac{5}{\sqrt{75}}\vec{k}\right)=\frac{11\sqrt{3}}{15}.$$

::: {#thm- } [Properties of the Gradient] Let f and g be differentiable functions. Then

- $\nabla c=0$ for any constant c
- $\nabla(af+bg)=a\nabla(f)+b\nabla(g)$ for any constants a and b
- $\nabla(fg)=f\cdot\nabla(g)+g\cdot\nabla(f)$
- $\nabla\left(\frac{f}{g}\right)=\frac{g\cdot\nabla(f)-f\cdot\nabla(g)}{g^2}$ provided $g^2\neq 0$
- $\nabla\left(f^n\right)=nf^{n-1}\cdot\nabla(f)$

:::

Proof. An outline of how each part can be proven in listed below.

- The (constant rule) is proved as follows by $\nabla c=(c)_x\vec{i}+(c)_y\vec{j}=0$.
- The (linearity rule) is proved as follows

$$\nabla(af+bg)=(af_x+bg_x)\,\vec{i}+\left(af_x+bg_y\right)\vec{j}=a\nabla(f)+b\nabla(g).$$

- The (product rule) is proved as follows

$$\begin{aligned}\nabla(fg)&=(fg)_x\vec{i}+(fg)_y\vec{j}=(f_xg+g_xf)\,\vec{i}+\left(f_yg+g_yf\right)\vec{j}\\&=\left[f\left(g_x\vec{i}+g_y\vec{j}\right)\right]+\left[g\left(f_x\vec{i}+f_y\vec{j}\right)\right]=f\cdot\nabla(g)+g\cdot\nabla(f).\end{aligned}$$

- The (quotient rule) is proved as follows

$$\begin{aligned}\nabla\left(\frac{f}{g}\right)&=\left(\frac{f}{g}\right)_x\vec{i}+\left(\frac{f}{g}\right)_y\vec{j}=\left(\frac{f_xg-g_xf}{g^2}\right)\vec{i}+\left(\frac{f_yg-g_yf}{g^2}\right)\vec{j}\\&=\frac{\left[g\left(f_x\vec{i}+f_y\vec{j}\right)\right]-\left[f\left(g_x\vec{i}+g_y\vec{j}\right)\right]}{g^2}=\frac{g\cdot\nabla(f)-f\cdot\nabla(g)}{g^2}.\end{aligned}$$

- The (power rule) is proved as follows

$$\nabla f^n=\left(f^n\right)_x\vec{i}+\left(f^n\right)_y\vec{j}=\left(nf^{n-1}\right)f_xb\vec{i}+\left(nf^{n-1}\right)f_yb\vec{j}=nf^{n-1}\cdot\nabla(f).$$

□

::: {#thm- } [Orthogonality of Gradient] Suppose the function f is differentiable at the point P and that the gradient at P satisfies $\nabla f_P \neq 0$. Then ∇f_P is orthogonal to the level surface (or curve) of f through P. :::

Proof. Let C be any smooth curve on the level surface $f(x, y, z) = K$ that passes through $P(a, b, c)$, and describe the curve C by the vector function $\vec{R}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ for all t in some interval I. We will show that the gradient ∇f_P is orthogonal to the tangent vector dR/dt at P. Because C lies on the level surface, any point $P(x(t), y(t), z(t))$ on C must satisfy $f[x(t), y(t), z(t)] = K$, and by applying the chain rule, we obtain

$$\frac{d}{dt}[f(x(t), y(t), z(t))] = f_x(x, y, z)\frac{dx}{dt} + f_y(x, y, z)\frac{dy}{dt} + f_z(x, y, z)\frac{dz}{dt}.$$

Suppose $t = t_0$ at P. Then

$$\begin{aligned} &\frac{d}{dt}\{f[x(t), y(t), z(t)]\}\bigg|_{t=t_0} \\ &\quad = f_x\left(x\left(t_0\right), y\left(t_0\right), z\left(t_0\right)\right)\frac{dx}{dt} + f_y\left(x\left(t_0\right), y\left(t_0\right), z\left(t_0\right)\right)\frac{dy}{dt} + f_z\left(x\left(t_0\right), y\left(t_0\right)\right. \\ &\quad = \nabla f_P \frac{d\vec{R}}{dt} \end{aligned}$$

since

$$\frac{d\vec{R}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}.$$

We also know that $f(x(t), y(t), z(t)) = K$ for all t in I. Thus, we have

$$\frac{d}{dt}\{f[x(t), y(t), z(t)]\} = \frac{d}{dt}(K) = 0$$

and it follows that $\nabla f_P \cdot \frac{d\vec{R}}{dt} = 0$. We are given that $\nabla f_P \neq 0$ and $d\vec{R}/dt \neq 0$ because the curve C is smooth. Therefore, ∇f_P is orthogonal to $d\vec{R}/dt$, as required.

□

4.3 Steepest Ascent and Steepest Descent

The direction of the greatest rate of increase (or decrease) of a given function at a specified point is called the direction of **steepest ascent** (or **steepest descent**).

::: {#thm- } [Steepest Ascent/Descent] Suppose f is differentiable at the point P and that the gradient of f at P satisfies $\nabla f_P \neq 0$. Then

- The largest value of the directional derivative $D_{\vec{u}}f$ at P is $\|\nabla f_P\|$ and occurs when the unit vector $\vec{u}$ points in the direction of ∇f_P.
- The smallest value of the directional derivative $D_{\vec{u}}f$ at P is $-\|\nabla f_P\|$ and occurs when the unit vector $\vec{u}$ points in the direction of $-\nabla f_P$.
:::

Proof. If $\vec{u}$ is any unit vector, then$D_{\vec{u}}f = \nabla f_P \cdot \vec{u} = \|\nabla f_P\| \|\vec{u}\| \cos\theta = \|\nabla f_P\| \cos\theta$ where θ is the angle between ∇f_P and $\vec{u}$. But $\cos\theta$ assumes its largest value of 1 at $\theta = 0$; that is, when $\vec{u}$ points in the direction ∇f_P. Thus, the largest possible value of $D_{\vec{u}}f$ is

$$D_{\vec{u}}f = \|\nabla f_P\| (1) = \|\nabla f_P\|.$$

Also $\cos\theta$ assumes its smallest value -1 when $\theta = \pi$. This value occurs when $\vec{u}$ points toward $-\nabla f_P$, and in this direction

$$D_{\vec{u}}f = \|\|\nabla f_P\| (-1) \ = \|\|\nabla f_P\| (-1) \ = -\|\nabla f_P\|.$$

$\square$

Example 4.8. Sketch the level curve corresponding to $C = 1$ for the function

$$f(x,y) = x^2 - y^2$$

and find a normal vector at the point $P(2,\sqrt{3})$.

Solution. The level curve for $C = 1$ is a hyperbola given by $x^2 - y^2 = 1$. The gradient vector is perpendicular to the level curve. We have

$$\nabla f = f_x\vec{i} + f_y\vec{j} = 2x\vec{i} - 2y\vec{j}$$

so at the point $\left(2,\sqrt{3}\right)$, $\nabla f_P = 4i - 2\sqrt{3}j$ is the required normal.

Example 4.9. In what direction is the function defined by

$$f(x,y) = xe^{2y-x}$$

increasing most rapidly at the point $P(2,1)$, and what is the maximum rate of increase? In what direction is f decreasing most rapidly?

Solution. We begin by finding the gradient of f,

$$\begin{aligned}
\nabla f &= f_x\vec{i} + f_y\vec{j} \\
&= \left[e^{2y-x} + xe^{2y-x}(-1)\right]\vec{i} + \left[xe^{2y-x}(2)\right]\vec{j} \\
&= e^{2y-x}(1-x)\vec{i} + 2x\vec{j}
\end{aligned}$$

At $P(2,1)$,

$$\nabla f_P = e^{2(1)-2}[(1-2)\vec{\imath} + 2(2)\vec{\jmath}] = -\vec{\imath} + 4\vec{\jmath}.$$

The most rapid rate of increase is $\|\nabla f_P\| = \sqrt{(-1)^2 + 4^2} = \sqrt{17}$ and occurs in the direction of $-\vec{\imath} + 4\vec{\jmath}$. The most rapid rate of decrease occurs in the direction of $-\nabla f_P = \vec{\imath} - 4\vec{\jmath}$ and is $-\sqrt{17}$.

Example 4.10. Let $f(x,y,z) = ye^{x+z} + ze^{y-x}$. At the point $P(2,2,-2)$, find the unit vector pointing in the direction of most rapid increase of f.

Solution. The gradient is

$$\nabla f = \left(ye^{x+z} - ze^{y-x}\right)\vec{\imath} + \left(e^{x+z} + xe^{y-x}\right)\vec{\jmath} + \left(ye^{x+z} + e^{y-z}\right)\vec{k}.$$

Thus, $\nabla f(2,2,-2) = 4\vec{\imath} - \vec{\jmath} + 3\vec{k}$ and so $\vec{u} = \frac{1}{\sqrt{26}}(4\vec{\imath} - \vec{\jmath} + 3\vec{k})$.

Example 4.11. Find the maximum rate of change of the function at the given point and the direction in which it occurs $f(x,y) = \sqrt{x^2 + 2y}$, $(4,10)$

Solution. We find

$$\nabla f(4,10) = \left.\frac{x}{\sqrt{x^2+2y}}\right|_{(4,10)} \vec{\imath} + \left.\frac{1}{\sqrt{x^2+2y}}\right|_{(4,10)} \vec{\jmath} = \frac{2}{3}\vec{\imath} + \frac{1}{6}\vec{\jmath}.$$

Thus the maximum rate of change is $\sqrt{(2/3)^2 + (1/6)^2} = \frac{\sqrt{17}}{6}$ and occurs in the direction of $\frac{2}{3}\vec{\imath} + \frac{1}{6}\vec{\jmath}$.

Example 4.12. Find the maximum rate of change of the function at the given point and the direction in which it occurs $f(x,y) = \sqrt{y^2 + 2x}$, $(-4,-10)$

Solution. We find

$$\nabla f(-4,-10) = \left.\frac{1}{\sqrt{2x+y^2}}\right|_{(-4,-10)} \vec{\imath} + \left.\frac{y}{\sqrt{2x+y^2}}\right|_{(-4,-10)} \vec{\jmath} = \frac{1}{2\sqrt{23}}\vec{\imath} - \frac{5}{\sqrt{23}}\vec{\jmath}.$$

Thus the maximum rate of change is

$$\sqrt{\left(\frac{1}{2\sqrt{23}}\right)^2 + \left(-\frac{5}{\sqrt{23}}\right)^2} = \frac{1}{2}\sqrt{\frac{101}{23}}$$

and occurs in the direction of $\frac{1}{2\sqrt{23}}\vec{\imath} - \frac{5}{\sqrt{23}}\vec{\jmath}$.

Example 4.13. Find the direction from $P(2,-1,2)$ in which the function $f(x,y,z)=(x+y)^2+(y+z)^2+(x+z)^2$ increases most rapidly and compute the magnitude of the greatest rate of increase.

Solution. We compute the gradient:

$$\nabla f=(4x+2y+2z)\vec{i}+(2x+4y+2z)\vec{j}+(2x+2y+4z)\vec{k}$$

and at $(2,-1,2)$ we have,
$\nabla f(2,-1,2)=10\vec{i}+4\vec{j}+10\vec{k}$. Therefore, $\|\nabla f\|=\sqrt{216}$ is the magnitude of the greatest rate of increase and occurs in the direction of $10\vec{i}+4\vec{j}+10\vec{k}$.

4.4 Tangent Planes

Suppose S is a surface with the equation $z=f(x,y)$ where f has continuous first partial derivatives f_x and f_y. Let $P(a,b,c)$ be a point on S and let C_1be the curve of intersection of S with the plane $x=a$ and C_2 the intersection of S with the plane $y=b$. The tangent lines T_1 and T_2 to C_1 and C_2, respectively determine a unique plane and this plane actually contains the tangent to every smooth curve C that passes through P. We call this plane the **tangent plane** to S at P.

::: {#thm- } [Equation of Tangent Plane] If $z=f(x,y)$ is differentiable at (a,b) then an equation of the tangent plane to the graph of f at (a,b) is

$$z-f(a,b)=f_x(a,b)(x-a)+f_y(a,b)(y-b).$$

:::

Proof. The proof is left for the reader. □

Example 4.14. Find an equation of the tangent plane to the surface

$$z=2x^2+y^2$$

at the point $(1,1,3)$.

Solution. Let $f(x,y)=2x^2+y^2$. Then

$$f_x(x,y)=4x,\quad f_x(1,1)=4,\quad f_y(x,y)=2y,\quad f_y(1,1)=2.$$

Then an equation of the tangent plane is $z-3=4(x-1)+2(y-1)$ or

$$z=4x+2y-3.$$

Example 4.15. Find an equation of the tangent plane to the surface

$$z = 4 - x^2 - y^2$$

at the point $(1, 1, 2)$.

Solution. Let $f(x, y) = 4 - x^2 - y^2$. Then

$$f_x(x, y) = -2x, \quad f_x(1, 1) = -2, \quad f_y(x, y) = -2y, \quad f_y(1, 1) = -2.$$

Then an equation of the tangent plane is $z - 2 = -2(x - 1) - 2(y - 1)$ or

$$z = -2x - 2y + 6$$

Example 4.16. Find an equation of the tangent plane to the surface

$$z = \sin x + \sin y$$

at the point $\left(\frac{\pi}{2}, \frac{\pi}{3}, 1 + \frac{\sqrt{3}}{2}\right)$.

Solution. Let $f(x, y) = \sin x + \sin y$. Then

$$f_x(x, y) = \cos x, \quad f_x\left(\frac{\pi}{2}, \frac{\pi}{3}\right) = 0, \quad f_y(x, y) = \cos y, \quad f_y\left(\frac{\pi}{2}, \frac{\pi}{3}\right) = \frac{1}{2}.$$

Then an equation of the tangent plane is

$$z - \left(1 + \frac{\sqrt{3}}{2}\right) = 0\left(x - \frac{\pi}{2}\right) + \frac{1}{2}\left(y - \frac{\pi}{3}\right)$$

or

$$z = \frac{1}{2}y - \frac{\pi}{6} + 1 + \frac{\sqrt{3}}{2}$$

::: {#thm- } [Tangent Plane To A Surface] Suppose S is a surface with the equation $F(x, y, z) = C$ and let $P(a, b, c)$ be a point on S where F is differentiable with $\nabla F_P \neq 0$. Then an equation of the tangent plane to S at P is

$$F_x(a, b, c)(x - a) + F_y(a, b, c)(y - b) + F_z(a, b, c)(z - c) = 0$$

and the normal line to S at P has parametric equations

$$x = a + F_x(a, b, c)t, \qquad y = b + F_y(a, b, c)t, \qquad z = c + F_z(a, b, c)t.$$

:::

Proof. Any plane that passes through $P(a,b,c)$ has an equation of the form

$$A(x-a)+B(y-b)+C(z-c)=0.$$

By dividing this equation by C and letting $a_1=-A/C$ and $b_1=-B/C$, we can write it in the form

$$z-c=a_1(x-a)+b_1(y-b).$$

If this equation represents the tangent plane at P, then its intersection with the plane $y=b$ must be the tangent line with slope $f_x(a,b)$. Therefore, $a_1=f_x(a,b)$. Similarly, putting $x=a$, we get $z-c=b_2(y-b)$, which must represent the tangent line with slope $f_y(a,b)$ and so $b_2=f_y(a,b)$. Thus, the equation of the tangent plane is $z-c=f_x(a,b)(x-a)+f_y(a,b)(y-b)$.

□

Example 4.17. Find an equation of the tangent plane to the surface

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

at the point (x_0,y_0,z_0).

Solution. Since $F_x=2x/a^2$, $F_y=2y/b^2$, and $F_z=2z/c^2$, the tangent plane at $P_0\,(x_0,y_0,z_0)$ has equation

$$\frac{2x_0}{a^2}\,(x-x_0)+\frac{2y_0}{b^2}\,(y-y_0)+\frac{2z_0}{c^2}\,(z-z_0)=0$$

$$\frac{x_0x}{a^2}-\frac{{x_0}^2}{a^2}+\frac{y_0y}{b^2}-\frac{y_0^2}{b^2}+\frac{z_0z}{c^2}-\frac{{z_0}^2}{c^2}=0$$

$$\frac{x_0x}{a^2}+\frac{y_0y}{b^2}+\frac{z_0z}{c^2}=1.$$

Example 4.18. Find an equation of the tangent plane to the surface

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

at the point (x_0,y_0,z_0).

Solution. Since $F_x=2x/a^2$, $F_y=2y/b^2$, and $F_z=-2z/c^2$, the tangent plane at $P_0\,(x_0,y_0,z_0)$ has equation

$$\frac{2x_0}{a^2}\,(x-x_0)+\frac{2y_0}{b^2}\,(y-y_0)-\frac{2z_0}{c^2}\,(z-z_0)=0$$

$$\frac{x_0 x}{a^2} - \frac{x_0^2}{a^2} + \frac{y_0 y}{b^2} - \frac{y_0^2}{b^2} + \frac{z_0 z}{c^2} + \frac{z_0^2}{c^2} = 0$$
$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} - \frac{z_0 z}{c^2} = 1.$$

Example 4.19. Find an equation of the tangent plane to the surface

$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

at the point (x_0, y_0, z_0).

Solution. Since $F_x = 2x/a^2$, $F_y = 2y/b^2$, and $F_z = -1/c$, the tangent plane at $P_0\,(x_0, y_0, z_0)$ has equation

$$\frac{2x_0}{a^2}(x - x_0) + \frac{2y_0}{b^2}(y - y_0) - \frac{1}{c}(z - z_0) = 0$$
$$\frac{x_0 x}{a^2} - \frac{x_0^2}{a^2} + \frac{y_0 y}{b^2} - \frac{{y_0}^2}{b^2} + \frac{z}{c} + \frac{z_0}{c} = 0$$
$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} - \frac{z}{c} = \frac{z_0}{c}.$$

4.5 Normal Lines

Example 4.20. Find the equations for the tangent plane and the normal line to the cone

$$z^2 = x^2 + y^2$$

at the point where $x = 3, y = 4$, and $z > 0$.

Solution. If $P(a, b, c)$ is the point of tangency and $a = 3, y = 4$, and $c > 0$, then $c = \sqrt{a^2 + b^2} = 5$. If we consider $F(x, y, z) = x^2 + y^2 - z^2$, then the cone can be regarded as the level surface $F(x, y, z) = 0$. The partial derivatives of F are $F_x = 2x$, $F_y = 2y$, and $F_z = -2z$ so at $P(3, 4, 5)$ we find $F_x(3, 4, 5) = 6$, $F_y(3, 4, 5) = 8$, and $F_z(3, 4, 5) = -10$. Thus the tangent plane has an equation

$$6(x - 3) + 8(y - 4) - 10(z - 5) = 0$$

or $3x + 4y - 5z = 0$ and the normal line is given parametrically by the equations $x = 3 + 6t$, $y = 4 + 8t$, and $z = 5 - 10t$.

Example 4.21. Find the equations of the tangent plane and the normal line at the point $(-2, 1, -3)$ to the ellipsoid

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3.$$

Solution. The ellipsoid is a level surface of the function

$$F(x,y,z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9}.$$

Therefore we have

$$F_x(x,y,z) = \frac{x}{2}, \quad F_y(x,y,z) = 2y, \quad F_z(x,y,z) = \frac{2z}{9}$$

Thus,

$$F_x(-2,1,3) = -1, \quad F_y(-2,1,3) = 2, \quad F_z(-2,1,3) = -2/3.$$

Then the tangent plane at $(-2,1,-3)$ is

$$-1(x+2) + 2(y-1) - (2/3)(z+3) = 0$$

which simplifies to

$$3x - 6y + 2z + 18 = 0$$

and parametric equations for the normal line are

$$x = -2 - t, \quad y = 1 + 2t, \quad z = -3 - (2/3)t.$$

4.6 Exercises

Exercise 4.1. Find the directional derivative of $f(x,y) = x^2y^2 - x^2 + 2y$ at the point $(2,2)$ in the direction of the unit vector $\vec{u} = \frac{1}{2}\vec{\imath} - \frac{\sqrt{3}}{2}\vec{\jmath}$.

Exercise 4.2. Find the gradient of the function $f(x,y) = y\tan x + \sin xy$.

Exercise 4.3. Find the directional directive of $f(x,y) = \frac{e^{-x}}{y}$ at $P(0,-1)$ in the direction of $\mathbf{v} = -\mathbf{i} + \mathbf{j}$.

Exercise 4.4. Find the gradient of the function.

- $f(x,y) = x^2 - 2xy$
- $f(x,y) = \ln\left(x^2 + y^2\right)$
- $f(x,y) = \sin(x + 2y)$
- $f(x,y,z) = \frac{xy-1}{z+x}$
- $f(x,y,z) = xyz^2$
- $g(x,y,z) = xe^{y+3z}$

Exercise 4.5. Compute the directional derivative of the function $f(x, y) = \ln\left(x^2 + 3y\right)$ at the point $(1, 1)$ in the direction of the vector $v = \vec{\imath} + \vec{\jmath}$.

Exercise 4.6. Compute the directional derivative of the function $f(x, y) = \sin xy$ at the point $(\sqrt{\pi}, \sqrt{\pi})$ in the direction of the vector $v = 3\pi\vec{\imath} - \pi\vec{\jmath}$.

Exercise 4.7. Find the directional derivative of the given function at the given point in the direction of the given vector.

- $f(x, y) = x^2 + xy$, $(1, -2)$, and $\vec{\imath} + \vec{\jmath}$
- $f(x, y) = \frac{e^{-x}}{y}$, $(2, -1)$, and $-\vec{\imath} + \vec{\jmath}$

Exercise 4.8. Find the directional derivative of the given function at the given point in the direction of the given vector.

- $f(x, y) = \ln\left(3x + y^2\right)$, $(0, 1)$, and $\vec{\imath} - \vec{\jmath}$
- $f(x, y) = \sec\left(xy - y^3\right)$, $(2, 0)$, and $-\vec{\imath} - 3\vec{\jmath}$

Exercise 4.9. Find the direction from P_0 in which the given function f increases most rapidly and compute the magnitude of the greatest rate of increase.

- $f(x, y, z) = (x + y)^2 + (y + z)^2 + (x + z)^2$ at $P_0(2, -1, 2)$
- $f(x, y, z) = z \ln\left(\frac{y}{z}\right)$ at $P_0(1, e, -1)$

4.7 Extreme Values of Two Variable Functions

4.8 Relative Extrema

Definition 4.3. Let f be a function of two variables x and y.

- The function f has a **relative maximum** at (x_0, y_0) if $f(x, y) \leq f(x_0, y_0)$ for all (x, y) in an open disk containing (x_0, y_0).

- The function f has a **relative minimum** at (x_0, y_0) if $f(x, y) \geq f(x_0, y_0)$ for all (x, y) in an open disk containing (x_0, y_0).

Collectively, relative maxima and relative minima are called **relative extrema** .

::: {#lem- } [Critical Points]

If f has a relative extremum at (x_0, y_0) and partial derivatives f_x and f_y both exist at (x_0, y_0), then

$$f_x(x_0, y_0) = f_y(x_0, y_0) = 0.$$

:::

Proof. Let $F(x) = f(x, y_0)$. Then $F(x)$ must have a relative extremum at $x = x_0$, so $F'(x_0) = 0$, which means that $f_x(x_0, y_0) = 0$. Similarly, $G(y) = f(x_0, y)$ has a relative extremum at $y = y_0$, so $G'(y_0) = 0$ and $f_y(x_0, y_0) = 0$. Thus, we must have both $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$.

□

Definition 4.4. A **critical point** of a function f defined on an open set D is a point (x_0, y_0) in D where either one of the following is true:

- $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$ or
- at least one of f_x or f_y does not exist at (x_0, y_0).

Example 4.22. Find the critical points for the function

$$f(x, y) = x^2 + y^2 - 2x - 6y + 14.$$

Solution. The first partial derivatives of f are

$$f_x(x, y) = 2x - 2 \qquad \text{and} \qquad f_y(x, y) = 2y - 6.$$

These partial derivatives are equal to 0 when $x = 1$ and $y = 3$ so the only critical point is $(1, 3)$. By completing the square we find that

$$f(x, y) = 4 + (x - 1)^2 + (y - 3)^2.$$

Since $(x - 1)^2 \geq 0$ and $(y - 3)^2 \geq 0$, we have $f(x, y) \geq 4$ for all values of x and y. Therefore, $f(1, 3) = 4$ is a local minimum, and in fact it is the absolute minimum of f. This can be confirmed geometrically from the graph of f, which is the elliptic paraboloid with vertex $(1, 3, 4)$ as shown in **??**.

Definition 4.5. A critical point $P_0(x_0, y_0)$ is called a **saddle point** of f if every open disk centered at P_0 contains points in the domain of f that satisfy $f(x,y) > f(x_0, y_0)$ as well as points in the domain of f that satisfy $f(x,y) < f(x_0, y_0)$.

Example 4.23. Find the extreme values of

$$f(x,y) = y^2 - x^2.$$

Solution. Since $f_x = -2x$ and $f_y = 2y$, the only critical point is $(0,0)$. Notice that for points on the x-axis we have $y = 0$, so $f(x,0) = -x^2 < 0$ (if $x \neq 0$.) However for points on the y-axis we have $x = 0$, so $f(0,y) = y^2 > 0$ (if $y \neq 0$). Thus every disk with center $(0,0)$ contains points where f takes positive values as well as points where f takes negative values. Therefore $f(0,0) = 0$ cannot be an extreme value for f, so f has no extreme values.

This example illustrates the fact that a function need not have a maximum or minimum value at a critical point. The graph of f is the hyperbolic paraboloid which has a horizontal tangent plane ($z = 0$) at the origin. You can see that $f(0,0) = 0$ is a maximum in the direction of the $x-$axis but not in the direction of the y-axis. Near the origin the graph has the shape of a saddle as shown in **??**.

::: {#thm- } [Second Partials Test] Assume f has a critical point at $P_0(x_0, y_0)$ and assume that f has continuous second order partial derivatives in a disk centered at (x_0, y_0). If $D := \left(f_{xx}f_{yy} - \left(f_{xy}\right)^2\right)(x_0, y_0) > 0$, then

- a relative maximum occurs at P_0 if $f_{xx}(x_0, y_0) < 0$
- a relative minimum occurs at P_0 if $f_{xx}(x_0, y_0) > 0$.

If $D(x_0, y_0) < 0$, then a saddle point occurs at P_0. :::

Example 4.24. Find the relative extrema of the function

$$f(x,y) = x^4 + y^4 - 4xy + 1.$$

Solution. We first locate the critical points

$$f_x = 4x^3 - 4y \qquad \text{and} \qquad f_y = 4y^3 - 4x.$$

Setting these partial derivatives to 0, we obtain the equations$x^3 - y = 0$ and $y^3 - x = 0$.

To solve these equations we substitute $y = x^3$ from the first equation into the second one. This gives

$$0 = x^9 - x = x\left(x^8 - 1\right) = x\left(x^4 - 1\right)\left(x^4 + 1\right) = x(x-1)(x+1)\left(x^2 + 1\right)\left(x^4 + 1\right)$$

So there are three real roots: $x = 0, 1, -1$. The three critical points are $(0,0)$, $(1,1)$, and $(-1,-1)$. Next we calculate the second partial derivatives and $D(x,y)$:

$$f_{xx} = 12x^2, \qquad f_{xy} = -4, \qquad f_{yy} = 12y^2.$$

Thus,

$$D(x,y) = f_{xx}f_{yy} - \left(f_{xy}\right)^2 = 144x^2y^2 - 16.$$

Since $D(0,0) = -16 < 0$, it follows that the origin is a saddle point; that is, f has no local extremum at $(0,0)$. Since $D(1,1) = 128 > 0$ and $f_{xx}(1,1) = 12 > 0$, we see that $f(1,1) = -1$ is a local minimum. Similarly, we have $D(-1,-1) = 128 > 0$ and $f_{xx}(-1, 1-1) = 12 > 0$, so $f(-1,-1) = -1$ is also a local minimum.

4.9 Absolute Extrema

Definition 4.6. Let f be a function of two variables x and y.

- The function f has a **relative maximum** at (x_0, y_0) if $f(x,y) \leq f(x_0, y_0)$ for all (x,y) in the domain D of f.
- The function f has a **relative minimum** at (x_0, y_0) if $f(x,y) \geq f(x_0, y_0)$ for all (x,y) in the domain D of f.

Collectively, absolute maxima and absolute minima are called **absolute extrema** .

::: {#thm- } [Extreme Value] A function f attains both an absolute maximum and an absolute minimum on any closed bounded set S where it is continuous. :::

Proof. The proof is left for the reader.

□

Example 4.25. Find the **shortest distance** from the point $(1,0,-2)$ to the plane

$$x + 2y + z = 4.$$

Solution. The distance from any point (x, y, z) to the point $(1, 0, -2)$ is

$$d = \sqrt{(x-1)^2 + y^2 + (z+2)^2}$$

but if (x, y, z) lies on the plane, then $z = 4 - x - 2y$ and so we have

$$d = \sqrt{(x-1)^2 + y^2 + (6-x-2y)^2}.$$

We can minimize d by minimizing the simpler expression

$$d^2 = f(x, y) = (x-1)^2 + y^2 + (6-x-2y)^2.$$

By solving the equations

$$f_x = 2(x-1)^2 - 2(6-x-2y) = 4x + 4y - 14 = 0$$

$$f_y = 2y - 4(6-x-2y) = 4x + 10y - 24 = 0$$

we find that the only critical point is $(11/3, 5/3)$. Since $f_{xx} = 4$, $f_{xy} = 4$, and $f_{yy} = 10$, we have

$$D(x, y) = f_{xx}f_{yy} - \left(f_{xy}\right)^2 = 24 > 0$$

and $f_{xx} > 0$ so f has a local minimum at $(11/6, 5/3)$. Intuitively, we can see that this local minimum is actually an absolute minimum because there must be a point on the given plane that is closest to $(1, 0, -2)$. If $x = 11/6$ and $y = 5/3$, then

$$d = \sqrt{\left(\frac{5}{6}\right)^2 + \left(\frac{5}{3}\right)^2 + \left(\frac{5}{6}\right)^2} = \frac{5\sqrt{6}}{6}.$$

which is the shortest distance.

Example 4.26. Find the absolute extrema of the function over the bounded region

$$f(x, y) = x^2 - 2xy + 2y$$

over the rectangle

$$D = \{(x, y) \mid 0 \leq x \leq 3, 0 \leq y \leq 2\}.$$

Solution. Since f is a polynomial it is continuous on the closed bounded rectangle D, therefore f has both absolute maximum and minimum values. We first find the critical points by solving the system

$$f_x = 2x - 2y = 0 \qquad \text{and} \qquad f_y = -2x + 2 = 0$$

The only critical point is $(1,1)$, and the value of f there is $f(1,1) = 1$. We look at the values of f on the boundary of D, which consists of four line segments L_1, L_2, L_3, and L_4 as follows.

On L_1 we have $y = 0$ and $f(x, 0) = x^2$ for $0 \le x \le 3$. This is an increasing function of x, so its minimum value is $f(0,0) = 0$ and its maximum value is $f(3,0) = 9$.

On L_2 we have $x = 3$ and $f(3, y) = 9 - 4y$ on $0 \le y \le 2$. This is a decreasing function of y, so its maximum value is $f(3,0) = 9$ and its minimum value is $f(3,2) = 1$.

On L_3 we have $y = 2$ and $f(x, 2) = x^2 - 4x + 4$ on $0 \le x \le 3$. By observing that $f(x, 2) = (x-2)^2$, we see that the minimum value of this function is $f(2,2) = 0$ and the maximum value is $f(0,2) = 4$.

On L_4 we have $x = 0$ and $f(0, y) = 2y$ on $0 \le y \le 2$ with maximum value $f(0,2) = 4$ and minimum value $f(0,0) = 0$.

Thus on the boundary, the minimum value of f is 0 and the maximum is 9. We compare these values with the value $f(1,1) = 1$ at the critical point and conclude that the absolute maximum value of f on D is $f(3,0) = 9$ and the absolute minimum value is $f(0,0) = f(2,2) = 0$.

Example 4.27. A rectangular box without a lid is to be made from 12 m^2 of cardboard. Find the maximum volume of such a box.

Solution. Let the length, width, and height of the box (in meters) be x, y, and z. Then the volume of the box is $V = xyz$. We can express V as a function of just two variables by using the fact that the surface area of the sides and the bottom of the box is $2xz + 2yz + xy = 12$. Solving these equation for z, we get

$$z = \frac{12 - xy}{2(x+y)},$$

so the expression for volume V becomes

$$V = \frac{12xy - x^2y^2}{2(x+y)}.$$

We compute the partial derivatives

$$V_x = \frac{y^2\left(12 - 2xy - x^2\right)}{2(x+y)} \qquad \text{and} \qquad V_y = \frac{x^2\left(12 - 2xy - y^2\right)}{2(x+y)^2}.$$

If V is a maximum, then $V_x = V_y = 0$, but $x = 0$ or $y = 0$ gives $V = 0$, so we must solve the equations

$$12 - 2xy - x^2 = 0 \qquad \text{and} \qquad 12 - 2xy - y^2 = 0.$$

These imply that $x^2 = y^2$ and so $x = y$. (Note that x and y must both be positive in this example.) If we put $x = y$ in either equation we get $12 - 3x^2 = 0$, which gives $x = 2$, $y = 2$, and $z = 1$. From the physical nature of this example there must be an absolute maximum volume that has to occur at a critical point of V, so it must be when $x = 2$, $y = 2$, and $z = 1$. Then $V = 2 \cdot 2 \cdot 1 = 4$, so the maximum volume of the box is $4m^3$.

Example 4.28. Find the absolute extrema of the function

$$f(x, y) = x^2 + 3y^2 - 4x + 2y - 3$$

over the rectangle

$$D = \{(x, y) \mid 0 \leq x \leq 3, -3 \leq y \leq 0\}.$$

Solution. We compute $f_x = 2x-4$ and $f_y = 6y+2$ and set $f_x = f_y = 0$ and find that $(2, -1/3)$ is the only critical point in the interior. On $x = t, y = 0$ for $0 \leq t \leq 3$, we have

$$F_1(t) = f(t, 0) = t^2 - 4t - 3 = 0$$

and so $F_1'(t) = 2t - 4$ yielding the point $(2, 0)$. On $x = 3, y = t$ for $-3 \leq t \leq 0$, we have

$$F_2(t) = f(3, t) = 3t^2 + 2t - 6$$

and so $F_2'(t) = 6t + 2 = 0$ yielding the point $(3, -1/3)$. On $x = t, y = -3$ for $0 \leq t \leq 3$, we have

$$F_3(t) = f(t, -3) = t^2 - 4t + 18$$

and so $F_3'(t) = 2t - 4 = 0$ yielding the point $(2, -3)$. On $x = 0, y = t$ for $-3 \leq t \leq 0$, we have

$$F_4'(t) = f(0, t) = 3t^2 + 2t - 3$$

and so $F_4'(t) = 6t + 2 = 0$ yielding the point $(0, -1/3)$.

Finally, we have $f(2, -1/3) = -22/3$ (the minimum), $f(3, -1/3) = -19/3$, $f(2, -3) = 14$, $f(0, -1/3) = -10/3$, $f(2, 0) = -7$, $f(0, 0) = -3$, $f(3, 0) = -6$, $f(3, -3) = 15$, and $f(0, -3) = 18$ (the maximum).

Example 4.29. Find the hottest and coldest points on the metal plate given as the region $R = [0, \pi] \times [0, \pi]$, whose temperature is given by $f(x, y) = \sin x + \cos 2y$.

::: {.proof }[Solution] Since f is continuous and R is closed and bounded, we know that the absolute maximum and minimum exist. We find that

$$\nabla f = \cos x\vec{i} + (-2\sin 2y)\vec{j}$$

and so the only critical point in the interior of R is $(\pi/2, \pi/2)$. This is a saddle point because the discriminant of f at $(\pi 2, \pi/2)$ is negative. The boundary of R consists of four line segments L_1, L_2, L_3, and L_4 as follows.

On L_1 and L_3, we have

$$f(x,0) = f(x,\pi) = \sin x + 1 \qquad \text{for } 0 \leq x \leq \pi$$

which achieves a maximum at $\pi/2$ and a minimum at 0 and π. Similarly, on L_2 and L_4 we have

$$f(0,y) = f(\pi,y) = \cos 2y \qquad \text{for } 0 \leq y \leq \pi$$

which achieves its maximum at 0 and π and its minimum at $\pi/2$. We see that the hottest points are $(\pi/2, 0)$ and $(\pi/2, \pi)$ and the coldest points are $(0, \pi/2)$ and $(\pi, \pi/2)$. :::

4.10 Exercises

Exercise 4.10. Find the absolute extrema for the function $f(x,y) = xy^2 - 2xy + 3y$ in the triangular domain with vertices $(0,0)$, $(1,0)$, and $(1,1)$.

Exercise 4.11. Find and classify the relative extrema and the saddle points of $f(x,y) = xy - 2x - 4y$.

Exercise 4.12. Find the critical points of the function $f(x,y) = x^3 + y^2 - 2xy + 7x - 8y + 2$.

Exercise 4.13. Find and classify the relative extrema and the saddle points of $f(x,y) = xy - 2x - 4y$.

Exercise 4.14. Find the absolute extrema for the function $f(x,y) = xy^2 - 2xy + 3y$ in the triangular domain with vertices $(0,0)$, $(1,0)$, and $(1,1)$.

Exercise 4.15. Find the critical points and classify each as a relative maximum, a relative minimum, or a saddle point.

- $f(x, y) = 2x^2 - 4xy + y^3 + 2$
- $f(x, y) = e^{-x} \sin y$
- $f(x, y) = (x - 2)^2 + (y - 3)^4$
- $f(x, y) = (x^2 + 2y^2)\, e^{1-x^2-y^2}$
- $f(x, y) = x^2 + y^2 + \frac{32}{xy}$
- $f(x, y) = (x - 4) \ln(xy)$
- $f(x, y) = 2x^3 + y^3 + 3x^2 - 18y^2 + 81y + 5$
- $f(x, y) = 3x^2 + 12x + 8y^3 - 12y^2 + 7$

Exercise 4.16. Find the absolute maximum and minimum values for each of the following functions.

- $f(x, y) = e^{x^2+2x+y^2}$ on the disk $x^2 + 2x + y^2 \leq 0$.
- $f(x, y) = x^2 + xy + y^2$ on the disk $x^2 + y^2 \leq 1$.
- $f(x, y) = xy - 2x - 5y$ on the triangular region S with vertices $(0, 0)$, $(7, 0)$, and $(7, 7)$.
- $f(x, y) = x^2 - 4xy + y^3 + 4y$ on the square $0 \leq x \leq 2$, $0 \leq y \leq 2$.
- $f(x, y) = x^2 + 3y^2 - 4x + 2y - 3$ on the square region S with vertices $(0, 0)$, $(3, 0)$, $(3, -3)$, and $(0, -3)$.

Exercise 4.17. Find three positive numbers whose sum is 54 and whose product is as large as possible.

Exercise 4.18. A wire of length L is cut into three pieces that are bent to form a circle, a square, and a equilateral triangle. How should the cuts be made to minimize the sum of the total area.?

Exercise 4.19. A rectangular box with no top is to have a fixed volume. What should its dimensions be if we want to use the least amount of material in its construction?

Exercise 4.20. Let R be the triangular region in the xy-plane with vertices $(-1, -2)$, $(-1, 2)$, and $(3, 2)$. A plate in the shape of R is heated so that the temperature at (x, y) is

$$T(x, y) = 2x^2 - x\ y + y^2 - 2y + 1$$

(in degrees Celsius). At what point in R or on its boundary is T maximized? What are the temperatures?

Exercise 4.21. Find the maximum and minimum values for the given function in the given closed region.

- $z = 8x^2 + 4y^2 + 4y + 5$, $x^2 + y^2 \leq 1$
- $z = 6x^2 + y^3 + 6y^2$, $x^2 + y^2 \leq 25$
- $z = 8x^2 - 24xy + y^2$, $x^2 + y^2 \leq 25$

Exercise 4.22. Find the volume of the largest box that cane inscribed in the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Exercise 4.23. Assuming that the function

$$F = 2x^2 + 6y^2 + 45z^2 - 4xy + 6yz + 12xz - 6y + 14$$

has a minimum, find it. Prove that this function has no maximum value.

4.11 Lagrange Multipliers

In many applied problems, the main focus is on optimizing a function subject to constraint; for example, finding extreme values of a function of several variables where the domain is restricted to a level curve (or surface) of another function of several variables. **Lagrange multipliers** are a general method which can be used to solve such optimization problems.

4.12 Lagrange's Theorem

The λ in 4.12 is called a **Lagrange multiplier.** Thus, Lagrange's Theorem gives necessary conditions for the existence of a Lagrange multiplier.

::: {#thm- } [Lagrange]

Assume that f and g have continuous first partial derivatives and that f has an extremum at $P_0\left(x_0, y_0\right)$ on the smooth constraint curve $g(x, y) = c$. If $\nabla g\left(x_0, y_0\right) \neq \vec{0}$, there is a number λ such that

$$\nabla f\left(x_0, y_0\right) = \lambda \nabla g\left(x_0, y_0\right). \tag{4.2}$$

:::

Proof. Denote the constraint curve $g(x, y) = c$ by C and note that C is smooth. We represent this curve by the vector function

$$\vec{R}(t) = x(t)\vec{i} + y(t)\vec{j}$$

for all t in an open interval I, including t_0 corresponding to P_0, where $x'(t)$ and $y'(t)$ exist and are continuous. Let $F(t) = f(x(t), y(t))$ for all t in I, and apply the chain rule to obtain

$$F'(t) = f_x(x(t), y(t))\frac{dx}{dt} + f_y(x(t), y(t))\frac{dy}{dt} = \nabla f(x(t), y(t)) \cdot \vec{R}'(t).$$

Because $f(x, y)$ has an extremum at P_0, we know that $F(t)$ has an extremum at t_0. Therefore, we have $F'(t_0) = 0$ and

$$F'(t_0) = \nabla f(x(t_0), y(t_0)) \cdot \vec{R}'(t_0) = 0.$$

If $\nabla f(x(t_0), y(t_0)) = \vec{0}$, then $\lambda = 0$, and the condition $\nabla f = \lambda \nabla g$ is satisfied trivially. If $\nabla f(x(t_0), y(t_0)) \neq \vec{0}$, then $\nabla f(x(t_0), y(t_0))$ is orthogonal to $\vec{R}'(t_0)$. Because $\vec{R}'(t_0)$ is tangent to the constraint curve C, it follows that $\nabla f(x_0, y_0)$ is normal to C. But $\nabla g(x(t_0), y(t_0))$ is also normal to C (because C is a level curve of g), and we conclude that ∇f and ∇g must be parallel at P_0. Thus, there is a scalar λ such that 4.2 holds.

□

Example 4.30. Find the extreme values of the function $f(x, y) = x^2 + y$ constrained to the circle $x^2 + y^2 = 1$.

Solution. Using 4.12, we solve the equations $\nabla f = \lambda \nabla g$, $g(x, y) = 1$, which can be written as $f_x = \lambda g_x$, $f_y = \lambda g_y$, and $g(x, y) = 1$ or written as

$$2x = 2x\lambda, \qquad 1 = 2y\lambda, \qquad \text{and} \qquad x^2 + y^2 = 1.$$

From the first equation we have $x = 0$ or $\lambda = 1$. If $x = 0$, then by the third equation $y = \pm 1$. If $\lambda = 1$, then $y = 1/2$ and we obtain $x = \pm\sqrt{3}/2$. Therefore, f has possible extreme values at the points

$$(0, 1), \qquad (0, -1), \qquad \left(\sqrt{3}/2, 1/2\right), \qquad \text{and} \qquad \left(-\sqrt{3}/2, 1/2\right).$$

Evaluating f at these four points, we find that

$$f(0, 1) = 1, \qquad f(0, -1) = -1, \qquad \text{and} \qquad f\left(\pm\sqrt{3}/2, 1/2\right) = 5/4.$$

Therefore the maximum value of f constrained to the circle $x^2 + y^2 = 1$ is $5/4$ and the minimum value is -1.

4.13 The Method of Lagrange Multipliers

::: {#thm- } Lagrange Multipliers

Suppose f and g satisfy the hypotheses of Lagrange's theorem, and that f has an extremum subject to the constraint $g(x, y) = c$. Then to find the extreme value, proceed as follows:

- Simultaneously solve the following three equations for x, y and λ:

$$f_x(x,y) = \lambda g_x(x,y), \quad f_y(x,y) = \lambda g_y(x,y), \quad \text{and} \quad g(x,y) = c$$

- Evaluate f at all points found in step (i). The extremum we seek must be among these values.

:::

%\begin{comment} We use the method of Lagrange multipliers to find the required constrained extrema. Suppose E is an extreme value of f subject to the constraint $g(x,y) = c$. Then the Lagrange multiplier λ is the rate of change of E with respect to c; that is $\lambda = dE/dc$. Note that at the extreme value (x,y) we have $f_x = \lambda g_x$, $f_y = \lambda g_y$, and $g(x,y) = c$. The coordinates of the optimal ordered pair (x,y) depend on c (because different constraint levels will generally lead to different optimal combinations of x and y). Thus, $E = E(x,y)$ where x and y are functions of c. By the chain rule for partial derivatives:

$$\frac{dE}{dc} = f_x\frac{dx}{dc} + f_y\frac{dy}{dc} = \lambda \mathrm{g}_x\frac{dx}{dc} + \lambda \mathrm{g}_y\frac{dy}{dc} = \lambda\left(\frac{dg}{dc}\right) = \lambda$$

%\end{comment}

Example 4.31. Maximize $f(x,y) = xy$ subject to $2x + 2y = 5$.

Solution. Let $g(x,y) = 2x + 2y$, then we have

$$f_x = y, \qquad f_y = x, \qquad g_x = 2, \qquad \text{and} \qquad g_y = 2.$$

We need to solve the system

$$y = 2\lambda, \qquad x = 2\lambda, \qquad \text{and} \qquad 2x + 2y = 5.$$

We find $x = y = \frac{5}{4}$. Therefore, $f\left(\frac{5}{4}, \frac{5}{4}\right) = \frac{25}{16}$ is the constrained maximum.

Example 4.32. Minimize $f(x,y,z) = x^2 + y^2 + z^2$ subject to $x - 2y + 3z = 4$.

Solution. Let $g(x,y,z) = x - 2y + 3z$, then we have

$$f_x = 2x, \quad f_y = 2y, \quad f_z = 2z, \quad g_x = 1, \quad g_y = -2, \quad \text{and} \quad g_z = 3.$$

We need to solve the system

$$2x = \lambda, \qquad 2y = -2\lambda, \qquad 2z = 3\lambda, \qquad \text{and} \qquad x - 2y + 3z = 4.$$

We find that $\lambda = \frac{4}{7}$ and then $x = \frac{2}{7}$, $y = -\frac{4}{7}$, and $z = \frac{6}{7}$. Therefore,

$$f\left(\frac{2}{7}, -\frac{4}{7}, \frac{6}{7}\right) = \frac{8}{7}$$

is the constrained minimum.

Example 4.33. A rectangular box with no top is to be constructed from 96 ft^2 of material. What should be the dimensions of the box if it is to enclose maximum volume?

Solution. Let x, y, and z be the length, width, and height of the rectangular box, respectively. We want to maximize the volume:

$$V = xyz \qquad \text{subject to} \qquad S(x, y, z) = xy + 2xz + 2yz = 96$$

which is obtained from the surface area of the rectangular box (with no lid). We have

$$V_x = yz, \quad V_y = xz, \quad V_z = xy, \quad S_x = y+2z, \quad S_y = x+2z, \quad S_z = 2x+2y.$$

Solve the system

$$yz = \lambda(y+2z), \quad xz = \lambda(x+2z), \quad xy = \lambda(2x+2y), \quad xy+2xz+2yz = 96.$$

We obtain $x = y = 2z$, and then find $x = y = 4\sqrt{2}$, $z = 2\sqrt{2}$. Therefore the maximum volume is

$$V\left(4\sqrt{2}, 4\sqrt{2}, 2\sqrt{2}\right) = 64\sqrt{2}\text{ft}^3.$$

Example 4.34. A cylindrical can is to hold $4\pi\text{in.}^3$ of orange juice. The cost per square inch of constructing the metal top and bottom is twice the cost per square inch of constructing the cardboard side. What are the dimensions of the least expensive can?

Solution. Let x and y be the radius and height of the cylinder, respectively. We want to minimize the cost

$$f(x, y) = 2\left(2\pi x^2\right)+2\pi xy \quad \text{subject to the constraint} \quad g(x, y) = \pi x^2 y = 4\pi.$$

We have

$$f_x = 8\pi x + 2\pi y, \qquad f_y = 2\pi x, \qquad g_x = 2\pi xy, \qquad \text{and} \qquad g_y = \pi x^2.$$

Solving the system

$$8\pi x + 2\pi y = 2\lambda\pi xy, \qquad 2\pi x = \lambda\pi x^2, \qquad \pi x^2 y = 4\pi.$$

we obtain $y = 4x$, and then find the radius $x = 1$ in. and the height $y = 4$ in.

4.14 Optimizing a Function Subject to Two Constraints

The method of Lagrange multipliers can also be applied in situations with more than one constraint equation. Suppose we wish to locate an extremum of a function defined by $f(x,y,z)$ subject to constraints $g(x,y,z)=c_1$ and $h(x,y,z)=c_2$, where g and h are also differentiable and ∇g and ∇h are not parallel. By generalizing Lagrange's theorem, it can be shown that if (x_0,y_0,z_0) is the desired extremum, then there are numbers λ and μ such that $g(x_0,y_0,z_0)=c_1, h(x_0,y_0,z_0)=c_2$, and

$$\nabla f(x_0,y_0,z_0)=\lambda\nabla g(x_0,y_0,z_0)+\mu\nabla h(x_0,y_0,z_0).$$

As in the case of one constraint, we proceed by first solving this system of equations simultaneously to find $\lambda,\mu,x_0,y_0,\text{and}z_0$ and then evaluating $f(x,y,z)$ at each solution and comparing to find the assumed extremum.

Example 4.35. Find the maximum value of the function

$$f(x,y,z)=x+2y+3z$$

on the curve of intersection of the plane $x-y+z=1$ and the cylinder $x^2+y^2=1$.

Solution. We maximize the function$f(x,y,z)=x+2y+3z$ subject to the constraints $g(x,y,z)=x-y+z=1$ and $h(x,y,z)=x^2+y^2=1$. The Lagrange condition is $\nabla f=\lambda\nabla g+\mu\nabla h$, so we solve the equations

$$1=\lambda+2x\mu,\quad 2=-\lambda+2,\quad y\mu 3=\lambda,\quad x-y+z=1,\quad \text{and}\quad x^2+y^2=1.$$

Putting $\lambda=3$, we get $2x\mu=-2$, so $x=-1/\mu$. Similarly, we have $y=5/(2\mu)$. Substitution yields

$$\frac{1}{\mu^2}+\frac{25}{4\mu^2}=1$$

and so $u^2=\frac{29}{4}$. Then

$$\mu=\pm\sqrt{29}\big/2,\qquad x=\mp 2\big/\sqrt{29}\,,\qquad \text{and}\qquad y=\pm 5\big/\sqrt{29}$$

and so we have $z=1-x+y=1\pm 7\big/\sqrt{29}$. The corresponding values of f are

$$\mp\frac{2}{\sqrt{29}}+2\left(\pm\frac{5}{\sqrt{29}}\right)+3\left(1\pm\frac{7}{\sqrt{29}}\right)=3\pm\sqrt{29}.$$

Therefore the maximum of f on the given curve is $3+\sqrt{29}$.

Example 4.36. Find the maximum of $f(x,y,z) = xyz$ subject to $x^2 + y^2 = 3$ and $y = 2z$.

Solution. We need to solve the system $\nabla f = \lambda \nabla g + \mu \nabla h$ where $g(x,y,z) = x^2 + y^2$ and $h(x,y,z) = y - 2z$. Therefore we need to solve the system

$$\begin{cases} f_x - \lambda g_x - \mu h_x = 0 \\ f_y - \lambda g_y - \mu h_y = 0 \\ f_z - \lambda g_z - \mu h_z = 0 \\ x^2 + y^2 - 3 = 0 \\ y - 2z = 0 \end{cases} \quad \text{which is written as} \quad \begin{cases} yz - 2\lambda x = 0 \\ xz - 2\lambda y + \mu = 0 \\ xy + 2\mu = 0 \\ y - 2z = 0 \\ x^2 + y^2 - 3 = 0. \end{cases}$$

The solutions are $(0, 0, \pm\sqrt{3})$, and

$$\left(1, \sqrt{2}, \frac{\sqrt{2}}{2}\right), \quad \left(1, -\sqrt{2}, -\frac{\sqrt{2}}{2}\right), \quad \left(-1, \sqrt{2}, \frac{\sqrt{2}}{2}\right), \quad \left(-1, -\sqrt{2}, -\frac{\sqrt{2}}{2}\right).$$

The maximum is $f\left(1, \sqrt{2}, \frac{\sqrt{2}}{2}\right) = f\left(1, -\sqrt{2}, -\frac{\sqrt{2}}{2}\right) = 1$.

Example 4.37. Find the minimum of

$$f(x,y,z) = x^2 + y^2 + z^2$$

subject to $x + y = 4$ and $y + z = 6$.

Solution. We want to minimize $x^2 + y^2 + z^2$ subject to the side conditions $x + y - 4 = 0$ and $y + z - 6 = 0$. We form

$$L(x,y,z,\lambda,\mu) = x^2 + y^2 + z^2 - \lambda(x + y - 4) - \mu(y + z - 6).$$

The conditions are $\frac{\partial L}{\partial x} = 2x - \lambda = 0$ and

$$\begin{aligned} \frac{\partial L}{\partial y} &= 2y - \lambda - \mu = 0 & \qquad \frac{\partial L}{\partial z} &= 2z - \mu = 0 \\ \frac{\partial L}{\partial \lambda} &= x + y - 4 = 0 & \qquad \frac{\partial L}{\partial \mu} &= y + z - 6 = 0. \end{aligned}$$

The first and third conditions give $\lambda = 2x$ and $\mu = 2z$, so the second condition becomes $2y - 2x - 2z = 0$. We then have

$$\begin{cases} x + y + z = 0 \\ x + y = 4 \\ y + z = 6 \end{cases}$$

The solution to this system is $P = (2/3, 10/3, 8/3)$. Therefore the minimum is $f(P) = 56/3$.

Example 4.38. Use Lagrange multipliers to find the point on the line of intersection of the planes $x - y = 2$ and $x - 2z = 4$ that is closest to the origin.

Solution. We want to minimize $x^2+y^2+z^2$ subject to the side conditions $x-y-2=0$ and $x-2z-4=0$. We form

$$L(x,y,z,\lambda,\mu) = x^2+y^2+z^2-\lambda(x-y-2)-\mu(x-2z-4).$$

The conditions are $\frac{\partial L}{\partial x} = 2x-\lambda-\mu = 0$ and

$$\begin{aligned} \frac{\partial L}{\partial y} &= 2y+\lambda = 0 & \frac{\partial L}{\partial z} &= 2z+2\mu = 0 \\ \frac{\partial L}{\partial \lambda} &= -x+y+2 = 0 & \frac{\partial L}{\partial \mu} &= -x+2z+4 = 0. \end{aligned}$$

The second and third conditions give $\lambda = -2y$ and $\mu = -z$, so the first condition becomes $2x+2y+z=0$. We then have $2x+2y+z=0$, $-x+y=-2$, $-x+2z=-4$. The last two equations may be written as $y=x-2$ and $z=(x-4)/2$. Substitution of these values into the first equation gives $x=4/3$. Consequently, $y=-2/3$ and $z=-4/3$. The desired point is therefore, $(4/3,-2/3,-4/3)$.

Example 4.39. Find the maximum and minimum values of

$$f(x,y,z) = 5x-y-6z$$

on the surface $2x^2+4y^2+6z^2=200$.

Solution. We set $g(x,y,z) = 2x^2+4y^2+6z^2$ and we use the Lagrange multiplier λ and solve the system

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ f_z = \lambda g_z \\ g = 200 \end{cases} \quad \text{which is written as} \quad \begin{cases} 5 = 4x\lambda \\ -1 = 8y\lambda \\ -6 = 12z\lambda \\ 2x^2+4y^2+6z^2 = 200 \end{cases}$$

to find

$$P = \left(-10\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}, 4\sqrt{\frac{2}{3}}\right) \quad \text{and} \quad Q = \left(10\sqrt{\frac{2}{3}}, -\sqrt{\frac{2}{3}}, -4\sqrt{\frac{2}{3}}\right)$$

Therefore,
$f(P) = -25\sqrt{6}$ is the minimum and $f(Q) = 25\sqrt{6}$
is the maximum value.

Example 4.40. Maximize

$$f(x,y) = \ln\left(xy^2\right)$$

subject to the constraint $2x^2+3y^2=8$ for $x>0$.

Solution. Let $g(x,y) = 2x^2 + 3y^2$ and we will use a Lagrange multiplier, say λ. We setup the system of equations

$$\begin{cases} \frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \\ \frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} \\ g(x) = 8 \end{cases} \quad \text{which is written as} \quad \begin{cases} \frac{y^2}{xy^2} = 4\lambda x \\ \frac{2xy}{xy^2} = 6\lambda y \\ 2x^2 + 3y^2 = 8 \end{cases}$$

Now to solve this we will try to eliminate λ from the first two equations. So we solve both of them for λ we have,

$$\lambda = \frac{1}{4x^2} \qquad \text{and} \qquad \lambda = \frac{1}{3x^2 y},$$

(since $x > 0$) respectively. Thus, these two expressions must equal yielding

$$\frac{1}{4x^2} = \frac{1}{3x^2 y}$$

we have $4x^2 = 3x^2 y$ which we write as $x^2(4-3y) = 0$ and so $y = 4/3$. So that

$$x^2 = \frac{8 - 3(4/3)^2}{2} = \frac{4}{3}.$$

However, since $x > 0$ we only use $x = \frac{2}{\sqrt{3}}$ with $y = 4/3$; and therefore the maximum value of $f(x,y) = \ln\left(xy^2\right)$ subject to the constraint $g(x,y) = 8$ is

$$f\left(\frac{2}{\sqrt{3}}, \pm\frac{4}{3}\right) = \ln\left(\frac{2}{\sqrt{3}}\left(\frac{4}{3}\right)^2\right) = \frac{5}{2}\ln\left(\frac{4}{3}\right).$$

4.15 Exercises

Exercise 4.24. Use Lagrange multipliers to find the point on the line of intersection of the planes $x - y = 2$ and $x - 2z = 4$ that is closest to the origin.

Exercise 4.25. Use the method of Lagrange multipliers to maximize or minimize each of the functions.

- maximize $f(x,y) = xy$ subject to $2x + 2y = 5$
- minimize $f(x,y) = xyz$ subject to $3x + 2y + z = 6$
- maximize $f(x,y) = e^{xy}$ subject to $x^2 + y^2 = 3$
- maximize $f(x,y) = \cos x + \cos y$ subject to $y = x + \frac{\pi}{4}$
- minimize $f(x,y) = x^2 - xy + 2y^2$ subject to $2x + y = 22$
- minimize $f(x,y) = x^2 - y^2$ subject to $x^2 + y^2 = 4$
- minimize $f(x,y,z) = 2x^2 + 3y^2 + 4z^2$ subject to $x + y + z = 4$ and $x - 2y + 5z = 3$

- minimize $f(x, y, z) = x^2 + y^2 + z^2$ subject to $x + y = 4$ and $y + z = 6$
- minimize $f(x, y, z) = xyz$ subject to $x^2 + y^2 = 3$ and $y = 2z$
- maximize $f(x, y, z) = xy + xz$ subject to $2x + 3z = 5$ and $xy = 4$

Exercise 4.26. Find the point on the plane $2x - 3y + 6z = 5$ that is closest to the origin.

Exercise 4.27. Find the point on the plane $3x - 2y + z = 5$ that is closet to $(4, -8, 5)$.

Exercise 4.28. A rectangular box has a square base and one top. Find the dimensions for minimum surface area the volume is to be 108 in^3.

Exercise 4.29. An open rectangular box has ends costing $\$6/\text{ft}^2$, sides costing $\$4/\text{ft}^2$, and a bottom costing $\$10/\text{ft}^2$. Find the dimensions for the minimum cost if the volume of the box is 120 ft^3.

Exercise 4.30. Find the maximum possible volume of a right circular cone inscribed in a sphere of radius a.

Exercise 4.31. The sides of a closed cylindrical container cost twice as much per square food as the ends. Find the ratio of the radius to the altitude of the cylinder for the cheapest such container having fixed volume.

Exercise 4.32. A pentagon consists of a rectangle surmounted by an isosceles triangle. Find the dimensions of the pentagon having the maximum area if the perimeter is to be P.

Exercise 4.33. Find the point on the curve of intersection of $x^2 + z^2 = 4$ and $x - y = 8$ that is farthest form the origin.

Exercise 4.34. Find the point on the line of intersection of the planes $x - y = 4$ and $y + 3z = 6$ that is closet to $(-1, 3, 2)$.

Chapter 5

Double Integrals

Multivariable calculus is a critical tool for solving problems in physics, engineering, and other sciences. However, before delving into the higher-dimensional world of multivariable calculus, it's important to have a strong foundation in the basics.

In this book, we focus on the fundamentals of multiple integration-the processes of integrating functions of several variables. We'll discuss both Cartesian and polar coordinate systems, along with the various techniques for evaluating integrals. With a solid understanding of these basic concepts, you'll be ready to tackle the more challenging problems in multivariable calculus.

Multivariable calculus is the study of functions of multiple variables. It allows for the determination of maxima and minima, as well as other important information about how a function behaves. In many cases, multiple variables can be treated as a single variable, allowing for the use of traditional calculus methods. However, in other cases, multiple variables require a more sophisticated approach.

This is where multivariable calculus comes in. Multivariable calculus is particularly important in physics and engineering, where multiple physical quantities often need to be considered simultaneously. It is also useful in economics, where multiple factors can affect the behavior of a market or individual consumers. In short, multivariable calculus is a powerful tool that can be used to analyze complicated situations involving multiple variables.

Double integrals are a type of multiple integration, meaning they allow you to integrate over multiple variables simultaneously. In other words, double integrals let you find the total amount of something (like area or volume) that is spread out over multiple dimensions.

To calculate a double integral, you first need to choose an appropriate coordinate system. Once you've done that, you can divide the region into smaller pieces and then add up the integrals over each of those pieces. Double integrals can be used to calculate all sorts of things, from the volume of an irregular solid to the probability of two random variables taking on certain values. So whatever it is you're trying to integrate, a double integral is probably the way to go!

Integrals are a fundamental tool in calculus, and iterated integrals are a powerful extension of this concept. With multiple integration, we can calculate the volume of a solid region bounded by multiple surfaces. This technique can also be used to calculate the moments of a 3D object, or to find the center of mass of an irregular shape.

In short, iterated integrals are a versatile tool that can be used in a wide range of applications. So if you're ever feeling lost in a sea of multiple integrals, just remember that each one is just an extension of the basic concept of integration. And with a little practice, you'll be iterating your way to success in no time!

Double integrals in polar coordinates can be a bit of a pain to wrap your head around. After all, multiple integration is already confusing enough - now you have to do it in a different coordinate system? But once you understand the basics, it's not so bad. And who knows, you might even find it fun.

To understand multiple integrals in polar coordinates, it helps to think about a two-dimensional coordinate system like a cake. The x-axis is like the diameter of the cake, and the y-axis is like the height. If we want to find the volume of the cake, we need to integrate both the diameter and the height. That's where multiple integrals come in.

With multiple integrals in polar coordinates, we're essentially doing the same thing - but with a twist. Instead of using a rectangular coordinate system, we're using a polar coordinate system. This means that our x-axis is now the angle, and our y-axis is now the radius. And just like before, if we want to find the volume (or more accurately, the area) of our cake, we need to integrate both the angle and the radius.

Integrals are a fundamental tool in calculus, and the double integral is a powerful extension of the single integral. Double integrals can be used to calculate the area of irregular shapes, the volume of solids with curved surfaces, and the amount of flow through an orientable surface. In fact, multiple integrals can be applied in countless ways to solve problems in physics, engineering, and other fields.

In many ways, multiple integrals are the natural extension of single integrals. Just as a single integral can be thought of as the area under a curve, a double integral can be thought of as the volume under a surface.

More generally, multiple integrals can be used to calculate the measure of any region in space. This makes them a powerful tool for solving problems in physics and engineering. In addition, multiple integrals can be used to calculate more complex quantities than their single-variable counterparts.

For example, double integrals can be used to calculate moments and centroids. As a result, multiple integrals are an essential tool for anyone looking to tackle problems in mathematics or the sciences.

So next time you're struggling with a difficult integral, remember that you're not alone - and that there's probably a multiple integral out there that can help.

Triple integrals are multiple integrals where the domain of integration is three-dimensional. In other words, they're integrals over three variables. These integrals can be difficult to calculate, but using cylindrical or spherical coordinates can make the process much simpler. Cylindrical coordinates are particularly well-suited for problems involving cylindrical objects, such as cylinders and cones. Spherical coordinates are best for problems involving spheres or other round objects.

We've all been there. We're given a multiple integral to solve, and we just don't know where to start. Should we use cylindrical coordinates? Or spherical coordinates? How do we even set up the integral in those coordinate systems? And what if the region of integration isn't nice and simple like a box or a cylinder?

Thanks to the magic of triple integrals, we can solve multiple integrals no matter what the coordinate system or the shape of the region. In this article, we'll look at how to set up triple integrals in both cylindrical and spherical coordinate systems.

With a little practice, you'll be able to tackle any multiple integral, no matter how complicated it might seem at first. Either way, these multiple integrals can be tricky, but with a little practice, you'll be a pro in no time!

Multiple integration is a powerful tool that can be used to calculate integrals of many different functions. However, it is sometimes necessary to change the variables in a multiple integral in order to make the calculation easier. This process is known as a change of variables.

There are many different ways to change the variables in a multiple integral, but the most common method is to use a substitution. This involves replacing the original variables with new ones that are more suitable for the particular problem.

For example, if we want to integrate a function that is only defined on a certain region of space, we can use a change of variables to transform the integral into one that is defined on all of space. This can be a very useful technique when dealing with multiple integrals.

Multivariable calculus may sound like a mouthful, but it's just a fancy way of saying "calculus with more than one variable." And as anyone who's taken calculus knows, one variable is already plenty. So what's the big deal with multiple variables? Well, for starters, multiple integration can be a real pain. You've got multiple integrals, double integrals, triple integrals... it can all get pretty confusing pretty quickly.

And then there's the issue of finding critical points. With multiple variables comes multiple partial derivatives, and keeping track of them all can be a challenge. But don't despair! These challenges can be overcome with practice and patience. After all, there's nothing wrong with taking things one step at a time. Just remember: take a deep breath, relax, and trust in the process. You'll get there eventually.

I hope that this tour has given you a better understanding of how I teach calculus in this book. I take a step-by-step approach, breaking down each concept and providing plenty of examples along the way. My goal is to make the process as easy and painless as possible, so you can focus on learning the material and not worrying about the mechanics of the math. With a little practice, you'll be able to tackle any calculus problem that comes your way!

5.1 The Volume Under a Surface

Consider the rectangle given by

$$R = [a, b] \times [c, d] = \{(x, y) \mid a \leq x \leq b, c \leq x \leq d\}.$$

We wish to construct a (regular) partition of R. To do so, let

$$a = x_0 < x_1 < x_2 < \cdots < x_{i-1} < x_i < \cdots < x_n = b$$

be a partition of $[a, b]$ into subintervals each of width $\Delta x = (b - a)/n$. Similarly let

$$c = y_0 < y_1 < y_2 < \cdots < y_{j-1} < y_j < \cdots < y_m = d$$

be a partition of $[c, d]$ into subintervals each of width $\Delta y = (d - c)/m$. The vertical lines $x = x_i$ for $0 \leq i \leq n$ and the horizontal lines $y = y_j$ for $0 \leq j \leq m$ form subrectangles which partition R as shown in **??**. Since each of the subrectangles have the same area, namely $\Delta A = \Delta x \, \Delta y$ we call this type of partition $\mathscr{P}$ a **regular partition** of the rectangle R.

For example, let $R = [-1, 2] \times [-2, 2]$. In **??** we have sketched the graph of the regular partition of R with $n = 6$ subintervals of $[-1, 2]$ and $m = 8$

subintervals of $[-2, 2]$. For this partition we have $\Delta x = 1/2$ and $\Delta y = 1/2$, making the area for each subrectangle $\Delta A = 1/4$.

If we increase the number of subintervals of both $[a, b]$ and $[c, d]$, then the number of subrectangles of R increases. This process is illustrated in **??**.

Now we assume that we have a function f of two variables whose domain contains a rectangle R and that we have a regular partition $\mathcal{P}$ of R as described above. From each subrectangle in $\mathcal{P}$ we choose a **representative point** $\left(x^*_{ij}, y^*_{ij}\right)$ and form the sum,

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f\left(x^*_{ij}, y^*_{ij}\right)\Delta A \tag{5.1}$$

where ΔA is the area of each subrectangle. A summation of this form is called a **Riemann sum** of f with respect to the partition $\mathcal{P}$ and the subrectangle representatives (x^*_{ij}, y^*_{ij}). Notice that a regular partition of R is completely determined by n and m.

The next three examples are an illustration of how Riemann sums can be used to estimate the volume of the 3-dimensional solid region over the xy-plane bound below by R and above by the surface $z = f(x, y)$ (see **??**).

Example 5.1. Consider the surface and the rectangle

$$z = 5 - \frac{1}{4}x^2 - \frac{1}{5}y^2 \qquad R = \{(x, y) \mid 0 \leq x \leq 2, 0 \leq y \leq 3\} \tag{5.2}$$

Using the lower left corners as subrectangle representatives, determine the Riemann sums when $n = 2$ and $m = 3$. :::

Solution. When $n = 2$ and $m = 3$ we have

$$\Delta x = \frac{2-0}{2} = 1, \qquad \Delta y = \frac{3-0}{2} = 1, \qquad \text{and} \qquad \Delta A = \Delta x \Delta y = 1.$$

We are choosing the lower left corners as subrectangle representatives (using $0 < 1 < 2$ and $0 < 1 < 2 < 3$), that is,

$$\begin{aligned}
&(x^*_{11}, y^*_{11}) = (0, 0) \qquad &&(x^*_{12}, y^*_{12}) = (0, 1) \qquad &&(x^*_{13}, y^*_{13}) = (0, 2)\\
&(x^*_{21}, y^*_{21}) = (1, 0) \qquad &&(x^*_{22}, y^*_{22}) = (1, 1) \qquad &&(x^*_{24}, y^*_{23}) = (1, 2)
\end{aligned}$$

as depicted in **??**. Now we find the Riemann sum for this partition and chosen representatives:

$$\sum_{i=1}^{2}\sum_{j=1}^{3} f\left(x_{ij}^*, y_{ij}^*\right)\Delta A = \sum_{i=1}^{2}\sum_{j=1}^{3} f\left(x_{ij}^*, y_{ij}^*\right)$$

$$= \sum_{i=1}^{2}\left[f\left(x_{i1}^*, y_{i1}^*\right) + f\left(x_{i2}^*, y_{i2}^*\right) + f\left(x_{i3}^*, y_{i3}^*\right)\right]$$

$$= f\left(x_{11}^*, y_{11}^*\right) + f\left(x_{12}^*, y_{12}^*\right) + f\left(x_{13}^*, y_{13}^*\right) + f\left(x_{21}^*, y_{21}^*\right) + f\left(x_{22}^*, y_{22}^*\right) + f\left(x_{23}^*, y_{23}^*\right)$$

$$= f(0,0) + f(0,1) + f(0,2) + f(1,0) + f(1,1) + f(1,2)$$

$$= 5 + \frac{24}{5} + \frac{21}{5} + \frac{19}{4} + \frac{91}{20} + \frac{79}{20} = \frac{109}{4} = 27.25.$$

Example 5.2. Consider the surface and the rectangle

$$z = 5 - \frac{1}{4}x^2 - \frac{1}{5}y^2 \qquad R = \{(x,y) \mid 0 \le x \le 2, 0 \le y \le 3\} \tag{5.3}$$

Using the lower left corners as subrectangle representatives, determine the Riemann sums when $n = 4$ and $m = 6$. :::

Solution. When $n = 4$ and $m = 6$ we have

$$\Delta x = \frac{2-0}{4} = \frac{1}{2}, \qquad \Delta y = \frac{3-0}{6} = \frac{1}{2}, \qquad \text{and} \qquad \Delta A = \Delta x \Delta y = \frac{1}{4}$$

Again using lower left corners as subrectangle representatives (see **??**) we find the Riemann sum for this partition to be:

$$\sum_{i=1}^{4}\sum_{j=1}^{6} f\left(x_{ij}^*, y_{ij}^*\right)\Delta A = \frac{1}{4}\left[\sum_{i=1}^{4}\sum_{j=1}^{6} f\left(x_{ij}^*, y_{ij}^*\right)\right]$$

$$= \frac{1}{4}\Bigg[f(0,0) + f\left(0,\frac{1}{2}\right) + f(0,1) + f\left(0,\frac{3}{2}\right) + f(0,2) + f\left(0,\frac{5}{2}\right)$$

$$+ f\left(\frac{1}{2},0\right) + f\left(\frac{1}{2},\frac{1}{2}\right) + f\left(\frac{1}{2},1\right) + f\left(\frac{1}{2},\frac{3}{2}\right) + f\left(\frac{1}{2},2\right) + f\left(\frac{1}{2},\frac{5}{2}\right)$$

$$+ f(0,1) + f\left(1,\frac{1}{2}\right) + f(1,1) + f\left(1,\frac{3}{2}\right) + f(1,2) + f\left(1,\frac{5}{2}\right)$$

$$+ f\left(\frac{3}{2},0\right) + f\left(\frac{3}{2},\frac{1}{2}\right) + f\left(\frac{3}{2},1\right) + f\left(\frac{3}{2},\frac{3}{2}\right) + f\left(\frac{3}{2},2\right) + f\left(\frac{3}{2},\frac{5}{2}\right)\Bigg]$$

$$= \left(\frac{1}{4}\right)\frac{415}{4} = 25.9375.$$

Example 5.3. Consider the surface and the rectangle

$$z = 5 - \frac{1}{4}x^2 - \frac{1}{5}y^2 \qquad R = \{(x, y) \mid 0 \leq x \leq 2, 0 \leq y \leq 3\} \tag{5.4}$$

Using the lower left corners as subrectangle representatives, determine the Riemann sums when $n = 6$ and $m = 12$. :::

Solution. When $n = 6$ and $m = 12$ we have

$$\Delta x = \frac{2-0}{6} = \frac{1}{3}, \qquad \Delta y = \frac{3-0}{12} = \frac{1}{4}, \qquad \text{and} \qquad \Delta A = \Delta x \Delta y = \frac{1}{12}$$

Using lower left corners as subrectangle representatives we find the Riemann sum for this partition to be

$$\begin{aligned}
\sum_{i=1}^{6}\sum_{j=1}^{12} f\left(x_{ij}^*, y_{ij}^*\right)\Delta A &= \frac{1}{12}\left[\sum_{i=1}^{6}\sum_{j=1}^{12} f\left(x_{ij}^*, y_{ij}^*\right)\right] \\
&= \frac{1}{12}\left[f(0,0) + \cdots + f\left(0, \frac{11}{4}\right) + f\left(\frac{1}{3}, 0\right) + \cdots + f\left(\frac{1}{3}, \frac{11}{4}\right)\right. \\
&\qquad \left. + \cdots + f\left(\frac{4}{3}, 0\right) + \cdots + f\left(\frac{4}{3}, \frac{11}{4}\right) + f\left(\frac{5}{3}, 0\right) + \cdots + f\left(\frac{5}{3}, \frac{11}{4}\right)\right] \\
&= \frac{1}{12}\left(\frac{18223}{60}\right) = \frac{14923}{60} = 25.3097.
\end{aligned}$$

As depicted in **??**, the volume of the solid region that lies directly above R and below the surface $z = f(x, y)$ is approximately 25.3097.

Definition 5.1. Let f be defined on the rectangle R and suppose that $f(x, y) \geq 0$ on R. Then the volume V of the solid region that lies directly above R and below the surface $z = f(x, y)$ is

$$V = \lim_{m,n\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_{ij}^*, y_{ij}^*)\,\Delta A \tag{5.5}$$

if this limit exists.

Midpoint Rule

It can be proven that if f is a continuous function on R, then the limit in 5.5 always exists, no matter how the subrectangle representatives (x_{ij}^*, y_{ij}^*) are chosen. For example, if $a = x_0 < x_1 < x_2 < \cdots < x_{i-1} < x_i < \cdots < x_n = b$ is a regular partition of $[a, b]$ and $c = y_0 < y_1 < y_2 < \cdots <$

$y_{j-1} < y_j < \cdots < y_m = d$ is a regular partition of $[c, d]$, then we can use midpoints, namely, from each subrectangle $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ we choose

$$(x_{ij}^*, y_{ij}^*) = \left(\frac{\overbrace{(a + \Delta x(i-1))}^{\text{left endpoint}} + \overbrace{(a + (\Delta x)i)}^{\text{right endpoint}}}{2}, \frac{\overbrace{(c + \Delta y(j-1))}^{\text{lower endpoint}} + \overbrace{(c + (\Delta y)j)}^{\text{upper endpoint}}}{2} \right)$$

as subrectangle representatives. This is called the **midpoint rule** .

Example 5.4. Consider the surface and the rectangle

$$z = 2 + \frac{1}{2}x^2 + \frac{1}{3}y^2 \qquad R = \{(x, y) \mid -1 \le x \le 2, -3 \le y \le 4\} \tag{5.6}$$

Use the midpoint rule to find the Riemann sum determined by the regular partition of R with $n = 12$ and $m = 28$ to estimate the volume of the solid that lies under the graph of the surface $z = f(x, y)$ and directly above the rectangle R.

Solution. When $n = 12$ and $m = 28$ we have

$$\Delta x = \frac{2 - (-1)}{12} = \frac{1}{4}, \quad \Delta y = \frac{4 - (-3)}{28} = \frac{1}{4}, \quad \text{and} \quad \Delta A = \Delta x \Delta y = \frac{1}{16}.$$

The subrectangle representations are:

$$\begin{aligned}(x_{ij}^*, y_{ij}^*) &= \left(\frac{(-1 + \frac{1}{4}(i-1)) + (-1 + \frac{1}{4}i)}{2}, \frac{-3 + \frac{1}{4}(j-1)) + (-3 + \frac{1}{4}j)}{2} \right) \\ &= \left(-\frac{9}{8} + \frac{1}{4}i, -\frac{25}{8} + \frac{1}{4}j \right)\end{aligned}$$

Using midpoints we find the Riemann sum for this partition to be:

$$\begin{aligned}\sum_{i=1}^{12}\sum_{j=1}^{28} f\left(x_{ij}^*, y_{ij}^*\right)\Delta A &= \frac{1}{16}\left[\sum_{i=1}^{12}\sum_{j=1}^{28} f\left(-\frac{9}{8} + \frac{1}{4}i, -\frac{25}{8} + \frac{1}{4}j\right)\right] \\ &= \frac{1}{16}\left[f\left(-\frac{7}{8}, -\frac{23}{8}\right) + f\left(-\frac{7}{8}, -\frac{21}{8}\right) + f\left(-\frac{7}{8}, -\frac{19}{8}\right)\right. \\ &\qquad \left. + \cdots + f\left(\frac{15}{8}, \frac{27}{8}\right) + f\left(\frac{15}{8}, \frac{29}{8}\right) + f\left(\frac{15}{8}, \frac{31}{8}\right)\right] \\ &= \frac{1}{16}\left(\frac{10591}{8}\right) = 82.7421875.\end{aligned}$$

Definition of Double Integral

In our previous examples, we used regular partitions to form Riemann sums for functions defined over rectangular regions. This is not necessity.

To measure the size of the rectangles in the partition P, we define the **norm** $\|P\|$ of the partition to be the length of the longest diagonal of any of the subrectangles in the partition. Refine a partition P by subdividing the cells in such a way that the norm decreases. When this process is applied to the Riemann sum and the norm decreases to zero we have the **double integral** of f over R.

Definition 5.2. If a function f is defined on a closed, bounded rectangular region R in the xy-plane, then the double integral of f over R is defined by

$$\iint_R f(x,y)dA = \lim_{\|P\|\to 0} \sum_{k=1}^{N} f\left(x_k^*, y_k^*\right) \triangle A_k$$

provided this limit exists, in which case, f is said to be **integrable** over R.

Let $f(x,y)$ be a function that is continuous on the region D that can be contained in a rectangle R. Define the function $F(x,y)$ on R as $f(x,y)$ if (x,y) is in D and 0 otherwise. If F is integrable over R, we say that f is **integrable** over D, and the double integral of f over D is defined as

$$\iint_D f(x,y)dA = \iint_R F(x,y)dA. \tag{5.7}$$

The function $F(x,y)$ may have discontinuities on the boundary of D, but if $f(x,y)$ is continuous on D and the boundary of D is fairly "well behaved", then it can be shown that the double integral of the right side of 5.7 exists and hence that the double integral of the left side of 5.7 exists.

Properties of Double Integrals

Theorem 5.1. *Assume that all the given integrals exist on a rectangular region R.*

- *For constants a and b,*

$$\iint_R \left[af(x,y) + bg(x,y)\right] dA = a\iint_R f(x,y)\, dA + b\iint_R g(x,y)\, dA.$$

- *If $f(x,y) \geq g(x,y)$ throughout a rectangular region R, then*

$$\iint_R f(x,y)dA \geq \iint_R g(x,y)dA.$$

- *If the rectangular region of integration R is subdivided into two (disjoint) subrectangles R_1 and R_2, then*

$$\iint_R f(x,y)dA = \iint_{R_1} f(x,y)dA + \iint_{R_2} f(x,y)dA.$$

Exercises

Exercise 5.1. If f is a constant function, say $f(x,y) = k$ and $R = [a,b] \times [c,d]$ show that

$$\iint_R kdA = k(b-a)(d-c).$$

Exercise 5.2. Approximate the given double integral by dividing the rectangle R with vertices $(0,0)$, $(4,0)$, $(4,2)$, and $(0,2)$ into eight squares and find the sum.

- $\iint_R (x+y)\,dA$
- $\iint_R xy\,dA$
- $\iint_R (x^2+y^2)\,dA$
- $\iint_R \frac{1}{(x+1)(y+1)}\,dA$

Exercise 5.3. Find an approximation for the volume V of the solid lying under the graph of the elliptic paraboloid $z = 8 - 2x^2 - y^2$ and above the rectangular region $R = [0,1] \times [0,2]$. Use a regular partition $\mathscr{P}$ of R with $m = n = 2$, and choose the subrectangular representation point (x_{ij}^*, y_{ij}^*) as indicated.

- The lower left-hand corner of R_{ij}.
- The upper left-hand corner of R_{ij}.
- The lower right-hand corner of R_{ij}.
- The upper right-hand corner of R_{ij}.
- The center of R_{ij}.

Exercise 5.4. Calculate the double Riemann sum of f for the partition of R given by the indicated lines and the given choice of (x_{ij}^*, y_{ij}^*). Also calculate the norm of the partition.

- $f(x,y) = x^2 + 4y$, $R = [0,2] \times [0,3]$, $x = 1, y = 1, y = 2$; (x_{ij}^*, y_{ij}^*) is the upper right corner of R_{ij}
- $f(x,y) = x^2 + 4y$, $R = [0,2] \times [0,3]$, $x = 1, y = 1, y = 2$; (x_{ij}^*, y_{ij}^*) is the center of R_{ij}

- $f(x,y) = xy - y^2$, $R = [0,5] \times [0,4]$, $x = 1, x = 2, x = 3, x = 4, y = 2$; (x_{ij}^*, y_{ij}^*) is the center of R_{ij}
- $f(x,y) = 2x + x^2 y$, $R = [-2,2] \times [-1,1]$, $x = -1, x = 0, x = 1, y = -1/2, y = 0, y = 1/2$; (x_{ij}^*, y_{ij}^*) is the lower left corner of R_{ij}
- $f(x,y) = x^2 - y^2$, $R = [0,5] \times [0,2]$, $x = 1, x = 3, x = 4, y = 1/2, y = 1$; (x_{ij}^*, y_{ij}^*) is the upper left corner of R_{ij}
- $f(x,y) = 5xy^2$, $R = [1,3] \times [1,4]$, $x = 1.8, x = 2.5, y = 2, y = 3$; (x_{ij}^*, y_{ij}^*) is the lower right corner of R_{ij}

Exercise 5.5. Find the Riemann sum

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A$$

of f over the region R with respect to the regular partition $\mathcal{P}$ with the indicated values of m and n.

- $f(x,y) = 2x + 3y$, $R = [0,1] \times [0,3]$, $m = 2, n = 3$; (x_{ij}^*, y_{ij}^*) is the lower left-hand corner of R_{ij}
- $f(x,y) = x^2 - 2y$, $R = [1,5] \times [1,3]$, $m = 4, n = 2$; (x_{ij}^*, y_{ij}^*) is the upper right-hand corner of R_{ij}
- $f(x,y) = x^2 + 2y^2$, $R = [-1,3] \times [0,4]$, $m = 4, n = 2$; (x_{ij}^*, y_{ij}^*) is the upper right-hand corner of R_{ij}
- $f(x,y) = 2xy$, $R = [-1,1] \times [-2,2]$, $m = 4, n = 4$; (x_{ij}^*, y_{ij}^*) is the center of R_{ij}

Exercise 5.6. Evaluate the integral $\iint_R (4 - 2y)\, dA$, where $R = [0,1] \times [0,1]$, by identifying it as the volume of a solid.

Exercise 5.7. The integral

$$\iint_R \sqrt{9 - y^2}\, dA,$$

where $R = [0,4] \times [0,2]$, represents the volume of a solid. Sketch the solid.

Exercise 5.8. If f is a constant function, $f(x,y) = k$, and $R = [a,b] \times [c,d]$, show that

$$\iint_R k\, dA = k(b-a)(d-c).$$

Exercise 5.9. If $R = [0,1] \times [0,1]$, show that

$$0 \leq \iint_R \sin(x+y)\, dA \leq 1.$$

Exercise 5.10. Evaluate $\iint_R y e^{xy}\, dA$ where $R = [0,2] \times [0,1]$.

Iterate Integrals Over Rectangular Regions

::: {#thm- } [Fubini's Theorem for Rectangular Regions] If f is a continuous function of x and y over the rectangle $R : a \leq x \leq b,\ c \leq y \leq d$, then the double integral of f over R may be evaluated by either iterated integral:

$$\iint_R f(x,y) dA = \int_a^b \left[\int_c^d f(x,y)\, dy \right] dx = \int_c^d \left[\int_a^b f(x,y)\, dx \right] dy.$$

Example 5.5. Evaluate the double integral

$$\iint_R x \sin xy\, dA$$

over the region $R = \{(x,y) \mid 0 \leq x \leq \pi,\ 0 \leq y \leq 1\}$.

Solution. Since the integrand is continuous over R we use 5.3 to find

$$\iint_R x \sin xy\, dA = \int_0^\pi \int_0^1 x \sin xy\, dy\, dx = -\int_0^\pi (\cos x - 1)\, dx = \pi. \quad (5.8)$$

See **??** for a visualization.

Example 5.6. Evaluate the double integral

$$\iint_R \left(\frac{4+x^2}{1+y^2} \right) dA$$

over the region $R = \{(x,y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$.

Solution. Since the integrand is continuous over R we use 5.3 to find

$$\iint_R \left(\frac{4+x^2}{1+y^2} \right) dA = \int_0^1 \int_0^1 \left(\frac{4+x^2}{1+y^2} \right) dx dy = \int_0^1 \frac{13}{3\,(y^2+1)}\, dy = \frac{13\pi}{12}. \quad (5.9)$$

See **??** for a visualization.

Example 5.7. Evaluate the double integral

$$\iint_R \frac{2xy}{x^2+1}\,dA$$

over the region $R = \{(x,y) \mid 0 \le x \le 1, 1 \le y \le 3\}$.

Solution. Since the integrand is continuous over R we use 5.3 to find

$$\iint_R \frac{2xy}{x^2+1}\,dA = \int_0^1 \int_1^3 \frac{2xy}{x^2+1}\,dy\,dx = \int_0^1 \frac{8x}{x^2+1}\,dx = 4\ln(2). \tag{5.10}$$

See **??** for a visualization.

Example 5.8. Evaluate the double integral

$$\iint_R \left(x^2 e^{xy}\right) dA$$

over the region $R = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le 1\}$.

Solution. Since the integrand is continuous over R we use 5.3 to find

$$\iint_R \left(x^2 e^{xy}\right) dA = \int_0^1 \int_0^1 x^2 e^{xy} dy dx = \int_0^1 \left(xe^x - x\right)\, dx = \frac{1}{2}.$$

See **??** for a visualization.

Example 5.9. Evaluate the double integral

$$\int_3^4 \int_1^2 \frac{x}{x-y}\,dy\,dx.$$

Solution. Since the integrand is continuous over R we use 5.3 to find

$$\int_3^4 \int_1^2 \frac{x}{x-y}\,dy\,dx = \int_3^4 x(\ln(1-x) - \ln(2-x))\,dx = \frac{1}{2} - 10\ln(2) + \frac{15}{2}\ln(3).$$

See **??** for a visualization.

Example 5.10. Evaluate the double integral

$$\int_2^3 \int_{-1}^2 \frac{1}{(x+y)^2}\,dy\,dx.$$

Solution. Since the integrand is continuous over R we use 5.3 to find

$$\int_2^3 \int_{-1}^2 \frac{1}{(x+y)^2}\, dy\, dx = \int_2^3 \left(\frac{1}{-x-2} - \frac{1}{1-x} \right) dx = \ln(2)+\ln(4)-\ln(5) = \ln\left(\frac{8}{5}\right).$$

See **??** for a visualization.

How does this example not contradiction 5.3?

Example 5.11. Show that the iterated integrals

$$\int_0^1 \int_0^1 \frac{y-x}{(x+y)^3}\, dy\, dx \quad \text{and} \quad \int_0^1 \int_0^1 \frac{y-x}{(x+y)^3}\, dx\, dy$$

have different values.

Solution. The first integral

$$\int_0^1 \int_0^1 \frac{y-x}{(x+y)^3}\, dy\, dx = \int_0^1 -\frac{1}{(x+1)^2}\, dx = -\frac{1}{2}. \tag{5.11}$$

and the second integral

$$\int_0^1 \int_0^1 \frac{y-x}{(x+y)^3}\, dx\, dy = \int_0^1 \frac{1}{(y+1)^2}\, dy = \frac{1}{2}. \tag{5.12}$$

do indeed have different values.

Iterated Integrals Over Non-Rectangular Regions

Let R be a region in the xy-plane. Then R is called a

- **vertically simple region** if R can be described by the inequalities

$$D_1: \quad a \leq x \leq b, \quad g_1(x) \leq y \leq g_2(x)$$

  where $g_1(x)$ and $g_2(x)$ are continuous functions of x on $[a,b]$.
- **horizontally simple region** if R can be described by the inequalities

$$D_2: \quad c \leq y \leq d, \quad h_1(y) \leq x \leq h_2(y)$$

  where $h_1(x)$ and $h_2(x)$ are continuous functions of y on $[c,d]$.

::: {#thm- } [Fubini's Theorem for Non-Rectangular Regions] If D_1 is a vertically simple region, then

$$\iint_{D_1} f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$$

whenever both integrals exist. Similarly, for a horizontally region D_2,

$$\iint_{D_2} f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

whenever both integrals exist.

Example 5.12. Evaluate the double integral

$$\iint_D xy\,dA$$

where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

Solution. The region D is both a vertically simple and a horizontally simple region, but the description of D as a vertically simple region is more complicated because the lower boundary consists of two parts. Therefore we express D as a horizontally simple region:

$$D = \left\{(x,y) \mid -2 \le y \le 4, \frac{y^2}{2} - 3 \le x \le y + 1\right\}.$$

as shown below. Then the double integral becomes

$$\iint_D xy\,dA = \int_{-2}^4 \int_{(1/2)y^2-3}^{y+1} xy\,dx\,dy = \frac{1}{2}\int_{-2}^4 \left(-\frac{y^5}{4} + 4y^3 + 2y^2 - 8y\right)\,dy = 36.$$

If we had expressed D as a vertically simple region, then we would have obtained

$$\iint_D xy\,dA = \int_{-3}^{-1} \int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} xy\,dy\,dx + \int_{-1}^{5} \int_{x-1}^{\sqrt{2x+6}} xy\,dy\,dx = 36.$$

See **??** for a visualization.

Example 5.13. Evaluate the iterated integral

$$\int_0^1 \int_{x^2}^{\sqrt{x}} xy^2\,dy\,dx.$$

Solution. We express the region of integration D as a vertically simple region:

$$D = \left\{(x,y) \mid 0 \le x \le 1, x^2 \le y \le \sqrt{x}\right\}.$$

We use 5.2 to obtain

$$\int_0^1 \int_{x^2}^{\sqrt{x}} xy^2 \, dy \, dx = \int_0^1 x \left(\frac{x^{3/2}}{3} - \frac{x^6}{3} \right) dx = \frac{3}{56}.$$

See **??** for a visualization.

Example 5.14. Evaluate the iterated integral

$$\int_0^{2\sqrt{3}} \int_{y^2/6}^{\sqrt{16-y^2}} dx \, dy.$$

Solution. We express the region of integration D as a horizontally simple region:

$$D = \left\{ (x,y) \mid 0 \leq y \leq 2\sqrt{3}, \frac{y^2}{6} \leq x \leq \sqrt{16 - y^2} \right\}.$$

We use 5.2 to obtain

$$\int_0^{2\sqrt{3}} \int_{y^2/6}^{\sqrt{16-y^2}} dx \, dy = \int_0^{2\sqrt{3}} \left(\sqrt{16 - y^2} - \frac{y^2}{6} \right) dy = \left(\frac{1}{18} \left(12\sqrt{3} + 48\pi \right) \right).$$

See **??** for a visualization.

Example 5.15. Evaluate the iterated integral

$$\int_0^1 \int_{-x^2}^{x^2} dy \, dx.$$

Solution. We express the region of integration D as a vertically simple region:

$$D = \left\{ (x,y) \mid 0 \leq x \leq 1, -x^2 \leq y \leq x^2 \right\}.$$

The region of integration is sketched in **??**. We find

$$\int_0^1 \int_{-x^2}^{x^2} dy \, dx = \int_0^1 2x^2 \, dx = \frac{2}{3}.$$

Example 5.16. Sketch the region of integration and evaluate

$$\int_{-2}^1 \int_{y^2+4y}^{3y+2} dx \, dy.$$

Solution. The region is horizontally simple region as $-2 \leq y \leq 1$ and $y^2 + 4y \leq x \leq 3y + 2$. We find

$$\int_{-2}^{1} \int_{y^2+4y}^{3y+2} dx\, dy = \int_{-2}^{1} \left(-y^2 - y + 2\right) dy = \frac{9}{2}.$$

This region could also be considered as a vertically simple region as $-4 \leq x \leq 5$ and $\frac{x-2}{3} \leq y \leq -2 + \sqrt{x+4}$. We find

$$\int_{-4}^{5} \int_{(x-2)/3}^{-2+\sqrt{x+4}} dy\, dx = \int_{-4}^{5} \left(\left(-2 + \sqrt{x+4}\right) - \left(\frac{x-2}{3}\right)\right) dx = \frac{9}{2}.$$

See **??** for a visualization.

Example 5.17. Sketch the region of integration and evaluate

$$\int_{0}^{\pi/2} \int_{0}^{\sin x} e^y \cos x\, dy\, dx.$$

Solution. The region is vertically simple region as $0 \leq x \leq \frac{\pi}{2}$ and $0 \leq y \leq \sin x$. We find

$$\int_{0}^{\pi/2} \int_{0}^{\sin x} e^y \cos x\, dy\, dx = \int_{0}^{\pi/2} \left(e^{\sin x} \cos x - \cos x\right) dx = e - 2.$$

The region is also horizontally simple region as $0 \leq y \leq 1$ and $\operatorname{Arcsin}(y) \leq x \leq \frac{\pi}{2}$. We find

$$\int_{0}^{1} \int_{\arcsin y}^{\pi/2} e^y \cos x\, dx\, dy = e - 2.$$

See **??** for a visualization.

Example 5.18. Sketch the region of integration and evaluate the iterated integral

$$\int_{0}^{1} \int_{x}^{1} \sin\left(y^2\right) dy\, dx.$$

Solution. If we try to evaluate the integral as it stands, we are faced with the task of first evaluating

$$\int \sin y^2\, dy. \tag{5.13}$$

We elect to change the order of integration. This is accomplished by first expressing the given iterated integral as a double integral. We have

$$\int_0^1 \int_x^1 \sin y^2 \, dy \, dx = \iint_D \sin y^2 \, dA$$

where $D = \{(x,y) \mid 0 \leq x \leq 1, x \leq y \leq 1\}$. This region has an alternate description:

$$D = \{(x,y) \mid 0 \leq y \leq 1, 0 \leq x \leq y\}.$$

Thus we can express the double integral as an iterated integral in the reverse order:

$$\iint_D \sin y^2 \, dA = \int_0^1 \int_0^y \sin y^2 \, dx \, dy = \int_0^1 \left[x \sin y^2\right]\Big|_0^y \, dy = \frac{1}{2}(1 - \cos 1).$$

Volume as a Double Integral

::: {#thm- } Volume as a Double Integral If $f(x,y) \geq 0$ on the rectangular region R, then the product $f\left(x_k^*, y_k^*\right) \triangle \mathrm{A}_k$ is the volume of a parallelepiped (a box) with height $f\left(x_k^*, y_k^*\right)$ and base area $\triangle \mathrm{A}_k$. The Riemann sum

$$\sum_{k=1}^{N} f\left(x_k^*, y_k^*\right) \triangle \mathrm{A}_k \tag{5.14}$$

provides an estimate of the total volume under the surface $z = f(x,y)$ over R, and if f is continuous, we expect the approximation to improve by using more refined partitions. That is, the volume under $z = f(x,y)$ over the domain R is given by

$$\lim_{\|P\| \to 0} \sum_{k=1}^{N} f\left(x_k^*, y_k^*\right) \triangle A_k = \iint_R f(x,y) dA \tag{5.15}$$

when $f(x,y) \geq 0$ on the rectangular region R.

Example 5.19. Find the volume of the bounded solid that lies inside both the cylinder $x^2 + y^2 = 3$ and the sphere $x^2 + y^2 + z^2 = 7$.

Solution. The volume is given by

$$V = 4 \int_0^{\sqrt{3}} \int_0^{\sqrt{3-y^2}} \sqrt{7 - x^2 - y^2} \, dx \, dy$$

or also

$$V = 4 \int_0^{\sqrt{3}} \int_0^{\sqrt{3-x^2}} \sqrt{7 - x^2 - y^2} \, dy \, dx \approx 22.0336.$$

Example 5.20. Find the volume of the solid region bounded below by the given rectangle in the xy-plane and above by the graph of the given surface

$$f(x,y) = \frac{xy}{\sqrt{x^2+y^2+1}}$$

on $0 \le x \le 1, 0 \le y \le 1$

Solution. The volume is given by

$$\begin{aligned}\int_0^1 \int_0^1 \frac{xy}{\sqrt{x^2+y^2+1}}\,dy\,dx &= \int_0^1 x\left(\sqrt{x^2+2} - \sqrt{x^2+1}\right)\,dx \\ &= \frac{1-2\sqrt{2}}{3} + \frac{-2\sqrt{2}+3\sqrt{3}}{3} = \frac{1-4\sqrt{2}+3\sqrt{3}}{3} \approx 0.179766\end{aligned}$$

Example 5.21. Find the volume of the solid region bounded below by the given rectangle in the xy-plane and above by the graph of the surface

$$f(x,y) = (x+y)^5$$

on $0 \le x \le 1, 0 \le y \le 1$.

Solution. The volume is

$$\int_0^1 \int_0^1 (x+y)^5 dy\,dx = \int_0^1 \left(\frac{1}{6}(x+1)^6 - \frac{x^6}{6}\right)\,dx = 3. \tag{5.16}$$

Example 5.22. Find the volume of the bounded solid between the two elliptic paraboloids

$$z = x^2 \big/ (9+y^2-4) \quad \text{and} \quad z = -x^2 \big/ (9-y^2+4).$$

Solution. Using symmetry, the volume is given by

$$\begin{aligned}V = 8 &\int_0^6 \int_0^{\sqrt{4-x^2/9}} \int_0^{4-x^2/9-y^2} \,dz\,dy\,dx \\ = 8 &\int_0^6 \int_0^{\sqrt{4-x^2/9}} \left(-\frac{x^2}{9} - y^2 + 4\right)\,dy\,dx \\ = 8 &\int_0^6 \left(\frac{8}{3}\sqrt{4-\frac{x^2}{9}} - \frac{2}{27}x^2\sqrt{4-\frac{x^2}{9}}\right)\,dx = 48 \approx 150.796.\end{aligned}$$

Example 5.23. Find the volume of the bounded solid bounded below by the rectangle $R: 1 \leq x \leq 2, 1 \leq y \leq 2$
in the xy-plane and above by the graph of

$$z = f(x,y) = \frac{x}{y} + \frac{y}{x}. \tag{5.17}$$

Solution. The volume is given by the following double integral and can be compute using Fubini's theorem.

$$\begin{aligned}
\iint_R \left(\frac{x}{y} + \frac{y}{x}\right)\, dA &= \int_0^2 \int_1^2 \left(\frac{x}{y} + \frac{y}{x}\right)\, dy\, dx \\
&= \int_0^2 \left[x \ln|y| + \frac{y^2}{2x}\right]\Bigg|_1^2 dx \\
&= \left[\frac{3}{2}\ln|x| \frac{+1}{2}(\ln 2)x^2\right]\Bigg|_1^2 = 3 \ln 2.
\end{aligned}$$

5.2 Exercises

Exercise 5.11. Sketch the region of integration and evaluate $\int_{-2}^{1}\int_{y^2+4y}^{3y+2} dx\, dy$.

Exercise 5.12. Evaluate $\iint_R (x^2+y^2)^{3/2}\, dA$ where R is the unit circle centered at $(0,0)$.

Exercise 5.13. Evaluate the following iterated integrals.

- $\int_0^2 \int_0^1 (x^2 + xy + y^2)\, dy\, dx$.
- $\int_1^2 \int_0^\pi x \cos y\, dy\, dx$.
- $\int_0^{\ln(2)} \int_0^1 e^{x+2y} dx\, dy$.
- $\int_3^4 \int_1^2 \frac{x}{x-y} dy\, dx$
- $\int_2^3 \int_{-1}^2 \frac{1}{(x+y)^2} dy\, dx$

Exercise 5.14. Use iterated integration to compute the double integral of the given rectangular region for each of the following.

- $R = \{(x,y) \mid 1 \leq x \leq 2, 0 \leq y \leq 1\}$

$$\iint_R x^2 y\, dA$$

- $R = \{(x,y) \mid -1 \le x \le 0, 0 \le y \le \ln(2)\}$
$$\iint_R 2xe^y \, dA$$
- $R = \left\{(x,y) \mid 0 \le x \le \frac{\pi}{4}, 0 \le y \le \frac{\pi}{2}\right\}$
$$\iint_R \sin(x+y) \, dA$$
- $R = \{(x,y) \mid 0 \le x \le \pi, 0 \le y \le 1\}$
$$\iint_R x\sin(xy) \, dA$$
- $R = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 2\}$
$$\iint_R x\sqrt{1-x^2}e^{3y} \, dA$$
- $R = \{(x,y) \mid 0 \le x \le 1, 1 \le y \le 2\}$
$$\iint_R xe^{xy} \, dA$$

Exercise 5.15. Find the volume of the solid bounded below by the given rectangular region in the xy-plane and above the graph of the given function.

- $z = 2x + 3y$,
  $ R={(x,y) 0 x 1,0 y 2} $
- $z = x\ln(xy)$, $ R={(x,y) 1 x 2,1 y e} $
- $z = x\cos y + y\sin x$.
$$R = \left\{(x,y) \mid 0 \le x \le \frac{\pi}{2}, 0 \le y \le \frac{\pi}{2}\right\}$$

Exercise 5.16. Find the volume of the solid region bounded below by the given rectangle in the xy-plane and above by the graph of the given surface.

- $f(x,y) = \frac{xy}{\sqrt{x^2+y^2+1}}$ on $0 \le x \le 1, 0 \le y \le 1$
- $f(x,y) = (x+y)^5$ on $0 \le x \le 1, 0 \le y \le 1$

Exercise 5.17. Sketch the region and evaluate the iterated integral over the non-rectangular region.

- $\int_0^4 \int_0^{4-x} xydy \, dx$.

- $\int_1^e \int_0^{\ln(x)} xy dy\, dx.$
- $\int_0^2 \int_0^{\sin(x)} y \cos x dy\, dx.$
- $\int_{-2}^1 \int_{y^2+4y}^{3y+2} dx\, dy.$
- $\int_0^{2\sqrt{3}} \int_{y^2/6}^{\sqrt{16-y^2}} dx\, dy.$
- $\int_0^1 \int_{-x^2}^{x^2} dy\, dx.$
- $\int_0^4 \int_{x^2}^{4x} dy\, dx.$
- $\int_0^1 \int_{x^2}^{\sqrt{x}} xy^2 dy\, dx$
- $\int_{-1}^3 \int_{\tan^{-1} x}^{\pi/4} xy dy\, dx.$
- $\int_0^4 \int_0^{4-x} xy dy\, dx.$
- $\int_0^7 \int_{x^2-6x}^{x} \sin x dy\, dx$

Exercise 5.18. Find the volume of the solid bounded by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

(Setup but do not evaluate the double integral)

Exercise 5.19. Find the volume of the solid that lies inside both the cylinder $x^2 + y^2 = 3$ and the sphere $x^2 + y^2 + z^2 = 7$. (Setup but do not evaluate the double integral).

Exercise 5.20. For each of the following sketch the region of integration and write an equivalent integral with the order of integration reversed for

- $\int_0^1 \int_0^{2y} f(x, y)\, dx\, dy.$
- $\int_0^1 \int_{x^2}^{\sqrt{x}} f(x, y)\, dy\, dx.$
- $\int_0^4 \int_{y/2}^{\sqrt{y}} f(x, y)\, dx\, dy.$
- $\int_1^2 \int_{\ln(x)}^2 f(x, y)\, dy\, dx.$
- $\int_0^3 \int_{y/3}^{\sqrt{4-y}} f(x, y)\, dx\, dy.$

Exercise 5.21. Evaluate the double integral

$$\iint_D xy\, dA$$

where D is the triangular region in the xy-plane with vertices $(0,0)$, $(1,0)$, and $(4,1)$.

Exercise 5.22. Evaluate the double integral

$$\iint_D \left(x^2 - xy - 1\right) \, dA$$

where D is the triangular region in the xy-plane bounded by the lines $x - 2y + 2 = 0$, $x + 3y - 3 = 0$, and $y = 0$.

Exercise 5.23.

(a) Write the iterated integral $I = \int_0^1 \int_y^1 \sin(x^2) \, dx \, dy$ as a double integral over a domain R.
(b) Sketch the domain R.
(c) Evaluate the double integral by reversing the order of integration.

5.3 Double Integrals In Polar Coordinates

The polar conversion formulas are used to convert from rectangular to polar coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r = \sqrt{x^2 + y^2}, \quad \tan\theta = \frac{y}{x}. \tag{5.18}$$

::: {#thm- } [Fubini's Theorem in Polar Coordinates] If f is continuous in the polar region R described by

$$0 \leq r_1(\theta) \leq r \leq r_2(\theta) \qquad \alpha \leq \theta \leq \beta$$

(with \$0 - 2 \$), then

$$\iint_R f(r,\theta) dA = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r,\theta) r \, dr \, d\theta. \tag{5.19}$$

:::

Example 5.24. Evaluate

$$\iint_R \left(3x + 4y^2\right) \, dA \tag{5.20}$$

where R is the upper-half plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Solution. The region of integration R is described as

$$R = \{(x, y) \mid y \geq 0, 1 \leq x^2 + y^2 \leq 4\}.$$

It is an upper ring with polar coordinates given by $1 \leq r \leq 2$, $0 \leq \theta \leq \pi$. Therefore

$$\begin{aligned}\iint_R \left(3x + 4y^2\right) dA &= \int_0^\pi \int_1^2 \left(3r\cos\theta + 4r^2\sin^2\theta\right) r\, dr\, d\theta \\ &= \int_0^\pi \left(7\cos\theta + 15\sin^2\theta\right) d\theta = \frac{15\pi}{2}\end{aligned}$$

by using the trigonometric identity $\sin^2\theta = \frac{1-\cos(2\theta)}{2}$.

Example 5.25. Evaluate

$$\int_0^{\pi/2} \int_1^3 re^{-r^2}\, dr\, d\theta. \tag{5.21}$$

Solution. In polar coordinates the region of integration is described as

$$R = \{(r, \theta) \mid 0 \leq \theta \leq \frac{\pi}{2}, 1 \leq r \leq 3\}.$$

Using Fubini's Theorem in polar coordinates the iterated integral is evaluated as

$$\int_0^{\pi/2} \int_1^3 re^{-r^2} drd\theta = \int_0^{\pi/2} \frac{e^8 - 1}{2e^9}\, d\theta = \frac{\left(e^8 - 1\right)\pi}{4e^9}.$$

Example 5.26. Evaluate

$$\int_0^{\pi/2} \int_1^2 \sqrt{4 - r^2}\, r\, dr\, d\theta. \tag{5.22}$$

Solution. In polar coordinates the region of integration is described as

$$R = \{(r, \theta) \mid 0 \leq \theta \leq \frac{\pi}{2}, 1 \leq r \leq 2\}.$$

Using polar coordinates the iterated integral is evaluated as

$$\int_0^{\pi/2} \int_1^2 \sqrt{4 - r^2} rdrd\theta = \int_0^{\pi/2} \sqrt{3}\, d\theta = \frac{\sqrt{3}\pi}{2}.$$

Example 5.27. Evaluate

$$\int_0^{\pi} \int_0^4 r^2 \sin^2 \theta \, dr \, d\theta. \tag{5.23}$$

Solution. In polar coordinates the region of integration is described as

$$R = \{(r, \theta) \mid 0 \leq \theta \leq \pi, 0 \leq r \leq 4\}.$$

Using polar coordinates the iterated integral is evaluated as

$$\int_0^{\pi} \int_0^4 r^2 \sin^2 \theta dr \, d\theta = \int_0^{\pi} \frac{64 \sin^2 \theta}{3} \, d\theta = \frac{32\pi}{3}$$

by using the trigonometric identity $\sin^2 \theta = \frac{1-\cos(2\theta)}{2}$.

Example 5.28. Evaluate

$$\int_0^{\pi/2} \int_1^3 r^2 \cos^2 \theta \, dr \, d\theta. \tag{5.24}$$

Solution. In polar coordinates the region is described as

$$R = \{(r, \theta) \mid 0 \leq \theta \leq \frac{\pi}{2}, 1 \leq r \leq 3\}.$$

Using polar coordinates the iterated integral is evaluated as

$$\int_0^{\pi/2} \int_1^3 r^2 \cos^2 \theta \, dr \, d\theta = \int_0^{\pi/2} \frac{26 \cos^2 \theta}{3} \, d\theta = \frac{13\pi}{6}$$

by using the trigonometric identity $\cos^2 \theta = \frac{1+\cos(2\theta)}{2}$.

Example 5.29. Evaluate

$$\int_0^{\pi} \int_0^{1+\sin\theta} dr \, d\theta. \tag{5.25}$$

Solution. In polar coordinates the region is described as

$$R = \{(r, \theta) \mid 0 \leq \theta \leq \pi, 0 \leq r \leq 1 + \sin\theta\}$$

Using polar coordinates the iterated integral is evaluated as

$$\int_0^{\pi} \int_0^{1+\sin\theta} dr d\theta = \int_0^{\pi} (\sin\theta + 1) \, d\theta = 2 + \pi.$$

Example 5.30. Evaluate

$$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \ln\left(x^2+y^2+9\right)\,dy\,dx. \tag{5.26}$$

Solution. In polar coordinates

$$\begin{aligned}\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \ln\left(x^2+y^2+9\right)\,dy\,dx &= \int_0^{2\pi}\int_0^3 \ln\left(r^2+9\right) rdrd\theta\\ &= \int_0^{2\pi}\left[-\frac{9}{2}(1+\ln(9)-2\ln(18))\right]d\theta\\ = (-9\pi(1+\ln(9)-2\ln(18))).\end{aligned}$$

Example 5.31. Evaluate

$$\int_0^2\int_0^{\sqrt{2x-x^2}} \frac{x-y}{x^2+y^2}\,dy\,dx \tag{5.27}$$

Solution. In polar coordinates

$$\begin{aligned}\int_0^2\int_0^{\sqrt{2x-x^2}} \frac{x-y}{x^2+y^2}\,dy\,dx &= \int_0^{\pi/2}\int_0^{2\cos\theta} (\sin\theta-\cos\theta)\,dr\,d\theta\\ &= \int_0^{\pi/2} 2\cos\theta(\sin\theta-\cos(\theta))\,d\theta\\ &= \frac{2-\pi}{2}. \approx -0.570796\end{aligned}$$

Example 5.32. Use a double integral to find the area enclosed by one loop of the four leaved rose $r=\cos 2\theta$.

Solution. From the sketch of the curve we see that a loop is given by the region

$$R=\left\{(r,\theta)\mid \frac{-\pi}{4}\leq\theta\leq\frac{\pi}{4}, 0\leq r\leq\cos 2\theta\right\}.$$

So the area is

$$\iint_R dA = \int_{-\pi/4}^{\pi/4}\int_0^{\cos 2\theta} rdr\,d\theta = \frac{1}{4}\int_{-\pi/4}^{\pi/4}(1+\cos 4\theta)\,d\theta = \frac{\pi}{8}.$$

Example 5.33. Find the volume of the bounded solid region bounded below by the rectangle $R : 0 \leq x \leq 1,\ 0 \leq y \leq 1$ in the xy-plane and above by the graph of

$$z = f(x, y) = \sqrt{x + y}.$$

Solution. The volume is

$$\begin{aligned}\int_0^1 \int_0^1 \sqrt{x+y}dy\, dx &= \int_0^1 \left(\frac{2}{3}\left(-x^{3/2} + (1+x)^{3/2}\right)\right) dx \\ &= \frac{8}{15}\left(-1 + 2\sqrt{2}\right).\end{aligned}$$

Example 5.34. Find the volume of the bounded solid region below by the rectangle $R : 1 \leq x \leq 2,\ 1 \leq y \leq e$ in the $xy-$ plane and above by the graph of

$$z = f(x, y) = x \ln(xy).$$

Solution. The volume is

$$\begin{aligned}\int_1^2 \int_1^e x \ln(xy) dy\, dx &= \int_1^2 (x(1 + (-1+e)\ln x))\, dx \\ &= -\frac{3}{4}(-3+e) + 2(-1+e)\ln 2.\end{aligned}$$

Note in this example we used the formulas

$$\int \ln u du = u \ln u - u + C, u > 0 \quad \text{and} \quad \int u \ln u du = \frac{-1}{4}u^2 + \frac{1}{2}u^2 \ln u.$$

Example 5.35. Find the volume of the bounded solid region bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.

Solution. If we put $z = 0$ in the equation of the paraboloid, we get $x^2 + y^2 = 1$. This means that the plane intersects the paraboloid in the circle $x^2 + y^2 = 1$, so the solid lies under the paraboloid and above the circular disk R given by $x^2 + y^2 \leq 1$. In polar coordinates the region of integration R is given by $0 \leq r \leq 1,\ 0 \leq \theta \leq 2\pi$. Since $1 - x^2 - y^2 = 1 - r^2$, the volume is

$$V = \iint_R (1 - x^2 - y^2)\, dA = \int_0^{2\pi} \int_0^1 (1 - r^2)\, r\, dr\, d\theta = \frac{\pi}{4} \int_0^{2\pi} d\theta = \frac{\pi}{2}.$$

Example 5.36. Find the volume of the bounded solid region under the paraboloid $z = x^2 + y^2$, above the $xy-$ plane, and inside the cylinder $x^2 + y^2 = 2x$

Solution. The solid lies above the disk R whose boundary circle has equation $x^2 + y^2 = 2x$ or $(x-1)^2 + y^2 = 1$. In polar coordinates the boundary is $r^2 = 2r\cos\theta$ or $r = 2\cos\theta$. Thus the region of integration is

$$R = \left\{(r,\theta) \mid -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2},\ 0 \leq r \leq 2\cos\theta\right\}$$

and so the volume is

$$V = \iint_R (x^2 + y^2)\, dA = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r^2 r dr d\theta = 4 \int_{-\pi/2}^{\pi/2} \cos^4\theta d\theta = \frac{3\pi}{2}$$

by using the trigonometric identity $\cos^4\theta = \left(\frac{1+\cos(2\theta)}{2}\right)^2$.

Example 5.37. Find the volume of the bounded solid common to the cylinder $x^2 + y^2 = 2$ and the ellipsoid $3x^2 + 3y^2 + z^2 = 7$.

Solution. In polar coordinates

$$\begin{aligned} V &= 2\int_0^{2\pi} \int_0^{\sqrt{2}} \sqrt{7 - 3r^2} r\, dr\, d\theta = 2\int_0^{2\pi} \left(-\frac{1}{9} + \frac{7\sqrt{7}}{9}\right) d\theta \\ &= \frac{4\pi}{9}\left(-1 + 7\sqrt{7}\right). \approx 24.4629. \end{aligned}$$

Example 5.38. Find the volume of the solid bounded above by the cone $z = x^2 + y^2$, below by the plane $z = 0$, and on both sides by the cylinder $x^2 + y^2 = y$.

Solution. In polar coordinates

$$V = \int_0^{\pi} \int_0^{\sin\theta} r^3\, dr\, d\theta = \int_0^{\pi} \frac{\sin^4\theta}{4}\, d\theta = \frac{3\pi}{32} \approx 0.294524.$$

5.4 Exercises

Exercise 5.24. Evaluate $\int_0^{\pi} \int_0^{1+\sin\theta}\ dr\, d\theta$.

Exercise 5.25. Sketch the region of integration and then evaluate the iterated integral in polar coordinates.

- $\int_0^{\pi/2} \int_1^3 re^{-r^2} drd\theta$
- $\int_0^{\pi/2} \int_1^2 \sqrt{4-r^2} drd\theta$
- $\int_0^{\pi} \int_0^4 r^2 \sin^2 \theta dr\, d\theta$
- $\int_0^{\pi/2} \int_1^3 r^2 \cos^2 \theta drd\theta$
- $\int_0^{\pi} \int_0^{1+\sin\theta} dr\, d\theta$

Exercise 5.26. Sketch the enclosed region and then use an iterated integral to find the area of the region.

- $r = 2\cos\theta$

- $r = 4\cos 3\theta$
- $r = 1$ and $r = 2\sin\theta$
- $r = 1$ and $r = 1 + \cos\theta$

Exercise 5.27. Use polar coordinates to evaluate the iterated integrals.

- $\int_0^1 \int_x^{\sqrt{x}} \left(x^2 + y^2\right)^{3/2} dydx$
- $\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x-y}{x^2+y^2} dydx.$

Exercise 5.28. Evaluate the following iterated integrals by converting to polar coordinates.

- $\int_0^3 \int_0^{\sqrt{9-x^2}} xdydx$
- $\int_0^2 \int_0^{\sqrt{4-y^2}} e^{x^2+y^2} dxdy$
- $\int_0^3 \int_0^{\sqrt{9-x^2}} xdydx$
- $\int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \ln\left(x^2 + y^2 + 9\right) dydx$
- $\int_0^2 \int_y^{\sqrt{8-y^2}} \frac{1}{\sqrt{1+x^2+y^2}} dxdy$
- $\int_0^3 \int_0^{\sqrt{9-x^2}} \cos\left(x^2 + y^2\right) dydx$
- $\int_0^4 \int_0^{\sqrt{4y-y^2}} \frac{1}{\sqrt{x^2+y^2}} dxdy$

Exercise 5.29. Find the volume of the solid region common to the cylinder $x^2 + y^2 = 2$ and the ellipsoid $3x^2 + 3y^2 + z^2 = 7$.

Exercise 5.30. Find the volume of the ice cream cone bounded by the hemisphere $z = \sqrt{8 - x^2 - y^2}$ and the cone $z = \sqrt{x^2 + y^2}$.

Exercise 5.31. Find the volume of the solid region bounded above by the cone $z = x^2 + y^2$, below by the plane $z = 0$, and on both sides by the cylinder $x^2 + y^2 = y$.

Exercise 5.32. Find the volume of the given solid region common to the cylinder $x^2 + y^2 = 2$ and the ellipsoid $3x^2 + 3y^2 + z^2 = 7$.

Exercise 5.33. Find the volume of the given solid region bounded above by the cone $z = x^2 + y^2$, below by the plane $z = 0$, and on both sides by the cylinder $x^2 + y^2 = y$.

Exercise 5.34. For a constant a where $0 \leq a \leq R$, the plane $z = R - a$ cuts off a "cap" from the hemisphere $z = \sqrt{R^2 - x^2 - y^2}$. Use a double integral in polar coordinates to find the volume of the cap. For what values of a is the volume of the cap half the volume of the hemisphere?

5.5 Applications of Double Integrals

We will consider the following applications: average value of a function over a region, mass of a lamina, electric charge, moments and center of mass, moments of inertia, and probability density functions.

5.6 Average Value

Recall the average value of an integrable function of one variable on a closed interval is the integral of the function over the interval divided by the length if the interval. For an integrable function of two variables defined on a bounded region in the plane, the **average value** is the integral over the region divided by the area of the region.

Example 5.39. Which do you think will be larger, the average value of $f(x, y) = xy$ over the square $0 \leq x \leq 1$, $0 \leq y \leq 1$, or over the quarter circle $x^2 + y^2 \leq 1$ in the first quadrant?

Solution. The average value of f over the square is

$$\int_0^1 \int_0^1 xydydx = \int_0^1 \frac{1}{2}xdx = \frac{1}{4}.$$

Using polar coordinates, the average value of f over the quarter circle is

$$\frac{1}{\pi/4}\int_0^{\pi/2}\int_0^1 r^3\cos\theta\sin\theta dr\,d\theta = \frac{4}{\pi}\int_0^{\pi/2}\frac{1}{4}\cos\theta\sin\theta d\theta = \frac{1}{2\pi}.$$

Therefore the average value of xy is larger over the square.

5.7 Mass of a Lamina

A **planar lamina** is a flat plate that occupies a region R in the plane that is so thin it can be regarded as two dimensional. If m is the lamina's mass and A is the area of the region R, then $\delta = m/A$ is the **density of the lamina** (in units of mass per unit area). A lamina is called **homogeneous** if its density $\delta(x,y)$ is constant over R and **nonhomogeneous** if $\delta(x,y)$ varies from point to point.

Definition 5.3. If δ is a continuous density function of the lamina corresponding to a plane region R then the (total) **mass** m of the planar lamina is given by

$$m = \iint_R \delta(x,y)\,dA.$$

Example 5.40. Find the mass of the planar lamina occupying the region R bounded by the parabola $y = 2 - x^2$ and the line $y = x$ if $\delta(x,y) = x^2$.

Solution. Begin by drawing the parabola and the line, and by finding their points of intersection $(-2,-2)$ and $(1,1)$. Considering the region R as vertically simple, the mass of the lamina is

$$m = \iint_R \int_x^{2-x^2} x^2\,dA = \int_{-2}^1\int_x^{2-x^2} x^2\,dy\,dx = \int_{-2}^1 x^2(2-x^2-x)\,dx = \frac{63}{20}.$$

5.8 Electric Charge

Physicists also consider other types of density that can be treated in the same manner. For example, if an electric charge is distributed over a region R and the charge density (in units of charge per unit area) is given by $\delta(x,y)$ at a point (x,y) in R, then the **total charge** Q is given by

$$Q = \iint_R \delta(x,y)dA.$$

Example 5.41. Charge is distributed over the triangular region R described by $(0,1)$, $(1,1)$, and $(1,0)$ so that the charge density at (x,y) is $\delta(x,y) = xy$, measured in coulombs per square meter. Find the total charge.

Solution. Considering the region R as a vertically simple region, we have

$$Q = \iint_R \delta(x,y)\,dA = \int_0^1 \int_{1-x}^1 xy\,dy\,dx = \frac{1}{2}\int_0^1 \left(2x^2 - x^3\right)\,dx = \frac{5}{24}.$$

Thus the total charge is 5/24 C.

5.9 Moments and Center of Mass of a Lamina

The **moment** of an object about an axis measures the tendency of the object to rotate about that axis. It is defined as the product of the object's mass and the signed distance from the axis. Let δ denote a continuous density function of a body. Then M_x and M_y, the **first moments** about the x-axis and y-axis, respectively are

$$M_x = \iint_R y\delta(x,y)\,dx\,dy \qquad \text{and} \qquad M_y = \iint_R x\delta(x,y)\,dx\,dy$$

The **center of mass** of the lamina covering R is the point $(\overline{x}, \overline{y})$ where the mass m can be concentrated without affecting the moments M_x and M_y; that is $m\overline{x} = M_y$ and $m\overline{y}$. If the density δ is constant, the point $(\overline{x}, \overline{y})$ is called the **centroid** of the region.

Example 5.42. A lamina occupies a region R in the xy-plane bounded by the parabola $y = x^2$ and the line $y = 1$. Find the center of mass of the lamina if its mass density at a point (x,y) is directly proportional to the distance between the point and the x-axis.

Solution. The mass density of the lamina is $\delta(x,y) = ky$ where k is a constant. Since the region R is symmetric with respect to the y-axis and the density of the lamina is directly proportional to the distance from the x-axis, we see that the center of mass is located on the y-axis and so $\overline{x} = 0$. To find $\overline{y}$ we view R as a vertically simple region and compute

$$m = \int\!\!\int_R \delta dA = \int_{-1}^1 \int_{x^2}^1 ky\,dy\,dx = \frac{k}{2}\int_{-1}^1 (1 - x^4)\,dx = \frac{4k}{5}.$$

We compute

$$\overline{y} = \frac{1}{m}\int_{-1}^1 \int_{x^2}^1 ky^2\,dy\,dx = \frac{5}{12}\int_{-1}^1 (1 - x^6)\,dx = \frac{5}{7}$$

Therefore, the center of mass is $\left(0, \frac{5}{7}\right)$.

5.10 Probability Density Functions

Quantities that range continuously over an interval of real numbers are called continuous random variables. Every continuous random variable X has a **probability density function** f with the property that the probability of X lying between the numbers a and b is given by the integral

$$P(a \leq X \leq b) = \int_a^b f(x)\, dx.$$

In general, $f(x) \geq 0$ for all x, and since the value of X is always some real number, it follows that $P(-\infty < X < \infty) = 1$, so $\int_{-\infty}^{\infty} f(x)dx = 1$. In geometric terms, the probability $P(a \leq X \leq b)$ is the area under the graph of f over the interval $a \leq x \leq b$.

If X and Y are both continuous random variables, then the **joint probability density** function for two random variables X and Y is a function of two variables $f(x, y)$ such that $f(x, y) \geq 0$ for all (x, y) and

$$P[(X, Y) \text{ in } D] = \int\int_D f(x, y)\, dA$$

where $P[(X, Y) \text{ in } D]$ denotes the probability that (X, Y) is in the region D. Note that

$$P\left[(X, Y) \text{ in } \mathbb{R}^2\right] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1.$$

Geometrically $P[(X, Y) \text{ in } D]$ may be thought of as the volume under the surface $z = f(x, y)$ above the region D.

Example 5.43. Suppose the joint probability density function for the random variable X and Y is modeled by

$$f(x, y) = \begin{cases} xe^{-x-y} & x \geq 0, y \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Find the probability that $X + Y \leq 1$.

Solution. The probability that $X + Y \leq 1$ is given as

$$\begin{aligned}
P(X+Y\le 1) &= \int_0^1\int_0^{1-x} xe^{-x-y}dy\,dx\\
&= \int_0^1\left(-x\int_0^{1-x} e^{-x-y}(-1)dy\right)\,dx\\
&= \int_0^1\left(-x\;e^{-x-y}\Big|_0^{1-x}\right)\,dx\\
&= \int_0^1\left[\left(-xe^{-x-(1-x)}\right)-\left(-xe^{-x-0}\right)\right]\,dx\\
&= \int_0^1\left(-\frac{1}{e}+e^{-x}\right)x\,dx\\
&= -\frac{1}{e}\int_0^1 x\,dx+\int_0^1 xe^{-x}dx
\end{aligned}$$

Let $u=x$ and $dv=e^{-x}dx$, then $du=dx$ and $v=-e^{-x}$; and so using integration by parts we find

$$\begin{aligned}
P(X+Y\le 1) &= -\frac{1}{e}\int_0^1 x\,dx - xe^{-x}\Big|_0^1+\int_0^1 e^{-x}\,dx\\
&= -\frac{1}{e}\,\frac{x^2}{2}\Bigg|_0^1 - x\;e^{-x}\Big|_0^1 - e^{-x}\Big|_0^1\\
&= -\frac{1}{e}\left(\frac{1}{2}\right)-\left(\frac{1}{e}\right)-\left(\frac{1}{e}\right)+1 = 1-\frac{5}{2e}\approx 0.0803014.
\end{aligned}$$

Example 5.44. Suppose the joint probability density function for the random variable X and Y is modeled by

$$f(x,y)=\begin{cases}2e^{-2x-y} & x\ge 0, y\ge 0\\ 0 & \text{otherwise.}\end{cases}$$

Find the probability that $X+Y\le 1$.

Solution. We find $P(X+Y\le 1)$ to be

$$\begin{aligned}
&2\int_0^1\int_0^{1-x} e^{-2x-y}\,dy\,dx = 2\int_0^1\left(-e^{-x-1}+e^{-2x}\right)\,dx\\
&=\frac{1}{e^2}-\frac{2-e}{e}\approx 0.399576.
\end{aligned}$$

Example 5.45. Suppose X measures the time (in minutes) that a person stands in line at a certain bank and Y, the duration (in minutes) of a routine transaction at the teller's window. You arrive at the bank to deposit a check. The joint probability density function for X and Y is modeled by

$$f(x,y) = \begin{cases} \frac{1}{8}e^{-x/2-y/4} & x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Solution. Find the probability that you will complete your business at the bank within 8 minutes.
The probability that you will complete your business at the bank within 8 minutes is

$$\begin{aligned}
P(X+Y \leq 8) &= \frac{1}{8}\int_0^8 \int_0^{8-x} e^{-x/2-y/4}\,dy\,dx \\
&= (-4)\frac{1}{8}\int_0^8 \int_0^{8-x} e^{-x/2-y/4}\left(-\frac{1}{4}\right)\,dy\,dx \\
&= \left(-\frac{1}{2}\right)\int_0^8 \left(e^{-x/2-y/4}\right)\Big|_0^{8-x}\,dx \\
&= \left(-\frac{1}{2}\right)\int_0^8 \left(e^{-x/2-(8-x)/4} - e^{-x/2}\right)\,dx \\
&= \left(-\frac{1}{2}\right)\int_0^8 \left(e^{\left(-\frac{x}{4}-2\right)} - e^{-x/2}\right)\,dx \\
&= \left(-\frac{1}{2}\right)\left((-4)\int_0^8 e^{\left(-\frac{x}{4}-2\right)}\left(\frac{-1}{4}\right)\,dx - (-2)\int_0^8 e^{\frac{-x}{2}}\left(\frac{-1}{2}\right)\,dx\right) \\
&= \left(-\frac{1}{2}\right)\left((-4)e^{\left(-\frac{x}{4}-2\right)}\Big|_0^8 + 2e^{\frac{-x}{2}}\Big|_0^8\right) \\
&= 2e^{\left(-\frac{x}{4}-2\right)}\Big|_0^8 - e^{\frac{-x}{2}}\Big|_0^8 \\
&= 2e^{\left(-\frac{8}{4}-2\right)} - 2e^{\left(-\frac{0}{4}-2\right)} - \left(e^{\frac{-8}{2}}\right) + \left(e^{\frac{-0}{2}}\right) \\
&= e^{-4} - 2e^{-2} + 1 = \left(\frac{1}{e^2} - 1\right)^2 \approx 0.747645.
\end{aligned}$$

5.11 Exercises

Exercise 5.35. Find the centroid for a lamina with $\delta = 4$ over the region bounded by the curve $y = \sqrt{x}$ and the line $x = 4$ in the first octant.

Exercise 5.36. Find the centroid for a lamina with $\delta = 2$ over the region between the line $y = 2x$ and the parabola $y = x^2$.

Exercise 5.37. Use double integration to find the center of mass of a lamina covering the region $x^2 + y^2 \leq 9$, $y \geq 0$ with density function $\delta(x, y) = x^2 + y^2$.

Exercise 5.38. Use double integration to find the center of mass of a lamina covering the region bounded by $y = 0$, $y = x^2$, and $x = 6$ with density function $\delta(x, y) = 3x$.

Exercise 5.39. Use double integration to find the center of mass of a lamina covering the region bounded by $y = \ln(x)$, $y = 0$, and $x = 2$ with density function $\delta(x, y) = 1/x$.

Exercise 5.40. A lamina has the shape of a semicircular region $x^2+y^2 \leq a^2$, $y \geq 0$. Find the center of mass of the lamina if the density at each point is directly proportional to the square of the distance from the point to the origin.

- Find the center of mass of the cardioid $r = 1 + \sin\theta$ if the density at each point (r, θ) is $\delta(r, \theta) = r$.

Exercise 5.41. Find the centroid of the loop of the lemniscate $r^2 = 2\sin 2\theta$ that lies in the first quadrant.

Exercise 5.42. Find the centroid of the part of the large loop of the limacon $r = 1 + 2\cos\theta$ that does not include the small loop.

Exercise 5.43. Find the center of mass of the lamina that covers the triangular region with vertices $(0, 0)$, $(a, 0)$, (a, b), if a and b are both positive and the density at $P(x, y)$ is directly proportional to the distance of P from the y-axis.

Exercise 5.44. For each of the following joint probability density functions with the random variables X and Y find the indicated probability.

- $f(x, y) = \begin{cases} 2e^{-2x}e^{-y} & x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$; $X + Y \leq 1$.

- $f(x,y) = \begin{cases} xe^{-x}e^{-y} & x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}; X + Y \leq 1.$
- $f(x,y) = \begin{cases} \frac{1}{6}e^{-x/2}e^{-y/3} & x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}; X + Y \leq 3.$
- $f(x,y) = \begin{cases} \frac{1}{300}e^{-x/30}e^{-y/10} & x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}; X + Y \leq 3.$

5.12 Surface Area of a Differentiable Function

We apply double integrals to the problem of computing the surface area over a region. We demonstrate a formula that is analogous to the formula for finding the arc length of a one variable function and detail how to evaluate a double integral to compute the surface area of the graph of a differentiable function of two variables.

::: {#thm- } [Surface Area] Assume that the function $f(x,y)$ has continuous partial derivatives f_x and f_y in a region R of the xy-plane. Then the portion of the surface $z = f(x,y)$ that lies over R has surface area

$$S = \iint_R \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1}\, dA.$$

:::

Proof. Consider a surface defined by $z = f(x,y)$ over a region R of the $xy -$ plane. Enclose the region R in a rectangle partitioned by a grid with lines parallel to the coordinate axes. This creates a number of cells, and we let R_1, R_2, ..., R_n denote those that lie entirely within R. For $m = 1, 2, 3, \dots, n$, let $P_m\,(x_m^*, y_m^*)$ be any corner of the rectangle R_m, and let T_m be the tangent plane above P_m on the surface of $z = f(x,y)$. Let $\triangle \mathrm{S}_m$ denote the area of the patch of the surface that lies directly above R_m. The rectangle R_m projects onto a parallelogram $ABDC$ in the tangent plane T_m, and if R_m is small we would expect the area of this parallelogram to approximate closely the element of the area $\triangle \mathrm{S}_m$. If $\triangle x_m$ and $\triangle y_m$ are the lengths of the sides of the rectangle R_m, the approximating parallelogram will have sides determined by the vectors

$$\vec{AB} = \triangle x_m \vec{i} + [f_x\,(x_m^*, y_m^*)\,\triangle x_m]\,\vec{k}$$

and

$$\vec{AC} = \triangle y_m \vec{j} + \left[f_y\,(x_m^*, y_m^*)\,\triangle y_m\right]\vec{k}.$$

If K_m is the area of the approximating parallelogram, we have

$$K_m = \|\vec{AB} \times \vec{AC}\|.$$

To determine K_m, we first find the cross product:

$$\begin{aligned}\vec{AB}\times\vec{AC} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \triangle x_m & 0 & f_x\left(x_m^*, y_m^*\right)\triangle x_m \\ 0 & \triangle y_m & f_y\left(x_m^*, y_m^*\right)\triangle y_m \end{vmatrix} \\ &= \triangle x_m \triangle y_m \left(-f_x\left(x_m^*, y_m^*\right)\vec{i} - f_y\left(x_m^*, y_m^*\right)\vec{j} + \vec{k}\right).\end{aligned}$$

Then we calculate the norm,

$$K_m = \|\vec{AB}\times\vec{AC}\| = \triangle x_m \triangle y_m \sqrt{\left[f_x\left(x_m^*, y_m^*\right)\right]^2 + \left[f_y\left(x_m^*, y_m^*\right)\right]^2 + 1}.$$

Finally, summing over the entire partition, we see that the surface area over R may be approximated by the sum

$$\triangle S_n = \sum_{m=1}^{n} \sqrt{\left[f_x\left(x_m^*, y_m^*\right)\right]^2 + \left[f_y\left(x_m^*, y_m^*\right)\right]^2 + 1}\,(\triangle A_m)$$

where $\triangle A_m = \triangle x_m \triangle y_m$. This is a Riemann sum, and by taking an appropriate limit (as the partition becomes more refined), we find that the surface area S, satisfies

$$\begin{aligned}S &= \lim_{n\to\infty}\sum_{m=1}^{n}\triangle A_m \sqrt{\left[f_x\left(x_m^*, y_m^*\right)\right]^2 + \left[f_y\left(x_m^*, y_m^*\right)\right]^2 + 1} \\ &= \iint_R \sqrt{\left[f_x(x,y)\right]^2 + \left[f_y(x,y)\right]^2 + 1}\,dA.\end{aligned}$$

□

Example 5.46. Find the surface area of the part of the surface $z = x^2 + 2y$ that lies above the triangular region T in the $xy-$plane with vertices $(0,0)$, $(1,0)$, and $(1,1)$.

Solution. The region T is described by $T = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le x\}$. We have

$$\begin{aligned}S &= \iint_R \sqrt{(2x)^2 + (2)^2 + 1}\,dA = \int_0^1 \int_0^x \sqrt{4x^2+5}\,dy\,dx \\ &= \int_0^1 x\sqrt{4x^2+5}\,dx = \frac{1}{12}\left(27 - 5\sqrt{5}\right).\end{aligned}$$

Example 5.47. Find the surface area of a sphere of radius a.

Solution. Let $z = f(x, y) = \sqrt{a^2 - x^2 - y^2}$. Then

$$f_x = \frac{-x}{\sqrt{a^2 - x^2 - y^2}}, \qquad f_y = \frac{-y}{\sqrt{a^2 - x^2 - y^2}},$$

and

$$\sqrt{f_x^2 + f_y^2 + 1} = \sqrt{\frac{x^2 + y^2 + (a^2 - x^2 - y^2)}{a^2 - x^2 - y^2}} = \frac{a}{\sqrt{a^2 - x^2 - y^2}}.$$

In polar form we have

$$\begin{aligned} &8\int_0^{\pi/2}\int_0^a \frac{a}{\sqrt{a^2 - r^2}} r\, dr d\theta \\ &= -4a\int_0^{\pi/2}\int_0^a \frac{(-2r)}{\sqrt{a^2 - r^2}}\, dr d\theta = -8a\int_0^{\pi/2} (-a)\, d\theta = 4\pi a^2. \end{aligned}$$

Example 5.48. Find the surface area of a cylinder of radius a and height h.

Solution. Let $z = f(x, y) = \sqrt{a^2 - x^2}$. Then

$$f_x = \frac{-x}{\sqrt{a^2 - x^2}}, f_y = 0,$$

and

$$\sqrt{f_x^2 + f_y^2 + 1} = \sqrt{\frac{x^2 + a^2 - x^2}{a^2 - x^2}} = \frac{a}{\sqrt{a^2 - x^2}}.$$

We find the surface area as

$$\begin{aligned} 4\int_0^a\int_0^h \frac{a}{\sqrt{a^2 - x^2}}\, dy\, dx &= 4a\int_0^a \frac{ha}{\sqrt{a^2 - x^2}} dx \\ &= 4ah \sin^{-1}\left(\frac{x}{a}\right)\Big|_0^a = 2\pi ah. \end{aligned}$$

Example 5.49. Find the surface area the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

Solution. The plane intersects the paraboloid in the circle $x^2 + y^2 = 9$ when $z = 9$. Therefore, the given surface lies above the disk D with center at the origin and radius 3. Converting to polar coordinates we have

$$\begin{aligned} S &= \iint_R \sqrt{(2x)^2 + (2y)^2 + 1}\, dA = \int_0^{2\pi}\int_0^3 \sqrt{4r^2 + 1} r\, dr\, d\theta \\ &= \int_0^{2\pi} d\theta \int_0^3 \frac{1}{8}\sqrt{4r^2 + 1}(8r)\, dr = \frac{\pi}{6}\left(37\sqrt{37} - 1\right). \end{aligned}$$

Example 5.50. Find the surface area the portion of the sphere $x^2+y^2+z^2=4$ that lies inside the cylinder $x^2+y^2=2y$.

Solution. Let $z=f(x,y)=\sqrt{4-x^2-y^2}$. Then

$$f_x=\frac{-x}{\sqrt{4-x^2-y^2}}, \qquad f_y=\frac{-y}{\sqrt{4-x^2-y^2}},$$

and

$$\sqrt{f_x^2+f_y^2+1}=\sqrt{\frac{x^2+y^2+(4-x^2-y^2)}{4-x^2-y^2}}=\frac{2}{\sqrt{4-x^2-y^2}}.$$

The projected region in polar form is $r=2\sin\theta$. Since half the surface is above the $xy-$ plane and half the surface is below we have

$$\begin{aligned} S&=2\int_0^{\pi}\int_0^{2\sin\theta}\frac{2r}{\sqrt{4-r^2}}drd\theta=-8\int_0^{\pi}\left(\sqrt{4-4\sin^2\theta}-2\right)d\theta \\ &=-8\int_0^{\pi/2}(2\cos\theta-2)\,d\theta-8\int_{\pi/2}^{\pi}(-2\cos\theta-2)\,d\theta=8(\pi-2). \end{aligned}$$

5.13 Surface Area Defined Parametrically

We show a way to find the surface area of the graph of a surface given a parameterization of the surface. We also illustrate how to find the surface area with two examples and then show how to find the surface area of a torus with given inner and outer radii.

::: {#thm- } [Surface Area (Parametrically)] Let S be a surface defined parametrically by

$$\vec{R}(u,v)=x(u,v)\vec{i}+y(u,v)\vec{j}+z(u,v)\vec{k}$$

on the region D in the uv-plane, and assume that S is smooth in the sense that $\vec{R}_u$ and $\vec{R}_v$ are continuous with $\vec{R}_u\times\vec{R}_v\neq\vec{0}$ on D. Then the surface area S is given by

$$S=\iint_D\left\|\vec{R}_u\times\vec{R}_v\right\|dudv.$$

The quantity $\left\|\vec{R}_u\times\vec{R}_v\right\|$ is called the **fundamental cross product** . :::

Proof. Suppose a surface S is defined parametrically by the vector function

$$\vec{R}(u,v)=x(u,v)\vec{i}+y(u,v)\vec{j}+z(u,v)\vec{k}$$

for parameters u and v. Let D be a region in the xy-plane on which x, y, and z, as well as their partial derivatives with respect to u and v are continuous. The partial derivatives of $R(u,v)$ are given by

$$\vec{R}_u = \frac{\partial x}{\partial u}\vec{i} + \frac{\partial y}{\partial u}\vec{j} + \frac{\partial z}{\partial u}\vec{k} \qquad \text{and} \qquad \vec{R}_v = \frac{\partial x}{\partial v}\vec{i} + \frac{\partial y}{\partial v}\vec{j} + \frac{\partial z}{\partial v}\vec{k}.$$

Suppose the region D is subdivided into cells. Consider a typical rectangle in this partition, dimension $\triangle x$ and $\triangle y$, where $\triangle x$ and $\triangle y$ are small. If we project this rectangle onto the surface $\vec{R}(u,v)$, we obtain a curvilinear parallelogram with adjacent sides $\vec{R}_u(u,v)\triangle u$ and $\vec{R}_v(u,v)\triangle v$. The area of this rectangle is approximated by

$$\triangle S = \left\|\vec{R}_u(u,v)\triangle u \times \vec{R}_v(u,v)\triangle v\right\| = \left\|\vec{R}_u(u,v) \times \vec{R}_v(u,v)\right\| \triangle u \triangle v.$$

By taking an appropriate limit, we find the surface area to be a double integral.

□

Example 5.51. Find the area of the surface given parametrically by the equation

$$\vec{R}(u,v) = uv\vec{i} + (u-v)\vec{j} + (u+v)\vec{k}$$

for $u^2 + v^2 \leq 1$.

Solution. We find

$$\vec{R}_u \times \vec{R}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v & 1 & 1 \\ u & -1 & 1 \end{vmatrix} = 2\vec{i} + (u-v)\vec{j} - (u+v)\vec{k}.$$

Therefore the fundamental cross product is

$$\left\|\vec{R}_u \times \vec{R}_v\right\| = \sqrt{4 + (u-v)^2 + (-u-v)^2} = \sqrt{4 + 2u^2 + 2v^2}.$$

Using polar coordinates the surface area is

$$\begin{aligned} S &= \int_0^{2\pi} \int_0^1 \sqrt{4 + 2r^2} r dr d\theta = \frac{1}{4} \int_0^{2\pi} \int_0^1 \sqrt{4 + 2r^2} 4r dr d\theta \\ &= \frac{1}{6} \int_0^{2\pi} \left(6\sqrt{6} - 8\right) d\theta = \frac{2\pi}{3} \left(3\sqrt{6} - 4\right). \end{aligned}$$

Example 5.52. Find the area of the surface given parametrically by the equation

$$\vec{R}(u,v) = (u \sin v)\vec{i} + (u \cos v)\vec{j} + v\vec{k}$$

for $0 \leq u \leq a$ and $0 \leq v \leq b$.

Solution. We find

$$\vec{R}_u = (\sin v)\vec{i} + (\cos v)\vec{j}, \quad \vec{R}_v = (u\cos v)\vec{i} + (-u\sin v)\vec{j} + \vec{k},$$

and

$$\vec{R}_u \times \vec{R}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \sin v & \cos v & 0 \\ u\cos v & -u\sin v & 1 \end{vmatrix} = (\cos v)\vec{i} - (\sin v)\vec{j} - u\vec{k}.$$

Therefore the fundamental cross product is $\left\|\vec{R}_u \times \vec{R}_v\right\| = \sqrt{1+u^2}$. So the surface area is

$$\begin{aligned} S &= \int_0^b \int_0^a \sqrt{1+u^2}\, du dv = \int_0^b \left[\frac{\ln\left|u+\sqrt{1+u^2}\right|}{2} + \frac{u\sqrt{1+u^2}}{2}\Big|_0^a \right] dv \\ &= \frac{b}{2}\left[\ln\left(a+\sqrt{1+a^2}\right) + a\sqrt{1+a^2}\right]. \end{aligned}$$

Example 5.53. Find the area of the torus which is given parametrically by

$$\vec{R}(u,v) = (a + b\cos v)\cos u\vec{i} + (a+b\cos v)\sin u\vec{j} + (b\sin v)\vec{k}$$

for $0 < b < a$, $0 \le u \le 2\pi$, and $0 \le v \le 2\pi$.

Solution. We find

$$\begin{aligned} \vec{R}_u &= -(a+b\cos v)\sin u\vec{i} + (a+b\cos v)\cos u\vec{j}, \\ \vec{R}_v &= -b\sin v\cos u\vec{i} - b\sin v\sin u\vec{j} + b\cos v\vec{k}, \text{ and} \\ \vec{R}_u \times \vec{R}_v &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -(a+b\cos v)\sin u & (a+b\cos v)\cos u & 0 \\ -b\sin v\cos u & -b\sin v\sin u & b\cos v \end{vmatrix} \\ &= \left(b^2\cos^2 v + ab\cos v\right)(\cos u)\vec{i} + \left(b^2\cos^2 v + ab\cos v\right)(\sin u)\vec{j} \\ &\qquad + \left(b^2\sin v\cos v + ab\sin v\right)\vec{k}. \end{aligned}$$

Therefore $\left\|\vec{R}_u \times \vec{R}_v\right\| = \left|ab + b^2\cos v\right|$. So the surface area is

$$S = \int_0^{2\pi}\int_0^{2\pi} \left|ab+b^2\cos v\right| du dv = \int_0^{2\pi} 2\pi\left(ab+b^2\cos v\right) dv = 4\pi^2 ab.$$

5.14 Exercises

Exercise 5.45. Find the surface area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

Exercise 5.46. Find the surface area of the surface of the portion of the plane $4x + y + z = 9$ that lies in he first octant.

Exercise 5.47. Find the surface area of the surface of the portion of the paraboloid $z = 3x^2 + 3y^2$ that lies inside the cylinder $x^2 + y^2 = 1$.

Exercise 5.48. Find the surface area of the surface of the portion of the plane $2x + 2y - z = 0$ that is above the square in the plane with vertices $(0, 1, 0)$, $(0, 0, 0)$, $(1, 0, 0)$, and $(1, 1, 0)$.

Exercise 5.49. Find the surface area of the surface of the portion of the surface $z = x^2$ that lies over the triangular region in the plane with vertices $(0, 0, 0)$, $(0, 1, 0)$, and $(1, 0, 0)$.

Exercise 5.50. Find the surface area of the surface of the portion of the sphere $x^2 + y^2 + z^2 = 25$ inside the cylinder $x^2 + y^2 = 9$.

Exercise 5.51. Find the surface area of the surface of the portion of the cone $z = 2\sqrt{x^2 + y^2}$ inside the cylinder $x^2 + y^2 = 4$.

Exercise 5.52. Find the surface area of the surface of the portion of the sphere $x^2 + y^2 + z^2 = a^2$ inside the cylinder $x^2 + y^2 = ax$ and above the xy-plane.

Exercise 5.53. Setup, but do not evaluate, the iterated integral for the surface area of the portion of the surface given by $z = e^{-x} \sin y$ over the triangle with vertices $(0, 0, 0)$, $(0, 1, 1)$, $(0, 1, 0)$.

Exercise 5.54. Setup, but do not evaluate, the iterated integral for the surface area of the portion of the surface given by $x = z^3 - yz + y^3$ over the square $(0, 0, 0)$, $(0, 0, 2)$, $(0, 2, 0)$, and $(0, 2, 2)$.

Exercise 5.55. Setup, but do not evaluate, the iterated integral for the surface area of the portion of the surface given by $z = \cos\left(x^2 + y^2\right)$ over the disk $x^2 + y^2 \leq \frac{\pi}{2}$.

Exercise 5.56. Setup, but do not evaluate, the iterated integral for the surface area of the portion of the surface given by $z = e^{-x} \cos y$ over the disk $x^2 + y^2 \leq 2$.

Exercise 5.57. Setup, but do not evaluate, the iterated integral for the surface area of the portion of the surface given by $z = x^2 + 5xy + y^2$ over the region in the xy-plane bounded by the curve $xy = 5$ and the line $x + y = 6$.

Exercise 5.58. Setup, but do not evaluate, the iterated integral for the surface area of the portion of the surface given by $z = x^2 + 3xy + y^2$ over the region in the xy-plane bounded by $0 \leq x \leq 4$, $0 \leq y \leq x$.

Exercise 5.59. Compute the magnitude of the fundamental cross product for the surface defined parametrically by $\vec{R}(u, v) = (2u \sin v)\,\vec{\imath} + (2u \cos v)\,\vec{\jmath} + u^2\,\vec{k}$.

Exercise 5.60. Compute the magnitude of the fundamental cross product for the surface defined parametrically by $\vec{R}(u, v) = (4 \sin u \cos v)\vec{\imath} + (4 \sin u \sin v)\,\vec{\jmath} + (5 \cos u)\vec{k}$.

Exercise 5.61. Compute the magnitude of the fundamental cross product for the surface defined parametrically by $\vec{R}(u, v) = u\,\vec{\imath} + v^2\,\vec{\jmath} + u^3\,\vec{k}$.

Exercise 5.62. Compute the magnitude of the fundamental cross product for the surface defined parametrically by $\vec{R}(u, v) = (2u \sin v)\,\vec{\imath} + (2u \cos v)\,\vec{\jmath} + \left(u^2 \sin 2v\right)\vec{k}$.

Exercise 5.63. Find the area of the surface given parametrically by the equation $\vec{R}(u, v) = uv\,\vec{\imath} + (u - v)\,\vec{\jmath} + (u + v)\vec{k}$ for $u^2 + v^2 \leq 1$.

Exercise 5.64. A spiral ramp has the vector parametric equation $\vec{R}(u,v) = (u\cos v)\,\vec{i} + (u\sin v)\,\vec{j} + v\,\vec{k}$ for $0 \le u \le 1$, $0 \le v \le \pi$. Find the surface area of this ramp.

Chapter 6

Triple Integrals

6.1 The Definition of a Triple Integral

Suppose $f(x, y, z)$ is defined on a closed bounded solid region R, which in turn is contained in a box B in space. We partition B into a finite number of smaller boxes, call this partition P, we choose a representative point (x_k^*, y_k^*, z_k^*) from each subdivision in the partition and we form the sum,

$$\sum_{k=1}^{N} f\left(x_k^*, y_k^*, z_k^*\right) \triangle V_k$$

where $\triangle V_k$ is the volume of the k-th representative subdivision. This sum is called the **Riemann sum** of $f(x, y, z)$ with respect to the partition P and the cell representation $\left(x_k^*, y_k^*, z_k^*\right)$. To measure the size of the rectangles in the partition P, we define the norm $\|P\|$ of the partition to be the length of the longest diagonal of any of the subdivisions in the partition. We refine the partition by subdividing the subdivisions in such a way that the norm decreases. When this process is applied to the Riemann sum and the norm decreases to zero, we write

$$\iiint_R f(x, y, z)\, dV = \lim_{\|P\| \to 0} \sum_{k=1}^{N} f\left(x_k^*, y_k^*, z_k^*\right) \triangle V_k.$$

If this limit exists, its value is called the **triple integral** of f over the closed bounded region R.

6.2 Basic Properties of Triple Integrals

::: {#thm- } [Properties of Triple Integrals] Assume that all the given integrals exist on a rectangular region R for given functions $f(x, y, z)$ and

$g(x, y, z)$.

- For constants a and b,

$$\iiint_R (af + bg)(x, y, z)\, dV = a \iiint_R f(x, y, z)\, dV + b \iiint_R g(x, y, z)\, dV.$$

- If $f(x, y, z) \geq g(x, y, z)$ throughout a closed bounded region R, then

$$\iiint_R f(x, y, z)\, dV \geq \iiint_R g(x, y, z)\, dV.$$

- If the closed bounded region of integration R is subdivided into two disjoint subdivisions R_1 and R_2 whose union is R, then

$$\iiint_R f(x, y, z)\, dV = \iiint_{R_1} f(x, y, z)\, dV + \iiint_{R_2} f(x, y, z)\, dV.$$

:::

6.3 Fubini's Theorem for Triple Integrals

::: {#thm- } Fubini's Theorem for Triple Integrals If $f(x, y, z)$ is continuous over a rectangular box $B = \{(x, y, z) \mid a \leq x \leq b, c \leq y \leq d, r \leq z \leq s\}$, then the triple integral may be evaluated by the iterated integral

$$\iiint_R f(x, y)\, dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\, dxdydz$$

The iterated integral can be performed in any order, with appropriate adjustments to the limits of integration. :::

Example 6.1. Evaluate the triple integral over B given

$$\iiint_B xyz^2\, dV \quad \text{and} \quad B = \{(x, y, z) \mid 0 \leq x \leq 1, -1 \leq y \leq 2, 0 \leq z \leq 3\}.$$

Solution. Since $f(x, y) = xyz^2$ is continuous on B, we can use Fubini's theorem for triple integrals

$$\int_0^3 \int_{-1}^2 \int_0^1 xyz^2\, dxdydz = \int_0^3 \int_{-1}^2 \frac{yz^2}{2}\, dydz = \int_0^3 \frac{3z^2}{4}\, dz = \frac{27}{4}.$$

Example 6.2. Evaluate the triple integral over B given

$$\iiint_B z^2 y e^x\, dV \quad \text{and} \quad B = \{(x, y, z) \mid 0 \leq x \leq 1, 1 \leq y \leq 2, -1 \leq z \leq 1\}.$$

Solution. Since $f(x,y) = z^2ye^x$ is continuous on B, we can use Fubini's theorem for triple integrals

$$\int_{-1}^{1}\int_{1}^{2}\int_{0}^{1} z^2ye^x\,dxdydz = \int_{-1}^{1}\int_{1}^{2}(-1+e)yz^2\,dydz$$
$$= \int_{-1}^{1}\frac{3}{2}(-1+e)z^2\,dz = -1+e$$

Example 6.3. Evaluate the iterated integral $\int_0^{2\pi}\int_0^4\int_0^1 zr\,dz\,dr\,d\theta$

Solution. We find

$$\int_0^{2\pi}\int_0^4\int_0^1 zr\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^4 \frac{r}{2}\,dr\,d\theta = \int_0^{2\pi} 4\,d\theta = 8\pi.$$

In many examples a sketch of the region of integration in the plane, can explain how to visualize the solid region of integration, and how to setup the limits of integration. Recall from studying double integrals, that a vertically simple region D_1 is a region of the plane that can be described by the inequalities

$$D_1 = \{(x,y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$$

where $g_1(x)$ and $g_2(x)$ are continuous functions of x on $[a,b]$. Similarly, a horizontally simple region D_2, in the plane is a region that can be described by the inequalities

$$D_2 = \{(x,y) \mid c \le y \le d, h_1(y) \le x \le h_2(y)\}$$

where $h_1(x)$ and $h_2(x)$ are continuous functions of y on $[c,d]$. In some cases it is possible to evaluate a triple integral by evaluating a triple iterated integral over a solid region that when projected onto the xy-plane can be described as a vertically simple or a horizontally simple region.

::: {#thm- } [Fubini's Theorem for z-Simple Regions]

Suppose R is a solid region bounded below by the surface $z = u(x,y)$ and above by the surface $z = v(x,y)$ that projects onto the region D in the xy-plane. If D is either a vertically simple or a horizontally simple region, then the triple integral of the continuous function $f(x,y,z)$ over R is

$$\iiint_R f(x,y)\,dV = \iint_D \left(\int_{u(x,y)}^{v(x,y)} f(x,y,z)\,dz\right)dA.$$

:::

Example 6.4. Evaluate

$$\iiint_D \frac{z}{\sqrt{x^2+y^2}}\,dx\,dy\,dz$$

and D is the solid bounded above by the plane $z = 2$ and below by the surface $x^2 + y^2 - 2z = 0$.

Solution. We consider the region of integration as being z-simple by projecting onto the xy-plane; and in the xy-plane the region is bounded by $x^2 + y^2 = 4$. Using polar coordinates the triple integral is evaluated as

$$\begin{aligned}\iiint_D \frac{z}{\sqrt{x^2+y^2}}\,dx\,dy\,dz &= 4\int_0^{\pi/2}\int_0^2\int_{r^2/2}^2 \frac{z}{\sqrt{r^2}} r\,dz\,dr\,d\theta\\ &= 4\int_0^{\pi/2}\int_0^2 \frac{r\left(2-\frac{r^4}{8}\right)}{\sqrt{r^2}}\,dr\,d\theta\\ &= 4\int_0^{\pi/2}\frac{16}{5}\,d\theta = \frac{32\pi}{5}.\end{aligned}$$

Example 6.5. Change the order of integration to show that

$$\int_0^x\int_0^v f(u)\,du\,dv = \int_0^x (x-u)f(u)\,du$$

Also, show that

$$\int_0^x\int_0^v\int_0^u f(w)\,dw\,du\,dv = \frac{1}{2}\int_0^x (x-w)2f(w)dw.$$

Solution. Let u be the horizontal axis, v be the vertical axis, and consider the triangular region T determined by $u = 0$, $v = x$, and $v = u$. Switching the order of integration, we obtain:

$$\int_0^x\int_0^v f(u)\,du\,dv = \int_0^x\int_u^x f(u)\,dv\,du = \int_0^x (x-u)f(u)\,du$$

as desired.

6.4 Volume as a Triple Integral

Example 6.6. Find the volume of the bounded solid bounded by the sphere $x^2 + y^2 + z^2 = 2$ and the paraboloid $x^2 + y^2 = z$.

Solution. By solving for the intersection, $x^2+y^2+z^2=2$ with $z=x^2+y^2$ and so $z^2+z-2=0$ leads to $z=1$. Therefore region of integration is $x^2+y^2=1$ and so we have a z-simple region with

$$\begin{aligned}
V &= 4\int_0^1\int_0^{\sqrt{1-x^2}}\int_{x^2+y^2}^{\sqrt{2-x^2-y^2}} dz\,dy\,dx \\
&= 4\int_0^1\int_0^{\sqrt{1-x^2}} \left(-x^2-y^2+\sqrt{-x^2-y^2+2}\right) dy\,dx \\
&= 4\int_0^{\pi/2}\int_0^1 \left(\sqrt{2-r^2}-r^2\right) r\,dr\,d\theta \\
&= 4\int_0^{\pi/2} \left(-\frac{7}{12}+\frac{2\sqrt{2}}{3}\right) d\theta = \left(\frac{8\sqrt{2}-7}{6}\right)\pi \approx 2.25865.
\end{aligned}$$

Example 6.7. Find the volume of the bounded solid bounded by the cylinders $y=z^2$ and $y=2-z^2$ and the planes $x=1$ and $x=-2$.

Solution. We represent this solid as a x-simple region. The curves intersect where $y=z^2$ and $y=2-z^2$ and so $z=\pm1$. Therefore the volume is given by,

$$\begin{aligned}
V &= 2\int_0^1\int_{z^2}^{2-z^2}\int_{-2}^1 dx\,dy\,dz = 2\int_0^1\int_{z^2}^{2-z^2} 3\,dy\,dz \\
&= 2\int_0^1 \left(6-6z^2\right) dz = 8
\end{aligned}$$

Example 6.8. Find the volume of the bounded solid the tetrahedron T bounded by the planes $x+2y+z=2$, $x=2y$, $x=0$, and $z=0$.

Solution. The vertices of the tetrahedron are $(0,0,2)$, $\left(1,\frac{1}{2},0\right)$, and $(0,1,0)$. So the region of integration in the xy-plane is bounded by the lines $x=0$, $y=\frac{1}{2}x$, and $y=1-\frac{1}{2}x$ and is described by the set

$$R=\{(x,y)\mid 0\le x\le 1, x/2\le y\le 1-x/2\}$$

This is determined by the vertices of the tetrahedron in the xy-plane and by determining the equations of the lines through these vertices. So the upper boundary is the plane $x+2y+z=2$; that is $z=2-x-2y$. Therefore, the volume of the tetrahedron T bounded by the planes $x+2y+z=2$,

$x = 2y$, $x = 0$, and $z = 0$ is

$$V = \iiint_R dV = \int_0^1 \int_{x/2}^{1-x/2} \int_0^{2-x-2y} dz\,dy\,dx$$
$$= \int_0^1 \int_{x/2}^{1-x/2} (-x-2y+2)\,dy\,dx = \int_0^1 \left(x^2-2x+1\right)\,dx = \frac{1}{3}.$$

6.5 Applications of Triple Integrals

Definition 6.1. The **average value** of a function f of three variables over a solid region R is defined to be

$$f_{\text{av}} = \frac{1}{V(R)} \int\int_R\int f(x,y,z)\,dV$$

where $V(R)$ is the volume of the solid R.

If a body in space occupies a region R then the mass of the body is the triple integral of the mass density. The first moment of the body about a plane is the triple integral of the product of the signed distance to the plane and the mass density function where the distance is from the differential element of volume $dx\,dydz$. The second moment (called **moment of inertia**) of a body about an axis is the triple integral of the product square of the distance from the axis and the mass density function where the distance is from the differential element of volume $dx\,dydz$. For example, a moment of inertia is used in computing kinetic energy of rotation $(1/2)I\omega^2$ where ω is the angular speed of rotation.

Suppose R is a region in space and that δ is a continuous density function of R. Then for a body occupying a region R in space the center of mass is located at the point $(\overline{x}, \overline{y}, \overline{z})$ where

$$\overline{x} = \frac{M_{yz}}{m}, \qquad \overline{y} = \frac{M_{xz}}{m}, \qquad \overline{z} = \frac{M_{xy}}{m}$$

and the **first moments** M_{xy}, M_{yx}, and M_{xz} about the xy-plane, yz-plane, and the xz-plane, respectively are

$$M_{xy} = \int\int_R\int z\delta(x,y,z)dx\,dydz,$$
$$M_{yz} = \int\int_R\int x\delta(x,y,z)dx\,dydz, \text{ and}$$
$$M_{xz} = \int\int_R\int y\delta(x,y,z)dx\,dydz, .$$

where

$$m = \iiint\limits_R \delta(x, y, z) dV$$

is the mass of the body. Further, I_x, I_y, and I_z, the **second moments** (or **moments of inertia**) about the coordinate axes are

$$I_x = \iiint\limits_R (y^2 + z^2)\, \delta(x, y, z) dx\, dy dz,$$
$$I_y = \iiint\limits_R (x^2 + z^2)\, \delta(x, y, z) dx\, dy dz, \text{ and}$$
$$I_z = \iiint\limits_R (x^2 + y^2)\, \delta(x, y, z) dx\, dy dz.$$

More generally the **moments of inertia** about a line L is

$$I_L = \iiint\limits_R r^2 \delta(x, y, z) dV$$

where $r(x, y, z)$ is the distance from the point (x, y, z) to the line L. Intuitively speaking, the moment of inertia I is a measure of the resistance of a body to rotational motion.

% Moreover, the **radius of gyration** about a line L is given as $R_L = \sqrt{I_L/M}$ where M is the total mass of the object.

Example 6.9. Find the moment of inertia about the z-axis of the solid tetrahedron S with vertices $(0, 0, 0)$, $(0, 1, 0)$, $(1, 0, 0)$, $(0, 0, 1)$, and density $\delta(x, y, z) = x$.

Solution. The solid S can be described as the set of all (x, y, z) such that $0 \le x \le 1$, $0 \le y \le 1 - x$, and $0 \le z \le 1 - x - y$. Thus,

$$\begin{aligned}
I_z &= \iiint\limits_S (x^2 + y^2)\delta(x, y, z)\, dV \\
&= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x(x^2 + y^2)\, dz\, dy\, dx \\
&= \int_0^1 \int_0^{1-x} x(x^2 + y^2)(1 - x - y) dy\, dx \\
&= \int_0^1 \left(\frac{x^3(1 - x^2)}{2} - \frac{x(1 - x^4)}{12} \right) dx = \frac{1}{90}.
\end{aligned}$$

6.6 Exercises

Exercise 6.1. Find the following iterated integrals.

- $\int_1^4 \int_{-2}^3 \int_2^5 dx\, dy\, dz$.
- $\int_{-1}^3 \int_0^2 \int_{-2}^2 dy\, dz\, dx$.
- $\int_1^2 \int_0^1 \int_{-1}^2 8x^2yz^3\, dx\, dy\, dz$.
- $\int_4^7 \int_{-1}^2 \int_0^3 x^2y^2z^2\, dx\, dy\, dz$.
- $\int_0^2 \int_0^x \int_0^{x+y} xyz\, dz\, dy\, dx$.
- $\int_0^1 \int_{\sqrt{x}}^{\sqrt{1+x}} \int_0^{xy} y^{-1}z\, dz\, dy\, dx$.
- $\int_{-1}^2 \int_0^\pi \int_1^4 yz\cos xy\, dz\, dx\, dy$.
- $\int_0^\pi \int_0^1 \int_0^1 x^2y\cos xyz\, dz\, dy\, dx$.
- $\int_0^1 \int_0^y \int_0^{\ln(y)} e^{z+2x}\, dz\, dx\, dy$.

Exercise 6.2. Evaluate the triple integral over the given region

- $\iiint\limits_D (x^2y+y^2z)\, dV$ where D is the boxed region defined by $1 \le x \le 3$, $-1 \le y \le 1$, and $2 \le z \le 4$.
- $\iiint\limits_D (xy+2yz)\, dV$ where D is the boxed region defined by $2 \le x \le 4$, $1 \le y \le 3$, and $-2 \le z \le 4$.
- $\iiint\limits_D xyz\, dV$ where D is the region bounded by the paraboloid$z = x^2+y^2$ and the plane $z=1$.
- $\iiint\limits_D yz\, dV$ where D is the solid in the first octant bounded by the hemisphere $x = \sqrt{9-y^2-z^2}$ and the coordinate planes.

Exercise 6.3. Find the volume of the region between the two elliptic paraboloids $z = x^2\big/(9+y^2-4)$ and $z = -x^2\big/(9-y^2+4)$.

Exercise 6.4. Change the order of integration to show that

$$\int_0^x \int_0^v f(u)\, du\, dv = \int_0^x (x-u)f(u)\, du.$$

Also, show that

$$\int_0^x \int_0^v \int_0^u f(w)dwdudv = \frac{1}{2}\int_0^x (x-w)2f(w)\, dw.$$

Exercise 6.5. Higher-dimensional multiple integrals can be defined and evaluated in essentially the same way as double integrals and triple integrals. Evaluate the multiple integrals

$$\iiiint_H xyw^2 \, dx \, dy \, dz \, dw$$

where H is the four-dimensional (hyperbox) defined by $0 \le x \le 1$, $0 \le y \le 2$, $-1 \le z \le 1$, and $1 \le w \le 2$.

Exercise 6.6. Find the volume V of the solids bounded by the graphs of the equations by using triple integration. The solid bounded by the sphere $x^2 + y^2 + z^2 = 2$ and the paraboloid $x^2 + y^2 = z$. The solid of the region bounded by the cylinders $y = z^2$ and $y = 2 - z^2$ and the planes $x = 1$ and $x = -2$.

Exercise 6.7. Evaluate the multiple integral

$$\iiiint_H e^{x-2y+z+w} \, dw \, dz \, dy \, dx$$

where H is the four-dimensional region bounded by the hyperplane $x + y + z + w = 4$ and the coordinate spaces $x = 0$, $y = 0$, $z = 0$, and $w = 0$ in the first hyperoctant (where $x \ge 0$, $y \ge 0$, $z \ge 0$, $w \ge 0$).

Exercise 6.8. A solid of constant density is bounded below by the plane $z = 0$, on the sides by the elliptical cylinder $x^2 + 4y^2 = 4$, and above by the plane $z = 2 - x$. Find $\bar{x}$, $\bar{y}$, and then evaluate the integral

$$M_{xy} = \int_{-2}^{2} \int_{-(1/2)\sqrt{4-x^2}}^{1/2\sqrt{4-x^2}} \int_{0}^{2-x} z \, dz \, dy \, dx$$

to determine $\bar{z}$.

Exercise 6.9. Find the centroid and the moments of inertia I_x, I_y, and I_z of the tetrahedron whose vertices are $(0, 0, 0)$, $(1, 0, 0)$, and $(0, 0, 1)$. Find the radius of gyration of the tetrahedron about the x-axis. Compare it with the distance from the centroid to the x-axis.

Exercise 6.10. A solid cube, 2 units on a side, is bounded by the planes $x = \pm 1$, $z = \pm 1$, $y = 3$ and $y = 5$. Find the center of mass, moments of inertia, and radii of gyration about the coordinate axes.

Exercise 6.11. A solid in the first octant is bounded by the coordinate lanes and the plane $x+y+z = 2$. The density of the solid is $\delta(x, y, z) = 2x$. Find the mass and the center of mass.

Exercise 6.12. Find the mass of the solid region bounded by the parabolic surface $z = 16 - 2x^2 - 2y^2$ and $z = 2x^2 + 2y^2$ if the density of the solid is $\delta(x, y) = \sqrt{x^2 + y^2}$.

Exercise 6.13. The container is in the shape of the region bounded by $y = 0$, $z = 0$, $z = 4 - x^2$, and $x = y^2$. The density of the liquid filling the region is $\delta(x, y) = kxy$ where k is a constant.

Exercise 6.14. Find the centroid for the part of the spherical solid with density $\delta = 2$ described by $x^2 + y^2 + z^2 \leq 9$, $x \geq 0$, $y \geq 0$, and $z \geq 0$.

Exercise 6.15. Find the centroid for the solid bounded by the surface $z = \sin x$, $x = 0$, $x = \pi$, $y = 0$, $z = 0$, and $y + z = 1$, where the density is $\delta = 1$.

6.7 Cylindrical Coordinates

Each point in three dimensions is uniquely represented in cylindrical coordinates by (r, θ, z) using $0 \leq r < \infty$, $0 \leq \theta < 2\pi$, and $-\infty < z < +\infty$. The conversion formulas involving rectangular coordinates (x, y, z) and cylindrical coordinates (r, θ, z) are

$$r = \sqrt{x^2 + y^2} \qquad \tan\theta = \frac{y}{x} \qquad z = z$$

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z.$$

A triple integral %

$$\iiint_R f(x, y, z)\, dV$$

over a region R can sometimes be evaluated by transforming to cylindrical coordinates if the region of integration R is z-simple and the projection of R onto the xy-plane is a region D that can be described naturally in terms of polar coordinates.

::: {#thm- } [Triple Integrals in Cylindrical Coordinates] Assume R is a solid region with continuous upper surface $z = v(r, \theta)$ and continuous lower surface $z = u(r, \theta)$ and assume is D be the projection of the solid onto the xy-plane expressed in polar coordinates:

$$D = \{(r, \theta) : \alpha \le r \le \beta, r_1(\theta) \le r(\theta) \le r_2(\theta)\}$$

where r_1 and r_2 are continuous functions of θ. If $f(x, y, z)$ is continuous on R, then the triple integral of f over R can be evaluated as follows

$$\iiint_R f(x, y, z)dV = \int_{\alpha}^{\beta} \int_{r_1(\theta)}^{r_2(\theta)} \int_{u(r,\theta)}^{v(r,\theta)} f(r\cos\theta, r\sin\theta, z)rdzdr\,d\theta.$$

:::

Example 6.10. Find the volume of the bounded solid bounded by the paraboloid $4x^2 + 4y^2 + z = 1$ and the xy-plane.

Solution. The projection of the solid region onto the xy-plane is the region enclosed by $x^2 + y^2 = \frac{1}{4}$. In cylindrical coordinates the volume is determined as,

$$\begin{aligned} 4\int_0^{\pi/2} \int_0^{1/2} \int_0^{1-4r^2} rdzdrd\theta &= 4\int_0^{\pi/2} \int_0^{1/2} \left(r - 4r^3\right) drd\theta \\ &= 4\int_0^{\pi/2} \frac{1}{16}\, d\theta = \frac{\pi}{8}. \end{aligned}$$

Example 6.11. Evaluate the iterated integral

$$I = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{x^2+y^2}^{\sqrt{2-x^2-y^2}} z\,dz\,dy\,dx.$$

Solution. In Cartesian coordinates the region of integration D is given as

$$\left\{(x, y, z) \mid x^2 + y^2 \le z \le \sqrt{2 - x^2 - y^2}, -\sqrt{1 - x^2} \le y \le \sqrt{1 - x^2}, -1 \le x \le 1\right\}$$

We use cylindrical coordinates to evaluate the triple iterated integral I as follows

$$\begin{aligned} I &= \int_0^{2\pi} \int_0^{1} \int_{r^2}^{\sqrt{2-r^2}} zrdzdrd\theta \\ &= \int_0^{2\pi} \int_0^{1} r\left(-\frac{r^4}{2} - \frac{r^2}{2} + 1\right) drd\theta = \int_0^{2\pi} \frac{7}{24}\, d\theta = \frac{7\pi}{12}. \end{aligned}$$

Example 6.12. Evaluate the triple integral over the given region

$$I = \iiint_R \left(x^4 + 2x^2y^2 + y^4\right)\, dx\, dy\, dz$$

where R is the cylindrical solid $x^2 + y^2 \le a^2$ with $0 \le z \le \frac{1}{\pi}$

Solution. We consider the region of integration as being z-simple by projecting onto the xy-plane; and in the xy-plane the region is bounded by $x^2 + y^2 = a^2$. Notice that

$$x^4 + 2x^2y^2 + y^4 = \left(x^2 + y^2\right)^2.$$

In cylindrical coordinates the triple integral is evaluated as,

$$\begin{aligned} I &= 4\int_0^{\pi/2}\int_0^a\int_0^{1/\pi} r^4 r dz dr d\theta \\ &= 4\int_0^{\pi/2}\int_0^a \frac{r^5}{\pi} dr d\theta = 4\int_0^{\pi/2} \frac{a^6}{6\pi}\, d\theta = \frac{a^6}{3}. \end{aligned}$$

Example 6.13. Evaluate the iterated integral

$$\int_0^{\pi}\int_0^2\int_0^{\sqrt{4-r^2}} r\sin\theta\, dz\, dr\, d\theta.$$

Solution. We find

$$\begin{aligned} \int_0^{\pi}\int_0^2\int_0^{\sqrt{4-r^2}} r\sin\theta\, dz\, dr\, d\theta &= \int_0^{\pi}\int_0^2 r\sqrt{4-r^2}\sin\theta\, dr\, d\theta \\ &= \int_0^{\pi} \frac{8\sin\theta}{3}\, d\theta = \frac{16}{3} \end{aligned}$$

Example 6.14. Evaluate the iterated integral

$$\int_0^{\pi/4}\int_0^1\int_0^{\sqrt{r}} r^2\sin\theta\, dz\, dr\, d\theta.$$

Solution. We find

$$\begin{aligned} \int_0^{\pi/4}\int_0^1\int_0^{\sqrt{r}} r^2\sin\theta\, dz\, dr\, d\theta &= \int_0^{\pi/4}\int_0^1 r^{5/2}\sin\theta\, dr\, d\theta \\ &= \int_0^{\pi/4} \frac{2\sin\theta}{7}\, d\theta = \frac{2}{7}\left(1 - \frac{1}{\sqrt{2}}\right) = \frac{2-\sqrt{2}}{7} \end{aligned}$$

6.8 Spherical Coordinates

Each point in three dimensions is uniquely represented in spherical coordinates by (ρ, θ, ϕ) using $0 \leq \rho < \infty$, $0 \leq \theta < 2\pi$, and $0 \leq \phi \leq \pi$. The conversion formulas from rectangular coordinates (x, y, z) to spherical coordinates (ρ, θ, ϕ) are

$$\begin{array}{lll} \rho = \sqrt{x^2 + y^2 + z^2} & \theta = \tan^{-1}\left(\frac{y}{x}\right) & \phi = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) \\ x = \rho \sin\phi \cos\theta & y = \rho \sin\phi \sin\theta & z = \rho \cos\phi \end{array}$$

Using Jacobians, we can show that the element of volume in spherical coordinates is

$$dV = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi.$$

::: {#thm- } [Triple Integrals in Spherical Coordinates] If $f(x, y, z)$ is continuous on the closed bounded region R, then the triple integral of f over R is given by

$$\begin{aligned} &\iiint_R f(x,y,z) dV \\ &= \iiint_S f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi) \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \end{aligned}$$

where S is the region R expressed in spherical coordinates. :::

Example 6.15. Find the volume of the bounded solid S inside the sphere of radius a.

Solution. Since an equation of the sphere is $\rho = a$ for $0 \leq \theta < 2\pi$ and $0 \leq \phi \leq \pi$, the volume is determined by evaluating a triple iterated integral in spherical coordinates

$$\begin{aligned} \iiint_S dV &= \int_0^{2\pi} \int_0^{\pi} \int_0^{a} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \\ &= \int_0^{2\pi} \int_0^{\pi} \frac{1}{3} a^3 \sin\phi \, d\phi \, d\theta = \frac{4a^3\pi}{3}. \end{aligned}$$

Example 6.16. Evaluate the triple integral over the given region

$$I = \iiint_D z^2 \, dx \, dy \, dz$$

where D is the solid hemisphere $x^2 + y^2 + z^2 \leq 1$ and $z \geq 0$

Solution. Using spherical coordinates to evaluate the triple integral

$$\begin{aligned} I =& 4\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \rho^4\cos^2\phi\sin\phi \; d\rho\, d\theta\, d\phi \\ &= 4\int_0^{\pi/2}\int_0^{\pi/2} \frac{1}{5}\cos^2\phi\sin\phi \; d\theta\, d\phi \\ &= 4\int_0^{\pi/2} \frac{1}{10}\pi\cos^2\phi\sin\phi \;\; d\phi \\ &= \frac{4\pi}{10}\left(-\frac{\cos\phi}{4} - \frac{1}{12}\cos(3\phi)\right) = \frac{2\pi}{15}. \end{aligned}$$

Example 6.17. Evaluate the triple integral over the given region

$$I = \iiint_D \frac{dx\, dydz}{\sqrt{x^2+y^2+z^2}}$$

where D is the solid sphere $x^2+y^2+z^2 \le 3$.

Solution. Since the integrand is symmetric about the origin we can use symmetry. In spherical coordinates the triple integral I evaluates to,

$$\begin{aligned} I &= 8\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{\sqrt{3}} \rho\sin\phi\, d\rho\, d\theta\, d\phi \\ &= 8\int_0^{\pi/2}\int_0^{\pi/2} \frac{3\sin\phi}{2}\, d\theta\, d\phi \\ &= 8\int_0^{\pi/2} \frac{3}{4}\pi\sin(\phi)\, d\phi = 6\pi. \end{aligned}$$

Example 6.18. Evaluate the iterated integral

$$I = \int_0^{\pi/2}\int_0^{2\pi}\int_0^2 \cos\phi\sin\phi\, d\rho\, d\theta d\phi.$$

Solution. We find

$$\begin{aligned} I &= \int_0^{\pi/2}\int_0^{2\pi} 2\cos\phi\sin\phi\, d\theta\, d\phi \\ &= \int_0^{\pi/2} 4\pi\cos\phi\sin\phi\, d\phi = 2\pi. \end{aligned}$$

Example 6.19. Evaluate the iterated integral

$$I = \int_0^{\pi/2} \int_0^{\pi/4} \int_0^{\cos\phi} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi.$$

Solution. We find

$$\begin{aligned} I &= \int_0^{\pi/2} \int_0^{\pi/4} \frac{1}{3} \cos^3\phi \sin\phi \, d\theta \, d\phi \\ &= \int_0^{\pi/2} \frac{1}{12}\pi \cos^3(\phi) \sin\phi \, d\phi = \frac{\pi}{48}. \end{aligned}$$

Example 6.20. Find the volume of the bounded solid in the spherical solid $\rho \leq 4$ after the solid cone $\phi \leq \pi/6$ has been removed.

Solution. We will use spherical coordinates and evaluate a triple iterated integral to find the volume

$$\begin{aligned} V &= \int_0^{2\pi} \int_{\pi/6}^{\pi} \int_0^{4} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_{\pi/6}^{\pi} \frac{64\sin\phi}{3} \, d\phi \, d\theta \\ &= \int_0^{2\pi} \frac{64}{3}\left(1 + \frac{\sqrt{3}}{2}\right) d\theta = \frac{64\pi}{3}\left(2 + \sqrt{3}\right). \end{aligned}$$

Example 6.21. Evaluate the triple iterated integral

$$I = \int_0^{3} \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} \left(x^2 + y^2 + z^2\right) dz \, dx \, dy.$$

Solution. Converting to spherical coordinates we evaluate I as follows,

$$\begin{aligned} I &= \int_0^{\pi/4} \int_0^{\pi/2} \int_0^{3\sqrt{2}} \rho^4 \sin\phi \, d\rho \, d\theta \, d\phi \\ &= \int_0^{\pi/4} \int_0^{\pi/2} \frac{972}{5}\sqrt{2} \sin\phi \, d\theta \, d\phi \\ &= \int_0^{\pi/4} \frac{486}{5}\sqrt{2}\pi \sin\phi \, d\phi = \frac{486\pi}{5}\left(\sqrt{2} - 1\right). \end{aligned}$$

6.9 Exercises

Exercise 6.16. Evaluate the following iterated integrals.

- $\int_0^{\pi} \int_0^2 \int_0^{\sqrt{4-r^2}} r \sin\theta \, dz \, dr \, d\theta$
- $\int_0^{\pi/4} \int_0^1 \int_0^{\sqrt{r}} r^2 \sin\theta \, dz \, dr \, d\theta$
- $\int_0^{2\pi} \int_0^4 \int_0^1 zr \, dz \, dr \, d\theta$
- $\int_{-\pi/4}^{\pi/3} \int_0^{\sin\theta} \int_0^{4\cos\theta} r \, dz \, dr \, d\theta$
- $\int_0^{\pi/2} \int_0^{\pi/4} \int_0^{\cos\phi} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$
- $\int_0^{\pi/2} \int_0^{2\pi} \int_0^2 \cos\phi \sin\phi \, d\rho \, d\theta \, d\phi$

6.10 Change of Variables In Multiple Integrals

Jacobians

If $x = x(u, v)$ and $y = y(u, v)$ then the **Jacobian** of x and y with respect to u and v is

$$\frac{\partial(x, y)}{\partial(u, v)} = J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}.$$

Example 6.22. Determine the Jacobian for the transformation from the rectangular plane to the polar plane.

Solution. The conversion formulas are $x = r\cos\theta$ and $y = r\sin\theta$. So the Jacobian is

$$J(r, \theta) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta - \left(-r\sin^2\theta\right) = r.$$

More generally for

$$x_1 = x_1\left(u_1, ..., u_n\right), \quad x_2 = x_2\left(u_1, \dots, u_n\right), \quad \dots \quad x_n = x_n\left(u_1, ..., u_n\right)$$

the **Jacobian** of $x_1, \dots, x_n$ with respect to $u_1, \dots, u_n$ is

$$J\left(u_1, ..., u_n\right) = \begin{vmatrix} \frac{\partial x_1}{\partial u_1} & ... & \frac{\partial x_1}{\partial u_n} \\ ... & ... & ... \\ \frac{\partial x_n}{\partial u_1} & ... & \frac{\partial x_n}{\partial u_n} \end{vmatrix}.$$

Example 6.23. Determine the Jacobian for the transformation from rectangular coordinates to cylindrical coordinates

Solution. The conversion formulas are $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$. So the Jacobian is

$$J(r,\theta,z) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$
$$= r\cos^2\theta - \left(-r\sin^2\theta\right) = r.$$

Example 6.24. Determine the Jacobian for the transformation from rectangular coordinates to spherical coordinates.

Solution. The conversion formulas are $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$ and $z = \rho\cos\phi$. So the Jacobian $J(\rho,\theta,\phi)$ is

$$\begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\phi\cos\theta & -\rho\mathrm{sin}\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix}$$
$$= -\rho^2\sin\phi.$$

Example 6.25. Find the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ given

$$u = \frac{x}{x^2+y^2} \quad \text{and} \quad v = \frac{y}{x^2+y^2}.$$

Solution. The Jacobian is

$$J(x,y) = \begin{vmatrix} \partial_x\left(\frac{x}{x^2+y^2}\right) & \partial_y\left(\frac{x}{x^2+y^2}\right) \\ \partial_x\left(\frac{y}{x^2+y^2}\right) & \partial_y\left(\frac{y}{x^2+y^2}\right) \end{vmatrix} = \begin{vmatrix} \frac{y^2-x^2}{(x^2+y^2)^2} & -\frac{2xy}{(x^2+y^2)^2} \\ -\frac{2xy}{(x^2+y^2)^2} & \frac{x^2-y^2}{(x^2+y^2)^2} \end{vmatrix}$$
$$= \frac{y^2-x^2}{(x^2+y^2)^2}\frac{x^2-y^2}{(x^2+y^2)^2} - \frac{2xy}{(x^2+y^2)^2}\frac{2xy}{(x^2+y^2)^2} = -\frac{1}{(x^2+y^2)^2}.$$

Since

$$u^2+v^2 = \left(\frac{x}{x^2+y^2}\right)^2 + \left(\frac{y}{x^2+y^2}\right)^2 = \frac{1}{x^2+y^2}$$

we find

$$J(u,v) = \frac{1}{J(x,y)} = \frac{-1}{(u^2+v^2)^2}.$$

6.11 Change of Variable in a Double Integral

::: {#thm- } Change of Variable in a Double Integral

Let $z = f(x, y)$ be a continuous function on a region R in the xy-plane, and let T be a one-to-one transformation that maps the region D in the uv-plane onto R under the change of variables $x = x(u, v)$ and $y = y(u, v)$, where x and y are continuously differentiable functions on R. If $J(u, v) \neq 0$, then

$$\iint_R f(x,y)dx\,dy = \iint_D f(x(u,v), y(u,v))|J(u,v)|dudv.$$

:::

Proof. Let T be the transformation from the uv-plane to the xy-plane given by $x = x(u, v)$ and $y = y(u, v)$. We will consider the effect that T has on the area of a small rectangular region D in the uv-plane with vertices (u_0, v_0), $(u_0+\Delta u, v_0)$, $(u_0, v_0+\Delta v)$, and $(u_0+\Delta u, v_0+\Delta v)$ where (u_0, v_0) is a given point and both Δu and Δv are increments of u and v respectively. Let L_1 be the line segment between the vertices (u_0, v_0) and $(u_0 + \Delta u, v_0)$ and also let L_2 be the line segment between the vertices (u_0, v_0) and $(u_0, v_0 + \Delta v)$.

Define the vector function $\vec{r}$ by

$$\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j}$$

and also

$$\vec{a} = \vec{r}(u_0 + \Delta u, v_0) - \vec{r}(u_0, v_0) \quad \text{and} \quad \vec{b} = \vec{r}(u_0, v_0 + \Delta v) - \vec{r}(u_0, v_0).$$

Notice that $T(D) = R$ is a region in the xy-plane whose area can be approximated by $|\vec{a} \times \vec{b}|$ because we are assuming Δu and Δv are small increments in u and v and T is a continuous one-to-one onto transformation. Let ΔA denote the area of R, it follows

$$\begin{aligned}
\Delta A \approx |\vec{a} \times \vec{b}| &= \left| \frac{\vec{r}(u_0 + \Delta u, v_0) - \vec{r}(u_0, v_0)\Delta u}{\Delta u} \times \frac{\vec{r}(u_0, v_0 + \Delta v) - \vec{r}(u_0, v_0)\Delta v}{\Delta v} \right| \\
&\approx |\vec{r}_u \Delta u \times \vec{r}_v \Delta v| = |\vec{r}_u \times \vec{r}_v|\,\Delta u \Delta v
\end{aligned}$$

If we consider the factor $|\vec{r}_u \times \vec{r}_v|$ in terms of the functions which define the transformation T we are lead to

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_u & y_u & 0 \\ x_v & y_v & 0 \end{vmatrix} = 0\vec{i} - 0\vec{j} + (x_u y_v - x_v y_u)\vec{k}$$

which implies

$$|\vec{r}_u \times \vec{r}_v| = x_u y_v - x_v y_u = \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \left| \frac{\partial(x,y)}{\partial(u,v)} \right|.$$

□

Example 6.26. Use a change of variables to evaluate the integral

$$\iint_R e^{(x+y)/(x-y)}\, dA,$$

where R is the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,-2)$, and $(0,-1)$.

Solution. Since it is not easy to integrate $f(x,y) = e^{(x+y)/(x-y)}$, we make a change of variables suggested by the form of f namely: $u = x + y$ and $v = x - y$. Solving for x and y

$$x = \frac{1}{2}(u+v) \qquad \text{and} \qquad y = \frac{1}{2}(u-v)$$

to find the Jacobian is

$$J(u,v) = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{1}{2}.$$

To find the region D in the uv-plane corresponding to R we note that the sides of R lie on the lines $y = 0$, $x - y = 2$, $x = 0$, $x - y = 1$ and using the rules for the transformation, $x = (u+v)/2$ and $y = (u-v)/2$ the images of the lines in the uv-plane are $u = v, v = 2, u = -v$, and $v = 1$. Thus the region D is the trapezoidal region with vertices $(1,1)$, $(2,2)$, $(-2,2)$ and $(-1,1)$; that is

$$D = \{(u,v) | 1 \le v \le 2, -v \le u \le v\}.$$

Therefore we can evaluate the integral as follows

$$\begin{aligned}
\iint_R e^{(x+y)/(x-y)}\, dA &= \iint_D e^{u/v} \left(\frac{1}{2}\right)\, dA \\
&= \int_1^2 \int_{-v}^{v} e^{u/v} \left(\frac{1}{2}\right)\, du\, dv \\
&= \frac{1}{2} \int_1^2 \left(e - e^{-1}\right) v dv = \frac{3}{4} \left(e - e^{-1}\right).
\end{aligned}$$

Example 6.27. Use a change of variables to evaluate the integral

$$\iint_R 3xy\, dA$$

where R is the region bounded by the lines $x - 2y = 0$, $x - 2y = -4$, $x + y = 4$, and $x + y = 1$.

Solution. Let $u = x+y$ and $v = x-2y$. Then solving for x and y produces $x = \frac{1}{3}(2u+v)$ and $y = \frac{1}{3}(u-v)$. The Jacobian of u and v is

$$J(u,v) = \begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{-1}{3} \end{vmatrix} = \left(\frac{2}{3}\right)\left(\frac{-1}{3}\right) - \left(\frac{1}{3}\right)\left(\frac{1}{3}\right) = -\frac{1}{3}.$$

The bounds under the transformation determined by the following equations

$$x+y=1 \Longrightarrow u=1 \qquad x+y=4 \Longrightarrow u=4$$
$$x-2y=0 \Longrightarrow v=0 \qquad x-2y=-4 \Longrightarrow v=-4$$

We consider the region D in the uv-plane as vertically simple. Then it follows

$$\begin{aligned}
\iint_R 3xy\,dA &= \iint_D 3\frac{1}{3}(2u+v)\frac{1}{3}(u-v)\,|J(u,v)|\,dv\,du \\
&= \int_1^4 \int_{-4}^0 \frac{1}{3}\left(2u^2 - uv - v^2\right)\left|-\frac{1}{3}\right|\,dv\,du \\
&= \int_1^4 \int_{-4}^0 \frac{1}{9}\left(2u^2 - uv - v^2\right)dvdu \\
&= \int_1^4 \frac{1}{9}\left(8u^2 + 8u - \frac{64}{3}\right)du = \frac{164}{9}.
\end{aligned}$$

Example 6.28. Use a change of variables to evaluate the integral

$$\iint_R (x+y)^2 \sin^2(x-y)\,dA,$$

where R is the region bounded by the square with vertices $(0,1)$, $(1,2)$, $(2,1)$, and $(1,0)$.

Solution. The region R is bounded by the lines $x-y=1$, $x-y=-1$, $x+y=1$, and $x+y=3$. Let $u=x+y$ and $v=x-y$, then solving for x and y produces $x=\frac{1}{2}(u+v)$ and $y=\frac{1}{2}(u-v)$. The Jacobian of u and v is

$$J(u,v) = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = \left(\frac{1}{2}\right)\left(\frac{-1}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{2}.$$

The bounds under the transformation determined by the following equations

$$x+y=1 \Longrightarrow u=1 \qquad x+y=3 \Longrightarrow u=3$$
$$x-y=-1 \Longrightarrow v=-1 \qquad x-y=1 \Longrightarrow v=1$$

Let's consider the region D in the uv-plane as horizontally simple. It follows

$$\begin{aligned}\iint_R (x+y)^2 \sin^2(x-y)\, dA &= \iint_D u^2 \sin^2 v |J(u,v)|\, du\, dv \\ &= \frac{1}{2}\int_{-1}^{1}\int_{1}^{3} u^2 \sin^2(v)\, du\, dv \\ &= \frac{1}{2}\int_{-1}^{1} \frac{26\sin^2(v)}{3}\, dv = \frac{13}{6}(2-\sin 2).\end{aligned}$$

Example 6.29. Use a change of variables to evaluate the integral

$$\iint_R y^3(2x-y)\cos(2x-y)\, dy\, dx.$$

where R is the region bounded by the parallelogram with vertices $(0,0)$, $(2,0)$, $(3,2)$, and $(1,2)$.

Solution. The boundary lines of the parallelogram are $y=0$, $y=2$, $y=2x$, and $y=2x-4$. Let $u=2x-y$ and $v=y$ with boundary lines $u=0$, $u=4$, $v=0$, and $v=2$. Solving for x and y produces $x=\frac{1}{2}(u+v)$ and $y=v$. Since the Jacobian is

$$J(u,v) = \begin{vmatrix} 1/2 & 1/2 \\ 0 & 1 \end{vmatrix} = \frac{1}{2},$$

we evaluate the integral as follows

$$\begin{aligned}\iint_D y^3(2x-y)\cos(2x-y)\, dy\, dx &= \int_0^4\int_0^2 v^3 u\cos(u)\frac{1}{2}\, dv\, du \\ = \int_0^4 2u\cos(u)\, du &= -2+2\cos(4)+8\sin(4).\end{aligned}$$

Example 6.30. Use a change of variables to evaluate the integral

$$\iint_R (x^4-y^4)\, e^{xy}\, dA$$

where R is the region bounded by the hyperbolas $xy=1$, $xy=2$, $x^2-y^2=1$, and $x^2-y^2=4$.

Solution. Let $u=xy$ and $v=x^2-y^2$. Then

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} y & x \\ 2x & -2y \end{vmatrix} = -2y^2-2x^2 = -2(x^2+y^2).$$

Notice % $v^2 = \left(x^2 - y^2\right)^2 = x^4 - 2x^2y^2 + y^4$ we have

$$\left(x^2 + y^2\right)^2 = x^4 - 2x^2y^2 + y^4 + 4x^2y^2 = v^2 + 4u^2.$$

Thus $x^2 + y^2 = \sqrt{v^2 + 4u^2}$ and so $J(u, v) = \frac{-1}{2\sqrt{v^2+4u^2}}$. We will make use of

$$\frac{\partial(u, v)}{\partial(x, y)} \frac{\partial(x, y)}{\partial(u, v)} = 1$$

because it is easier not to solve for x and y in terms of u and v. So we evaluate the integral as follows

$$\begin{aligned}
\iint_R \left(x^4 - y^4\right) e^{xy}\, dA &= \int_1^4 \int_1^2 e^u v\sqrt{v^2 + 4u^2} \left|\frac{-1}{2\sqrt{v^2 + 4u^2}}\right| dudv \\
&= \frac{1}{2} \int_1^4 \int_1^2 ve^u \, du\, dv \\
&= \frac{1}{2} \int_1^4 \left(-e + e^2\right) v\, dv = \frac{15}{4} e(e - 1).
\end{aligned}$$

Example 6.31. Use a change of variables to evaluate the integral

$$I = \iint_R \ln\left(\frac{x - y}{x + y}\right) dy\, dx$$

where R is the triangular region bounded by the vertices $(1, 0)$, $(4, -3)$, $(4, 1)$

Solution. Let $u = x - y$ and $v = x + y$ so that $x = \frac{1}{2}(u + v)$ and $y = \frac{-1}{2}u + \frac{1}{2}$. Then the Jacobian is

$$J(u, v) = \begin{vmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{vmatrix} = \frac{1}{2}.$$

The given the region R is bounded by the lines $x - 3y = 1$, $x + y = 1$, and $x = 4$ which transform into $2u - v = 1$, $v = 1$, and $u + v = 8$. Therefore we evaluate the integral as follows,

$$\begin{aligned}
I &= \frac{1}{2} \int_1^5 \int_{(v+1)/2}^{8-v} \ln\left(\frac{u}{v}\right) dudv \\
&= \frac{1}{4} \int_1^5 \left[-(v + 1) \ln\left(\frac{v + 1}{2v}\right) + 2(8 - v) \ln\left(\frac{8 - v}{v}\right) + 3(v - 5)\right] dv \\
&= \frac{1}{4} \left(49 \ln 7 - \frac{75}{2} \ln 5 - 27 \ln 3 + 6\right).
\end{aligned}$$

Example 6.32. Use a change of variables to evaluate the integral

$$I = \iint_D \exp\left(-\frac{x^2}{a^2} - \frac{y^2}{b^2}\right)\, dy\, dx,$$

where D is the region bounded by the quarter ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, in the first octant.

Solution. Let's try the change of variables $x = ar\cos\theta$ and $y = br\sin\theta$ with $0 \le r < \infty$ and $0 \le \theta < 2\pi$; to see if we can simply the region of integration and the integrand. Notice

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} = -\left[\frac{x^2}{a^2} + \frac{y^2}{b^2}\right] = -\left[\frac{(ar\cos\theta)^2}{a^2} + \frac{(br\sin\theta)^2}{b^2}\right] = -r^2$$

using $\sin^2\theta + \cos^2\theta = 1$. Further the region of becomes $r^2 = 1$ or $r = 1$ which is the circle of radius 1. We wish to use

$$\iint_D f(x,y)dx\, dy = \iint_R f(x(u,v), y(u,v))|J(u,v)|\, du\, dv$$

so we compute the Jacobian,

$$J(r,\theta) = \begin{vmatrix} a\cos\theta & -ar\text{sin}\theta \\ b\sin\theta & br\cos\theta \end{vmatrix} = abr\cos\theta + abr\sin^2\theta = abr.$$

Now then we have,

$$\begin{aligned} I &= \int_0^{\pi/2}\int_0^1 e^{-r^2}|abr|\, dr\, d\theta = \int_0^{\pi/2}\int_0^1 e^{-r^2} abr\, dr\, d\theta \\ &= \int_0^{\pi/2} ab\left(\frac{1}{2} - \frac{1}{2e}\right)\, d\theta = \frac{ab\pi}{4}\left(1 - e^{-1}\right). \end{aligned}$$

Example 6.33. Use a change of variables to evaluate the integral

$$\iint_D \left(\frac{x-y}{x+y}\right)^4\, dy\, dx,$$

where D be the region in the xy-plane that is bounded by the coordinate axes and the line $x + y = 1$

Solution. The region is transformed into $0 \le v \le 1$ and $-v \le u \le v$ and so

$$\iint_R \left(\frac{x-y}{x+y}\right)^4\, dy\, dx = \int_0^1\int_{-v}^{v} \frac{u^4}{v^4}\left(\frac{1}{2}\right)\, du\, dv = \int_0^1 \frac{v}{5}\, dv = \frac{1}{10}.$$

Example 6.34. Use integration and a change of variables determine the area of an ellipse

Solution. Assume the ellipse is given in standard form by

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1.$$

Let $u = \frac{x}{a}$ and $v = \frac{y}{b}$, then the ellipse in the xy-plane corresponds to the unit circle $u^2 + v^2 = 1$ in the uv-plane. Since $x = au$ and $y = bv$, the Jacobian is %$\begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab$ and so the area of an ellipse is given by

$$A = 4\int_0^a \int_0^{b\sqrt{1-x^2/a^2}} dy\, dx = 4\int_0^1 \int_0^{\sqrt{1-u^2}} abdvdu = ab\pi$$

since π is the area of the unit circle.

Example 6.35. A rotation of the xy-plane through the fixed angle θ is given by

$$x = u\cos\theta - v\sin\theta \qquad \text{and} \qquad y = u\sin\theta + v\cos\theta$$

Compute the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$. Let E denote the region bounded by the ellipse $x^2 + xy + y^2 = 3$. Use a rotation of $\pi/4$ to obtain an integral that is equivalent to

$$\iint_E y\, dy\, dx.$$

Evaluate the transformed integral.

Solution. We find $J(u,v) = 1$ and so $dx\, dy = du\, dv$. A rotation of $\frac{\pi}{4}$ eliminates the uv-term so we use the transformation

$$x = \frac{\sqrt{2}}{2}(u - v) \qquad \text{and} \qquad y = \frac{\sqrt{2}}{2}(u + v)$$

Then $x^2 + xy + y^2$ becomes $\left(\frac{u}{\sqrt{2}}\right)^2 + \left(\frac{v}{\sqrt{6}}\right)^2 = 1$. Therefore,

$$\begin{aligned}\iint_E y\, dy\, dx &= 4\int_0^{\sqrt{2}} \int_0^{\sqrt{6-3u^2}} \frac{\sqrt{2}}{2}(u+v)(1)\, dv\, du \\ &= 4\int_0^{\sqrt{2}} \frac{-\frac{3u^2}{2} + \sqrt{6-3u^2}u + 3}{\sqrt{2}}\, du = 8 + \frac{8}{\sqrt{3}} = \frac{8\sqrt{3}+8}{\sqrt{3}}\end{aligned}$$

6.12 Change of Variable in a Triple Integral

::: {#thm- } Change of Variable in a Triple Integral Let f be a continuous function on a region R in the xyz-space, and let T be a one-to-one transformation that maps the region D in the uvw-space onto R under the change of variables $x = x(u, v, w)$, $y = y(u, v, w)$, and $z = z(u, v, w)$ where functions x, y, and z are continuously differentiable functions on D. If $J(u, v, w) \neq 0$, then

$$\begin{aligned}&\iiint_R f(x, y, z)\,dx\,dy\,dz \\ &\qquad = \iiint_D f(x(u, v, w), y(u, v, w), z(u, v, w))\,|J(u, v, w)|\;du\,dv\,dw.\end{aligned}$$

:::

Example 6.36. Use integration and a change of variables the volume of an ellipsoid

Solution. Assume the ellipsoid is given in standard form by

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1.$$

Let $u = \frac{x}{a}$, $v = \frac{y}{b}$, and $w = \frac{z}{c}$ then the ellipsoid corresponds to the unit sphere $u^2 + v^2 + w^2 = 1$. Since $x = au$, $y = bv$, $z = cw$, the Jacobian is abc

$$J(u, v, w) = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$$

and so the volume of an ellipsoid is given by

$$\begin{aligned}V &= 8\int_0^a \int_0^{\sqrt{1-x^2/a^2}} \int_0^{\sqrt{1-x^2/a^2-y^2/b^2}} dz\,dy\,dx \\ &= 8\int_0^1 \int_0^{\sqrt{1-u^2}} \int_0^{\sqrt{1-u^2-v^2}} abc\,dw\,dv\,du = \frac{4}{3}abc\pi.\end{aligned}$$

6.13 Exercises

In Exercises 6.17-6.22, find the Jacobian of the change of variables.

Exercise 6.17. $x = u^2$ and $y = u + v$

Exercise 6.18. $x = u \cos v$ and $y = u \sin v$

Exercise 6.19. $x = \frac{u}{v}$, $y = \frac{v}{w}$, and $z = \frac{w}{u}$

Exercise 6.20. $u = ye^{-x}$ and $v = e^x$

Exercise 6.21. $u = \frac{x}{x^2+y^2}$ and $v = \frac{y}{x^2+y^2}$

Exercise 6.22.

$x = u+v-w$, $y = 2u-v+3w$,
and $z = -u + 2v - w$

In Exercises 6.23-6.28, use a change of variables to compute the following integrals.

Exercise 6.23. $\iint_D \left(\frac{x-y}{x+y}\right)^5 dy\,dx$ where D is the region in the xy-plane bounded by the coordinate axes and the line $x + y = 1$

Exercise 6.24. $\iint_D (x - y)e^{x^2+y^2}\,dy\,dx$ where D is the region in the xy-plane bounded by the coordinate axes and the line $x + y = 1$

Exercise 6.25. $\iint_D \left(\frac{2x+y}{x-2y+5}\right)^2 dy\,dx$ where D is the square in the xy-plane with vertices $(0, 0)$, $(1, -2)$, $(3, -1)$, and $(2, 1)$

Exercise 6.26. $\iint_D \sqrt{(2x + y)(x - 2y)}\,dy\,dx$ where D is the square in the xy-plane with vertices $(0, 0)$, $(1, -2)$, $(3, -1)$, and $(2, 1)$

Exercise 6.27. $\iint_D e^{(2y-x)(y+2x)}\,dA$ where D is the trapezoid with vertices $(0, 2)$, $(1, 0)$, $(4, 0)$, and $(0, 8)$

Exercise 6.28. $\iint_R y^3(2x - y)\cos(2x - y)\,dy\,dx$ where D is the region bounded by the parallelogram with vertices $(0, 0)$, $(2, 0)$, $(3, 2)$, and $(1, 2)$

Exercise 6.29. Under the change of variables $x = s^2 - t^2$, $y = 2st$, the quarter circle region in the st-plane given by $s^2 + t^2 \leq 1$, $s \geq 0$, $t \geq 0$ is mapped onto a certain region D of the xy-plane. Evaluate

$$\iint_R \frac{1}{\sqrt{x^2+y^2}}\, dy\, dx.$$

Exercise 6.30. A rotation of the xy-plane through the fixed angle θ is given by $x = u\cos\theta - v\sin\theta$ and $y = u\sin\theta + v\cos\theta$. Determine the Jacobian $J(u,v)$. Let E denote the region bounded by the ellipse $x^2+x+y^2=3$. Use a rotation of $\pi/4$ to obtain an integral that is equivalent to

$$\iint_E y\, dy\, dx.$$

Evaluate the transformed integral.

Chapter 7

Vector Fields

So you want to learn vector calculus? Well, you've come to the right place! In this book, we will provide a gentle introduction to vector calculus for beginners. We will start with the basics and work our way up to some of the more advanced topics. By the end of this book, you should have a basic understanding of vectors and vector operations, as well as how they are used in calculus. Let's get started!

Vector calculus is a powerful tool for solving problems in physics and engineering. It allows us to model physical phenomena more accurately and to find solutions to problems that would be otherwise intractable. vector calculus is a vast and complex subject, but it is also incredibly versatile.

In this book, we will explore some of the ways in which vector calculus can be used to solve problems in physics and engineering. We will see how vector calculus can be used to model physical phenomena, to find solutions to difficult problems, and simplify complex equations. vector calculus is a powerful tool that should be in the arsenal of every physics and engineering student.

Just as calculus is used to calculate the rate of change of a function, vector calculus is used to calculate the rate of change of a vector. Vector calculus is a powerful tool that can be used to calculate the motion of objects in three-dimensional space. In particular, vector calculus can be used to find the velocity and acceleration of an object at any given point in time. Additionally, vector calculus can be used to find the force exerted on an object by another object.

Vector calculus is also used to calculate the electric and magnetic fields of objects in three-dimensional space. Additionally, vector calculus can be used to calculate the electric and magnetic forces exerted on an object by another object. Finally, vector calculus can be used to solve problems in physics and engineering that involve differential equations.

Vector fields are a type of mathematical object that can be used to model various physical phenomena. In vector calculus, a vector field is simply a function that assigns a vector to every point in space. The vector at each point represents the direction and magnitude of the force at that point.

In vector calculus, vector fields are used to represent the flow of fluids, the movement of electrically charged particles, and even the direction of gravitational forces. Vector fields can be used to model everything from electric and magnetic fields to the flow of fluids. They're also pretty handy for navigation; if you know the vector field of a current, you can use it to figure out where you'll end up if you start floating downstream.

In addition, vector fields can be used to create models of complex systems, such as weather patterns or the motion of planets in space. Ultimately, vector fields are a crucial tool for studying the world around us. By understanding vector fields, we can gain a deeper insight into the behavior of these physical phenomena.

In vector calculus, divergence and curl are two important operations. They are used to measure how vector fields change in space. Divergence measures how a vector field diverges from a point, while curl measures how the vector field curls around a point. These operations are essential for understanding how vector fields behave in different situations. For example, they can be used to determine the flow of fluids in a given space. In addition, they can be used to study the behavior of electromagnetic fields. Without divergence and curl, our understanding of these phenomena would be greatly limited.

Line integrals may sound like something out of a math textbook, but they can actually be quite fun to calculate. In vector calculus, a line integral is an integral that uses a vector function to describe the path of a curve. To calculate a line integral, you need to first find a vector function that describes the curve. Then, you need to integrate that function along the curve. Sounds simple enough, right? But don't worry. Just make sure you keep your vector function pointing in the right direction, and you should be good to go.

However, the real power of line integrals comes from the fact that they can be used to calculate some of the most important quantities in vector calculus, such as the flux of a vector field. In addition, line integrals can be used to solve differential equations, making them an essential tool for mathematical modeling. Despite their seeming complexity, line integrals are actually quite easy to understand and use in practice. With a little practice, anyone can master this powerful technique.

In vector calculus, a vector field is a construction that associates a vector to every point in a vector space. The vector at each point is specified by giving its magnitude and direction. Vector fields are often used to model physical phenomena such as the flow of fluids or the distribution of electric charge. In this context, the vector can be thought of as representing the velocity or force at each point.

A vector field is said to be independent of path if the vector at each point is unaffected by the choice of path taken to reach that point. In other words, the vector field represents a consistent flow or distribution throughout the vector space. A vector field is said to be conservative if it is independent of path and if there is a scalar function (known as a potential function) that defines the vector field throughout the space.

Conservative vector fields are often used to model physical phenomena such as gravity or electromagnetism. In this context, the potential function can be thought of as representing the energy associated with each point in the vector space.

Vector calculus is the mathematics of change. It's the study of how things move around in space, and how those motions can be described using mathematical equations. Green's Theorem is a vector calculus theorem that allows us to calculate the rate of change of a vector field.

In other words, it allows us to figure out how fast something is moving in a particular direction. The theorem is named after British mathematician George Green, who first published it in an 1828 paper. Green's Theorem is a powerful tool that has many applications in physics and engineering.

For example, it can be used to calculate the electric field around a charge, or the gravitational field around a mass. It can also be used to study fluid flow, or to understand the behavior of waves. In short, Green's Theorem is a versatile tool that can be used to tackle a wide range of problems.

In essence, it allows you to integrate a vector field over a closed curve. Sounds simple enough, right? But the implications of Green's Theorem are far-reaching. It can be used to solve problems in fluid dynamics, electromagnetism, and even general relativity. So next time you see something green, take a moment to appreciate the power of vector calculus. You might be surprised at what this humble color can do.

Integrals are a fundamental part of vector calculus, and the surface integral is no exception. Put simply, a surface integral is used to calculate the amount of vector flow through a closed surface. This might sound like a daunting task, but it's actually relatively straightforward.

To calculate a surface integral, you first need to choose a vector field that you want to integrate over. This vector field can be defined explicitly, or it can be derived from another field such as the gradient of a scalar field. Once you have your vector field, you need to choose a closed surface that bounds it. This surface could be something like a sphere or a cylinder.

Finally, you need to evaluate the integral on this surface. The result of this calculation will give you the total amount of vector flow through the surface. Surface integrals may seem like a complicated concept, but once you understand the basics, they're actually quite simple. So go ahead and give them a try!

In vector calculus, the divergence theorem states that the flux of a vector field through a closed surface is equal to the volume integral of the divergence of the vector field over the region enclosed by the surface. In other words, if you have a vector field and you want to know how much of it is flowing out of a given region, you can calculate it by taking the divergence of the vector field and integrating it over the region.

The theorem is named after Joseph-Louis Lagrange, who first proved it in 1762. However, its modern form was not published until 1882, when Oliver Heaviside finally put all the pieces together. The divergence theorem is an important tool in physics and engineering, and it has applications in many different fields.

For example, it can be used to calculate the flow of electric charge or the movement of heat energy. It can also be used to study the behavior of fluids and plasmas. The theorem is really just a special case of Gauss's law, which states that the flux of a vector field through any closed surface is zero. However, the divergence theorem provides a way to calculate the flux without having to worry about what's happening on the other side of the surface.

This makes it a very powerful tool for studying vector fields.

In vector calculus, Stokes' theorem is a statement that relates the curl of a vector field to its divergence. In other words, it tells us that if we have a vector field with a lot of curl, then it must also have a lot of divergence. Similarly, if we have a vector field with very little curl, then it must also have very little divergence. The theorem is named after George Stokes, who first proved it in 1857.

The theorem is actually pretty easy to understand intuitively. Think about a vector field as something like a flowing river. If the water is flowing in a straight line, then the river has very little curl. But if the water is flowing in a spiral, then the river has a lot of curl. Now think about what happens when the water reaches the end of the river. If it just keeps flowing straight, then the river has very little divergence. But if it starts spilling out in all directions, then the river has a lot of divergence.

So, intuitively speaking, Stokes' theorem tells us that vector fields with lots of curl must also have lots of divergence.

This book is aimed at anyone who wants to learn about vector calculus. It doesn't matter if you're a student, a physics enthusiast, or just someone who's curious about the world around them. If you're willing to put in the effort, then this book will help you learn everything you need to know about vector calculus.

In this book, I take a step-by-step approach to teaching vector calculus. I start with the basics and gradually build up to more advanced topics. I also include plenty of example calculations to help you understand the material.

- Introduce the basics of vector calculus
- Explain how each of the theorems relates to each other
- Demonstrate how to apply these concepts through example calculations
- Include exercises for readers to practice what they've learned
- Wrap up with a summary of key points

I believe that the best way to learn anything is by doing it yourself. So in each chapter, I include exercises for you to work on. These exercises will help you practice what you've learned and solidify your understanding of the material.

7.1 Introduction to Vector Fields

Definition 7.1. A **vector field** in $\mathbb{R}^n$ is a function $\mathbf{V}$ that assigns a vector from each point in its domain. A vector field with domain D in $\mathbb{R}^n$ has the form

$$\vec{V}(x_1, \dots, x_n) = \langle u_1(x_1, \dots, x_n), \dots, u_n(x_1, \dots, x_n) \rangle$$

where the scalar functions $u_1,,...,u_n$ are called the components of $\mathbf{V}$.

For example a vector field in $\mathbb{R}^2$ has the form

$$\vec{V}(x, y) = u(x, y)\vec{i} + v(x, y)\vec{j} = \langle u, v \rangle$$

and in $\mathbb{R}^3$ has the form

$$\vec{V}(x, y, z) = u(x, y, z)\vec{i} + v(x, y, z)\vec{j} + w(x, y, z)\vec{k} = \langle u, v, w \rangle.$$

Common examples of vector fields include force fields, velocity fields, gravitational fields, magnetic fields, and electric fields. Vector fields can be used to quantify the amount of work done by a variable force acting on a moving body. Measuring the amount of force (fluid flow, electric charge, etc.) can sometimes be achieved by computing an integral of a vector field with respect to an orientable curve or surface.

7.2 Gradient Fields

Definition 7.2. Let f be a differentiable function. The vector field obtained by applying the del operator to f is called the **gradient field** of f.

Example 7.1. Find the gradient field of the function $f(x, y) = \sin x + e^{xy}$.

Solution. Since
$$\nabla f(x, y) = \langle \cos x + ye^{xy}, xe^{xy} \rangle$$
the gradient field of f is
$$\vec{V}(x, y) = (\cos x + ye^{xy})\,\vec{i} + xe^{xy}\vec{j}.$$

7.3 Conservative Vector Fields

Definition 7.3. A vector field $\vec{V}$ is said to be **conservative** in a region D if $\vec{V} = \nabla f$ for some scalar function f in D. The function f is called a **scalar potential** of $\vec{V}$ in D.

Example 7.2. Determine whether the vector field is conservative and if so, find a scalar potential function
$$\vec{V}(x, y, z) = y^2\vec{i} + (2xy + e^{3z})\,\vec{j} + (3ye^{3z})\,\vec{k}$$

Solution. If there is such a function f then $f_x(x, y, z) = y^2$, $f_y(x, y, z) = 2xy + e^{3z}$, and $f_z(x, y, z) = 3ye^{3z}$. Integrating f_x with respect to x, $f(x, y, z) = xy^2 + g(y, z)$. Then differentiating f with respect to y, we have $f_y(x, y, z) = 2xy + g_y(y, z)$ and this yields $g_y(y, z) = e^{3z}$. Thus $g(y, z) = ye^{3z} + h(z)$ and we have
$$f(x, y, z) = xy^2 + ye^{3z} + h(z).$$

Finally, differentiating f with respect to z and comparing, we obtain $h'(z) = 0$ and therefore, $h(z) = K$, a constant. The desired scalar function is
$$f(x, y, z) = xy^2 + ye^{3z} + K$$
with $\vec{V} = \nabla f$.

Definition 7.4. A region D in the plane is called **connected** (one piece) if it has the property: (i) any two points in the region can be connected by a piecewise smooth curve lying entirely within D; and a **simply connected region** (no holes) is a connected region D that has the property: (ii) every closed curve in D encloses only points that are in D.

::: {#thm- } [Conservative in Space] Suppose that the vector field $\vec{V}$ and curl$\vec{V}$ are both continuous in the simply connected region D of $\mathbb{R}^3$. Then $\vec{V}$ is conservative in D if and only if curl$\vec{V} = \vec{0}$. :::

::: {#thm- } [Conservative in the Plane]

Consider the vector field

$$\vec{V}(x, y) = u(x, y)\vec{\imath} + v(x, y)\vec{\jmath}$$

where u and v have continuous first partials in the open simply connected region D in the plane. Then $\vec{V}(x, y)$ is conservative in D if and only if

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

on D. :::

Proof. Note that a vector field $\vec{V} = u(x, y)\vec{\imath} + v(x, y)\vec{\jmath}$ in $\mathbb{R}^2$ can be regarded as the vector field $\vec{U} = u(x, y, 0)\vec{\imath} + v(x, y, 0)\vec{\jmath} + 0\vec{k}$ in $\mathbb{R}^3$. Since

$$\operatorname{curl} \vec{U} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u(x, y, 0) & v(x, y, 0) & 0 \end{vmatrix} = 0\,\vec{\imath} + 0\,\vec{\jmath} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}$$

we have $\operatorname{curl} \vec{U} = 0$ if and only if $\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$.

□

Example 7.3. Determine whether the vector field is conservative and if so, find a scalar potential function $\vec{V}(x, y) = 2xy\vec{\imath} + xy^3\vec{\jmath}$.

Solution. Since $u(x, y) = 2xy$, $v(x, y) = xy^3$, and $\frac{\partial u}{\partial x} = 2x \neq y^3 = \frac{\partial v}{\partial y}$, we see that $\vec{V}$ is not a conservative vector field.

Example 7.4. Show that the vector field

$$\vec{V}(x, y, z) = \left(\frac{y}{1 + x^2} + \tan^{-1} z\right)\vec{\imath} + \left(\tan^{-1} x\right)\vec{\jmath} + \left(\frac{x}{1 + z^2}\right)\vec{k}$$

is conservative and find a scalar potential function.

Solution. Since $\operatorname{curl}\vec{V} = 0$, it follows $\vec{V}$ is conservative. Now we set out to find the scalar potential function f. Since

$$\frac{\partial f}{\partial x} = \frac{y}{1+x^2} + \tan^{-1} z$$

we set

$$f(x,y,z) = y\tan^{-1} x + x\tan^{-1} z + c(y,z).$$

Since

$$\frac{\partial f}{\partial y} = \tan^{-1} z = \frac{\partial}{\partial y}\left(y\tan^{-1} x + x\tan^{-1} z + c\right) = \tan^{-1} x + \frac{\partial c}{\partial y},$$

we find $\frac{\partial c}{\partial y} = 0$ and $c = c_1(z)$ and so we set

$$f(x,y,z) = y\tan^{-1} x + x\tan^{-1} z + c_1(z).$$

Since

$$\frac{\partial f}{\partial z} = \frac{x}{1+z^2} = \frac{\partial}{\partial z}\left[y\tan^{-1} x + x\tan^{-1} z + c_1(z)\right] = \frac{x}{1+z^2} + c_1'(z),$$

we then find $c_1'(z) = 0$, $c_1 = 0$ and so we obtain

$$f(x,y,z) = y\tan^{-1} x + x\tan^{-1} z$$

as desired.

7.4 The Divergence and Curl of a Vector Field

Definition 7.5. Let $\mathbf{V}$ be a given vector field. The **divergence** of $\mathbf{V}$ is defined by $\operatorname{div}\mathbf{V} = \nabla\cdot\mathbf{V}$ and the **curl** of $\mathbf{V}$ is defined by $\operatorname{curl}\mathbf{V} = \nabla\times\mathbf{V}$ where

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}k$$

is the **del operator**.

Example 7.5. Find the divergence and curl of a constant vector field.

Solution. Let $\vec{V} = a\vec{i} + b\vec{j} + c\vec{k}$ for constants a, b, and c. Then

$$\operatorname{div}\vec{V} = \frac{\partial}{\partial x}(a) + \frac{\partial}{\partial y}(b) + \frac{\partial}{\partial z}(c) = \vec{0}$$

and

$$\operatorname{curl}\vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a & b & c \end{vmatrix} = 0\vec{i} - 0\vec{j} + 0\vec{k} = 0.$$

Example 7.6. Find the divergence and curl of the vector field

$$\vec{V}(x, y, z) = xz\vec{i} + xyz\vec{j} - y^2\vec{k}.$$

Solution. The divergence of $\mathbf{V}$ is

$$\text{div}\mathbf{V} = \nabla \cdot \mathbf{V} = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}\left(-y^2\right) = z + xz.$$

and the curl of $\mathbf{V}$ is

$$\begin{aligned}
\text{curl}\mathbf{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix} &= \left[\frac{\partial}{\partial y}\left(-y^2\right) - \frac{\partial}{\partial z}(xyz)\right]\vec{i} \\
&- \left[\frac{\partial}{\partial x}\left(-y^2\right) - \frac{\partial}{\partial z}(xz)\right]\vec{j} + \left[\frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial y}(xz)\right]\vec{k} \\
&= (-2y - xy)\vec{i} - (0 - x)\vec{j} + (yz - 0)\vec{k} = -y(2 + x)\vec{i} + x\vec{j} + yz\vec{k}.
\end{aligned}$$

::: {#thm- } [Properties of Divergence and Curl] Let $\vec{U}$ and $\vec{V}$ be vector fields with component functions that have continuous first and second partial derivatives. Then - $\text{div}(c\vec{U}) = c\,\text{div}\vec{U}$ - $\text{div}(\vec{U} + \vec{V}) = \text{div}\vec{U} + \text{div}\vec{V}$ - $\text{div}(f\vec{U}) = f\text{div}\vec{U} + (\nabla f \cdot \vec{U})$ - $\text{div}(f\nabla g) = f\text{div}\nabla g + \nabla f \cdot \nabla g$ - $\text{curl}(c\vec{U}) = c\text{curl}\vec{U}$ - $\text{curl}(\vec{U} + \vec{V}) = \text{curl}\vec{U} + \text{curl}\vec{V}$ - $\text{curl}(f\vec{U}) = f\text{curl}\vec{U} + (\nabla f \times \vec{U})$ - $\text{curl}(\nabla f + \text{curl}\vec{U}) = \text{curl}(\nabla f) + \text{curl}(\text{curl}\vec{U})$ - $\nabla \times (\nabla f) = 0$ - $\text{div}(\text{curl}\vec{U}) = 0$
:::

Example 7.7. Show that the divergence of the curl of a vector field is 0.

Solution. Let $\vec{V} = u(x, y, z)\vec{i} + v(x, y, z)\vec{j} + w(x, y, z)\vec{k}$ be a vector field, then

$$\text{curl}\vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \left(w_y - v_z\right)\vec{i} - \left(w_x - u_z\right)\vec{j} + \left(v_x - u_y\right)\vec{k}$$

Therefore

$$\nabla \cdot \vec{V} = \partial_x\left(w_y - v_z\right) - \partial_y\left(w_x - u_z\right) + \partial_z\left(v_x - u_y\right) = 0$$

as desired.

Example 7.8. Show that the curl of the gradient of a function is always $\vec{0}$.

Solution. Let $\nabla f = f_x\vec{i} + f_y\vec{j} + f_z\vec{k}$ then

$$\text{curl}\nabla f = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix} = \left(f_{zy} - f_{yz}\right)\vec{i} - \left(f_{xz} - f_{zx}\right)\vec{j} + \left(f_{yx} - f_{xy}\right)\vec{k} = \vec{0}.$$

Example 7.9. Let $\vec{F} = \langle x^2y, yz^2, zy^2\rangle$. Either find a vector field $\vec{G}$ such that $\vec{F} = \text{curl}\vec{G}$ or show that no such $\vec{G}$ exists.

Solution. If a vector field $\vec{G}$ does exist then, then $\text{div}\vec{F} = \text{div}(\text{curl}\vec{G})$ but

$$\text{div}\vec{F} = \partial_x\left(x^2y\right) + \partial_y\left(yz^2\right) + \partial_y\left(zy^2\right) = 2xy + z^2 + 2yz$$

and since the div of the curl of a vector field is always zero we see there can be no vector $\vec{G}$ with $\vec{F} = \text{curl}\vec{G}$ for this vector field F.

Definition 7.6. Let $f(x, y, z)$ define a function with continuous first and second partial derivatives. Then the **Laplacian** of f is

$$\nabla^2 f = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

The equation $\nabla^2 f = 0$ is called **Laplacian's equation** and a function that satisfies it in a region D is said to be **harmonic** on D.

7.5 Exercises

Exercise 7.1. If $\vec{F}(x, y) = u(x, y)\vec{i} + v(x, y)\vec{j}$ where u and v are differentiable functions, show that curl $\vec{F} = 0$ if and only if $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$.

Exercise 7.2. Show that $\text{div}(f\nabla g) = f\text{div}\nabla g + \nabla f \cdot \nabla g$.

Exercise 7.3. Show that the curl of the gradient of a function is always 0.

Exercise 7.4. Show that the divergence of the curl of a vector field is 0.

Exercise 7.5. Let $\vec{F} =< x^2y, yz^2, zy^2 >$. Either find a vector field $\vec{G}$ such that $\vec{F} = \text{curl}\,\vec{G}$ or show that no such $\vec{G}$ exists.

Exercise 7.6. Find the divergence and the curl for the following vector field. - $\vec{F}(x,y) = x^2\vec{i} + xy\vec{j} + z^3\vec{k}$. - $\vec{F}(x,y,z) = z\,\vec{i} - \vec{j} + 2y\,\vec{k}$. - $\vec{F}(x,y,z) = xyz\,\vec{i} + y\,\vec{j} + x\,\vec{k}$. - $\vec{F}(x,y,z) = e^{-x}\sin y\,\vec{i} + e^{-x}\cos y\,\vec{j} + \vec{k}$. - $\vec{F}(x,y) = x\,\vec{i} + y\,\vec{j}$. - $\vec{F}(x,y) = x^2\,\vec{i} - y^2\,\vec{j}$. - $\vec{F}(x,y,z) = x^2\,\vec{i} + y^2\,\vec{j} + z^2\,\vec{k}$. - $\vec{F}(x,y,z) = 2xz\,\vec{i} + yz^2\,\vec{j} - \vec{k}$. - $\vec{F}(x,y,z) = z^2e^{-x}\,\vec{i} + y^3\ln z\,\vec{j} + xe^{-y}\,\vec{k}$.

Exercise 7.7. Find the divergence of $\vec{F}$ given that $\vec{F} = \nabla f$ where $f(x,y,z) = xy^3z^2$.

Exercise 7.8. If $\vec{F}(x,y,z) = xy\,\vec{i} + yz\,\vec{j} + z^2\,\vec{k}$ and $\vec{G}(x,y,z) = x\,\vec{i} + y\,\vec{j} - z\,\vec{k}$ find $\operatorname{curl}(\vec{F}\times\vec{G})$.

Exercise 7.9. Determine whether or not the following vector fields are conservative. - $\vec{F}(x,y) = y^2\,\vec{i} + 2xy\,\vec{j}$ - $\vec{F}(x,y) = 2xy^3\,\vec{i} + 3y^2x^2\,\vec{j}$ - $\vec{F}(x,y) = xe^{xy}\sin y\,\vec{i} + (e^{xy}\cos xy + y)\,\vec{j}$ - $\vec{F}(x,y) = (-y + e^x\sin y)\,\vec{i} + ((x+2)e^x\cos y)\,\vec{j}$ - $\vec{F}(x,y) = (y - x^2)\,\vec{i} + (2x + y^2)\,\vec{j}$ - $\vec{F}(x,y) = e^{2x}\sin y\,\vec{i} + e^{2x}\cos y\,\vec{j}$

Chapter 8

Line Integrals

Let C be a smooth curve, with parametric equations $x = x(t)$, $y = y(t)$, and $z = z(t)$ for $a \leq t \leq b$, that lies within the domain of a function $f(x, y, z)$. We say that C is **orientable** if it is possible to describe direction along the curve for increasing t. Partition C into n sub-arcs, the kth of which has length $\triangle s_k$. Let (x_k^*, y_k^*, z_k^*) be a point chosen (arbitrarily) from the kth sub-arc. Form the Riemann sum

$$\sum_{k=1}^{n} f\left(x_k^*, y_k^*, z_k^*\right) \triangle s_k$$

and let $\|\triangle s\|$ denote the largest sub-arc length in the partition. The following limit

$$\lim_{\|\triangle \mathrm{s}\| \to 0} \sum_{k=1}^{n} f\left(x_k^*, y_k^*, z_k^*\right) \triangle s_k$$

is called the **line integral** of f over C and is denoted by

$$\int_C f(x, y, z)\, dS.$$

Additionally, if C is a closed curve, then we denote the line integral by

$$\oint_C f\, dS.$$

Theorem 8.1. *Let f, f_1, and f_2 be continuous scalar functions defined on a piecewise smooth orientable curve C. Then for any constants k_1 and k_2,*

- $\int_C (k_1 f_1 + k_2 f_2)\; dS = k_1 \int_C f_1\, dS + k_2 \int_C f_2\, dS$

- *$\int_C f\,dS = \int_{C_1} f\,dS + \cdots + \int_{C_n} f\,dS$ where C is the union of smooth orientable sub-arcs $C = C_1 \cup C_2 \cup \cdots \cup C_n$ with only endpoints in common.*
- *$\int_{-C} f\,dS = -\int_C f\,dS$.*

8.1 Evaluating Line Integrals Using Parametrization

Theorem 8.2. *Suppose that the function f is continuous at each point on a smooth curve C, with parametric equations $x = x(t)$, $y = y(t)$, and $z = z(t)$ for $a \leq t \leq b$, that lies within the domain of f. Then*

$$\int_C f(x,y,z)\,dS = \int_a^b f(x(t),y(t),z(t))\sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\,dt.$$

where $dS = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\,dt$.

The definition of a line integral can be extended to curves that are piecewise smooth in the sense that they are the union of a finite number of smooth curves with only endpoints in common. In particular, if C is comprised of a number of smooth sub-arcs $C_1, C_2, \dots, C_n$, then

$$\int_C f(x,y,z)\,dS = \int_{C_1} f(x,y,z)\,dS + \cdots + \int_{C_n} f(x,y,z)\,dS.$$

Example 8.1. Evaluate the line integral

$$\int_C y \sin z\,dS,$$

where C is the circular helix given by the equations $x = \cos t, y = \sin t, z = t$, and $0 \leq t \leq 2\pi$.

Solution. We determine

$$\begin{aligned}
\int_C y\sin z\,dS &= \int_0^{2\pi} (\sin t)\sin t\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt \\
&= \int_0^{2\pi} \sin^2 t\sqrt{\sin^2 t + \cos^2 t + 1}dt \\
&= \sqrt{2}\int_0^{2\pi} \frac{1}{2}(1 - \cos 2t)dt \\
&= \frac{\sqrt{2}}{2}\left[t - \frac{1}{2}\sin 2t\right]_0^{2\pi} = \sqrt{2}\pi.
\end{aligned}$$

Theorem 8.3. *Let C be a smooth curve and let $f(x,y,z)$ be a continuous function with domain containing the trace of C. Then the value of the integrals*

$$\int_C f\,dS,\quad \int_C f dx,\quad \int_C f dy,\quad \int_C f dz$$

depen, dS only on the initial point A, terminal point B, and the trace of C. That is, two different parameterizations having the same trace from A to B yield the same values for these integrals.

Example 8.2. Suppose the smooth curves C_1 and C_2 are given by

$$\begin{aligned} C_1 : &\quad x = t,\ y = t^2 \quad \text{for } 0 \le t \le 1, \text{ and} \\ C_2 : &\quad x = \sin t,\ y = \sin^2 t \quad \text{for } 0 \le t \le \frac{\pi}{2}. \end{aligned}$$

Evaluate

$$\int_{C_1} x\,dS \quad \text{and} \quad \int_{C_2} x\,dS.$$

Solution. Both C_1 and C_2 are smooth curves from $(0,0)$ to $(1,1)$ with the same trace which is the portion of the parabola $y = x^2$ for $0 \le x \le 1$. For C_1, we have $x = t$ and $dS = \sqrt{1+4t^2}dt$, therefore

$$\int_{C_1} x\,dS = \int_0^1 t\sqrt{1+4t^2}dt = \frac{1}{12}\left(17^{3/2} - 1\right).$$

For C_2, we have $x = \sin t$ and $dS = \sqrt{\cos^2 t + 4\sin^2 t\ \cos^2 t}dt$, therefore

$$\int_{C_2} x\,dS = \int_0^{\pi/2} \sin t\ \cos t\sqrt{1+4\sin^2 t}dt = \frac{1}{12}\left(17^{3/2} - 1\right).$$

8.2 Line Integrals with Respect to Coordinate Variables

Other line integrals are obtained by replacing $\triangle s_k$ by $\triangle x_k = x_k - x_{k-1}$. This is called the **line integral** of f along C with respect to x:

$$\int_C f(x,y,z)\,dx = \lim_{\|\triangle x\| \to 0} \sum_{k=1}^{n} f\left(x_k^*, y_k^*, z_k^*\right) \triangle x_k.$$

Similarly with respect to y and z, we define

$$\int_C f(x,y,z)\,dy = \lim_{\|\triangle y\|\to 0} \sum_{k=1}^{n} f\left(x_k^*, y_k^*, z_k^*\right) \triangle y_k$$

$$\int_C f(x,y,z)\,dz = \lim_{\|\triangle z\|\to 0} \sum_{k=1}^{n} f\left(x_k^*, y_k^*, z_k^*\right) \triangle z_k.$$

When we want to distinguish the original line integral $\int_C f\ dS$ from these, we call it the **line integral** with respect to arc length.

The following formulas say that the line integrals with respect to x, y, or z can also be evaluated by expressing everything in terms of t: $x = x(t)$, $y = y(t)$, $z = z(t)$, $dx = x'(t)$, $dy = y'(t)$, and $dz = z'(t)$ yielding:

$$\int_C f(x,y,z)\,dx = \int_a^b f(x(t), y(t), z(t))x'(t)\,dt$$

$$\int_C f(x,y,z)\,dy = \int_a^b f(x(t), y(t), z(t))y'(t)\,dt$$

and

$$\int_C f(x,y,z)\,dz = \int_a^b f(x(t), y(t), z(t))z'(t)\,dt$$

assuming f is continuous and C lies within the domain of f.

Example 8.3. Evaluate the line integral

$$\int_C y\,dx + z\,dy + x\,dz,$$

where C consists of the line segment C_1 from $(2,0,0)$ to $(3,4,5)$ followed by the vertical line segment C_2 from $(3,4,5)$ to $(3,4,0)$

Solution. The curve C is the union of the curves

$$C_1 : \vec{R}_1(t) = (1-t)\langle 2,0,0\rangle + t\langle 3,4,5\rangle = \langle 2+t, 4t, 5t\rangle \text{ for } 0 \le t \le 1$$
$$C_2 : \vec{R}_2(t) = (1-t)\langle 3,4,5\rangle + t\langle 3,4,0\rangle = \langle 3,4,5-5t\rangle \text{ for } 0 \le t \le 1.$$

Thus

$$\begin{aligned}\int_C y\,dx + z\,dy + x\,dz &= \int_{C_1} y\,dx + z\,dy + x\,dz + \int_{C_2} y\,dx + z\,dy + x\,dz\\ &= \int_0^1 ((4t) + 5t(4) + (2+t)5)\,dt + \int_0^1 (0+0+3(-5))\,dt\\ &= \int_0^1 (10+29t)\,dt + \int_0^1 (-15)\,dt = 9.5\end{aligned}$$

8.3 Line Integral of Vector Field Along a Curve

Theorem 8.4. *Let*

$$\vec{V}(x,y,z) = u(x,y,z)\vec{i} + v(x,y,z)\vec{j} + w(x,y,z)\vec{k}$$

be a continuous vector field, and let C be a piecewise smooth orientable curve with parametric representation

$$\vec{R}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$$

for $a \le t \le b$. Using $d\vec{R} = dx\vec{i} + dy\vec{j} + dz\vec{k}$ we define the ***line integral*** *of $\vec{V}$ along C by*

$$\begin{aligned}\int_C \vec{V}\cdot d\vec{R} &= \int_C (u\,dx + v\,dy + w\,dz) = \int_C \left(\vec{V}[\vec{R}(t)]\cdot \vec{R}'(t)\right)dt \\ &= \int_a^b \left[u[x(t),y(t),z(t)]\frac{dx}{dt} + v[x(t),y(t),z(t)]\frac{dy}{dt} + w[x(t),y(t),z(t)]\frac{dz}{dt}\right]dt.\end{aligned}$$

Theorem 8.5. *Let $\vec{F}$ be a continuous force field over a domain D. Then the work W performed as an object moves along a smooth curve C in D is given by the integral*

$$W = \int_C \vec{F}\cdot\vec{T}\,dS = \int_C \vec{F}\cdot d\vec{R}$$

where $\vec{T}$ is the unit tangent at each point on C and $\vec{R}$ is the position vector of the object moving on C.

Example 8.4. Evaluate the line integral

$$\oint \vec{F}\cdot d\vec{R}$$

where $\vec{F} = y^2\vec{i} + x^2\vec{j} - (x+z)\vec{k}$ and C is the boundary of the triangle with vertices $(0,0,0)$, $(1,0,0)$, and $(1,1,0)$, transversed once clockwise, as viewed from above.

Solution. First let C_1 be the line segment from $(1,0,0)$ to $(0,0,0)$. Then C_1 can be represented by the vector function

$$\vec{R}_1(t) = (1-t)\langle 1,0,0\rangle + t\langle 0,0,0\rangle = \langle 1-t,0,0\rangle$$

with $0 \leq t \leq 1$. Let C_2 be the line segment from $(0,0,0)$ to $(1,1,0)$ and so can be represented by the vector function,

$$\vec{R}_2(t) = (1-t)\langle 0,0,0\rangle + t\langle 1,1,0\rangle = \langle t,t,0\rangle$$

with $0 \leq t \leq 1$. Let C_3 be the line segment from $(1,1,0)$ to $(1,0,0)$ and so can be represented by the vector function,

$$\vec{R}_3(t) = (1-t)\langle 1,1,0\rangle + t\langle 1,0,0\rangle = \langle 1,1-t,0\rangle$$

with $0 \leq t \leq 1$. Then

$$\begin{aligned} \vec{F}_1 &= (1-t^2)\,\vec{j} - (1-t)\vec{k} & d\vec{R}_1 &= \langle -dt,0,0\rangle \\ \vec{F}_2 &= t^2\vec{i} + t^2\vec{j} - t\vec{k} & d\vec{R}_2 &= \langle dt,dt,0\rangle \\ \vec{F}_3 &= (1-t)^2\vec{i} + \vec{j} - \vec{k} & d\vec{R}_3 &= \langle 0,-dt,0\rangle \end{aligned}$$

We find

$$\begin{aligned} \int_C \vec{F}\cdot d\vec{R} &= \int\limits_{C_1} \vec{F}_1\cdot d\vec{R}_1 + \int\limits_{C_2} \vec{F}_2\cdot d\vec{R}_2 + \int\limits_{C_3} \vec{F}_3\cdot d\vec{R}_3 \\ &= \int_0^1 0\,dt + \int_0^1 2t^2\,dt + \int_0^1 (-1)\,dt = -\frac{1}{3}. \end{aligned}$$

8.4 Independence of Path

Definition 8.1. The line integral is called **independent of path** if in a region D, if for any two points P and Q in D then the line integral along every piecewise smooth curve in D from P to Q has the same value.

::: {#thm- } Independence of Path If $\vec{V}$ is a continuous vector field on the open connected set D, then the following three conditions are either all true or all false:

- $\vec{V}$ is conservative on D
- $\int_C \vec{V}\cdot d\vec{R} = 0$ for every piecewise smooth closed curve C in D.
- $\int_C \vec{V}\cdot d\vec{R}$ is independent of path within D.

:::

Example 8.5. Let $F = \langle y,-x\rangle$ and let C_1 and C_2 be the following two paths joining $(0,0)$ to $(1,1)$; C_1: $y = x$ for $0 \leq x \leq 1$ and C_2: $y = x^2$ for

$0 \leq x \leq 1$. Show that

$$\int_{C_1} \vec{F} \cdot d\vec{R} \neq \int_{C_2} \vec{F} \cdot d\vec{R}.$$

Explain what this means? On $C_1 : x = t,\ y = t$ with $0 \leq t \leq 1$,

$$\int_{C_1} y\,dx - x\,dy = \int_0^1 (t - t)\,dt = 0.$$

On $C_2 : x = t,\ y = t^2$, with $0 \leq t \leq 1$,

$$\int_{C_2} y\,dx - x\,dy = \int_0^1 \left(t^2 - t(2t)\right)\,dt = -\frac{1}{3}.$$

Solution. Since these two line integrals do not have the same value we see that not all line integrals are independent of path.

8.5 Fundamental Theorem of Line Integrals

::: {#thm- } Fundamental Theorem of Line Integrals Let C be a piecewise smooth curve that is parametrized by the vector function $\vec{R}(t)$ for $a \leq t \leq b$ and let $\vec{V}$ be a vector field that is continuous on C. If f is a scalar function such that $\vec{V} = \nabla f$, then

$$\int_C \vec{V} \cdot d\vec{R} = f(Q) - f(P)$$

where $Q = R(b)$ and $P = R(a)$ are the endpoints of C. :::

Proof. Suppose $\vec{R}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ and let G be the composite function $G(t) = f(x(t), y(t), z(t))$. We have

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{R} = \int_C \nabla f \cdot d\vec{R} &= \int_C \left[\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz\right] \\
&= \int_a^b \left[\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\right] dt \\
&= \int_a^b \frac{dG}{dt}\,dt = G(b) - G(a) = f(Q) - f(P).
\end{aligned}$$

□

Example 8.6. Show that the vector field

$$\vec{F}(x,y,z) = \left(\frac{y}{1+x^2} + \tan^{-1} z\right)\vec{\imath} + \left(\tan^{-1} x\right)\vec{\jmath} + \left(\frac{x}{1+z^2}\right)\vec{k}$$

is conservative and find a scalar potential f for F. Then evaluate the line integral

$$\int_C \vec{F}\cdot d\vec{R}.$$

where C is any piecewise smooth path connecting $A(1,0,-1)$ to $B(0,-1,1)$.

Solution. Since curl$\vec{V} = \vec{0}$, $\vec{V}$ is conservative. Now we set out to find f. Since

$$\frac{\partial f}{\partial x} = \frac{y}{1+x^2} + \tan^{-1} z$$

we set

$$f(x,y,z) = y\tan^{-1} x + x\tan^{-1} z + c(y,z).$$

Since

$$\frac{\partial f}{\partial y} = \tan^{-1} z = \frac{\partial}{\partial y}\left(y\tan^{-1} x + x\tan^{-1} z + c\right) = \tan^{-1} x + \frac{\partial c}{\partial y},$$

so $\frac{\partial c}{\partial y} = 0$ and $c = c_1(z)$ and so we set
$f(x,y,z) = y\tan^{-1} x + x\tan^{-1} z + c_1(z)$. Since

$$\frac{\partial f}{\partial z} = \frac{x}{1+z^2} = \frac{\partial}{\partial z}\left[y\tan^{-1} x + x\tan^{-1} z + c_1(z)\right] = \frac{x}{1+z^2} + c_1'(z),$$

so $c_1'(z) = 0$, $c_1 = 0$ and so we set
$f(x,y,z) = y\tan^{-1} x + x\tan^{-1} z$.

$$\int_C \vec{F}\cdot d\vec{R} = f(1,0,-1) - f(0,-1,1) = \frac{3\pi}{4} + \frac{\pi}{4} = \pi.$$

Example 8.7. Show that the vector field

$$\vec{V}(x,y,z) = \left(xy^2 + yz\right)\vec{\imath} + \left(x^2y + xz + 3y^2z\right)\vec{\jmath} + \left(xy + y^3\right)\vec{k}$$

is conservative and find a scalar potential f for $\vec{V}$. Then evaluate the line integral

$$\int_C \vec{F}\cdot d\vec{R}.$$

where C is any piecewise smooth path joining $A(8,6,1)$ to $B(1,3,5)$.

Solution. First we determine curl$\vec{V}$. We find

$$\operatorname{curl}\vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy^2 + yz & x^2y + xz + 3y^2z & xy + y^3 \end{vmatrix}$$
$$= (x + 3y^2 - x - 3y^2)\,\vec{i} - (y - y)\vec{j} + (2xy + z - 2xy - z)\vec{k} = \vec{0}$$

and so the vector field is conservative and we can find the scalar potential function. Now we set out to find f with $\nabla f = \vec{V}$. Since $f_x = xy^2 + yz$ we know that

$$f(x, y, z) = \frac{x^2}{2}y^2 + xyz + C_1(y, z).$$

Then we find

$$f_y = x^2y + xz + \frac{\partial C_1}{\partial y};$$

and so comparing this with the given $x^2y + xz + 3y^2z$ we determine that

$$\frac{\partial C_1}{\partial y} = 3y^2z.$$

So $C_1 = y^3z + C_2(z)$. So far we have

$$f = \frac{x^2}{2}y^2 + xyz + y^3z + C_2(z).$$

Also since $f_z = xy + y^3 + \frac{dC_2}{dz}$ and comparing this to the given $xy + y^3$ we determine $\frac{dC_2}{dz} = 0$. So C_2 is a constant with respect to x, y, and z. Therefore a scalar potential function is

$$f(x, y, z) = \frac{x^2}{2}y^2 + xyz + y^3z$$

(taking the constant to be zero). Finally

$$\int_C \vec{V} \cdot d\vec{R} = f(1, 3, 5) - f(8, 6, 1) \tag{8.1}$$

Example 8.8. Show that the vector field is conservative and find a scalar potential f for F. Then evaluate the line integral

$$\int_C \vec{F} \cdot d\vec{R} = \vec{F}(x, y) = \frac{(y + 1)\vec{i} - x\vec{j}}{(y + 1)^2}$$

where C is any smooth path connecting $A(0, 0)$ to $B(1, 1)$.

Solution. Let $\vec{F}(x,y) = u(x,y)\vec{i} + v(x,y)\vec{j}$, then since

$$\frac{\partial u}{\partial y} = \frac{-1}{(y+1)^2} = \frac{\partial v}{\partial x},$$

we know that F is conservative. By definition, we know

$$f_x(x,y) = \frac{1}{y+1} \qquad \text{and} \qquad f_y(x,y) = \frac{-x}{(y+1)^2}.$$

Integrating with respect to x, $f(x,y) = \frac{x}{y+1} + c(y)$. Since,

$$f_y(x,y) = -\frac{x}{(y+1)^2} + c'(y) = -\frac{x}{(y+1)^2},$$

so $c'(y) = 0$.
For $c(y) = 0$; and then $f(x,y) = \frac{x}{y+1}$ is a scalar potential function for $\vec{F}$. By the fundamental theorem of line integrals,

$$\int_C \vec{F} \cdot d\vec{R} = f(1,1) - f(0,0) = \frac{1}{2} - 0 = \frac{1}{2}.$$

8.6 Work

Example 8.9. Find the work done on an object moves in the force field

$$\vec{F}(x,y,z) = x\vec{i} + y\vec{j} + (xz - y)\vec{k}$$

along the curve C defined parametrically by $\vec{R}(t) = t^2\vec{i} + 2t\vec{j} + 4t^3\vec{k}$ for $0 \le t \le 1$.

Solution. We determine

$$d\vec{R} = \left(2t\vec{i} + 2\vec{j} + 12t^2\vec{k}\right) dt, \qquad \vec{F}(t) = t^2\vec{i} + 2t\vec{j} + \left(4t^5 - 2t\right)\vec{k},$$

(from $x(t) = t^2, y(t) = 2t$, and $z = 4t^3$) and

$$\vec{F} \cdot d\vec{R} = \left(2t^3 + 4t + 48t^7 - 24t^3\right) dt.$$

Thus,

$$W = \int_C \vec{F} \cdot d\vec{R} = \int_0^1 \left(2t^3 + 4t + 48t^7 - 24t^3\right) \, dt = \frac{5}{2}.$$

Example 8.10. Find the work done when an object moves along a closed path in a connected domain where the force field is conservative.

Solution. In such a force field $\vec{F}$, where f is a scalar potential of $\vec{F}$ and because the path of motion is closed, it begins and ends at the same point P. Thus, the work is given by

$$W = \oint_C \vec{F} \cdot d\vec{R} = f(P) - f(P) = 0.$$

Example 8.11. Let

$$\vec{F}(x, y) = \frac{-y\vec{i} + x\vec{j}}{x^2 + y^2}.$$

Evaluate the line integral

$$\int_{C_1} \vec{F} \cdot d\vec{R}$$

where C_1 is the upper semicircle $y = \sqrt{1 - x^2}$ transversed counterclockwise. What is the value of

$$\int_{C_2} \vec{F} \cdot d\vec{R}$$

if C_2 is the lower semicircle $y = -\sqrt{1 - x^2}$ also transversed counterclockwise?

Solution. On the upper semi-circle C_1, we let $x = \cos(t)$ and $y = \sin(t)$ for $0 \leq t \leq \pi$, and then

$$\int_{C_1} \frac{-y\,dx + x\,dy}{x^2 + y^2} = \int_0^\pi \frac{-\sin(t)(-\sin(t)) + \cos(t)(\cos(t))}{\sin^2(t) + \cos^2(t)}\,dt = \int_0^\pi dt = \pi.$$

For $\pi \leq t \leq 2\pi$, we have the lower semi-circle C_2:

$$\int_{C_2} \frac{-y\,dx + x\,dy}{x^2 + y^2} = \int_\pi^{2\pi} dt = \pi.$$

8.7 Finding Area with Line Integral

Theorem 8.6. *If R is a region bounded by a piecewise smooth simple closed curve C oriented counterclockwise, then the area of R is given by*

$$A = \oint_C xdy = -\oint_C ydx = \frac{1}{2}\oint_C xdy - ydx.$$

8.8 Exercises

Exercise 8.1. Evaluate the line integral

$$\int_C \frac{1}{3+y}\, dS$$

where C is the curve with parametric equations $x = 2t^{3/2}$ and $y = 3t$ with $0 \le t \le 1$.

Exercise 8.2. Evaluate the line integral

$$\int_C (x^2 + y^2)\ dS$$

where C is the curve with parametric equations $x = e^{-t}\cos t$ and $y = e^{-t}\sin t$ with $0 \le t \le \frac{\pi}{2}$.

Exercise 8.3. Evaluate the line integral

$$\int_C xdy - ydx$$

where C is the curve defined by $2x - 4y = 1$ with $4 \le x \le 8$.

Exercise 8.4. Evaluate the line integral

$$\int_C -xdy + (y^2 - x^2)\, dx$$

where C is the quarter-circle $x^2 + y^2 = 4$ from $(0, 2)$ to $(2, 0)$.

Exercise 8.5. Evaluate the line integral

$$\int_C (x+y)^2 dx - (x-y)^2 dy$$

where C is the curve defined by $y = |2x|$ from $(-1, 2)$ to $(1, 2)$.

Exercise 8.6. Evaluate the line integral

$$\oint \vec{F} \cdot d\vec{R}$$

where $\vec{F} = y^2\vec{\imath} + x^2\vec{\jmath} - (x+z)\vec{k}$ and C is the boundary of the triangle with vertices $(0,0,0)$, $(1,0,0)$, and $(1,1,0)$, transversed once clockwise, as viewed from above.

Exercise 8.7. Let $\vec{F} = y\,\vec{\imath} - x\,\vec{\jmath}$ and let C_1 and C_2 be the following two paths joining $(0,0)$ to $(1,1)$.

$$C_1 : y = x \text{ for } 0 \le x \le 1 \quad \text{and} \quad C_2 : y = x^2 \text{ for } 0 \le x \le 1.$$

Show that

$$\int_{C_1} \vec{F} \cdot d\vec{R} \ne \int_{C_2} F \cdot d\vec{R}.$$

Explain what this means?

Exercise 8.8. Evaluate the closed line integral

$$\oint_C \vec{F} \cdot dS$$

where $\vec{F} = -x\,\vec{\imath} + 2\,\vec{\jmath}$ and C is the boundary of the trapezoid with vertices $(0,0)$, $(1,0)$, $(2,1)$ and $(0,1)$ transversed once clockwise, as viewed from above.

Exercise 8.9. Find the work done by the force field

$$\vec{F}(x,y,z) = (y^2 - z^2)\,\vec{\imath} + 2yz\vec{\jmath} - x^2\vec{k}$$

on any object moving along the curve C where C is the path given by $x(t) = t$, $y(t) = t^2$, and $z(t) = t^3$, for $0 \le t \le 1$.

Exercise 8.10. Find the work done by the force field

$$\vec{F}(x,y,z) = 2xy\,\vec{\imath} + \left(x^2 + 2\right)\,\vec{\jmath} + y\,\vec{k}$$

on any object moving along the curve C where C is the line segment from $(1,0,2)$ to $(3,4,1)$.

Exercise 8.11. Show that the vector field

$$\vec{F}(x,y) = (x+2y)\,\vec{\imath} + (2x+y)\,\vec{\jmath}$$

is conservative and find a scalar potential function f for F. Then evaluate the line integral $\int_C \vec{F} \cdot d\vec{R}$ where C is any smooth path connecting $A(0,0)$to $B(1,1)$.

Exercise 8.12. Show that the vector field

$$\vec{F}(x,y) = (y - x^2)\,\vec{\imath} + (x + y^2)\,\vec{\jmath}$$

is conservative and find a scalar potential function f for $\vec{F}$. Then evaluate the line integral

$$\int_C \vec{F} \cdot dR$$

where C is any smooth path connecting $A(0,0)$to $B(1,1)$.

Exercise 8.13. Show that the vector field

$$\vec{F}(x,y) = e^{-y}\,\vec{\imath} - xe^{-y}\,\vec{\jmath}$$

is conservative and find a scalar potential function f for $\vec{F}$. Then evaluate the line integral

$$\int_C \vec{F} \cdot d\vec{R}$$

where C is any smooth path connecting $A(0,0)$to $B(1,1)$.

Exercise 8.14. Show that the vector field

$$\vec{F}(x,y,z) = e^{xy}yz\,\vec{\imath} + e^{xy}xz\,\vec{\jmath} + e^{xy}\,\vec{k}$$

is conservative and find a scalar potential function f for $\vec{F}$.

Exercise 8.15. Show that the vector field $\vec{F}$ with component functions $f(x,y,z) = (2xz^3 - e^{-xy}y\sin z)$, $g(x,y,z) = -xe^{-xy}\sin z$, and $h(x,y,z) = (3x^2z^2 + e^{-xy}\cos z)$ is conservative and find a scalar potential function f for $\vec{F}$. Then evaluate the line integral

$$\int_C \vec{F} \cdot d\vec{R}$$

where C is any smooth path connecting $A(1,0,-1)$to $B(0,-1,1)$.

Exercise 8.16. Show that the vector field

$$\vec{F}(x,y) = \frac{(y+1)\vec{\imath} - x\vec{\jmath}}{(y+1)^2}$$

is conservative and find a scalar potential f for $\vec{F}$. Then evaluate the line integral

$$\int_C \vec{F} \cdot d\vec{R}$$

where C is any smooth path connecting $A(0,0)$ to $B(1,1)$.

8.9 Green's Theorem

Suppose C is a piecewise smooth closed curve oriented counterclockwise in the Cartesian plane whose bounded region R is **simply connected** . Suppose further that we have a **continuously differentiable** vector field of the form $\vec{F} = M(x,y)\vec{i}$.
Let $[a,b]$ be the projection of R onto the x-axis, let $y_1(x)$ represent the lower part of the curve C, and let $y_2(x)$ represent the upper part of C. We can write

$$\begin{aligned}
\oint_C \vec{F} \cdot d\vec{r} &= \int_a^b M(x, y_1(x))\,dx + \int_b^a M(x, y_2(x))\,dx \\
&= \int_a^b \left[M(x, y_1(x)) - M(x, y_2(x))\right]\,dx \\
&= -\int_a^b M(x,y)\Big|_{y_1(x)}^{y_2(x)}\,dx \\
&= -\int_a^b \int_{y_1(x)}^{y_2(x)} \frac{\partial M}{\partial y}\,dy\,dx \\
&= -\iint_R \frac{\partial M}{\partial y}\,dy\,dx
\end{aligned} \tag{8.2}$$

Now let us suppose that we have a continuously differentiable vector field of the form $\vec{F} = N(x,y)\vec{j}$.
Let $[a,b]$ be the projection of R onto the y-axis, let $x_1(y)$ represent the lower part of the curve C, and let $x_2(y)$ represent the upper part of C.

We can write

$$\begin{aligned}
\oint_C \vec{F} \cdot d\vec{r} &= \int_a^b N(x_1(x), y)\, dy + \int_b^a N(x_2(x), y)\, dy \\
&= \int_a^b \left[N(x_1(y), y) - N(x_2(y), y) \right]\, dy \\
&= -\int_a^b N(x, y)|_{x_1(y)}^{x_2(y)}\, dy \\
&= \int_a^b \int_{x_2(y)}^{x_1(y)} \frac{\partial N}{\partial x}\, dx\, dy \\
&= \iint_R \frac{\partial N}{\partial x}\, dx\, dy \qquad (8.3)
\end{aligned}$$

Finally, suppose that $\vec{F}$ is a continuously differentiable vector field in the plane, say

$$\vec{F}(x, y) = M(x, y)\vec{i} + N(x, y)\vec{j}.$$

Using 8.2 and 8.3 we find

$$\begin{aligned}
\oint_C \vec{F} \cdot d\vec{r} &= \oint_C M(x, y)\, dx + N(x, y)\, dy \\
&= \oint_C M(x, y)\, dx + \oint_C N(x, y)\, dy \\
&= -\iint_R \frac{\partial M}{\partial y}\, dy\, dx + \iint_R \frac{\partial N}{\partial x}\, dx\, dy \\
&= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA
\end{aligned}$$

as needed to prove the following theorem.

Theorem 8.7 (Green's Theorem)**.** *Let R be a simply connected region with a piecewise smooth boundary curve C oriented counterclockwise and let*

$$\vec{F} = M\vec{i} + N\vec{j} + 0\vec{k}$$

be a continuously differentiable vector field on R, then

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA. \qquad (8.4)$$

Example 8.12. Use Green's theorem to evaluate the line integral

$$\oint_C 4y\, dx - 3x\, dy$$

around the curve C defined by the ellipse $2x^2 + y^2 = 4$ oriented counterclockwise.

Solution. Let $\vec{F}(x, y) = 4y\vec{\imath} + (-3x)\vec{\jmath}$ and let R be the region enclosed by C. Notice that $\vec{F}$ and R satisfy the hypothesis of Green's theorem, so that

$$\oint_C 4y\,dx - 3x\,dy = \iint_R (-3-4)dA = -7(2)\left(\sqrt{2}\right)\pi.$$

Example 8.13. Use Green's theorem to evaluate the line integral

$$\oint_C 4xy\,dx$$

around the curve C defined by the unit circle oriented clockwise.

Solution. Let $\vec{F}(x, y) = 4xy\vec{\imath} + 0\vec{\jmath}$ and let R be the region enclosed by C. Notice that $\vec{F}$ and R satisfy the hypothesis of Green's theorem, so that

$$\oint_C 4xy\,dx = 4\iint_R x\,dA = 4\int_0^{2\pi}\int_0^1 r^2\cos\theta dr d\theta = 0$$

Example 8.14. Use Green's theorem to evaluate the line integral

$$\oint_C y^2\,dx + x\,dy$$

around the curve C defined by the square with vertices $(0, 0)$, $(2, 0)$, $(2, 2)$, $(0, 2)$ oriented counterclockwise.

Solution. Let $\vec{F}(x, y) = y^2\vec{\imath} + x\vec{\jmath}$, then $M(x, y) = y^2$ which is continuously differentiable over the square as well as $N(x, y) = x$. Therefore $\vec{F}$ and R satisfy the hypothesis of Green's theorem and

$$\oint_C y^2\,dx + x\,dy = \int_0^2\int_0^2 (1-2y)\,dy\,dx = \int_0^2 (-2)\,dx = -4.$$

Example 8.15. Find the work done on an object that moves in the force field

$$\vec{F}(x, y) = y^2\vec{\imath} + x^2\vec{\jmath}$$

once counterclockwise around the circular path $x^2 + y^2 = 2$.

Solution. Let $\vec{F}(x, y) = y^2\vec{\imath} + x^2\vec{\jmath}$, then $M(x, y) = y^2$ which is continuously differentiable over the circle as well as $N(x, y) = x^2$. Let R be the region bounded by the curve $x^2 + y^2 = 2$. Then $\vec{F}$ and R satisfy the hypothesis of Green's theorem and

$$\begin{aligned} W &= \oint_C \vec{F} \cdot d\vec{R} = \oint_C \left(y^2\vec{\imath} + x^2\vec{\jmath}\right) \cdot (dx\vec{\imath} + dy\vec{\jmath}) = \oint_C (y^2 dx + x^2 dy) \\ &= \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = 2\iint_R (x-y)\,dA \\ &= 2\int_0^{2\pi}\int_0^{\sqrt{2}} r^2(\cos\theta - \sin\theta)\,dr\,d\theta = \frac{4\sqrt{2}}{3}\int_0^{2\pi} (\cos\theta - \sin\theta)\;d\theta = 0. \end{aligned}$$

Example 8.16. Find the work done on an object that moves in the force field

$$\vec{F}(x,y) = (x + 2y^2)\,\vec{\jmath}$$

once counterclockwise around the circular path $(x-2)^2 + y^2 = 1$.

Solution. Let $\vec{F}(x,y) = 0\vec{\imath} + (x+2y^2)\,\vec{\jmath}$. Then $M(x,y) = 0$ which is continuously differentiable over the circle as well as $N(x,y) = x + 2y^2$. Let R be the region bounded by the curve $(x-2)^2 + y^2 = 1$. Then $\vec{F}$ and R satisfy the hypothesis of Green's theorem and

$$\begin{aligned} W &= \oint_C \vec{F} \cdot d\vec{R} = \oint_C (x+2y^2)\,\vec{\jmath} \cdot (dx\vec{\imath} + dy\vec{\jmath}) = \oint_C (x+2y^2)\,dy \\ &= \oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \iint_R dA = \pi. \end{aligned}$$

Example 8.17. Evaluate the closed line integral

$$\oint_C \frac{-ydx + (x-1)dy}{(x-1)^2 + y^2}$$

where C is any Jordan curve whose interior does not contain the point $(1,0)$ transversed counterclockwise.

Solution. Let

$$\vec{F}(x,y) = \frac{-y}{(x-1)^2 + y^2}\vec{\imath} + \frac{x-1}{(x-1)^2 + y^2}\vec{\jmath},$$

then

$$M(x,y) = \frac{-y}{(x-1)^2 + y^2}$$

which is continuously differentiable over the circle as well as

$$N(x,y) = \frac{x-1}{(x-1)^2 + y^2}.$$

Let R be the region bounded by the given curve C. Then $\vec{F}$ and R satisfy the hypothesis of Green's theorem and

$$\begin{aligned}
&\oint_C \frac{-ydx+(x-1)dy}{(x-1)^2+y^2} \\
&\quad = \oint_C \frac{-y}{(x-1)^2+y^2}dx + \frac{x-1}{(x-1)^2+y^2}dy \\
&\quad = \iint_R \left(\partial_x\left(\frac{x-1}{(x-1)^2+y^2}\right) - \partial_y\left(\frac{-y}{(x-1)^2+y^2}\right)\right) dA \\
&\quad = \iint_R 0\, dA = 0.
\end{aligned}$$

8.10 Doubly-Connected Regions

Definition 8.2. A **Jordan curve** is a closed curve C that does not intersect itself and a simply connected region R has the property that it is connected and the interior of every Jordan curve C in R also lies in R.

::: {#thm- } [Green's Theorem for Doubly-Connected Regions] Let R be a doubly-connected region with a piecewise smooth outer boundary curve C_1 oriented counterclockwise and a piecewise smooth inner boundary curve C_2 oriented clockwise and let $\vec{F} = M\vec{i} + N\vec{j} + 0\vec{k}$ be a continuously differentiable vector field on R, then

$$\oint_{C_1} M\,dx + N\,dy + \oint_{C_2} M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA.$$

:::

Example 8.18. Evaluate the closed line integral

$$\oint_C \frac{xdx+ydy}{x^2+y^2}$$

where C is any Jordan curve whose interior contains the point $(0,0)$ transversed counterclockwise

Solution. Let C_1 be a circle centered at $(0,0)$ with radius r so small that all of C_1 is contained within C. Let C_1 be oriented clockwise and let R be the region between C_1 and C. Then by Green's theorem for doubly-connected regions,

$$\begin{aligned}\oint_C \frac{x}{x^2+y^2}dx + \frac{y}{x^2+y^2}dy + \oint_{C_1} \frac{x}{x^2+y^2}dx + \frac{y}{x^2+y^2}dy \\ = \iint_R \left(\frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)\right) dA \\ = \iint_R \left(\frac{-2xy}{x^2+y^2} - \frac{-2xy}{x^2+y^2}\right) dA = 0.\end{aligned}$$

Thus,

$$\oint_C \frac{xdx+ydy}{x^2+y^2} = -\oint_{C_1} \frac{xdx+ydy}{x^2+y^2}$$

which is something that can be easily evaluated, using say, the parametrization $C_1 : x = r\sin\theta, y = r\cos\theta; 0 \le \theta \le 2\pi$. Therefore

$$\begin{aligned}\oint_C \frac{xdx+ydy}{x^2+y^2} &= -\oint_{C_1} \frac{xdx+ydy}{x^2+y^2} \\ &= \int_0^{2\pi} \frac{(r\sin\theta)(r\cos\theta) + (r\cos\theta)(-r\sin\theta)}{\sin^2\theta + \cos^2\theta} d\theta = 0.\end{aligned}$$

Example 8.19. Evaluate the closed line integral

$$\oint_C (x^2ydx - y^2xdy)$$

where C is the boundary of the region between the x-axis and the semi-circle $y = \sqrt{a^2 - x^2}$, traversed counterclockwise (including the x-axis).

Solution. Using Green's Theorem, we find

$$\begin{aligned}\oint_C (x^2ydx - y^2xdy) &= \int\int (-y^2) - (x^2)\, dA = -\int\int (x^2+y^2)\, dA \\ &= -\int_0^{\pi}\int_0^{a} r^3 dr d\theta = -\int_0^{\pi} \frac{a^4}{4}\, d\theta = -\frac{a^4\pi}{4}\end{aligned}$$

Example 8.20. Evaluate the closed line integral

$$\oint_C \frac{-(y+2)dx + (x-1)dy}{(x-1)^2 + (y+2)^2}$$

where C is any Jordan curve whose interior does not contain the point $(1, -2)$.

Solution. Green's theorem applies because we are using any Jordan curve which does not contain the point $(1,-2)$.

$$\begin{aligned}
&\oint_C \frac{-(y+2)dx+(x-1)dy}{(x-1)^2+(y+2)^2} \\
&\quad = \oint_C -\frac{(y+2)}{(x-1)^2+(y+2)^2}dx + \frac{(x-1)}{(x-1)^2+(y+2)^2}dy \\
&\quad = \iint_D \left(\partial_x\left(\frac{(x-1)}{(x-1)^2+(y+2)^2}\right)\right) - \left(\partial_y\left(-\frac{(y+2)}{(x-1)^2+(y+2)^2}\right)\right) dA \\
&\quad = \iint_D \left(\frac{-x^2+2x+y^2+4y+3}{(x^2-2x+y^2+4y+5)^2}\right) - \left(\frac{-x^2+2x+y^2+4y+3}{(x^2-2x+y^2+4y+5)^2}\right) dA = 0
\end{aligned}$$

Example 8.21. Evaluate the closed line integral

$$\oint_C \left[(x-3y)dx + \left(2x-y^2\right)dy\right]$$

where A is the region D enclosed by a Jordan curve C.

Solution. Green's theorem applies and

$$\begin{aligned}
\oint_C (x-3y)dx + \left(2x-y^2\right)dy &= \iint_D \left(\partial_x\left(2x-y^2\right)\right) - \left(\partial_y(x-3y)\right) dA \\
&= \iint_D (2+3)dA = 5A.
\end{aligned}$$

Example 8.22. Use a line integral to find the area enclosed by the region R defined by the circle $x^2+y^2=4$

Solution. We can parametrize the circle by $x=2\cos t$ and $y=2\sin t$ for $0 \le t \le 2\pi$. Then the area is

$$\frac{1}{2}\oint_C xdy - ydx = \frac{1}{2}\int_0^{2\pi} [4\cos(t)\cos(t) + 4\sin(t)\sin(t)]\, dt = 4\pi.$$

- We can parametrize the line segments by

$$\begin{aligned}
&C_1 : x=t, y=t; 0 \le t \le 1 \\
&C_2 : x=1-t, y=1+t; 0 \le t \le 1 \\
&C_3 : x=0, y=2(1-t); 0 \le t \le 1
\end{aligned}$$

Then the area is

$$\begin{aligned}\oint_C xdy &= \int_{C_1} xdy + \int_{C_2} xdy + \int_{C_3} xdy \\ &= \int_0^1 t\,dt + \int_0^1 (1-t)\,dt + \int_0^1 0(-2)dt = \frac{1}{2} + \frac{1}{2} + 0 = 1.\end{aligned}$$

Example 8.23. Use a line integral to find the area enclosed by the region R defined by the curve $C : x = \cos^3 t$, $y = \sin^3 t$ for $0 \le t \le 2\pi$

Solution. The required area is

$$\begin{aligned}A = \oint_C xdy &= \int_0^{2\pi} \cos^3 t\left(3\sin^2 t\cos t\right) dt \\ &= \int_0^{2\pi} \cos^4 t\,(1-\cos^2 t)\,dt = \int_0^{2\pi} (\cos^4 t - \cos^6 t)\,dt\end{aligned}$$

by using the reduction formula,

$$\int \cos^n x\,dx = \frac{\cos^{n-1} x\sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} xdx.$$

8.11 Exercises

Exercise 8.17. Use Green's theorem to evaluate the closed line integral

$$\oint_C y^2dx + x^2dy$$

where C is the boundary of the square with vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$ traversed counterclockwise.

Exercise 8.18. Use Green's theorem to evaluate the closed line integral

$$\oint_C 4xydx$$

where C is the boundary of the circle $x^2 + y^2 = 1$ traversed clockwise.

Exercise 8.19. Use Green's theorem to evaluate the closed line integral

$$\oint_C x\sin xdx - e^{y^2}dy$$

where C is the boundary of the triangle with vertices $(-1,-1)$, $(1,-1)$, and $(2,5)$ traversed counterclockwise.

Exercise 8.20. Use Green's theorem to evaluate the closed line integral

$$\oint_C \sin x \cos y \, dx + \cos x \sin y \, dy$$

where C is the boundary of the square with vertices $(0,0)$, $(2,0)$, $(2,2)$, and $(0,2)$ traversed clockwise.

Exercise 8.21. Use Green's theorem to evaluate a closed line integral that represents the area enclosed by the region defined by the curve $x^2 + y^2 = 4$.

Exercise 8.22. Use Green's theorem to evaluate a closed line integral that represents the area enclosed by the region defined by the trapezoid with vertices $(0,0)$, $(4,0)$, $(1,3)$, and $(0,3)$.

Exercise 8.23. Evaluate the closed line integral

$$\oint_C \frac{xdx + ydy}{x^2 + y^2}$$

where C is any piecewise smooth Jordan curve enclosing the origin, traversed counterclockwise.

Exercise 8.24. Evaluate the closed line integral

$$\oint_C x^2 y \, dx - y^2 x \, dy,$$

where C is the boundary of the region between the x-axis and the semicircle $y = \sqrt{a^2 - x^2}$, traversed counterclockwise (including the x-axis).

Exercise 8.25. Evaluate

$$\oint_C \frac{-(y+2)dx + (x-1)dy}{(x-1)^2 + (y+2)^2}$$

where C is any Jordan curve whose interior does not contain the point $(1, -2)$.

Exercise 8.26. If C is a Jordan curve, show that

$$\oint_C (x-3y)dx + \left(2x - y^2\right) dy = 5A$$

where A is the region D enclosed by C.

Exercise 8.27. Suppose $\vec{F} = M(x,y)\,\vec{i} + N(x,y)\,\vec{j}$ is continuously differentiable in a doubly-connected region R and that

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

throughout R. How many distinct values of I are there for the integral

$$I = \oint_C M(x,y)\,dx + N(x,y)\,dy$$

where C is a piecewise smooth Jordan curve in R?

Exercise 8.28. Evaluate the line integral

$$\oint_C x^2 y\,dx - y^2 x\,dy,$$

where C is the boundary of the region between the x-axis and the semicircle $y = \sqrt{a^2 - x^2}$, transversed counterclockwise (including the x-axis).

Exercise 8.29. Find the work done if an object that moves in the force field $\vec{F}(x,y) = y^2\vec{i} + x^2\vec{j}$ once counterclockwise around the circular path $x^2 + y^2 = 2$.

References

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