Single Variable Calculus

To my wife, Sally, and our children,

Courtney, Rebecca, and Benjamin,

whose support made this book possible

David A. Smith

Single Variable Calculus

With Python

First Edition

Direct Knowledge

Cover photo attribution: *Stephan's Quintet (NIRCam and MIRI Composite Image)* courtesy of NASA, ESA, CSA, STScI. Published on July 12, 2022. For more information visit: https://webbtelescope.org

ISBN: 978-1-951290-03-0

Typeset with LaTeX and written in the United States of America.

David A. Smith
B.S. Mathematics, M.S. Mathematics
https://directknowledge.com

Direct Knowledge

Preface

Writing Philosophy

Writing mathematics well requires more than just a knack for creativity: critical thinking is necessary to construct convincing arguments and effectively communicate ideas. Additionally, the technical aspects of writing must be mastered to ensure quality work that adheres to my high standards - learn more at https://directknowledge.com/writing

Acknowledgements

Behind every printed word lies those who helped make it possible. So, I'm profoundly thankful to all the generous contributors whose efforts have enabled this book's fruition and would like to earnestly express my gratitude for their invaluable role in its completion.

David A. Smith \ Fort Worth, Texas

Table of contents

Chapter 1

Limits and Continuity

In mathematics, both limits and continuity play a very important role in proofs and calculations. In this book, you will learn all about limits: what they are, how to find them, and how to use them in mathematical proofs. You will also learn about the different types of discontinuities and how to deal with them. With this knowledge, you will be able to perform complex calculations with ease and confidence.

Limits are the bread and butter of calculus. Without limits, we wouldn't be able to find derivatives or integrals. In short, limits are essential for understanding calculus. So what are limits? Put simply, limits are a way of describing what happens when a function gets closer and closer to a certain value.

For example, if we have a function that is always increasing but never reaches a specific number, we say that the function approaches that number as its limit. This may not seem like a big deal, but limits are actually incredibly important. They allow us to define things like continuity and differentiability, which are essential concepts in calculus. So next time you're struggling with a calculus problem, remember that it all comes down to limits!

Intuitively, one can think of limits as boundaries. Just as there are physical limits to what a person can do, there are limits to what a function can do. By understanding these limits, we can better understand the nature of continuity and how to take advantage of it in our calculations.

In short, limits are important in calculus because they allow us to understand how functions behave at specific points within their domains.

Limits are important because they help us understand continuity, derivatives, and integrals. And continuity is important because, well, the world is continuous (so it seems)! So limits are pretty important.

There are a few techniques that can be helpful for finding limits. One is to think about what the function is doing approaching the point in question. Is it getting closer and closer to a certain value? If so, that value is probably the limit. Another technique is to plug in values that are close to the point in question and see what happens. This can be helpful for getting an intuition for how the function behaves near the point in question. And finally, sometimes it can be helpful to use some algebraic manipulation to simplify the expression before taking the limit.

As with all things mathematical, practice makes perfect, so don't be discouraged if limits give you some trouble at first. Keep at it and you'll get the hang of it in no time!

Limits are important in calculus because they allow us to determine whether a function is continuous or not. Intuitively, a function is continuous if given any two points within its domain, there exists a smooth curve that connects those points. Limits enable us to test for continuity by seeing if the function's value approaches the same number as we approach a specific point within the domain.

If the function's value does approach the same number, then we say that the function is continuous at that point. If the function's value does not approach the same number, then we say that the function is discontinuous at that point. Continuity is important because it allows us to determine how a function will behave as we approach certain values within its domain.

Continuity describes how a function behaves over a range of values. A function is continuous if it is smooth and uninterrupted over that range. In other words, there are no sudden jumps or discontinuities. Most functions you encounter in everyday life are continuous. For example, the position of a car on a road is continuous - there are no gaps or discontinuities in the car's position. On the other hand, a function like 1/x is not continuous, because it has a discontinuity at x=0.

Together, limits and continuity allow us to understand and predict the behavior of functions at both the micro and macro levels. At the micro level, limits allow us to understand what happens to a function as x approaches a certain value. And at the macro level, continuity allows us to understand how a function behaves over a range of values. These concepts are essential for understanding and manipulating functions in calculus.

The existence of limits is one of the fundamental properties distinguishing calculus from ordinary algebra and geometry. Limits are basic to differential and integral calculus and play an important role in many other branches of mathematics. They also occur in physics and other sciences in connection with limiting cases and approximations. Informally, limits describe the behavior of a function as its arguments "approach" certain defined values; they provide a precise foundation for calculus, enabling earlier work with limits to be rigorously justified.

In most cases limits can be found by algebraic manipulations; more sophisticated means may sometimes be required, but even in difficult cases it is almost always possible to find some numerical indication of the desired limit. Many functions have limits at infinity that can be expressed in simple algebraic form, while others grow so rapidly that no finite limit exists.

Conversely, some functions have limits that do not exist when approached from either direction (the right or left), while others that have left and right limits may not have a limit as x approaches any real number other than these two specific values. The concept of a limit is thus seen to be closely related both to that of continuity (which deals with single-valued functions only) and to that of an infinite series, which enables discontinuities and multivalued functions also to be treated rigorously by means of limits.

One might wonder, “Why do we need a precise definition of limits? Can’t we just say that a limit is when a function gets close to a certain value?” While this informal definition suffices in many cases, there are situations where a more precise definition is required. For example, consider the function f(x) = 1/x. As x approaches 0, this function gets closer and closer to infinity. However, it never actually reaches infinity.

So if we were to use the informal definition of limits, we would say that the limit of f(x) as x approaches 0 is infinity. However, this is not very useful, since it doesn’t tell us anything about what happens to the function at x = 0. To get around this problem, mathematicians have devised a more precise definition of limits.

This definition allows us to say that the limit of f(x) as x approaches 0 is actually undefined. While this might not seem like a very satisfying answer, it does give us important information about the behavior of the function at x = 0. So in some cases, a more precise definition of limits can be quite helpful.

However, the definition of limits is actually quite technical, and it turns out to be very useful in calculus and other areas of mathematics. Without a precise definition of limits, many important theorems would be impossible to prove. So next time you’re feeling frustrated with your math homework, remember that your struggles could be helping to further the field of mathematics!

Have you ever been driving down the highway and suddenly had to brake for a stop sign or red light? If so, then you’ve experienced limits firsthand. Whenever you’re driving, you’re constantly changing speeds, and your car’s speedometer is measuring the rate of change of your car’s velocity.

But what happens when your car comes to a stop? The speedometer still measures a nonzero rate of change, but it's obvious that your car's velocity has changed dramatically. This is because the speedometer is measuring the instantaneous rate of change, or the limit of the average rate of change as the time interval approaches zero. This may seem like a lot of math jargon, but limits are actually quite intuitive.

In essence, they tell us how things are changing at a given moment. And limits are not just useful for calculus; they're also essential for understanding rates of change in everyday life. So next time you're stuck in traffic, just think of it as an opportunity to learn about limits!

One of the most important concepts in calculus is continuity. limits are a fundamental tool in calculus that allow us to determine whether a function is continuous at a point. Intuitively, continuity means that a function is "smooth" and doesn't have any sharp jumps or abrupt changes.

Continuity is important because many of the most useful functions in calculus, such as the derivative and integral, are only defined for continuous functions. As a result, being able to identify and understand limits is essential for anyone who wants to study calculus. Luckily, limits are not as difficult as they may seem at first. With a little practice, anyone can learn to find them. And once you've mastered limits, you'll be well on your way to understanding one of the most important branches of mathematics.

Limits and continuity may seem like dry, theoretical concepts, but they actually have a lot of real-world applications. For example, limits are used in calculus to determine the rate of change of a function, and they also play a role in physics when studying things like motion and acceleration. Similarly, continuity is important in many fields, including economics and computer programming. In fact, virtually any time you're dealing with change or movement, limits and continuity are likely to be involved.

For example, limits can be used to understand how a car accelerates or how a projectile moves through the air. Continuity can be used to determine whether a path is smooth or whether a financial market is stable. In each of these cases, limits and continuity provide valuable insight that can help us to make better decisions.

So next time you're stuck in traffic or trying to figure out why your computer keeps crashing, remember that you're actually dealing with some pretty complex math!

Limits, the cornerstone of calculus, are a measure of how close a function gets to a certain value as it approaches that value from either direction. In other words, limits tell us how a function behaves near a point, such as whether it approaches the point from above or below, or oscillates around the point.

Continuity is a related concept: it deals with how well a function can be defined at a point, and whether it behaves predictably near that point. A function is continuous at a point if its limit exists at that point and if the function's value at that point equals the limit. Intuitively, this means that a continuous function can be drawn without lifting one's pencil from the paper. Many of the most important results in calculus depend on continuity; in fact, one could argue that calculus is really about continuity.

The idea of limits is used to define all sorts of important geometric objects such as tangent lines and curves, and these in turn are used to study functions and their behavior. Continuity is also used extensively in physics: for example, when solving problems involving fluids or electricity, one often assumes that the underlying functions are continuous. In short, continuity is a fundamental concept in calculus with far-reaching implications.

In the world of mathematics, limits are important because they help to define continuity. In other words, limits tell us when a function is "continuous" or "smooth." Without limits, it would be difficult to calculate things like derivatives and integrals. So, in a sense, limits are the foundation of calculus. Without them, many of the important ideas in calculus would simply not be possible. Of course, limits can be a bit tricky to understand at first. But once you get the hang of it, you'll see that they're not so bad after all. And who knows? You might even come to enjoy working with them.

1.1 An Intuitive Introduction to Limits

This lecture illustrates when a limit does not exist by giving three case examples:

- A limit does not exist because the **one-sided limits** do not agree in value.
- A limit does not exist because of an oscillating behavior of a function, and
- a limit does not exist (no finite value) because of an unbounded behavior of a function.

1.2 Limits Using Tables

A **limit** is used to describe the behavior of a function near a point but not at the point. The function need not even be defined at the point. If it is defined there, the value of the function at the point does not affect the limit. Intuitively,

$$\lim_{x \to c} f(x) = L \tag{1.1}$$

means we can make$f(x)$ as close to L as we wish by taking any x sufficiently close to, but different from c.

Example 1.1. Find the limit of $f(x)$ as x **approaches** c using a table of functional values for

$$f(x) = \frac{4x-9}{x^2-4} \tag{1.2}$$

and $c = 3$.

Solution. We compute,

x	$f(x)$
2.997	0.599758
2.998	0.599839
2.999	0.59992

x	$f(x)$
3.003	0.600238
3.002	0.600159
3.001	0.600079

Thus as x approaches 3 from the left we estimate that $f(x)$ approaches $3/5$; and as x approaches 3 from the right we estimate that $f(x)$ approaches $3/5$. Therefore, we estimate

$$\lim_{x\to 3} \frac{4x-9}{x^2-4} = \frac{3}{5}. \tag{1.3}$$

Example 1.2. We will use a guessing method to show why the formal definition of a limit is a necessity.
Use a table to guess the values of

$$L = \lim_{x\to 0} \frac{2\sqrt{x+1}-x-2}{x^2}. \tag{1.4}$$

Solution. From the table

x	-0.5	-0.1	-0.01	-0.001	0	0.001	0.005
$f(x)$	-0.3431	-0.2633	-0.2513	-0.2501	-	-0.2499	-0.2494

The number L is suggested to be -0.25. Interestingly, if you try

$$x = 0.0000001 \tag{1.5}$$

just to make sure you have taken numbers close enough to 0, you may find that the calculator gives the value 0. Does this mean that the limit is 0? No, the calculator may give you a false answer because when x is small enough (like 0.0000001) then $2\sqrt{x+1}-x-2$ seems like 0. But in fact $f(0.0000001)$ is not equal to 0.

The point is, using technology to verify a computation can lead to misunderstanding; and in fact, a formal definition of a limit is needed. Using the formal definition of a limit, we can prove what the value of the limit is without any doubt. This type of proof is usually called an epsilon-delta proof since the formal definition is usually stated with the greek letters $ $ (epsilon) and $ $ (delta).

Example 1.3. We will use a guessing method to show why the formal definition of a limit is a necessity.
Use tables of values to find the limit

$$\lim_{x\to 0}\left(x^3+\frac{\cos 5x}{10,000}\right). \tag{1.6}$$

Solution. We construct a table of values.

x	1	0.5	0.1	0.05	0.01
$f(x)$	1.000028	0.124920	0.001088	0.000222	0.000101

From the table it appears that

$$\lim_{x\to 0}\left(x^3+\frac{\cos 5x}{10,000}\right)=0. \tag{1.7}$$

However, if we persevere with smaller values of x, the next table suggests

$$\lim_{x\to 0}\left(x^3+\frac{\cos 5x}{10,000}\right)=0.0001000=\frac{1}{10,0000} \tag{1.8}$$

(1.9)

x	0.005	0.001
$f(x)$	1.00010009	0.00010000

In fact,

$$\lim_{x\to 0}\left(x^3+\frac{\cos 5x}{10,000}\right)=\frac{1}{10,0000} \tag{1.10}$$

which is easily proven once the formal limit definition is used to prove some interesting limit rules and continuity is discussed.

In summary, a three-pronged approach to solving limits is often:

- numerical approach by constructing tables of values,
- graphical approach by sketching a graph by hand or using technology,
- analytic approach by using algebra or calculus.

1.3 One-Sided and Two-Sided Limits

Consider, for example, a piecewise function with a jump where the function is pieced together (defined or not). Even though the one-sided limits might exist, they must agree in value for the **two-sided limit** to exist. In short, if the one-sided limits do not agree then the two-sided limit does not exist.

Theorem 1.1. *The two-sided limit* $\lim_{x\to c} f(x)$ *exists if and only if the one-sided limits* $\lim_{x\to c^+} f(x)$ *and* $\lim_{x\to c^-} f(x)$ *both exist and* $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x)$. *In which case,* $\lim_{x\to c} f(x) = \lim_{x\to c^+} f(x)$.

Example 1.4. Sketch the graph of the **piecewise function** f defined by

$$f(x) = \begin{cases} 2x^2+1 & x<2 \\ 4 & x=2 \\ 3x & x>2. \end{cases} \tag{1.11}$$

Evaluate the following limits,

- $\lim_{x\to 2^-} f(x)$.
- $\lim_{x\to 2^+} f(x)$.
- $\lim_{x\to 2} f(x)$.
- $\lim_{x\to 1^-} f(x)$.
- $\lim_{x\to 1^+} f(x)$.
- $\lim_{x\to 1} f(x)$.
- $\lim_{x\to 4^-} f(x)$.
- $\lim_{x\to 4^+} f(x)$.
- $\lim_{x\to 4} f(x)$.

Example 1.5. Sketch the graph of the piecewise function $g(x)$ where g is defined by

$$g(x) = \begin{cases} 3x-2 & x<-1 \\ 4 & x=-1 \\ x+5 & -1<x<3 \\ 4 & x=3 \\ 2-x & x>3. \end{cases} \tag{1.12}$$

Evaluate the following limits,

- $\lim_{x\to -1^-} g(x)$
- $\lim_{x\to -1^+} g(x)$
- $\lim_{x\to -1} g(x)$
- $\lim_{x\to 3^-} g(x)$

- $\lim_{x\to 3^+} g(x)$
- $\lim_{x\to 3} g(x)$
- $\lim_{x\to 4} g(x)$
- $\lim_{x\to 0} g(x)$
- $\lim_{x\to 1} g(x)$

1.4 Oscillating Behavior

Also, consider another case where a function has an **oscillating behavior** . On one hand, the trigonometric functions all have an oscillating (**periodic**) behavior. However, imagine a function where the oscillation becomes much more pronounced as the variable approaches a fixed point; this type of oscillating behavior is where the function may not have a limit.

Example 1.6. Find $\lim\limits_{x\to 0} \cos\left(\dfrac{1}{x}\right)$.

Solution. The limit does not exist because $\cos\frac{1}{x} = 1$ and $\cos\frac{1}{x} = -1$ for

$$x = \frac{1}{2\pi}, \frac{1}{4\pi}, \frac{1}{6\pi}, \dots \quad \text{and} \quad x = \frac{1}{\pi}, \frac{1}{3\pi}, \frac{1}{5\pi}, \dots$$

respectively.

The graph of $y = \cos\left(\frac{1}{x}\right)$ is oscillating around $x = 0$, so we infer that the limit does not exist because$f(x) = \cos\frac{1}{x}$ does not approach a number, but rather oscillates, as x approaches 0.

1.5 Unbounded Behavior

Finally, we illustrate the case where a function becomes unbounded as the variable approaches a fixed point; for example, a function with a vertical asymptote. Without a finite number to assign the limit, we sometimes say that the limit does not exist.

Example 1.7. Determine $\lim\limits_{x\to 0} \dfrac{1}{x}$.

Solution. Since f decreases without bound as $x \to 0^-$ and f increases without bound as $x \to 0^+$, we say that $\lim_{x\to 0} f(x)$ does not exist.

Example 1.8. Determine $\lim\limits_{x\to 0} \dfrac{1}{x^2}$.

Solution. Since f increases without bound as $x \to 0^-$ and f increases without bound as $x \to 0^+$, we say that $\lim_{x\to 0} f(x) = +\infty$.

1.6 Exercises

Exercise 1.1. Estimate the limits, if they exist, by using a table of values to two decimal places.

- $\lim_{x\to 2^+} \frac{x^2-4}{x-4}$
- $\lim_{x\to 4^+} \frac{\frac{1}{\sqrt{x}}-\frac{1}{2}}{x-4}$
- $\lim_{x\to 0} \frac{\sin 2x}{x}$
- $\lim_{x\to 0} |x| \sin \frac{1}{x}$

Exercise 1.2. Sketch the graph of f and g. Then identify the values of c for which $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist given

- $f(x) = \begin{cases} x^2 & x \le 2 \\ 8-2x & 2 < x < 4 \\ 4 & x \ge 4 \end{cases}$
- $g(x) = \begin{cases} \sin x & x < 0 \\ 1-\cos x & 0 \le x \le \pi \\ \cos x & x > \pi \end{cases}$

Exercise 1.3. Find a so that the function $f(x) = \begin{cases} ax+3 & x \le 2 \\ 3-x & x > 2 \end{cases}$ satisfies $\lim_{x\to 2} f(x) = 1$.

Exercise 1.4. Estimate the limits by using tables of values for

- $\lim_{x\to 13} \frac{x^3-9x^2-45x-91}{x-13}$
- $\lim_{x\to 13} \frac{x^3-9x^2-39x-86}{x-13}$

Then using long division (or *synthetic division* if you know it) explain why one of the limits exists and the other does not.

Exercise 1.5. Consider the function $f(x) = \frac{|x+1|-|x-1|}{x}$. Estimate $\lim_{x\to 0} f(x)$ by evaluating f at x-values near 0. Sketch the graph of f.

Exercise 1.6. Explain why $\lim_{x\to 2} \frac{|x-2|}{x-2}$ does not exist.

Exercise 1.7. Evaluate the function $f(x) = x^2 - \frac{2^x}{1000}$ for $x = 1$, 0.8, 0.6, 0.4, 0.2, 0.1, and 0.05. Guess the value of $\lim_{x\to 0} f(x)$. Evaluate the function $f(x) = x^2 - \frac{2^x}{1000}$ for $x = 0.04, 0.02, 0.01, 0.005, 0.003$, and 0.001. Guess again.

Exercise 1.8. The tabular approach is a convenient device for discussing limits informally, but if it is not used carefully, it can be misleading. For example, for $x > 0$, let $f(x) = \sin\left(\frac{\pi}{\sqrt{x}}\right)$

- Construct a table showing the value of x and $f(x)$ for $x = 4$, 4/25, 4/81, 4/169, and 4/289. Based on this table what would you say about $\lim_{x\to 0^+} f(x)$?
- Construct a table showing the value of x and $f(x)$ for $x = 4$, 4/49, 4/121, 4/225, 4/361. Based on this table what would you say about $\lim_{x\to 0^+} f(x)$?
- Based on your results in (a) and (b) what do you conclude about $\lim_{x\to 0^+} f(x)$?

Exercise 1.9. Sketch the graph of the function

$$f(x) = \begin{cases} -(x+1)^2 + 1 & -1 \le x \le 0 \\ x^2 & 0 < x < 1 \\ 1 & 1 < x < 2 \\ 2 & x = 2 \\ 1 & 2 < x \le 3. \end{cases}$$

and then use the graph to determine which the following statements about the function $y = f(x)$ are true and which are false?

- $\lim_{\to -1^+} f(x) = 1$
- $\lim_{x\to 2} f(x)$ does not exist
- $\lim_{x\to 2} f(x) = 2$
- $\lim_{x\to 1^-} f(x) = 2$(e) $\lim_{x\to 1^+} f(x) = 1$
- $\lim_{x\to 1} f(x)$ does not exist
- $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$
- $\lim_{x\to c} f(x)$ exists at every c in the open interval $(-1, 1)$.
- $\lim_{x\to c} f(x)$ exists at every c in the open interval $(1, 3)$.
- $\lim_{x\to -1^-} f(x) = 0$
- $\lim_{x\to 3^+} f(x)$ does not exist

Exercise 1.10. Sketch the graph of the function

$$f(x) = \begin{cases} 3-x & x<2 \\ 2 & x=2 \\ \frac{x}{2} & x>2. \end{cases}$$

and then use the graph to determine the following?

- Find $\lim_{x\to 2^+} f(x)$, $\lim_{x\to 2^-} f(x)$, and $f(2)$.
- Does $\lim_{x\to 2} f(x)$ exist? If so, what is it? If not, why not?
- Find $\lim_{x\to -1^-} f(x)$ and $\lim_{x\to -1^+} f(x)$.
- Does $\lim_{x\to -1} f(x)$ exist? If so, what is it? If not, why not?

Exercise 1.11. Let $g(x) = \sqrt{x}\sin\left(\frac{1}{x}\right)$. Use the graph of g to determine the following.

- Does $\lim_{x\to 0^+} g(x)$ exist? If so, what is it? If not, why not?
- Does $\lim_{x\to 0^-} g(x)$ exist? If so, what is it? If not, why not?
- Does $\lim_{x\to 0} g(x)$ exist? If so, what is it? If not, why not?

Exercise 1.12. Graph $f(x) = \begin{cases} x^3 & x\neq 1 \\ 0 & x=1. \end{cases}$ Find $\lim_{x\to 1^-} f(x)$ and $\lim_{x\to 1^+} f(x)$. Does $\lim_{x\to 1} f(x)$ exist? If so, what is it? If not, why not?

Exercise 1.13. Graph $f(x) = \begin{cases} 1-x^2 & x\neq 1 \\ 2 & x=1. \end{cases}$ Find $\lim_{x\to 1^-} f(x)$ and $\lim_{x\to 1^+} f(x)$. Does $\lim_{x\to 1} f(x)$ exist? If so, what is it? If not, why not?

Exercise 1.14. Graph $f(x) = \begin{cases} x & -1\leq x<0 \quad \text{or} \quad 0<x\leq 1 \\ 1 & x=0 \\ 0 & x<-1 \quad \text{or} \quad x>1. \end{cases}$

- What is the domain and range of f?
- At what points c, if any does $\lim_{x\to c} f(x)$ exist?
- At what points does only the left-hand limit exist?
- At what points does only the right-hand limit exist?

Exercise 1.15. Find the following limits.

- $\lim_{x\to -7}(2x+5)$.
- $\lim_{x\to -2}\left(x^3-2x^2+4x+8\right)$.
- $\lim_{x\to 2}\left(\frac{x+3}{x+6}\right)$.

- $\lim_{y\to 2}\left(\frac{y+2}{y^2+5y+6}\right)$.
- $\lim_{y\to -3}(5-y)^{4/3}$.
- $\lim_{h\to 0}\left(\frac{5}{\sqrt{5h+4}+2}\right)$.
- $\lim_{x\to 5}\left(\frac{x-5}{x^2-25}\right)$.
- $\lim_{x\to 2}\left(\frac{x^2-7x+10}{x-2}\right)$.
- $\lim_{x\to -2}\left(\frac{-2x-4}{x^3+2x^2}\right)$.
- $\lim_{x\to 1}\left(\frac{x-1}{\sqrt{x+3}-2}\right)$.
- $\lim_{x\to -2}\left(\frac{x+2}{\sqrt{x^2+5}-3}\right)$.

Exercise 1.16. Suppose $\lim_{x\to 4} f(x) = 0$ and $\lim_{x\to 4} g(x) = 3$. Using limit laws find the limits $\lim_{x\to 4}(g(x)+3)$, $\lim_{x\to 4} xf(x)$, $\lim_{x\to 4}(g(x))^2$, and $\lim_{x\to 4}\frac{g(x)}{f(x)+1}$.

Exercise 1.17. Suppose that $\lim_{x\to -2} p(x) = 4$, $\lim_{x\to -2} r(x) = 0$, and $\lim_{x\to -2} s(x) = -3$. Using limit laws find the limits, $\lim_{x\to -2}(p(x)+r(x)+s(x))$, $\lim_{x\to -2} p(x)r(x)s(x)$, and $\lim_{x\to -2}\left(\frac{-4p(x)+5r(x)}{s(x)}\right)$.

Exercise 1.18. Using limit laws evaluate the limit, $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ where $f(x) = x^2$ and $x = 1$.

Exercise 1.19. Using limit laws evaluate the limit, $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ where $f(x) = 3x-4$ and $x = 2$.

Exercise 1.20. If $\lim_{x\to -2}\frac{f(x)}{x^2} = 1$, find $\lim_{x\to -2} f(x)$ and $\lim_{x\to -2}\frac{f(x)}{x}$.

Exercise 1.21. If $\lim_{x\to 2}\frac{f(x)-5}{x-2} = 3$, find $\lim_{x\to 2} f(x)$. Also if $\lim_{x\to 2}\frac{f(x)-5}{x-2} = 4$ find $\lim_{x\to 2} f(x)$.

1.7 Techniques for Finding Limits

1.8 Calculating Limits Using Limit Theorems

In this topic we concentrate not on the formal definition of a limit of a function of one variable but rather give several examples which emphasis algebra and trigonometry techniques to evaluate limits of functions using basic limit theorems.

::: {#thm- } [Limit Theorems] For any real number c, suppose the functions f and g both have finite limits at $x = c$. Then

- (Constant) $\lim_{x\to c} k = k$ for any constant k
- (Limit of x) $\lim_{x\to c} x = c$
- (Multiple) $\lim_{x\to c} kf(x) = k \lim_{x\to c} f(x)$
- (Sum) $\lim_{x\to c}[f(x) + g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$
- (Difference) $\lim_{x\to c}[f(x) - g(x)] = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$
- (Product) $\lim_{x\to c}[f(x)g(x)] = (\lim_{x\to c} f(x)) (\lim_{x\to c} g(x))$
- (Quotient) $\lim_{x\to c}[f(x)/g(x)] = (\lim_{x\to c} f(x)) / (\lim_{x\to c} g(x))$
- (Power) $\lim_{x\to c}[f(x)]^n = (\lim_{x\to c} f(x))^n$ where n is a rational number and whenever the limit on the right exists
- (Polynomial) $\lim_{x\to c} P(x) = P(c)$ for any polynomial P
- (Rational) $\lim_{x\to c} R(x) = R(c)$ for any rational function R where c is in the domain of R. :::

Example 1.9. Find the limit of $f(x) = \frac{2x^3-5x+8}{x^2-3}$ at $x = 3$.

Solution. By using several limit rules, we have

$$\begin{aligned}
\lim_{x\to 3} \frac{2x^3 - 5x + 8}{x^2 - 3} &= \frac{\lim_{x\to 3} (2x^3 - 5x + 8)}{\lim_{x\to 3} (x^2 - 3)} \\
&= \frac{\lim_{x\to 3} (2x^3) - \lim_{x\to 3}(5x) + \lim_{x\to 3}(8)}{\lim_{x\to 3} (x^2) - \lim_{x\to 3}(3)} \\
&= \frac{2 \lim_{x\to 3} (x^3) - 5 \lim_{x\to 3}(x) + 8}{\lim_{x\to 3} (x^2) - 3} \\
&= \frac{2 (\lim_{x\to 3} x)^3 - 5(3) + 8}{(\lim_{x\to 3} x)^2 - 3} = \frac{2(3)^3 - 5(3) + 8}{(3)^2 - 3} = \frac{47}{6}
\end{aligned}$$

Notice this is the same as evaluating the rational function $f(x)$ at $x = 3$.

Example 1.10. Compute the limit of $f(x) = \frac{2x^3+x^2-16x+12}{x^2-4}$ at $x = 2$.

Solution. By using several limit rules, we have

$$\begin{aligned}
\lim_{x\to 2} \frac{2x^3 + x^2 - 16x + 12}{x^2 - 4} &= \lim_{x\to 2} \frac{(x-2)(2x^2 + 5x - 6)}{(x-2)(x+2)} \\
&= \lim_{x\to 2} \frac{(2x^2 + 5x - 6)}{(x+2)} = \frac{(2(2)^2 + 5(2) - 6)}{((2) + 2)} = 3.
\end{aligned}$$

In the previous example notice that we used

$$\lim_{x\to 2} \frac{(x-2)(2x^2 + 5x - 6)}{(x-2)(x+2)} = \lim_{x\to 2} \frac{(2x^2 + 5x - 6)}{(x+2)}. \tag{1.13}$$

Now it is not true that the functions

$$f(x) = \frac{(2x^2 + 5x - 6)}{(x + 2)} \qquad \text{and} \qquad g(x) = \frac{(2x^2 + 5x - 6)}{(x + 2)}$$

are the same function because they have different domains. But the above equality is true because x is approaching 2, and not equal to 2. So the point is,because $f(x) = g(x)$ when $x \neq 2$ we can indeed say (1.13) holds. This is an important part of understanding limits.

Example 1.11. Suppose $\lim_{x\to -2^-} f(x) = 2$, $\lim_{x\to -2^+} f(x) = 4$, $\lim_{x\to -2^-} g(x) = 0$, and $\lim_{x\to -2^+} g(x) = 0$, find

$$\lim_{x\to -2}[f(x) + g(x)] \quad \text{and} \quad \lim_{x\to -2}[f(x)g(x)].$$

Solution. Since

$$\lim_{x\to -2^-} f(x) \neq \lim_{x\to -2^+} f(x)$$

we know $\lim_{x\to -2} f(x)$ does not exist, but this does not imply anything about $\lim_{x\to -2}[f(x) + g(x)]$ nor $\lim_{x\to -2}[f(x)g(x)]$.
To find these limits we first find the two one-sided limits,

$$\lim_{x\to -2^-}[f(x) + g(x)] = \lim_{x\to -2^-} f(x) + \lim_{x\to -2^-} g(x) = 2 + 0 = 2$$

$$\lim_{x\to -2^+}[f(x) + g(x)] = \lim_{x\to -2^+} f(x) + \lim_{x\to -2^+} g(x) = 4 + 0 = 4$$

and since

$$\lim_{x\to -2^-}[f(x) + g(x)] \neq \lim_{x\to -2^+}[f(x) + g(x)]$$

we can now say the two-sided limit $\lim_{x\to -2}[f(x) + g(x)]$ does not exist. Similarly,

$$\lim_{x\to -2^-}[f(x)g(x)] = \left(\lim_{x\to -2^-} f(x)\right)\left(\lim_{x\to -2^-} g(x)\right) = 2(0) = 0$$

$$\lim_{x\to -2^+}[f(x)g(x)] = \left(\lim_{x\to -2^+} f(x)\right)\left(\lim_{x\to -2^+} g(x)\right) = 4(0) = 0$$

and therefore, $\lim_{x\to -2}[f(x)g(x)] = 0$.

1.9 Special Trigonometric Limits

Theorem 1.2. *The following trigonometric limits hold:*

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad \textit{and} \qquad \lim_{x\to 0} \frac{\cos x - 1}{x} = 0$$

Example 1.12. Find $\lim_{x\to 0} \frac{\sin^2 x}{2x}$.

Solution. We have $\lim_{x\to 0} \frac{\sin^2 x}{2x} = \lim_{x\to 0} \frac{\sin x}{2} \frac{\sin x}{x} = \frac{0}{2}(1) = 0$.

Example 1.13. Find $\lim_{x\to 0} \frac{\sin x(1-\cos x)}{2x^2}$.

Solution. We have

$$\begin{aligned}\lim_{x\to 0} \frac{\sin x(1-\cos x)}{2x^2} &= \lim_{x\to 0} \frac{1}{2} \frac{\sin x}{x} \frac{1-\cos x}{x} \\ &= \frac{1}{2} \lim_{x\to 0} \left(\frac{\sin x}{x}\right) \lim_{x\to 0} \left(\frac{1-\cos x}{x}\right) = \frac{1}{2}(1)(0) = 0.\end{aligned}$$

Example 1.14. Find $\lim_{x\to 0} \frac{\sin 5x}{\sin 4x}$.

Solution. We have

$$\begin{aligned}\lim_{x\to 0} \frac{\sin 5x}{\sin 4x} &= \lim_{x\to 0} \left(\frac{\sin 5x}{x}\right)\left(\frac{x}{\sin 4x}\right) \\ &= \lim_{x\to 0} \left(\frac{5\sin 5x}{5x}\right)\left(\frac{4x}{4\sin 4x}\right) \\ &= \lim_{x\to 0} \left(\frac{5\sin 5x}{5x}\right) \lim_{x\to 0} \left(\frac{4x}{4\sin 4x}\right) = \frac{5}{4}.\end{aligned}$$

Example 1.15. Given $g(x) = -27 - 9x - 6x^2 + x^3 + x^4$ and $h(x) = 21 - 16x - 3x^2 + 2x^3$, compute the limit $g(x)/h(x)$ at $x = 3$.

Solution. Try a factor of $(x-3)$ from $g(x)$ obtaining

$$g(x) = (x-3)\left(9 + 6x + 4x^2 + x^3\right)$$

and a graph of h is which also inspires to try to factor of $(x-3)$ from $h(x)$ obtaining $h(x) = (x-3)\left(2x^2 + 3x - 7\right)$.
Therefore,

$$\begin{aligned}\lim_{x\to 3} \frac{-27 - 9x - 6x^2 + x^3 + x^4}{21 - 16x - 3x^2 + 2x^3} &= \lim_{x\to 3} \frac{(x-3)\left(9 + 6x + 4x^2 + x^3\right)}{(x-3)\left(2x^2 + 3x - 7\right)} \\ &= \lim_{x\to 3} \frac{9 + 6x + 4x^2 + x^3}{2x^2 + 3x - 7} = \frac{9 + 6(3) + 4(3)^2 + (3)^3}{2(3)^2 + 3(3) - 7} = \frac{9}{2}.\end{aligned}$$

Theorem 1.3. *If c is any real number in the domain of the given function, then*

$$\begin{array}{lll} \lim_{x\to c} \cos x = \cos c & \lim_{x\to c} \sin x = \sin c & \lim_{x\to c} \tan x = \tan c \\ \lim_{x\to c} \sec x = \sec c & \lim_{x\to c} \csc x = \csc c & \lim_{x\to c} \cot x = \cot c. \end{array}$$

Example 1.16. Compute the limit of $f(x) = \frac{x}{\sin x - 2\cos x}$ at $x = \pi$.

Solution. By using several limit rules, we have

$$\lim_{x\to\pi} \frac{x}{\sin x - 2\cos x} = \frac{\pi}{\sin\pi - 2\cos\pi} = \frac{\pi}{0 - 2(-1)} = \frac{\pi}{2}.$$

Example 1.17. Compute $\lim_{x\to\pi/4} \frac{1-\tan x}{\sin x - \cos x}$.

Solution. We have

$$\begin{aligned}\lim_{x\to\pi/4} \frac{1-\tan x}{\sin x - \cos x} &= \lim_{x\to\pi/4} \frac{1-\frac{\sin x}{\cos x}}{\sin x - \cos x}\\ &= \lim_{x\to\pi/4} \frac{\frac{\cos x - \sin x}{\cos x}}{\sin x - \cos x} = \lim_{x\to\pi/4} \frac{\cos x - \sin x}{\cos x} \frac{1}{\sin x - \cos x}\\ &= \lim_{x\to\pi/4} \frac{-1}{\cos x} = \lim_{x\to\pi/4} \frac{-1}{\frac{1}{\sqrt{2}}} = -\sqrt{2}\end{aligned}$$

Example 1.18. Compute the limit of $f(x) = \frac{\tan x}{\sin x}$ as $x \to 0$.

Solution. We have $\lim_{x\to 0} \frac{\tan x}{\sin x} = \lim_{x\to 0} \left(\frac{\sin x}{\cos x}\right)\left(\frac{1}{\sin x}\right) = \lim_{x\to 0} \frac{1}{\cos x} = 1$.

1.10 Using Rationalization

Example 1.19. Compute: $\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1}$.

Solution. We have

$$\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} = \lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} \frac{\sqrt{x}+1}{\sqrt{x}+1} = \lim_{x\to 1} \frac{1}{\sqrt{x}+1} = \frac{1}{\sqrt{1}+1} = \frac{1}{2}.$$

Example 1.20. Compute the limit of $f(x) = \frac{x-\pi}{\sqrt{x}-\sqrt{\pi}}$ as $x \to \pi$.

Solution. We have

$$\begin{aligned}\lim_{x\to\pi} \frac{x-\pi}{\sqrt{x}-\sqrt{\pi}} &= \lim_{x\to\pi} \left(\frac{x-\pi}{\sqrt{x}-\sqrt{\pi}}\right)\left(\frac{\sqrt{x}+\sqrt{\pi}}{\sqrt{x}+\sqrt{\pi}}\right)\\ &= \lim_{x\to\pi} \frac{(x-\pi)\left(\sqrt{x}+\sqrt{\pi}\right)}{x-\pi} = 2\sqrt{\pi}\end{aligned}$$

Example 1.21. Compute the limit of $f(x) = \frac{x-1}{\sqrt[3]{x}-1}$ as $x \to 1$.

Solution. We have

$$\lim_{x\to1} \frac{x-1}{\sqrt[3]{x}-1} = \lim_{x\to1} \left(\frac{x-1}{\sqrt[3]{x}-1}\right)\left(\frac{\sqrt[3]{x^2}+\sqrt[3]{x}+1}{\sqrt[3]{x^2}+\sqrt[3]{x}+1}\right)$$
$$= \lim_{x\to1} \frac{(x-1)\left(\sqrt[3]{x^2}+\sqrt[3]{x}+1\right)}{x-1} = \lim_{x\to1}\left(\sqrt[3]{x^2}+\sqrt[3]{x}+1\right) = 3.$$

1.11 Limits of Piecewise Functions

Example 1.22. Find

$$\lim_{x\to0} \begin{cases} 2(x+1) & x<3 \\ 4 & x=3 \\ x^2-1 & x>3 \end{cases} \tag{1.14}$$

Solution. Since $x \to 0$ we know that $x < 3$ and so we use $2(x+1)$ to evaluate the limit of the piecewise function, as follows,

$$\lim_{x\to0} \begin{cases} 2(x+1) & x<3 \\ 4 & x=3 \\ x^2-1 & x>3. \end{cases} = \lim_{x\to0} 2(x+1) = 2 \tag{1.15}$$

Example 1.23. At $x = 7$, compute the limit of

$$f(x) = \begin{cases} 2\left(x-x^2\right) & x<7 \\ -83 & x=7 \\ (x-7)^2-84 & x>7 \end{cases} \tag{1.16}$$

Solution. Since the function is pieced together at $x = 7$ we will evaluate two one sided limit. First the limit from the left, we have

$$\lim_{x\to7^-} f(x) = -84$$

and for the limit from the right we have

$$\lim_{x\to7^+} f(x) = -84.$$

Since

$$\lim_{x\to7^-} f(x) = \lim_{x\to7^+} f(x)$$

we know the two-sided limit must exist and we have

$$\lim_{x\to 7} f(x) = -84$$

even though $f(7) = -83$.

1.12 Squeeze Theorem

Next we state the squeeze theorem and through an example show how to use it. Basically, the idea is to bound a function on both sides by functions whose limits can be more easily computed; and thus in the process squeeze the value of the limit of the original function out.

::: {#thm- } Squeeze Theorem Let f, g and h be functions of x . If both of the following conditions hold

- $g(x) \leq f(x) \leq h(x)$ on an open interval containing c and
- $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$

then $\lim_{x\to c} f(x) = L$.
:::

Example 1.24. Use the squeeze rule to find the limit of $f(x) = x\cos\left(\frac{1}{x}\right)$ as $x \to 0^+$.

Solution. We are interested in this function around $x > 0$. Knowing that the cosine function is always less than or equal to one, we see that when $-1 \leq x \leq 1$ we have

$$-x \leq f(x) = x\cos\left(\frac{1}{x}\right) \leq x.$$

Since $\lim_{x\to 0} -x = \lim_{x\to 0} x = 0$ we have $\lim_{x\to 0} x\cos\left(\frac{1}{x}\right) = 0$ by the squeeze theorem.

1.13 Exercises

Exercise 1.22. Find the following limits.

- $\lim_{x\to -7}(2x+5)$
- $\lim_{x\to -2}\left(x^3 - 2x^2 + 4x + 8\right)$
- $\lim_{x\to 2}\left(\frac{x+3}{x+6}\right)$
- $\lim_{y\to 2}\left(\frac{y+2}{y^2+5y+6}\right)$
- $\lim_{y\to -3}(5-y)^{4/3}$
- $\lim_{h\to 0}\left(\frac{5}{\sqrt{5h+4}+2}\right)$

- $\lim_{x\to 5}\left(\frac{x-5}{x^2-25}\right)$
- $\lim_{x\to 2}\left(\frac{x^2-7x+10}{x-2}\right)$
- $\lim_{x\to -2}\left(\frac{-2x-4}{x^3+2x^2}\right)$
- $\lim_{x\to 1}\left(\frac{x-1}{\sqrt{x+3}-2}\right)$
- $\lim_{x\to -2}\left(\frac{x+2}{\sqrt{x^2+5}-3}\right)$

Exercise 1.23. Suppose $\lim_{x\to 4} f(x) = 0$ and $\lim_{x\to 4} g(x) = 3$. Using limit laws find the limits $\lim_{x\to 4}(g(x)+3)$, $\lim_{x\to 4} xf(x)$, $\lim_{x\to 4}(g(x))^2$, and $\lim_{x\to 4}\frac{g(x)}{f(x)+1}$.

Exercise 1.24. Suppose that $\lim_{x\to -2} p(x) = 4$, $\lim_{x\to -2} r(x) = 0$, and $\lim_{x\to -2} s(x) = -3$. Using limit laws find the limits, $\lim_{x\to -2}(p(x)+r(x)+s(x))$, $\lim_{x\to -2} p(x)r(x)s(x)$, and $\lim_{x\to -2}\left(\frac{-4p(x)+5r(x)}{s(x)}\right)$.

Exercise 1.25. Using limit laws evaluate the limit, $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ where $f(x) = x^2$ and $x = 1$.

Exercise 1.26. Using limit laws evaluate the limit, $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ where $f(x) = 3x-4$ and $x = 2$.

Exercise 1.27. If $\lim_{x\to -2}\frac{f(x)}{x^2} = 1$, find $\lim_{x\to -2} f(x)$ and $\lim_{x\to -2}\frac{f(x)}{x}$.

Exercise 1.28. If $\lim_{x\to 2}\frac{f(x)-5}{x-2} = 3$, find $\lim_{x\to 2} f(x)$. Also if $\lim_{x\to 2}\frac{f(x)-5}{x-2} = 4$ find $\lim_{x\to 2} f(x)$.

1.14 Continuous Functions

1.15 Continuity at a Point

Definition 1.1. A function is **continuous** at a point c means

- $f(c)$ is defined,
- $\lim_{x\to c} f(x)$ exists, and
- $\lim_{x\to c} f(x) = f(c)$.

1.16 Discontinuity

Example 1.25. Find three examples of how a **discontinuity** might arise.

Solution. First, the function

$$f(x) = \frac{x^2 - 2x + 1}{x - 1}$$

is discontinuous at $x = 1$ because $f(1)$ is not defined. So one type of discontinuity is a **hole** in the function.

Secondly, the function

$$f(x) = \begin{cases} x^2 + 1 & x \geq 0 \\ -x^2 - 2 & x < 0 \end{cases}$$

is discontinuous at $x = 0$ because $\lim_{x \to 0^-} f(x) = -2$ and $\lim_{x \to 0^+} f(x) = 1$; thus $\lim_{x \to 0} f(x)$ does not exist and so f has a discontinuity at $x = 0$. This type of discontinuity is called a **jump**.

Thirdly, the function

$$f(x) = \frac{x - 1}{x - 2}$$

is discontinuous at $x = 2$ because $f(2)$ is not defined and this type of discontinuity is called a **pole** because \$f(x)→+∞\$ as $x \to 2^+$.

1.17 Continuous Functions

Theorem 1.4. *If f and g are functions that are continuous at $x = c$ then $f \pm g$, fg, f/g,and $f \circ g$ are continuous at $x = c$, provided that c is in the domain of the function.*

Example 1.26. Give some examples of continuous functions.

Solution. For example, the functions $2x^2 - 2x + 5$, (polynomial), $\dfrac{x-1}{x}$ (rational), $\csc x$ (trigonometric), and $\sec^{-1} x$ (inverse trigonometric) are continuous on their domains. Also the functions

$$2x^2 - 2x + 5 + \frac{x - 1}{x}(\csc x),$$

$\sec^{-1}\left(\dfrac{x-1}{x}\right)$, and $(2x^2 - 2x + 5)\left(\dfrac{x-1}{x}\right)$ are continuous functions on their domains.

Theorem 1.5. *If f is a*

- *polynomial function,*
- *rational function,*
- *trigonometric function, or*
- *inverse trigonometric function,*

then f is continuous where it is defined.

::: {#thm- } [Composition Limit Theorem] If $\lim_{x\to c} g(x) = L$ and f is a continuous function at L, then

$$\lim_{x\to c}(f\circ g)(x) = f(L).$$

:::

Example 1.27. Use the **composition** limit theorem to evaluate the following limits.

$$\lim_{x\to 3}\left(x^2+3\right)^2 \quad \text{and} \quad \lim_{x\to \pi/4}\sin^4 x.$$

Solution. By the composition limit theorem, we have

$$\lim_{x\to 3}\left(x^2+3\right)^2 = \left(\lim_{x\to 3}\left(x^2+3\right)\right)^2 = 12^2 = 144.$$

By the composition limit theorem, we have

$$\lim_{x\to \pi/4}\sin^4 x = \left(\lim_{x\to \pi/4}\sin x\right)^4 = \left(\frac{\sqrt{2}}{2}\right)^4 = \left(\frac{1}{4}\right).$$

1.18 One-Sided Continuity

Definition 1.2. The function f is **continuous from the right** at c if and only if

$$\lim_{x\to c^+} f(x) = f(c)$$

and it is **continuous from the left** at c if and only if

$$\lim_{x\to c^-} f(x) = f(c).$$

Example 1.28. Give an example of a function that is continuous from the right (or **right continuous**) at $x = 0$.

Solution. The function $f(x) = \sqrt{x}$ is continuous from the right at $x = 0$ because $\lim_{x\to 0^+} \sqrt{x} = 0$.

1.19 Determining Parameters for Continuity

Example 1.29. Find constants a and b so that

$$\begin{cases} ax^2 + b & x > 2 \\ 4 & x = 2 \\ x^2 - ax + b & x < 2 \end{cases}$$

is continuous on $\mathbb{R}$.

Solution. Since f is defined on $\mathbb{R}$, and f is continuous for all $x \neq 2$ for any a and b that we choose, itis left to find an a and b such that $\lim_{x\to 2^+} ax^2 + b = 4$ and $\lim_{x\to 2^-} (x^2 - ax + b) = 4$. Thus we have the system $4a + b = 4$ and $4 - 2a + b = 4$. Solving this system we have, $a = 2/3$ and $b = 4/3$. :::

Example 1.30. Find constants a and b such that f is continuous at $x = 1$ where

$$\begin{cases} ax + b & x > 1 \\ 3 & x = 1 \\ x^2 - 4x + b + 3 & x < 1 \end{cases}$$

Solution. To have continuity at $x = 1$ we must have $\lim_{x\to 1^-} f(x) = 3$ and $\lim_{x\to 1^+} f(x) = 3$, thus $a(1) + b = 3$ and $(1)^2 - 4(1) + b + 3 = 3$. Therefore, $a + b = 3$ and $b = 3$. So $a + 3 = 3$ and $a = 0$.

Removable Continuity

Example 1.31. Determine the value for which $f(2)$ should be assigned, if any, to have

$$f(x) = \sqrt{\frac{x^2 - 4}{x - 2}}$$

continuous at $x = 2$.

Solution. Since $\lim_{x\to 2^-} f(x) = 2$ and $\lim_{x\to 2^+} f(x) = 2$ we have $\lim_{x\to 2} f(x) = 2$. Therefore, if we define $f(2) = 2$ the function f will be continuous at $x = 2$.

Intermediate Value Theorem

::: {#thm- } Intermediate Value Theorem If f is a continuous function on the closed interval $[a, b]$ and L is some number strictly between$f(a)$ and $f(b)$, then there exists at least one number c on the open interval (a, b) such that $f(c) = L$.

Example 1.32. The population (in thousands) of a colony of bacteria t minutes after the application of a toxin is given by the function

$$P(t) = \begin{cases} t^2+1 & \text{if } 0 \le t < 5 \\ -8t+66 & \text{if } t \ge 5 \end{cases}$$

- When does the colony die out? - Show that at some time between $t = 2$ and $t = 7$, the population is $9,000$.

Solution. Since

$$\lim_{t\to 5^+} P(t) = \lim_{t\to 5^+} (-8t+66) = 26$$

and

$$\lim_{t\to 5^-} P(t) = \lim_{t\to 5^-} (t^2+1) = 26$$

we know that P is continuous at $t = 5$ and thus, also for all $t \ge 0$. (a) The colony dies out when $-8t + 66 = 0$ which means $t = 33/4 \approx 8.25$. Therefore, the colony dies out in 8 minutes and 15 seconds.

(b) Since $P(2) = 5$, $P(7) = 10$, and P is continuous on $(2, 7)$, the intermediate value theorem yields at least one number c between 2 and 7 such $P(c) = 9$. Therefore, there is some time $t = c$ between $t = 2$ and $t = 7$ such that the population is 9,000.

1.20 Exercises

Exercise 1.29. Sketch the graph of the function

$$f(x) = \begin{cases} x^2-1 & -1 \le x < 0 \\ 2x & 0 < x < 1 \\ 1 & x = 1 \\ -2x+4 & 1 < x < 2 \\ 0 & 2 < x < 3. \end{cases}$$

Use the graph of the function f to answer the following. - Does $f(-1)$ exist? - Does the limit, $\lim_{x\to -1^+} f(x)$ exist? - Does the limit, $\lim_{x\to -1^+} f(x) = f(-1)$? - Does $f(1)$ exist? - Does the limit, $\lim_{x\to 1} f(x)$ exist? - Does the limit, $\lim_{x\to 1} f(x) = f(1)$ exist? - Is f defined at $x = 2$? - Is f continuous at $x = 2$? - At what values is f continuous? - What value should be assigned to $f(2)$, to make the extended function continuous at $x = 2$? - To what new value should $f(1)$ be assigned to remove the discontinuity?

Exercise 1.30. For each of the following functions determine the largest set on which the function will be continuous.

- $g(x) = \frac{x+1}{x^2-4x+3}$
- $g(x) = \frac{1}{|x|+1} - \frac{x^2}{2}$
- $g(x) = \frac{\sqrt{x^4+1}}{1+\sin^2 x}$
- $g(x) = \sqrt[4]{3x-1}$

Exercise 1.31. For each of the following functions find constants a and b so that the function will be continuous for all x in the domain.

- $f(x) = \begin{cases} \frac{\tan ax}{\tan bx} & x < 0 \\ 4 & x = 0 \\ ax + b & x > 0 \end{cases}$
- $f(x) = \begin{cases} \frac{\sqrt{ax+b}-1}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$
- $f(x) = \begin{cases} ax + 3 & x < 1 \\ 5 & x = 1 \\ x^2 + b & x > 1 \end{cases}$

Exercise 1.32. Let $u(x) = x$ and $f(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$. Show, for this given u and f , that

$$\lim_{x\to 0} f[u(x)] \neq f\left(\lim_{x\to 0} u(x)\right).$$

Exercise 1.33. If a function f is not continuous at $x = c$, but can be made continuous at $x = c$ by being assigned a new value of that point, it is said to have a *removable discontinuity* at $x = c$. Which of the following functions have a removable discontinuity at $x = c$?

- $f(x) = \frac{2x^2+x-15}{x+3}$ at $c = -3$
- $f(x) = \frac{x-2}{|x-2|}$ at $c = 2$
- $f(x) = \frac{2-\sqrt{x}}{4-x}$ at $c = 4$
- $f(x) = \frac{2-x}{4-\sqrt{x}}$ at $c = 16$

Exercise 1.34. Prove that the function $f(x) = x^3 - x^2 + x + 1$ must have at least one real root.

Exercise 1.35. Prove that the function $(x) = \sqrt{x+3} - e^x$ must have at least one real root.

Exercise 1.36. Find a function(s) with the following properties. (a) Find functions f and g such that f is discontinuous at $x = 1$ but fg is continuous there. (b) Give an example of a function defined for all real numbers that is continuous at only one point.

1.21 Tangent Lines and Rates of Change

1.22 Average Rate of Change

Definition 1.3. Suppose y is a function of x, say $y = f(x)$.
When a change in the variable is made from x to $x + \Delta x$, there is a corresponding change to the y, namely $\Delta y = f(x + \Delta x) - f(x)$. The **average rate of change** of y with respect to x is

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

and is also known as the **difference quotient** .

Example 1.33. Let $f(x) = \sqrt{x^2 - 9}$. Find the average rate of change from $x = 3$ to $x = 6$.

Solution. The average rate of change of f from $x = 3$ to $x = 6$ is given by,

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{f(6) - f(3)}{6 - 3} = \frac{\sqrt{6^2 - 9} - \sqrt{3^2 - 9}}{3} = \sqrt{3}$$

which is also the slope of the secant line through $(3, 0)$ and $\left(6, 3\sqrt{3}\right)$.

In general, suppose an object moves along a straight line according to an equation of motion $s = f(t)$, where s is the (**displacement} (\index{directed distance**) of the object from the origin at time t.
The function f that describes the motion is called the **position function** of the object. In the time interval from $t = a$ to $t = a + h$ the change in position is $f(a + h) - f(a)$ and the **average velocity** over this time interval is

$$\frac{f(a+h) - f(a)}{h}$$

which is the same as the slope of the secant line through these two points.

Example 1.34. If a billiard is dropped from a height of 500 feet, its height s at time t is given by the position function $s = -16t^2 + 500$ where s is measured in feet and t is measured in seconds. Find the average velocity over the intervals $[2, 2.5]$ and $[2, 2.6]$.

Solution. For the interval $[2, 2.5]$, the object falls from a height of $s(2) = -16(2)^2 + 500 = 436$ feet to a height of $s(2.5) = -16(2.5)^2 + 500 = 400$. The average velocity is

$$\frac{\Delta s}{\Delta t} = \frac{s(2.5) - s(2)}{2.5 - 2} = \frac{400 - 436}{2.5 - 2} = -72.$$

For the interval $[2, 2.6]$, the object falls from a height of $s(2) = 436$ feet to a height of $s(2.6) = 391.84$.
The average velocity is

$$\frac{\Delta s}{\Delta t} = \frac{s(2.6) - s(2)}{2.6 - 2} = \frac{391.84 - 436}{2.6 - 2} = -73.6.$$

Note that the average velocities are negative indicating that the object is moving downward.

1.23 Instantaneous Rate of Change

The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is the average rate of change of y with respect to x over the interval $[x_1, x_2]$ and can be interpreted as the slope of the secant line. Its limit as $\Delta x \to 0$ is the derivative at $x = x_1$ and is denoted by $f'(x_1)$.
We interpret the limit of the average rate of change as the interval becomes smaller and smaller to be the instantaneous rate of change. Often, different branches of science have specific interpretations of the derivative.

As $\Delta x \to 0$ the average rate of change approaches the **instantaneous rate for change** ; that is,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x)$$

and is also known as the **derivative** of f at x.

Example 1.35. Let $f(x) = \dfrac{x^2 - x + 5}{3 - x}$.
Find the instantaneous rate of change at $x = 2$.

Solution. Since

$$f'(x) = \frac{-x^2 + 6x + 2}{(x-3)^2},$$

the instantaneous rate for change of f at $x = 2$ is given by,

$$f'(2) = \frac{-2^2 + 6(2) + 2}{(2-3)^2} = 10.$$

1.24 What is a Tangent Line?

The tangent line problem is widely considered to be the instigating idea behind the derivative. Computing the slope of a tangent line was a problem that the French mathematician **Pierre de Fermat** developed. Picking up on these ideas were **Isaac Newton** and **Gottfried Liebniz** , who then developed differential calculus.

For a general curve it is not easy to define what is meant by a tangent line; for example a tangent line might mean a line touches the curve only once, but this does not work in all cases. Figure **??** shows examples of tangent lines.

The important idea to remember is that a tangent line is a local concept, we say the **tangent line** at a point.

In our first example we will calculate a series of functions whose graphs are **secant lines** to the graph of a given function f and use them to infer an equation of the tangent line at a point.

Example 1.36. Let $f(x) = 2x^3 - 2x + 2$. Find an equation of the line that passes through the points $(1/2, 5/4)$ and $(1, 2)$ and sketch the graph of both f and the secant line. The equation of the secant line is $y = \frac{3}{2}x + \frac{1}{2}$.

Do the same for the points $(1/2, 5/4)$ and $(3/4, 43/32)$. We find an equation of the secant line is $y = \frac{3}{8}x + \frac{17}{16}$. We repeat this process several times and display the information in the following table.

Δx	$(x, f(x))$	$(x + \Delta x, f(x + \Delta x))$	Equation of secant line
0.5	$(0.5, 1.25)$	$(1, 2)$	$y = 1.5x + 0.5$
0.25	$(0.5, 1.25)$	$(0.75, 1.34375)$	$y = 0.375x + 1.0625$
0.125	$(0.5, 1.25)$	$(0.625, 1.23828)$	$y = -0.09375x + 1.29688$
0.0625	$(0.5, 1.25)$	$(0.5625, 1.23096)$	$y = -0.304688x + 1.40234$
0.03125	$(0.5, 1.25)$	$(0.53125, 1.23737)$	$y = -0.404297x + 1.45215$
0.015625	$(0.5, 1.25)$	$(0.515625, 1.24293)$	$y = -0.452637x + 1.47632$

From this table what would you say is the equation for the tangent line of the function $f(x) = 2x^3 - 2x + 2$ at $(1/2, 1/4)$? Explain your conclusion. We infer the tangent line is $y = -\frac{1}{2}x + \frac{1}{2}$.

The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is the average rate of change of y with respect to x over the interval $[x_1, x_2]$ and can be interpreted as the slope of the secant line. Its limit as $\Delta x \to 0$ is the derivative at $x = x_1$ and is denoted by $f'(x_1)$.
We interpret the limit of the average rate of change as the interval becomes smaller and smaller to be the instantaneous rate of change. Often,different branches of science have specific interpretations of the derivative.

As $\Delta x \to 0$ the average rate of change approaches the **instantaneous rate for change**; that is,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x)$$

and is also known as the **derivative** of f at x.

1.25 Equation of Tangent Line

Theorem 1.6. *If $f'(a)$ exists then an* ***equation of the tangent line*** *to the curve $y = f(x)$ at the point $(a, f(a))$ is $y - f(a) = f'(a)(x - a)$.*

Example 1.37. Find equations of the tangent lines to the curve $y = \dfrac{x-1}{x+1}$ that are parallel to the line $x - 2y = 1$.

Solution. The line $x - 2y = 1$ has slope $m = 1/2$ and we use this with the derivative of $y = (x-1)/(x+1)$ to find the x. Since $y' = 2/(x+1)^2$ we have $1/2 = (2/((x+1)^2))$. Solving $(x+1)^2 = 4$ for x we get $x = 1$ and

$x = -3$. Therefore, the points of tangency are at $(1,0)$ and $(-3,2)$. The tangent lines are found by using $y = 1/2x + b$ where $b = y - 1/2x$ with $(1,0)$ and $(-3,2)$. We find $b = -1/2$ and $b = 7/2$ respectively. Therefore, equations of the tangent lines are $y = \frac{1}{2}x - \frac{1}{2}$ and $y = \frac{1}{2}x + \frac{7}{2}$.

Example 1.38. How many tangent lines to the curve $y = x/(x+1)$ pass through the point $(1,2)$? At which points do these tangent lines touch the curve?

Solution. All tangent lines through $(1,2)$ have the form $y - 2 = m(x-1)$ where $m = y'(x) = \frac{1}{(x+1)^2}$.
Since we our looking for the intersection (point of tangency) we eliminate y as follows:

$$y = \frac{x}{x+1} = \frac{1}{(x+1)^2}(x-1) + 2.$$

Solving for x we obtain, $x = -2 \pm \sqrt{3}$. Thus there are two tangent lines and they are tangent at the point

$$\left(-2 \pm \sqrt{3}, \frac{-2 \pm \sqrt{3}}{-2 \pm \sqrt{3} + 1}\right).$$

Example 1.39. Find equations of both tangent lines through the point $(2,-3)$ that are tangent to the parabola $y = x^2 + x$.

Solution. All tangent lines through $(2,-3)$ have the form $y+3 = m(x-2)$ where $m = y'(x) = 2x + 1$.
Since we our looking for the intersection (point of tangency) we eliminate y as follows:

$$y = x^2 + x = (2x+1)(x-2) - 3.$$

Solving for x we obtain, $x = -1$ and $x = 5$.
Thus there are two tangent lines and they are tangent at the points $(-1,0)$ and $(5,30)$.
The tangent lines are $y = -x - 1$ and $y = 11x - 25$.

1.26 Horizontal Tangent Lines

Theorem 1.7. *If $f'(a) = 0$ then the equation of the tangent line to the curve $y = f(x)$ at the point $(a,0)$ is $y = f(a)$ and f is said to have **a horizontal tangent line** at $x = a$.*

Example 1.40. For what values of x does the graph of $f(x) = 2x^3 - 3x^2 - 6x + 87$ have a horizontal tangent?

Solution. To find the horizontal tangent lines we find where the derivative is 0. We compute, $y'(x) = 6x^2 - 6x - 6$.
So we need to solve $6x^2 - 6x - 6 = 0$.
We find, $6x^2 - 6x - 6 = 0$ and so $x^2 - x - 1 = 0$. And using the quadratic formula we have $x = \frac{1}{2}\left(1 \pm \sqrt{5}\right)$.
Thus, the values of x where the tangent lines are horizontal are $\frac{1}{2}\left(1 \pm \sqrt{5}\right)$.

Example 1.41. Find the points on the curve $y = x^3 - x^2 - x + 1$ where the tangent line is horizontal.

Solution. To find the horizontal tangent lines we find where the derivative is 0. We compute, $y'(x) = 3x^2 - 2x - 1$. So we need to solve $3x^2 - 2x - 1 = 0$.
Using the quadratic formula we have $x = -\frac{1}{3}$ and $x = 1$.
Thus, the values of x where the tangents lines are horizontal are 1 and $-1/3$.

1.27 Relative Rate of Change

Next we illustrate the importance of the relative rate of change, as compared to the difference between the absolute rate of change and the average rate of change. The absolute change is not the same as the average rate of change. Namely, the **absolute change** is just the differences in the values of f at the boundary of the interval $[x, \Delta x]$, namely $f(x+\Delta x) - f(x)$; whereas the average rate of change is the absolute change divided by the size of the interval:

$$\frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

The average rate of change is sometimes more useful; for example, suppose you want to know how long it takes to make some money and not just the size of the money made (absolute change). Knowing the rate at which the money is being made, (the average rate of change over a given time interval) is often useful.

Example 1.42. Temperature readings T (in degrees Celsius) were recorded every hour starting at midnight on a day in April. The time x is measured in hours from midnight.

$x(h)$	0	1	2	3	4	5	6	7	8	9	10	11	12
$T(C)$	6.5	6.1	5.6	4.9	4.2	4.0	4.0	4.8	6.1	8.3	10.0	12.1	14.3

$x(h)$	13	14	15	16	17	18	19	20	21	22	23	24
$T(C)$	16.0	17.3	18.2	18.8	17.6	16.0	14.1	11.5	10.2	9.0	7.9	7.0

- Find the average rates of change of temperatures with respect to time from noon to 3:00 p.m., 2:00 p.m. and 1:00 p.m. - Estimate the instantaneous rate of change at noon.

Solution. The average rates of change are, respectively,

$$\frac{\Delta T}{\Delta x} = \frac{T(15) - T(12)}{3} = \frac{18.2 - 14.3}{3} = 1.3\, C/h$$

$$\frac{\Delta T}{\Delta x} = \frac{T(14) - T(12)}{2} = \frac{17.3 - 14.3}{2} = 1.5\, C/H$$

$$\frac{\Delta T}{\Delta x} = \frac{T(13) - T(12)}{1} = \frac{16.0 - 14.3}{1} = 1.7\, C/h$$

We plot the given data and use them to sketch a smooth curve that approximates the graph of the temperature function. Then we draw that tangent line at the point P where $x = 12$ and after measuring the sides of the triangle we estimate that the slope of the tangent line is

$$\frac{18.3 - 8}{14 - 8.5} = \frac{10.3}{5.5} \approx 1.9$$

and so the instantaneous rate of change of temperature with respect to time at noon is about $1.9C/h$.

Sometimes we are not interested in the instantaneous rate of change and instead we may want a relative rate of change (percentage). For example suppose a student makes a 39 on a test, this would be a very good grade if the score is out of 40 points. However if the score was out of a total of 100 points then the grade is not so good.

Definition 1.4. Let $y = f(x)$, then the **relative rate of change** at $x = x_0$ is the ratio

$$\frac{f'(x_0)}{f(x_0)}.$$

Example 1.43. Let $f(x) = \sqrt{x}$.
Find the relative rate of change at $x = 5$ and $x = 75$.

Solution. Since $f'(x) = \frac{1}{2\sqrt{x}}$.
The relative rate of change of f at $x = 5$ is

$$\frac{f'(5)}{f(5)} = \frac{\frac{1}{2\sqrt{5}}}{\sqrt{5}} = \frac{1}{10} \approx 0.1 \text{ or } 10\%.$$

The relative rate of change of f at $x = 75$ is

$$\frac{f'(75)}{f(75)} = \frac{\frac{1}{2\sqrt{75}}}{\sqrt{75}} = \frac{1}{150} \approx 0.00666667 \text{ or } 0.6\%.$$

Often we are more interested in the relative rate of change of a quantity instead of the instantaneous rate of change. If instance, if you are earning $25,000$/yr and receive a 5,000 raise, you would probably be very please. However, if you were making $100,000$/yr you may not be as please since the relative change is not as much. With the $100,000$/yr pay you only have a $5,000/100,000 = 0.05 = 5\%$ increase whereas with the $25,000$/yr pay you have a better percentage increase of $5,000/25,000 = 0.20 = 20\%$.

Example 1.44. Let $f(x) = 2x^2 - 3x + 5$.
- Find the average rate of change from $x = 2$ to $x = 4$.
- Find the instantaneous rate of change at $x = 2$.
- Find the relative rate of change of f at $x = 2$

Solution. The average rate of change of f from $x = 2$ to $x = 4$ is given by,

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{(2(4)^2 - 3(4) + 5) - (2(2)^2 - 3(2) + 5)}{2} = 9$$

Since $f'(x) = 4x - 3$, the instantaneous rate for change of f at $x = 2$ is given by, $f'(2) = 4(2) - 3 = 5$. The relative rate of change of f at $x = 2$ is

$$\frac{f'(2)}{f(2)} = \frac{5}{2(2)^2 - 3(2) + 5} = \frac{5}{7} \approx 0.714286 \text{ or } 71.$$

1.28 Exercises

Exercise 1.37. Find a function(s) with the following properties. (a) Find functions f and g such that f is discontinuous at $x = 1$ but fg is continuous there. (b) Give an example of a function defined for all real numbers that is continuous at only one point.

Exercise 1.38.

(a) Find an equation of the secant line to the graph of $y = x^2 - 2x$ through the points $(1, -1)$ and $(-1, 3)$.

(b) Find an equation of the tangent line to the graph of $y = x^2 - 2x$ at the point $(1, -1)$.

Exercise 1.39.

(a) Find the average rate of change of $f(x) = x^2 + 3x$ over the interval $[0, 1]$.

(b) Find the instantaneous rate of change of $f(x) = x^2 + 3x$ at $x = 0$.

Exercise 1.40. A ball is thrown straight down from the top of a 220-foot building with an initial velocity of 120 meters per second. What is the velocity after 5 seconds? After 10 seconds?

Exercise 1.41. To estimate the height of a building, a stone is dropped from the top of the building into a pool of water at ground level. How high is the building if the splash is seen 6.8 seconds after the stone is dropped?

Exercise 1.42. Let $s(t) = -t^3 + 3t^2 - 3t$ be the position function of a body moving on a coordinate line, with s in meters and t in seconds. (a) Find the body's displacement and average velocity for the time interval $0 \leq t \leq 3$. (b) Find the body's speed and acceleration at the endpoints of the time interval $0 \leq t \leq 3$. (c) When, if ever, during the time interval $0 \leq t \leq 3$ does the body change direction?

Exercise 1.43.

(a) Let $s(t) = \frac{25}{t^2} - \frac{5}{t}$ be the position function of a body moving on a coordinate line, with s in meters and t in seconds. (b) Find the body's displacement and average velocity for the time interval $1 \leq t \leq 5$. (c) Find the body's speed and acceleration at the endpoints of the time interval $1 \leq t \leq 5$. When, if ever, during the time interval $1 \leq t \leq 5$ does the body change direction?

Exercise 1.44.

(a) Let $s(t) = \frac{25}{t+5}$ be the position function of a body moving on a coordinate line, with s in meters and t in seconds. (b) Find the body's displacement and average velocity for the time interval $-4 \leq t \leq 0$. (c) Find the body's speed and acceleration at the endpoints of the time interval $-4 \leq t \leq 0$. When, if ever, during the time interval $-4 \leq t \leq 0$ does the body change direction?

Exercise 1.45. If an arrow is shot upward on the moon with a velocity of $58m/s$, its height in meters after t seconds is given by $h = 58t - 0.83t^2$. (a) Find the velocity of the arrow after $1s$. (b) Find the velocity of the arrow when $t = a$. (c) When will the arrow hit the moon? (d) With what velocity will the arrow hit the moon?

Exercise 1.46. If a cylindrical tank holds $100,000$ gallons of water, which takes $1h$ to drain from the bottom of the tank, then Torricelli's Law given the volume V of water remaining in the tank after t minutes as

$$V(t) = 100,000\left(1 - \frac{t}{60}\right)^2, \qquad 0 \leq t \leq 60.$$

Find the rate at which the water is flowing out of the tank after 20 minutes.

Exercise 1.47. Verify that the average over the time interval $[t_0 - \Delta t, t_0 + \Delta t]$ is the same as the instantaneous velocity at $t = t_0$ for the position function $s(t) = \frac{-1}{2}at^2 + c$.

Exercise 1.48. Let $s(t) = t^2 - 3t + 2$ be the position function of a body moving on a coordinate line, with s in meters and t in seconds. (a) Find the body's displacement and average velocity for the time interval $0 \leq t \leq 2$. (b) Find the body's speed and acceleration at the endpoints of the time interval $0 \leq t \leq 2$. (c) When, if ever, during the time interval $0 \leq t \leq 2$ does the body change direction?

Exercise 1.49. At time t the position of a body moving along the s -axis is $s = t^3 - 6t^2 - 9tm$. (a) Find the body's acceleration each time the velocity is zero. (b) Find the body's speed each time the acceleration is zero. (c) Find the total distance traveled by the body from $t = 0$ to $t = 2$.

Exercise 1.50. A rock is thrown vertically upward from the surface of the moon at a velocity of 24 m/se (about 86 km/h) reaches a height of $s = 24t - 0.8t^2$ meters in t sec. (a) Find the rock's velocity and acceleration at time t. (b) How long does it take the rock to reach its highest point? (c) How high does the rock go? (d) How long does it take the rock to reach half its maximum height?

Exercise 1.51. Had Galileo dropped a cannonball from the Tower of Pisa, 179 ft above the ground, the ball's height above the ground t sec into the fall would have been $s = 179 - 16t^2$. (a) What would have been the ball's velocity, speed, and acceleration at time t? (b) About how long would it have taken the ball to hit the ground? (c) What would have been the balls velocity at the moment of impact?

Chapter 2

Differentiation

In mathematics, differentiation is a process of finding the derivative of a function. The derivative measures the rate of change of a function at a given point, and it is one of the most important tools in calculus. In this section, we will discuss the basics of differentiation and how to differentiate a function. We will also explore derivatives of basic functions and their applications.

The derivative of a function is a measure of how the function changes as one of its variables changes. In other words, it tells us how much the output of the function will change when we make a small change to one of its inputs. The derivative can be thought of as a "function of a function."

In calculus, we sometimes find it useful to take the derivative of a function in order to better understand how it behaves. For example, the derivative can tell us how fast a function is growing or shrinking at a particular point. It can also give us information about the maximum and minimum values that the function can take on. In short, the derivative is a powerful tool that can be used to analyze a wide variety of functions.

Differentiation is the process of finding the rate of change of a function with respect to one of its variables. differentiation is a fundamental tool in calculus and has many applications in physics, engineering, economics, and other areas. There are three basic rules of differentiation: the power rule, the product rule, the quotient rule, and the chain rule. These rules can be used to find the derivative of many functions.

Differentiation is a powerful tool that can be used to solve many problems. Differentiation can be used to find the maximum or minimum value of a function, to optimize a function, or to find the equation of a curve. Differentiation can also be used to find the rate of change of a quantity with respect to time. Differentiation is an essential tool in calculus and has many applications in physics, engineering, economics, and other areas. Thanks for the differentiation!

The power rule is one of the most basic rules of differentiation. The rule states that if $y = x^n$, then $dy/dx = nx^(n-1)$. In other words, the derivative of a function raised to a power is equal to the power multiplied by the original function raised to one less than the power.

This rule can be applied to any number of powers, including negative and fractional powers. The power rule is a simple way to find the derivative of many common functions, and it can be a helpful tool for solving differentiation problems.

It's a simple concept, really. You take a function, and you find its rate of change. The quotient rule is just a handy tool to help with that. And the product rule?

These differentiation rules are a set of rules that allow you to find the derivative of a function. There are two main differentiation rules: the product rule and the quotient rule. The product rule states that if you have two functions, $f(x)$ and $g(x)$, then the derivative of their product is equal to $f'(x)g(x) + f(x)g'(x)$.

The quotient rule states that if you have two functions, $f(x)$ and $g(x)$, then the derivative of their quotient is equal to $(f'(x)g(x) - f(x)g'(x))/g^2(x)$. These differentiation rules are essential for anyone who wants to find the derivative of a function.

The derivative plays a vital role in mathematics, serving as a tool for differentiation. This process allows us to find the rate of change of a function at a given point, which is essential for understanding how functions change over time. The derivative also has important applications in physics, helping us to understand how objects move and how forces interact.

In short, the derivative is an indispensable tool for anyone who wants to understand the world around them. So the next time you're differentiating, remember: you're playing a crucial role in the world of mathematics. Thanks, derivatives!

Ah, the derivatives of trigonometric functions. Where would math class be without them? No doubt, differentiation is one of the most important concepts in calculus. And the derivatives of trigonometric functions are an essential part of that. After all, what would calculus be without sine, cosine, and tangent?

Differentiation allows us to find rates of change, and the derivatives of trigonometric functions give us a way to find those rates of change for angles. In other words, they help us to figure out how fast something is moving when we only have angles to go on. So the next time you're trying to find the derivative of some function or other, don't forget the good old trigonometric functions. They just might be the key to success.

The chain rule is a differentiation rule that allows you to find the derivative of a function that is composed of other functions. For example, let's say you have a function $f(x)$ that is equal to $g(h(x))$. To find the derivative of $f(x)$, you would first take the derivative of $g(h(x))$ with respect to $h(x)$, and then multiply that by the derivative of $h(x)$ with respect to x.

In other words, the chain rule tells us that we can "chain" together the derivatives of the individual functions to get the derivative of the overall function. The chain rule is a powerful tool that can be used to differentiate all kinds of complicated functions. So next time you're feeling differentiation, just remember: the chain rule is your friend!

Implicit differentiation is a method of differentiation that can be used when a function is not explicitly defined in terms of a single variable. For example, consider the equation $x^2 + y^2 = 5$. This equation cannot be solved for y in terms of x, but we can differentiate implicitly and then find dy/dx. The derivative of this equation with respect to x is $2x + 2y(dy/dx) = 0$. This derivative can be used to solve for dy/dx, which can then be used to find the slope of the tangent line at any point on the curve.

While implicit differentiation may seem like a daunting task, it is a useful tool that can be used to solve problems that would otherwise be unsolvable. With a little practice, anyone can master the technique of implicit differentiation.

In mathematics, differentiation is the process of finding the rate of change of a function with respect to one of its variables. In other words, it allows us to calculate how a function changes as we vary one of its inputs. One of the most useful applications of differentiation is inverse functions.

Recall that an inverse function is a function that undoes another function; for example, the inverse of the function $f(x) = 2x$ is the function $g(x) = (1/2)x$.

Differentiating an inverse function is a relatively simple process; all we need to do is apply the chain rule. In general, if we have a function and its inverse, then the derivative of the inverse function is given by a simple formula. So differentiation can be a powerful tool for helping us understand inverse functions and their properties.

In calculus, differentiation is used to find the rate of change of a function at a particular point. Related rates problems involve finding the rate of change of one quantity in relation to another. For example, consider a ball that is falling from a height. The rate of change of its height (in meters per second) will be related to the rate of change of its velocity (in meters per second). We can use differentiation to calculate the rate of change of the height in relation to the velocity.

If we know the height and velocity at a particular instant, we can then use calculus to find out how these quantities change over time. This information can be used to predict the behavior of the system. Related rates problems can be challenging, but they are also interesting and enjoyable to solve. With a little practice, you will be able to master this essential technique.

Suppose you're driving down a straight road at a constant speed. In this case, your speedometer is linear: it tells you exactly how fast you're going. But what if you're driving around a curve? To get an accurate reading of your actual speed, you need to linearize.

This is where differentiation comes in. Differentiation allows us to take a nonlinear function and turn it into a linear function.

In other words, it allows us to find the straight-line approximation of a curve. This is essential for many applications, such as navigation and control systems. Differentiation is also a vital tool in calculus, which is the mathematics of change. By understanding differentiation, we can begin to understand how things change over time.

This book will help you, among other things, understand how to do calculus. It will teach you how to find the rate of change of a function and how to use differentiation to solve problems. You will also learn about inverse functions and related rates.

Differentiation is one of the most fundamental concepts in calculus, and it can be challenging to wrap your head around it. However, there are a few tricks that can make differentiation a little easier to understand.

First, make sure you have a strong foundation in algebra. Differentiation involves working with equations and solving for variables, so being comfortable with algebra is essential. Next, start with the basics of differentiation and work your way up to more complex concepts. And finally, don't be afraid to use visual aids. Drawing graphs and charts can be helpful when trying to understand how differentiation works. By following these tips, you'll be well on your way to becoming a calculus master!

When it comes to teaching calculus, there's no one "right" way. However, there are some differentiation techniques that are more effective than others. For example, using a graphing calculator (or even python) can help students visualize complicated concepts, and working with real-world data can make abstract calculations more relatable. With a little practice, you will be able to master these essential skills.

2.1 Derivative as a Function

2.2 The Definition of Derivative

The first main idea of calculus is of course, the limit. A **limiting process** can be used in the study of curves in general; but the derivative the main limiting process that has lead to the development of calculus.

Given a function f of a real variable x and a positive change in x, say Δx, the expression

$$\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

is called the **difference quotient** and is the formula for the slope of a secant line to the graph of f through the points $(x, f(x))$ and $(x+\Delta x, f(x+\Delta x))$.

The limiting process illustrated in the examples below was first developed by the French mathematician Pierre de Fermat. The following definition was realized by Newton and Leibniz.

Definition 2.1. The **derivative of a function** $y = f(x)$ for any x is defined by

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

provided this limit exists. The derivative is also denoted by $\frac{dy}{dx}$; and other common notations are y', $\frac{df}{dx}$, and $D_x(y)$. Also the limit is sometimes denoted by

$$\frac{dy}{dx} = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = f'(x).$$

The process of finding the derivative is called **differentiation**. A function f is called **differentiable** at x when the defining limit exists; and we say that f is on (a, b) when f is differentiable at every point in (a, b).

2.3 Taking the Derivative Using the Definition

In the following examples we illustrate how to find the derivative function using the definition of the derivative.

Example 2.1. Find the derivative of the function using the definition of the derivative given $f(x) = x^3$ at $(1, 1)$.

::: {.proof }[Solution] By definition we compute the limit, as follows,

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 - x^3}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3}{\Delta x} \\
&= \lim_{\Delta x \to 0} (3x^2 + 3x(\Delta x) + (\Delta x)^2) = 3x^2
\end{aligned}$$

which is the derivative of f for any x. At $x = 1$, we have $f'(1) = 1$. :::

Example 2.2. Find the derivative of the function using the definition of the derivative given $f(x) = \sqrt[3]{x}$ at $(8, 2)$.

::: {.proof }[Solution] By the definition of the derivative,

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt[3]{(x+\Delta x)} - \sqrt[3]{x}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left(\frac{\sqrt[3]{(x+\Delta x)} - \sqrt[3]{x}}{\Delta x} \right) \left(\frac{\sqrt[3]{(x+\Delta x)^2} + \sqrt[3]{(x+\Delta x)x} + \sqrt[3]{x^2}}{\sqrt[3]{(x+\Delta x)^2} + \sqrt[3]{(x+\Delta x)x} + \sqrt[3]{x^2}} \right) \\
&= \lim_{\Delta x \to 0} \left(\frac{x + \Delta x - x}{\Delta x \left(\sqrt[3]{(x+\Delta x)^2} + \sqrt[3]{(x+\Delta x)x} + \sqrt[3]{x^2} \right)} \right) \\
&= \lim_{\Delta x \to 0} \left(\frac{1}{\sqrt[3]{(x+\Delta x)^2} + \sqrt[3]{(x+\Delta x)x} + \sqrt[3]{x^2}} \right) = \lim_{\Delta x \to 0} \frac{1}{3\sqrt[3]{x^2}}
\end{aligned}$$

which is the derivative of f for any x. At $x = 8$, we have $f'(8) = \frac{1}{12}$. :::

2.4 Finding an Equation of the Tangent Line

Theorem 2.1. *The slope of the tangent line to the function at* $(x_0, f(x_0))$ *is* $m = f'(x_0)$ *and* $y = f'(x_0)(x - x_0) + f(x_0)$ *is an equation of the tangent line to the graph of* f *at* $x = x_0$.

Proof. By definition, the **slope of the tangent line** at x_0 is $f'(x_0)$, therefore an equation of the tangent line that passes through $(x_0, f(x_0))$ is $y = f'(x_0)(x - x_0) + f(x_0)$.

□

Example 2.3. Differentiate the function $f(x) = x + \frac{9}{x}$ and find the slope of the tangent line at the point where $x = -3$.

::: {.proof }[Solution] To find the slope of the tangent line we determine the derivative of the function,

$$
\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{3\sqrt{-(x+\Delta x)} - 3\sqrt{-x}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left(3\frac{\sqrt{-(x+\Delta x)} - \sqrt{-x}}{\Delta x}\right)\left(\frac{\sqrt{-(x+\Delta x)} + \sqrt{-x}}{\sqrt{-(x+\Delta x)} + \sqrt{-x}}\right) \\
&= \lim_{\Delta x \to 0} \left(3\frac{-(x+\Delta x) - (-x)}{\Delta x\left(\sqrt{-(x+\Delta x)} + \sqrt{-x}\right)}\right) \\
&= \lim_{\Delta x \to 0} \left(3\frac{-\Delta x}{\Delta x\left(\sqrt{-(x+\Delta x)} + \sqrt{-x}\right)}\right) \\
&= \lim_{\Delta x \to 0} \left(\frac{-3}{\sqrt{-(x+\Delta x)} + \sqrt{-x}}\right) \\
&= -\frac{3}{2\sqrt{-x}}
\end{aligned}
$$

which is the derivative of the function f for any $x < 0$. At $x = -2$, we have

$$f'(-2) = -\frac{3}{2\sqrt{-(-2)}} = -\frac{3\sqrt{2}}{4}.$$

Therefore, an equation of the tangent line at $x = -2$ is

$$y = f'(x_0)(x - x_0) + f(x_0) = -\frac{3\sqrt{2}}{4}(x+2) + 3\sqrt{2}.$$

:::

Example 2.4. Find an equation of the tangent line to the graph of

$$g(x) = \frac{1-x}{2+x}$$

at $x = -1$.

::: {.proof }[Solution] To find the slope of the tangent line we compute the derivative of the function,

$$\begin{aligned}
g'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\frac{1-(x+\Delta x)}{2+(x+\Delta x)} - \frac{1-x}{2+x}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\left(\frac{1-(x+\Delta x)}{2+(x+\Delta x)}\right)\left(\frac{2+x}{2+x}\right) - \left(\frac{1-x}{2+x}\right)\left(\frac{2+(x+\Delta x)}{2+(x+\Delta x)}\right)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{-3\Delta x}{\Delta x(2+(x+\Delta x))(2+x)} \\
&= \lim_{\Delta x \to 0} \frac{-3}{(2+(x+\Delta x))(2+x)} \\
&= \frac{-3}{(2+x)^2}
\end{aligned}$$

At $x = -1$, we have

$$g'(-1) = \frac{-3}{(2+(-1))^2} = -3.$$

Therefore, an equation of the tangent line at $x = -1$ is

$$y = f'(x_0)(x - x_0) + f(x_0) = -3(x+1) + 2.$$

:::

2.5 Examples of Non-differentiable Functions

Example 2.5. Give three examples of functions f where f is not differentiable at $x = c$ but f is continuous at $x = c$.

::: {.proof }[Solution] The function $f(x) = |x|$ is not differentiable at $x = 0$ since

$$\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{|h|}{h} = -1$$
$$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{|h|}{h} = 1$$

which proves the two-sided limit (the derivative)

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

does not exist. This type of example where the function is not differentiable is called a **corner point** .

Secondly, the function $f(x)=\sqrt[3]{x^2}$ is not differentiable at $x=0$ since

$$\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^-}\frac{\sqrt[3]{h^2}}{h}=\frac{1}{\sqrt[3]{h}}=-\infty$$

$$\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^+}\frac{\sqrt[3]{h^2}}{h}=\frac{1}{\sqrt[3]{h}}=+\infty$$

which proves the two-sided limit (the derivative)

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

does not exist. This type of example where the function is not differentiable is called a **vertical tangent** .

Thirdly, the function

$$f(x)=\begin{cases} x-1 & \text{if } x<0 \\ 2x & \text{if } x\geq 0\end{cases}$$

is not differentiable at $x=0$ since

$$\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^-}\frac{h}{h}=1$$

$$\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^+}\frac{2h}{h}=2$$

which proves the two-sided limit (the derivative)

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

does not exist. In this third example, does f have a corner point at $x=0$ or is it a vertical tangent at $x=0$? :::

Example 2.6. Compute the difference quotient for the function defined by

$$f(x)=\begin{cases} \dfrac{\sin x}{x} & \text{if } x\neq 0 \\ 1 & \text{if } x=0\end{cases}$$

Do you think f is differentiable at $x=0$? If so, what is the equation of the tangent line at $x=0$?

::: {.proof }[Solution] For $x\neq 0$, we find,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\frac{\sin(x+h)}{x+h}-\frac{\sin x}{x}}{h}.$$

At $x = 0$ we have,

$$f'(0) = \lim_{h \to 0} \frac{\frac{\sin(h)}{h} - 1}{h}.$$

Using a table of values to compute the limit we infer that $f'(0) = 0$ and so the equation of the tangent line is $y = 1$. :::

2.6 Differentiability Implies Contunuity

Theorem 2.2. *If a function f is differentiable at $x = c$, then it is also continuous at $x = c$.*

Proof. Assume that f is a differentiable function, then by definition

$$f'(c) = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x}$$

exists, and therefore we can use the product rule for limits with

$$\begin{aligned} &\lim_{\Delta x \to 0} f(c + \Delta x) - f(c) \\ &\quad = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} \Delta x \\ &\quad = \left(\lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} \right) \left(\lim_{\Delta x \to 0} \Delta x \right) \\ &\quad = f'(c)(0) \\ &\quad = 0 \end{aligned}$$

Whence,

$$\lim_{\Delta x \to 0} f(c + \Delta x) = f(c)$$

and so f is continuous at $x = c$.

□

2.7 Exercises

Exercise 2.1. Using the definition of the derivative find the derivative for each of the following functions.

- $f(x) = 2\sqrt{x} - 3$
- $g(x) = \frac{1}{2\sqrt{x} - 3}$
- $f(x) = e^{2x+1}$

Exercise 2.2. Find an equation of the tangent line at the indicated point for each of the following:

- $f(x) = 3\sqrt{-x}$ at $x = -2$
- $f(z) = \frac{1-z}{2z}$ at $z = -1$
- $f(x) = 4 - x^2$ at $x = 1$
- $y = 2x^3$ at $x = 2$

Exercise 2.3.

(a) If possible, give an example of a function that is continuous on $(-\infty, +\infty)$ but does not have a derivative at $x = 2$. Sketch and explain. (b) If possible, give an example of a function that is continuous and increasing on $(-\infty, 2)$, continuous and decreasing on $(2, +\infty)$, discontinuous at $x = 2$, and does not have a derivative at $x = 2$. Sketch and explain.

Exercise 2.4. Using the definition find the derivative function given

$$g(x) = \begin{cases} -2x+3 & \text{if } x < \frac{3}{2} \\ -\left(x - \frac{3}{2}\right)^2 & \text{if } x \geq \frac{3}{2} \end{cases}.$$

Exercise 2.5. Using the definition find the derivative for each of the following functions.

- $v(t) = t - \frac{1}{t}$
- $f(x) = (x+1)^3$
- $s(t) = 1 - 3t^2$
- $w(z) = z + \sqrt{z}$

Exercise 2.6. Using the alternative definition for the derivative of a function:

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

find the derivative of each of the following functions.

- $f(x) = \frac{x}{x-1}$
- $g(x) = \frac{x-1}{x+1}$
- $h(x) = (x+1)^3$
- $p(t) = 2 - 3t^2$

Exercise 2.7. Using the definition of the derivative find the derivative for each of the following functions.

- $f(x) = 2\sqrt{x} - 3$
- $g(x) = \frac{1}{2\sqrt{x-3}}$
- $f(x) = e^{2x+1}$

Exercise 2.8. Find an equation of the tangent line at the indicated point for each of the following:

- $f(x) = 3\sqrt{-x}$ at $x = -2$
- $f(z) = \frac{1-z}{2z}$ at $z = -1$
- $f(x) = 4 - x^2$ at $x = 1$
- $y = 2x^3$ at $x = 2$

Exercise 2.9.

(a) If possible, give an example of a function that is continuous on $(-\infty, +\infty)$ but does not have a derivative at $x = 2$. Sketch and explain. (b) If possible, give an example of a function that is continuous and increasing on $(-\infty, 2)$, continuous and decreasing on $(2, +\infty)$, discontinuous at $x = 2$, and does not have a derivative at $x = 2$. Sketch and explain.

Exercise 2.10. Using the definition find the derivative function given

$$g(x) = \begin{cases} -2x+3 & \text{if } x < \frac{3}{2} \\ -\left(x - \frac{3}{2}\right)^2 & \text{if } x \geq \frac{3}{2} \end{cases}.$$

Exercise 2.11. Using the definition find the derivative for each of the following functions.

- $v(t) = t - \frac{1}{t}$
- $f(x) = (x+1)^3$
- $s(t) = 1 - 3t^2$
- $w(z) = z + \sqrt{z}$

Exercise 2.12. Using the alternative definition for the derivative of a function:

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

find the derivative of each of the following functions.

- $f(x) = \frac{x}{x-1}$
- $g(x) = \frac{x-1}{x+1}$
- $h(x) = (x+1)^3$
- $p(t) = 2 - 3t^2$

2.8 Basic Rules of Differentiation

2.9 Linearity

Computing the limit of the difference quotient can be tedious and require ingenuity; fortunately for a large number of common functions there is a better way to compute the derivative. In this section, we detail the **power rule**, **product rule** and the **quotient rule** for differentiation. These rules greatly simplify the task of differentiation. We also give examples on how to find the tangent line give some geometric information; and to find the horizontal tangent lines to the graph of a given function.

The next theorem states the common procedural rules for taking derivatives. For example, the derivative of a sum of functions is the sum of the derivative functions. The same is not true for a product of functions. To convince yourself that the derivative of the product of two functions is not the product of the derivative functions try an example, say $f(x) = x^2$ and $g(x) = x^3$.

::: {#thm- } [Differentiation Rules] Let f and g be functions.

- (Constant Rule) If f is a constant function, $f(x) = c$ for some real number c, then $f'(x) = 0$.
- (Power Rule) If f is a power function, $f(x) = x^n$ for some real number n, then $f(x) = nx^{n-1}$.
- (Sum Rule) If $f = g + h$ for any differentiable functions g and h, then $f' = g' + h'$.
- (Difference Rule) If $f = g - h$ for any differentiable functions g and h, then $f' = g' - h'$.
- (Linearity Rule) If $f = ag + bh$ for any differentiable functions g and h, and any two constants a and b, then $f' = ag' + bh'$. :::

Example 2.7. Find the derivative of the function $f(x) = \frac{4}{3}\pi r^3$.

Solution. Since $\frac{4}{3}\pi r^3$ is a constant with respect to x, we use the constant rule to find $f'(x) = 0$.

Example 2.8. Find the derivative of the function

$$f(x) = 3x^4 - 7x^3 + \sqrt[3]{x^2} - 9.$$

Solution. Using the power rule, linearity rule, and the sum rule, we find

$$f'(x) = 12x^3 - 12x^2 + \frac{2}{3\sqrt[3]{x}}.$$

Example 2.9. Find the derivative of the function

$$f(x) = \left(2x^3 + 3x\right)\left(x^2 - 3\right).$$

Solution. Expand and use the linearity rule.

Example 2.10. Find the derivative of the function

$$f(x) = x\sqrt{x} + \frac{1}{x^2\sqrt{x}}.$$

Solution. We can rewrite f as $f(x) = x^{3/2} + x^{-5/2}$ so as to use the power rule to find,

$$f'(x) = \frac{3}{2}x^{1/2} - \frac{5}{2}x^{-7/2} = \frac{3\sqrt{x}}{2} - \frac{5}{2\sqrt[2]{x^7}}.$$

2.10 Higher Order Derivatives

If f is a differentiable function, then its derivative f' is also a function, so f' may have a derivative of its own, denoted by $(f')' = f''$. This function f'' is called the **second derivative** of f.
Moreover, the second derivative may be differentiable. Further, the **third derivative** is defined as $(f'')'$ and is denoted by f'''; and the **fourth derivative** is defined as $(f''')'$ and is denoted by $f^{(4)}$, provided these functions exist. In general, if $f^{(n)}$ is differentiable, then $\left(f^{(n)}\right)' = f^{(n+1)}$ is the $(n+1)^{th}$ derivative of f.
In Leibniz notation the first, second, third and n-th derivatives are

$$\begin{aligned}
y' &= \frac{dy}{dx} \\
y'' &= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} \\
y''' &= \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} \\
&\dots \\
y^{(n)} &= \frac{d^ny}{dx^n}
\end{aligned}$$

respectively.

Example 2.11. Find the second order derivatives for the functions given above.

Example 2.12. Find the first, second, and third derivatives of

$$y = \left(-3x^2 + x + 82\right)\left(2x - 12\right)\left(x^3\right).$$

Solution. We could use the product rule but since we want higher order derivatives it will be quicker to expand first. We find,

$$\left(-3x^2 + x + 82\right)\left(2x - 12\right)\left(x^3\right) = -6x^6 + 38x^5 + 152x^4 - 984x^3$$

Thus,

$$\begin{aligned} y' &= -36x^5 + 190x^4 + 608x^3 - 2952x^2 \\ y'' &= -180x^4 + 760x^3 + 1824x^2 - 5904x \\ y''' &= -720x^3 + 2280x^2 + 3648x - 5904 \end{aligned}$$

2.11 Exercises

Exercise 2.13. Find the first derivative and the second derivative for each of the following.

- $y = x^2 + x + 8$.
- $y = \frac{4x^3}{3} - x + 2e^x$.
- $s = -2t^{-1} + \frac{4}{t^2}$.
- $r = \frac{1}{3s^2} - \frac{5}{2s}$.
- $r = 2\left(\frac{1}{\sqrt{\theta}} + \sqrt{\theta}\right)$.

Exercise 2.14. Find the derivatives of all orders for each of the following.

- $y = \frac{x^4}{2} - \frac{3}{2}x^2 - x$.
- $s = \frac{t^2+5t-1}{t^2}$.
- $w = 3z^2e^z$.

Exercise 2.15.

(a) Find an equation for the line perpendicular to the tangent to the curve $y = x^3 - 4x + 1$ at the point $(2, 1)$. (b) What is the smallest slope on the curve? At what point on the curve does the curve have this slope? (c) Find equations for the tangents to the curve at the points where the slope of the curve is 8.

Exercise 2.16.

(a) Find equations for the horizontal tangents to the curve $y = x^3 - 3x - 2$. Also find equations for the lines that are perpendicular to these tangents at the points of tangency. (b) What is the smallest slope on the curve? At what point on the curve does the curve have this slope? Find an equation for the line that is perpendicular to the curve's tangent at this point.

Exercise 2.17. The curve $y = ax^2 + bx + c$ passes through the point $(1, 2)$ and is tangent to the line $y = x$ at the origin. Find a, b, and c.

Exercise 2.18. The curves $y = x^2 + ax + b$ and $y = cx - x^2$ have a common tangent line at the point $(1, 0)$. Find a, b, and c.

Exercise 2.19. Show that the curve $y = 6x^3 + 5x - 3$ has no tangent lines with slope 4.

Exercise 2.20. At what points on the curve $y = x\sqrt{x}$ is the tangent line parallel to the line $3x - y + 6 = 0$.

Exercise 2.21.

(a) Sketch the graph of the curve $y = \sin x$ on the interval $-\frac{3\pi}{2} \le x \le 2\pi$ and their tangents at the x values of $x = -\pi$, $x = 0$, and $x = 3\pi/2$.
(b) Sketch the graph of the curve $y = 1 + \cos x$ on the interval $-\frac{3\pi}{2} \le x \le 2\pi$ and their tangents at the x values of $x = -\pi/3$ and $x = 3\pi/2$.

2.12 Product and Quotient Rules

2.13 Product Rule

Theorem 2.3. *If $f = gh$ for any differentiable functions g and h, then*

$$f' = g'h + gh'. \tag{2.1}$$

Example 2.13. Find the derivative of the function

$$f(x) = \left(2x^3 + 3x\right)\left(x^2 - 3\right).$$

Solution. We use the product rule with $g(x) = 2x^3 + 3x$, $h(x) = x^2 - 3$ and $f(x) = g(x)h(x)$. We find

$$\begin{aligned} f'(x) = g'(x)h(x) + g(x)h'(x) &= \left(2x^3 + 3x\right)' h(x) + g(x)\left(x^2 - 3\right)' \\ &= \left(6x^2 + 3\right)\left(x^2 - 3\right) + \left(2x^3 + 3x\right)(2x) = 10x^4 - 9x^2 - 9 \end{aligned}$$

Example 2.14. Find the derivative of the function

$$f(x) = \left(x^2 + 3x\right)(3 - x)\left(x^4 + 5x - 2\right).$$

Solution. We use the product rule with $g(x) = \left(x^2 + 3x\right)(3 - x)$, $h(x) = x^4 + 5x - 2$ and $f(x) = g(x)h(x)$. We find

$$\begin{aligned} f'(x) &= g'(x)h(x) + g(x)h'(x) \\ &= \left[\left(x^2 + 3x\right)(3 - x)\right]' h(x) + \left(x^2 + 3x\right)(3 - x)\left(x^4 + 5x - 2\right)' \\ &= \left[\left(x^2 + 3x\right)(3 - x)\right]' \left(x^4 + 5x - 2\right) + \left(x^2 + 3x\right)(3 - x)\left(4x^3 + 5\right) \end{aligned}$$

Since

$$\begin{aligned} \left[\left(x^2 + 3x\right)(3 - x)\right]' &= \left(x^2 + 3x\right)'(3 - x) + \left(x^2 + 3x\right)(3 - x)' \\ &= (2x + 3)(3 - x) + \left(x^2 + 3x\right)(-1) = 9 - 3x^2 \end{aligned}$$

Thus,

$$f'(x) = \left(9 - 3x^2\right)\left(x^4 + 5x - 2\right) + \left(x^2 + 3x\right)(3 - x)\left(4x^3 + 5\right)$$

which simplifies to,

$$f'(x) = -7x^6 + 45x^4 - 20x^3 + 6x^2 + 90x - 18.$$

2.14 Quotient Rule

Theorem 2.4. *If $f = g/h$ for any differentiable functions g and h, then*

$$f' = \frac{hg' - gh'}{h^2}. \tag{2.2}$$

Example 2.15. Find the derivative of the function

$$f(x) = \frac{(x^3 - 3x)(x^2 - 3)}{(x - 4)x^2}.$$

Solution. We use the quotient rule with $g(x) = (x^3 - 3x)(x^2 - 3)$ and $h(x) = (x - 4)x^2$. But first we compute $g'(x) = 5x^4 - 18x^2 + 9$ and $h'(x) = 3x^2 - 8x$. Thus,

$$f'(x) = \frac{((x-4)x^2)(5x^4 - 18x^2 + 9) - (x^3 - 3x)(x^2 - 3)(3x^2 - 8x)}{(x-4)^2x^4}$$

which simplifies to

$$f'(x) = \frac{9}{4x^2} - \frac{169}{4(x-4)^2} + 4 + 2x = \frac{2(18 - 9x + 12x^2 - 6x^4 + x^5)}{(-4+x)^2x^2}.$$

Example 2.16. Find the first and second derivatives of the function

$$f(x) = \frac{ax+b}{cx+d}.$$

Solution. Using the product rule with $f(x) = (ax+b)(cx+d)^{-1}$ we find

$$f'(x) = \frac{a}{d+cx} - \frac{c(b+ax)}{(d+cx)^2}.$$

Using the quotient rule with $f(x) = g(x)/h(x)$, $g(x) = ax+b$, and $h(x) = cx+d$ we find

$$f'(x) = \frac{ad - bc}{(d+cx)^2}.$$

The second expression for f' is easier to work with.

2.15 Exercises

Exercise 2.22. Determine a so that $f'\left(\pm\sqrt{\frac{2}{3}}\right) = 0$ given $f(x) = x^3 - 2ax + 1$. Determine a so that $f'\left(1 \pm \sqrt{2}\right) = 0$ given $f(x) = \frac{x^2+1}{x-a}$.

Exercise 2.23. Find the first derivative and the second derivative for each of the following.

- $y = (x-1)(x^2 + x + 1)$.

- $y = \frac{2x+5}{3x-2}$.
- $f(t) = \frac{t^2-1}{t^2+t-2}$.

Exercise 2.24. Find the derivatives of all orders for each of the following.

- $u = \frac{(x^2+x)(x^2-x+1)}{x^4}$.
- $p = \frac{q^2+3}{(q-1)^3+(q+1)^3}$.
- $w = \frac{3z^2e^z}{z+1}$.

Exercise 2.25. Suppose u and v are differentiable functions of x and that $u(1) = 2$, $u'(1) = 0$, $v(1) = 5$, and $v'(1) = -1$. Find the values of $\frac{d}{dx}(uv)$, $\frac{d}{dx}\left(\frac{u}{v}\right)$, $\frac{d}{dx}\left(\frac{v}{u}\right)$, and $\frac{d}{dx}(7v - 2u)$ at $x = 1$.

Exercise 2.26. The curves $y = x^2 + ax + b$ and $y = cx - x^2$ have a common tangent line at the point $(1, 0)$. Find a, b, and c.

Exercise 2.27. If gas in a cylinder is maintained at a constant temperature T, the pressure P is related to the volume V by a formula of the form

$$P = \frac{nRT}{V - nb} - \frac{an^2}{V^2},$$

in which a, b, n, and R are constants. Find $\frac{dP}{dV}$.

Exercise 2.28. Find the first, second, and third derivatives of the following functions.

- $y = \left(x^4 + 3x^2 + 17x + 82\right)^3$
- $y = \frac{x^4+3x^2+17x+82}{\sqrt{x}}$

2.16 Role of the Derivative

2.17 Average Rate of Change

In general, suppose an object moves along a straight line according to an equation of motion $s = f(t)$, where s is the **displacement** (**directed distance**) of the object from the origin at time t.
The function f that describes the motion is called the **position function** of the object. In the time interval from $t = a$ to $t = a + h$ the change

in position is $f(a+h)-f(a)$ and the **average velocity** over this time interval is

$$\frac{f(a+h)-f(a)}{h}$$

which is the same as the slope of the secant line through these two points.

Example 2.17. If a billiard is dropped from a height of 500 feet, its height s at time t is given by the position function $s=-16t^2+500$ where s is measured in feet and t is measured in seconds. Find the average velocity over the intervals $[2,2.5]$ and $[2,2.6]$.

Solution. For the interval $[2,2.5]$, the object falls from a height of $s(2)=-16(2)^2+500=436$ feet to a height of $s(2.5)=-16(2.5)^2+500=400$. The average velocity is

$$\frac{\Delta s}{\Delta t}=\frac{s(2.5)-s(2)}{2.5-2}=\frac{400-436}{2.5-2}=-72.$$

For the interval $[2,2.6]$, the object falls from a height of $s(2)=436$ feet to a height of $s(2.6)=391.84$.
The average velocity is

$$\frac{\Delta s}{\Delta t}=\frac{s(2.6)-s(2)}{2.6-2}=\frac{391.84-436}{2.6-2}=-73.6.$$

Note that the average velocities are negative indicating that the object is moving downward.

2.18 Instantaneous Rate of Change

The difference quotient

$$\frac{\Delta y}{\Delta x}=\frac{f\left(x_2\right)-f\left(x_1\right)}{x_2-x_1}$$

is the average rate of change of y with respect to x over the interval $[x_1,x_2]$ and can be interpreted as the slope of the secant line. Its limit as $\Delta x\to 0$ is the derivative at $x=x_1$ and is denoted by $f'\left(x_1\right)$.
We interpret the limit of the average rate of change as the interval becomes smaller and smaller to be the instantaneous rate of change. Often, different branches of science have specific interpretations of the derivative.

As $\Delta x\to 0$ the average rate of change approaches the **instantaneous rate for change** ; that is,

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=f'(x)$$

and is also known as the **derivative** of f at x.

2.19 Free-Falling Body

Basically, **rectilinear motion** refers to the motion of an object that can be modeled along a straight line; and the so-called **falling body problems** are a special type of rectilinear motion where the motion of an object is falling (or propelled) in a vertical direction. Another type of rectilinear motion is the **free-falling body problem**.

The position of a free-falling body (neglect air resistance) under the influence of gravity can be represented by the function

$$s(t) = \frac{1}{2}gt^2 + v_0 t + s_0$$

where g is the acceleration due to gravity (on earth $g \approx -32\text{ft}\left/s^2\right)$ and s_0 and v_0 are the initial height and velocity of the object (when $t = 0$).

Example 2.18. A ball is thrown vertically upward from the ground with an initial velocity of 160 ft/s. - When will it hit the ground? - With what velocity will the ball hit the ground? - When will the ball reach its maximum height and what is the maximum height?

Solution.

- We can determine when the ball will hit the ground by solving

  $$s(t) = \frac{1}{2}gt^2 + v_0 t + s_0 = 0$$

  for t. Using $g = -32$, $v_0 = 160$, and $s_0 = 0$. We find

  $$s(t) = -16t^2 + 160t = 16t(-t + 10) = 0$$

  when $t = 0$ and $t = 10$. Thus the ball will hit the ground 10 seconds after it is thrown upwards.
- The velocity of the ball at time t is given by the first derivative of s, namely

  $$v(t) = s'(t) = gt + v_0.$$

  When $t = 10$ we find, $v(10) = -32(10) + 160 = -160$ and so the velocity of the ball is -160 ft$/s$ when it hits the ground.
- The ball reaches it's maximum height when the velocity is zero, thus we solve

  $$v(t) = gt + v_0 = -32t + 160 = 0$$

  yielding $t = 2$. Its position at $t = 2$ is the maximum height which is $s(2) = 16(2)(-2 + 10) = 256$. Therefore, 2 seconds after the ball is thrown, the ball reaches it's maximum height of 256 ft.

Definition 2.2. An object that moves along a straight line with **position** $s(t)$ has **velocity** $v(t) = \frac{ds}{dt}$.

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

when these derivatives exist. The **speed** of an object at time t is $|v(t)|$.

Example 2.19. A particle moving along the x-axis has position

$$x(t) = 2t^3 + 3t^2 - 36t + 40$$

after an elapsed time of t seconds.
- Find the velocity of the particle at time t. - Find the acceleration at time t. - What is the **total distance travelled** by the particle during the first 3 seconds?

Solution. The velocity is given by $v(t) = x'(t) = 6t^2 + 6t - 36$.
The acceleration is given by $a(t) = v'(t) = x''(t) = 12t + 6$.
Since $v(t) = 0$ when

$$6t^2 + 6t - 36 = 6(t-2)(t+3) = 0$$

so $t = 2, -3$ but -3 is not on $[0, 3]$. Therefore, the distance covered is

$$|x(2) - x(0)| + |x(3) - x(2)| = |-4 - 40| + |13 - (-4)| = 61.$$

2.20 Exercises

Exercise 2.29. The number of gallons of water in a tank t minutes after the tank has started to drain is $Q(t) = 200(30-t)^2$. How fast is the water running out at the end 19 min? What is the average rate at which the water flows out during the first 10 minutes?

Exercise 2.30. Suppose that the distance an aircraft travels along a runway before takeoff is given by $D = \frac{10}{9}t^2$, where D is measured in meters from the starting point an t is measured in seconds from the time the brakes are released. The aircraft will become airborne when its speed reached 200 km/h. How long will it take to become airborne, and what distance will it travel to that time?

Exercise 2.31. Suppose that the dollar cost of producing x washing machines is $c(x) = 2000+100x-0.1x^2$. (a) Find the average cost per machine of producing the first 100 washing machines. (b) Find the marginal cost when 100 washing machines are produced. (c) Show that the marginal cost when 100 washing machines are produced is approximately the cost of producing one more washing machine after the first 100 have been made, by calculating the latter cost directly.

Exercise 2.32. Suppose the revenue from selling x washing machines is

$$r(x) = 20000\left(1 - \frac{1}{x}\right)$$

dollars. (a) Find the marginal revenue when 100 machines are produced. (b) Use the function $r'(x)$ to estimate the increase in revenue that will result from increasing production from 100 machines a week to 101 machines a week. (c) Find the limit of $r'(x)$ a $x \to \infty$. How would you interpret this number?

Exercise 2.33. How fast does the area of a circle change with respect to its diameter? Its circumference?

Exercise 2.34. Given a position function $s(t)$ where t represents time, define the displacement, average velocity, instantaneous velocity, speed, and acceleration of an object whose motion along a line is modeled by $s(t)$.

Exercise 2.35. A particle moving along the x-axis has position $x(t) = 2t^3+3t^2-36t+40$ after an elapsed time of t seconds. (a) Find the velocity of the particle at time t . (b) Find the acceleration at time t . (c) What is the total distance traveled by the particle during the first 3 seconds?

Exercise 2.36. At time $t \geq 0$, the velocity of a body moving along the s -axis is $v = t^2 - 4t + 3$. (a) Find the body's acceleration each time the velocity is zero. (b) When is the body moving forward? Backward? (c) When is the body's velocity increasing? Decreasing?

2.21 Derivatives of Trigonometric Functions

Theorem 2.5. *The trigonometric functions sine, cosine, tangent, cotangent, cosecant, and secant are all differentiable functions on their domain* *and their derivative functions are:*

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\cos x = -\sin x \qquad \frac{d}{dx}\tan x = \sec^2 x$$
$$\frac{d}{dx}\cot x = -\csc^2 x \qquad \frac{d}{dx}\sec x = \sec x \tan x \qquad \frac{d}{dx}\csc x = -\csc x \cot x$$

Proof. For the derivative of the **cosine function** , we use the formula

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

along with the definition of the derivative:

$$\begin{aligned}
\frac{d}{dx}\cos x &= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} \\
&= \lim_{h\to 0}\frac{\cos x\cos h - \sin x \sin h - \cos x}{h} \\
&= \lim_{h\to 0}\frac{\cos x(\cos h - 1) - \sin x \sin h}{h} \\
&= \lim_{h\to 0}\frac{\cos x(\cos h - 1)}{h} + \lim_{h\to 0}\frac{-\sin x \sin h}{h} \\
&= (\cos x)\lim_{h\to 0}\frac{(\cos h - 1)}{h} - \sin(x)\lim_{h\to 0}\frac{\sin h}{h} \\
&= (\cos x)(0) - (\sin x)(1) \\
&= -\sin x.
\end{aligned}$$

For the derivative of the **sine function** , we use the formula

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

along with the definition of the derivative:

$$\begin{aligned}
\frac{d}{dx}\sin x &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} \\
&= \lim_{h\to 0}\frac{\sin x\cos h + \cos x \sin h - \sin x}{h} \\
&= \lim_{h\to 0}\frac{\sin x(\cos h - 1) + \cos x \sin h}{h} \\
&= \lim_{h\to 0}\frac{\sin x(\cos h - 1)}{h} + \lim_{h\to 0}\frac{\cos x \sin h}{h} \\
&= (\sin x)\lim_{h\to 0}\frac{(\cos h - 1)}{h} + \cos(x)\lim_{h\to 0}\frac{\sin h}{h} \\
&= (\sin x)(0) + \cos(x)(1) \\
&= \cos x.
\end{aligned}$$

For the derivative of the **tangent function** , we use the formula

$$\tan x = \frac{\sin x}{\cos x}$$

along with the quotient rule:

$$\frac{d}{dx}\tan x = \frac{(\cos x)(\cos x) - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \sec^2 x.$$

For the derivative of the **cotangent function** , we use the formula

$$\cot x = \frac{\cos x}{\sin x}$$

along with the quotient rule:

$$\frac{d}{dx}\cot x = \frac{\sin x(-\sin x) - (\cos x)(\cos x)}{\sin^2 x} = -\csc^2 x.$$

For the derivative of the **secant function** , we use the formula

$$\sec x = \frac{1}{\cos x}$$

along with the quotient rule:

$$\frac{d}{dx}\sec x = \frac{(\cos x)(0) - 1(-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \sec x \tan x.$$

For the derivative of the **cosecant function** , we use the formula

$$\csc x = \frac{1}{\sin x}$$

along with the quotient rule:

$$\frac{d}{dx}\csc x = \frac{(\sin x)(0) - 1(\cos x)}{\sin^2 x} = -\csc x \cot x.$$

□

2.22 Derivatives of Trigonometric Functions and Simplification

Since the trigonometric functions are differentiable functions on their domains they are also continuous functions on their domain.

Example 2.20. Find the derivative of the function

$$g(x) = \frac{x^2 + \tan x}{3x + 2\tan x}.$$

Solution. For the function g we use the quotient rule and the derivative rules for sine and cosine, we determine,

$$g'(x) = \frac{(3x + 2\tan x)\left(2x + \sec^2 x\right) - \left(x^2 + \tan x\right)\left(3 + 2\sec^2 x\right)}{3x + 2\tan x}.$$

Example 2.21. Find the derivative of the function

$$f(x) = \frac{\sin x + \cos x}{\sin x - \cos x}.$$

Solution. For the function f we use the quotient rule, derivative rules for sine and cosine, and a few trigonometric identities, we determine,

$$f'(x) = \frac{(\sin x - \cos x)(\cos x - \sin x) - (\sin x + \cos x)(\cos x + \sin x)}{(\sin x - \cos x)^2}.$$

After expanding and simplifying,

$$f'(x) = \frac{-2}{1 - \sin 2x}.$$

2.23 Exercises

Exercise 2.37. Find all points on the curve $y = \cot x$, $0 < x < \pi$, where the tangent line is parallel to the line $y = -x$. Sketch the curve and tangent(s) together, labeling each with its equation.

Exercise 2.38. Find an equation for the tangent to the curve $y = 1 + \sqrt{2}\csc x + \cot x$ at the point $\left(\frac{\pi}{4}, 4\right)$ and find an equation for the horizontal tangent line.

Exercise 2.39. Suppose the function given by $s = \sin t + \cos t$ represents the position of a body moving on a coordinate line (s in meters, t in seconds). Find the body's velocity, speed, and acceleration at time $t = \pi/4\,\mathrm{sec}$.

Exercise 2.40. Is there a value of b that will make $g(x) = \begin{cases} x+b & x<0 \\ \cos x & x \geq 0 \end{cases}$ continuous at $x = 0$? Differentiable at $x = 0$? Give reasons for your answer.

Exercise 2.41. Find an equation of the tangent line to $y = 3\tan x - 2\csc x$ at $x = \frac{\pi}{3}$.

Exercise 2.42. Show that the function

$$f(x) = \begin{cases} x^3 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

has a continuous first derivative.

Exercise 2.43. Find the derivative of each of the following.

- $y = -10x + 3\cos x$
- $y = \csc x - 4\sqrt{x} + 7$
- $y = (\sin x + \cos x)\sec x$
- $y = \dfrac{4}{\cos x} + \dfrac{1}{\tan x}$
- $y = x^2 \cos x - 2x \sin x - 2\cos x$
- $s = \dfrac{1 + \csc t}{1 - \csc t}$
- $r = \theta \sin \theta + \cos \theta$
- $p = (1 + \csc q)\cos q$

Exercise 2.44.

(a) Sketch the graph of the curve $y = \sin x$ on the interval $-\frac{3\pi}{2} \leq x \leq 2\pi$ and their tangents at the x values of $x = -\pi$, $x = 0$, and $x = 3\pi/2$.

(b) Sketch the graph of the curve $y = 1 + \cos x$ on the interval $-\frac{3\pi}{2} \leq x \leq 2\pi$ and their tangents at the x values of $x = -\pi/3$ and $x = 3\pi/2$.

Exercise 2.45. Find all points on the curve $y = \cot x$, $0 < x < \pi$, where the tangent line is parallel to the line $y = -x$. Sketch the curve and tangent(s) together, labeling each with its equation.

Exercise 2.46. Find an equation for the tangent to the curve $y = 1 + \sqrt{2}\csc x + \cot x$ at the point $\left(\frac{\pi}{4}, 4\right)$ and find an equation for the horizontal tangent line.

Exercise 2.47. Suppose the function given by $s = \sin t + \cos t$ represents the position of a body moving on a coordinate line (s in meters, t in seconds). Find the body's velocity, speed, and acceleration at time $t = \pi/4\,\text{sec}$.

Exercise 2.48. Is there a value of b that will make $g(x) = \begin{cases} x+b & x<0 \\ \cos x & x \geq 0 \end{cases}$ continuous at $x = 0$? Differentiable at $x = 0$? Give reasons for your answer.

Exercise 2.49. Find an equation of the tangent line to $y = 3\tan x - 2\csc x$ at $x = \frac{\pi}{3}$.

Exercise 2.50. Show that the function

$$f(x) = \begin{cases} x^3 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

has a continuous first derivative.

Exercise 2.51. Find the derivative of each of the following.

- $y = -10x + 3\cos x$
- $y = \csc x - 4\sqrt{x} + 7$
- $y = (\sin x + \cos x)\sec x$
- $y = \frac{4}{\cos x} + \frac{1}{\tan x}$
- $y = x^2 \cos x - 2x \sin x - 2\cos x$
- $s = \frac{1+\csc t}{1-\csc t}$
- $r = \theta \sin\theta + \cos\theta$
- $p = (1 + \csc q)\cos q$

2.24 Chain Rule

2.25 The Chain Rule and Its Proof

With a lot of work, we can sometimes find derivatives without using the chain rule either by expanding a polynomial, by using another differentiation rule, or maybe by using a trigonometric identity. The derivative would be the same in either approach; however, the chain rule allows us to find derivatives that would otherwise be very difficult to handle. This section gives plenty of examples of the use of the chain rule as well as an easily understandable proof of the chain rule.

::: {#thm- } Chain Rule Suppose f is a differentiable function of u which is a differentiable function of x. Then $f(u(x))$ is a differentiable function of x and

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx}.$$

:::

Proof. We wish to show $\frac{df}{dx} = \frac{df}{du}\frac{du}{dx}$ and will do so by using the definition of the derivative for the function f with respect to x, namely,

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f[u(x+\Delta x)] - f[u(x)]}{\Delta x}$$

To better work with this limit let's define an auxiliary function:

$$g(t) = \begin{cases} \frac{f[u(x)+t]-f[u(x)]}{t} - \frac{df}{du} & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases}$$

Let $\Delta u = u(x+\Delta x) - u(x)$, then three properties of the function g are

- $g(\Delta u) = \frac{f[u(x)+\Delta u]-f[u(x)]}{\Delta u} - \frac{df}{du}$ provided $\Delta u \neq 0$
- $\left[g(\Delta u) + \frac{df}{du}\right] \Delta u = f[u(x)+\Delta u] - f[u(x)]$
- g is continuous at $t = 0$ since $\lim_{t\to 0}\left[\frac{f[u(x)+t]-f[u(x)]}{t}\right] = \frac{df}{du}$

Now we can rewrite $\frac{df}{dx}$ as follows:

$$\begin{aligned}
\frac{df}{dx} &= \lim_{\Delta x \to 0} \frac{f[u(x+\Delta x)] - f[u(x)]}{\Delta x} = \lim_{\Delta x \to 0} \frac{f[u(x)+\Delta u] - f[u(x)]}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\left(g(\Delta u) + \frac{df}{du}\right)\Delta u}{\Delta x} = \lim_{\Delta x \to 0}\left(g(\Delta u) + \frac{df}{du}\right) \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \\
&= \left[\lim_{\Delta x \to 0} g(\Delta u) + \lim_{\Delta x \to 0} \frac{df}{du}\right] \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \\
&= \left[g\left(\lim_{\Delta x \to 0} \Delta u\right) + \lim_{\Delta x \to 0} \frac{df}{du}\right] \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \\
&= \left[g(0) + \lim_{\Delta x \to 0} \frac{df}{du}\right] \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \left[0 + \lim_{\Delta x \to 0} \frac{df}{du}\right] \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \frac{df}{du}\frac{du}{dx}.
\end{aligned}$$

□

2.26 Examples Using The Chain Rule

Example 2.22. Find the derivative of the function

$$y = \frac{x}{\sqrt{x^4+4}}.$$

Solution. Using the chain rule and the quotient rule,

$$\frac{dy}{dx} = \frac{\sqrt{x^4+4}(1) - x\frac{d}{dx}\left(\sqrt{x^4+4}\right)}{\left(\sqrt{x^4+4}\right)^2} = \frac{\sqrt{x^4+4}(1) - x\left(\frac{2x^3}{\sqrt{4+x^4}}\right)}{\left(\sqrt{x^4+4}\right)^2}$$

which simplifies to

$$\frac{dy}{dx} = \frac{4-x^4}{\left(4+x^4\right)^{3/2}}.$$

Example 2.23. Find the derivative of the function

$$g(x) = \left(\frac{3x^2-2}{2x+3}\right)^3.$$

Solution. Using the chain rule and the quotient rule, we determine,

$$\frac{dg}{dx} = 3\left(\frac{3x^2-2}{2x+3}\right)^2 \left(\frac{(2x+3)6x - \left(3x^2-2\right)2}{(2x+3)^2}\right)$$

which simplifies to

$$\frac{dg}{dx} = \frac{6\left(2-3x^2\right)^2\left(2+9x+3x^2\right)}{(3+2x)^4}.$$

Example 2.24. Find the derivative of the function

$$h(t) = 2\cot^2(\pi t + 2).$$

Solution. Using the chain rule and the formula $\frac{d}{dx}(\cot u) = -u' \csc^2 u$,

$$\frac{dh}{dt} = 4\cot(\pi t+2)\frac{d}{dx}[\cot(\pi t+2)] = -4\pi\cot(\pi t+2)\csc^2(\pi t+2).$$

Example 2.25. Find the derivative of the function

$$y = \sin\sqrt[3]{x} + \sqrt[3]{\sin x}$$

Solution. Using the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= \cos\sqrt[3]{x}\frac{d}{dx}\left(\sqrt[3]{x}\right) + \frac{1}{3}(\sin x)^{-2/3}\frac{d}{dx}(\sin x)\\ &= \frac{1}{3x^{2/3}}\cos\sqrt[3]{x} + \frac{\cos x}{3(\sin x)^{2/3}}.\end{aligned}$$

Example 2.26. Show that

$$\frac{d}{dx}(\ln|\cos x|) = -\tan x \qquad \text{and} \qquad \frac{d}{dx}(\ln|\sec x + \tan x|) = \sec x.$$

Solution. Using the differentiation rule $\frac{d}{dx}[\ln u] = \frac{u'}{u}$; we have,

$$\frac{d}{dx}(\ln|\cos x|) = \frac{1}{\cos x}\frac{d}{dx}(\cos x) = \frac{\sin x}{\cos x} = \tan x$$

and

$$\begin{aligned}\frac{d}{dx}((\ln|\sec x + \tan x|)) &= \frac{1}{|\sec x + \tan x|}\frac{d}{dx}(|\sec x + \tan x|)\\ &= \frac{1}{|\sec x + \tan x|}\frac{\sec x + \tan x}{|\sec x + \tan x|}\frac{d}{dx}(\sec x + \tan x)\\ &= \frac{1}{|\sec x + \tan x|}\frac{\sec x + \tan x}{|\sec x + \tan x|}(\sec x\tan x + \sec^2 x)\\ &= \frac{\sec x\tan x + \sec^2 x}{\sec x + \tan x} = \sec x\end{aligned}$$

using $\frac{d}{dx}[|u|] = \frac{u}{|u|}(u'), u \neq 0$.

Example 2.27. Find the derivative of the function

$$y = \sin^4\left(x^2 - 3\right) - \tan^2\left(x^2 - 3\right).$$

Solution. Using the quotient rule with the chain rule,

$$y' = 4\sin^3\left(x^2-3\right)\cos\left(x^2-3\right)(2x) - 2\tan\left(x^2-3\right)\sec^2\left(x^2-3\right)(2x)$$

which simplifies to

$$y' = 4x\left[2\sin^3\left(x^2-3\right)\cos\left(x^2-3\right) - \tan\left(x^2-3\right)\sec^2\left(x^2-3\right)\right].$$

In the following examples we continue to illustrate the chain rule.

Example 2.28. Let f be a function for which $f'(x) = \dfrac{1}{x^2+1}$. If $g(x) = f(3x-1)$, what is $g'(x)$? Also, if $h(x) = f\left(\dfrac{1}{x}\right)$, what is $h'(x)$?

Solution. By the chain rule

$$g'(x) = f'(3x-1)\frac{d}{dx}(3x-1) = 3f'(3x-1) = \frac{3}{(3x-1)^2+1}.$$

Also, by the chain rule

$$h'(x) = f'\left(\frac{1}{x}\right)\frac{d}{dx}\left(\frac{1}{x}\right) = -f'\left(\frac{1}{x}\right)\left(\frac{1}{x^2}\right) = \frac{-1}{\left(\frac{1}{x}\right)^2+1}\left(\frac{1}{x^2}\right) = \frac{-1}{x^2+1}.$$

Example 2.29. Let f be a function for which $f(2) = -3$ and $f'(x) = \sqrt{x^2+5}$. If $g(x) = x^2 f\left(\dfrac{x}{x-1}\right)$, what is $g'(2)$?

Solution. Using the chain rule and the product rule we determine,

$$\begin{aligned} g'(x) &= 2xf\left(\frac{x}{x-1}\right) + x^2 f'\left(\frac{x}{x-1}\right)\frac{d}{dx}\left(\frac{x}{x-1}\right) \\ &= 2xf\left(\frac{x}{x-1}\right) + x^2 f'\left(\frac{x}{x-1}\right)\left(\frac{-1}{(x-1)^2}\right). \end{aligned}$$

Therefore,

$$g'(2) = 2(2)f\left(\frac{2}{2-1}\right) + 2^2 f'\left(\frac{2}{2-1}\right)\left(\frac{-1}{(2-1)^2}\right) = -24.$$

Example 2.30. Assuming that the following derivatives exists, find

$$\frac{d}{dx}f'[f(x)] \qquad \text{and} \qquad \frac{d}{dx}f[f'(x)].$$

Solution. Using the chain rule,

$$\frac{d}{dx}f'[f(x)] = f''[f(x)]f'(x)$$

which is the second derivative evaluated at the function multiplied by the first derivative; while,

$$\frac{d}{dx}f[f'(x)] = f'[f'(x)]f''(x)$$

is the first derivative evaluated at the first derivative multiplied by the second derivative. When will these derivatives be the same?

Example 2.31. Show that if a particle moves along a straight line with position $s(t)$ and velocity $v(t)$, then its acceleration satisfies $a(t) = v(t)\frac{dv}{ds}$. Use this formula to find $\frac{dv}{ds}$ in the case where $s(t) = -2t^3 + 4t^2 + t - 3$.

Solution. By the chain rule,

$$a(t) = \frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = v(t)\frac{dv}{ds}$$

In the case where $s(t) = -2t^3 + 4t^2 + t - 3$; we determine,

$$\frac{ds}{dt} = v(t) = -6t^2 + 8t + 1 \qquad \text{and} \qquad a(t) = -12t + 8.$$

Thus,

$$\frac{dv}{ds} = \frac{-12t + 8}{-6t^2 + 8t + 1}.$$

What does this rate of change represent?

Example 2.32. Find an equation of the tangent line to the graph of the function $f(x) = \left(9 - x^2\right)^{2/3}$ at the point $(1, 4)$.

Solution. By using the chain rule we determine,

$$f'(x) = \frac{2}{3}\left(9 - x^2\right)^{-1/3}(-2x) = \frac{-4x}{3\sqrt[3]{9 - x^2}}$$

and so $f'(1) = \frac{-4}{3\sqrt[3]{9-1^2}} = \frac{-2}{3}$. Therefore, an equation of the tangent line is $y - 4 = \left(\frac{-2}{3}\right)(x - 1)$ which simplifies to $y = \frac{-2}{3}x + \frac{14}{3}$.

Example 2.33. Determine the point(s) at which the graph of

$$f(x) = \frac{x}{\sqrt{2x - 1}}$$

has a horizontal tangent.

Solution. By using the chain rule we determine,

$$f'(x) = \frac{\sqrt{2x-1}(1) - x\frac{d}{dx}\left(\sqrt{2x-1}\right)}{\left(\sqrt{2x-1}\right)^2} = \frac{\sqrt{2x-1}(1) - x\left(\frac{1}{\sqrt{-1+2x}}\right)}{\left(\sqrt{2x-1}\right)^2}$$

which simplifies to $f'(x) = \dfrac{-1 + x}{(-1 + 2x)^{3/2}}$. Thus the only point where f has a horizontal tangent line is $(1, 1)$. See Figure **??**.

2.27 Exercises

Exercise 2.52. Given $y = 6u - 9$ and find $\frac{dy}{dx}$ for (a) $u = (1/2)x^4$, (b) $u = -x/3$, and (c) $u = 10x - 5$.

Exercise 2.53. For each of the following functions, write the function $y = f(x)$ in the form $y = f(u)$ and $u = g(x)$, then find $\frac{dy}{dx}$.

- $y = \left(\frac{x^2}{8} + x - \frac{1}{x}\right)^4$
- $y = \sec(\tan x)$
- $y = 5\cos^{-4} x$
- $y = e^{5-7x}$
- $y = \sqrt{2x - x^2}$
- $y = e^x \sqrt{2x - x^2}$

Exercise 2.54. Find the derivative of the following functions.

- $r = -(\sec\theta + \tan\theta)^{-1}$
- $y = \frac{1}{x}\sin^{-5} x - \frac{x}{3}\cos^3 x$
- $y = (4x+3)^4(x+1)^{-3}$
- $y = (1+2x)e^{-2x}$
- $h(x) = x\tan\left(2\sqrt{x}\right) + 7$
- $g(t) = \left(\frac{1+\cos t}{\sin t}\right)^{-1}$
- $q = \sin\left(\frac{t}{\sqrt{t+1}}\right)$
- $y = \theta^3 e^{-2\theta}\cos 5\theta$
- $y = (1 + \cos 2t)^{-4}$
- $y = \left(e^{\sin(t/2)}\right)^3$
- $y = \left(1 + \tan^4\left(\frac{t}{12}\right)\right)^3$
- $y = 4\sin\left(\sqrt{1+\sqrt{t}}\right)$
- $y = \frac{1}{9}\cot(3x - 1)$
- $y = \sin\left(x^2 e^x\right)$
- $y = e^x \sin\left(x^2 e^x\right)$

Exercise 2.55. Suppose that the functions f, g, and their derivatives with respect to x have the following values at $x = 0$ and $x = 1$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
0	1	1	5	1/3
1	3	−4	−1/3	−8/3

Find the derivatives with respect to x of the following combinations at a given value of x,

- $5f(x) - g(x), x = 1$
- $f(x)g^3(x), x = 0$
- $\frac{f(x)}{g(x)+1}, x = 1$
- $f(g(x)), x = 0$
- $g(f(x)), x = 0$
- $\left(x^{11} + f(x)\right)^{-2}, x = 1$
- $f(x + g(x)), x = 0$
- $f(xg(x)), x = 0$
- $f^3(x)g(x), x = 0$

Exercise 2.56. Find dy/dt when $x = 1$ if $y = x^2 + 7x - 5$ and $dx/dt = 1/3$.

Exercise 2.57.

(a) Find the tangent to the curve $y = 2\tan(\pi x/4)$ at $x = 1$. (b) What is the smallest value the slope of the curve can ever have on the interval $-2 < x < 2$? Give reasons for you answer.

Exercise 2.58. Suppose that $u = g(x)$ is differentiable at $x = -5$, $y = f(u)$ is differentiable at $u = g(-5)$, and $(f \circ g)'(-5)$ is negative. What, if anything, can be said about the values of $g'(-5)$ and $f'(g(-5))$?

Exercise 2.59. Differentiate the functions given by the following equations

- $y = \cos^2\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)$
- $y = \sqrt{1 + \tan\left(x + \frac{1}{x}\right)}$
- $n = \left(y + \sqrt[3]{y + \sqrt{2y - 9}}\right)^8$

Exercise 2.60. If $g(t) = [f(\sin t)]^2$, where f is a differentiable function, find $g'(t)$.

Exercise 2.61. Suppose f is a differentiable function on $\mathbb{R}$. Let $F(x) = f(\cos x)$ and $G(x) = \cos(f(x))$. Find expressions for $F'(x)$ and $G'(x)$.

Exercise 2.62. Determine if the following statement is true or false. Then justify your claim. If y is a differentiable function of u, u is a differentiable function of v, and v is a differentiable function of x, then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx}$.

Exercise 2.63. Let u be a differentiable function of x. Use $|u| = \sqrt{u^2}$ to prove that $\frac{d}{dx}(|u|) = \frac{u'u}{|u|}$ when $u \neq 0$. Use the formula to find h' given $h(x) = x|2x-1|$.

Exercise 2.64. Show that $\frac{d}{d\theta}(\sin\theta^\circ) = \frac{\pi}{180}\cos\theta$. What do you think is the importance of the exercise?

2.28 Implicit Differentiation

2.29 Implicit Differentiation as a Procedure

In this section the procedure of implicit differentiation is outlined and many examples are given. Proofs of the derivative formulas for the inverse trigonometric functions are provided and several examples of using them are given. Also detailed is the logarithmic differentiation procedure which can simplify the process of taking derivatives of equations involving products and quotients. Finding the slope of a tangent line is a local process; for example, a circle locally around a point, can have a tangent line even though it is not a function. In fact, every circle has a tangent line at every point.

The following process allows us to find derivatives of more general curves (not just functions); and in particular for an implicitly defined function. Notice that the process relies heavily on the chain rule.

::: {#thm- } Implicit Differentiation **implicit differentiation** Suppose that $f(x,y) = 0$ is a given equation involving both x and y; and that $\frac{dy}{dx}$ exists at (x_0, y_0). Then $\frac{dy}{dx}|_{(x_0,y_0)}$ can be found using the following procedure:

- Using the chain rule where appropriate, differentiate both sides of the equation with respect to x.
- If possible, solve the differentiated equation algebraically for $\frac{dy}{dx}$ and evaluate at (x_0, y_0). :::

Example 2.34. Use implicit differentiation to find $\frac{dy}{dx}$ given

$$\sin(x+y) = y^2\cos x.$$

Solution. We will use implicit differentiation, and in doing so we use the chain rule on the right hand and the product rule together with the chain rule on the left hand side of the equation:

$$\begin{aligned}
\cos(x+y)\frac{d}{dx}(x+y) &= 2y\cos x\frac{dy}{dx} - y^2\sin x \\
\cos(x+y)\left(1+\frac{dy}{dx}\right) - 2y\cos x\frac{dy}{dx} &= -y^2\sin x \\
\cos(x+y) + \frac{dy}{dx}\cos(x+y) - 2y\cos x\frac{dy}{dx} &= -y^2\sin x \\
\frac{dy}{dx}(\cos(x+y) - 2y\cos x) &= -y^2\sin x - \cos(x+y) \\
\frac{dy}{dx} &= -\frac{y^2\sin x + \cos(x+y)}{\cos(x+y) - 2y\cos x}
\end{aligned}$$

Example 2.35. Use implicit differentiation to find the tangent to the folium of Descartes $x^3 + y^3 = 6xy$ at $(3,3)$.

Solution. Using implicit differentiation we have,

$$3x^2 + 3y^2\frac{dy}{dx} = 6y + 6x\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x} = \frac{x^2 - 2y}{2x - y^2}$$

So an equation of the tangent line at $(3,3)$ is $y - 3 = \left.\frac{dy}{dx}\right|_{(3,3)}(x-3)$ which simplifies to $y = -x + 6$.

Example 2.36. Use implicit differentiation to find the tangent to the lemniscate of Bernoulli

$$2\left(x^2+y^2\right)^2 = 25\left(x^2-y^2\right).$$

Solution. Using implicit differentiation we have,

$$4\left(x^2+y^2\right)\left(2x + 2y\frac{dy}{dx}\right) = 50x - 50y\frac{dy}{dx}$$

and so

$$\frac{dy}{dx} = \frac{25x - 4x\left(x^2+y^2\right)}{25y + 4y\left(x^2+y^2\right)}$$

So an equation of the tangent line at $(3,1)$ is $y - 1 = \frac{dy}{dx}|_{(3,1)}(x-3)$ which simplifies to $y = \frac{-9}{13}x + \frac{40}{13}$.

Example 2.37. Use implicit differentiation to find all points on the lemniscate of Bernoulli $\left(x^2+y^2\right)^2 = 4\left(x^2-y^2\right)$ where the tangent line is horizontal.

Solution. Using implicit differentiation we have,

$$2\left(x^2+y^2\right)\left(2x+2y\frac{dy}{dx}\right) = 8x-8y\frac{dy}{dx}$$

and so

$$\frac{dy}{dx} = -\frac{x\left(-2+x^2+y^2\right)}{y\left(2+x^2+y^2\right)}$$

we need to find all (x,y) where $\frac{dy}{dx}=0$. Clearly, the point $(0,0)$ is ruled out and so $-2+x^2+y^2=0$; that is $x^2+y^2=2$. Using $x^2+y^2=2$ with the original we see $x^2-y^2=1$ also. Therefore, $2x^2=3$ and so $x=\pm\sqrt{\frac{3}{2}}$ and $y=\pm\sqrt{\frac{1}{2}}$.

Example 2.38. Use implicit differentiation to find two points on the curve whose equation is $x^2-3xy+2y^2=-2$, where the tangent line is vertical.

Solution. Using implicit differentiation we determine,

$$2x-3y-3x\frac{dy}{dx}+4y\frac{dy}{dx}=0$$

and so, $\frac{dy}{dx}=\frac{3y-2x}{4y-3x}$. Since we want vertical tangent lines we need $4y-3x=0$; that is, $y=\frac{3}{4}x$ and with the original equation this means; $x^2-3x\left(\frac{3}{4}x\right)+2\left(\frac{3}{4}x\right)^2=-2$ which is solved as $x=\pm 4$. So the points where the tangent line is vertical are $(-4,-3)$ and $(4,3)$.

2.30 Logarithmic Differentiation

Logarithmic differentiation is a procedure that uses the chain rule and implicit differentiation. Basically the idea is to apply an appropriate logarithmic function to both sides of the given equation and then use some properties of logarithms to simplify before using implicit differentiation.

::: {#thm- } Logarithmic Differentiation **logarithmic differentiation** Suppose that $f(x,y)=0$ is a given equation involving both x and y; and that $\frac{dy}{dx}$ exists at (x_0,y_0). Then $\frac{dy}{dx}|_{(x_0,y_0)}$ can be found using the following procedure:

- Apply a logarithmic function with the appropriate base to both sides.

- Use properties of logarithms to simplify.

- Differentiate both sides of the equation with respect to x.
- If possible, solve the differentiated equation algebraically for $\frac{dy}{dx}$. :::

Example 2.39. Find the derivative of $y = \frac{1}{(x-1)^3-(x+1)^3}$ without using the quotient rule.

Example 2.40. Use logarithmic differentiation to find $\frac{dy}{dx}$ given

$$y = \frac{e^{2x}}{\left(x^2-3\right)^2 \ln \sqrt{x}}.$$

Solution. Using the natural logarithmic function, we have $\ln y = 2x - 2\ln\left(x^2-3\right) - \ln\left(\ln\left(\sqrt{x}\right)\right)$ and applying implicit differentiation we have,

$$\begin{aligned}\frac{1}{y}\frac{dy}{dx} =& 2 - \frac{4x}{x^2-3} - \frac{1}{\ln\left(\sqrt{x}\right)}\frac{1}{\sqrt{x}}\frac{1}{2\sqrt{x}}\frac{dy}{dx} \\ &= \frac{e^{2x}}{\left(x^2-3\right)^2 \ln\sqrt{x}}\left(2 - \frac{4x}{x^2-3} - \frac{1}{\ln\left(\sqrt{x}\right)}\frac{1}{\sqrt{x}}\frac{1}{2\sqrt{x}}\right)\end{aligned}$$

as desired.

Example 2.41. Use logarithmic differentiation to find $\frac{dy}{dx}$ given $y = x^{\sin x}$.

Solution. Using the natural logarithmic function, we have $\ln y = (\sin x)(\ln x)$ and applying implicit differentiation we have,

$$\frac{1}{y}\frac{dy}{dx} = (\cos x)(\ln x) + \frac{\sin x}{x}\frac{dy}{dx} = x^{\sin x}(\cos x)(\ln x) + \frac{\sin x}{x}$$

as desired.

2.31 Exercises

Exercise 2.65. Find the derivative $\frac{dy}{dx}$ given each of the following.

- $y = x^{-3/5}$
- $y = 7\sqrt{x+6}$
- $y = (1-6x)^{2/3}$
- $y = \sqrt[3]{x^2}$

- $y = \cos(1-6x)^{2/3}$
- $y = \sqrt[3]{1+\cos(2x)}$

Exercise 2.66. Use implicit differentiation to find $\frac{dy}{dx}$ given each of the following.

- $x^3 + y^3 = 18xy$
- $x^2(x-y)^2 = x^2 - y^2$
- $x^2 = \frac{x-y}{x+y}$
- $e^{2x} = \sin(x+3y)$
- $e^{x^2 y} = 2x + 2y$
- $\sin(xy) = \frac{1}{2}$

Exercise 2.67. Use implicit differentiation to find $\frac{dy}{dx}$ at $(1,1)$ for each of the following.

- $(x+y)^3 = x^3 + y^3$
- $y^2\left(x^2+y^2\right) = 2x^2$
- $x\sqrt{1+y} + y\sqrt{1+2x} = 2x$
- $x^2 = \frac{y^2}{y^2-1}$

Exercise 2.68. Use implicit differentiation to find $\frac{dy}{dx}$ and then find $\frac{d^2y}{dx^2}$ given each of the following.

- $x^{2/3} + y^{2/3} = 1$
- $2\sqrt{y} = x - y.$

Exercise 2.69. Verify that the given point is on the given curve and find equations for the lines that are tangent and normal to the curve at this point.

- $x^2 + xy - y^2 = 1$ at $(2,3)$
- $y^2 - 2x - 4y - 1 = 0$ at $(-2,1)$
- $2xy + \pi \sin y = 2\pi$ at $(1, \pi/2)$
- $x^2 \cos^2 y - \sin y = 0$ at $(0, \pi)$

Exercise 2.70. Find points on the curve $x^2 + xy + y^2 = 7$ (a) where the tangent is parallel to the x-axis and (b) where the tangent is parallel to the y-axis. In the latter case, $\frac{dy}{dx}$ is not defined, but $\frac{dx}{dy}$ is? What value does $\frac{dx}{dy}$ have at these points?

Exercise 2.71. Find two points on the curve whose equation is $x^2 - 3xy + 2y^2 = -2$, where the tangent line is vertical.

Exercise 2.72. Use logarithmic differentiation to find $\frac{dy}{dx}$ for each of the following.

- $y = x^{x^3-2x+7}$
- $y = x^{\cos x}$
- $y = \frac{\sin x}{x^2-3x} e^{x^2}$
- $y = (3x-1)^{3x-1}$
- $y = \frac{e^{-3x}(x-2)}{(x^2+3)^2\sqrt{2x-1}}$
- $y = x^{\tan x} + x^{\cot x}$

Exercise 2.73. Show that the sum of the x- and y-intercepts of any tangent line to the curve $\sqrt{x} + \sqrt{y} = \sqrt{c}$ is equal to c.

Exercise 2.74.

(a) The equations $x^2 - 2tx + 2t^2 = 4$ and $2y^3 - 3t^2 = 4$ define x and y implicitly as differentiable functions $x = f(t)$ and $y = g(t)$, find the slope of the curve $x = f(t)$, $y = g(t)$ at $t = 2$.

(b) The equations $x \sin t + 2x = t$ and $t \sin t - 2t = y$ define x and y implicitly as differentiable functions $x = f(t)$ and $y = g(t)$, find the slope of the curve $x = f(t)$, $y = g(t)$ at $t = \pi$.

Exercise 2.75. Which of the following could be true if $f''(x) = x^{-1/3}$?

- $f(x) = \frac{3}{2}x^{2/3} - 3$
- $f(x) = \frac{9}{10}x^{5/3} - 7$
- $f'''(x) = \frac{-1}{3}x^{4/3}$
- $f(x) = \frac{3}{2}x^{2/3} + 6$

Exercise 2.76. Use implicit differentiation to show that tangent line at (x_0, y_0) to any ellipse centered at the origin $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ has the form $\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$. Also use implicit differentiation to show that tangent line at (x_0, y_0) to any hyperbola centered at the origin $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ has the form $\frac{x_0 x}{a^2} - \frac{y_0 y}{b^2} = 1$.

Exercise 2.77. Find the derivative $\frac{dy}{dx}$ for each of the following.

- $y = \cos^{-1}\left(x^2\right)$.

- $y = \sin^{-1}(1-t)$.
- $y = \csc^{-1}\left(x^2+1\right),\ x > 0$.
- $y = \sin^{-1}\left(\frac{3}{t^2}\right)$.
- $y = \ln\left(\tan^{-1} x\right)$.
- $y = \cos^{-1}\left(e^{-t}\right)$.

2.32 Derivatives of Inverse Functions

2.33 Inverse Functions

In this section we state the derivative rules for the natural exponential function and the general exponential function. We also go over several examples of the chain rule and the exponential derivative rules.

Theorem 2.6. ***derivative of inverse*** *If f has an interval I as domain and $f'(x)$ exists and is never zero on I, then f^{-1} is differentiable at every point in its domain. The value of $\left(f^{-1}\right)'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at $a = f^{-1}(b)$ given by*

$$\left(f^{-1}\right)'(b) = \frac{1}{f'\left(f^{-1}(b)\right)}.$$

Example 2.42. Let $f(x) = \dfrac{2x-3}{7-5x}$. Sketch the graph of f and state whether or not the graph of f passes the horizontal line test. If so, find a rule for f^{-1} and then use it to find $\left(f^{-1}\right)'(0)$. Verify the formula

$$\left(f^{-1}\right)'(0) = \frac{1}{f'\left(f^{-1}(0)\right)}.$$

Solution. Notice that the graph of f passes the horizontal line test, which implies that f is a one-to-one function and thus must have an inverse function. To find the inverse function of f, set $y = f(x)$. Then simply switch the variables x and y and solve for y. The resulting equation will be a rule for the function f^{-1}. To see this, let $y = \frac{2x-3}{7-5x}$ and then switching the variables x and y: $x = \frac{2y-3}{7-5y}$. Solving for y yields: $y = \frac{3+7x}{2+5x}$. Thus, $f^{-1}(x) = \frac{3+7x}{2+5x}$. Of course since f and $f^{-1}(x)$ are inverse function it must happen that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$ for each x in the domains. Notice that

$$f'(x) = -\frac{1}{(7-5x)^2} \qquad \text{and} \qquad (f^{-1})'(x) = -\frac{1}{(2+5x)^2}.$$

Also notice that

$$(f^{-1})'(0) = -\frac{1}{(2+5(0))^2} = -\frac{1}{4} = \frac{1}{f'\left(\frac{3}{2}\right)} = \frac{1}{f'\left(f^{-1}(0)\right)}.$$

Example 2.43. Let $f(x) = \dfrac{5x}{1-2x}$. Without finding a rule for $f^{-1}(x)$ determine $\left(f^{-1}\right)'(1)$.

Solution. Since $f'(x) = \dfrac{5}{(1-2x)^2}$ and $1 = \dfrac{5x}{1-2x}$ when $x = \frac{1}{7}$

$$\left(f^{-1}\right)'(1) = \frac{1}{f'\left(f^{-1}(1)\right)} = \frac{1}{f'(1/7)} = \frac{1}{\frac{49}{5}} = \frac{5}{49}.$$

2.34 Derivatives of Exponential Functions

Theorem 2.7. *The derivative of the* ***exponential function*** $f(x) = b^x$ *is* $f'(x) = (\ln b)b^x$. *In the special case when* $b = e$ *we have* $f(x) = e^x$ *and* $f'(x) = e^x$. *Therefore,*

$$\frac{d}{dx}(b^x) = (\ln b)b^x \qquad \textit{and} \qquad \frac{d}{dx}(e^x) = e^x.$$

Example 2.44. Find the derivative of the function given $y(x) = x^2 - 3e^x$.

Solution. The derivative is $\frac{dy}{dx} = 2x - 3e^x$.

Example 2.45. Find the derivative of the function given $y(x) = 1 - 2e^{-x^2}$.

Solution. Using the chain rule the derivative is,

$$\frac{dy}{dx} = -2e^{-x^2}\frac{d}{dx}(-x^2) = -2e^{-x^2}(-2x) = 4xe^{-x^2}.$$

Example 2.46. What is the slope of the tangent line to $y = \dfrac{e^{-x}}{1+e^{-x}}$ at $x = 0$?

Solution. Using the quotient rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{(1+e^{-x})(-e^{-x}) - e^{-x}(-e^{-x})}{(1+e^{-x})^2} \\ &= \frac{-e^{-x}(1+e^{-x}-e^{-x})}{(1+e^{-x})^2} = \frac{-e^{-x}}{(1+e^{-x})^2} = -\frac{e^{x}}{(1+e^{x})^2}\end{aligned}$$

Therefore, the slope of the tangent line to y at $x=0$ is $\frac{-e^{-(0)}}{(1+e^{-(0)})^2} = -\frac{1}{4}$.

2.35 Derivatives of Logarithmic Functions

Theorem 2.8. *The derivative of the **logarithmic function** $f(x) = \log_b x$ is $f'(x) = 1/(\ln b)x$. In the special case when $b = e$ we have $f(x) = \ln x$ and $f'(x) = 1/x$. Therefore,*

$$\frac{d}{dx}(\log_b x) = \frac{1}{(\ln b)x} \qquad \textit{and} \qquad \frac{d}{dx}(\ln x) = \frac{1}{x}.$$

Example 2.47. Find the equation of the tangent line to the curve $y = x^2 \ln x$ at $x = e^2$.

Solution. The derivative of y is $y' = 2x\ln x + x$ and at $(e^2, 2e^4)$ we have the slope of the tangent line as $y'(e^2) = 2(e^2)\ln(e^2) + (e^2) = 5e^2$. Therefore, the equation of the tangent line is $y - 2e^4 = 5e^2(x - e^2)$ which simplifies to $y = 5e^2x - 3e^4$.

Example 2.48. For what values of A and B does $y = Ax\ln x + Be^x$ satisfy $y'' - y = 0$?

Solution. We determine, $y' = A\ln x + A + Be^x$ $y" = \frac{A}{x} + Be^x$. Since $y'' - y = \frac{A}{x} - Ax\ln x = 0$ we find that $A = 0$ and that B can be any real number.

Example 2.49. Find the derivative of the function given by $y(x) = 5^{2x^2}\ln(4x)$.

Solution. Using the product rule and the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(5^{2x^2}\right)\ln(4x) + 5^{2x^2}\frac{d}{dx}(\ln(4x)) \\ &= \left(5^{2x^2}\ln(5)(4x)\right)\ln(4x) + 5^{2x^2}\left(\frac{1}{4x}(4)\right)\end{aligned}$$

Example 2.50. Find the derivative of the function given by $y(x) = \ln(4x+9)$.

Solution. Using the chain rule,

$$\frac{dy}{dx} = \frac{1}{4x+9}\frac{d}{dx}(4x+9) = \frac{4}{4x+9}.$$

Example 2.51. Find the derivative of the function given by

$$y(x) = \ln\left(\frac{x^3}{x+1}\right).$$

Solution. Here we simplify first to have,

$$y(x) = \ln\left(\frac{x^3}{x+1}\right) = \ln\left(x^3\right) - \ln(x+1) = 3\ln(x) - \ln(x+1)$$

and then using derivative rules,

$$\frac{dy}{dx} = \frac{3}{x} - \frac{1}{x+1} = \frac{2x+3}{x(x+1)}.$$

Example 2.52. Find the derivative of the function given by

$$y(x) = \log_2\left(\frac{3x+2}{x^2-5}\right)^{1/4}.$$

Solution. Here we simplify first to have,

$$y(x) = \log_2\left(\frac{3x+2}{x^2-5}\right)^{1/4} = \frac{1}{4}\log_2(3x+2) - \frac{1}{4}\log_2\left(x^2-5\right)$$

and then using derivative rules,

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{4(3x+2)\ln(2)}\frac{d}{dx}(3x+2) - \frac{1}{4\left(x^2-5\right)\ln(2)}\frac{d}{dx}\left(x^2-5\right)\\ &= \frac{3}{4(3x+2)\ln(2)} - \frac{2x}{4\left(x^2-5\right)\ln(2)} = \frac{-3x^2-4x-15}{4(3x+2)\left(x^2-5\right)\ln(2)}\end{aligned}$$

Example 2.53. Find the derivative of the function given

$$s(t) = \log_5\left(\frac{t^2+3}{\sqrt{1-t}}\right).$$

::: {.solution } Here we simplify first to have,

$$y(x) = \log_5\left(\frac{t^2+3}{\sqrt{1-t}}\right) = \log_5\left(t^2+3\right) - \frac{1}{2}\log_5(1-t)$$

and then using derivative rules,

$$\frac{dy}{dx} = \frac{2t}{(t^2+3)\ln(5)} - \frac{(-1)}{2\ln(5)(1-t)} = \frac{3t^2-4t-3}{2(t-1)\left(t^2+3\right)\log(5)}$$

:::

2.36 Derivatives of Inverse Trigonometric Functions

Theorem 2.9. *The* ***inverse trigonometric functions*** *arcsine, arccosine, arctangent, arccotangent, arccosecant, and arcsecant are all differentiable functions on their domain and their derivative functions are:*

$$\begin{aligned}
\frac{d}{dx}\sin^{-1}x &= \frac{1}{\sqrt{1-x^2}} & \frac{d}{dx}\cos^{-1}x &= -\frac{1}{\sqrt{1-x^2}}\\
\frac{d}{dx}\tan^{-1}x &= \frac{1}{1+x^2} & \frac{d}{dx}\cot^{-1}x &= -\frac{1}{1+x^2}\\
\frac{d}{dx}\sec^{-1}x &= \frac{1}{|x|}\frac{1}{\sqrt{x^2-1}} & \frac{d}{dx}\csc^{-1}x &= -\frac{1}{|x|}\frac{1}{\sqrt{x^2-1}}
\end{aligned}$$

Example 2.54. Find the derivative of the function $y = \sin^{-1}x\cos^{-1}x$.

Solution. Using the product rule and the derivative formulas for arcsine and arccosine we determine:

$$y' = \left(\frac{1}{\sqrt{1-x^2}}\right)\left(\cos^{-1}x\right) - \left(\frac{1}{\sqrt{1-x^2}}\right)\left(\sin^{-1}x\right) = \frac{\cos^{-1}x - \sin^{-1}x}{\sqrt{1-x^2}}$$

Example 2.55. Find the derivative of the function $y = \tan^{-1}x\cot^{-1}x$

Solution. Using the product rule and the derivative formulas for arctangent and arccotangent we determine:

$$y' = \left(\frac{1}{1+x^2}\right)\cot^{-1}x - \left(\frac{1}{1+x^2}\right)\tan^{-1}x = \frac{\cot^{-1}x - \tan^{-1}x}{1+x^2}$$

Example 2.56. Find the derivative of the function $y = \tan^{-1}x - x\sec^{-1}x$

Solution. Using the product rule and the derivative formulas for arctangent and arcsecant we determine:

$$y' = \left(\frac{1}{1+x^2}\right) - \sec^{-1} x - x\left(\frac{1}{|x|}\frac{1}{\sqrt{x^2-1}}\right)$$

Recall, the **cofunction theorem** from trigonometry: if $\alpha = \cos^{-1} x$ and $\beta = \sin^{-1} x$ then $\cos\alpha = \sin\beta$ if and only if $\alpha + \beta = \pi/2$.

Example 2.57. Find the derivative of the function $y = \dfrac{\sin^{-1} x}{\cos^{-1} x}$.

Solution. Using the quotient rule, the derivative formulas for arcsine and arccosine and some trigonometric identities,

$$\begin{aligned} y' &= \frac{(\cos^{-1} x)\left(\frac{1}{\sqrt{1-x^2}}\right) - \left(\sin^{-1} x\right)\left(\frac{-1}{\sqrt{1-x^2}}\right)}{\left(\cos^{-1} x\right)^2} \\ &= \frac{\cos^{-1} x + \sin^{-1} x}{\left(\cos^{-1} x\right)^2\sqrt{1-x^2}} = \frac{\pi}{2\sqrt{1-x^2}\left(\cos^{-1} x\right)^2} \end{aligned}$$

2.37 Exercises

Exercise 2.78. Let $f(x) = \frac{2x-3}{7-5x}$. Sketch the graph of f and state whether or not the graph of f passes the *horizontal line test*. If so, find a rule for f^{-1} and then use it to find $\left(f^{-1}\right)'(0)$. Verify the formula $\left(f^{-1}\right)'(0) = \frac{1}{f'(f^{-1}(0))}$.

Exercise 2.79. Let $f(x) = \frac{5x}{1-2x}$. Without finding a rule for $f^{-1}(x)$ determine $\left(f^{-1}\right)'(1)$.

Exercise 2.80. Find $\frac{dy}{dx}$ given $y = \frac{e^{2x}}{\left(x^2-3\right)^2 \ln\sqrt{x}}$.

Exercise 2.81. Find $\frac{dy}{dx}$ given $y = x^{\sin x}$.

Exercise 2.82. Determine whether the following functions are one-to-one.

- $f(x) = x^2 + 4$
- $f(x) = 2x^3 - 4$

- $f(x) = -54 + 54x - 15x^2 + 2x^3$
- $f(x) = \frac{2x-3}{x-7}$

Exercise 2.83. Determine whether or not the given functions are inverses of each other or not.
 - $f(x) = x^3 - 4$ and $g(x) = \sqrt[3]{x+4}$ - $f(x) = x^2 + 5, x \leq 0$ and $g(x) = -\sqrt{x-5}, x \geq 5$

Exercise 2.84. Find the inverse of the given function, it if exists.

- $f(x) = \frac{4x}{x-2}$
- $f(x) = \left(x^3 + 1\right)^5$
- $f(x) = x^2 - 6x, x \geq 3$
- $f(x) = \frac{x}{x^2-2}$

Exercise 2.85. Solve the following equations for real x and give the number of solutions you find for each one. Also clearly state any extraneous solutions that you find for each one.

- $\log_2 x = \log_4 5 + 3\log_2 3$
- $\left(\sqrt[3]{5}\right)^{x+2} = 5^{x^2}$
- $\log_5 |x+3| + \log_5 |x-3| = 1$
- $\frac{1}{e^{x+3}+6e^{-x-3}} = \frac{1}{5}$

Exercise 2.86. Given $f(x) = 5 - 4x$ and $a = 1/2$ find $f^{-1}(x)$ and graph f and f^{-1} together. Then evaluate $\frac{df}{dx}$ at $x = a$ and $\frac{df^{-1}}{dx}$ at $x = f(a)$ to show that at these points $\frac{df^{-1}}{dx} = \frac{1}{\left(\frac{df}{dx}\right)}$.

Exercise 2.87. Show that $h(x) = x^3 \big/ 4$ and $k(x) = (4x)^{1/3}$ are inverses of one another. Graph h an k over an x-interval large enough to show the graphs intersecting at $(2,2)$ and $(-2,-2)$. Find the slopes of the tangents to the graphs at h and k at $(2,2)$ and $(-2,-2)$. What lines are tangent to the graphs at h and k at $(2,2)$ and $(-2,-2)$.

Exercise 2.88. Suppose that the differentiable function $y = f(x)$ has an inverse and that the graph of f passes through the point $(2,4)$ and has a slope of $1/3$ there. Find the value of $\frac{df^{-1}}{dx}$ at $x = 4$.

2.38 Related Rates

In this topic we show how implicit differentiation and the chain rule can be used to calculate the rate of change of one variable in terms of the rate of change of another variable (which may be more easily measured).

The procedure of solving a **related rates** problem is to find an equation that relates two quantities and then use the chain rule to differentiate both sides with respect to time. Then, from knowing the rate of change of one value at a point in time, we can calculate the rate of change of another quantity at that moment in time.

The equation which describes the application is often derived from a verbal description and is called a **mathematical model** of the problem. For these equations we can relate different rates of change by using the chain rule and implicit differentiation.

More precisely, given a function $y = f(x)$ where both x and y are functions of time t, we have $x = x(t)$ and $y = y(t)$; and so we can use the chain rule and implicit differentiation to determine $\frac{dy}{dt}$ and $\frac{dx}{dt}$. Knowing one of these values we can calculate the other as demonstrated in the following examples.

Example 2.58. Find $\frac{dy}{dt}$ when $x = 7$ given $y = 5\sqrt{x+9}$ and $\frac{dx}{dt} = 2$.

Solution. Differentiating with respect to t and using the chain rule,

$$\frac{dy}{dt} = \frac{5}{2\sqrt{x+9}}\frac{dx}{dt}$$

and so when $x = 7$,

$$\left.\frac{dy}{dt}\right|_{x=7} = \frac{5}{2\sqrt{7+9}}(2) = \frac{5}{4}.$$

Example 2.59. Find $\frac{dy}{dt}$ when $x = 1$ given $5xy = 10$ and $\frac{dx}{dt} = -2$.

Solution. Differentiating with respect to t and using the chain rule,

$$x\frac{dy}{dt} + y\frac{dx}{dt} = 0$$

and so when $x = 1$, then $y = 2$ and

$$(1)\frac{dy}{dt} + (2)(-2) = 0.$$

Therefore,

$$\left.\frac{dy}{dt}\right|_{x=1} = 4.$$

2.39 General vs Specific Situation

Every **related rates** problem has a general situation (properties that hold true at every instant in time) and a specific situation (properties that hold true at a particular instant in time). Distinguishing between these two situations is often the key to successfully solving a related-rates problem. Here are a few guidelines for solving related rate problems:

- Read the problem carefully and identify all given quantities and unknown quantities to be determined.
- Introduce notation, make a sketch, and label the quantities.
- Write equations involving the variables whose rates of change either are given or are to be determined.
- Use implicit differentiation, by differentiating both sides with respect to he appropriate variable.
- Substitute known variables and known rates of change into the resulting equation to determine the missing rate of change.

Example 2.60. Model a water tank by a cone 40 ft high with a circular base of radius 20 ft at the top. Water is flowing into the tank at a constant rate of 80 $\mathrm{ft}^3 \big/ \min$. How fast is the water level rising when the water is 12 feet deep?

Solution. Let x be the radius of the top circle of the body of water and y its height. The radius of the top circle is 20, the height of the cone is 40 ft. By similar triangles, $20/40 = x/y$ and so $x = \frac{1}{2}y$. The volume of the body of water is

$$V = \frac{1}{3}\pi x^2 y = \frac{1}{12}\pi y^3.$$

Then

$$\frac{dV}{dt} = \frac{1}{4}\pi y^2 \frac{dy}{dt}.$$

When $y = 12$, $x = 6$, and $\frac{dV}{dt} = 80$, we have

$$80 = \frac{1}{4}\pi(12)^2 \frac{dy}{dt}$$

and thus

$$\frac{dy}{dt} = \frac{80(4)}{\pi(144)}\mathrm{ft.}/\min = \frac{20}{9\pi}\mathrm{ft.}/\min \approx 0.71\mathrm{ft.}/\min.$$

Example 2.61. Air is being pumped into a spherical balloon at a rate of 4.5 cubic feet per minute. Find the rate of change of the radius when the radius is 2 feet.

Solution. Let V be the volume of the balloon and let r be the radius. Because the volume is increasing at a rate of 4.5 cubic feet per minute, we know that at time t the rate of change of the volume is $\frac{dV}{dt} = \frac{9}{2}$. The equation that relates the radius r to the volume V is $V = \frac{4}{3}\pi r^3$. So we have

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

and by solving for $\frac{dr}{dt}$ we have

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\left(\frac{dV}{dt}\right).$$

Finally, when $r = 2$, the rate of change of the radius is

$$\frac{dr}{dt} = \frac{1}{16\pi}\left(\frac{9}{2}\right) \approx 0.09$$

foot per minute.

Example 2.62. A pebble is dropped into a calm pond, causing ripples in the form of concentric circles. The radius of the outer ripple is increasing at a constant rate of 1 foot per second. When the radius is 4 feet, at what rate is the total area A of the disturbed water changing?

Solution. For the area of a circle we use $A = \pi r^2$ and we differentiate implicitly with respect to time t leading to

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

by using the chain rule. Since $\frac{dr}{dt} = 1$ and when $r = 4$ we have

$$\frac{dA}{dt} = 2\pi(4)(1) = 8\pi.$$

Example 2.63. A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?

Solution. Let x meters be the distance from the bottom of the ladder to the wall and y meters the distance from the top of the ladder to the ground. Since x and y are functions of time and $\frac{dx}{dt} = 1$ ft/s we are asked to find

$\frac{dy}{dt}$ when $x = 6$ ft. The relationship between x and y is the Pythagorean Theorem, namely $x^2 + y^2 = 100$. Using implicit differentiation and the chain rule we have

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0.$$

By solving for $\frac{dy}{dt}$ we have

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}.$$

When $x = 6$ then \$y=8 \$,

$$\frac{dy}{dt} = \frac{-6}{8}(1) = \frac{-3}{4} \text{ ft/s}.$$

Example 2.64. Car A is going west at 50 mi/h and car B is going north at 60 mi/h. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 0.3 mi and car B is 0.4 mi from the intersection?

Solution. At a given time t, let x be the distance from car A to the point of intersection P and let y be the distance from car B to P and let z be the distance between the cars, where x, y, and z are measured in miles. Since x and y are decreasing we take the derivatives to be negative and so we are given

$$\frac{dx}{dt} = -50 \text{ mi/h} \qquad \text{and} \qquad \frac{dy}{dt} = -60 \text{ mi/h}.$$

To find $\frac{dz}{dt}$ we use the Pythagorean Theorem, namely $x^2 + y^2 = z^2$ and differentiate with respect to time t. We have

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

and by solving for $\frac{dz}{dt}$ we have

$$\frac{dz}{dt} = \frac{1}{z}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right).$$

Now when $x = 0.3$ mi and $y = 0.4$ mi we have $z = 0.5$ mi and so

$$\frac{dz}{dt} = \frac{1}{0.5}[0.3(-50) + 0.4(-60)] = -78 \text{ mi/h}.$$

Therefore, the cars are approaching each other at a rate of 78 mi/h.

Example 2.65. A person 6 ft tall walks away from a streetlight at the rate of 5 ft/s. If the light is 18 ft above ground level, how fast is the person's shadow lengthening?

Solution. Let x be the length of the shadow and y be the distance of the person from the street light. Using similar triangles, $x/6 = (x+y)/18$ and by solving for y we have $y = 2x$. Thus, $dy/dt = 2(dx/dt)$ and given that $dy/dt = 5\text{ft}/s$ we find that $dx/dt = 2.5\text{ft}/s$ is the rate the shadow is lengthening.

Example 2.66. At noon, a ship sails due north from a point P at 8 knots. Another ship, sailing at 12 knots, leaves the same point 1 h later on a course 60° east of north. How fast is the distance between the ships increasing at 5 P.M.?

Solution. Let A be the distance travelled by the first ship, and B for the distance travelled by the second ship, D for the distance between them, and θ the constant angle of 60°. We need to find $\frac{dD}{dt}$ at $t = 5$. The equation that relates all the variables is the law of cosines and is $D^2 = A^2 + B^2 - 2AB\cos\ 60^\circ$.
Using the chain rule and implicit differentiation we have

$$2D\frac{dD}{dt} = 2A\frac{dA}{dt} + 2B\frac{dB}{dt} - 2\left(A\frac{dB}{dt} + B\frac{dA}{dt}\right)\left(\frac{1}{2}\right)$$

since $\cos 60^\circ = \frac{1}{2}$. Then at $t = 5$, $A = 5(8) = 40$, $B = 12(4) = 48$, $\frac{dA}{dt} = 8$, $\frac{dB}{dt} = 12$, and

$$D = \sqrt{40^2 + 48^2 - 2(40)(48)\left(\frac{1}{2}\right)} = \sqrt{1984}.$$

So we have,

$$\begin{aligned} \frac{dD}{dt} &= \frac{2(40)(8) + 2(48)(12) - (40)(12) - (48)(8)}{2\sqrt{1984}} \\ &= \frac{58}{\sqrt{31}} \text{ knots} \approx 10.4171 \text{ knots.} \end{aligned}$$

2.40 Exercises

Exercise 2.89. Suppose that the radius r and surface area $S = 4\pi r^2$ of a sphere are differentiable functions of t. Write an equation that relates $\frac{dS}{dt}$ to $\frac{dr}{dt}$.

Exercise 2.90. If x, y, and z are lengths of the edges of a rectangular box, the common length of the box's diagonals is $s = \sqrt{x^2 + y^2 + z^2}$. (a) Assuming that x, y, and z are differentiable functions of t, how is ds/dt related to dx/dt, dy/dt, and dz/dt.

Exercise 2.91. The length l of a rectangle is decreasing at the rate of cm/ sec while the width w is increasing at the rate of 2cm/ sec. When $l = 12$cm and $w = 5$cm, find the rates of change of (a) the area, (b) the perimeter, (c) the lengths of the diagonals of the rectangle. Which of these quantities are decreasing and which are increasing?

Exercise 2.92. The coordinates of a particle in the metric xy-plane are differentiable functions of time t with $dx/dt = -1m/s$ and $dy/dt = -5m/s$. How fast is the particle's distance from the origin changing as it passes through the point $(5, 12)$?

Exercise 2.93. A 13-ft ladder is leaning against a house when its base starts to slide away. By the time the base is 12 ft from the house, the base is moving at the rate of 5ft/ sec. (a) How fast is the top of the ladder sliding down the wall then? (b) At what rate is the area of the triangle formed by the ladder, wall, and ground changing then? (c) At what rate is the angle θ between the ladder and the ground changing then?

Exercise 2.94. Sand falls from a conveyor belt at the rate of 10 $m^3 \big/$ min onto the top of a conical pile. The height of the pile is always three-eights of the base diameter. How fast are the (a) height and (b) radius changing when the pile is $4m$ high?

Exercise 2.95. Suppose that a drop of mist is a perfect sphere and that, through condensation, the drop picks up moisture at a rate proportional to is surface area. Show that under thee circumstances the drop's radius increase at a constant rate.

Exercise 2.96. A balloon is rising vertically above a level straight road at a constant rate of 1ft/ sec. Just when the balloon is 65ft above the ground, a bicycle moving at a constant rate of 17ft/ sec passes under it. How fast is the distance $s(t)$ between the bicycle and balloon increasing 3 sec later?

Exercise 2.97. A man 6 ft all walks at a rate of 5ft/ sec toward a street-light that is 16 ft above the ground. At what rates the tip of his shadow moving? At what rate is the length of his shadow changing when he is 10 ft from the base of the light?

Exercise 2.98. Two commercial airplanes are flying at 40,000 ft along straight-line courses that intersect at right angles. Plane A is approaching the intersection point at a speed of 442 knots. Plane B is approaching the intersection at 481 knots. At what rate is the distance between the planes changing when A is nautical miles from the intersection point and B is 12 nautical miles from the intersection point?

Exercise 2.99. All edges of a cube are expanding at a rate of 3 centimeters per second. How fast is the volume changing when each side is (a) 1 centimeter and (b) 10 centimeters?

Exercise 2.100. A conical tank (with vertex down) is 10 feet across the top and 12 feet deep. If water is flowing into the tank at a rate of 10 cubic feet per minute, find the rate of change of the depth of the water when the water is 8 feet deep?

Exercise 2.101. A balloon rises at a rate of 3 meters per second from a point on the ground 30 meters from an observer. What is the rate of change of the angle of elevation of the balloon from the observer when the balloon is 30 meters above the ground?

Exercise 2.102. A trough is 10 feet long and its ends have the shape of an isosceles triangles that are 3 feet across at the top and have a height of 1 foot. If the trough is filled with water at a rate of 2 $\mathrm{ft}^3 \big/ \mathrm{min}$, how fast does the water level rise when the water is 6 inches deep?

2.41 Linearization and Differentials

2.42 Differentials

In this section differentials are motivated, defined, and then used to approximate real numbers. The linearization of a function around a point (via the tangent line) is illustrated.

Let $y = f(x)$ where f is a differentiable function. The **differential** dy represents the amount that the tangent line rises or falls, whereas Δy represents the amount that the curve $y = f(x)$ rises or falls when x changes by an amount dx. Since

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

we have

$$\frac{\Delta y}{\Delta x} \approx \frac{dy}{dx} \qquad \text{whenever} \qquad \Delta x \approx 0$$

If we take $dx = \Delta x$, then we have $\Delta y \approx dy$ which says that the actual change in y is approximately equal to the differential dy.

If $f(x_1)$ is a known number and it is desired to calculate an approximate value for $f(x_1 + \Delta x)$ where Δx is small, then

$$f(x_1 + \Delta x) \approx f(x_1) + dy.$$

Definition 2.3. If $y = f(x)$ is a differentiable function then the **differential** dy is defined by the equation $dy = f'(x)dx$ where dx is an independent variable.

Example 2.67. Find the differential for $y = \sqrt[4]{x}$.

Solution. For $y = \sqrt[4]{x}$ the derivative is $\frac{dy}{dx} = \frac{1}{4x^{3/4}}$ and so the differential of y is

$$dy = \left(\frac{1}{4x^{3/4}}\right) dx.$$

Example 2.68. Find the differential for $y = \left(x^2 - 2x - 3\right)^{10}$.

Solution. For $y = \left(x^2 - 2x - 3\right)^{10}$ the derivative is

$$\frac{dy}{dx} = 10(-2 + 2x)\left(-3 - 2x + x^2\right)^9$$

and so the differential of y is

$$dy = \left(10(-2 + 2x)\left(-3 - 2x + x^2\right)^9\right) dx.$$

Example 2.69. Find the differential for $y = \sqrt{x + \sqrt{2x - 1}}$.

Solution. For $y = \sqrt{x + \sqrt{2x - 1}}$ the derivative is

$$\frac{dy}{dx} = \frac{1 + \sqrt{-1 + 2x}}{2\sqrt{-1 + 2x}\sqrt{x + \sqrt{-1 + 2x}}}$$

and so the differential of y is

$$dy = \left(\frac{1 + \sqrt{-1 + 2x}}{2\sqrt{-1 + 2x}\sqrt{x + \sqrt{-1 + 2x}}} \right) dx.$$

2.43 Approximating Decimals

Example 2.70. Use differentials to approximate the real number $\sqrt[3]{218}$.

Solution. If $y = f(x) = \sqrt[3]{x}$, then $dy = \left(\frac{1}{3x^{2/3}}\right) dx$ and using $dx = \Delta x$ $=$ 218-216 $=$ 2 the linear approximation is,

$$\sqrt[3]{218} = f(216{+}2) \approx f(216){+}dy = 6{+}\left(\frac{1}{3(216)^{2/3}} \right)(2) = \frac{325}{54} = 6.01852.$$

Example 2.71. Use differentials to approximate the real number $\sqrt[3]{1.02} + \sqrt[4]{1.02}$.

Solution. If $y = f(x) = x^{1/3} + x^{1/4}$, then $dy = \left(\frac{1}{4x^{3/4}} + \frac{1}{3x^{2/3}}\right) dx$ and using $dx = \Delta x =$ 1.02-1 $=$ 0.02 the linear approximation is,

$$\sqrt[3]{1.02}{+}\sqrt[4]{1.02} = f(1{+}0.02) \approx f(1){+}dy = 2{+}\left(\frac{1}{4} + \frac{1}{3}\right)(0.02) = \frac{1207}{600} = 2.011\overline{6}$$

2.44 Linearization

Definition 2.4. The approximation

$$f(x) \approx f(a) + f'(a)(x - a)$$

is called the **linear approximation** (or sometimes the **tangent line approximation**) of f at a and the function

$$L(x) = f(a) + f'(a)(x - a)$$

is called the **linearization** of f at a.

The equation of the tangent line to the curve $y = f(x)$ at $(a, f(a))$ is $y = f(a) + f'(a)(x - a)$ which is $y = f(a) + f'(a)dx$ so that in fact we have $y = f(a) + dy$. Thus when using differentials to approximate, that is, when using $f(x+\Delta x) \approx f(x) + dy$ to approximate we are using the tangent line at $(a, f(a))$ as an approximation to the curve $y = f(x)$ when a is near x.

Next we find the linearization for a given function at a given value.

Example 2.72. Find the linearization of $f(x) = \dfrac{1}{\sqrt{2+x}}$ at $x_1 = 0$.

Solution. The linearization of the function $f(x) = \dfrac{1}{\sqrt{2+x}}$at $x_1 = 0$ is

$$
\begin{aligned}
L(x) &= f(x_1) + f'(x_1)(x - x_1) \\
L(x) &= f(0) + \left[-\frac{1}{2(2+0)^{3/2}}\right](x-0) \\
L(x) &= \frac{1}{\sqrt{2}} + \left(-\frac{1}{4\sqrt{2}}\right)x
\end{aligned}
$$

Therefore, we have the linear approximation

$$
\frac{1}{\sqrt{2+x}} \approx \frac{\sqrt{2}}{2} + \left(-\frac{\sqrt{2}}{8}\right)x
$$

for when x is near 0.

Example 2.73. Find the linearization of $f(x) = \dfrac{1}{(1+2x)^4}$ at $x_1 = 0$.

Solution. The linearization of the function $f(x) = \dfrac{1}{(1+2x)^4}$at $x_1 = 0$ is

$$
L(x) = f(x_1) + f'(x_1)(x - x_1) = f(0) + \left[-\frac{8}{(1+2(0))^5}\right](x-0) = 1 - 8x.
$$

Therefore, we have the linear approximation

$$
\frac{1}{(1+2x)^4} \approx 1 - 8x
$$

for when x is near 0.

2.45 Exercises

Exercise 2.103.

(a) Given $f(x)=x^2+2x$, $x_0=1$, and $\Delta x=0.1$, find the change $\Delta f=f(x_0+\Delta x)-f(x_0)$ and the value of the estimate $df=f'(x_0)\,dx$.
(b) Given $f(x)=x^4$, $x_0=1$, and $\Delta x=0.1$, find the change $\Delta f=f(x_0+\Delta x)-f(x_0)$ and the value of the estimate $df=f'(x_0)\,dx$.
(c) Given $f(x)=x^3-2x+3$ $x_0=2$, and $\Delta x=0.1$, find the change $\Delta f=f(x_0+\Delta x)-f(x_0)$ and the value of the estimate $df=f'(x_0)\,dx$.

Exercise 2.104. Write a differential formula that estimates the change in the volume $V=\frac{4}{3}\pi r^3$ of a sphere when the radius changes from r_0 to r_0+dr.

Exercise 2.105. Write a differential formula that estimates the change in the lateral surface area $S=\pi r\sqrt{r^2+h^2}$ of a right circular cone when the radius changes from r_0 to r_0+dr.

Exercise 2.106. Find the differential dy of the functions

- $y=f(x)=\frac{1}{3}\cos\left(\frac{6\pi x-1}{2}\right)$
- $y=f(x)=\sqrt{\sqrt{x}+\frac{1}{\sqrt{x}}}$.

Exercise 2.107. Find the differential dy of each of the following functions

- $y=f(x)=\frac{2x}{1+x^2}$
- $xy^2-4x^{3/2}-y=0$
- $y=f(x)=4\tan\left(\frac{x^2}{3}\right)$
- $y=f(x)=2\cot\left(\frac{1}{\sqrt{x}}\right)$
- $y=f(x)=xe^{-x}$
- $y=f(x)=\ln\left(1+x^2\right)$

Exercise 2.108. Use differentials to find an approximate value for the following real numbers
- $\cos 31.5^\circ$ - $\sqrt[4]{624}$ - $(2.99)^3$.

Exercise 2.109. Find the linearization for each of the following functions at the given value of x.

- $f(x)=\frac{1}{\sqrt{2+x}}$ at $x_1=0$
- $f(x)=\frac{1}{(1+2x)^4}$ at $x_1=0$
- $f(x)=x+\frac{1}{x}$ at $x=a$

- $f(x) = x^2$ at $x = a$
- $f(x) = \cos x$ at $x = a$
- $f(x) = \tan x$ at $x = a$
- $f(x) = e^x$ at $x = a$
- $f(x) = \ln(1 + x)$ at $x = a$
- $f(x) = x + \frac{1}{x}$ at $x = a$
- $f(x) = 2x^2 + 4x - 3$ at $x_0 = -0.9$
- $f(x) = \frac{x}{x+1}$ at $x_0 = 1.3$

Exercise 2.110. Show that the linearization of $f(x) = (1+x)^k$ at $x = 0$ is $L(x) = 1 + kx$ for a constant k.

Exercise 2.111. Find the linearization of $f(x) = \sqrt{x+1} + \sin x$ at $x = 0$. How is it related to the individual linearization of $\sqrt{x+1}$ and $\sin x$ at $x = 0$?

Chapter 3

Applications Of Derivatives

This book explores different applications of differentiation in order to provide a better understanding of how it can be used to solve problems. It is written for those who have an understanding of basic calculus but want to learn more about how differentiation can be applied in other areas. The author provides detailed examples and explanations which make the material easy to understand. Overall, this book provides a comprehensive look at differentiation and its many applications.

Differentiation is a process that allows us to find the derivative of a function. The derivative can be used to calculate how a function changes at any given point, and it's an essential tool for understanding how physical systems work. In this book, we'll discuss what differentiation is and how it works, as well as some of its most common applications.

Differentiation is a fundamental tool of calculus that has countless applications in the real world. Optimization problems are one type of problem where differentiation can be used to great effect. In an optimization problem, the goal is to find the maximum or minimum value of a function. To do this, we take the derivative of the function and set it equal to zero. This will give us the critical points of the function, which are the points where the function is either at a maximum or minimum. From there, we can use some more calculus to determine which of these critical points is actually the point of maximum or minimum for the function. Optimization problems can be applied in many different fields, such as engineering, physics, and economics. In each case, using calculus to find the optimum solution can save time and resources.

One of the most common applications of differentiation is finding the extreme values of a function. Extreme values can tell us things like the highest and lowest points on a graph, or the fastest and slowest points in a motion. To find extreme values, we need to take the derivative of the function and set it equal to zero. This will give us the points where the function is either increasing or decreasing at its fastest rate.

Applications of Differentiation include finding extreme values of a function. The derivative of a function at a point measures the rate of change of the function at that point. So, the derivative can be used to find points where the function is increasing or decreasing at its fastest. These points are known as local extrema, and they can be found by setting the derivative equal to zero and solving. Differentiation can also be used to find points where the function changes from increasing to decreasing (or vice versa). These points are called inflection points, and they can be found by setting the second derivative equal to zero and solving. In addition, differentiation can be used to find absolute extrema, which are the largest and smallest values that a function can take on over a given domain. To find absolute extrema, we need to take into account both the function's values and its derivatives. So, applications of differentiation are not just limited to finding extreme values; they also allow us to understand the behavior of a function at different points.

The Mean Value Theorem is another important application of differentiation. This theorem states that if a function is continuous on a closed interval, then there exists at least one point in the interval where the function's derivative is equal to the average rate of change of the function over the interval. In other words, if we take any two points in an interval and find the average rate of change of the function between those points, there will be at least one point in the interval where the function's derivative is equal to that average rate of change. This theorem has a number of important implications, and we'll discuss some of them in this chapter.

The Mean Value Theorem is one of the most important results in calculus. And it's not just because it's on the AP Calculus exam. The theorem has countless applications in mathematics, science, and engineering. In fact, it's often referred to as the "workhorse" of calculus. Differentiation is all about finding rates of change. The Mean Value Theorem tells us that for any function that is continuous on an interval, there exists a point where the rate of change is equal to the average rate of change over the entire interval. In other words, the theorem provides a way to find the "average" rate of change. Of course, the theorem is much more than just a simple averaging result. It has profound implications for how we understand and work with functions. In particular, it gives us a way to estimate values and understand relationships between different quantities. As one famous mathematician once said, the Mean Value Theorem is "the most useful single theorem in all of mathematics." With so many applications, it's easy to see why!

A function is monotonic if it either always increases or always decreases as we move from left to right on the graph. In other words, a function is monotonic if its derivative is either always positive or always negative. To see why this is true, recall that the derivative at a point measures the rate of change of the function at that point. So, if the derivative is always positive, then the function is always increasing; if the derivative is always negative, then the function is always decreasing. It's easy to see that a constant function has a derivative of zero, so it is neither increasing nor decreasing.

There are two types of monotonic functions: Increasing functions are those whose values always get larger as we move from left to right. Decreasing functions are those whose values always get smaller as we move from left to right. It's important to note that a function can be increasing or decreasing without being continuous. In fact, many discontinuous functions are monotonic.

Applications of Differentiation can be broadly classified into two categories: Monotonic Functions and Non-monotonic Functions. A function is said to be monotonic if it either always increases or always decreases as we move along the x-axis. On the other hand, a function is non-monotonic if it changes direction at least once. Examples of monotonic functions include linear functions and exponential functions. On the other hand, examples of non-monotonic functions include polynomial functions and trigonometric functions. Applications of differentiation to monotonic functions are typically concerned with finding the maximum or minimum values of the function, while applications to non-monotonic functions are typically concerned with finding points of inflection.

In our approach, we will dicuss functions that are monotonic with repsect to an interval. In this approach, we see the full power of the derivative.

We say that a function is concave up on an interval if it lies entirely above its tangent lines on that interval. Similarly, we say that a function is concave down on an interval if it lies entirely below its tangent lines on that interval. The points at which a function changes from concave up to concave down (or vice versa) are called inflection points.

Geometrically, concavity corresponds to the second derivative. More precisely, we say that a function is concave up on an interval if its second derivative is positive on that interval, and we say that a function is concave down on an interval if its second derivative is negative on that interval. In other words, the second derivative can be used to determine concavity. This is not too surprising, since the second derivative measures the rate of change of the first derivative. So, if the first derivative is increasing, then the function is concave up; if the first derivative is decreasing, then the function is concave down.

Applications of Differentiation aren't just limited to finding the slope of a tangent line at a point. In fact, one of the more interesting applications is finding what's known as an inflection point. An inflection point is a point on a curve where the curve changes from concave up to concave down, or vice versa. To find one, we take the second derivative of a function and set it equal to zero. The second derivative can tell us concavity, so if we set it equal to zero, we're saying that the concavity changes at that point. And that's an inflection point! Now that we know how to find them, what are they good for? Well, they're often used in optimization problems. If we're trying to maximize or minimize an object's volume, for example, we might want to know where its inflection points are. That way, we can be sure that we're not accidentally making the object too big or too small. So next time you see an inflection point, don't just ignore it - it could be just what you need!

If you've ever wondered how your math teacher can tell what a graph is going to look like just by looking at an equation, the answer lies in differentiation. Differentiation is a process of finding the rate of change of a function, and it turns out that this rate of change can be used to sketch the shape of a graph. For example, if a function is increasing at a constant rate, then its graph will be a straight line. If the rate of change is increasing, then the graph will be curved upwards; if the rate of change is decreasing, then the curve will bend downwards. By understanding how differentiation works, we can gain insight into the shape of many mathematical functions.

Differentiation is one of the most powerful tools in mathematics, with applications in everything from physics to engineering. It can also be used to sketch the graphs of functions, a process known as curve sketching. By studying the derivatives of a function, we can learn about its local behavior and how it changes over time. This information can then be used to sketch a rough graph of the function, without having to calculate any points. Curve sketching is a valuable tool for visualizing functions and understanding their behavior. It can also be used to estimate things like turning points and asymptotes, which can be difficult to calculate exactly. In short, curve sketching is a powerful tool that everyone should know how to do.

Differentiation is a powerful tool that can be used to solve a wide variety of optimization problems. Whether you're trying to find the maximum or minimum value of a function or to optimize some other objective, differentiation can help you find the answer. In this article, we'll take a look at some of the most common applications of differentiation in optimization. Hopefully, by the end, you'll see just how powerful this tool can be!

If you're anything like me, you love nothing more than a good optimization problem. You know, the kind where you have to find the maximum or minimum of some function by differentiating it and setting the derivative equal to zero. But what's even better than solving those kinds of problems is finding applications for them in the real world. And that's exactly what applied optimization is all about. By understanding the principles of differentiation, we can figure out how to optimize all sorts of things in the world around us. For example, we can use optimization to find the fastest route between two points, or the least expensive way to produce a product. We can even use it to help design more efficient algorithms. So next time you're struggling with a difficult optimization problem, just remember that there's a good chance someone out there has already found an application for it.

An indeterminate form is an expression that takes on different values depending on the values of the variables involved. Indeterminate forms arise in many applications of differentiation, including the determination of maxima and minima. In general, indeterminate forms can be difficult to work with, but there are a few simple rules that can be used to simplify them. One famous example is the expression $0/0$, which is known as an indeterminate form because it can take on any value depending on further analysis.

Differentiation is a powerful tool that can be used in a variety of ways to solve mathematical problems. In this book, we learn about several of the most common applications of differentiation. By understanding how to differentiate functions, we can gain insight into their behavior and solve optimization problems more easily. We've also seen how indeterminate forms can arise in different situations and learned some simple rules for dealing with them. So if you're looking to become skilled, it's important to understand the principles and applications of differentiation inside and out.

3.1 Extreme Values

3.2 Relative Extreme Values

If a function is defined on an open interval and if at some point in that interval the function reaches a maximum or minimum value (relative to that interval), then we say that the function has a relative extrema on that interval. A maximum or minimum value that occurs at an endpoint is not, by definition, a relative maximum nor a relative minimum. A relative maximum or relative minimum must occur in the interior of an interval.
Now we define relative extrema and state the relative extrema theorem.

Definition 3.1. Let f be a function defined on an open interval I with $c \in I$.

- If $f(c) \geq f(x)$ for all x in I, then $f(c)$ is called a **relative maximum** of f on I.
- If $f(c) \leq f(x)$ for all x in I, then $f(c)$ is called a **relative minimum** of f on I.
- If $c \in I$ and $f(c)$ is either a relative maximum or a relative minimum then $f(c)$ is a relative extrema, and we say that $f(c)$ is a **relative extreme value** of f.

A relative maximum is sometimes called a **local maximum** .
A relative minimum is sometimes called a **local minimum** .

The following proposition is sometimes called Fermat's theorem due to acknowledgment that Fermat realized the result first. The following examples show that even when $f'(c)=0$ there need not be a maximum or minimum at c. In other words, the converse of Fermat's Theorem is false in general. Furthermore, there may be an extreme value when $f'(c) \neq 0$ or when $f'(c)$ does not exist. We state Fermat's wonderful observation as the **Relative Extrema Theorem** .

::: {#thm- } [Relative Extrema Theorem] If f has a relative extremum at c and $f'(c)$ exists then $f'(c)=0$. :::

Proof. Since f is differentiable at c, $f'(c)$ must be positive, zero, or negative. If

$$f'(c)=\lim_{x \to c} \frac{f(x)-f(c)}{x-c}>0$$

then there exists an interval (a,b) containing c such that $\frac{f(x)-f(c)}{x-c}>0$ for all $x \neq c$ in (a,b). This produces the following inequalities for x-values in the interval (a,b). If $x<c$ then $f(x)<f(c)$ and so $f(c)$ is not a relative minimum. If $x>c$ then $f(x)>f(c)$ and so $f(c)$ is not a relative maximum. So the assumption that $f'(c)>0$ leads to a contradiction. Assuming that $f'(c)<0$ will also lead to a similar contradiction. Thus it must be the case $f'(c)=0$ as desired.

□

Example 3.1. Determine if the Relative Extrema Theorem applies and if so find the relative extrema for the function

$$f(x)=\begin{cases} 2x-3 & x<2/3 \\ 3-7x & x \geq 2/3 \end{cases}$$

Solution. The function f has its maximum value (local and absolute) at $x=2/3$, but we can not find this absolute maximum by setting $f'(x)=0$ because f' is not defined at $x=2/3$. Since $f'(2/3)$ does not exist the Relative Extrema Theorem does not apply.

Example 3.2. Determine if the Relative Extrema Theorem applies and if so find the relative extrema for the function $h(x) = x^3 + 4$.

Solution. Since $h'(x) = 3x^2$ we have $h'(0) = 0$. However, since h does not have a relative extremum at $x = 0$ the Relative Extrema Theorem does not hold.

3.3 Critical Numbers

In general, the critical numbers divide the domain of a function into intervals on which the sign of the derivative remains the same, either positive or negative. Therefore, if a function is defined on that interval it is either increasing or decreasing on that interval. In particular the graph can not change directions on that interval. This is the crucial idea behind using the derivative to analyze graphs of function.

Definition 3.2. A real number c is called a **critical number** of the function f provided:

- c is in the domain of f and $f'(c) = 0$ or
- c is in the domain of f and c is not in the domain of f'.

Or said differently, a critical number of a function f is a real number c in the domain of f such that $f'(c) = 0$ or $f'(c)$ does not exist.

Theorem 3.1. *If f is a continuous function and has a relative extremum at c, then c is a critical number of f.*

Example 3.3. Find the critical numbers, if there are any, for the function

$$f(x) = |x + 2| - 3.$$

Solution. The function $f(x) = |x + 2| - 3$ can be rewritten as a piecewise function as

$$f(x) = \begin{cases} x - 1 & x \geq -2 \\ -x - 5 & x < -2 \end{cases}$$

The graph of the function f has a sharp corner at $x = -2$ which can be seen by evaluating the left derivative of f at $x = -2$ and the right derivative of f at $x = -2$. Therefore f' is not defined at $x = -2$. Since f' is defined everywhere else and f' is not 0 anywhere the only critical number is -2.

Example 3.4. Find the critical numbers, if there are any, for the function

$$g(x) = x^3 + 2.$$

Solution. The function g has derivative $g'(x) = 3x^2$ which is defined for all real numbers. Notice that at $x = 0$, $g'(0) = 0$ and so 0 is a critical number. Also notice that $x = 0$ is not a relative extrema nor an absolute extrema. In summary, critical numbers allow us to check if there are any extrema at that point, but not conversely.

Example 3.5. Find the critical numbers, if there are any, for the function

$$h(x) = \sin x.$$

Solution. The function $h(x) = \sin x$ has derivative $h'(x) = \cos x$ which is defined for all real numbers. So to check for extrema we will need to determine where $h'(x) = 0$. This occurs at $x = \frac{\pi}{2} + \pi k$ where k is any integer. In fact, in this case the absolute extrema of h occurs at these values.

Example 3.6. Find the critical numbers, if there are any, for the function $f(x) = x^{2/3}(1 - x)$.

Solution. The function f has derivative,

$$f'(x) = \frac{2}{3}x^{-1/3}(1 - x) + x^{2/3}(-1) = \frac{2 - 5x}{3x^{1/3}}$$

and so we need to check for any x in the domain of f such that $f'(x) = 0$ or $f'(x)$ is undefined. We see that f' is undefined for $x = 0$; and $f'(x) = 0$ for $x = \frac{2}{5}$. So for this function f there are two critical numbers namely $x = 0$ and $x = \frac{2}{5}$.

Example 3.7. Find the critical numbers, if there are any, for the function

$$f(x) = \sqrt{9 - x^2}.$$

Solution. The function f has derivative,

$$f'(x) = \frac{1}{2}\left(9 - x^2\right)^{-1/2}(-2x) = -\frac{x}{\sqrt{9 - x^2}}$$

and so we need to check for any x in the domain of f such that $f'(x) = 0$ or $f'(x)$ is undefined. We see that f' is undefined for $x = \pm 3$; and $f'(x) = 0$ for $x = 0$. So there are three critical numbers of f namely, $x = 0$ and $x = \pm 3$.

Example 3.8. Find the critical numbers, if there are any, for the function

$$f(x) = \frac{x^2}{x^2 - 3}.$$

Solution. The function f has derivative,

$$f'(x) = \frac{(x^2 - 3)\,2x - x^2(2x)}{(x^2 - 3)^2} = \frac{-6x}{(-3 + x^2)^2}$$

and so we need to check for any x in the domain of f such that $f'(x) = 0$ or if $f'(x)$ is undefined. We see that f' is undefined for $x = \pm\sqrt{3}$; and $f'(x) = 0$ for $x = 0$. So there are three critical numbers of f namely $x = 0$ and $x = \pm\sqrt{3}$.

Example 3.9. Find the critical numbers, if there are any, for the function

$$f(x) = \ln\sqrt{x - 2}.$$

Solution. The function f has derivative,

$$f'(x) = \frac{1}{2(-2 + x)}$$

and so we need to check for any x in the domain of f such that $f'(x) = 0$ or $f'(x)$ is undefined. We see that f' is undefined for $x = 2$; and there are no real numbers with $f'(x) = 0$. One might be tempted to say that there is one critical numbers of $x = 2$. However, $x = 2$ is not a critical number because $x = 2$ is not in the domain of $f(x) = \ln\sqrt{x - 2}$.

3.4 Absolute Extrema

Now we turn our attention to absolute extrema which take into account the whole domain of a given f and not just an open interval in the domain as do relative extrema.
Absolute extrema are defined and a procedure for finding absolute extrema on a given closed bounded interval is given. We give a couple of examples to illustrate this procedure.

Definition 3.3. Let f be a function with domain D.

- If $f(c) \geq f(x)$ for all x in D, then f has an **absolute maximum** at c.

- If $f(c) \leq f(x)$ for all x in D, then f has an **absolute minimum** at c.
- If f has either an absolute maximum or a absolute minimum at c, then we say f has an **absolute extrema** at c, and we say that $f(c)$ is an **extreme value**.

Sometimes an absolute maximum is called a **global maximum** . Sometimes an absolute minimum is called a **global minimum** .

Example 3.10. State whether linear functions, quadratic functions, and the six trigonometric functions have absolute extrema on their domains.

Solution. Linear functions $f(x) = ax + b$ do not have absolute extrema on their natural domain of $\mathbb{R}$ unless in the trivial case of $a = 0$. Quadratic functions $f(x) = ax^2 + bx + c$ have an absolute maximum or absolute minimum depending on the sign of a. The vertex is given by

$$\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$$

and is always an absolute maximum (if $a > 0$) or absolute minimum (if $a < 0$). The trigonometric functions sine and cosine have global maximum and global minimum; and since these functions are periodic they attain these values periodically. However, the functions secant, cosecant, tangent and cotangent do not have any absolute maximum and absolute minimum values on their natural domain. The reader should verify the last statement with either a graph of each of these functions or by analyzing these four functions in terms of sine and cosine.

::: {#thm- } [Extreme Value Theorem] If f is a continuous function on a closed bounded interval $[a, b]$, then f must attain an absolute maximum value $f(s)$ and an absolute minimum value $f(t)$ at some numbers s and t in $[a, b]$. :::

The extreme value theorem is an existence theorem because the theorem tells of the existence of maximum and minimum values but does not show how to find it.

Example 3.11. State whether the function has absolute extrema on its domain

$$f(x) = \begin{cases} -x^2 & x \neq 0 \\ -1 & x = 0 \end{cases}$$

Solution. The function f does not have a global maximum since it takes on all values less than, but arbitrarily close to 0. However, it never reaches the value of 0. Notice that this function is not continuous on a closed bounded interval containing 0 and so the Extreme Value Theorem does not apply.

Example 3.12. State whether the function has absolute extrema on its domain

$$g(x) = \begin{cases} x & x > 0 \\ 3 & x \le 0 \end{cases}$$

Solution. The function g does not have a global minimum since it takes on all values greater than, but arbitrarily close to 0. However, it never reaches the value of 0. Notice that this function is not continuous on a closed bounded interval containing 0 and so the Extreme Value Theorem does not apply.

Theorem 3.2. *Let f be a continuous function whose domain contains $[a,b]$. Then to find the absolute extrema of f on $[a,b]$ perform the following steps.*

- *Find all critical numbers of f on $[a,b]$ and evaluate f for each of these numbers.*
- *Evaluate f at the boundary; that is, find $f(a)$ and $f(b)$.*
- *Select the largest and smallest values those listed in (1) and (2).*

The largest value is the absolute maximum and the smallest value is the absolute minimum.

Example 3.13. Find the absolute extrema of $f(x) = x^4 - 4x^2 + 2$ on the interval $[-3, 2]$.

Solution. The function f has derivative, $f'(x) = 4x^3 - 8x$. Since $f'(x) = 0$ only when $x = 0, \pm\sqrt{2}$ and f' is defined for all x, the only critical numbers are $x = 0, \pm\sqrt{2}$ (see Figure ~**??**). The following table determines the absolute extrema of the function f.

x	$f(x)$	$f'(x)$	Conclusion
-3	47	—	boundary, absolute maximum
$-\sqrt{2}$	-2	0	critical number, absolute minimum
0	2	0	critical number
$\sqrt{2}$	-2	0	critical number, absolute minimum
2	2	—	boundary

Example 3.14. Find the absolute extrema of $f(x) = x^{4/5}$ on the interval $[-32, 1]$.

Solution. The function f has derivative, $f'(x) = 4/(5x^{1/5})$. Since $f'(x) \neq 0$ for every x in the domain of f and $f'(x)$ is undefined only when $x = 0$, the only critical numbers are $x = 0$ (see Figure ~**??**). The following table determines the absolute extrema of the function f.

x	$f(x)$	$f'(x)$	Conclusion
-32	16	—	boundary, absolute maximum
0	0	0	critical number, absolute minimum
1	1	—	boundary

Example 3.15. Find the absolute extrema of $f(x) = \dfrac{x}{x+1}$ on the interval $[1, 2]$.

Solution. The function f has derivative, $f'(x) = \frac{1}{(1+x)^2}$ and so there are no critical numbers because $x = -1$ is not in our domain of $[1, 2]$ (see Figure ~**??**) and f' is defined for all x in $[1, 2]$. The following table determines the absolute extrema of the function f.

x	$f(x)$	$f'(x)$	Conclusion
1	1/2	—	absolute minimum
2	2/3	—	absolute maximum

Example 3.16. Find the absolute extrema of $f(x) = \sin x + \cos x$ on the interval $[0, \pi/3]$.

Solution. The function f has derivative, $f'(x) = \cos x - \sin x$ and so to find our critical number we need to solve $\sin x = \cos x$ or $\tan x = 1$ which is of course $x = \frac{\pi}{4} + \pi k$ where k is any integer. However, for our given domain of $[0, \pi/3]$ the only critical number is $\pi/4$ (see Figure ~**??**). The following table determines the absolute extrema of the function f.

x	$f(x)$	$f'(x)$	Conclusion
0	1	—	boundary, absolute minimum
$\pi/4$	$\sqrt{2}$	0	critical number, absolute maximum
$\pi/3$	1	—	boundary

3.5 Exercises

Exercise 3.1. Find the critical numbers of the given functions.

- $f(x) = x^2(2x-1)^{2/3}$
- $g(x) = \frac{x+1}{x^2+x+1}$
- $h(x) = \sqrt[3]{x^2 - x}$

Exercise 3.2. Find the absolute maximum and minimum values of the function $f(x) = x^2 - 1$ on the interval $-1 \leq x \leq 2$. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

Exercise 3.3. Find the absolute maximum and minimum values of the function $f(x) = \frac{-1}{x}$ on the interval $-2 \leq x \leq -1$. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

Exercise 3.4. Find the absolute maximum and minimum values of the function $f(x) = \tan x$ on the interval $\frac{-\pi}{3} \leq x \leq \frac{\pi}{4}$. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

Exercise 3.5. Find the absolute maximum and minimum values of the function $f(x) = 2 - |x|$ on the interval $-1 \leq x \leq 3$. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

Exercise 3.6. Find the absolute maximum and minimum values of the function $f(x) = xe^{-x}$ on the interval $-1 \leq x \leq 1$. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

Exercise 3.7. Find the absolute maximum and minimum values of the function $f(x) = x^{4/3}$ on the interval $-1 \leq x \leq 8$. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

Exercise 3.8. Find the absolute maximum and minimum values of the function $f(x) = 3x^{2/3}$ on the interval $-27 \leq x \leq 8$. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

Exercise 3.9. Find the absolute maximum and absolute minimum values of the given function on the given interval.

- $h(x) = \frac{x}{x+1}$ on $[1, 2]$
- $i(x) = \cos^{-1} x \tan^{-1} x$ on $[0, 1]$
- $j(x) = e^{-x}(\cos x + \sin x)$ on $[0, 4\pi]$
- $k(x) = \sqrt[3]{x}\sqrt[3]{(x-3)^2}$ on $[-1, 4]$

Exercise 3.10. Find the extreme values of the function $y = x^3 + x^2 - 8x + 5$ and where they occur.

Exercise 3.11. Find the extreme values of the function $y = \frac{1}{\sqrt{1-x^2}}$ and where they occur.

Exercise 3.12. Find the extreme values of the function $y = \frac{x}{x^2+1}$ and where they occur.

Exercise 3.13. Find the extreme values of the function $y = e^x - e^{-x}$ and where they occur.

Exercise 3.14. Find the extreme values of the function $y = \cos^{-1}\left(x^2\right)$ and where they occur.

Exercise 3.15. Find the derivative at each critical point and determine the local extreme values for the function $y = x^{2/3}\left(x^2 - 4\right)$.

Exercise 3.16. Find the derivative at each critical point and determine the local extreme values for the function

$$y = \begin{cases} 4 - 2x & x \leq 1 \\ x + 1 & x > 1 \end{cases}$$

Exercise 3.17. Find the derivative at each critical point and determine the local extreme values for the function

$$y = \begin{cases} \frac{-1}{4}x^2 - \frac{1}{2}x + \frac{15}{4} & x \leq 1 \\ x^3 - 6x^2 + 8x & x > 1 \end{cases}$$

Exercise 3.18. Let $f(x) = \left|x^3 - 9x\right|$. Does $f'(0)$ exist? Does $f'(3)$ exist? Does $f'(-3)$ exist? Determine all extrema of f.

Exercise 3.19. What is the largest possible area for a right triangle whose hypotenuse is 5cm long?

Exercise 3.20. The height of a body moving vertically is given by $s = \frac{-1}{2}gt^2 + v_0t + s_0$, $g > 0$ with s in meters and t in seconds. Find the body's maximum height.

Exercise 3.21. Show that 5 is a critical number of the function $g(x) = 2 + (x-5)^3$ but g does not have a relative extremum at $x = 5$.

Exercise 3.22. Consider the cubic function $f(x) = ax^3 + bx^2 + cx + d$ where $a \neq 0$. Show that f can have zero, one, or two critical numbers and give examples of each.

Exercise 3.23. Explain why the function $f(x) = \frac{8}{\sin x} + \frac{27}{\cos x}$ must attain a minimum in the open interval $\left(0, \frac{\pi}{2}\right)$.

3.6 Mean Value Theorem

3.7 Rolle's Theorem

The Extreme Value Theorem guarantees the existence of a maximum and minimum value of a continuous function on a closed bounded interval. The next theorem is called Rolle's Theorem and it guarantees the existence of an extreme value on the interior of a closed interval, under certain conditions. Basically Rolle's theorem states that if a function is differentiable on an open interval, continuous at the endpoints, and if the function values are equal at the endpoints, then it has at least one horizontal tangent. Of course if the function is constant this is automatically true for all points in the interval. So the point is that Rolle's theorem guarantees us at least one point in the interval where there will be a horizontal tangent. Rolle's theorem is a special case of the Mean Value Theorem for when the values of the function are the same at the endpoints of the interval.

::: {#thm- } Rolle's Theorem Let f be a function that is continuous on $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$. Then there exists at least one number c in (a, b) such that $f'(c) = 0$. :::

Proof. If f is a constant function, then the statement is true; in fact $f'(c) = 0$ for all c in (a, b). If $f(x) > f(a) = f(b)$ for some x in (a, b), then by the Extreme Value Theorem, f attains its absolute maximum value somewhere in the open interval (a,b). But precisely at this c we have, $f'(c)=0$. If $f(x)<f(a)=f(b)$ for some x in (a,b), then by the Extreme Value Theorem, f attains its absolute minimum value somewhere in the open interval (a,b). But precisely at this c we have, $f'(c)=0$.

$\square$

Example 3.17. Verify Rolle's theorem for $f(x)=4-4x-x^2+x^3$ on $[-2,2]$.

Solution. Notice that f is continuous and differentiable for all real numbers. Also, $f(-2)=\mathrm{f}(2)=0$ and therefore Rolle's theorem applies and so there is at least one c in $(-2,2)$ such that $f'(c)=0$. We can find it by solving $f'(c)=-4-2c+3c^2=0$. In fact we find two, namely $c=\frac{1}{3}\left(1\pm\sqrt{13}\right)$ (see Figure ~**??**).

Example 3.18. Show that the equation $x^3+x-1=0$ has exactly one real root.

Solution. Since the function $f(x)=x^3+x-1$ is a polynomial it is continuous and differentiable for all real numbers. Thus, the Intermediate Value Theorem and Rolle's Theorem applies. Since $f(0)=-1<0$ and $f(1)=1>0$, by the Intermediate Value Theorem there is a c in $(0,1)$ such that $f(x)=0$. Therefore, the equation has at least one solution. To prove that $f(x)=0$ for only one x, we assume that there are two roots namely, x_1 and x_2; and we prove that this can not happen. Thus, assume x_1 and x_2 are solutions, that is $f\left(x_1\right)=f\left(x_2\right)=0$ with $x_1<x_2$. Then by Rolle's Theorem there exists a c in (x_1,x_2) such that $f'(c)=0$. Notice that $f'(c)=3x^2+1>0$ so that in fact such a c can not exist. Therefore, there can not be $x_1<x_2$ and in fact the equation has exactly one real root (see Figure ~**??**).

3.8 The Mean Value Theorem

Given a function that is differentiable on an open interval and continuous at the endpoints the Mean Value Theorem states there exists a number in the open interval where the slope of the tangent line at this point on the graph is the same as the slope of the line through the two points on the graph determined by the endpoints of the interval. The **mean** in the Mean Value Theorem is referring to the mean (average) rate of change of f in the interval.

Next we detail Rolle's Theorem and the Mean Value Theorem. We provide examples and illustrate why the hypotheses of these two theorems are necessary. We also give applications and detail two other theorems which are consequences of the Mean Value Theorem. We also emphasize that the Mean Value Theorem tells us that between two fixed points of time, the instantaneous velocity is equal to the average velocity.

::: {#thm- } Mean Value Theorem Let f be a function that is continuous on $[a, b]$ and differentiable on (a, b). Then there exists at least one number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

:::

Proof. The equation of the secant line through $(a, f(a))$ and $(b, f(b))$ is

$$y = \left[\frac{f(b) - f(a)}{b - a}\right](x - a) + f(a).$$

Let $g(x)$ be the difference between $f(x)$ and y. Then

$$g(x) = f(x) - \left[\frac{f(b) - f(a)}{b - a}\right](x - a) - f(a).$$

We can see that $g(a) = g(b) = 0$. Because f is continuous on $[a, b]$ and differentiable on (a, b), so is g. By Rolle's Theorem, there exists a number c in (a, b) such that $g'(c) = 0$, which means

$$g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} = 0$$

and so $f'(c) = \frac{f(b)-f(a)}{b-a}$ as desired.

□

Example 3.19. Find all numbers c in the interval $[a, b]$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

for the function $f(x) = 1 + \dfrac{1}{x}$ on $[1, 4]$.

Solution. Since f is continuous on $[0, 1]$ and differentiable on $(0, 1)$ we apply the Mean Value Theorem,

$$f'(c) = -\frac{1}{c^2} = \frac{f(b) - f(a)}{b - a} = \frac{\left(1 + \frac{1}{4}\right) - \left(1 + \frac{1}{1}\right)}{4 - 1} = -0.25$$

Solving for c we obtain, $c = 2$ since -2 is not in $[1, 4]$ (see Figure ~**??**).

Example 3.20. Find all numbers c in the interval $[a, b]$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

for the function $f(x) = \tan^{-1} x$ on $[0, 1]$.

Solution. Since f is continuous on $[0, 1]$ and differentiable on $(0, 1)$ we apply the Mean Value Theorem,

$$f'(c) = \frac{1}{1 + c^2} = \frac{f(b) - f(a)}{b - a} = \frac{\left(\tan^{-1} 1\right) - \left(\tan^{-1} 0\right)}{1 - 0} = \frac{\pi}{4}$$

Solving for c we obtain, $c = \sqrt{\frac{4-\pi}{\pi}}$ since $-\sqrt{\frac{4-\pi}{\pi}}$ is not in $[0, 1]$ (see Figure ~??).

3.9 Using the Mean Value Theorem

Example 3.21. Using the Mean Value Theorem evaluate,

$$\lim_{x \to 0} \frac{(1 + x)^n - 1}{x}$$

where n is a natural number.

Solution. We will use the Mean Value Theorem to find all numbers c in the interval $[a, b]$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

for the function $f(x) = (1 + x)^n$ on $[0, x]$. Since f is continuous and differentiable on $[0, x]$, we apply the Mean Value Theorem,

$$f'(c) = n(1 + c)^{n-1} = \frac{f(b) - f(a)}{b - a} = \frac{(1 + x)^n - (1 + 0)^n}{x - 0} = \frac{(1 + x)^n - 1}{x}$$

Since f is continuous on $[0, x]$ and differentiable on $(0, x)$ we know this c must exist. In fact since $0 < c < x$, as $x \to 0$ we see that $c \to 0$ because the Mean Value Theorem says that c is in the open interval $(0, x)$. Thus we can evaluate the limit

$$\lim_{x \to 0} \frac{(1 + x)^n - 1}{x} = \lim_{c \to 0} n(1 + c)^{n-1} = n$$

as desired.

If an object moves in a straight line with position function $s = f(t)$, then the average velocity between $t = a$ and $t = b$ is

$$\frac{f(b) - f(a)}{b - a}$$

and the velocity at $t = c$ is $f'(c)$. Thus the Mean Value Theorem tells us that at some time $t = c$ between a and b the instantaneous velocity $f'(c)$ is equal to that average velocity.

Example 3.22. Two stationary patrol cars equipped with radar are 1.2 miles apart on a street. As a truck passes the first patrol car, its speed is clocked at 35 miles per hour. One and half minutes later, when the truck passes the second patrol car, its speed is clocked at 30 miles per hour. Prove that the truck must have exceeded the speed limit (of 35 miles per hour) at some time during the one and half minutes.

Solution. Let $t = 0$ be the time when the truck passes the first patrol car. The time when it passes the second patrol car is $1.5/60$ hour. By letting $s(t)$ represent the distance (in miles) travelled by the truck, we have $s(0) = 0$ and $s(1.5/60) = 1.2$. So the average velocity is

$$\frac{s\left(\frac{1.5}{60}\right) - s(0)}{\frac{1.5}{60} - 0} = \frac{1.2}{\frac{1.5}{60}} = 48.0 \text{ mph}$$

Assuming that the position function is differentiable, we can apply the Mean Value Theorem to conclude that the truck must have been traveling at a rate of 48 miles per hour sometime during the one and half minutes.

3.10 Exercises

Exercise 3.24. Determine the values of the constants a, b, c, and d such that the functions

$$f(x) = \begin{cases} 1 & x = 0 \\ ax + b & 0 < x \leq 1 \\ x^2 + 4x + c & 1 < x \leq 3 \end{cases}$$

$$g(x) = \begin{cases} a & x = -1 \\ 2 & 0 < x \leq 0 \\ bx^2 + c & 0 < x \leq 1 \\ dx + 4 & 1 < x \leq 2 \end{cases}$$

satisfies the hypotheses of the Mean Value Theorem on the intervals $[0, 3]$ and $[-1, 2]$ respectively.

Exercise 3.25. Find the value(s) of c that satisfy the equation $\frac{f(b)-f(a)}{b-a} = f'(c)$ for $f(x) = x^{2/3}$ in the conclusion of the Mean Value Theorem on $[0, 1]$.

Exercise 3.26. Find the value(s) of c that satisfy the equation $\frac{f(b)-f(a)}{b-a} = f'(c)$ for $f(x) = \ln(x-1)$ in the conclusion of the Mean Value Theorem on $[2, 4]$.

Exercise 3.27. Does the function $f(x) = x^{2/3}$ satisfy the hypotheses of the Mean Value Theorem on the interval $[-1, 8]$? State why or why not.

Exercise 3.28. Does the function

$$f(x) = \begin{cases} \frac{\sin x}{x} & -\pi \leq x \leq 0 \\ 0 & x = 1 \end{cases}$$

satisfy the hypotheses of the Mean Value Theorem on the interval $[-1, 8]$? State why or why not.

Exercise 3.29. Assume $a_1 \neq 0$. Let $f(x) = a_1x^2 + a_2x + a_3$. Prove that for any interval $[a, b]$ the value of c guaranteed by the Mean Value Theorem for f is the midpoint of the interval.

Exercise 3.30. Show that a cubic can have at most three zeros.

Exercise 3.31. Show that the function $g(t) = \sqrt{t}+\sqrt{1+t}-4$ has exactly one zero in the interval $(0, +\infty)$.

Exercise 3.32. Show that the function $g(t) = 2t-\cos^2 t+\sqrt{2}$ has exactly one zero in the interval $(-\infty, +\infty)$.

Exercise 3.33. Suppose that $f(-1) = 3$ and that $f'(x) = 0$ for all x. Must $f(x) = 3$ for all x? Why or why not?

Exercise 3.34. Suppose that $f'(x) = 2x$ for all x. Find $f(2) = 3$ if $f(-2) = 3$.

Exercise 3.35. Show that the equations $x^5+10x+3=0$ and $x^7+5x^3+x-6=0$ each have exactly one real root.

Exercise 3.36. Use the Mean Value Theorem to show the following

$$\lim_{x\to\pi}\frac{\cos x+1}{x-\pi}=0$$

and

$$\frac{1}{2x+1}>\frac{1}{5}+\frac{2}{25}(2-x)$$

when $0<x<2$.

Exercise 3.37. Let $f(x)=\frac{1}{x}$ and

$$g(x)=\begin{cases}\frac{1}{x} & x>0\\ 1+\frac{1}{x} & x<0\end{cases}$$

Show that $f'(x)=g'(x)$ for all x in their domains. Can we conclude that $f-g$ is constant?

3.11 Monotonic Functions

3.12 Increasing and Decreasing Functions

To determine where a function f is increasing or decreasing, we begin by finding the critical numbers. These numbers divide the x-axis into intervals, and we test the sign of $f'(x)$ in each of these intervals. This procedure is often called the first derivative test and can be used to determine local extrema and intervals of monotonicity.

Definition 3.4. A function f is called **increasing** on an interval I if $f(x_1)<f(x_2)$ whenever $x_1<x_2$ in I. A function f is called **decreasing** on an interval I if $f(x_1)>f(x_2)$ whenever $x_1<x_2$ in I.

Definition 3.5. A function f is called **monotonic** on an interval I if it is either increasing or decreasing on I.

Theorem 3.3. *Suppose f is continuous on $[a,b]$ and differentiable on (a,b).*

- *If $f'(x) > 0$ for all x in (a,b), then f is increasing on (a,b).*
- *If $f'(x) < 0$ for all x in (a,b), then f is decreasing on (a,b).*

Example 3.23. Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and decreasing; that is determine the intervals where f is monotonic.

Solution. The derivative of f is

$$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x-2)(x+1).$$

Since f is continuous and differentiable, to test where the function is monotonic we divide the x-axis according to the sign of $f'(x)$ which depending on the signs of $12x$, $x-2$, and $x+1$. We put our results into the following table:

Interval	$12x$	$x-2$	$x+1$	$f'(x)$	f
$x < -1$	$-$	$-$	$-$	$-$	decreasing on $(-\infty, -1)$
$-1 < x < 0$	$-$	$-$	$+$	$+$	increasing on $(-1, 0)$
$0 < x < 2$	$+$	$-$	$+$	$-$	decreasing on $(0, 2)$
$x > 2$	$+$	$+$	$+$	$+$	increasing on $(2, \infty)$

(see Figure ~**??**). :::

The First Derivative Test

::: {#thm- } [First Derivative Test] Suppose that c is a critical number of a function that is continuous on $[a,b]$. Then the following statements hold:

- If $f'(x) > 0$ for $a < x < c$ and $f'(x) < 0$ for $c < x < b$, then f has a relative (local) minimum at c.
- If $f'(x) < 0$ for $a < x < c$ and $f'(x) > 0$ for $c < x < b$, then f has a relative (local) maximum at c.
- If neither hold then f has no relative (local) extremum at c.

Example 3.24. Apply the First Derivative Test to find the local extrema of the function $f(x) = x(1-x)^{2/5}$ and sketch its graph.

Solution. First we find the critical numbers of f by solving $f'(x) = 0$ and determining where $f'(x)$ is undefined but $f(x)$ is defined. The derivative of f is

$$f'(x) = (1-x)^{2/5} + \frac{2x}{5}(1-x)^{-3/5}(-1) = \frac{5(1-x) - 2x}{5(1-x)^{3/5}} = \frac{5-7x}{5(1-x)^{3/5}}.$$

Solving $f'(x) = 0$ we find $x = 5/7$. Also $f'(1)$ does not exist but $f(1) = 0$; and therefore the only critical numbers of f are $x = 5/7$ and $x = 0$. We determine the local extrema using the following table

Interval	$5-7x$	$(1-x)^{3/5}$	$f'(x)$	$f(x)$
$x < \frac{5}{7}$	$+$	$+$	$+$	increasing on $\left(-\infty, \frac{5}{7}\right)$
$\frac{5}{7} < x < 1$	$-$	$+$	$-$	decreasing on $\left(\frac{5}{7}, 1\right)$
$x > 1$	$-$	$-$	$+$	increasing on $(1, +\infty)$

\ Therefore, $\left(\frac{5}{7}, f\left(\frac{5}{7}\right)\right)$ is a local maximum and $(1, f(1))$ is a local minimum. Here is the graph of the function $f(x) = x(1-x)^{2/5}$. Notice there is a corner at $(1, f(1))$ because f is defined there but f' is not (see Figure ~**??**).

Example 3.25. Find the local and absolute extrema values of the function $f(x) = x^3(x-2)^2$ on the interval $-1 \leq x \leq 3$. Sketch the graph.

Solution. First we notice that f is a polynomial and so is continuous and differentiable for all real numbers. Next we find the critical numbers of f by solving $f'(x) = 0$ and determining where $f'(x)$ is undefined, but $f(x)$ is defined. The derivative of f is

$$f'(x) = 3x^2(x-2)^2 + 2x^3(x-2) = x^2(x-2)(5x-6).$$

To find the critical numbers we set $f'(x) = 0$ and obtain $x = 0, 2, 6/5$. We determine the local extrema and absolute extrema using the following table:

Interval	x^2	$x-2$	$5x-6$	$f'(x)$	f
$-1 < x < 0$	$+$	$-$	$-$	$+$	increasing on$(-1, 0)$
$0 < x < \frac{6}{5}$	$+$	$-$	$-$	$+$	increasing on $\left(0, \frac{6}{5}\right)$
$\frac{6}{5} < x < 2$	$+$	$-$	$+$	$-$	decreasing on $\left(\frac{6}{5}, 2\right)$
$2 < x < 3$	$+$	$+$	$+$	$+$	increasing on $(2, 3)$

\ The function f does not have a local extrema at $x = 0$. The local maximum is

$$\left(\frac{6}{5}, f\left(\frac{6}{5}\right)\right)$$

and the local minimum is $(2, f(2))$.
Since f is a continuous function we can use the extreme value theorem to determine absolute extrema. We compute the functional values at the endpoints, namely $f(-1) = -9$ and $f(3) = 27$. Therefore, the absolute maximum is $f(3) = 27$ and the absolute minimum is $f(-1) = -9$ (see Figure ~**??**).

3.13 Exercises

Exercise 3.38. For each of the following functions determine the critical points and apply the first derivative test to determine the intervals where the function is increasing or decreasing, and all local extrema.

- $f(x) = (x-1)^2(x+2)$
- $f(x) = (x-1)e^{-x}$
- $f'(x) = x^{-1/3}(x+2)$
- $f(\theta) = 3\theta^2 - 4\theta^3$
- $h(r) = (r+7)^3$
- $g(x) = x^2\sqrt{5-x}$
- $f(x) = \frac{x^3}{3x^2+1}$
- $h(x) = x^{1/3}\left(x^2-4\right)$
- $f(x) = e^{2x} + e^{-x}$
- $f(x) = x \ln x$
- $f(x) = (x+1)^2$
- $f(x) = -x^2 - 6x - 9$
- $k(x) = x^3 + 3x^2 + 3x + 1$

Exercise 3.39. Sketch the graph of a differentiable function $y = f(x)$ through the point $(1,1)$ such that

- $f'(1) = 0$
- $f'(x) > 0$ for $x < 1$
- $f'(x) < 0$ for $x > 1$
- $f'(x) > 0$ for $x \neq 1$
- $f'(x) < 0$ for $x \neq 1$

Exercise 3.40. Sketch the graph of a differentiable function $y = f(x)$ that satisfies all of the following conditions.

- a local minimum at $(1,1)$
- a local maximum at $(3,3)$
- local maximum at $(1,1)$
- a local minimum at $(3,3)$

Exercise 3.41. Sketch the graph of a differentiable function $y = h(x)$ that satisfies all of the following conditions.

- $h(0) = 0$
- for all x, $-2 \leq h(x) \leq 2$
- $h’(x)→+∞$ as $x \to 0^-$

- \$h'(x)→+∞\$ as $x \to 0^+$

3.14 Concavity and Inflection Points

3.15 First and Second Order Critical Points

Definition 3.6. Suppose c is in the domain of a function f. We will call c a **first-order critical number** of f when $f'(c) = 0$ or $f'(c)$ does not exist and a **second-order critical number** of f when $f''(c) = 0$ or $f''(c)$ does not exist.

3.16 Concavity and Inflection Points

Definition 3.7. Suppose a function f is differentiable on an open interval I.

- If f' is increasing on I, then the graph of f is called **concave upward** on I.
- If f' is decreasing on I, then the graph of f is called **concave downward** on I.

If the graph of f lies above all of its tangents on an interval I, it is concave upward on I. If the graph of f lies below all of its tangents, it is concave downward on I.

Definition 3.8. A point $P(c, f(c))$ on a curve is called an **inflection point** of the graph of f provided f has a tangent line at c and the concavity of f changes at $x = c$.

3.17 Concavity Test

::: {#thm- } Concavity Test Suppose a function f is twice differentiable on an interval I.

- If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.
- If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I. :::

Example 3.26. Determine where the curve $y = x^4 - 4x^3$ is concave upward, where it is concave downward, and where the points of inflection are.

Solution. The first and second derivatives of the function f are

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$$

and

$$f''(x) = 12x^2 - 24x = 12x(x-2).$$

Since f'' is defined for all real numbers and $f''(x) = 0$ only when $x = 0$ and $x = 2$ the second-order critical numbers of f are $x = 0$ and $x = 2$. We summarize the Concavity Test in the following table:

Interval	f	f''	Conclusion
$x < 0$		$+$	concave up
$x = 0$	0	0	inflection point
$0 < x < 2$		$-$	concave down
$x = 2$	-16	0	inflection point
$x > 2$		$+$	concave up

Therefore, f is concave up on $(-\infty, 0) \cup (2, +\infty)$ and concave down on $(0, 2)$. The points $(0, 0)$ and $(2, -16)$ are inflection points (see Figure ~**??**).

Example 3.27. Determine where the curve $y = x^3 - 3x + 1$ is concave upward and where it is concave downward. Find all inflection points, local extrema, and sketch the curve.

Solution. The first and second derivatives of the function f are

$$f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$$

and $f''(x) = 6x$. Since f' and f'' are polynomials they are both defined for all real numbers and since $f'(x) = 0$ and $f''(x) = 0$ only when $x = \pm 1$ and $x = 0$, respectively, the first-order critical numbers are $x = \pm 1$ and the second-order critical number is $x = 0$. We summarize the First Derivative Test and the Concavity Test in the following table.

Interval	f	f'	f''	Conclusion
$x < -1$		$+$		increasing
$x = -1$	3	0		relative maximium
$-1 < x < 1$		$-$		decreasing
$x = 1$	-1	0		relative minimum
$x > 1$		$+$		increasing
$x < 0$			$-$	concave down
$x = 0$	1		0	inflection point
$x > 0$			$+$	concave up

Therefore, the function f has a local maximum at $(3,0)$ and a local minimum at $(1,-1)$. The function f is increasing on $(-\infty,-1)\cup(1,+\infty)$ and decreasing on $(-1,1)$. The point $(0,1)$ is an inflection point because f is concave up on the interval $(0,+\infty)$ and concave down on $(-\infty,0)$.
(see Figure ~**??**).

3.18 Second Derivative Test

::: {#thm- } Second Derivative Test Suppose f'' in continuous on an open interval that contains c with $f'(c)=0$.

- If $f''(c)>0$, then f has a relative (local) minimum at c.
- If $f''(c)<0$, then f has a relative (local) maximum at c. :::

The Second Derivative Test is inconclusive when both $f'(c) = 0$ and $f''(c)=0$. For example, if $f(x)=x^3$ and $g(x)=x^4$, both $f'(0)=g'(0)=0$ and $f''(0) = g''(0) = 0$. The point $x = 0$ is a minimum for g but is neither a maximum nor a minimum for f. For some functions the Second Derivative Test might be a straightforward method for determining whether a point is a local extrema, however for some functions the First Derivative Test is a necessity.

Example 3.28. Use the Second Derivative Test to determine whether each first-order critical number of the function
$f(x) = 3x^5 - 5x^3 + 2$ corresponds to a relative maximum, a relative minimum, or neither.

Solution. Since $f'(x)=15x^4-15x^2=15(x-1)x^2(x+1)$ is defined for all real numbers and $f'(x) = 0$ when $x = 0$ and $x = \pm 1$ the first-order critical numbers are $x=0$ and $x=\pm 1$. Since

$$f''(x)=60x^3-30x=30x\left(2x^2-1\right)$$

is continuous for all real numbers we can apply the Second Derivative Test. Since $f''(-1)<0$ the point $(-1,4)$ is a local maximum and since $f''(1)>0$ the point $(1,0)$ is a local minimum. Since $f''(0)=0$ the Second Derivative Test is inconclusive at $x=0$. (see Figure ~**??**).

Example 3.29. Use the Second Derivative Test to determine whether each critical number of the function $f(x) = x + 4/x$ corresponds to a relative maximum, a relative minimum, or neither.

Solution. The first derivative of the function f is

$$f'(x)=1-\frac{4}{x^2}=\frac{x^2-4}{x^2}=\frac{(x-2)(x+2)}{x^2}.$$

The first derivative is defined for all real numbers except $x = 0$ and $f'(x) = 0$ only when $x = \pm 2$ the first-order critical numbers of $f $ are $x = \pm 2$. Note that, even though $f'(0)$ is undefined, so is $f(0)$ and so 0 is not a critical number. The second derivative of the function f is $f''(x) = \frac{4}{x^3}$ and since f'' is continuous for all real numbers except $x = 0$ we can apply the Second Derivative Test. Since $f''(2) > 0$ the point $(2, 4)$ is a local minimum and since $f''(-2) < 0$ the point $(-2, -4)$ is a local maximum (see Figure ~**??**).

Example 3.30. For the function $f(x) = x^2 e^{-3x}$ find all first-order and second-order critical numbers. Apply the First Derivative Test, Concavity Test, and the Second Derivative Test. Sketch the graph of the function.

Solution. The first and second derivatives of the function f are

$$f'(x) = 2xe^{-3x} - 3x^2 e^{-3x} = e^{-3x}\left(2x - 3x^2\right) = x(2 - 3x)e^{-3x}$$

and

$$f''(x) = e^{-3x}\left(9x^2 - 12x + 2\right).$$

Since both the first and second derivatives are continuous for all real numbers the first-order and second-order critical numbers will be found by solving $f'(x) = 0$ and $f''(x) = 0$, respectively. Solving $f'(x) = 0$ we find the first-order critical numbers to be $x = 0$ and $x = \frac{2}{3}$. Solving $f''(x) = 0$ we find the second-order critical numbers to be $x = \frac{2}{3} \pm \frac{\sqrt{2}}{3}$. Since f is continuous we can apply the First Derivative Test as follows:

Interval	f	f'	Conclusion
$x < 0$		$-$	decreasing
$x = 0$	0	0	relative minimum
$0 < x < \frac{2}{3}$		$+$	increasing
$x = \frac{2}{3}$	$\frac{4}{9e^2}$	0	relative maximum
$x > \frac{2}{3}$		$-$	decreasing

Since f'' is continuous we can apply the Second Derivative Test as follows: since $f''(0) > 0$ the point $(0, 0)$ is a local minimum and since $f''\left(\frac{2}{3}\right) < 0$ the point $\left(\frac{2}{3}, f\left(\frac{2}{3}\right)\right)$ is a local maximum. Applying the Concavity Test:

Interval	f	f''	Conclusion
$x < \frac{2}{3} - \frac{\sqrt{2}}{3}$		$+$	concave up
$x = \frac{2}{3} - \frac{\sqrt{2}}{3}$	$\frac{1}{9}\left(6 - 4\sqrt{2}\right)e^{-2+\sqrt{2}}$	0	inflection point
$\frac{2}{3} - \frac{\sqrt{2}}{3} < x < \frac{2}{3} + \frac{\sqrt{2}}{3}$		$-$	concave down
$x = \frac{2}{3} + \frac{\sqrt{2}}{3}$	$\frac{2}{9}\left(3 + 2\sqrt{2}\right)e^{-2-\sqrt{2}}$	0	inflection point
$x > \frac{2}{3} + \frac{\sqrt{2}}{3}$		$+$	concave up

(see Figure ~**??**).

3.19 Exercises

Exercise 3.42. For each of the following functions identify the inflection points and local maxima and local minima. Identify the intervals on which the function is concave up and concave down. Sketch the graph showing these specific features.

- $f(x) = \frac{x^4}{4} - 2x^2 + 4$
- $f(x) = \frac{9}{14}x^{1/3}\left(x^2 - 7\right)$
- $f(x) = \tan x - 4x$ on $-\frac{\pi}{2} < x < \frac{\pi}{2}$
- $f(x) = 2\cos x - \sqrt{2}x$ on $-\pi < x < \frac{3\pi}{2}$
- $f(x) = x^2 - 4x + 3$
- $f(x) = x^4 - 2x^2$
- $f(x) = x - \sin x$ on $0 x 2 $
- $f(x) = \sqrt{|x|}$
- $f(x) = e^x - 2e^{-x} - 3x$
- $f(x) = \frac{\ln x}{\sqrt{x}}$

Exercise 3.43. Sketch a smooth curve $y = f(x)$ with $f(-2) = 8$, $f(0) = 4$, $f(2) = 0$, $f'(x) > 0$ for $|x| > 2$, $f'(2) = f'(-2) = 0$, $f'(x) < 0$ for $|x| < 2$, $f''(x) < 0$ for $x < 0$, and $f''(x) > 0$ for $x > 0$.

Exercise 3.44. Sketch the graph of the following function $f(x) = x^{2/3}(x - 7)$. Find all vertical and horizontal asymptotes of the graph of each function. Determine intervals of increasing and decreasing, determine concavity, and locate all critical points and points of inflection. Show all special features such as cusps or vertical tangents.

Exercise 3.45. Sketch the graph of the following function

$$h(x) = \frac{3x - 2}{\sqrt{2x^2 + 1}}.$$

Find all vertical and horizontal asymptotes of the graph of each function. Determine intervals of increasing and decreasing, determine concavity, and locate all critical points and points of inflection. Show all special features such as cusps or vertical tangents.

Exercise 3.46. Find constants a and b that guarantee that the graph of the function defined by

$$f(x) = \frac{ax + 5}{3 - bx}$$

will have a vertical asymptote at $x = 5$ and a horizontal asymptote at $y = -3$. Sketch the graph of the function.

Exercise 3.47. Find all vertical tangents and cusps for the function $f(x) = \sqrt{4 - x^2}$. Justify your work. Sketch the graph of the function.

Exercise 3.48. Find all vertical tangents and cusps for the function

$$f(x) = \begin{cases} x^{1/3} + 3 & x \le 0 \\ 3 - x^{1/5} & x \ge 0 \end{cases}.$$

Justify your work. Sketch the graph of the function.

Exercise 3.49. Explain why the function

$$f(x) = \begin{cases} 1 & x \le 0 \\ \frac{1}{x} & x > 0 \end{cases}$$

has a vertical asymptote but no vertical tangent. Sketch the graph of the function.

Exercise 3.50. Sketch the graph of the curve

$$y^2 = \frac{x^3}{2a - x}$$

for $[0, 2a)$. Show all special features such as vertical asymptotes, horizontal asymptotes, cusps, vertical tangents, and intercepts. Sketch the graph of the function.

Exercise 3.51. Sketch the graph of the following functions. Find all vertical and horizontal asymptotes of the graph of each function. Determine intervals of increasing and decreasing, determine concavity, and locate all critical points and points of inflection. Show all special features such as cusps or vertical tangents.

- $f(x) = x^{2/3}(x - 7)$
- $g(x) = \frac{72 - 54x + x^2 + 2x^3}{15 - 17x + 4x^2}$
- $h(x) = \frac{3x - 2}{\sqrt{2x^2 + 1}}$

3.20 Limits Involving Infinity

3.21 Infinite Limits

Infinite limits are used to described unbounded behavior of a function near a given real number which is not necessarily in the domain of the function. They are particularly useful for showing the intentions of the graph of a function by drawing dashed lines representing unbounded growth which are called vertical asymptotes.

Definition 3.9. Let f be a function defined on both sides of c, except possible at c itself. Then $\lim_{x\to c} f(x) = +\infty$ means that the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to c $(x \neq c\ )$.

Definition 3.10. Let f be a function defined on both sides of c, except possible at c itself. Then $\lim_{x\to c} f(x) = -\infty$ means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to c $(x \neq c\ )$.

Example 3.31. Determine $\displaystyle\lim_{x\to 0} \frac{1}{x}$.

Solution. Since $f(x) = \frac{1}{x}$ decreases without bound as $x \to 0^-$, we notice $\lim_{x\to 0^-} f(x) = -\infty$. Also since f increases without bound as $x \to 0^+$, we notice $\lim_{x\to 0^+} f(x) = +\infty$. Thus $\lim_{x\to 0} f(x)$ does not exist.

Compare the previous example with the following.

Example 3.32. Determine $\displaystyle\lim_{x\to 2} \frac{2x^3 + x^2 - 16x + 12}{x^2 - 4}$.

Solution. By factoring,

$$\begin{aligned}\lim_{x\to 2} \frac{2x^3 + x^2 - 16x + 12}{x^2 - 4} &= \lim_{x\to 2} \frac{(x-2)\left(2x^2 + 5x - 6\right)}{(x-2)(x+2)} \\ &= \lim_{x\to 2} \frac{(2x^2 + 5x - 6)}{(x+2)} = \frac{(2(2)^2 + 5(2) - 6)}{((2)+2)} = 3.\end{aligned}$$

Theorem 3.4. *Let A be a positive real number.*

- *If n is a positive even integer, then* $\lim_{x\to c} \frac{A}{(x-c)^n} = +\infty$.

- *If n is a positive odd integer, then*

$$\lim_{x\to c^+} \frac{A}{(x-c)^n} = +\infty \qquad \textit{and} \qquad \lim_{x\to c^-} \frac{A}{(x-c)^n} = -\infty.$$

Example 3.33. Evaluate $\displaystyle\lim_{x\to 2^+} \frac{-3}{\sqrt[3]{x-2}}$.

Solution. Notice that $\dfrac{1}{\sqrt[3]{x-2}}$ increases without bound as $x \to 2^+$ and therefore, $\displaystyle\lim_{x\to 2^+} \frac{-3}{\sqrt[3]{x-2}} = -\infty$.

Example 3.34. Evaluate $\displaystyle\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{x^2}\right)$.

Solution. Note that

$$f(x) = \frac{1}{x} - \frac{1}{x^2} = \frac{-1+x}{x^2}$$

and that $\frac{-1+x}{x^2}$ decreases without bound as $x \to 0^+$ and therefore,

$$\lim_{x\to 0^+} \frac{-1+x}{x^2} = -\infty.$$

This is true because when $x > 0$ is close to 0 we know that $-1+x$ is negative. Similarly, $\lim_{x\to 0^-} \frac{-1+x}{x^2} = -\infty$ because when $x < 0$ and close to 0 we know that $-1+x$ is negative. Therefore it follows

$$\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{x^2}\right) = -\infty$$

3.22 Vertical Asymptotes

Definition 3.11. The line $x = c$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

- $\lim_{x\to c} f(x) = +\infty$ $\lim_{x\to c^-} f(x) = +\infty$ $\lim_{x\to c^+} f(x) = +\infty$
- $\lim_{x\to c} f(x) = -\infty$ $\lim_{x\to c^-} f(x) = -\infty$ $\lim_{x\to c^+} f(x) = -\infty$

The following example demonstrates that not all rational functions have vertical asymptotes.

Example 3.35. Determine the vertical asymptotes of the function

$$g(x) = \frac{x^3}{x^2+3x+10}.$$

Solution. The function g is continuous on its domain which is $\mathbb{R}$ and therefore there are no vertical asymptotes for this function.

Example 3.36. Determine the vertical asymptotes of the function

$$f(x) = \frac{x^3}{x^2 + 3x - 10}.$$

Solution. Since

$$f(x) = \frac{x^3}{x^2 + 3x - 10} = \frac{x^3}{(x-2)(x+5)}$$

and therefore the vertical asymptotes are $x = 2$ and $x = -5$ because

$$\lim_{x \to 2^+} \frac{x^3}{(x-2)(x+5)} = +\infty \qquad \text{and} \qquad \lim_{x \to 5^+} \frac{x^3}{(x-2)(x+5)} = +\infty.$$

The following example demonstrates that there can be an unlimited number of vertical asymptotes for a function.

Example 3.37. Determine the vertical asymptotes of the function

$$h(x) = \tan x - \cot x.$$

Solution. We can rewrite this function as

$$h(x) = \tan x - \cot x = \frac{\sin x}{\cos x} - \frac{\cos x}{\sin x} = \frac{\sin^2 x - \cos^2 x}{\sin x \cos x} = \frac{\cos 2x}{\sin x \cos x}.$$

Therefore the zeros of the sine and cosine functions yield the vertical asymptotes of $x = \pm\frac{\pi}{2} + k$ for all integers k, since $\cos 2x$ is not zero for these values of x and because, for any $c = \pm\frac{\pi}{2} + k$, $\lim_{x \to c} h(x)$ is either $\pm\infty$. See Figure (**??**).

3.23 Limits at Infinity

Definition 3.12. Let f be a function defined on some interval $(a, +\infty)$. Then

$$\lim_{x \to \infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large; or more precisely, for every $\epsilon > 0$ there exists an N such that if $x > N$ then $|f(x) - L| < \epsilon$.

Definition 3.13. Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x\to-\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large negative; or more precisely, for every$\epsilon > 0$ there exists an N such that if $x < N$ then $|f(x) - L| < \epsilon$.

The following limit rules are similar to the limit rules used for when $x \to c$ but instead use $x \to +\infty$; and they are also valid for when $x \to +\infty$ is replaced by $x \to -\infty$.

Theorem 3.5. *If* $\lim_{x\to\infty} f(x)$ *and* $\lim_{x\to\infty} g(x)$ *exist, then*

- $\lim_{x\to\infty} k = k$ *for any constant* k
- $\lim_{x\to\infty} kf(x) = k\lim_{x\to\infty} f(x)$
- $\lim_{x\to\infty}[f(x) + g(x)] = \lim_{x\to\infty} f(x) + \lim_{x\to\infty} g(x)$
- $\lim_{x\to\infty}[f(x) - g(x)] = \lim_{x\to\infty} f(x) - \lim_{x\to\infty} g(x)$
- $\lim_{x\to\infty}[f(x)g(x)] = (\lim_{x\to\infty} f(x))\,(\lim_{x\to\infty} g(x))$
- $\lim_{x\to\infty}[f(x)/g(x)] = (\lim_{x\to\infty} f(x))\,/\,(\lim_{x\to\infty} g(x))$
- $\lim_{x\to\infty}[f(x)]^n = (\lim_{x\to\infty} f(x))^{\,n}$ *where n is a rational number and whenever the limits exist.*

Theorem 3.6. *Let A be a real number.*

- *If $r > 0$ is a rational number, then*

$$\lim_{x\to\infty} \frac{A}{x^r} = 0.$$

- *If $r > 0$ is a rational number with x^r defined for all x, then*

$$\lim_{x\to-\infty} \frac{A}{x^r} = 0.$$

Example 3.38. Evaluate $\displaystyle\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$.

Solution. To evaluate this limit we divide both the numerator and the denominator by the highest power of x that occurs. So we have

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \lim_{x\to\infty} \frac{\frac{3x^2}{x^2} - \frac{x}{x^2} - \frac{2}{x^2}}{\frac{5x^2}{x^2} + \frac{4x}{x^2} + \frac{1}{x^2}} = \lim_{x\to\infty} \frac{3 - \frac{1}{x} - \frac{2}{x^2}}{5 + \frac{4}{x} + \frac{1}{x^2}} = \frac{3}{5}.$$

Example 3.39. Evaluate $\lim_{x\to\infty}\left(x-\sqrt{x^2+1}\right)$.

Solution. We use the conjugate radical as follows

$$\begin{aligned}\lim_{x\to\infty}\left(x-\sqrt{x^2+1}\right) &= \lim_{x\to\infty}\frac{\left(x-\sqrt{x^2+1}\right)\left(x+\sqrt{x^2+1}\right)}{\left(x+\sqrt{x^2+1}\right)}\\ &= \lim_{x\to\infty}\frac{x^2-(x^2+1)}{\left(x+\sqrt{x^2+1}\right)} = \lim_{x\to\infty}\frac{-1}{x+\sqrt{x^2+1}}\\ &= \lim_{x\to\infty}\frac{\frac{-1}{x}}{\frac{x}{x}+\sqrt{\frac{x^2+1}{x^2}}} = \frac{0}{1+\sqrt{1+0}} = 0.\end{aligned}$$

3.24 Horizontal Asymptotes

In mathematics, the symbol ∞ is not a number, rather it is used to describe the process of unrestricted growth or the result of such a growth.

Definition 3.14. The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$.

::: {#thm- } If f is a rational functions of the form:

$$f(x) = \frac{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n}{b_0 + b_1x + b_2x^2 + \cdots + b_mx^m}$$

where n is the degree of the polynomial in the numerator and m is the degree of the polynomial in the denominator, then the horizontal asymptote of the curve $y = f(x)$ is determined by the following.

- If $n = m$, then $y = \frac{a_n}{b_m}$ is the horizontal asymptote.
- If $n < m$, then $y = 0$ is the horizontal asymptote.
- If $n > m$, then there is no horizontal asymptote, but rather a slant (oblique) asymptote and can be found be using long division. :::

Example 3.40. Find the horizontal asymptote of the graph of the function

$$f(x) = \frac{(1-x)(2+x)}{(1+2x)(2-3x)}.$$

Solution. The degree of the numerator $(1-x)(2+x) = -x^2 - x + 2$ is 2 and the degree of the denominator $(1+2x)(2-3x) = -6x^2+x+2$ is 2 we divide both numerator and denominator by x^2 and using the properties of limits, we have

$$\lim_{x\to\pm\infty}\frac{-x^2-x+2}{-6x^2+x+2} = \lim_{x\to\pm\infty}\frac{\frac{-x^2}{x^2}-\frac{x}{x^2}+\frac{2}{x^2}}{\frac{-6x^2}{x^2}+\frac{x}{x^2}+\frac{2}{x^2}} = \frac{-1-0+0}{-6+0+0} = 6$$

Therefore the only horizontal asymptote of the graph of $f(x)$ is $y = 6$.

Example 3.41. Find the horizontal asymptote of the graph of the function

$$f(x) = \frac{\sqrt{2x^2+1}}{3x-5}.$$

Solution. Dividing both numerator and denominator by x and using the properties of limits, we have

$$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to\infty} \frac{\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}} = \frac{\sqrt{\lim_{x\to\infty}\left(2+\frac{1}{x^2}\right)}}{\lim_{x\to\infty}\left(3-\frac{5}{x}\right)} = \frac{\sqrt{2+0}}{3-5(0)} = \frac{\sqrt{2}}{3}$$

Therefore the line $y = \sqrt{2}\big/3$ is a horizontal asymptote. It is also important to realize that

$$\begin{aligned}\lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5} &= \lim_{x\to-\infty} \frac{\frac{1}{x}\sqrt{2x^2+1}}{3-\frac{5}{x}} \\ &= \lim_{x\to-\infty} \frac{\left(\frac{-1}{\sqrt{x^2}}\right)\sqrt{2x^2+1}}{3-\frac{5}{x}} = \lim_{x\to-\infty} \frac{-\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}} = -\frac{\sqrt{2}}{3}\end{aligned}$$

Therefore, the line $y = -\sqrt{2}\big/3$ is another horizontal asymptote.

Example 3.42. Find all horizontal asymptotes of the graph of the function

$$f(x) = \frac{x}{\sqrt[4]{x^4+1}}.$$

Solution. Dividing both numerator and denominator by x and using the properties of limits, we have

$$\begin{aligned}\lim_{x\to\infty} \frac{x}{\sqrt[4]{x^4+1}} &= \lim_{x\to\infty} \frac{\frac{x}{x}}{\frac{1}{x}\sqrt[4]{x^4+1}} = \lim_{x\to\infty} \frac{1}{\sqrt[4]{\frac{x^4+1}{x^4}}} \\ &= \lim_{x\to\infty} \frac{1}{\sqrt[4]{1+\frac{1}{x^4}}} = \frac{1}{\sqrt[4]{\lim_{x\to\infty}\left(1+\frac{1}{x^4}\right)}} = \frac{1}{\sqrt[4]{1+0}} = 1.\end{aligned}$$

Therefore, the line $y = 1$ is a horizontal asymptote. In computing the limit $x \to -\infty$ we must remember that for $x < 0$, we have $\sqrt[4]{x^4} = -x$, so when we divide the numerator by x, when $x < 0$ we have,

$$\begin{aligned}\lim_{x\to-\infty} \frac{x}{\sqrt[4]{x^4+1}} &= \lim_{x\to-\infty} \frac{\frac{x}{x}}{\frac{1}{x}\sqrt[4]{x^4+1}} \\ &= \lim_{x\to\infty} \frac{1}{\frac{1}{\sqrt[4]{x^4}}\sqrt[4]{\frac{x^4+1}{x^4}}} = \lim_{x\to\infty} \frac{1}{-\sqrt[4]{1+\frac{1}{x^4}}} = -1.\end{aligned}$$

Therefore, the horizontal asymptotes are $y = \pm 1$.

3.25 Curve Sketching

3.26 Curve Sketching as a Procedure

The following steps can be used to graph many commonly used functions.

- If given a function, determine the domain and range.
- If possible algebraically simplify the function.
- Test for symmetry with respect to the x-axis, y-axis, and the origin.
- Determine any x-intercepts and y-intercepts.
- Determine any vertical asymptotes.
- Determine any horizontal asymptotes.
- Determine the first order critical numbers.
- Apply the First Derivative Test.
- Apply the Second Derivative Test.
- Determine the second order critical numbers.
- Apply the Concavity Test.
- Determine any vertical tangents.
- Determine any cusps.
- Plot Points.
- Sketch the curve.

3.27 Vertical Tangents and Cusps

There are four possibilities for unbounded behavior of a derivative $f'(x)$ around a given real number c.

Definition 3.15. Suppose the function f is continuous at the point $P(c, f(c))$.

- The graph of f has a **vertical tangent** at P if one of the following holds.
- $\lim_{x\to c^+} f'(x) = \lim_{x\to c^-} f'(x) = +\infty$
- $\lim_{x\to c^+} f'(x) = \lim_{x\to c^-} f'(x) = -\infty$
- The graph of f has a **cusp** at P if one of the following holds.
- $\lim_{x\to c^+} f'(x) = +\infty$ and $\lim_{x\to c^-} f'(x) = -\infty$
- $\lim_{x\to c^+} f'(x) = -\infty$ and $\lim_{x\to c^-} f'(x) = +\infty$

Example 3.43. Sketch the graph of

$$f(x) = 3x^{3/5}\left(5 - x - 4x^2\right)$$

and explain why there is a vertical tangent at $x = 0$.

Solution. Using the product rule the derivative of the function f is

$$\begin{aligned} f'(x) &= \frac{9}{5}x^{-2/5}\left(5 - x - 4x^2\right) + 3x^{3/5}(-1 - 8x) \\ &= \frac{9\left(5 - x - 4x^2\right)}{5x^{2/5}} + \frac{\left(5x^{2/5}\right)\left(3x^{3/5}\right)(-1 - 8x)}{5x^{2/5}} \\ &= \frac{9\left(5 - x - 4x^2\right)}{5x^{2/5}} + \frac{15x(-1 - 8x)}{5x^{2/5}} \\ &= \frac{45 - 24x - 156x^2}{5x^{2/5}}. \end{aligned}$$

To determine any vertical tangents we consider where $f'(x)$ is undefined. Notice that at $x = 0$ the derivative is undefined but $f(0) = 0$. Thus, the point $(0, 0)$ is a candidate for a being a vertical tangent to the graph of f. We check the following limits to determine if $(0, 0)$ is a vertical tangent. Since $45 - 24x - 156x^2 \to 45$ and $5x^{2/5} \to -\infty$ as $x \to 0^-$,

$$\lim_{x \to 0^-} \frac{45 - 24x - 156x^2}{5x^{2/5}} = +\infty.$$

Since $45 - 24x - 156x^2 \to 45$ and $5x^{2/5} \to -\infty$ as $x \to 0^-$,

$$\lim_{x \to 0^+} \frac{45 - 24x - 156x^2}{5x^{2/5}} = +\infty.$$

Therefore the function f has a vertical tangent at $(0, 0)$ which can be seen from the sketch of the graph of f. (see Figure ~**??**).

Example 3.44. Sketch the graph of $f(x) = x^{2/3}\left(x^2 + 5x - 20\right)$ and explain why there is a cusp at $x = 0$.

Solution. Using the product rule we find the derivative as

$$\begin{aligned} f'(x) &= \frac{2}{3}x^{-1/3}\left(x^2 + 5x - 20\right) + x^{2/3}(2x + 5) \\ &= \frac{2\left(x^2 + 5x - 20\right)}{3x^{1/3}} + \frac{3x^{1/3}x^{2/3}(2x + 5)}{3x^{1/3}} \\ &= \frac{2x^2 + 10x - 40}{3x^{1/3}} + \frac{3x(2x + 5)}{3x^{1/3}} \\ &= \frac{2x^2 + 10x - 40}{3x^{1/3}} + \frac{6x^2 + 15x}{3x^{1/3}} \\ &= \frac{8x^2 + 25x - 40}{3\sqrt[3]{x}}. \end{aligned}$$

To determine any cusps we consider where $f'(x)$ is undefined. Notice that at $x = 0$ the derivative is undefined but $f(0) = 0$. Thus the point $(0, 0)$ is a candidate for a being a cusp for the graph of the function f. We check the following limits to determine if $(0, 0)$ is a cusp. Since $8x^2 + 25x - 40 \to -40$ and $3\sqrt[3]{x} \to -\infty$ as $x \to 0^-$,

$$\lim_{x \to 0^-} \frac{8x^2 + 25x - 40}{3\sqrt[3]{x}} = +\infty.$$

Since $8x^2 + 25x - 40 \to -40$ and $3\sqrt[3]{x} \to +\infty$ as $x \to 0^+$,

$$\lim_{x \to 0^+} \frac{8x^2 + 25x - 40}{3\sqrt[3]{x}} = -\infty.$$

Therefore the function f has a cusp at $(0, 0)$ which can be seen from the sketch of the graph of f. (see Figure ~**??**).

Example 3.45. Determine any vertical tangents and cusps for the function

$$f(x) = x^{2/3}(x - 1)^{1/3}.$$

Solution. To find the vertical tangents and cusps we check where the first derivative is undefined. The derivative of f is

$$f'(x) = \frac{x^{2/3}}{3(x-1)^{2/3}} + \frac{2\sqrt[3]{x-1}}{3\sqrt[3]{x}} = \frac{3x - 2}{3(x-1)^{2/3}\sqrt[3]{x}}.$$

Therefore the points $(0, 0)$ and $(1, 0)$ are candidates for where a vertical tangent and cusp might occur. The function f has a vertical asymptote at $x = 1$ since

$$\lim_{x \to 1^-} f'(x) = \lim_{x \to 1^-} \frac{3x - 2}{3(x-1)^{2/3}\sqrt[3]{x}} = +\infty$$

The function f has a cusp at $x = 0$ since

$$\lim_{x \to 0^-} f'(x) = \lim_{x \to 0^-} \frac{3x - 2}{3(x-1)^{2/3}\sqrt[3]{x}} = +\infty$$

(see Figure ~**??**).

Example 3.46. Sketch the graph of the rational function

$$f(x) = \frac{2x^2}{x^2 - 1}$$

showing all special features.

Solution. The domain of f is

$$\{x\,|x^2 - 1 \neq 0\} = \{x| \neq \pm 1\} = (-\infty, -1) \cup (-1, 1) \cup (1, +\infty).$$

The x and y intercepts are both 0. Since $f(-x) = f(x)$ the function is even and so the curve is symmetric about the y-axis. Since

$$\lim_{x \to \pm\infty} \frac{2x^2}{x^2 - 1} = \lim_{x \to \pm\infty} \frac{2}{1 - \frac{1}{x^2}} = 2$$

the line $y = 2$ is a horizontal asymptote. Since the denominator is 0 when $x \pm 1$, we compute the following limits:

$$\begin{array}{ll} \displaystyle\lim_{x \to 1^+} \frac{2x^2}{x^2 - 1} = +\infty & \displaystyle\lim_{x \to 1^-} \frac{2x^2}{x^2 - 1} = -\infty \\ \displaystyle\lim_{x \to -1^+} \frac{2x^2}{x^2 - 1} = -\infty & \displaystyle\lim_{x \to -1^-} \frac{2x^2}{x^2 - 1} = +\infty \end{array}$$

Therefore, the lines $x = 1$ and $x = -1$ are vertical asymptotes. Next we find the derivative function.

$$f'(x) = \frac{4x\left(x^2 - 1\right) - 2x^2(2x)}{\left(x^2 - 1\right)^2} = \frac{-4x}{\left(x^2 - 1\right)^2}.$$

Since $f'(x) > 0$ when $x < 0$ $(x \neq -1)$ and $f'(x) < 0$ when $x > 0$ $(x \neq 1)$, f is increasing on $(-\infty, -1)$ and $(-1, 0)$ and decreasing on $(0, 1)$ and $(1, +\infty)$. The only critical number is $x = 0$. Since f' changes sign from positive to negative at 0, $f(0) = 0$ is a local maximum by the First Derivative Test. Also,

$$f''(x) = \frac{-4\left(x^2 - 1\right)^2 + 4x(2)\left(x^2 - 1\right)(2x)}{\left(x^2 - 1\right)^4} = \frac{12x^2 + 4}{\left(x^2 - 1\right)^3}$$

Since $12x^2 + 4 > 0$ for all x, we have $f''(x) > 0 \Longleftrightarrow x^2 - 1 > 0 \Longleftrightarrow |x| > 1$ and $f''(x) < 0 \Longleftrightarrow |x| < 1$. Thus the curve is concave downward on the intervals $(-\infty, -1)$ and $(1, +\infty)$ and concave downward on $(-1, 1)$. There is no point of inflection since 1 and -1 are not in the domain of f. (see Figure ~**??**).

Example 3.47. Sketch the graph of the trigonometric function

$$f(x) = 2\cos x + \sin 2x$$

showing all special features.

Solution. The domain of the function f is $\mathbb{R}$. The y-intercept is $(0, 2)$ since $f(0) = 2$. The x-intercepts occur when $2\cos x + \sin 2x = 2\cos x + 2\sin x \cos x = 2\cos x(1 + \sin x) = 0$ which is precisely when $x = \pi/2$ and

$x = 3\pi/2$, because we need only consider $[0, 2\pi]$ since function is periodic via,

$$f(x + 2\pi) = 2\cos(x + 2\pi) + \sin[2(x + 2\pi)] = 2\cos x + \sin 2x = f(x).$$

There are no vertical asymptotes nor horizontal asymptotes. The first derivative of f is,

$$f'(x) = -2\sin x + 2\cos 2x = -2(2\sin x - 1)(\sin x + 1).$$

Thus, $f'(x) = 0$ when $\sin x = 1/2$ or when $\sin x = -1$, so in $[0, 2\pi]$ we only consider the critical numbers $x = \pi/6$, $x = 5\pi/6$, and $x = 3\pi/2$. Applying the First Derivative Test we find:

Interval	f	f'	Conclusion
$0 < x < \pi/6$		$+$	increasing
$x = \pi/6$	$3\sqrt{3}/2$	0	relative maximum
$\pi/6 < x < 5\pi/6$		$-$	decreasing
$x = 5\pi/6$	$-3\sqrt{3}/2$	0	relative minimum
$5\pi/6 < x < 3\pi/2$		$+$	increasing
$x = 3\pi/2$	0	0	horizontal tangent
$3\pi/2 < x < 2\pi$		$+$	increasing

The second derivative of f is,

$$f''(x) = -2\cos x - 4\sin 2x = -2\cos x(1 + 4\sin x)$$

so the second order critical numbers are $x = \pi/2, 3\pi/2, \alpha_1$, and α_2 where $\sin\alpha_1 = -\frac{1}{4}$ and $\sin\alpha_2 = -\frac{1}{4}$. Applying the Concavity Test we find,

Interval	f	f''	Conclusion
$0 < x < \pi/2$		$-$	concave down
$x = \pi/2$	0	0	inflection point
$\pi/2 < x < \alpha_1$		$+$	concave up
$x = \alpha_1$	$f(\alpha_1)$	0	inflection point
$\alpha_1 < x < 3\pi/2$		$-$	concave down
$x = 3\pi/2$	0	0	inflection point
$3\pi/2 < x < \alpha_2$		$+$	concave up
$x = \alpha_2$	$f(\alpha_2)$	0	inflection point
$\alpha_2 < x < 2\pi$		$-$	concave down

3.28 Exercises

Exercise 3.52. Evaluate the following limits.

- $\lim_{x\to 0^-} \frac{5}{2x}$

- $\lim_{x\to -8^+} \frac{2x}{x+8}$
- $\lim_{x\to 0} \frac{-1}{x^2(x+1)}$
- $\lim_{x\to 0} \frac{-1}{x^2(x-1)}$
- $\lim_{x\to 0^+} \frac{2}{x^{1/5}}$
- $\lim_{x\to \frac{\pi}{2}^-} \tan x$

Exercise 3.53. Find the limits for the following functions. - $f(x) = \frac{1}{x^2-4}$ as $x \to 2^+$, $x \to 2^-$, $x \to -2^+$, and $x \to -2^-$.
- $f(x) = \frac{x^2-1}{2x+4}$ as $x \to -2^+$, $x \to -2^-$, $x \to 1^+$, and $x \to 0^-$.
- $f(t) = 2 - \frac{3}{t^{1/3}}$ as $t \to 0^+$, and $t \to 0^-$.
- $f(x) = \left(\frac{1}{x^{1/3}} - \frac{1}{(x-1)^{4/3}}\right)$ as $x \to 0^+$, $x \to 0^-$, $x \to 1^+$, and $x \to 1^-$.

Exercise 3.54. Graph the following rational functions and include the graphs and equations of their asymptotes.

- $f(x) = \frac{-3}{x-3}$
- $f(x) = \frac{x^2}{x-1}$
- $f(x) = \frac{x^2-1}{2x+4}$
- $f(x) = \frac{x^3-x^2+x-1}{(2x+3)(2x-1)}$

Exercise 3.55. For each of the following sketch the graph of a function $y = f(x)$ that satisfies the conditions. - $f(0) = 0$, $f(1) = 2$, $f(-1) = -2$, $\lim_{x\to -\infty} f(x) = -1$, and $\lim_{x\to\infty} f(x) = 1$ - $f(2) = 1$, $f(-1) = 0$, $\lim_{x\to\infty} f(x) = 0$, $\lim_{x\to 0^+} f(x) = \infty$, $\lim_{x\to 0^-} f(x) = -\infty$, and $\lim_{x\to -\infty} f(x) = 1$
- $\lim_{x\to\pm\infty} g(x) = 0$, $\lim_{x\to 3^-} g(x) = -\infty$, and $\lim_{x\to 3^+} g(x) = +\infty$

Exercise 3.56. Sketch the graph of the function

$$f(x) = \begin{cases} -(x+1)^2+1 & -1 \le x \le 0 \\ x^2 & 0 < x < 1 \\ 1 & 1 < x < 2 \\ 2 & x = 2 \\ 1 & 2 < x \le 3 \end{cases}$$

and then use the graph to determine which the following statements about the function $y = f(x)$ are true and which are false?

- $\lim_{x\to -1^+} f(x) = 1$
- $\lim_{x\to 2} f(x)$ does not exist
- $\lim_{x\to 2} f(x) = 2$
- $\lim_{x\to 1^-} f(x) = 2$
- $\lim_{x\to 1^+} f(x) = 1$
- $\lim_{x\to 1} f(x)$ does not exist
- $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$
- $\lim_{x\to c} f(x)$ exists at every c in the open interval $(-1, 1)$
- $\lim_{x\to c} f(x)$ exists at every c in the open interval $(1, 3)$
- $\lim_{x\to -1^-} f(x) = 0$
- $\lim_{x\to 3^+} f(x)$ does not exist

Exercise 3.57. Sketch the graph of the function

$$f(x) = \begin{cases} 3-x & x<2 \\ 2 & x=2 \\ \frac{x}{2} & x>2 \end{cases}$$

and then use the graph to determine the following? - Find $\lim_{x\to 2^+} f(x)$, $\lim_{x\to 2^-} f(x)$, and $f(2)$. - Does $\lim_{x\to 2} f(x)$ exist? If so, what is it? If not, why not? - Find $\lim_{x\to -1^-} f(x)$ and $\lim_{x\to -1^+} f(x)$. - Does $\lim_{x\to -1} f(x)$ exist? If so, what is it? If not, why not?

Exercise 3.58. Let $g(x) = \sqrt{x}\sin\left(\frac{1}{x}\right)$. Use the graph of g to determine the following. - Does $\lim_{x\to 0^+} g(x)$ exist? If so, what is it? If not, why not? - Does $\lim_{x\to 0^-} g(x)$ exist? If so, what is it? If not, why not? - Does $\lim_{x\to 0} g(x)$ exist? If so, what is it? If not, why not?

Exercise 3.59. Sketch the graph of the function

$$f(x) = \begin{cases} x^3 & x \neq 1 \\ 0 & x = 1 \end{cases}.$$

Find $\lim_{x\to 1^-} f(x)$ and $\lim_{x\to 1^+} f(x)$. Does $\lim_{x\to 1} f(x)$ exist? If so, what is it? If not, why not?

Exercise 3.60. Sketch the graph of the function

$$f(x) = \begin{cases} 1-x^2 & x \neq 1 \\ 2 & x = 1 \end{cases}.$$

Find $\lim_{x\to 1^-} f(x)$ and $\lim_{x\to 1^+} f(x)$. Does $\lim_{x\to 1} f(x)$ exist? If so, what is it? If not, why not?

Exercise 3.61. Sketch the graph of the function

$$f(x) = \begin{cases} x & -1 \le x < 0 \quad \text{or} \quad 0 < x \le 1 \\ 1 & x = 0 \\ 0 & x < -1 \quad \text{or} \quad x > 1. \end{cases}$$

- What is the domain and range of f? - At what points c, if any does $\lim_{x\to c} f(x)$ exist? - At what points does only the left-hand limit exist? - At what points does only the right-hand limit exist?

Exercise 3.62. Find the following one-sided limits algebraically.

- $\lim_{x\to 2^+} \frac{x(2x+5)}{(x+1)(x^2+x)}$
- $\lim_{h\to 0^+} \frac{\sqrt{h^2+4h+5}-\sqrt{5}}{h}$
- $\lim_{x\to 1^+} \frac{\sqrt{2x}(x-1)}{|x-1|}$

Exercise 3.63. Find the following two-sided limits.

- $\lim_{t\to 0} \left(\frac{\sin kt}{t}\right)$ where k is a constant
- $\lim_{x\to 0} \left(\frac{\tan 2x}{x}\right)$
- $\lim_{x\to 0} \left(6x^2(\cot x)(\csc 2x)\right)$
- $\lim_{h\to 0} \left(\frac{\sin(\sin h)}{\sin h}\right)$
- $\lim_{\theta\to 0} \left(\frac{\sin\theta}{\sin 2\theta}\right)$

Exercise 3.64. Find the limits for the following functions for both $x \to +\infty$ and $x \to -\infty$.

- $f(x) = \pi - \frac{2}{x^2}$
- $f(x) = \frac{-5+\left(\frac{7}{x}\right)}{3-\left(\frac{1}{x}\right)}$
- $f(\theta) = \frac{\cos\theta}{3\theta}$
- $f(x) = e^{-x}\sin x$
- $f(x) = \frac{2x+3}{5x+7}$
- $f(x) = \frac{x+1}{x^2+3}$

- $f(x) = \frac{9x^4+x}{2x^4+5x^2-x+6}$
- $f(x) = \frac{2\sqrt{x}+x^{-1}}{3x-7}$
- $f(x) = \frac{x^{-1}+x^{-4}}{x^{-2}-x^{-3}}$

3.29 Applied Optimization Problems

3.30 Optimization Procedures

In this section we give a few examples on how to set up a function to be optimized using its derivative. In general, the first step in solving an application problem is to understand the problem; maybe ask what are the unknowns? and what are the given quantities? Then the next best step is usually to draw a picture, labeling the unknowns and introducing notation. The final step should always be to check the solution to see that it makes sense for the given questions.

An optimization procedure is a step-by-step procedure to solve an application problem using calculus; and in particular, using derivatives. Here is a typical optimization procedure:

- Sketch a graph or draw a diagram.
- Introduce mathematical notation.
- Express information as expressions, equations and functions.
- Formulate the problem mathematically by stating the knowns and unknowns.
- State the knowns and unknowns.
- Identify needed theorems and validate their hypotheses. Apply the theorem.
- Solve the mathematical problem.
- Answer the original problem in terms of the given language.

We will illustrate this optimization procedure in the following examples.

3.31 Optimizing with Numbers

Example 3.48. Find two nonnegative numbers whose sum is 8 and the product of whose squares is as large as possible.

Solution. We are looking for two nonnegative numbers, say x and y with $x \geq 0$, $y \geq 0$, and $x + y = 8$. We want to maximize the function

$$P(x) = x^2y^2 = x^2(8-x)^2$$

with $x \geq 0$. Since the derivative of P,

$$P'(x) = 128x - 48x^2 + 4x^3 = 4(-8+x)(-4+x)x$$

is continuous, the only critical numbers are $x = 0, 4, 8$. We evaluate P to find $P(0) = 0 = P(8)$ and so the largest possible value is $P(4) = 256$ with $x = y = 4$.

Example 3.49. Under the condition that $2x - 5y = 18$, minimize $x^2 y$ when $x \geq 0$ and $y \geq 0$.

Solution. We want to minimize

$$P(x) = x^2 y = x^2 \left(\frac{1}{5}\right)(-18 + 2x).$$

Since the derivative of P,

$$P'(x) = \frac{6}{5}\left(-6x + x^2\right) = \frac{6}{5}(-6+x)x$$

is continuous, the only critical numbers are $x = 0$ and $x = 6$. We evaluate to find $P(0) = 0$ and the smallest possible value to be $P(6) = -\frac{216}{5}$. Thus the values are $x = 6$ and $y = \frac{-6}{5}$.

3.32 Optimizing Volume

Example 3.50. Find the dimensions of the right circular cylinder of largest volume that can be inscribed in a sphere of radius R.

Solution. Let R be the radius of the sphere and r the radius of the cylinder with height $2h$ so that $0 \leq h \leq R$. Since $r^2 + h^2 = R^2$, the volume of the cylinder is given by

$$V(h) = \pi r^2 2h = 2\pi h\left(R^2 - h^2\right) = 2\pi\left(R^2 h - h^3\right).$$

Using the Extreme Value Theorem, we wish to maximize the continuous function $V(h)$ on the closed interval $[0, R]$. The derivative of V is $V'(h) = 2\pi\left(R^2 - 3h^2\right)$. Since V' is continuous the only critical numbers are found by solving $V'(h) = 0$ for $0 \leq h \leq R$. Thus the only critical number is $h = R\big/\sqrt{3}$. Since $V(0) = V(R) = 0$, $h = 0$ and $h = R$ give minima, it follows by the Extreme Value Theorem that $h = R\big/\sqrt{3}$ must be a maximum. After solving for r we find the dimensions of the cylinder are, height: $2R\frac{\sqrt{3}}{3}$ and radius: $\frac{R}{3}\sqrt{6}$.

3.33 Optimizing with Geometry

Example 3.51. Find all points on the circle $x^2 + y^2 = a^2$ such that the product of the x-coordinate and the y-coordinate is as large as possible.

Solution. In the first quadrant we have $y = \sqrt{a^2 - x^2}$ and so we want to maximize $f(x) = x\sqrt{a^2 - x^2}$ subject to $x > 0$. The derivative of f is,

$$f'(x) = \frac{a^2 - 2x^2}{\sqrt{a^2 - x^2}}.$$

Thus the critical number is $x = \frac{a}{\sqrt{2}}$. The maximum value of the function f is

$$f\left(\frac{a}{\sqrt{2}}\right) = \frac{a}{\sqrt{2}}\sqrt{a^2 - \frac{a^2}{2}} = \frac{a^2}{2}.$$

Thus the points are

$$\left(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right) \text{ and } \left(\frac{-a}{\sqrt{2}}, \frac{-a}{\sqrt{2}}\right).$$

3.34 Optimizing Area

Example 3.52. A woman plans to fence off a rectangular garden whose area is 64 ft^2. What should be the dimensions of the garden if she wants to minimize the amount of fencing used?

Solution. Let x and y be the dimensions of the rectangular plot. The fencing (perimeter) is $P = 2x + 2y$ and the area is $A = xy = 64$ with domain $x > 0$. We want to minimize P so we write P as a function of one variable, say

$$P = 2x + \frac{2(64)}{x}.$$

Since the derivative of P is,

$$P' = 2 - \frac{128}{x^2} = \frac{2x^2 - 128}{x^2}$$

and is continuous the only critical number is when $P' = 0$ with $x > 0$ which is when $x = 8$. In this example we apply the First Derivative Test to verify that the critical number $x = 8$ is useful as follows

Interval	f	f'	Conclusion
$0 < x < 8$		$-$	decreasing
$x = 8$	8	0	local minimum
$x > 8$		$+$	increasing

Since P is decreasing on $(0,8)$ and increasing on $(0,\infty)$, the function P has an absolute minimum at $x=8$; and the dimensions of the garden should be 8 ft by 8 ft.

::: {#exm- } [Optimizing with Area] Someone with 750 ft of fencing wants to enclose a rectangular area and then divide it into four pens with fencing parallel to one side of the rectangle. What is the largest possible total area of the four pens? :::

Solution. Let x be the lengths of the 2 sides and let y be the lengths of the other 5 sides. Since there are three divides making the four pens $2x+2y+3y=750$. The area of the four pens is $A=xy$, thus we can solve $2x+2y+3y=750$ for y obtaining $y=\frac{1}{5}(750-2x)$. So a function of the area is

$$A(x)=\frac{1}{5}x(750-2x).$$

The derivative of A is, $A'(x)=\frac{1}{5}(750-4x)$ and so the critical number is $x=750/4$. The largest possible area is $A\left(\frac{750}{4}\right)=14062.5$ square feet.

3.35 Optimizing Angles

Example 3.53. The bottom of an 8-ft-high mural painted on a vertical wall is 13 ft above the ground. The lens of a camera fixed to a tripod is 4 ft above the ground. How far from the wall should the camera be placed to photograph the mural with the largest possible angle?

Solution. Let the horizontal distance from the camera to the wall be x. Let $ $ be the angle of elevation from the camera lens to the top of the mural and let $ $ the angle of elevation from the camera to the bottom of the mural. Also, let $\theta=\alpha-\beta$. Then

$$\theta(x)=\tan^{-1}\frac{17}{x}-\tan^{-1}\frac{9}{x}.$$

Since the first derivative of θ is,

$$\begin{aligned}\frac{d\theta}{dx}&=\frac{1}{1+\left(\frac{17}{x}\right)^2}\left(\frac{-17}{x^2}\right)-\frac{1}{1+\left(\frac{9}{x}\right)^2}\left(\frac{-9}{x^2}\right)\\&=\frac{-17}{x^2+289}+\frac{9}{x^2+81}\\&=-\frac{8\left(x^2-153\right)}{\left(x^2+81\right)\left(x^2+289\right)}\end{aligned}$$

we see that $\frac{d\theta}{dx}=0$ and $x>0$ when $x=\sqrt{153}$. Applying the First Derivative Test, the largest possible angle is when $x=\sqrt{153}=3\sqrt{17}$ or approximately 12.4 feet.

3.36 Optimizing Distance

Example 3.54. A truck is 250 mi due east of a sports car and is traveling west at a constant speed of 60 mi/h. Meanwhile, the sports car is going north at 80 mi/h. When will the truck and the car be closest to each other? What is the minimum distance between them?

Solution. Draw a figure with the car at the origin of a Cartesian coordinate system and the truck at $(250, 0)$. At time t, (in hours) the truck is at position $(250-x, 0)$, while the car is at $(0, y)$. Let D be the distance that separates them. Then $\frac{dx}{dt} = 60$ and $\frac{dy}{dt} = 80$ so that $x = 60t$ and $y = 80t$. We will minimize the square of the distance, $D^2 = (250 - x)^2 + y^2$,

$$D^2 = (250 - 60t)^2 + (80t)^2 = 2500\left(25 - 12t + 4t^2\right)$$

Since $\frac{dD^2}{dt} = 10,000(2t - 3)$ the derivative of the distance squared is 0 when $t = 1.5$hr. Substituting into the equation for D^2 produces the shortest distance: $x = 60(1.5) = 90$ and $y = 80(1.5) = 120$.
Thus, $D^2 = (250-90)^2 + 120^2 = 1600(25)$ and so $D = 40(5) = 200$ which is the minimum distance (because there is no maximum distance and the Extreme Value Theorem applies).

3.37 Optimizing Time

Example 3.55. A jeep is on the desert at a point P located 40 km from a point Q, which lies on a long straight road. The driver can travel at 45 km/h on the desert and 75 km/h on the road. The driver will win a prize if he arrives at the finish line at point F, 50 km from Q, in 84 minutes or less. What route should he travel to minimize the time of travel? Does he win the prize?

Solution. Suppose that the driver heads for a point S located x km down the road from Q towards his destination. We want to minimize the time. We will need to remember the formula $d = rt$, or in terms of time $t = d/r$. Since the distance between P and S is $\sqrt{x^2 + 1600}$ and the distance between S and F is $50 - x$, the total time is given by

$$T(x) = \frac{\sqrt{x^2 + 1600}}{45} + \frac{50 - x}{75} \text{where } 0 \le x \le 50.$$

Since

$$T'(x) = \frac{x}{45\sqrt{x^2 + 1600}} - \frac{1}{75} = \frac{5x - 3\sqrt{1600 + x^2}}{225\sqrt{1600 + x^2}}$$

we find that $x = 30$ is the only critical number of T. To find the extreme values we evaluate T at the endpoints, we find

$$T(30) = \frac{\sqrt{(30)^2 + 1600}}{45} + \frac{50 - 30}{75} = \frac{62}{45} = 1.37778 \text{ hr}$$

$$T(0) = \frac{\sqrt{(0)^2 + 1600}}{45} + \frac{50 - 0}{75} = \frac{14}{9} = 1.55556 \text{ hr}$$

$$T(50) = \frac{\sqrt{(50)^2 + 1600}}{45} + \frac{50 - 50}{75} = \frac{2\sqrt{41}}{9} = 1.42292 \text{ hr}$$

Therefore, the driver can minimize the total driving time by heading for a point that is 30km from the point Q and then traveling on the road to point D. He wins the prize because the minimal route is only 83 minutes.

3.38 Marginal Analysis

Marginal analysis is concerned with the way quantities such as price, cost, revenue, and profit vary with small changes in the level of production. The demand function $p(x)$ is defined to be the price that consumers will pay for each unit of the commodity when x units are brought to market. Then $R(x) = xp(x)$ is the total revenue function derived from the sale of the x units and $P(x) = R(x) - C(x)$ is the total profit function where $C(x)$ is the total cost function for producing x units.

::: {#exm- } [Optimizing Profits] A toy manufacturer produces an inexpensive doll (Dolly) and an expensive doll (Polly) in units of x hundred and y hundred, respectively. Suppose it is possible to produce the dolls in such a way that $y = \frac{82-10x}{10-x}$ with $0 \leq x \leq 8$ and that the company receives twice as much for selling a Polly doll as for selling a Dolly doll. Find the level of production for both x and y for which total revenue derived from selling these dolls is maximized. What vital assumption must be made about sales in the model? :::

::: {#exm- } [Optimizing Revenue] A business manager estimates that when p dollars are charged for every unit of a product, the sales will be $x = 380 - 20p$ units. At this level of production, the average cost is modeled by

$$A(x) = 5 + \frac{x}{30}.$$

- Find the total revenue and total cost functions, and express the profit as a function of x. - What price should the manufacturer charge to maximize profit? - What is the maximum profit? :::

::: {#exm- } [Optimizing Costs] Suppose the total cost (in dollars) of manufacturing x units of a certain commodity is $C(x) = 3x^2 + 5x + 75$.
- At what level of production is the average cost per unit the smallest?
- At what level of production is the average cost per unit equal to the

marginal cost? - Graph the average cost and the marginal cost on the same set of axes, for $x > 0$. :::

3.39 Exercises

Exercise 3.65. You are planning to close off a corner of the first quadrant with a line segment 20 units long running from $(a, 0)$ to $(0, b)$. Show that the area of the triangle enclosed by the segment is largest when $a = b$.

Exercise 3.66. Your iron work has contracted to design and build a 500ft^3, square-based, open-top, rectangular steel holding tank for a paper company. The tank is to be made by welding thin stainless steel plates together along their edges. As the production engineer, your job is to find dimensions for the base and height that will make the tank weigh as little as possible. (a) What dimensions do you tell the shop to use? (b) Briefly describe how you took weight into account.

Exercise 3.67. The bottom of an 8-ft-high mural painted on a vertical wall is 13 ft above the ground. The lens of a camera fixed to a tripod is 4 ft above the ground. How far from the wall should the camera be placed to photograph the mural with the largest possible angle?

Exercise 3.68. A 1125 ft^3 open-top rectangular tank with a square base x ft on a side and y ft deep is to be built with its top flush with the ground to catch runoff water. The costs associated with the tank involve not only the material from which the tank is made but also an excavation charge proportional to the product xy. (a) If the total cost is $c = 5\left(x^2 + 4xy\right) + 10xy$ what values of x and y will minimize it? (b) Give a possible scenario fro the cost function in part (a).

Exercise 3.69. Two sides of a triangle have lengths a and b, and the angle between then θ. What value of \$ \$ will maximize the triangle's area?

Exercise 3.70. The height of an object moving vertically is given by $s = -16t^2 + 96t + 112$ with s in feet and t in seconds. Find the object's velocity when $t = 0$. Find its maximum height and when it occurs. Also find its velocity when $s = 0$.

Exercise 3.71. Find the dimensions of the right circular cylinder of largest volume that can be inscribed in a sphere of radius R.

Exercise 3.72. Jane is 2 mi offshore in a boat and wishes to reach a coastal village 6 mi down a straight shoreline from the point nearest the boat. She can now row 2 mph can walk 5 mph. Where should she land her boat to reach the village in the least amount of time?

Exercise 3.73. The positions of two particles on the s-axis are $s_1 = \sin t$ and $s_2 = \sin(t + \pi/3)$ with s_1 and s_2 in meters and t in seconds. (a) At what time(s) in the interval $0 t 2 $ do the particles meet? (b) What is the farthest apart that the particles ever get? (c) When in the interval $0 t 2 $ is the distance between the particles chaining the fastest?

Exercise 3.74. A truck is 250 mi due east of a sports car and is traveling west at a constant speed of 60 mi/h. Meanwhile, the sports car is going north at 80 mi/h. When will the truck and the car be closest to each other? What is the minimum distance between them?

Exercise 3.75. A jeep is on the desert at a point P located 40 km from a point Q, which lies on a long straight road. The driver can travel at 45 km/h on the desert and 75 km//h on the road. The driver will win a prize if he arrives at the finish line at point F, 50 km from Q, in 84 minutes or less. What route should he travel to minimize the time of travel? Does he win the prize?

3.40 Indeterminate Forms

We say that $\lim_{x\to\infty} \frac{3^x-1}{x^3}$ has the intermediate form **intermediate form** $\frac{\infty}{\infty}$ because $3^x \to \infty$ and $x^3 \to \infty$ as $x \to \infty$

In this section we will consider the following seven indeterminate forms

$$\frac{\infty}{\infty}, \qquad \frac{0}{0}, \qquad \infty - \infty, \qquad 0 \cdot \infty, \qquad \infty^0, \qquad 0^0, \qquad 1^\infty.$$

3.41 l'H^opital's Rule

Our main investigative tool will be l'H^opital's rule which says that the limit of a quotient of functions f and g is equal to the limit of the quotient of their derivatives f' and g' provided some conditions are satisfied. It is especially important to verify the conditions regarding the limits of f and g before applying l'H^opital's rule.

::: {#thm- } l'H^opital's Rule Suppose f and g are differentiable functions and $g'(x) \neq 0$ on an open interval I that contains c. Suppose

$$\lim_{x\to c} \frac{f(x)}{g(x)}$$

produces an intermediate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and that

$$\lim_{x\to c} \frac{f'(x)}{g'(x)}$$

exists or is infinite, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

:::

Remark. Keep in mind that l'Hôpital's' rule also holds if $x \to c$ is replaced by $x \to c^+$, $x \to c^-$, $x \to \infty$, $x \to -\infty$.

To illustrate l'H^opital's Rule we will evaluate several limits.

3.42 Indeterminate Forms 0/0 and ∞/∞

Example 3.56. Evaluate $\lim\limits_{x\to\infty} \dfrac{\ln x}{x}$ using l'H^opital's rule.

Solution. The given limit has the indeterminate form $\frac{\infty}{\infty}$ since $\lim_{x\to\infty} \ln x = +\infty$ and $\lim_{x\to+\infty} x = +\infty$. We apply l'H^opital's rule to evaluate the limit as follows

$$\lim_{x\to+\infty} \frac{\ln x}{x} = \lim_{x\to+\infty} \frac{\frac{1}{x}}{1} = 0.$$

Example 3.57. Evaluate

$$\lim_{x\to-\infty} \frac{x^2}{e^{-x}}$$

using l'H^opital's rule.

Solution. The given limit has the indeterminate form $\frac{\infty}{\infty}$ since $\lim_{x\to-\infty} x^2 = +\infty$ and $\lim_{x\to-\infty} e^{-x} = +\infty$. We apply l'H^opital's rule to evaluate the limit as follows

$$\lim_{x\to-\infty} \frac{x^2}{e^{-x}} = \lim_{x\to-\infty} \frac{2x}{-e^{-x}} = \lim_{x\to-\infty} \frac{2}{e^{-x}} = 0.$$

Thus illustrating that L'Hopital's rule can be used multiple times.

We are justified in calling $\frac{\infty}{\infty}$ an indeterminate form since

$$\lim_{x\to+\infty}\frac{\ln x}{x}=0\neq 1=\lim_{x\to 0}\frac{\sin x}{x}.$$

Example 3.58. Evaluate

$$\lim_{x\to 0}\frac{\sin x}{x}$$

using l'H^opital's rule.

Solution. The given limit has the indeterminate form $\frac{0}{0}$ since $\lim_{x\to 0}\sin x = 0$ and $\lim_{x\to 0} x = 0$. We apply l'H^opital's rule to evaluate the limit as follows

$$\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{\cos x}{1}=1.$$

Example 3.59. Evaluate

$$\lim_{x\to 0}\frac{e^{2x}-1}{x}$$

using l'H^opital's rule.

Solution. The given limit has the indeterminate form $\frac{0}{0}$ since $\lim_{x\to 0} e^{2x}-1=0$ and $\lim_{x\to 0} x = 0$. We apply l'H^opital's rule to evaluate the limit as follows

$$\lim_{x\to 0}\frac{e^{2x}-1}{x}=\lim_{x\to 0}\frac{2e^{2x}}{1}=2.$$

We are justified in calling $\frac{0}{0}$ an indeterminate form since

$$\lim_{x\to 0}\frac{\sin x}{x}=1\neq 2=\lim_{x\to 0}\frac{e^{2x}-1}{x}.$$

Example 3.60. Evaluate

$$\lim_{x\to 0}\frac{a^x-b^x}{x}$$

using l'H^opital's rule, where $a, b > 0$.

Solution. The given limit has the indeterminate form $\frac{0}{0}$ since $\lim_{x\to 0} a^x - b^x = 0$ and $\lim_{x\to 0} x = 0$. We apply l'H^opital's rule to evaluate the limit as follows

$$\lim_{x\to 0}\frac{a^x-b^x}{x}=\lim_{x\to 0}\frac{(\ln a)a^x-(\ln b)b^x}{1}=\ln a-\ln b=\ln\frac{a}{b}.$$

3.43 Indeterminate Forms $\infty - \infty$ and $0 \cdot \infty$

Example 3.61. Evaluate

$$\lim_{x\to+\infty} e^{-x}\sqrt{x}$$

using l'Hˆopital's rule.

Solution. The given limit has indeterminate form $0 \cdot \infty$ since $\lim_{x\to+\infty} e^{-x} = 0$ and $\lim_{x\to+\infty} \sqrt{x} = +\infty$. We apply l'Hˆopital's rule to evaluate the limit as follows

$$\lim_{x\to+\infty} \frac{\sqrt{x}}{e^x} = \lim_{x\to+\infty} \frac{\frac{1}{2\sqrt{x}}}{e^x} = \lim_{x\to+\infty} \frac{1}{2\sqrt{x}e^x} = 0.$$

Example 3.62. Evaluate

$$\lim_{x\to 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$$

using l'Hˆopital's rule.

Solution. The given limit has indeterminate form $\infty - \infty$ since $\lim_{x\to 1^+} \frac{1}{\ln x} = +\infty$ and $\lim_{x\to 1^+} \frac{1}{x-1} = +\infty$. We apply l'Hˆopital's rule to evaluate the limit as follows

$$\lim_{x\to 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1}\right) = \lim_{x\to 1^+} \frac{x-1-\ln x}{(x-1)\ln x} = \lim_{x\to 1^+} \frac{1-\frac{1}{x}}{\ln x + \frac{x-1}{x}}$$
$$= \lim_{x\to 1^+} \frac{x-1}{x\ln x + x - 1} = \lim_{x\to 1^+} \frac{1}{\ln x + x\left(\frac{1}{x}\right) + 1} = \frac{1}{2}$$

3.44 Indeterminate Forms ∞^0, 0^0, and 1^∞

The limit

$$\lim_{x\to a} f(x)^{g(x)}$$

is said to an indeterminate form of the type

- [] ∞^0 if $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$, $$-5pt]
- [] 0^0 if $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, and $$-5pt]
- [] 1^∞ if $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \pm\infty$.

With these indeterminate forms we can apply the identity $f(x)^{g(x)} = e^{g(x)\ln f(x)}$. Since we know that the exponential function is continuous, we can use

$$\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a} e^{g(x)\ln f(x)} = e^{\left(\lim\limits_{x\to a} g(x)\ln f(x)\right)}$$

and then possibly apply l'Hˆopital's rule, of course we must first check whether or not $\lim_{x\to a} g(x)\ln f(x)$ exists first.

Example 3.63. Evaluate

$$\lim_{x\to 0^+}(\sin x)^x$$

using l'Hˆopital's rule.

Solution. The given limit has the indeterminate form 0^0 since $\lim_{x\to 0^+}\sin x = 0$ and $\lim_{x\to 0^+} x = 0$. Since $\lim_{x\to 0^+}\ln(\sin x) = +\infty$ and $\lim_{x\to 0^+}\frac{1}{x} = +\infty$ we have indeterminate form of $\frac{\infty}{\infty}$ and so we apply l'Hˆopital's rule to evaluate the following limit

$$\lim_{x\to 0^+} x\ln\sin x = \lim_{x\to 0^+}\frac{\ln(\sin x)}{\frac{1}{x}} = \lim_{x\to 0^+}\frac{\frac{\cos x}{\sin x}}{\frac{-1}{x^2}} = \lim_{x\to 0^+}\frac{-x^2}{\tan x} = \lim_{x\to 0^+}\frac{-2x}{\sec^2 x} = 0.$$

Therefore, since the exponential function is continuous,

$$\lim_{x\to 0^+}(\sin x)^x = \lim_{x\to 0^+} e^{x\ln\sin x} = e^{\left(\lim\limits_{x\to 0^+} x\ln\sin x\right)} = e^0 = 1.$$

Example 3.64. Evaluate

$$\lim_{x\to +\infty} x^{1/x}$$

using l'Hˆopital's rule.

Solution. The given limit has the indeterminate form ∞^0 since $\lim_{x\to\infty}\frac{1}{x} = 0$ and $\lim_{x\to +\infty}\frac{1}{x} = +\infty$.
Since $\lim_{x\to\infty}\ln x = +\infty$ and $\lim_{x\to\infty} x = +\infty$ the following limit has indeterminate form $\frac{\infty}{\infty}$ and so we apply l'Hˆopital's rule as follows

$$\lim_{x\to +\infty}\frac{1}{x}\ln x = \lim_{x\to +\infty}\frac{\frac{1}{x}}{1} = 0$$

Since the exponential function is continuous

$$\lim_{x\to +\infty} x^{1/x} = \lim_{x\to +\infty} e^{\frac{1}{x}\ln x} = e^{\left(\lim\limits_{x\to +\infty}\frac{1}{x}\ln x\right)} = e^0 = 1.$$

Example 3.65. Evaluate

$$L = \lim_{x\to 0^+}(1+\sin 4x)^{\cot x}$$

using l'Hˆopital's rule.

Solution. The given limit has the indeterminate form 1^∞ since $\lim_{x\to 0^+} 1+\sin 4x = 1$ and $\lim_{x\to 0^+} \cot x = +\infty$. Notice $\lim_{x\to 0^+} \frac{\ln(1+\sin 4x)}{\tan x}$ has indeterminate form $\frac{0}{0}$ since $\lim_{x\to 0^+} \ln(1+\sin 4x) = 0$ and $\lim_{x\to 0^+} \tan x = 0$. We apply l'H^opital's rule to find

$$\lim_{x\to 0^+} \frac{\ln(1+\sin 4x)}{\tan x} = \lim_{x\to 0^+} \frac{\frac{4\cos 4x}{1+\sin 4x}}{\sec^2 x} = \frac{\frac{4}{1+0}}{1} = 4.$$

Since the exponential function is continuous

$$\begin{aligned}\lim_{x\to 0^+} (1+\sin 4x)^{\cot x} &= \lim_{x\to 0^+} e^{\cot x \ln(1+\sin 4x)} \\ &= e^{\left(\lim_{x\to 0^+} \cot x \ln(1+\sin 4x)\right)} = e^4.\end{aligned}$$

3.45 Change of Variable

In the next example we show how making a change of variable can simplify the process of evaluating a limit.

Example 3.66. Evaluate

$$\lim_{x\to +\infty} x^5 \left[\sin\left(\frac{1}{x}\right) - \frac{1}{x} + \frac{1}{6x^3}\right]$$

using l'H^opital's rule.

Solution. We make a change of variable to simplify the expression, namely $u = \frac{1}{x}$. Since $u \to 0$ as $x \to +\infty$, we have

$$\lim_{x\to +\infty} x^5 \left[\sin\left(\frac{1}{x}\right) - \frac{1}{x} + \frac{1}{6x^3}\right] = \lim_{u\to 0} \frac{\sin(u) - u + \frac{1}{6}u^3}{u^5}$$

Now we have indeterminate form $\frac{0}{0}$ and so applying l'H^opital rule several times yields

$$\begin{aligned}&= \lim_{u\to 0} \frac{\sin(u) - u + \frac{1}{6}u^3}{u^5} = \lim_{u\to 0} \frac{(\cos u) - 1 + \frac{1}{2}u^2}{5u^4} = \lim_{u\to 0} \frac{(-\sin u) + u}{20u^3} \\ &= \lim_{u\to 0} \frac{(-\cos u) + 1}{60u^2} = \lim_{u\to 0} \frac{\sin u}{120u} = \lim_{u\to 0} \frac{\cos u}{120} = \frac{1}{120}.\end{aligned}$$

3.46 l'H^opital's Rule Fails

In this final example we illustrate one way in which l'H^opital's rule can fail even though the value of the limit is finite.

Example 3.67. Try to evaluate

$$\lim_{x\to+\infty} \frac{x+\sin x}{x-\cos x}$$

using l'H^opital's rule.

Solution. This limit has indeterminate form since

$$\lim_{x\to+\infty}(x+\sin x) = +\infty \quad \text{and} \quad \lim_{x\to+\infty}(x+\cos x) = +\infty.$$

If we try to apply L'Hospitals's Rule we find,

$$\lim_{x\to+\infty} \frac{x+\sin x}{x-\cos x} = \lim_{x\to\infty} \frac{1+\cos x}{1+\sin x}$$

but the limit $\lim_{x\to\infty} \frac{1+\cos x}{1+\sin x}$ does not exist because of osculating behavior, so we can not use l'H^opital's rule. To correctly find this limit we divide by x as follows

$$L = \lim_{x\to+\infty} \frac{x+\sin x}{x-\cos x} = \lim_{x\to+\infty} \frac{\frac{x}{x}+\frac{\sin x}{x}}{\frac{x}{x}-\frac{\cos x}{x}} = \frac{1+0}{1-0} = 1.$$

3.47 Exercises

Exercise 3.76. Use l'H^opital's rule to find the following limits.

- $\lim_{x\to2} \frac{x-2}{x^2-4}$.
- $\lim_{x\to1} \frac{x^3-1}{4x^3-x-3}$.
- $\lim_{x\to-5} \frac{x^2-25}{x+5}$.
- $\lim_{x\to0} \frac{\sin x^2}{x}$.
- $\lim_{x\to0} \frac{\sin x - x}{x^3}$.
- $\lim_{x\to\pi/2} \frac{1-\sin x}{1+\cos 2x}$.
- $\lim_{x\to\pi/2} \frac{\ln(\csc x)}{\left(x-\frac{\pi}{2}\right)^2}$.
- $\lim_{x\to\left(\frac{\pi}{2}\right)^-} \left(x-\frac{\pi}{2}\right)\sec x$.
- $\lim_{x\to0} \frac{\left(\frac{1}{2}\right)^x-1}{x}$.
- $\lim_{x\to0^+} \frac{\ln(e^x-1)}{\ln x}$.
- $\lim_{x\to\infty}(\ln 2x - \ln(x+1))$.
- $\lim_{x\to0^+}\left(\frac{3x+1}{x} - \frac{1}{\sin x}\right)$.
- $\lim_{x\to0} \frac{\cos x-1}{e^x-x-1}$.
- $\lim_{x\to\infty} x^2 e^{-x}$.
- $\lim_{x\to\infty}(\ln x)^{1/x}$.
- $\lim_{x\to\infty} x^{1/\ln x}$.
- $\lim_{x\to\infty}(1+2x)^{1/2\ln x}$.

- $\lim_{x\to 0^+} \frac{\sqrt{x}}{\sqrt{\sin x}}$.

Exercise 3.77. Let

$$f(x) = \begin{cases} x+2 & x \neq 0 \\ 0 & x = 0 \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} x+1 & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Show that $\lim_{x\to 0} \frac{f'(x)}{g'(x)} = 1$ but $\lim_{x\to 0} \frac{f(x)}{g(x)} = 2$. Explain why this does not contradict l'H^opital's Rule.

Exercise 3.78. Find constant a and b so that $\lim_{x\to 0} \left(\frac{\sin 2x}{x^3} + \frac{a}{x^2+b} \right) = 1$.

Exercise 3.79. Find all values of a and b so that $\lim_{x\to 0} \frac{\sin ax + bx}{x^3} = 36$.

Exercise 3.80. Find the values of a so that $\lim_{x\to 0} \frac{a - \cos bx}{x^2} = 2$.

Exercise 3.81. For a certain value of a, the limit $\lim_{x\to+\infty} \left(x^4 + 5x^3 + 3\right)^a - x$ is finite and nonzero. Find a and then use l'H^opital's rule to compute the limit.

Exercise 3.82. Evaluate $\lim_{x\to a} \frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}}$.

Exercise 3.83. Determine which values of constants a and b is it true that

$$\lim_{x\to 0} \left(x^{-3} \sin 7x + ax^{-2} + b\right) = -2?$$

Chapter 4

Integration

This book will help readers understand the fundamentals of integration, and how to apply them in practical situations.

This book is for anyone who wants to learn more about integration theory and how to apply it in practical situations. Whether you are a student preparing for a calculus class, or an engineer working on a complex problem, this book has something for you.

Integration is one of the fundamental operations in calculus. It allows us to find the area under a curve, the volume of a solid of revolution, and many other applications. integration is basically finding the sum of an infinite number of infinitesimal pieces. It's crazy to think about, but it turns out to be incredibly useful.

So integration is important because it helps us calculate all sorts of things that we wouldn't be able to calculate without it. And that's why it's such an integral part of the calculus. Get it? Integral? Anyway, that's why integration is important.

integration can be used to find the area under a curve. It is essentially the inverse of differentiation. Whereas differentiation allows you to find the slope of a curve at any given point, integration allows you to find the area under a curve between two points. Indefinite integration is integration without limits, which means that you are finding the area under the curve for all points between two given points.

As with differentiation, there are many rules and properties that you can use to simplify integration problems. However, it is still generally a complex process. Nevertheless, indefinite integration is a powerful tool that can be used to solve a variety of problems in mathematics and physics. And who knows, maybe one day you'll be able to use it to find the area under your favorite curve!

There are two main types of integration: indefinite and definite. Indefinite integration is when we don't have bounds on our integrals, while definite integration has specific boundaries. Both are important, and each has its own uses.

So whether you're just getting started with integration or you're a seasoned pro, I hope this article gives you a better understanding of what indefinite integration is and how it works.

Integration by substitution is a technique for finding the integral of a function by making an appropriate substitution. The idea is to find a new function, f(x), whose derivative is equal to the original function, g(x). Then, using the known formula for the integration of f(x), we can find the integral of g(x). This technique can be used to find integrals that would be difficult or impossible to calculate using other methods.

In addition, it can be used to simplify complex integrals. Integration by substitution is a powerful tool that can be used to solve a variety of problems.

If you've ever had to integrate a function, then you know that it can be a pain. However, there is a method that can make integration easier: Riemann sums. Riemann sums involve taking the area under the curve and dividing it into small intervals, each of which can be approximated by a rectangle. The heights of the rectangles are determined by the function values at the endpoint of each interval. Then, all of the rectangles are added together to approximate the area under the curve.

Although Riemann sums may seem like a lot of work, they can actually make integration easier because they provide a way to break up the area into manageable pieces. Plus, once you get the hang of it, integration using Riemann sums can be quite satisfying. So next time you're stuck trying to integrate a function, give Riemann sums a try!

So what's the big deal with area and Riemann sums? In a nutshell, they're used to calculate the area under a curve. To do this, we divide the curve into a bunch of small rectangles, calculate the area of each rectangle, and then add them all up. The more rectangles we use, the more accurate our answer will be.

But what if we want to be really precise? That's where integration comes in. Integration is just a fancy way of saying "area under the curve," and it gives us a way to calculate the exact area of any shape, no matter how complex. So if you're ever stuck trying to find the area of something, remember: integration is your friend!

More formally, integration is the process of finding a function that represents the rate of change of a quantity over time.

The definite integral is a tool that allows us to calculate the exact area under a curve. It is defined as the limit of the sum of an infinite number of small rectangles, each infinitesimally close to the curve. The height of each rectangle is equal to the value of the function at its x-coordinate, and the width of each rectangle is infinitesimally small. As the number of rectangles approaches infinity, the sum of their areas approaches the true value of the integral.

This definition may seem confusing at first, but it provides a precise way to calculate the area under any smooth curve. In practice, integration is used to solve problems in physics, engineering, and finance. It can be used to calculate things like displacement, velocity, and acceleration. It can also be used to find out how much money will be required to fund a project over time.

In short, integration is a powerful tool that can be used to solve a wide variety of problems.

The Fundamental Theorem of Calculus is a pretty amazing result. It states that integration and differentiation are inverse operations. In other words, if you know how to do one, you can do the other. And that's not all! The theorem has two versions, each of which is pretty darn useful.

The first version says that if you have a function f(x), and you want to find its integral, all you need to do is find another function F(x) such that F'(x)=f(x). This is often called the "differential form" of the theorem, and it's very handy when you're trying to find an antiderivative but you're having trouble coming up with a closed-form solution.

The second version says that if you have a function f(x), and you want to find its derivative, all you need to do is find another function F(x) such that $F(a) - F(b) = \int_a^b f(x)dx$ for any values of a and b. This is sometimes called the "integral form" of the theorem, and it's very handy when you're trying to find a derivative but you don't have access to the original function.

So there you have it: the Fundamental Theorem of Calculus in all its glory. Whether you need to find an antiderivative or a derivative, this theorem has got your back!

The Fundamental Theorem of Calculus is a theorem that states that integration and differentiation are inverse operations. In other words, if a function is integrated, the resulting function will be the derivative of the original function.

The theorem has two versions: the first version states that integration is the antiderivative of a function, while the second version states that integration is the area under a curve. The theorem is named after Isaac Newton and Gottfried Leibniz, who are credited with its discovery. Today, the theorem is an important part of calculus and has many applications in mathematics and physics.

Without it, we would not be able to solve differential equations or to perform statistical inference. So if you ever wondered what calculus is good for, wonder no more: it's good for pretty much everything!

Integration can be used to find the area of a region, the volume of a solid, the rate of change of a function, and much more. Integration is a powerful tool that can be used to solve many problems in mathematics and physics. However, it is not always easy to calculate an integral. In many cases, integration must be done numerically by approximating the area under the curve. This can be done using integration formulas or by using numerical integration methods such as trapezoidal rule or Simpson's rule. Integration is an important topic in calculus and is essential for understanding many concepts in physics and engineering.

Teaching integration can be tough. Some students love the challenge of finding a function that represents the area under a curve, while others find it tedious. In this book, I've tried to find a balance between the two.

We'll start with the basics of integration, and gradually work our way up to more difficult concepts. And along the way, we'll apply what we've learned to practical situations.

In addition, the book will include worked examples and practice problems to help readers master the material. Whether you're a student learning integration for the first time, or a seasoned mathematician looking for a refresher, this book will give you the tools you need to succeed.

So whether you're a student who loves integration or one who just wants to get through it, this book is for you.

4.1 Indefinite Integrals

4.2 Antiderivatives

We go over antidifferentation by defining an antiderivative function and working out examples on finding antiderivatives. We also concentrate on the following problem: if F is an antiderivative of the continuous function f, then any other antiderivative of f must have the form $F(x)+C$ where C is some constant.
Thus showing, if a function $F(x)$ is an antiderivative of the function $f(x)$ then so is $F(x)+C$ where C is called an arbitrary constant. The mean value theorem will be used to show that all derivatives of $f(x)$ are of the form $F(x)+C$ and that there are no others.

Definition 4.1. A function F is called an **antiderivative** of a given function f on an interval I if $F'(x)=f(x)$ for all x in I.

Theorem 4.1. *If F is an antiderivative of the continuous function f, then any other antiderivative G, of f must have the form $G(x) = F(x) + C$ where C is some constant.*

Proof. We show that if F is differentiable in $[a, b]$ and $F'(x) = 0$ for all x in $[a, b]$, then $F(x) = F(a)$ for all x in $[a, b]$. By the mean-value theorem applied to $[a, x]$ for any x such that $a < x \leq b$,

$$\frac{F(x) - F(a)}{x - a} = F'(c) = 0$$

where $a < c < x$. Thus $F(x) = F(a) = 0$, so $F(x) = F(a)$, and $F(x)$ is constant in $[a, b]$. So in fact, for $F'(x) = f(x)$ with all x in $[a, b]$, if we suppose $G'(x) = f(x)$ also for x in $[a, b]$ then

$$\frac{d(G(x) - F(x))}{dx} = G'(x) - F'(x) = 0.$$

Thus $G(x) - F(x) = G(a) - F(a)$. If we let $C = G(a) - F(a)$, then $G(x) = F(x) + C$.

□

The notation $\int f(x)\,dx = F(x) + C$ where C is an arbitrary constant means that F is an antiderivative of f. The function F is called the **indefinite integral** of f and satisfies the condition that $F'(x) = f(x)$ for all x in the domain of f. It is important to remember that $F(x) + C$ represents a family of functions.

4.3 Integral Notation

Example 4.1. Find the family of antiderivatives of the function $f(x) = \sin x$ and write an equation using the indefinite integral notation.

Solution. If $F(x) = -\cos x$, then $F'(x) = \sin x$, and so an antiderivative of sine is $-\cos x$. Thus the general antiderivative is $G(x) = -\cos x + C$. Therefore, $\int \sin x dx = -\cos x + C$ where C is an arbitrary constant.

Example 4.2. Find the family of antiderivatives of the function $f(x) = x^n$, $n \geq 0$, and write an equation using the indefinite integral notation.

Solution. Since

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{(n+1)x^n}{n+1} = x^n$$

the general antiderivative of f is

$$F(x) = \frac{x^{n+1}}{n+1} + C,$$

where C is a constant, which is valid for $n \geq 0$ because $f(x) = x^n$ is defined on the interval $(-\infty, +\infty)$. Therefore,

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$$

where C is an arbitrary constant and $n \geq 0$.

Theorem 4.2. *Suppose f and g are integrable functions and a, b, and c are constants. Then*

- $\displaystyle\int cf(u)du = c\int f(u)\,du$
- $\displaystyle\int [f(u) + g(u)]\,du = \int f(u)\,du + \int g(u)\,du$
- $\displaystyle\int [f(u) - g(u)]\,du = \int f(u)\,du - \int g(u)\,du$
- $\displaystyle\int [af(u)du + bg(u)]\,du = a\int f(u)du + b\int g(u)\,du$

Theorem 4.3. *Suppose C is an arbitrary constant. Then*

$$\int x^n \, dx = \begin{cases} \dfrac{x^{n+1}}{n+1} + C & n \neq -1 \\ \ln|x| + C & n = -1. \end{cases}$$

Theorem 4.4. *Suppose u is a differentiable function of x and C is an arbitrary constant.*

- $\displaystyle\int 0\,du = 0 + C$
- $\displaystyle\int e^u\,du = e^u + C$
- $\displaystyle\int \sin u\,du = -\cos u + C$
- $\displaystyle\int \cos u\,du = \sin u + C$
- $\displaystyle\int \sec^2 u\,du = \tan u + C$
- $\displaystyle\int \sec u \tan u\,du = \sec u + C$

- $\displaystyle\int \csc u \cot u \, du = -\csc u + C$
- $\displaystyle\int \csc^2 u \, du = -\cot u + C$
- $\displaystyle\int \frac{1}{\sqrt{1-u^2}} \, du = \sin^{-1} u + C$
- $\displaystyle\int \frac{1}{1+u^2} \, du = \tan^{-1} u + C$
- $\displaystyle\int \frac{1}{|u|\sqrt{u^2-1}} \, du = \sec^{-1} u + C$

Example 4.3. Find the general antiderivative of the function

$$f(x) = \frac{x^2}{x^2+1}.$$

Solution. Since

$$\frac{x^2}{x^2+1} = \frac{x^2+1-1}{x^2+1} = 1 - \frac{1}{x^2+1}$$

we find

$$\begin{aligned} \int \frac{x^2}{x^2+1} \, dx &= \int \left(1 - \frac{1}{x^2+1}\right) \, dx \\ &= \int dx - \int \frac{1}{x^2+1} \, dx = x - \tan^{-1} x + C. \end{aligned}$$

Example 4.4. Find the general antiderivative of the function

$$f(x) = \left(1 + \frac{1}{x}\right)\left(1 - \frac{4}{x^2}\right).$$

Solution. We want to evaluate $\int \left(1 + \frac{1}{x}\right)\left(1 - \frac{4}{x^2}\right) dx$. Since

$$\left(1 + \frac{1}{x}\right)\left(1 - \frac{4}{x^2}\right) = 1 + \frac{1}{x} - \frac{4}{x^2} - \frac{4}{x^3}$$

we use the linearity rule and the power rule to find

$$\begin{aligned} \int \left(1 + \frac{1}{x}\right)\left(1 - \frac{4}{x^2}\right) dx &= \int \left(1 + x^{-1} - 4x^{-2} - 4x^{-3}\right) \, dx \\ &= x + \ln x + \frac{4}{x} + \frac{2}{x^2} + C. \end{aligned}$$

Example 4.5. Find $f(x)$ given $f''(x) = x + \sqrt{x}$, $f(1) = 1$, and $f'(1) = 2$.

Solution. First we find f' by

$$f'(x) = \int (x + \sqrt{x})\ dx = \frac{x^2}{2} + \frac{2x^{3/2}}{3} + C$$

where C is some constant which can be determined with $f'(1) = 2$. So

$$f'(1) = \frac{(1)^2}{2} + \frac{2(1)^{3/2}}{3} + C = \frac{7}{6} + C = 2 \qquad \implies \qquad C = \frac{5}{6}.$$

So in fact

$$f'(x) = \frac{x^2}{2} + \frac{2x^{3/2}}{3} + \frac{5}{6}.$$

Now to find f we follow the same procedure

$$f(x) = \int \left(\frac{x^2}{2} + \frac{2x^{3/2}}{3} + \frac{5}{6}\right)\ dx = \frac{x^3}{6} + \frac{4x^{5/2}}{15} + \frac{5x}{6} + K$$

where K is a constant which can be determined with $f(1) = 1$. So

$$f(1) = \frac{(1)^3}{6} + \frac{4(1)^{5/2}}{15} + \frac{5(1)}{6} + K = \frac{19}{15} + K = 1 \qquad \implies \qquad K = -\frac{4}{15}.$$

Therefore

$$f(x) = \frac{x^3}{6} + \frac{4x^{5/2}}{15} + \frac{5x}{6} - \frac{4}{15}$$

as desired.

4.4 Finding Area

::: {#thm- } [Area Function] If f is a continuous function such that $f(x) \geq 0$ for all x on the closed interval $[a, b]$, then the area bounded by the curve $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = t$, viewed as a function of t, is an antiderivative of $f(t)$ on $[a, b]$. :::

Example 4.6. Find the area under the parabola $y = x^2$ over the interval $[0, 1]$.

Solution. Since $f(x) = x^2$ is a continuous function with $f(x) \geq 0$ for all x. The area function is given by

$$A(t) = \int t^2\ dt = \frac{1}{3}t^3 + C$$

and we can determine C using $A(0) = 0$ and so

$$A(0) = \frac{1}{3}(0)^3 + C = 0 \qquad \implies \qquad C = 0$$

which means $A(t) = \frac{1}{3}t^3$. Therefore the area under the curve from $[0, 1]$ is $A(1) = \frac{1}{3}(1)^3 = \frac{1}{3}$.

4.5 Applications of Integration

In the next example we find the demand function given the marginal revenue.

Example 4.7. A manufacturer estimates that the marginal revenue of a certain commodity is $R'(x) = 240 + 0.1x$ when x units are produced. Find the demand function $p(x)$.

Solution. Since

$$R(x) = \int R'(x)dx = \int (240 + 0.1x)\, dx = 240x + 0.05x^2 + C$$

and because $R(x) = xp(x)$, where $p(x)$ is the demand function, we must have $R(0) = 0$ so that $240(0) + 0.05(0) + C + 0$ yielding $C = 0$ and

$$p(x) = \frac{R(x)}{x} = \frac{240x + 0.05x^2}{x} = 240 + 0.05x.$$

Example 4.8. A ball is thrown upward with a speed of 48 ft/s from the edge of a cliff 432 feet above the ground. Find its height above the ground t seconds later. When does it reach its maximum height? When does it hit the ground?

Solution. The motion is vertical and we choose the positive direction to be upward. At time t the distance above the ground is $s(t)$ and the velocity $v(t)$ is decreasing. Therefore the acceleration must be negative $a(t) = \frac{dv}{dt} = -32$. Taking the antiderivative

$$v(t) = \int a(t)\, dt = -32t + C.$$

To determine C we use the given information of $v(0) = 48$. Thus $v(0) = -32(0) + C = 48$ and so $C = 48$ and $v(t) = -32t + 48$. It follows the ball reaches its maximum height at $v(t) = 0$ which means $t = \frac{48}{32}s = 1.5s$. Taking the antiderivative

$$s(t) = v'(t) = \int (-32t + 48)\, dt = -16t^2 + 48t + K$$

and using $s(0) = 432$ we find that $s(0) = -16(0)^2 + 48(0) + K = 432$ and so $K = 432$. Therefore the height function is

$$s(t) = -16t^2 + 48t + 432$$

and so the ball hits the ground when $s(t) = 0$ meaning $t = \frac{3+3\sqrt{13}}{2} \approx 6.9s$ by using the quadratic formula.

Example 4.9. A company has found that the rate of change of its average cost for a product is

$$\bar{C}'(x) = \frac{1}{4} - \frac{100}{x^2}$$

where x is the number of units and cost is in dollars. The average cost of producing 20 units is 40,000 dollars.

- Find the average cost function for the product.
- Find the average cost of 100 units of the product.

Solution. To find $\bar{C}(x)$ we integrate, so

$$\bar{C}(x) = \int \left(\frac{1}{4} - \frac{100}{x^2}\right) dx = \frac{x}{4} + \frac{100}{x} + K$$

to find the constant K we use the given that the average cost of producing 20 units is 40,000. So we find the constant K by $\bar{C}(20) = \frac{20}{4} + \frac{100}{20} + K = 10 + K = 40000$ which means $K = 40000 - 10 = 39990$. (a) So the average cost function for the product is $\bar{C}(x) = \frac{x}{4} + \frac{100}{x} + 39990$. (b) The average cost of 100 units of the product is

$$\bar{C}(10) = \frac{10}{4} + \frac{100}{10} + 39990 = \frac{80005}{2} = 40002.5$$

dollars.

Example 4.10. An excellent film with a very small advertising budget must depend largely on world-of-mouth advertising. In this case, the rate at which weekly attendance might grow can be given by

$$\frac{dA}{dt} = \frac{-100}{(t+10)^2} + \frac{2000}{(t+10)^3}$$

where t is in the time in weeks since release and A is attendance in millions.

- Find the function that describes weekly attendance at this film.
- Find the attendance at this film in the tenth week.

Solution. The function that describes weekly attendance at this film is found by integration

$$\begin{aligned} A(t) &= \int \frac{dA}{dt}\, dt = \int \left(\frac{-100}{(t+10)^2} + \frac{2000}{(t+10)^3}\right) dt \\ &= \int \left(\frac{-100}{(t+10)^2} + \frac{2000}{(t+10)^3}\right) dt = \frac{100}{t+10} - \frac{1000}{(t+10)^2} + K \end{aligned}$$

where t is in the time in weeks since release and A is attendance in millions. When $t = 0$ the attendance was 0 and so we find K by

$$A(0) = \frac{100}{0+10} - \frac{1000}{(0+10)^2} + K = K = 0.$$

Thus

$$A(t) = \frac{100}{t+10} - \frac{1000}{(t+10)^2}$$

and so the attendance at this film in the tenth week is

$$A(10) = \frac{100}{10+10} - \frac{1000}{(10+10)^2} = \frac{5}{2} = 2.5$$

million people.

Example 4.11. Suppose the marginal cost for a product is $\overline{MC} = 60\sqrt{x+1}$ and its fixed cost is 340.00. If the marginal revenue for the product is $\overline{MR} = 80x$, find the profit or loss from the production and sale of (a) 3 units (b) 8 units.

Solution. The marginal cost is $\overline{MC} = 60\sqrt{x+1}$ and so the cost function is fond by integration $C(x) = \int 60\sqrt{x+1}dx = 40(1+x)^{3/2} + K$ where K is a constant. Since $C(0) = 340$ we find $K = 300$ and so the cost function is $C(x) = 40(1+x)^{3/2} + 30$. The marginal revenue is $\overline{MR} = 80x$ and so the revenue function is found by integration $R(x) = \int 80x dx = 40x^2$. Thus the profit function for this product is

$$P(x) = R(x) - C(x) = 40x^2 - 40(1+x)^{3/2} - 300.$$

(a) The loss from the sale of 3 units is $P(3) = 40(3)^2 - 40(1+(3))^{3/2} - 300 = -260$ dollars. (b) The profit from the sale of 8 units is $P(8) = 40(8)^2 - 40(1+(8))^{3/2} - 300 = 1180$ dollars.

4.6 Constant-Difference Theorem

The following theorem says that two functions with equal derivatives on an open interval differ by a constant on that interval.

::: {#thm- } Constant-Difference Theorem Let f and g be functions that are continuous on $[a, b]$ and differentiable on (a, b). If $f'(x) = g'(x)$ for all x in (a, b), then $f - g$ is constant on (a, b); that is, $f(x) = g(x) + k$ where k is a constant. :::

Proof. Let $F(x) = f(x) - g(x)$. Then $F'(x) = f'(x) - g'(x) = 0$ for all x in (a, b). Thus by the Zero Derivative Theorem, $F(x) = k$ for some constant k and so $f(x) = g(x) + k$ as desired.

□

Example 4.12. Let $g(x) = \sqrt{x^2+5}$. Find a function f with $f'(x) = g'(x)$ and $f(2) = 1$.

Solution. Let $f(x) = \sqrt{x^2+5} + k$ where k is some constant to be determined. Then $f'(x) = g'(x)$ and to determine k we use $f(2) = 1$ to obtain $f(2) = \sqrt{2^2+5} + k = 3 + k = 1$. Therefore, $f(x) = \sqrt{x^2+5} - 2$ is the function we desire.

Example 4.13. Show that $f(x) = \frac{x+4}{5-x}$ and $g(x) = \frac{-9}{x-5}$ differ by a constant. Are the conditions of the constant difference theorem satisfied? Does $f'(x) = g'(x)$.

Solution. We simplify $f'(x) = g'(x) = \frac{9}{(x-5)^2}$ which is valid on any interval not containing $x = 5$. Thus on any interval not containing $x = 5$, the constant difference theorem applies. In fact, when $x \neq 5$, we have

$$f(x) - g(x) = \frac{x+4}{5-x} - \frac{-9}{x-5} = -1$$

Example 4.14. Let $f(x) = (x-2)^3$ and $g(x) = \left(x^2+12\right)(x-6)$. Use f and g to demonstrate the Constant-Difference Theorem.

Solution. The functions f and g are polynomial functions so they are continuous and differentiable for all real numbers. Also,

$$f'(x) = 3(x-2)^2 = 3x^2 - 12x + 4 = (2x)(x-6) + \left(x^2+12\right) = g'(x)$$

for all real numbers. By the Constant Difference Theorem, we have $f(x) = g(x) + k$ for some real number k.

The following theorem is a partial converse to the statement that the derivative of a constant is 0.

::: {#thm- } [Zero-Derivative Theorem] Let f be a function that is continuous on $[a, b]$ and differentiable on (a, b). If $f'(c) = 0$ for all c in (a, b) then f is constant on $[a, b]$. :::

Proof. If x_1 and x_2 are different points in $[a, b]$ then by the Mean Value Theorem there exists a c in (x_1, x_2) such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

By hypothesis $f'(c) = 0$ and so $f(x_2) - f(x_1) = 0$. Since x_1 and x_2 were chosen arbitrarily, f is a constant function on $[a, b]$.

□

Example 4.15. Consider $f(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$. Notice that $f'(x) = 0$ for all x in the domain, but f is not a constant. Does this example contradict the Zero-Derivative Theorem?

Solution. No it does not, rather it shows that the assumptions of the zero-derivative theorem are necessary.

4.7 Exercises

Exercise 4.1. Find an antiderivative for the following sets of functions $f(x)$, $g(x)$, and $h(x)$. Write out an equation using an indefinite integral and each of these functions.

- $f(x) = 2x$, $g(x) = x^2$, and $h(x) = x^2 - 2x + 1$
- $f(x) = -3x^{-4}$, $g(x) = x^{-4}$, and $h(x) = x^{-4} + 2x + 3$
- $f(x) = \frac{-2}{x^3}$, $g(x) = \frac{1}{2x^3}$ and $h(x) = x^3 - \frac{1}{x^3}$
- $f(x) = \frac{2}{3}x^{-1/3}$, $g(x) = \frac{1}{3}x^{-2/3}$ and $h(x) = \frac{-1}{3}x^{-4/3}$
- $f(x) = \frac{1}{3x}$, $g(x) = \frac{2}{5x}$, and $h(x) = 1 + \frac{4}{3x} - \frac{1}{x^2}$
- $f(x) = \sec^2 x$, $g(x) = \frac{2}{3}\sec^2 \frac{x}{3}$ and $h(x) = -\sec^2 \frac{3x}{2}$
- $f(x) = e^{-2x}$ $g(x) = e^{4x/3}$, and $h(x) = e^{-x/5}$.
- $f(x) = 3^x$, $g(x) = 2^{-x}$, and $h(x) = \left(\frac{5}{3}\right)^x$.
- $f(x) = x - \left(\frac{1}{2}\right)^x$ $g(x) = x^2 + 2^x$, and $h(x) = \pi^x - x^{-1}$.

4.8 Integration by Substitution

4.9 Understanding the Integration by Substitution Rule

Suppose it is known that in a certain country the life expectancy at birth of a female is changing at the rate of

$$g'(t) = \frac{5.45218}{(1 + 1.09t)^{0.9}}$$

years per year. Here, t is measured in years, with $t = 0$ corresponding to the beginning of 1900. Can we find an expression $g(t)$ giving the life expectancy at birth (in years) of a female in that country if the life expectancy at the beginning of 1900 is 50.02 years. If so, for example what is the life expectancy at birth of a female born at the beginning of 2000 in that country?

We use integration to find $g(t)$ as follows,

$$g(t) = \int g'(t)\,dt = \int \frac{5.45218}{(1+1.09t)^{0.9}}\,dt$$

by making a change of variables. Let $u = 1 + 0.09t$. Then $du = 1.09\,dt$ and so

$$g(t) = \frac{1}{1.09}\int \frac{5.45218}{u^{0.9}}\,du \approx 50.02(1+1.09t)^{0.1} + C.$$

Since $g(0) = 50.02$ we find $C = 0$, and then $g(t) = 50.02(1+1.09t)^{0.1}$. The life expectancy at birth of a female in the year 2000 is $g(100) = 50.02\left(110^{0.1}\right) \approx 80.04$ years.

We turn our attention to the substitution rule which (as seen below) is basically the reverse of the chain rule. To see this let f, g, and u be differentiable functions of x such that

$$f(x) = g(u)\frac{du}{dx}$$

Then

$$\int f(x)\,dx = \int g(u)\frac{du}{dx}\,dx = \int g(u)\,du = G(u) + C$$

where G is the antiderivative of g. Indeed, if G is an antiderivative of g, then $G'(u) = g(u)$ and by the chain rule

$$f(x) = \frac{d}{dx}[G(u)] = G'(u)\frac{du}{dx} = g(u)\frac{du}{dx}.$$

Now integrating both sides of this equation we obtain

$$\int f(x)\,dx = \int \left[g(u)\frac{du}{dx}\right]\,dx = \int \left[\frac{d}{dx}G(u)\right]\,dx = G(u) + C$$

where C is a constant.

4.10 The Integration by Substitution Rule

::: {#thm- } [Substitution Rule] If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

:::

Proof. By the chain rule, $F(g(x))$ is an antiderivative of $f(g(x))g'(x)$ whenever F is an antiderivative of f since

$$\frac{d}{dx}F(g(x)) = F(g(x))g'(x) = f(g(x))g'(x).$$

If we make the substitution $u = g(x)$ then

$$\begin{aligned}\int f(g(x))g'(x)\,dx &= \int \frac{d}{dx}F(g(x))\,dx = F(g(x)) + C = F(u) + C\\ &= \int F'(u)\,du = \int f(u)\,du.\end{aligned}$$

$\square$

The idea behind the substitution rule is to choose $u = g(x)$. Then find $du = g'(x)\,dx$. Next, make the substitution to obtain $\int f(u)\,du$. If it is possible to integrate the last integral involving u only, then the change of variable, using the substitution rule can successful.

Example 4.16. Evaluate $\displaystyle\int \frac{\cos\sqrt{\theta}}{\sqrt{\theta}\sin^2\sqrt{\theta}}\,d\theta.$

Solution. Let $u = \sin\sqrt{\theta}$. Then $du = \cos\sqrt{\theta}\left(\frac{1}{2\sqrt{\theta}}\right)d\theta$. By substitution

$$\int \frac{\cos\sqrt{\theta}}{\sqrt{\theta}\sin^2\sqrt{\theta}}\,d\theta = 2\int \frac{1}{u^2}du = \frac{-2}{u} + C = -\frac{2}{\sin\sqrt{\theta}} + C = -2\csc\sqrt{\theta} + C$$

where C is a constant.

Example 4.17. Evaluate $\displaystyle\int r^4\left(7 - \frac{r^5}{10}\right)^3 dr.$

Solution. Let $u = 7 - \frac{r^5}{10}$. Then $du = \frac{-r^4}{2}\,dr$. By substitution

$$\begin{aligned}\int r^4\left(7 - \frac{r^5}{10}\right)^3 dr &= -2\int\left(7 - \frac{r^5}{10}\right)^3\left(-\frac{1}{2}r^4\right)dr\\ &= -2\left(\frac{x^4}{4}\right) + C = -\frac{1}{2}\left(7 - \frac{r^5}{10}\right)^4 + C\end{aligned}$$

where C is a constant.

Example 4.18. Evaluate $\displaystyle\int \sqrt{\frac{x-1}{x^5}}\,dx.$

Solution. Let $u = 1 - \frac{1}{x}$. Then $du = \frac{1}{x^2}\,dx$. By substitution

$$\begin{aligned}\int \sqrt{\frac{x-1}{x^5}}\,dx &= \int \sqrt{1 - \frac{1}{x}}\left(\frac{1}{x^2}\right)dx\\ &= \int \sqrt{u}\,du = \frac{u^{3/2}}{3/2} + C = \frac{2}{3}\left(1 - \frac{1}{x}\right)^{3/2} + C\end{aligned}$$

where C is a constant.

Example 4.19. Evaluate $\displaystyle\int e^{\sin^2\theta}\sin 2\theta\,dx$.

Solution. Let $u = \sin^2\theta$. Then $du = 2\sin\theta\cos\theta\,d\theta = \sin 2\theta\,d\theta$. By substitution

$$\int e^{\sin^2\theta}\sin 2\theta\,dx = \int e^u\,du = e^u + C = e^{\sin^2\theta} + C$$

where C is a constant.

Example 4.20. Evaluate $\displaystyle\int \frac{1}{x\sqrt{x^4-1}}\,dx$.

Solution. Let $u = x^2$. Then $du = 2x\,dx$. By substitution

$$\begin{aligned}\int \frac{1}{x\sqrt{x^4-1}}\,dx &= \int \frac{2x}{2x^2\sqrt{x^4-1}}\,dx = \int \frac{1}{2u\sqrt{u^2-1}}\,du \\ &= \frac{1}{2}\int \frac{1}{|u|\sqrt{u^2-1}}\,du = \frac{1}{2}\sec^{-1}u + C = \frac{1}{2}\sec^{-1}x^2 + C\end{aligned}$$

where C is a constant.

Example 4.21. Evaluate $\displaystyle\int \frac{e^{\cos^{-1}x}}{\sqrt{1-x^2}}\,dx$.

Solution. Let $u = \cos^{-1}x$. Then $du = -\frac{1}{\sqrt{1-x^2}}\,dx$. By substitution

$$\int \frac{e^{\cos^{-1}x}}{\sqrt{1-x^2}}\,dx = -\int e^u\,du + C = -e^{\cos^{-1}x} + C$$

where C is a constant.

Example 4.22. Evaluate $\displaystyle\int \frac{1}{\sin^{-1}y\sqrt{1-y^2}}\,dy$.

Solution. Let $u = \sin^{-1}y$. Then $du = \frac{1}{\sqrt{1-y^2}}\,dy$. By substitution

$$\int \frac{1}{\sin^{-1}y\sqrt{1-y^2}}\,dy = \int \frac{1}{u}\,du = \ln|u| + C = \ln|\sin^{-1}y| + C$$

where C is a constant.

Example 4.23. Evaluate $\displaystyle\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta\cos^3\sqrt{\theta}}}\,d\theta.$

Solution. Let $u = \cos\sqrt{\theta}$. Then $du = -\sin\sqrt{\theta}\frac{1}{2\sqrt{\theta}}\,d\theta$. By substitution

$$\begin{aligned}\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta\cos^3\sqrt{\theta}}}\,d\theta &= -\int \frac{-2\sin\sqrt{\theta}}{2\sqrt{\theta}\cos^{3/2}\sqrt{\theta}}\,d\theta \\ &= -2\int \frac{1}{u^{3/2}}\,du = \frac{4}{\sqrt{u}} + C = \frac{4}{\sqrt{\cos\sqrt{\theta}}} + C\end{aligned}$$

where C is a constant.

Example 4.24. Solve the initial value problem:

$$y''(x) = 4\sec^2 2x\tan 2x, \qquad y'(0) = 4, \qquad y(0) = -1.$$

Solution. First we find

$$y' = \int 4\sec^2 2x\tan 2x\,dx = \int 2u\,du = u^2 + C = \tan^2 2x + C$$

using $u = \tan 2x$ and $du = 2\sec^2 2x\,dx$ and where C can be determine using $y'(0) = 4$. We find $C = 4$ and so $y' = \tan^2 2x + 4$. Using $\tan^2 x = \sec^2 x - 1$, we find

$$\begin{aligned}y &= \int \left(\tan^2 2x + 4\right)\,dx = \int \left(\sec^2 2x + 3\right)\,dx \\ &= \int \sec^2 2x\,dx + \int 3\,dx = \frac{1}{2}\tan 2x + 3x + C\end{aligned}$$

where C can be determined by $y(0) = 1$. We find $C = -1$ and therefore

$$y(x) = \frac{1}{2}\tan 2x + 3x - 1$$

as desired.

Example 4.25. Evaluate $\int 2\sin x\cos x\,dx$.

Solution. Since $2\sin x\cos x = \sin 2x$, we use $u = 2x$. Then $du = 2dx$ so

$$\int 2\sin x\cos x\,dx = \int \sin 2x\,dx = \int \frac{1}{2}\sin u\,du = -\frac{1}{2}\cos u + C = -\frac{1}{2}\cos 2x + C$$

where C is a constant. Alternatively, let $v = \sin x$. Then $dv = \cos x\,dx$ and so

$$\int 2\sin x\cos x\,dx = 2\int v\,dv = v^2 + K = \sin^2 x + K$$

where K is a constant. Alternatively, let $w = \cos x$. Then $dv = -\sin x\, dx$ and so

$$\int 2\sin x \cos x\, dx = -2\int w\, dw = -w^2 + L = -\cos^2 x + L$$

where L is a constant.

Example 4.26. The slope at each point (x, y) on the graph of $y = F(x)$ is given by $x(x^2-1)^{1/3}$ and the graph passes through the point $(3, 1)$. Find F.

Solution. Since $\frac{dy}{dx} = x(x^2-1)^{1/3}$ we find

$$F(x) = \int x(x^2-1)^{1/3}\, dx = \frac{1}{2}\int u^{1/3}\, du = \frac{1}{2}\cdot\frac{3}{4}(x^2-1)^{1/3} + C$$

Using, $F(3) = \frac{3}{8}(2)^4 + C = 1$ implies $C = -5$, it follows $F(x) = \frac{3}{8}(x^2-1)^{4/3} - 5$.

4.11 An Application to Special Relativity

Example 4.27. According to Einstein's special theory of relativity, the mass of a particle is given by

$$m = \frac{m_0}{\sqrt{1-\dfrac{v^2}{c^2}}}$$

where m_0 is the rest mass of the particle, v is its velocity, and c is the speed of light. Suppose that a particle starts from rest at $t = 0$ and moves along a straight line under the action of a constant force F. Then, according to Newton's second law of motion, the equation of motion is

$$F = m_0\frac{d}{dt}\left(\frac{v}{\sqrt{1-\frac{v^2}{c^2}}}\right).$$

Find the velocity and position functions of the particle. What happens to the velocity of the particle as time goes by?

Solution. From $F = m_0\frac{d}{dt}\left(\frac{v}{\sqrt{1-\frac{v^2}{c^2}}}\right)$ we find

$$\frac{d}{dt}\left[\frac{v}{\sqrt{1-\frac{v^2}{c^2}}}\right] = \frac{F}{m_0}$$

which implies

$$\frac{v}{\sqrt{1-\frac{v^2}{c^2}}} = \int \frac{F}{m_0}\,dt = \frac{Ft}{m_0} + k$$

where k is a constant. But $v(0) = 0$, so $k = 0$. Therefore,

$$\frac{v}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{Ft}{m_0}$$

and after solving for velocity

$$v = \frac{cFt}{\sqrt{m_0^2c^2 + F^2t^2}}.$$

Next we find the position function,

$$s = \int v(t)\,dt = \int \frac{cFt}{\sqrt{m_0^2c^2 + F^2t^2}}\,dt$$

Let $u = m_0^2c^2 + F^2t^2$ then $du = 2F^2t\,dt$. Then by substitution,

$$s = \frac{cF}{2F^2}\int \frac{1}{u^{1/2}}\,du = \frac{c}{F}\left(m_0^2c^2 + F^2t^2\right)^{1/2} + C$$

where C is the constant of integration. Then $s(0) = 0$ which shows $C = -\frac{c^2m_0}{F}$, so

$$s(t) = \frac{c}{F}\left(m_0^2c^2 + F^2t^2\right)^{1/2} - \frac{c^2m_0}{F}.$$

Finally,

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} \frac{cFt}{\sqrt{m_0^2 + F^2t^2}} = \lim_{t\to\infty} \frac{cF}{\sqrt{\frac{m_0^2c^2}{t^2} + F^2}} = \frac{cF}{F} = c$$

showing that its velocity approaches the speed of light.

4.12 Exercises

Exercise 4.2. Evaluate the following integrals.

- $\int \frac{\ln x}{x}\,dx$
- $\int \sqrt{3x-5}\,dx$
- $\int (11-2x)^{-4/5}\,dx$
- $\int \csc^2 5x\,dx$
- $\int \cot\left[\ln(x^2+1)\right]\frac{2x}{x^2+1}\,dx$

- $\int \frac{6x-9}{(x^2-3x+5)^3}\,dx$
- $\int \sin^3 x \cos x\,dx$
- $\int \frac{x^2}{x^3+1}\,dx$
- $\int \frac{4x}{2x+1}\,dx$
- $\int \frac{e^{\sqrt[3]{x}}}{x^{2/3}}\,dx$
- $\int x^3(x^2+4)^{1/2}\,dx$
- $\int \frac{\ln(x+1)}{x+1}\,dx$
- $\int \frac{1}{x^{2/3}(\sqrt[3]{x}+1)}\,dx$
- $\int \frac{e^{\sqrt{x}}}{\sqrt{x}(e^{\sqrt{x}}+1)}\,dx$

Exercise 4.3. The slope at each point (x, y) on the graph of $y = F(x)$ is given by $\frac{2x}{1-3x^2}$. What is $F(x)$ if the graph passes through $(0, 5)$?

Exercise 4.4. A particle moves along the t-axis in such a way that at time t, its velocity is $v(t) = t^2(t^3 - 8)^{1/3}$. At what time does the particle turn around? If the particle starts at a position which we denote as 1, where does it turn around?

Exercise 4.5. A rectangular storage tank has a square base 10 ft on a side. Water is flowing into the tank at the rate modeled by the function

$$R(t) = t(3t^2 + 1)^{-1/2}$$

in units ft^3/s at time t seconds. If the tank is empty at time $t = 0$, how mush water does it contain 4 sec later? What is the depth of the water at that time?

Exercise 4.6. Evaluate the following integrals.

- $\int \frac{1}{(x+1)\sqrt{(x+1)^2-9}}\,dx$
- $\int \frac{1}{x[9+(\ln x)^2]}\,dx$
- $\int \frac{\sqrt{a^2-x^2}}{x^4}\,dx$
- $\int \left[(x^2-1)(x+1)\right]^{-2/3}\,dx$

Exercise 4.7. Find the indefinite integral for the following then check you answer by differentiation.

- $\int \left(3t^2 + \frac{t}{2}\right)\,dt$
- $\int \left(1 - x^2 - 3x^5\right)\,dx$

- $\int \left(x^{-1/3}\right) dx$
- $\int \left(\frac{\sqrt{x}}{2} + \frac{2}{\sqrt{x}}\right) dx$
- $\int 2x \left(1 - x^{-3}\right) dx$
- $\int \left(\frac{4+\sqrt{t}}{t^3}\right) dt$
- $\int \left(7 \sin \frac{\theta}{3}\right) d\theta$
- $\int \left(-\frac{\sec^2 x}{3}\right) dx$
- $\int \left(e^{3x} + 5e^{-x}\right) dx$
- $\int (1.3)^x \, dx$
- $\int \left(\frac{1+\cos 4t}{2}\right) dt$
- $\int \left(\frac{2}{\sqrt{1-y^2}} - \frac{1}{y^{1/4}}\right) dy$
- $\int \left(1 + \tan^2 \theta\right) d\theta$

Exercise 4.8. Verify the following formulas by differentiation where C is an arbitrary constant.

- $\int (7x - 2)^3 \, dx = \frac{(7x-2)^4}{28} + C$
- $\int \csc^2 \left(\frac{x-1}{3}\right) dx = -3 \cot \left(\frac{x-1}{3}\right) + C$
- $\int xe^x dx = xe^x - e^x + C$

Exercise 4.9. Determine which of the following formulas are right and which are wrong. Write an explanation for each. Assume C is an arbitrary constant.

- $\int \tan \theta \sec^2 \theta d\theta = \frac{\sec^3 \theta}{3} + C$
- $\int \tan \theta \sec^2 \theta d\theta = \frac{1}{2} \tan^2 \theta + C$
- $\int \tan \theta \sec^2 \theta d\theta = \frac{1}{2} \sec^2 \theta + C$
- $\int \sqrt{2x + 1} \, dx = \sqrt{x^2 + x + C}.$
- $\int \sqrt{2x + 1} \, dx = \sqrt{x^2 + x} + C.$
- $\int \sqrt{2x + 1} \, dx = \frac{1}{3} \left(\sqrt{2x + 1}\right)^3 + C.$

Exercise 4.10. Solve the following initial value problems.

- $\frac{dy}{dx} = \frac{1}{x^2} + x$, $x > 0$; $y(2) = 1$.
- $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$, $y(4) = 0$.
- $\frac{d^2 r}{dt^2} = \frac{2}{t^3}$, $\frac{dr}{dt}\Big|_{t=1} = 1$, and $\{\ \}r(1) = 1$.
- $\frac{d^3 y}{dx^3} = 6$, $y''(0) = -8$, $y'(0) = 0$, and $\{\ \}y(0) = 5$.

Exercise 4.11. Find a curve $y = f(x)$ with $d^2y/dx^2 = 6x$ and its graph passes through the point $(0, 1)$ and has a horizontal tangent there. How many curves like this are there? How do you know?

Exercise 4.12. Given $f(x) = \frac{d}{dx}\left(1 - \sqrt{x}\right)$ and $g(x) = \frac{d}{dx}(x+2)$, find each of the following.

- $\int f(x)\, dx$,
- $\int g(x)\, dx$,
- $\int [-f(x)]\, dx$,
- $\int [-g(x)]\, dx$,
- $\int [f(x) + g(x)]\, dx$,
- $\int [f(x) - g(x)] dx$.

4.13 Area

4.14 Sigma Notation

In order to understand Riemann sums and integration theory correctly it is important to understand summations using sigma notation.

Definition 4.2. If a_m, a_{m+1}, ... , a_n are real numbers such that $m \leq n$, then the summation of these numbers written in **sigma notation** is

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \cdots + a_{n-1} + a_n$$

and also using functional notation,

$$\sum_{i=m}^{n} f(i) = f(m) + f(m+1) + \cdots + f(n-1) + f(n)$$

where $f(i) = a_i$. The i is called the **index of summation**, the a_i are called the ith term of the sum, and the upper and lower bounds of the summation are n and m, respectively.

Example 4.28. Write the sum
$\displaystyle\sum_{k=0}^{4} \frac{2k-1}{2k+1}$ in expanded form.

Solution. In expanded form the sum is

$$\begin{aligned} \sum_{k=0}^{4} \frac{2k-1}{2k+1} &= \frac{2(0)-1}{2(0)+1} + \frac{2(1)-1}{2(1)+1} + \frac{2(2)-1}{2(2)+1} + \frac{2(3)-1}{2(3)+1} + \frac{2(4)-1}{2(4)+1} \\ &= -1 + \frac{1}{3} + \frac{3}{5} + \frac{5}{7} + \frac{7}{9} = \frac{449}{315}. \end{aligned}$$

Example 4.29. Write the sum
$\left(\sum_{i=1}^{6} \frac{i}{i+1}\right) - 40$ in expanded form.

Solution. In expanded form, the sum is

$$\left(\sum_{i=1}^{6} \frac{i}{i+1}\right) - 40 = \frac{1}{1+1} + \frac{2}{2+1} + \frac{3}{3+1} + \frac{4}{4+1} + \frac{5}{5+1} + \frac{6}{6+1} - 40$$
$$= \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6} + \frac{6}{7} - 40 = -\frac{4983}{140}.$$

Example 4.30. Write the sum $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{7225}$ in sigma notation.

Solution. In sigma notation, the sum is

$$\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{7225} = \sum_{k=1}^{85} \frac{1}{k^2}.$$

Example 4.31. Write the sum
$\frac{3}{7} + \frac{4}{8} + \frac{5}{9} + \frac{6}{10} + \cdots + \frac{23}{27}$ in sigma notation.

Solution. In sigma notation the sum is

$$\sum_{k=7}^{27} \frac{k-4}{k}.$$

Example 4.32. Write the sum

$$\left[1 - \left(\frac{1}{4}\right)^2\right] + \left[1 - \left(\frac{2}{4}\right)^2\right] + \left[1 - \left(\frac{3}{4}\right)^2\right] + \cdots + \left[1 - \left(\frac{4}{4}\right)^2\right]$$

in sigma notation.

Solution. In sigma notation the sum is

$$\sum_{k=1}^{4} \left[1 - \left(\frac{k}{4}\right)^2\right].$$

Example 4.33. Write the sum

$$\left(\frac{1}{n}\right)\sqrt{1-\left(\frac{0}{n}\right)^2}+\left(\frac{1}{n}\right)\sqrt{1-\left(\frac{1}{n}\right)^2}+\cdots+\left(\frac{1}{n}\right)\sqrt{1-\left(\frac{n-1}{n}\right)^2}$$

in sigma notation.

Solution. In sigma notation the sum is

$$\sum_{k=0}^{n-1}\left(\frac{1}{n}\right)\sqrt{1-\left(\frac{k}{n}\right)^2}$$

4.15 Properties of Finite Sums

The linearity rule, subtotal rule, and the dominance rule are the basic properties of summations illustrated below. We also give summation formulas for

$$\sum_{k=1}^{n} k^i$$

when $i = 1, 2, \ldots, 6$ followed by some examples on taking limits of sums.

Theorem 4.5. *If c and d are real numbers that do not depend on integers m and n, then*

- $\displaystyle\sum_{k=1}^{n} c = nc$
- $\displaystyle\sum_{k=1}^{n}(a_k+b_k)=\sum_{k=1}^{n}a_k+\sum_{k=1}^{n}b_k$
- $\displaystyle\sum_{k=1}^{n}ca_k = c\left(\sum_{k=1}^{n}a_k\right)$
- $\displaystyle\sum_{k=1}^{n}(ca_k+db_k)=c\left(\sum_{k=1}^{n}a_k\right)+d\left(\sum_{k=1}^{n}b_k\right)$
- *If* $1 < m < n$, *then* $\displaystyle\sum_{k=1}^{n}a_k=\sum_{k=1}^{m}a_k+\sum_{k=m+1}^{n}a_k$
- *If* $a_k \le b_k$ *for all* k, *then* $\displaystyle\sum_{k=1}^{n}a_k \le \sum_{k=1}^{n}b_k.$

The following formulas will be necessary when determining limits of Riemann sums to find the area under a curve of a polynomial.

Theorem 4.6. *The summation formulas for* $\sum_{k=1}^{n} k^i$ *when* $i = 1, 2, \ldots, 7.$

- $\displaystyle\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$
- $\displaystyle\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$
- $\displaystyle\sum_{k=1}^{n} k^3 = \frac{1}{4}n^2(n+1)^2$
- $\displaystyle\sum_{k=1}^{n} k^4 = \frac{1}{30}n(n+1)(2n+1)\left(3n^2+3n-1\right)$
- $\displaystyle\sum_{k=1}^{n} k^5 = \frac{1}{12}n^2(n+1)^2\left(2n^2+2n-1\right)$
- $\displaystyle\sum_{k=1}^{n} k^6 = \frac{1}{42}n(n+1)(2n+1)\left(3n^4+6n^3-3n+1\right)$
- $\displaystyle\sum_{k=1}^{n} k^7 = \frac{1}{24}n^2(n+1)^2\left(3n^4+6n^3-n^2-4n+2\right)$

Example 4.34. Find the value of the sum $\displaystyle\sum_{i=1}^{100} i\left(i^2+1\right)$.

Solution. The value of the sum is

$$\begin{aligned}\sum_{i=1}^{100} i\left(i^2+1\right) = \sum_{i=1}^{100}\left(i^3+i\right) &= \sum_{i=1}^{100} i^3 + \sum_{i=1}^{100} i \\ &= \frac{1}{4}100^2(100+1)^2 + \frac{1}{2}100(100+1) \\ &= 25502500 + 5050 = 25507550.\end{aligned}$$

Example 4.35. Find the value of the sum $\displaystyle\sum_{i=1}^{4}\left(2^i+i^2\right)$.

Solution. The value of the sum is

$$\begin{aligned}\sum_{i=1}^{4}\left(2^i+i^2\right) &= \left(2^1+1^2\right)+\left(2^2+2^2\right)+\left(2^3+3^2\right)+\left(2^4+4^2\right) \\ &= 3+8+17+32 = 60.\end{aligned}$$

Example 4.36. Find the value of the sum $\displaystyle\sum_{i=1}^{n} i^2\left(i^2-i+1\right)$.

Solution. The value of the sum is

$$\begin{aligned}\sum_{i=1}^{n} i^2\left(i^2-i+1\right) &= \sum_{i=1}^{n}\left(i^4-i^3+i^2\right) = \sum_{i=1}^{n} i^4 - \sum_{i=1}^{n} i^3 + \sum_{i=1}^{n} i^2 \\ &= \frac{1}{30}n(n+1)(2n+1)\left(3n^2+3n-1\right) - \frac{1}{4}n^2(n+1)^2 + \frac{1}{6}n(n+1)(2n+1) \\ &= \frac{1}{60}n(n+1)\left(12n^3+3n^2+7n+8\right).\end{aligned}$$

4.16 Definition of Riemann Sum

In this topic we illustrate how a Riemann sum can be used to approximate the area under a curve and in doing so, we anticipate the notion of definite integral. We will investigate the area under the curve $y=2+6x-5x^2+x^3$, above the x-axis and between the vertical lines $x=0$ and $x=5$.

Definition 4.3. A **Riemann sum** for a function f on the closed bounded interval $[a,b]$ is a sum of the form

$$\sum_{k=1}^{n} f\left(x_i{}^*\right)\Delta x_i$$

where $a=x_0<x_1<\cdots<x_n=b$ and $x_{i-1}\leq x_i{}^*\leq x_i$ for $i=1,\ldots,n$. Let $\Delta x_i = x_i - x_{i-1}$ denote the width of each subinterval. The set

$$\mathcal{P}=\{x_0,x_1,\ldots,x_n\}$$

is called a partition of $[a,b]$, the largest of the Δx_i is called the *norm* of $\mathcal{P}$, and the $x_i{}^*$ are called the *subinterval representatives* for the Riemann sum.

Example 4.37. Given the function $f(x)=x^3$, the closed bounded interval $[0,1]$, and the partition $\left\{0,\frac{1}{2},\frac{3}{4},\frac{5}{6},1\right\}$, compute a Riemann sum.

Solution. Organizing into a table we determine the values,

i	subinterval	$x_i{}^*$	$f\left(x_i{}^*\right)$	Δx_i
1	$[0,\frac{1}{2}]$	$\frac{1}{3}$	$\left(\frac{1}{3}\right)^3=\frac{1}{27}$	$\frac{1}{2}-0=\frac{1}{2}$
2	$[\frac{1}{2},\frac{3}{4}]$	$\frac{2}{3}$	$\left(\frac{2}{3}\right)^3=\frac{8}{27}$	$\frac{3}{4}-\frac{1}{2}=\frac{1}{4}$
3	$[\frac{3}{4},\frac{5}{6}]$	$\frac{8}{10}$	$\left(\frac{8}{10}\right)^3=\frac{64}{125}$	$\frac{5}{6}-\frac{3}{4}=\frac{1}{12}$
4	$[\frac{5}{6},1]$	$\frac{9}{10}$	$\left(\frac{9}{10}\right)^3=\frac{729}{1000}$	$1-\frac{5}{6}=\frac{1}{6}$

So the Riemann sum for these x_i^* is

$$\begin{aligned}\sum_{i=1}^{4} f\left(x_i{}^*\right)\Delta x_i &= f\left(x_1{}^*\right)\Delta x_1 + f\left(x_2{}^*\right)\Delta x_2 + f\left(x_3{}^*\right)\Delta x_3 + f\left(x_4{}^*\right)\Delta x_4 \\ &= \left(\frac{1}{27}\right)\left(\frac{1}{2}\right) + \left(\frac{8}{27}\right)\left(\frac{1}{4}\right) + \left(\frac{64}{125}\right)\left(\frac{1}{12}\right) + \left(\frac{729}{1000}\right)\left(\frac{1}{6}\right) \\ &= \frac{2773}{10800}.\end{aligned}$$

4.17 Estimating the Area Under a Curve

Example 4.38. Given the function $f(x) = 2 + 6x - 5x^2 + x^3$, the closed bounded interval $[0, 5]$, and the partition $\{0, 1, 2, 3, 4, 5\}$, compute a Riemann sum to approximate the area.

Solution. Organizing into a table we determine the values,

i	subinterval	$x_i{}^*$	$f\left(x_i{}^*\right)$	Δx_i
1	$[0, 1]$	0	$2 + 6(0) - 5(0)^2 + (0)^3 = 2$	1
2	$[1, 2]$	1	$2 + 6(1) - 5(1)^2 + (1)^3 = 4$	1
3	$[2, 3]$	2	$2 + 6(2) - 5(2)^2 + (2)^3 = 2$	1
4	$[3, 4]$	3	$2 + 6(3) - 5(3)^2 + (3)^3 = 2$	1
5	$[4, 5]$	4	$2 + 6(4) - 5(4)^2 + (4)^3 = 10$	1

So the Riemann sum for these $x_i{}^*$ is

$$\sum_{i=1}^{5} f\left(x_i{}^*\right)\Delta x_i = (2)(1) + (4)(1) + (2)(1) + (2)(1) + (10)(1) = 20.$$

Figure **??** shows the Riemann sum as the approximate area under the given curve.

4.18 Refining Partitions

For our next example we will use a finer partition, say a partition using 14 subintervals between $x = 0$ and $x = 5$ (also with uniform width). We will still use left-endpoints for our subinterval representatives. Our second estimate for the area is 28.1481.

Example 4.39. Given the function $f(x) = 2 + 6x - 5x^2 + x^3$, the closed bounded interval $[0, 5]$, and the partition

$$\left\{0, \frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}, 2, \frac{7}{3}, \frac{8}{3}, 3, \frac{10}{3}, \frac{11}{3}, 4, \frac{13}{3}, \frac{14}{3}, 5\right\},$$

compute a Riemann sum to approximate the area.

Solution. Organizing into a table we determine the values,

i	subinterval	$x_i{}^*$	$f\left(x_i{}^*\right)$	Δx_i
1	$\left[0, \frac{1}{3}\right]$	0	$2+6(0)-5(0)^2+(0)^3=2$	$\frac{1}{3}$
2	$\left[\frac{1}{3}, \frac{2}{3}\right]$	$\frac{1}{3}$	$2+6\left(\frac{1}{3}\right)-5\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^3=\frac{94}{27}$	$\frac{1}{3}$
3	$\left[\frac{2}{3}, 1\right]$	$\frac{2}{3}$	$2+6\left(\frac{2}{3}\right)-5\left(\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)^3=\frac{110}{27}$	$\frac{1}{3}$
4	$\left[1, \frac{4}{3}\right]$	1	$2+6(1)-5(1)^2+(1)^3=4$	$\frac{1}{3}$
5	$\left[\frac{4}{3}, \frac{5}{3}\right]$	$\frac{4}{3}$	$2+6\left(\frac{4}{3}\right)-5\left(\frac{4}{3}\right)^2+\left(\frac{4}{3}\right)^3=\frac{94}{27}$	$\frac{1}{3}$
6	$\left[\frac{5}{3}, 2\right]$	$\frac{5}{3}$	$2+6\left(\frac{5}{3}\right)-5\left(\frac{5}{3}\right)^2+\left(\frac{5}{3}\right)^3=\frac{74}{27}$	$\frac{1}{3}$
7	$\left[2, \frac{7}{3}\right]$	2	$2+6(2)-5(2)^2+(2)^3=2$	$\frac{1}{3}$
8	$\left[\frac{7}{3}, \frac{8}{3}\right]$	$\frac{7}{3}$	$2+6\left(\frac{7}{3}\right)-5\left(\frac{7}{3}\right)^2+\left(\frac{7}{3}\right)^3=\frac{40}{27}$	$\frac{1}{3}$
9	$\left[\frac{8}{3}, 3\right]$	$\frac{8}{3}$	$2+6\left(\frac{8}{3}\right)-5\left(\frac{8}{3}\right)^2+\left(\frac{8}{3}\right)^3=\frac{38}{27}$	$\frac{1}{3}$
10	$\left[3, \frac{10}{3}\right]$	3	$2+6(3)-5(3)^2+(3)^3=2$	$\frac{1}{3}$
11	$\left[\frac{10}{3}, \frac{11}{3}\right]$	$\frac{10}{3}$	$2+6\left(\frac{10}{3}\right)-5\left(\frac{10}{3}\right)^2+\left(\frac{10}{3}\right)^3=\frac{94}{27}$	$\frac{1}{3}$
12	$\left[\frac{11}{3}, 4\right]$	$\frac{11}{3}$	$2+6\left(\frac{11}{3}\right)-5\left(\frac{11}{3}\right)^2+\left(\frac{11}{3}\right)^3=\frac{164}{27}$	$\frac{1}{3}$
13	$\left[4, \frac{13}{3}\right]$	4	$2+6(4)-5(4)^2+(4)^3=10$	$\frac{1}{3}$
14	$\left[\frac{13}{3}, \frac{14}{3}\right]$	$\frac{13}{3}$	$2+6\left(\frac{13}{3}\right)-5\left(\frac{13}{3}\right)^2+\left(\frac{13}{3}\right)^3=\frac{418}{27}$	$\frac{1}{3}$
15	$\left[\frac{14}{3}, 5\right]$	$\frac{14}{3}$	$2+6\left(\frac{14}{3}\right)-5\left(\frac{14}{3}\right)^2+\left(\frac{14}{3}\right)^3=\frac{614}{27}$	$\frac{1}{3}$

So the Riemann sum $\sum_{i=1}^{15} f\left(x_i{}^*\right)\Delta x_i$ for these $x_i{}^*$ is $\frac{760}{27}$.

Example 4.40. Use a Riemann sum to approximate the area under the graph of $f(x)=6x^2+2x+4$ on $[1,3]$ with 8 subintervals.

Solution. As a partition we choose $\mathcal{P}=\left\{1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2, \frac{9}{4}, \frac{5}{2}, \frac{11}{4}, 3\right\}$. Organizing our computations and choices for our subinterval representations we have

i	subinterval	$x_i{}^*$	$f\left(x_i{}^*\right)$	Δx_i	$f\left(x_i{}^*\right)\Delta x_i$
1	$\left[1, \frac{5}{4}\right]$	1	$6(1)^2+2(1)+4=12$	$\frac{1}{4}$	3
2	$\left[\frac{5}{4}, \frac{3}{2}\right]$	$\frac{5}{4}$	$6\left(\frac{5}{4}\right)^2+2\left(\frac{5}{4}\right)+4=\frac{127}{8}$	$\frac{1}{4}$	$\frac{127}{32}$
3	$\left[\frac{3}{2}, \frac{7}{4}\right]$	$\frac{3}{2}$	$6\left(\frac{3}{2}\right)^2+2\left(\frac{3}{2}\right)+4=\frac{41}{2}$	$\frac{1}{4}$	$\frac{41}{8}$
4	$\left[\frac{7}{4}, 2\right]$	$\frac{7}{4}$	$6\left(\frac{7}{4}\right)^2+2\left(\frac{7}{4}\right)+4=\frac{207}{8}$	$\frac{1}{4}$	$\frac{207}{32}$
5	$\left[2, \frac{9}{4}\right]$	2	$6(2)^2+2(2)+4=32$	$\frac{1}{4}$	8
6	$\left[\frac{9}{4}, \frac{5}{2}\right]$	$\frac{9}{4}$	$6\left(\frac{9}{4}\right)^2+2\left(\frac{9}{4}\right)+4=\frac{311}{8}$	$\frac{1}{4}$	$\frac{311}{32}$
7	$\left[\frac{5}{2}, \frac{11}{4}\right]$	$\frac{5}{2}$	$6\left(\frac{5}{2}\right)^2+2\left(\frac{5}{2}\right)+4=\frac{93}{2}$	$\frac{1}{4}$	$\frac{93}{8}$
8	$\left[\frac{11}{4}, 3\right]$	$\frac{11}{4}$	$6\left(\frac{11}{4}\right)^2+2\left(\frac{11}{4}\right)+4=\frac{439}{8}$	$\frac{1}{4}$	$\frac{439}{32}$

So the Riemann sum for this partition $\mathcal{P}$ and these x_i^* is $\sum_{i=1}^{8} f\left(x_i{}^*\right)\Delta x_i = 493/8$. Thus, the approximate the area under the graph of $f(x) = 6x^2+2x+4$ on $[1,3]$ with 8 subintervals is $\frac{493}{8}$ as shown in Figure **??**.

Example 4.41. Use a Riemann sum to approximate the area under the graph of $f(x) = \sqrt{1+x^2}$ on $[0,1]$ with 10 subintervals.

Solution. As a partition we choose

$$\mathcal{P} = \left\{0, \frac{1}{10}, \frac{1}{5}, \frac{3}{10}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{7}{10}, \frac{4}{5}, \frac{9}{10}, 1\right\}.$$

Organizing our computations and choices for our subinterval representations we have

i	subinterval	$x_i{}^*$	$f(x_i{}^*)$	Δx_i	$f(x_i{}^*)\,\Delta x_i$
1	$[0, \frac{1}{10}]$	$\frac{1}{10}$	$\sqrt{1+\left(\frac{1}{10}\right)^2} = \frac{\sqrt{101}}{10}$	$\frac{1}{10}$	$\frac{\sqrt{101}}{100}$
2	$[\frac{1}{10}, \frac{1}{5}]$	$\frac{1}{5}$	$\sqrt{1+\left(\frac{1}{5}\right)^2} = \frac{\sqrt{26}}{5}$	$\frac{1}{10}$	$\frac{\sqrt{\frac{13}{2}}}{25}$
3	$[\frac{1}{5}, \frac{3}{10}]$	$\frac{3}{10}$	$\sqrt{1+\left(\frac{3}{10}\right)^2} = \frac{\sqrt{109}}{10}$	$\frac{1}{10}$	$\frac{\sqrt{109}}{100}$
4	$[\frac{3}{10}, \frac{2}{5}]$	$\frac{2}{5}$	$\sqrt{1+\left(\frac{2}{5}\right)^2} = \frac{\sqrt{29}}{5}$	$\frac{1}{10}$	$\frac{\sqrt{29}}{50}$
5	$[\frac{2}{5}, \frac{1}{2}]$	$\frac{1}{2}$	$\sqrt{1+\left(\frac{1}{2}\right)^2} = \frac{\sqrt{5}}{2}$	$\frac{1}{10}$	$\frac{1}{4\sqrt{5}}$
6	$[\frac{1}{2}, \frac{3}{5}]$	$\frac{3}{5}$	$\sqrt{1+\left(\frac{3}{5}\right)^2} = \frac{\sqrt{34}}{5}$	$\frac{1}{10}$	$\frac{\sqrt{\frac{17}{2}}}{25}$
7	$[\frac{3}{5}, \frac{7}{10}]$	$\frac{7}{10}$	$\sqrt{1+\left(\frac{7}{10}\right)^2} = \frac{\sqrt{149}}{10}$	$\frac{1}{10}$	$\frac{\sqrt{149}}{100}$
8	$[\frac{7}{10}, \frac{4}{5}]$	$\frac{4}{5}$	$\sqrt{1+\left(\frac{4}{5}\right)^2} = \frac{\sqrt{41}}{5}$	$\frac{1}{10}$	$\frac{\sqrt{41}}{50}$
9	$[\frac{4}{5}, \frac{9}{10}]$	$\frac{9}{10}$	$\sqrt{1+\left(\frac{9}{10}\right)^2} = \frac{\sqrt{181}}{10}$	$\frac{1}{10}$	$\frac{\sqrt{181}}{100}$
10	$[\frac{9}{10}, 1]$	1	$\sqrt{1+(1)^2} = \sqrt{2}$	$\frac{1}{10}$	$\frac{1}{5\sqrt{2}}$
				Total $\approx$	1.16909

So the approximate the area under the graph of $f(x) = \sqrt{1+x^2}$ on $[0,1]$ using 10 subintervals is 1.16909 as shown in Figure **??**.

4.19 Area using Limits of Riemann Sums

It is important to be able to evaluate limits of sums so here are a couple of examples.

Theorem 4.7. *Suppose f is continuous and $f(x) \geq 0$ throughout the interval $[a,b]$. Then the area of the region under the curve $y = f(x)$ over this interval is*

$$A = \lim_{\Delta x \to 0} \sum_{k=1}^{n} f(a + k\Delta x)\Delta x$$

where $\Delta x = (b-a)/n$.

Example 4.42. Evaluate the limit $\lim_{n\to+\infty} \sum_{k=1}^{n} \frac{k}{n^2}$.

Solution. The value of the limit is

$$\begin{aligned}\lim_{n\to+\infty} \sum_{k=1}^{n} \frac{k}{n^2} &= \lim_{n\to+\infty} \frac{1}{n^2}\left(\sum_{k=1}^{n} k\right) \\ &= \lim_{n\to+\infty} \frac{1}{n^2}\left(\frac{1}{2}n(n+1)\right) = \lim_{n\to+\infty} \frac{n+1}{2n} = \frac{1}{2}.\end{aligned}$$

Example 4.43. Evaluate the limit $\lim_{n\to+\infty} \sum_{k=1}^{n} \left(1+\frac{2k}{n}\right)^2 \left(\frac{2}{n}\right)$.

Solution. The value of the limit is

$$\begin{aligned}&\lim_{n\to+\infty} \sum_{k=1}^{n} \left(1+\frac{2k}{n}\right)^2 \left(\frac{2}{n}\right) = \lim_{n\to+\infty} \left(\frac{2}{n}\right) \sum_{k=1}^{n} \left(1+\frac{2k}{n}\right)^2 \\ &\quad = \lim_{n\to+\infty} \left(\frac{2}{n}\right) \sum_{k=1}^{n} \left(\frac{4k^2}{n^2}+\frac{4k}{n}+1\right) = \lim_{n\to+\infty} \left(\frac{2}{n}\right)\left(\frac{13n^2+12n+2}{3n}\right) \\ &\quad = \lim_{n\to+\infty} \frac{2\left(13n^2+12n+2\right)}{3n^2} = \frac{26}{3}.\end{aligned}$$

Example 4.44. Find the exact area under the curve $y = x^2+4$ on $[2, 10]$.

Solution. We will use the formula

$$A = \lim_{\Delta x\to 0} \sum_{k=1}^{n} f(a+k\Delta x)\Delta x$$

with $b = 10$ and $a = 2$. We see $\Delta x = \frac{10-2}{n} = \frac{8}{n}$; and we notice that

$n \to +\infty$ as $\Delta x \to 0$ and so the area A is given by

$$\begin{aligned}
A &= \lim_{\Delta x \to 0} \sum_{k=1}^{n} f(a + k\Delta x)\Delta x \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} f\left(2 + k\frac{8}{n}\right)\frac{8}{n} \\
&= \lim_{n\to\infty} \frac{8}{n} \sum_{k=1}^{n} \left[\left(2 + \frac{8k}{n}\right)^2 + 4\right] \\
&= \lim_{n\to\infty} \frac{8}{n} \sum_{k=1}^{n} \left(8 + 4\left(\frac{8k}{n}\right) + \frac{64k^2}{n^2}\right) \\
&= \lim_{n\to\infty} \frac{8}{n} \sum_{k=1}^{n} \left(8 + 4\left(\frac{8k}{n}\right) + \frac{64k^2}{n^2}\right) \\
&= \lim_{n\to\infty} \frac{8}{n} \left[\sum_{k=1}^{n} 8 + \frac{32}{n}\sum_{k=1}^{n} k + \frac{64}{n^2}\sum_{k=1}^{n} k^2\right] \\
&= \lim_{n\to\infty} \frac{8}{n} \left[8n + \frac{32}{n}\left(\frac{1}{2}n(n+1)\right) + \frac{64}{n^2}\left(\frac{1}{6}n(n+1)(2n+1)\right)\right] \\
&= \lim_{n\to\infty} \frac{64\left(4 + 18n + 17n^2\right)}{3n^2} \qquad \text{(after algebraic simplification)} \\
&= \frac{64(17)}{3} \\
&= \frac{1088}{3}.
\end{aligned}$$

Therefore, the exact area under the curve $y = x^2 + 4$ bounded by the lines $y = 0$, $x = 2$, and $x = 10$ is $\frac{1088}{3}$.

Example 4.45. Find the exact area under the curve $y = 4x^3 + 3x^2$ on $[0, 1]$.

Solution. We will use the formula

$$A = \lim_{\Delta x \to 0} \sum_{k=1}^{n} f(a + k\Delta x)\Delta x$$

with $b = 1$ and $a = 0$. We see $\Delta x = \frac{1}{n}$; and we notice that $n \to +\infty$ as

$\Delta x \to 0$ and so the area A is given by

$$
\begin{aligned}
A &= \lim_{\Delta x \to 0} \sum_{k=1}^{n} f(a + k\Delta x)\Delta x \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\frac{1}{n} \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left[4\left(\frac{k}{n}\right)^3 + 3\left(\frac{k}{n}\right)^2\right] \\
&= \lim_{n\to\infty} \frac{1}{n} \left[\frac{4}{n^3}\sum_{k=1}^{n} k^3 + \frac{3}{n^2}\sum_{k=1}^{n} k^2\right] \\
&= \lim_{n\to\infty} \frac{1}{n} \left[\frac{4}{n^3}\left(\frac{1}{4}n^2(n+1)^2\right) + \frac{3}{n^2}\left(\frac{1}{6}n(n+1)(2n+1)\right)\right] \\
&= \lim_{n\to\infty} \frac{3 + 7n + 4n^2}{2n^2} \qquad \text{(after algebraic simplification)} \\
&= 2.
\end{aligned}
$$

Therefore, the exact area under the curve $y = 4x^3 + 3x^2$ bounded by the lines $y = 0$, $x = 0$, and $x = 1$ is 2.

4.20 Exercises

Exercise 4.13. Use finite approximations to estimate the area under the graph of $f(x) = x^3$ between $x = 0$ and $x = 1$.

Exercise 4.14. Use finite approximations to estimate the area under the graph of $f(x) = 4 - x^2$ between $x = -2$ and $x = 2$.

Exercise 4.15. Using rectangles whose height is given by the value of the function at the midpoint of the rectangle's base (the midpoint rule) estimate the area under the graph of the function $f(x) = x^2$ between $x = 0$ and $x = 1$, using first two and then four rectangles.

Exercise 4.16. Using rectangles whose height is given by the value of the function at the midpoint of the rectangle's base (the midpoint rule) estimate the area under the graph of the function $f(x) = 4 - x^2$ between $x = -2$ and $x = 2$, using first two and then four rectangles.

Exercise 4.17. An object is shot straight upward from sea level with an initial velocity of $400 \mathrm{ft}/\sec$. (a) Assuming that gravity is the only force acting on the object, give an upper estimate for its velocity after 5 sec have elapsed. Use $g = -32 \mathrm{ft}\big/\sec^2$ for the gravitational acceleration. (b) Find a lower estimate for the height attained after 5 sec. :::

Exercise 4.18. Use a finite sum to estimate the average of $f(x) = \left(\frac{1}{2}\right) + \sin^2 \pi t$ on the closed bounded interval $[0, 2]$ by partitioning the interval into four subintervals of equal length and evaluating f at the subinterval midpoints.

Exercise 4.19. Write the sum $\sum_{k=1}^{2} \frac{6k}{k+1}$ without sigma notation and then evaluate the sum.

Exercise 4.20. Write the sum $\sum_{k=1}^{4} (-1)^k \cos k\pi$ without sigma notation and then evaluate the sum.

Exercise 4.21. Rewrite the expression $1 - 2 + 4 - 8 + 16 - 32$ in sigma notation.

Exercise 4.22. Which formula is not equivalent to the other two?

- $\sum_{k=2}^{4} \frac{(-1)^{k-1}}{k-1}$
- $\sum_{k=0}^{2} \frac{(-1)^{k}}{k+1}$
- $\sum_{k=-1}^{1} \frac{(-1)^{k}}{k+2}$

Exercise 4.23. Rewrite the sum using sigma notation $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}$.

Exercise 4.24. Rewrite the sum using sigma notation $\frac{-1}{5} + \frac{2}{5} - \frac{3}{5} + \frac{4}{5} - \frac{5}{5}$.

Exercise 4.25. Suppose that $\sum_{k=1}^{n} a_k = 0$ and $\sum_{k=1}^{n} b_k = 1$. Find the values of

- $\sum_{k=1}^{n} 8a_k$
- $\sum_{k=1}^{n} 250b_k$
- $\sum_{k=1}^{n} (a_k + 1)$
- $\sum_{k=1}^{n} (b_k - 1)$.

Exercise 4.26. Evaluate the following sums.

- $\sum_{k=1}^{7}(-2k)$
- $\sum_{k=1}^{6}\left(k^2-5\right)$
- $\sum_{k=1}^{5}\frac{k^3}{225}+\left(\sum_{k=1}^{5}k\right)^3$

Exercise 4.27. Graph the function $f(x) = -x^2$ over the interval $[0, 2]$. Partition the interval into four subintervals of equal length. Then add to your sketch the rectangles associated with the Riemann sum $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$, given that c_k is the (a) left-hand endpoint, (b) right-hand endpoint, (c) the midpoint of the k-th subinterval . Make a separate sketch for each set of rectangles.

Exercise 4.28. Graph the function $f(x) = \sin x\ + 1$ over the interval $[-\pi, \pi]$. Partition the interval into four subintervals of equal length. Then add to your sketch the rectangles associated with the Riemann sum $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$, given that c_k is the (a) left-hand endpoint, (b) right-hand endpoint, (c) the midpoint of the k-th subinterval . Make a separate sketch for each set of rectangles.

Definite Integral

The Definition of the Definite Integral

Suppose a bounded function f is given along with a closed interval $[a, b]$ on which f is defined. First we will partition the interval $[a, b]$ into n subintervals by choosing points $\{x_0, x_1, \ldots, x_n\}$ arranged in such a way that

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

Call this partition P. Now for $i = 1, 2, \ldots, n$ the i^{th} subinterval width is $\Delta_i = x_i - x_{i-1}$. The largest of these widths is called the **norm** of the partition P and is denoted by $||P||$; that is, $||P|| = \max_{i=1,2,\ldots,n}\{\Delta_i\}$. Choose a number (arbitrarily) from each subinterval, say x_i^*. For $i = i, 2, \ldots, n$ the number x_i^* chosen from the i^{th} is called the i^{th} **subinterval representative** of the partition P. Then the sum

$$\sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

is called the **Riemann sum** associated with f, the partition P, the chosen subinterval representatives.

Definition 4.4. Let f be a bounded function defined on an interval $[a, b]$. Then the **definite integral** of f over $[a, b]$ is defined as

$$\int_a^b f(x)\,dx := \lim_{||P||\to 0} \sum_{k=1}^{n} f(c_k)\Delta x.$$

if the limit exists for all partitions P of $[a, b]$ and all choices of c_k in $[x_{k-1}, x_k]$.
When the above limit exists, we say the Riemann sums of f on $[a, b]$ converge to the definite integral $I = \int_a^b f(x)\,dx$ and that f is **integrable** over $[a, b]$.

For special cases we can define particular kinds of Riemann sums. For example, if we choose equally spaced subintervals using $\Delta x_i = \Delta x = \frac{b-a}{n}$ and $x_i^* = a + k\Delta x$ for $i = 1, 2, \dots, n$ then the Riemann sum formed is

$$\sum_{i=1}^{n} f(a + k\Delta x)\Delta x$$

and the partition P used to form this Riemann sum is called a **regular partition** . For regular partitions, $||P|| \to 0$ if and only if $n \to \infty$ and so the definition of the definite integral sometimes is given as

$$\int_a^b f(x)\,dx := \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x.$$

Theorem 4.8. *If a function f is continuous on an interval $[a, b]$ then its definite integral over $[a, b]$ exists.*

Example 4.46. Express

$$\lim_{||P||\to 0} \sum_{k=1}^{n} \sqrt{4 - c_k^2}\Delta x_k$$

where P is a partition of $[0, 1]$, as a definite integral and find its value using geometry.

Solution. See Figure **??**. Recall the area of a circle with radius 2 is 4π. Let $f(x) = \sqrt{4 - x^2}$. Since f is continuous on $[0, 1]$,

$$\pi = \int_0^1 \sqrt{4 - x^2}\,dx = \lim_{||P||\to 0} \sum_{k=1}^{n} \sqrt{4 - c_k^2}\Delta x_k.$$

Example 4.47. Express

$$\lim_{||P||\to 0} \sum_{k=1}^{n} (\tan c_k)\Delta x_k$$

where P is a partition of $[0, \pi/4]$, as a definite integral and find its approximate value using geometry.

Solution. Let $f(x) = \tan x$. Since f is continuous on $[0, \pi/4]$,

$$\frac{\pi}{8} \approx \int_0^{\pi/4} \tan x \, dx = \lim_{||P||\to 0} \sum_{k=1}^{n} (\tan c_k)\Delta x_k$$

since the area under the curve is approximately equal to the area under the triangle with vertices $(0, 0)$, $(\pi/4, 0)$, and $(\pi/4, 1)$.

Properties of the Definite Integral

Being familiar with the basic properties of finite sums and limits many of the properties listed below are straightforward to understand.

Theorem 4.9. *Suppose f and g are integrable functions on $[a, b]$. Then*

- $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx,$
- $\displaystyle\int_a^a f(x)\,dx = 0,$
- $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx,$
- $\displaystyle\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx,$ *and*
- $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$
- *If f has a maximum value M and a minimum value m, then*

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

- *If $f(x) \leq g(x)$ on $[a, b]$, then*

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

Example 4.48. Suppose $\int_0^2 f(x)\,dx = 3$, $\int_0^2 g(x)\,dx = -1$, and $\int_0^2 h(x)\,dx = 3$. Evaluate

$$\int_0^2 [2f(x) + 5g(x) - 6h(x)]\,dx$$

and then find a constant s such that

$$\int_0^2 [5f(x) + sg(x) - 7h(x)]\,dx = 0.$$

Solution. Using the basic properties of the definite integral

$$\begin{aligned}\int_0^2 [2f(x) + 5g(x) - 7h(x)]\,dx &= 2\int_0^2 f(x)\,dx + 5\int_0^2 g(x)\,dx - 7\int_0^2 h(x)\,dx \\ &= 2(3) + 5(-1) - 7(3) = -20.\end{aligned}$$

Also

$$\begin{aligned}\int_0^2 [5f(x) + s\,g(x) - 6h(x)]\,dx &= 5\int_0^2 f(x)\,dx + s\int_0^2 g(x)\,dx - 7\int_0^2 h(x)\,dx \\ &= 5(3) + s(-1) - 6(3)\,dx = -s - 3 = 0\end{aligned}$$

which implies $s = -3$.

Example 4.49. Sketch the graph of f and use it to evaluate $\int_{-1}^5 f(x)\,dx$ given

$$f(x) = \begin{cases} 2 & \text{for } -1 \le x \le 1 \\ 3 - x & \text{for } 1 \le x \le 4 \\ 2x - 9 & \text{for } 4 \le x \le 5. \end{cases}$$

Solution. Figure **??** is a sketch of the graph of f. Using the basic properties of the definite integral

$$\begin{aligned}\int_{-1}^5 f(x)\,dx &= \int_{-1}^1 f(x)\,dx + \int_1^4 f(x)\,dx + \int_4^5 f(x)\,dx \\ &= \int_{-1}^1 2\,dx + \int_1^4 (3 - x)\,dx + \int_4^5 (2x - 9)\,dx \\ &= 2(2) + \frac{3}{2} - 0 = \frac{11}{2}.\end{aligned}$$

Example 4.50. Suppose that f and h are integrable and that $\int_1^9 f(x)\,dx = -1$, $\int_7^9 f(x)\,dx = 5$, and $\int_7^9 h(x)\,dx = 4$. Find

- $\int_1^9 -2f(x)\,dx$
- $ _7^9 [f(x)+h(x)] , dx $
- $ _7^9 [2f(x)-3h(x)] , dx $
- $ _1^7 f(x) , dx $
- $ _9^1 f(x) , dx $
- $ _9^7 [h(x)-f(x)] , dx $

Solution. Using the basic properties of the definite integral - $\int_1^9 -2f(x)\,dx = -2\int_1^9 f(x)\,dx = -2(-1) = 2$ - $\int_7^9 [f(x)+h(x)]\,dx = \int_7^9 f(x)\,dx + \int_7^9 h(x)\,dx = 5+4 = 9$ - $\int_7^9 [2f(x) - 3h(x)]\,dx = 2\int_7^9 f(x)\,dx - 3\int_7^9 h(x)]\,dx = 2(5)-3(4) = -2$ - $\int_1^7 f(x)\,dx = \int_1^9 f(x)\,dx - \int_7^9 f(x)\,dx = -1-5 = -6$ - $\int_9^1 f(x)\,dx = -\int_1^9 f(x)\,dx = -(-1) = 1$ - $\int_9^7 [h(x)-f(x)]\,dx = \int_7^9 f(x)\,dx - \int_7^9 h(x)]\,dx = 5-4 = 1$

Example 4.51. What values of a and b minimize the value of $\int_a^b (x^4 - 2x^2)\,dx$?

Solution. By solving $x^4 - 2x^2 = 0$ we determine that $x^4 - 2x^2 \leq 0$ on $[-\sqrt{2}, \sqrt{2}]$. Thus we find that $a = -\sqrt{2}$ and $b = \sqrt{2}$ will minimize the value of $\int_a^b (x^4 - 2x^2)\,dx$. See Figure **??**.

Example 4.52.

(a) Find upper and lower bounds for $\int_0^1 \frac{1}{1+x^2}\,dx$.

(b) Find upper and lower bounds for $\int_0^{1/2} \frac{1}{1+x^2}\,dx$ and $\int_{1/2}^1 \frac{1}{1+x^2}\,dx$. Use (b) to find another estimate of the integral in (a).

Solution. For (a) Since $f(x) = \frac{1}{1+x^2}$ is decreasing on $[0,1]$, it follows the maximum of f, M occurs at 0 with $M = 1$ and the minimum value of f, m occurs at 1 with $m = 1/2$. Therefore,

$$\frac{1}{2}(1-0) = \frac{1}{2} \leq \int_0^1 \frac{1}{1+x^2}\,dx \leq 1(1-0) = 1.$$

For (b) with $[0, 1/2]$ we find, $M = 1$ and $m = 4/5$ and so

$$\frac{4}{5}\left(\frac{1}{2} - 0\right) = \frac{2}{5} \leq \int_0^{1/2} \frac{1}{1+x^2}\,dx \leq 1(\frac{1}{2} - 0) = \frac{1}{2}.$$

For (b) with $[0, 1/2]$ we find, $M = 4/5$ and $m = 1/2$ and so

$$\frac{1}{2}\left(1 - \frac{1}{2}\right) = \frac{1}{4} \leq \int_{1/2}^1 \frac{1}{1+x^2}\,dx \leq \frac{4}{5}(1 - \frac{1}{2}) = \frac{2}{5}$$

Then for part (c), we have a better estimate on the bounds of the original integral

$$\frac{13}{20} = \frac{1}{4} + \frac{2}{5} \leq \int_0^{1/2} \frac{1}{1+x^2}\,dx + \int_{1/2}^{1} \frac{1}{1+x^2}\,dx \leq \frac{1}{2} + \frac{2}{5} = \frac{9}{10}.$$

namely,

$$\frac{13}{20} \leq \int_0^1 \frac{1}{1+x^2}\,dx \leq \frac{9}{10}.$$

Example 4.53. Show that the value of $\int_0^1 \sqrt{x+8}\,dx$ lies between $2\sqrt{2} \approx 2.8$ and 3.

Solution. See Figure **??**. Since $f(x) = \sqrt{x+8}$ is increasing on $[0,1]$ it follows the maximum of f is $f(1) = \sqrt{1+8} = 3$ and the minimum of f is $f(0) = \sqrt{0+8} = 2\sqrt{2}$. Therefore,

$$2\sqrt{2} = 2\sqrt{2}(1-0) \leq \int_0^1 \sqrt{x+8}\,dx \leq 3(1-0) = 3.$$

Displacement

Consider an object moving along a straight line. Assume the position of the object at time t is given by the position function $s(t)$ and that its velocity at time t is given by $v(t) = s'(t)$. If we happen to know that the object is always moving forward from $t = a$ to $t = b$, that is, $v(t) > 0$ on $[a, b]$, then the total distance travelled is $s(b) - s(a)$.

Theorem 4.10. *The **total distance travelled** by an object with continuous velocity $v(t)$ along a straight line from $t = a$ to $t = b$ is $S = \int_a^b |v(t)|\,dt$.*

Example 4.54. Suppose $v(t) = \frac{1}{t+1}$ is the velocity of an object moving along a straight line. Use the formula $S_n = \sum_{k=1}^{n} |v(a + k\Delta t)|\Delta t$ where $\Delta t = \frac{b-a}{n}$ to estimate (using right endpoints) the total distance travelled by the object during the time interval $[0, 1]$.

Solution. Let $a = 0$ with $\Delta t = \frac{1}{4}$. Then

$$v(a + k\Delta t) = v\left(\frac{k}{4}\right) = \frac{1}{1+\frac{k}{4}} = \frac{4}{k+4}$$

Thus the total distance travelled by the object during the time interval $[0, 1]$ is

$$S_4 = \sum_{k=1}^{4} \frac{4}{k+4}\left(\frac{1}{4}\right) \approx 0.635.$$

Area Under a Curve

Theorem 4.11. *Suppose f is a continuous function and $f(x) \geq 0$ on the closed interval $[a, b]$. Then the area under the curve $y = f(x)$ on $[a, b]$ is given by the definite integral of f on $[a, b]$.*

Example 4.55. Use the definition of the definite integral to find the area of the region between the curve $y = 3x^2$ and the x-axis on the interval $[0, b]$.

Solution. Let $\Delta x = \frac{b-0}{n} = \frac{b}{a}$ and let $x_0 = 0$, $x_1 = \Delta x$, $x_2 = 2\Delta x$, ..., $x_{n-1} = \Delta x$, $x_n = n\Delta x = b$. Let the subinterval representatives, the $c_k's$ be the right endpoints of the subintervals, so $c_1 = x_1$, $c_2 = x_2$, and so on. The rectangles defined have areas:

$$
\begin{aligned}
&f(c_1)\Delta x = f(\Delta x)\Delta x = 3(\Delta x)^2\Delta x = 3(\Delta x)^3 \\
&f(c_2)\Delta x = f(2\Delta x)\Delta x = 3(2\Delta x)^2\Delta x = 3(2)^2(\Delta x)^3 \\
&f(c_3)\Delta x = f(3\Delta x)\Delta x = 3(3\Delta x)^2\Delta x = 3(3)^2(\Delta x)^3 \\
&\vdots \\
&f(c_n)\Delta x = f(n\Delta x)\Delta x = 3(n\Delta x)^2\Delta x = 3(n)^2(\Delta x)^3
\end{aligned}
$$

Then forming the Riemann sum

$$
\begin{aligned}
S_n &= \sum_{k=1}^{n} f(c_k)\Delta x = \sum_{k=1}^{n} 3k^2(\Delta x)^3 \\
&= 3(\Delta x)^3 \sum_{k=1}^{n} k^2 = 3\left(\frac{b^3}{n^3}\right)\left(\frac{n(n+1)(2n+1)}{6}\right) = \frac{b^3}{2}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right).
\end{aligned}
$$

Therefore, applying the definition of the definite integral

$$
\int_0^b 3x^2\,dx = \lim_{n\to\infty} \frac{b^3}{2}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) = b^3.
$$

Average Value

Let f be a continuous function defined on $[a, b]$. We will ask the question: what are the average values that f takes on over the interval $[a, b]$? To answer this, we will sample a finite number of values of f. For example, say we sample n values of f by dividing $[a, b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$ and evaluating f at a point c_k in each subinterval. The average of the n sampled values is

$$\begin{aligned}\frac{f(c_1)+f(c_2)+\cdots+f(c_n)}{n} &= \frac{1}{n}\sum_{k=1}^{n} f(c_k)\\ &= \frac{\Delta x}{b-a}\sum_{k=1}^{n} f(c_k) = \frac{1}{b-a}\sum_{k=1}^{n} f(c_k).\end{aligned}$$

If we pass the limit, if possible, as $n \to \infty$ we arrive at the following definition.

Definition 4.5. If f is integrable on $[a, b]$ then its *averagevalue* on $[a, b]$, denoted by the function $\operatorname{av}(f)$, sometimes called its **mean value** , is

$$\operatorname{av}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Example 4.56. Find the average value of the function $f(x) = 3x^2 - 3$ over the interval $[0, 1]$.

Solution. The average value of the function $f(x) = 3x^2 - 3$ over the interval $[0, 1]$ is

$$\begin{aligned}\left(\frac{1}{1-0}\right)\int_0^1 (3x^2-3)\,dx &= 3\int_0^1 x^2\,dx - \int_0^1 3\,dx\\ &= 3\left(\frac{1}{3}\right) - 3(1-0) = -2.\end{aligned}$$

Example 4.57. If $\operatorname{av}(f)$ really is a typical value of the integrable function $f(x)$ on $[a, b]$, then the number $\operatorname{av}(f)$ should have the same integral over $[a, b]$ that f does. Show that $\int_a^b \operatorname{av}(f)\,dx = \int_a^b f(x)\,dx$.

Solution. Notice that

$$\operatorname{av}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx := K$$

is a constant, say K. Then

$$\int_a^b K\,dx = K\int_a^b dx = K(b-a) = (b-a)\frac{1}{b-a}\int_a^b f(x)\,dx = \int_a^b f(x)\,dx$$

and so yes the number $\operatorname{av}(f)$ has the same integral over $[a, b]$ that f does.

Exercises

Exercise 4.29. For each of the following you are given a function f defined on an interval $[a, b]$, then number n of subintervals of equal length $\Delta x = (b - a)/n$, and the evaluation points c_k in $[x_{k-1}, x_k]$. For each of the following, (a) sketch the graph of f and the rectangles associated with the Riemann sum for f on $[a, b]$, and (b) find the Riemann sum. - $f(x) = 2x - 3, [0, 2], n = 4, c_k$ is the midpoint - $f(x) = -2x + 1, [-1, 2], n = 6, c_k$ is the left endpoint - $f(x) = \sqrt{x} - 1, [0, 3], n = 6, c_k$ is the right endpoint - $f(x) = 2 \sin x, [0, 5\pi/4], n = 5, c_k$ is the right endpoint

Exercise 4.30. Use the definition of the definite integral to evaluate the following integrals.

- $\int_{-1}^{2} x^2 \, dx$
- $\int_{-1}^{3} (x - 2) \, dx$
- $\int_{-1}^{1} (2x + 1) \, dx$
- $\int_{-2}^{1} (x^3 + 2x) \, dx$

Exercise 4.31. Each of the following is given as the limit of a Riemann sum of a function f on $[a, b]$. Write this expression as a definite integral on $[a, b]$.

- $\lim_{n\to\infty} \sum_{k=1}^{n} (4c_k - 3)\Delta x, [-3, -1]$
- $\lim_{n\to\infty} \sum_{k=1}^{n} 2c_k(1 - c_k)^2 \Delta x, [0, 3]$
- $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{2c_k}{c_k^2+1} \Delta x, [1, 2]$
- $\lim_{n\to\infty} \sum_{k=1}^{n} c_k(\cos c_k)\Delta x, [0, \frac{\pi}{2}]$

Exercise 4.32. For each of the following make a sketch of f on $[a, b]$ and then use the geometric interpretation of the integral to evaluate it.

- $\int_{-2}^{3} (2x + 1) \, dx$
- $\int_{-1}^{2} |x| \, dx$
- $\int_{-2}^{2} \sqrt{4 - x^2} \, dx$
- $\int_{0}^{2} \sqrt{-x^2 + 2x} \, dx$

Exercise 4.33. Given that $\int_{-1}^{3} f(x) \, dx = 4$ and $\int_{3}^{6} f(x) \, dx = 2$, evaluate the following integrals.

- $\int_{-1}^{3}[f(x)+g(x)]\,dx$
- $\int_{-1}^{3}[g(x)-f(x)]\,dx$
- $\int_{-1}^{3}[3f(x)-2g(x)]\,dx$
- $\int_{3}^{-1}[f(x)+5g(x)]\,dx$

Exercise 4.34. Given that $\int_{-2}^{2} f(x)\,dx = 3$ and $\int_{0}^{2} f(x)\,dx = 2$, evaluate the following integrals.

- $\int_{2}^{0} f(x)\,dx$
- $\int_{-2}^{0}[f(x)+3]\,dx$
- $\int_{2}^{0} 3f(x)\,dx - \int_{0}^{-2} 2f(x)\,dx$

Exercise 4.35. For each of the following use the properties of the integral to prove that the inequality without evaluating the integral.

- $\int_{0}^{1} \frac{\sqrt{x^3+x}}{x^2+1}\,dx \geq 0$
- $\int_{0}^{1} x^2\,dx \leq \int_{0}^{1} \sqrt{x}\,dx$
- $\int_{0}^{\pi/4} \sin^2 x\cos x\,dx \leq \int_{0}^{\pi/4} \sin^2 x\,dx$
- $\int_{0}^{\pi/2} \cos x\,dx \leq \int_{0}^{\pi/2} (x^2+1)\,dx$

Fundamental Theorem of Calculus

Before we begin, let's review the Intermediate Value Theorem and the Extreme Value Theorem.

::: {#thm- } Intermediate Value Theorem If f is a continuous function on a closed interval $[a,b]$ and M is a number between $f(a)$ and $f(b)$, inclusive, then there is at least one number c in $[a,b]$ such that $f(c)=M$.

::: {#thm- } [Extreme Value Theorem] If f is a continuous function on a closed interval $[a,b]$, then f attains an absolute maximum value $f(c)$ for some number c in $[a,b]$ and an absolute minimum value $f(d)$ for some number d in $[a,b]$. :::

Definition 4.6. If f is integrable on $[a,b]$, then the **average value of** f over $[a,b]$ is the number

$$f_{\text{av}} = \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{4.1}$$

If we assume that f is nonnegative, then we have the following geometric interpretation for the average value of a function over $[a, b]$. We see that f_{av} is the height of the rectangle with base lying on the interval $[a, b]$ and having the same area as the area of the region under the graph of f on $[a, b]$.

Example 4.58. Find the average value of $f(x) = \frac{x}{\sqrt{x^2+1}}$ over the interval $[0, 3]$. Answer: $\frac{\sqrt{10}-1}{3}$

4.21 Mean Value Theorem for Integrals

::: {#thm- } Mean Value Theorem for Integrals If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

:::

Proof. Since f is continuous on the interval $[a, b]$, the Extrema Value Theorem tells us that f attains an absolute minimum value m at some number in $[a, b]$ and an absolute maximum value M at some number in $[a, b]$. So $m \leq f(x) \leq M$ for all x in $[a, b]$. Hence the average value holds:

$$m \leq \frac{1}{b-a} \int_a^b f(x)\, dx \leq M$$

Because the average value lies between m and M, the Intermediate Value Theorem there exists at least one number c in $[a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

as needed.

$\square$

Example 4.59. Show that the inequality

$$0 \leq \int_0^1 \frac{x^5}{\sqrt[3]{1+x^4}}\, dx \leq \frac{1}{6} \tag{4.2}$$

holds.

Solution. Let $f(x) = \frac{x^5}{\sqrt[3]{1+x^4}}$. Since f is continuous on $[0, 1]$, we know that f must attain a maximum M and a minimum value m. We find,

$$f'(x) = \frac{x^4(19x^4 + 15)}{3(x^4 + 1)^{2/3}}$$

Hence the only critical number is 0. We test and find $m = f(0) = 0$ and $M = f(1) = 1/\sqrt[3]{2} \approx 0.79$. However, we can do better, by observing that $1 + x^4 \geq 1$ for x in $[0, 1]$. Then

$$0 \leq \frac{x^5}{\sqrt[3]{1+x^4}} \leq x^5 \implies 0 = \int_0^1 0\,dx \leq \int_0^1 \frac{x^5}{\sqrt[3]{1+x^4}}\,dx \leq \int_0^1 x^5\,dx = \frac{1}{6}$$

as needed.

4.22 The First Fundamental Theorem of Calculus

Theorem 4.12 (First Fundamental Theorem of Calculus)**.** *If f is continuous on $[a, b]$, then the function F defined by*

$$F(x) = \int_a^x f(t)\,dt, \quad a \leq x \leq b$$

is differentiable on (a, b) and

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Proof. Fix x in (a, b) and suppose that $x + h$ is in (a, b), where $h \neq 0$. Then

$$\begin{aligned} F(x+h) - F(x) &= \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt \\ &= \int_a^x f(t)\,dt + \int_x^{x+h} f(t)\,dt - \int_a^x f(t)\,dt \\ &= \int_x^{x+h} f(t)\,dt. \end{aligned}$$

By the Mean Value Theorem for Integrals there exists a number c between x and $x + h$ such that

$$f(c) = \int_x^{x+h} f(t)\,dt$$

Therefore,

$$\frac{F(x+h)-F(x)}{h}=\frac{1}{h}\int_x^{x+h} f(t)\,dt=f(c).$$

Next, observe that as h approaches 0, the number c, which is squeezed between x and $x+h$ approaches x, and by continuity, $f(c)$ approaches $f(x)$. Therefore,

$$F'(x)=\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}=\lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)\,dt=\lim_{h\to 0} f(c)=f(x).$$

as needed.

$\square$

Example 4.60. Find the derivative of the following functions.

- $G(x)=\int_{-1}^{x} t\sqrt{t^2+1}\,dt$
- $G(x)=\int_{0}^{x^2} t\sin t\,dt$
- $G(x)=\int_{2}^{\sqrt{x}} \frac{\sin t}{t}\,dt$
- $G(x)=\int_{x^2}^{x^3} \ln t\,dt,\ x>0$

Solution. • $G'(x)=x\sqrt{x^2+1}$

- $G'(x)=(x\sin x)(2x)=2x^2\sin x$
- $G'(x)=\frac{\sin x}{x}\frac{d}{dx}\left(\frac{1}{x}\right)$
- $G'(x)=\frac{\sin x}{x}\left(\frac{-1}{x^2}\right)$ First we write,

$$G(x)=\int_{x^2}^{c}\ln t\,dt+\int_{c}^{x^3}\ln t\,dt=-\int_{c}^{x^2}\ln t\,dt+\int_{c}^{x^3}\ln t\,dt$$

and so we find,

$$G'(x)=-2x\ln x^2+3x^2\ln x^3=-4x\ln x+9x^2\ln x=x(9x-4)\ln x$$

as requested.

Example 4.61. Evaluate $\lim_{h\to 0}\frac{1}{h}\int_{2}^{2+h}\sqrt{5+t^2}\,dt$.

Solution. Let $F(x) = \int_2^x \sqrt{5+t^2}\,dt$. Then

$$\begin{aligned} F'(2) &= \lim_{h\to 0} \frac{F(2+h)-F(2)}{h} \\ &= \lim_{h\to 0} \frac{\int_0^{2+h}\sqrt{5+t^2}\,dt - \int_0^2 \sqrt{5+t^2}\,dt}{h} \\ &= \lim_{h\to 0} \frac{\int_2^{2+h}\sqrt{5+t^2}\,dt}{h}. \end{aligned}$$

Next using the Fundamental Theorem, we have

$$F'(x) = \frac{d}{dx}\int_2^x \sqrt{5+t^2}\,dt = \sqrt{5+x^2}$$

and so $F'(2) = 3$, hence

$$\lim_{h\to 0}\frac{1}{h}\int_2^{2+h}\sqrt{5+t^2}\,dt = 3.$$

as requested.

4.23 The Second Fundamental Theorem of Calculus

::: {#thm- } The Second Fundamental Theorem of Calculus If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, that is $F' = f$. :::

Proof. Let $G(x) = \int_a^x f(t)\,dt$. By **??**, we know that G is an antiderivative of f. If F is any other antiderivative of f, then F and G must differ by a constant. In other words, $F(x) = G(x) + C$. To determine C, we put $x = a$ to obtain

$$F(a) = G(a) + C = \int_a^a f(t)\,dt + C = C.$$

Therefore, evaluating F at b, we have

$$F(b) = G(b) + C = \int_a^b f(t)\,dt + F(a)$$

from which we conclude that

$$F(b) - F(a) = \int_a^b f(x)\,dx.$$

as needed.

□

Example 4.62. Evaluate the following integrals.

- $\int_{-2}^{0}(2x-3)\,dx$
- $\int_{0}^{2}(2-4u+u^2)\,du$
- $\int_{1}^{2} \frac{3x^4-2x^2+1}{2x^2}\,dx$
- $\int_{2}^{0} \sqrt{x}(x+1)(x-2)\,dx$
- $\int_{0}^{\pi} \sin 2x \cos x\,dx$
- $\int_{0}^{\pi} |\cos x|\,dx$
- $\int_{0}^{4\sqrt{3}} \frac{1}{x^2+16}\,dx$
- $\int_{1}^{e} \frac{\ln x}{x} e^{(\ln x)^2}\,dx$

4.24 Exercises

Exercise 4.36. Find the derivative of the following functions.

- $G(x) = \int_{-1}^{x} t\sqrt{t^2+1}\,dt$
- $G(x) = \int_{0}^{x^2} t \sin t\,dt$
- $G(x) = \int_{2}^{\sqrt{x}} \frac{\sin t}{t}\,dt$
- $G(x) = \int_{x^2}^{x^3} \ln t\,dt$

Exercise 4.37. Evaluate the following integrals.

- $\int_{-2}^{0}(2x-3)\,dx$
- $\int_{0}^{2}(2-4u+u^2)\,du$
- $\int_{1}^{2} \frac{3x^4-2x^2+1}{2x^2}\,dx$
- $\int_{2}^{0} \sqrt{x}(x+1)(x-2)\,dx$
- $\int_{0}^{\pi} \sin 2x \cos x\,dx$
- $\int 0^{\pi} |\cos x|\,dx$
- $\int_{0}^{4\sqrt{3}} \frac{1}{x^2+16}\,dx$

- $\int_1^e \frac{\ln x}{x} e^{(\ln x)^2}\, dx$

Exercise 4.38. Show that each of the following inequalities hold.

- $0 \leq \int_0^1 \frac{x^5}{\sqrt[3]{1+x^4}}\, dx \leq \frac{1}{6}$
- $0 \leq \int_0^1 \frac{1}{\sqrt{4-3x+x^2}}\, dx \leq \frac{2}{3}$

Exercise 4.39. For each of the following, find the area of the region under the graph of f on $[a,b]$.

- $f(x) = x^2 - 2x + 2; [-1, 2]$
- $f(x) = \frac{1}{x^2}; [1, 2]$
- $f(x) = 2 + \sqrt{x+1}; [0, 3]$
- $f(x) = \frac{1}{4+x^2}; [0, 1]$

Exercise 4.40. Find $\frac{dx}{dy}$ if

$$\int_0^x \sqrt{3 + 2\cos t}\, dt + \int_0^y \sin t\, dt = 0.$$

Exercise 4.41. Find all functions f on $[0,1]$ such that f is continuous on $[0,1]$ and $\int_0^x f(t)\, dt = \int_x^1 f(t)\, dt$ for every x in $(0,1)$.

Exercise 4.42. Evaluate $\lim_{h\to 0} \frac{1}{h} \int_2^{2+h} \sqrt{5+t^2}\, dt$.

Exercise 4.43. Evaluate $\int_{-1}^1 \frac{2x^5+x^4-3x^3+2x^2+8x+1}{x^2+1}\, dx$.

Exercise 4.44. Use the identity

$$\frac{x \sin\left(n+\frac{1}{2}\right)}{2\sin\frac{x}{2}} = \frac{1}{2} + \cos x + \cos 2x + \cdots + \cos nx$$

to show that

$$\int_0^\pi \frac{x \sin\left(n+\frac{1}{2}\right)}{2\sin\frac{x}{2}}\, dx = \pi.$$

4.25 Numerical Integration

4.26 Left and Right Endpoints

The left and right rules are the most straight-forward to learn. They can be applied with either uniform width subintervals or varying width subintervals. They consist of using a Riemann sum where the subinterval representatives are chosen as the left-endpoints or the right-endpoints, respectively.

Theorem 4.13. *Let f be a continuous function on $[a, b]$ and let*

$$\mathcal{P} = \{a = x_0, x_1, \cdots, x_{k-1}, x_k, \cdots, x_n = b\}$$

be a partition of the interval $[a, b]$. Then the Riemann sum formed by using left-endpoints as the subinterval representatives ${x_k}^*$ is

$$\int_a^b f(x)\,dx \approx \sum_{k=1}^{n} f\left({x_k}^*\right)\Delta x_k = \sum_{k=1}^{n} f\left(x_{k-1}\right)\Delta x_k$$

and the Riemann sum formed by using right-endpoints as the subinterval representatives ${x_k}^*$ is

$$\int_a^b f(x)\,dx \approx \sum_{k=1}^{n} f\left({x_k}^*\right)\Delta x_k = \sum_{k=1}^{n} f\left(x_k\right)\Delta x_k$$

*where in both cases $\Delta x = x_k - x_{k-1}$. The first formula is called the **left rule** and the second formula the **right rule**.*

Example 4.63. Consider $f(x) = e^{-x^2}$ on $[0, 2]$ and let $\{\ \}\mathcal{P} = \{0, \frac{1}{2}, 1, \frac{3}{2}, 2\}$ be a partition of the interval $[0, 2]$ (so $\Delta x = \frac{1}{2}$).

Solution. Then the Riemann sum formed by using left-endpoints as the subinterval representatives ${x_k}^*$ is

$$\int_0^2 e^{-x^2}\,dx \approx \frac{1}{2}e^{-(0)^2} + \frac{1}{2}e^{-\left(\frac{1}{2}\right)^2} + \frac{1}{2}e^{-(1)^2} + \frac{1}{2}e^{-\left(\frac{3}{2}\right)^2} \approx 1.12604$$

The Riemann sum formed by using right-endpoints as the subinterval representatives x_k^* is

$$\int_0^2 e^{-x^2}\,dx \approx \frac{1}{2}e^{-\left(\frac{1}{2}\right)^2} + \frac{1}{2}e^{-(1)^2} + \frac{1}{2}e^{-\left(\frac{3}{2}\right)^2} + \frac{1}{2}e^{-(2)^2} \approx 0.635198.$$

Consider that as n increases then so does the estimation of the area. Note that a somewhat accurate estimation is 0.882081. However, because a fairly large number of rectangles are needed for a good approximation there are other common techniques which do not use rectangles.

4.27 Midpoint Rule

The midpoint rule uses a Riemann sum where the subinterval representatives are the midpoints of the subintervals. For some functions it may be easy to choose a partition that more closely approximates the definite integral using midpoints.

::: {#thm- } Midpoint Rule Let f be a continuous function on $[a, b]$ and let

$$\mathcal{P} = \left\{a = x_0, x_1, \cdots, x_{k-1,} x_k, \cdots, x_n = b\right\}$$

be a partition of the interval $[a, b]$. Then the Riemann sum formed by using midpoints as the subinterval representatives ${x_k}^*$ is

$$\int_a^b f(x)\, dx \approx \sum_{k=1}^{n} f\left({x_k}^*\right) \Delta x_k = \sum_{k=1}^{n} f\left(\frac{x_k + x_{k-1}}{2}\right) \Delta x_k$$

where $\Delta x_k = x_k - x_{k-1}$. :::

Example 4.64. Consider $f(x) = e^{-x^2}$ on $[0, 2]$ and let $\mathcal{P} = \left\{0, \frac{1}{2}, 1, \frac{3}{2}, 2\right\}$ be a partition of the interval $[0, 2]$ (so $\Delta x = \frac{1}{2}$). Use the midpoint rule.

Solution. Then the Riemann sum formed by using midpoints as the subinterval representatives x_k^* is

$$\int_0^2 e^{-x^2}\, dx \approx \frac{1}{2} e^{-\left(\frac{1}{4}\right)^2} + e^{-\left(\frac{3}{4}\right)^2} + \frac{1}{2} e^{-\left(\frac{5}{4}\right)^2} + \frac{1}{2} e^{-\left(\frac{7}{4}\right)^2} \approx 1.16768$$

Consider that as n increases then so does the estimation of the area. Note that a somewhat accurate estimation is 0.882081. However, because a fairly large number of rectangles are needed for a good approximation there are other common techniques which do not use rectangles.

4.28 Trapezoidal Rule

The trapezoidal rule uses trapezoids instead of rectangles to approximate the definite interval over a closed bounded interval. By using points on the graph of the function determined by a uniform width partition of the interval the upper boundary of the trapezoid is formed. Recall that the area of a trapezoid is one half the width times the sum of the two heights; and by using this formula we can add up the areas to approximate the definite integral. Of course the more subintervals, (or said another way: the more trapezoids) the better accuracy of the estimation.

Consider $y = -x^2 + 1$ on $[0, 1]$ and say we want to find the area. We could use rectangles as in a Riemann sum; however from inspection of the following diagram it might seem more reasonable to use trapezoids: The area is underestimated but not as much as using rectangles would (with the same partition).

::: {#thm- } Trapezoidal Rule The trapezoidal rule estimates the definite integral of f over $[a, b]$ using the formula

$$\int_a^b f(x)\,dx \approx \frac{1}{2}\left[f\left(x_0\right)+2 f\left(x_1\right)+2 f\left(x_2\right)+\cdots+2 f\left(x_{n-1}\right)+f\left(x_n\right)\right] \Delta x$$

where $x_k = a + k\Delta x$ and $\Delta x = \frac{b-a}{n}$. The trapezoidal rule formula can be simplified as follows

$$\begin{aligned}
\int_a^b f(x)\,dx &\approx \frac{1}{2}\left[f\left(x_0\right)+2 f\left(x_1\right)+2 f\left(x_2\right)+\cdots .+2 f\left(x_{n-1}\right)+f\left(x_n\right)\right] \Delta x \\
&= \frac{1}{2}[f(a)+2f(a+\Delta x)+2f(a+2\Delta x)+\cdots+2f(a+(n-1)\Delta x)+f(b)]\frac{b-a}{n} \\
&= \left(\frac{f(a)+f(b)}{2}\right)\left(\frac{b-a}{n}\right)+\left(\frac{b-a}{n}\right)\sum_{k=1}^{n-1} f\left(a+k\frac{b-a}{n}\right)
\end{aligned}$$

which is particularly nice if you want to program or pass a limit. :::

Example 4.65. Consider $f(x) = e^{-x^2}$ on $[0, 2]$ and let $\{\ \}\mathcal{P} = \{0, \frac{1}{2}, 1, \frac{3}{2}, 2\}$ be a partition of the interval $[0, 2]$ (so $\Delta x = \frac{1}{2}$). Then

$$\int_0^2 e^{-x^2}\,dx \approx \left(\frac{e^{-0^2}+e^{-2^2}}{4}\right)+\frac{1}{2}\sum_{k=1}^{3} e^{-\left(\frac{k}{2}\right)^2} \approx 0.880619$$

which is a much better estimation than using rectangles with the same partition.

Theorem 4.14. *[Error in the Trapezoidal Rule] If $f$has a continuous second derivative on $[a, b]$, then the error E_n in approximating*

$$\int_a^b f(x)\,dx$$

by the trapezoidal rule satisfies:

$$|E_n| \leq \frac{(b-a)^3}{12n^2} M$$

where M is the maximum value of $|f\,''(x)|$ on $[a, b]$.

4.29 Simpson's Rule

Instead of rectangles and trapezoids Simpson's rule uses parabolic arcs. The formula obtained for Simpson's rule requires using uniform width subintervals and an even number of them.

Consider $y = -x^2 + 1$ on $[0,1]$ and say we want to find the area. We could use rectangles as in a Riemann sum; however many graphs have curvature as so maybe using strips with parabolic arc will yield better estimates with less work. That is instead if rectangles or trapezoids: let's use "strips" with parabolic arcs. If $\mathcal{P} = \{x_0, x_1, \cdots, x_n\}$ is a partition of $[a,b]$ with $x_0 = a$ and $x_n = b$ and if we pass a parabolic arc through the points, three at a time say, the points with x-coordinates x_0, x_1, x_2 then those with $x_2, x_{3,} x_4$ and so on. It can be shown that the area of the region under the parabolic curve $y = f(x)$ on the interval $[x_{2k-2}, x_{2k}]$ has area given by

$$\frac{1}{3}\left[f\left(x_{2k-2}\right) + 4f\left(x_{2k-1}\right) + f\left(x_{2k}\right)\right]\Delta x$$

where $\Delta x = \frac{b-a}{n}$. Thus, to estimate $\int_a^b f(x)\,dx$ we can add up the strips as follows.

::: {#thm- } Simpson's Rule Simpson's rule estimates the definite integral of f over $[a,b]$ using the formula

$$\begin{aligned}\int_a^b f(x)\,dx \approx \frac{1}{3}\left[f\left(x_0\right) + 4f\left(x_1\right) + 2f\left(x_2\right) + 4f\left(x_3\right) + \right.\\ \left. 2f\left(x_4\right) + \cdots + 4f\left(x_{n-1}\right) + f\left(x_n\right)\right]\Delta x\end{aligned}$$

where $x_k = a + k\Delta x$, $\Delta x = \frac{b-a}{n}$, and n is an even integer. :::

Example 4.66. Consider $f(x) = e^{-x^2}$ on $[0,2]$ and let $\{\ \}\mathcal{P} = \{0, \frac{1}{2}, 1, \frac{3}{2}, 2\}$ be a partition of the interval $[0,2]$ (so $\Delta x = \frac{1}{2}$).

$$\int_0^2 e^{-x^2}\,dx \approx \frac{1}{6}\left(e^{-0^2} + 4e^{-\left(\frac{1}{2}\right)^2} + 2e^{-(1)^2} + 4e^{-\left(\frac{3}{2}\right)^2} + e^{-(2)^2}\right) \approx 0.881812$$

which is a much better estimation than using rectangles with the same partition.

::: {#thm- } [Error in the Simpson Rule] If $f$has a continuous fourth derivative on $[a,b]$, then the error E_n (n even) in approximating

$$\int_a^b f(x)\,dx$$

by the Simpson rule satisfies:

$$|E_n| \leq \frac{(b-a)^5}{180n^4}M$$

where M is the maximum value of $\left|f^{(4)}(x)\right|$ on $[a,b]$. :::

Example 4.67. Consider $f(x) = e^{-x^2}$ on $[0,2]$ and let $\mathcal{P} = \left\{0, \frac{1}{2}, 1, \frac{3}{2}, 2\right\}$ be a partition of the interval $[0,2]$. Find the error in the approximation

$$\int_0^2 e^{-x^2}\,dx \approx \frac{1}{6}\left(e^{-0^2} + 4e^{-\left(\frac{1}{2}\right)^2} + 2e^{-(1)^2} + 4e^{-\left(\frac{3}{2}\right)^2} + e^{-(2)^2}\right) \approx 0.881812$$

Solution. We compute

- $f'(x) = -2e^{-x^2}x$,
- $f''(x) = -2e^{-x^2} + 4e^{-x^2}x^2$,
- $f'''(x) = 12e^{-x^2}x - 8e^{-x^2}x^3$
- $f^{(4)}(x) = 12e^{-x^2} - 48e^{-x^2}x^2 + 16e^{-x^2}x^4$

We plot the fourth derivative obtaining: and so there is a maximum at $x = 0$. We obtain $f^{(4)}(0) = 12 = M$. Therefore, the error is less than $\frac{32}{1804^4}(12) = 0.00833333$.

Chapter 5

Applications of Integrals

Integration is a mathematical process that allows us to find the area under a curve, or to solve other problems involving functions. In this book, we will explore the applications of integration in detail, and learn how to apply it in various situations.

We will start with the basics of finding volumes, including the washer method and the shell method. From there, we will move on to more advanced applications. With practice, you will be able to apply these ideas and use integration to solve problems yet to be discovered.

Integrals are one of the most important concepts in mathematics, with applications in calculus, physics, engineering, and more. Simply put, an integral is a way to calculate the area under a curve. This can be used to determine the velocity of an object, the volume of a three-dimensional object, or even the length of a curve. In essence, integrals allow us to determine things that would otherwise be impossible to calculate.

Of course, integrals are just one part of calculus; there are also derivatives, limits, and so on. But without integrals, many of the things we take for granted would be impossible. So next time you see a complicated equation involving an integral, remember that it just represents someone's attempt to calculate something important.

The applications of integration are vast and include finding the volume or mass of an object, calculating work done or energy transferred, and solving many types of differential equations.

In this book, we will explore the applications of integration and how to apply it in various situations. So if you're ready to learn about the applications of integration, let's get started!

Calculus is all about finding the area between curves. And while that may sound like a rather esoteric focus, the applications of integration are actually quite practical.

For example, consider a firm that wants to determine how much material to order for manufacturing a product. The company knows the length of the product and the amount of material required per unit length. To find out how much material to order, they need to integrate the function that represents the amount of material required over the length of the product.

In other words, they need to find the area between the curve representing the amount of material required and an axis. Similarly, if a company wants to calculate how much money it will save by reducing its energy consumption, it needs to find the area between the two functions representing its current and proposed energy consumption rates.

As these examples illustrate, integration is a powerful tool with numerous applications in the business world. So next time you see a calculus problem involving the area between curves, don't be discouraged - remember that there's a practical application for all that math!

Have you ever wondered how engineers calculate the volume of irregularly shaped objects? The answer lies in a branch of mathematics called integral calculus. One of the applications of integration is calculating the volume of a solid object by revolving a two-dimensional shape around an axis. This technique is known as the method of volumes of solids of revolution. To use this method, you first need to identify a suitable axis of revolution. This is typically a line that passes through the center of the object.

Once you have chosen an axis, you need to determine the equation that describes the shape. For example, if you were calculating the volume of a cylindrical container, you would use the equation for a circle. You then need to integrate this equation to find the area enclosed by the shape. Finally, you need to multiply this area by the length of the axis of revolution to find the volume. Although it may seem like a lot of work, the method of volumes of solids of revolution can be used to calculate the volumes of all sorts of objects, from coffee cups to oil tanks.

The Disk Method is a simple way to find volumes, but it has its limitations. First, it only works for solids that are rotationally symmetric about one axis. Second, it only works for solids with flat or circular bases. However, despite these limitations, the disk method is still a useful tool for finding volumes and has applications in many different fields.

The Washer Method is a volume of solids of revolution technique that can be applied to a wide variety of applications of integration. In essence, the washer method calculates the volume of an object that has been revolved around an axis by cleverly dividing the object into infinitely thin slices, and then calculating the volume of each slice.

The key to understanding the washer method is to remember that each slice must have a constant cross-sectional area; otherwise, the volume calculation will be inaccurate. Once this concept is understood, the applications of the washer method are nearly endless. From calculating the volumes of simple objects like cylinders and spheres to more complex objects like cones and toruses, the Washer Method is a powerful tool.

The Shell Method is also one of the applications of integration that can be used to calculate the volumes of certain solids. This method is most often used when calculating the volumes of solids of revolution, which are created when a curve is rotated about an axis. The shell method involves slicing the solid into thin shells and then calculating the volume of each individual shell. This method can be used to calculate the volumes of solids with a variety of shapes, including cylinders, spheres, and cones. While the shell method can be somewhat tricky to master, it is a powerful tool for calculating the volumes of solids of revolution.

As every math student knows, integration is a powerful tool with a wide range of applications. One of the most common applications of integration is finding the arc length of a curve. By using a small angle approximation, you can break up the arc into a series of straight line segments. To do this, you need to first break up the arc into small segments. Then, you take the length of each segment and multiply it by the corresponding width. Finally, you add up all of the lengths to get the total length of the arc.

It might sound complicated, but it's actually quite simple. And once you know how to do it, you can apply this technique to all sorts of applications. So why not give it a try? Who knows, you might just find yourself getting addicted to integration!

One of the most practical applications is in the calculation of surface area. When an object is rotated about an axis, the resulting shape is known as a surface of revolution. By using integration, it's possible to calculate the surface area of this type of object without having to take measurements of the individual components. This can be a particularly useful tool in engineering applications, where accurate measurements are crucial.

When calculating the surface area of an object, we are essentially finding the amount of material required to cover the object. This process can be applied to both regular and irregular objects. For example, when we calculate the surface area of a cylinder, we are finding the amount of material needed to cover the curved sides. In general, however, calculating the surface area of an arbitrary object can be a very difficult task.

However, by breaking the object down into smaller pieces, we can use integration to more easily calculate its surface area. This technique is particularly helpful for surfaces of revolution, which can be difficult to visualize. By using integration, we can more easily calculate the amount of material needed to cover these complex shapes.

These are just a few of the applications of integration. As you can see, this powerful tool can be used in a wide variety of situations. Whether you're calculating the volume of an object or finding the arc length of a curve, integration is a powerful tool that every math student should master. So why not give it a try? You might just be surprised at how useful it can be!

In this book, I will take the applications of integration one by one, explaining in detail how to use them with the help of examples. I will also provide tips on when and how to use each application. By the end of this book, you will have a solid understanding of the applications of integration and how to apply it in various situations. So let's get started!

5.1 Area Between Curves

We explain, through several examples, how to find the area between curves (as a bounded region) using integration. We demonstrate both vertical and horizontal strips and provide several exercises.

When applying the definition for **the area between curves** , finding the intersection points of the curves and sketching their graphs is crucial. These graphs often reveal whether we should use vertical or horizontal strips by determining which curve is the upper curve and which is the lower.

5.2 Area Between Curves Using Vertical Strips

Definition 5.1. Let f and g be continuous functions on $[a, b]$, and suppose that $f(x) \geq g(x)$ for all x in $[a, b]$. Then the area of the region between the graphs of f and g and the vertical lines $x = a$ and $x = b$ is

$$A = \int_a^b [f(x) - g(x)]\, dx. \tag{5.1}$$

In particular, if $f(x) \geq g(x) = 0$, then (5.1) yields the area of the region bounded by: the graph of f, above the x-axis, and between the lines $x = a$ and $x = b$ as

$$A = \int_a^b f(x)\, dx. \tag{5.2}$$

The reader is expected to be familiar with (5.2) and also with the Fundamental Theorem of Calculus.

Example 5.1. Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

Solution. First we find the intersection points $(-1, 1)$ and $(2, -2)$. Then we see that we have $f(x) = 2 - x^2 \geq g(x) = -x$ for all $x \in [-1, 2]$ as required.

Therefore the area between the curves is

$$\begin{aligned}
\int_a^b [f(x) - g(x)]\, dx &= \int_{-1}^2 [(2 - x^2) - (-x)]\, dx \\
&= \int_{-1}^2 (2 + x - x^2)\, dx = \left[2x + \frac{x^2}{x} - \frac{x^3}{3}\right]_{-1}^2 \\
&= \left(4 + \frac{4}{2} - \frac{8}{3}\right) - \left(-2 + \frac{1}{2} + \frac{1}{3}\right) = \frac{9}{2}
\end{aligned}$$

as desired.

Example 5.2. Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.

Solution. First we find the intersection point $(4, 2)$. From a sketch of the graphs we see that we should consider the bounded region as two separate regions.

We have $f(x) = \sqrt{x} \geq g(x) = 0$ for all x in $[0, 2]$ and in the other region we have $f(x) = \sqrt{x} \geq g(x) = x - 2$ for all x in $[2, 4]$. The total area is

$$\begin{aligned}
&\int_0^2 \sqrt{x}\, dx + \int_2^4 (\sqrt{x} - x + 2)\, dx \\
&= \left[\frac{2}{3}x^{2/3}\right]_0^2 + \left[\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x\right]_2^4 \\
&= \left[\frac{2}{3}2^{2/3}\right] + \left[\frac{2}{3}4^{3/2} - \frac{4^2}{2} + 2(4)\right] - \left[\frac{2}{3}2^{3/2} - \frac{2^2}{2} + 2(2)\right] \\
&= \frac{10}{3}
\end{aligned}$$

as desired.

Example 5.3. Find the area of the shaded region between the curves $y = 2x^2$ and $y = x^4 - 2x^2$.

Solution. First we find the intersection points $(0, 0)$ and $(\pm 2, 8)$. From a sketch of the graph we see that we can use symmetry about the y-axis. We have

$$f(x) = 2x^2 \geq g(x) = x^4 - 2x^2$$

for all x in $[0, 2]$.

The total area is

$$2\int_0^2 [2x^2 - (x^4 - 2x^2)]\,dx = 2\int_0^2 (4x^2 - x^4)\,dx$$
$$= 2\left(\frac{4x^3}{3} - \frac{x^5}{5}\Big|_0^2\right) = 2\left(\frac{4(2)^3}{3} - \frac{(2)^5}{5}\right) = \frac{96}{5}$$

as desired.

Example 5.4. Find the area of the shaded region between the curves $y = 2x^3 - x^2 - 5x$ and $y = -x^2 + 3x$.

Solution. First we find the intersection points $(-2, -10)$, $(0, 0)$, and $(2, 2)$. From a sketch of the graphs wee see that we should break the bounded region into two regions.

Notice that for all x in $[-2, 0]$ we have

$$f(x) = 2x^3 - x^2 - 5x \geq g(x) = -x^2 + 3x$$

and that for all x in $[0, 2]$ we have $f(x) = -x^2+3x \geq g(x) = 2x^3-x^2-5x$. The total area is 16 given by

$$\int_{-2}^0 [(2x^3 - x^2 - 5x) - (-x^2 + 3x)]\,dx$$
$$+ \int_0^2 [(-x^2 + 3x) - (2x^3 - x^2 - 5x)]\,dx$$
$$= 8 + 8 = 16$$

as desired.

5.3 Area Between Curves Using Horizontal Strips

If a region's bounding curves are described by functions of y, the approximating rectangles are horizontal instead of vertical and the basis formula has y instead of x.

Definition 5.2. Let f and g be continuous functions on $[c, d]$, and suppose that $f(y) \geq g(y)$ for all y in $[c, d]$. Then the area of the region between the graphs of f and g and the horizontal lines $y = c$ and $y = d$ is

$$A = \int_c^d [f(y) - g(y)]\,dy. \tag{5.3}$$

In this equation f always denotes the right-hand curve and g the left-hand curve, so that $f(y) - g(y)$ is nonnegative.

Example 5.5. Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x-2$.

Solution. In this example we consider horizontal strips. Here we have

$$f(y) = y + 2 \geq g(y) = y^2$$

for all y in $[0, 2]$ as is seen in the sketch of the region.

The total area is

$$\int_0^2 [y + 2 - y^2]\, dy = \left[2y + \frac{y^2}{2} - \frac{y^3}{3}\right]_0^2 = 4 + \frac{4}{2} - \frac{8}{3} = \frac{10}{3}$$

as desired.

5.4 Exercises

Exercise 5.1. Find the areas of the regions enclosed by the lines and curves. - $x = 2y^2$, $x = 0$, and $y = 3$ - $y^2 - 4x = 4$, $4x - y = 16$ - $y = \frac{x}{x^2+1}$, $y = -\frac{1}{2}x^2$, $x = 1$ - $x = y^3 - y^2$ and $x = 2y$ - $y = x\sqrt{4 - x^2}$, $y = 0$ - $y = \sqrt{x + 3}$, $y = (x + 3)/2$ - $x + y^2 = 3$ and $4x + y^2 = 0$ - $\sqrt{x} + \sqrt{y} = 1$, $x + y = 1$ - $y = -x^3 + x$, $y = x^4 - 1$ - $y = \sin x$, $y = x$, $x = \pi/2$, $x = \pi$ - $y = \cos x$, $y = 2 - \cos x$, $0 \leq x \leq 2\pi$ - $x = 1 - y^2$, $x = y^2 - 1$

Exercise 5.2. Find the are of the region in the first quadrant bounded on the left by the y-axis and on the right by the curves $y = \sin x$ and $y = \cos x$.

Exercise 5.3. Find the area of the region enclosed by the curve $y^2 = x^2(1 - x^2)$.

Exercise 5.4. Find the area, integrating with respect to x and then y, of the region between the curve $y = 3 - x^2$ and the line $y = -1$.

Exercise 5.5. Use integration to find the area of the triangle with the given vertices $(-2, 4)$, $(0, -2)$, and $(6, 2)$.

Exercise 5.6. Find the area of the region in the first quadrant bounded on the left by the y-axis, below by the curve $x = 2\sqrt{y}$, above left by the curve $x = (y-1)^2$, and above right by the line $x = 3 - y$.

Exercise 5.7. Find the area of the region bounded by the graph of the curve $y^2 = x^3 - x^2$ and the line $x = 2$.

Exercise 5.8. Find the value of c such that the parabola $y = cx^2$ divides the region by the parabola $y = \frac{1}{9}x^2$, and the lines $y = 2$, and $x = 0$ into two subregions of equal area.

Exercise 5.9. Find the area of the region bounded by the parabola $y = x^2$, the tangent line to this parabola at $(1, 1)$ and the x-axis.

Exercise 5.10. Find the values of c such that at the area of the region bounded by the parabolas $ y=x^2 - c^2$ and $y = c^2 - x^2$ is 576.

Exercise 5.11. Find a such that the line $x = a$ divides the region bounded by the graphs of the equations into two regions of equal area given $y = x$, $y = 4$, and $x = 0$.

5.5 Volumes of Solids of Revolution

Finding the volume of the solid generated by rotating a bounded planar region about an axis of rotation is discussed. We cover the disk method, the washer method, and method of cylindrical shells. We provide several examples of solids generated by revolving around both vertical and horizontal lines.

The solid generated by rotating a plane region about an axis in its plane is called a **solid of revolution** .

Definition 5.3. The **volume** of a solid of known integrable cross-sectional area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b,

$$V = \int_a^b A(x)\,dx \tag{5.4}$$

Throughout our discussion we assume that $A(x)$ is integrable and in particular, we assume that the reader is familiar with the Fundamental Theorem of Calculus.

5.6 The Disk Method

::: {#thm- } Let $A(x)$ be the cross-sectional area of a disk of radius $R(x)$, the distance of the planar regions's boundary from the axis of revolution. The volume of the solid of revolution about the x-axis is

$$V = \int_a^b A(x)\,dx = \int_a^b \pi[R(x)]^2\,dx. \tag{5.5}$$

:::

Example 5.6. Find the volume of the solid that results when the region enclosed by $y = 9 - x^2$ and $y = 0$ is revolved about the x-axis.

Solution. First we sketch the region bounded by the given curves. We use $R(x) = 9 - x^2$ and find the volume

$$\begin{aligned} V &= \pi \int_{-3}^{3} (9 - x^2)^2\,dx \\ &= \pi \int_{-3}^{3} (81 - 18x^2 + x^4)\,dx = \frac{1296\pi}{5} \end{aligned}$$

as desired.

Example 5.7. Find the volume of the solid generated when the region bounded by the graph of $f(x) = \sqrt{x} + 1$ and the line $y = 2$ on the interval $[0, 1]$, is revolved about the line $y = 2$.

Solution. First we sketch the bounded region. The radius is given by

$$R(x) = 2 - (\sqrt{x} + 1) = 1 - \sqrt{x} \tag{5.6}$$

We find the volume using the Disk Method

$$V = \pi \int_0^1 (1 - \sqrt{x})^2\,dx = \pi \int_0^1 (1 - 2\sqrt{x} + x)\,dx = \frac{\pi}{6}$$

as desired.

::: {#thm- } Let $A(y)$ be the cross-sectional area of a disk of radius $R(y)$, the distance of the planar regions's boundary from the axis of revolution. The volume of the solid of revolution about the y-axis is

$$V = \int_a^b A(y)\, dy = \int_a^b \pi[R(y)]^2\, dy. \tag{5.7}$$

:::

Example 5.8. Find the volume of the solid that results when the region enclosed by the curves $y = 0$, $y = \ln x$, $y = 2$, and $x = 0$ is revolved about the y-axis.

Solution. First we sketch the bounded region. The radius is given by $R(y) = e^y$.

We find the volume using the Disk Method

$$V = \pi \int_0^2 e^{2y}\, dy = \frac{\pi(e^4 - 1)}{2} \tag{5.8}$$

as desired.

Example 5.9. Find the volume of the solid that results when the region enclosed by $y = \sqrt{x}$, $y = 0$, and $x = 9$ is revolved about the line $x = 9$.

Solution. First we sketch the bounded region. The radius is given by $R(y) = 9 - y^2$.

We find the volume using the Disk Method

$$\begin{aligned} V &= \pi \int_0^3 (9 - y^2)^2\, dy \\ &= \pi \int_0^3 (81 - 18y^2 + y^4)\, dy = \frac{648\pi}{5} \end{aligned}$$

as desired.

5.7 The Washer Method

::: {#thm- } Let $A(x)$ be the cross-sectional area of a disk of outer radius $R(x)$ and inner radius r(x).
The volume of the solid of revolution about the x-axis is

$$V = \int_a^b A(x)\, dx = \int_a^b \pi\left([R(x)]^2 - [r(x)]^2\right)\, dx. \tag{5.9}$$

:::

Example 5.10. Use the washer method to find the volume of the solid generated when the region bounded by $y = x$ and $y = 2\sqrt{x}$ is revolved about the x-axis.

Solution. First we sketch the bounded region. The outer radius is $R(x) = 2\sqrt{x}$ and the inner radius is $r(x) = x$.

We find the volume using the Washer Method

$$\begin{aligned} V &= \pi \int_0^4 ((2\sqrt{x})^2 - x^2)\, dx \\ &= \pi \left(2x^2 - x^3/3\right)\Big|_0^4 = \pi(32 - 64/3) = \frac{32\pi}{3}. \end{aligned}$$

as desired.

Example 5.11. Find the volume for the region bounded by the graphs of $y = 2\sin x$ and the x-axis on $[0, \pi]$ is revolved around the line $y = -2$.

Solution. First we sketch the bounded region.

The washer has outer radius $R(x) = 2 + 2\sin x$ and inner radius 2, so using the Washer Method the volume is

$$\begin{aligned} V &= \pi \int_0^\pi ((2 + 2\sin x)^2 - 2^2)\, dx = \pi \int_0^8 (8\sin x + 4\sin^2 x)\, dx \\ &= \pi \int_0^8 (8\sin x + 2(1 - \cos 2x))\, dx = \pi \left(-8\cos x + 2x - sin 2x\right)\Big|_0^\pi \\ &= \pi(8 + 2\pi + 8) = 2\pi(\pi + 8) \end{aligned}$$

as desired.

::: {#thm- } Let $A(y)$ be the cross-sectional area of a disk of outer radius $R(y)$ and inner radius r(y).
The volume of the solid of revolution about the y-axis is

$$V = \int_c^d A(y)\, dy = \int_c^d \pi\left([R(y)]^2 - [r(y)]^2\right)\, dy. \tag{5.10}$$

:::

Example 5.12. Find the volume of the solid that results when the region enclosed by the curves $x = 1 - y^2$, $x = 2 + y^2$, $y = 1$, $y = -1$ is revolved about the y-axis.

Solution. First we sketch the bounded region.

The washer has outer radius $R(y) = 2 + y^2$ and inner radius $r(y) = 1 - y^2$, so using the Washer Method the volume is

$$\begin{aligned} V &= \pi \int_{-1}^{1} [(2 + y^2)^2 - (1 - y^2)^2] \, dy \\ &= \pi \int_{-1}^{1} (3 + 6y^2) \, dy = 10\pi \end{aligned}$$

as desired.

Example 5.13. Find the volume of the solid generated by the region bounded by $y = \ln x$ and the y-axis on the interval $0 \leq y \leq 1$ and is revolved about the line $x = -1$.

Solution. First we sketch the bounded region.

The washer has outer radius $R(y) = 1 + e^y$ and inner radius $r(y) = 1$, so using the Washer Method the volume is

$$\begin{aligned} V &= \pi \int_{0}^{1} ((e^y + 1)^2 - 1^2) \, dy \\ &= \pi \int_{0}^{1} (2e^y + e^{2y}) \, dy = \pi \left(2e^y + \frac{1}{2} e^{2y} \right) \Big|_0^1 \\ &= \pi \left(2e + \frac{1}{2} e^2 - \frac{5}{2} \right) = \frac{\pi}{2} (e^2 + 4e - 5). \end{aligned}$$

as desired.

5.8 The Shell Method

::: {#thm- } Let f be a continuous nonnegative function of x on $[a, b]$, where $0 \leq a \leq b$, and let R be the region under the graph of f on the interval $[a, b]$. The volume V of the solid of revolution generated by revolving R about the y-axis is

$$V = \lim_{n \to \infty} \sum_{k=1}^{n} 2\pi c_k f(c_k) \Delta x = \int_a^b 2\pi x f(x) \, dx. \tag{5.11}$$

:::

Example 5.14. Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.

Solution. First we sketch the bounded region.

The circumference is $2\pi x$ and height is $f(x) = 2x^2 - x^3$ so by the Shell Method the volume is

$$V = \int_0^2 (2\pi x)(2x^2 - x^3)\,dx = 2\pi \int_0^2 (2x^3 - x^4)\,dx = \frac{16}{5}\pi$$

as desired.

Example 5.15. Find the volume of the solid formed by revolving the region bounded by the graphs of $y = x^3 + x + 1$, $y = 1$, and $x = 1$ about the line $x = 2$.

Solution. First we sketch the bounded region.

Notice the variable of integration should be x and so we find the volume using the Shell Method,

$$\begin{aligned} V &= 2\pi \int_0^1 (2-x)(x^3 + x + 1 - 1)\,dx \\ &= 2\pi \int_0^1 (-x^4 + 2x^3 - x^2 + 2x)\,dx = \frac{29\pi}{15} \end{aligned}$$

as desired.

::: {#thm- } Let f be a continuous nonnegative function of y on $[c, d]$, where $0 \leq c \leq d$, and let R be the region bounded by the graph of f on the interval $[c, d]$ and the y-axis. The volume V of the solid of revolution generated by revolving R about the x-axis is

$$V = \lim_{n\to\infty} \sum_{k=1}^{n} 2\pi a_k f(a_k)\Delta y = \int_c^d 2\pi y f(y)\,dy. \tag{5.12}$$

:::

Example 5.16. The region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 4$ is revolved about the x axis to generate a solid. Find the volume of the solid.

Solution. First we sketch the bounded region.

We find the volume using the Shell Method

$$V = \int_0^2 2\pi y(4 - y^2)\,dy = \int_0^2 2\pi(4y - y^3)\,dy = 8\pi$$

as desired.

Example 5.17. The region bounded by the curve $y = x^2$, the x-axis, and the line $x = 1$ is revolved about the line $y = 2$ to generate a solid. Find the volume of the solid.

Solution. First we sketch the bounded region.

$$
\begin{aligned}
V &= 2\pi \int_0^1 (2-y)(1-\sqrt{y})\,dy \\
&= 2\pi \int_0^1 (2-y-2y^{1/2}+y^{3/2})\,dy = \frac{17\pi}{15}
\end{aligned}
$$

as desired.

5.9 Exercises

Exercise 5.12. Find the volumes of the solids generated by revolving the regions bounded by the lines and curves about the the given axis. - $y = x^2+1$, $y = x+3$, about the x-axis - $y = 4-x^2$, $y = 2-x$, about the x-axis - $x = 1/y$, $x = 0$, $y = 1$, $y = 2$, about the y-axis - $x = \sqrt{4-y^2}$, $x = 0$, $y = 0$, about the y-axis

Exercise 5.13. Use the disk method to verify that the volume of a sphere is $\frac{4}{3}\pi r^2$, where r is the radius.

Exercise 5.14. Use the disk method to verify that the volume of a cone is $\frac{1}{3}\pi r^2 h$, where r is the radius of the base and h is the height.

Exercise 5.15. Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 2$ and $x = 0$ about

- the x-axis
- the line $y = 2$
- the y-axis
- the line $x = 4$

Exercise 5.16. Find the volume of the solid region. The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 4$. The cross-sections perpendicular to the axis on the interval $0 \le x \le 4$ are squares whose diagonals run form the parabola $y - \sqrt{x}$ to the parabola $y = \sqrt{x}$.

Exercise 5.17. Find the volume of the solid generated by revolving the region bounded by $y = 4 + 2x - x^2$ and the line $y = 4 - x$.

- the x-axis
- the line $y = 1$

Exercise 5.18. Using integration, find the volume of the solid generated by revolving the triangular region with vertices $(0, 0), (b, 0), (0, h)$ about

- the x-axis
- the y-axis

Exercise 5.19. Find the volume of the solid generated by revolving the region bounded by the graphs of the equations about the indicated line. - $y = x$, $y = x^2$, the line $y = 2$ - $y = x^2$, $y = \frac{1}{2}x^2 + 2$, about the line $y = 5$ - $x = \sqrt{9 - y^2}$, $x = 0$, $y = 0$, about the x-axis - $y = \sqrt{x - 1}$, $y = x - 1$, about the y-axis - $x = y - y^2$, $x = 0$; about the y-axis - $y = 1/x$, $y = 0$, $x = 1$, $x = 3$, about the line $y = -1$ - $y = x$, $y = 0$, $x = 2$, $x = 4$, about the line $x = 1$

Exercise 5.20. Use the shell method to find the volume of the solids generated by revolving the regions bounded by the curves and lines. - $x = 1 + y^2$, $x = 0$, $y = 1$, $y = 2$, about the x-axis - $x = 1 + (y - 2)^2$, $x = 2$, about the x-axis - $y = \sqrt{9 - x^2}$, $y = \frac{2}{3}\sqrt{9 - x^2}$, $x \geq 0$, about the y-axis - $y = x^2 + 1$, $y = 0$, $x = 0$, $x = 2$, about the line $x = 3$ - $y = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$, $y = 0$, $x = 0$, $x = 1$, about the y-axis - $y = \frac{1}{3}x^2$, $y = 6x - x^2$, about the line $x = 3$

Exercise 5.21. Let f be the function defined by

$$f(x) = \begin{cases} \sqrt{x} & \text{if } 0 \leq x \leq 1 \\ x^2 - 2x + 2 & \text{if } 1 < x \leq 2. \end{cases} \tag{5.13}$$

Find the volume of the solid generated by revolving the region under the graph of f on $[0, 2]$ about the x-axis.

Exercise 5.22. Compute the volume of the solid generated by revolving the region bounded by $y = x$ and $y = x^2$ about each coordinate axis using

- the shell method
- the washer method

Exercise 5.23. Compute the volume of the solid generated by revolving the triangular region bounded by the lines $2y = x + 4$, $y = x$, and $x = 0$ about

- the x-axis using the washer method
- the y-axis using the shell method
- the line $x = 4$ using the shell method
- the line $y = 8$ using the washer method

Exercise 5.24. Each integral represents the volume of a solid. Describe the solid. - $2\pi \int_0^2 \frac{y}{1+y^2}\, dy$ - $\int_0^{\pi/4} 2\pi(\pi - x)(\cos x - \sin x)\, dx$ - $\pi \int_{-r}^{r} [(R + \sqrt{r^2 - x^2})^2 - (R - \sqrt{r^2 - x^2})^2]\, dx$

Exercise 5.25. Use the disk method or the shell method to find the volumes of the solid generated by revolving the region bounded by the graphs of $y = 10/x^2$, $y = 0$, $x = 1$, $x = 5$.

- the x-axis
- the y-axis
- the line $y = 10$

Exercise 5.26. Use the disk method or the shell method to find the volumes of the solid generated by revolving the region bounded by the graphs of $x^{1/2} + y^{1/2} = a^{a^1/2}$, $x = 0$, $y = 0$.

- the x-axis
- the y-axis
- the line $x = a$

Exercise 5.27. A solid has a circular base of radius 2, and its parallel cross-sections perpendicular to its base are isosceles right triangles oriented so that the endpoints of the hypotenuse of a triangle lie on the circle. Find the volume of the solid.

Exercise 5.28. The region in the first quadrant that is bounded above by the curve $y = 1/\sqrt{x}$ on the left by the line $x = 1/4$, and below by the line $y = 1$ is revolved about the y-axis to generate a solid. Find the volume of the solid by

- the washer method
- the shell method

Exercise 5.29. Use the method of cylindrical shells to find the volume of the ellipsoid obtained by revolving the elliptical region enclosed by the graph of

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad x \geq 0 \tag{5.14}$$

about the y-axis.

Exercise 5.30. Set up an integral for the volume of a solid **torus** (doughnut) with radii r and R. By interpreting the integral as an area, find the volume of the torus.

Exercise 5.31. A bowl is shaped like a hemisphere with diameter 30 cm. A ball with diameter 10 cm is placed in the bowl and water is poured into the bowl to a depth of h centimeters. Find the volume of water in the bowl.

Exercise 5.32. A hole of radius r is bored through the center of a sphere of radius $R > r$. Find the volume of the remaining portion of the sphere.

5.10 Arc Length and Surfaces of Revolution

We motivate an integration formula for finding the arc length of a smooth curve in a plane. Similarly, we use a formula for finding the surface area of the solid obtained by revolving a curve about an axis. These formulas are demonstrated with several examples and exercises are provided at the end to develop integration skills.

5.11 Arc Length

To find a nice usable formula for finding the *arc length of a curve* we will work with *smooth curves.*

Definition 5.4. Let $y = f(x)$ be a plane curve over an interval $[a, b]$. We say that f is **smooth** whenever f' is continuous on $[a, b]$.

To find the arc length of a smooth curve we begin by dividing the curve into small segments, or **subcurves** .

Then we approximate the curve segments by line segments and sum the lengths of the line segments to form a Riemann sum. We then let the number of segments increase, the corresponding Riemann sums approach a definite integral (under our assumption of smooth curve) whose value we will take to be the **arc length** L of the curve. Let's see this in more detail.

Consider the k-th subinterval $[x_{k-1}, x_k]$ and the line segment between the points

$$(x_{k-1}, f(x_{k-1})) \qquad \text{and} \qquad (x_k, f(x_k)). \tag{5.15}$$

The change in the y-coordinate is denoted by

$$\Delta y_k = f(x_k) - f(x_{k-1}).$$

Using the Pythagorean theorem, the length of each line segment is then

$$\sqrt{(\Delta x)^2 + (\Delta y_k)^2} \qquad \text{for } k = 1, 2, \dots, n. \tag{5.16}$$

Summing these lengths we obtain an approximation for the arc length L as

$$L \approx \sum_{k=1}^{n} \sqrt{(\Delta x)^2 + (\Delta y_k)^2}. \tag{5.17}$$

Manipulating the inside expression we obtain

$$L \approx \sum_{k=1}^{n} \sqrt{(\Delta x)^2 \left(1 + \left(\frac{\Delta y_k}{\Delta x}\right)^2\right)} = \sum_{k=1}^{n} \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x}\right)^2}\, \Delta x. \tag{5.18}$$

Using the Mean Value Theorem we obtain

$$L \approx \sum_{k=1}^{n} \sqrt{1 + [f'(x_k^*)]^2}\, \Delta x \tag{5.19}$$

where x_k^* is in the k-th subinterval $[x_{k-1}, x_k]$.

::: {#thm- } If $y = f(x)$ is a smooth curve on the interval $[a, b]$, then the arc length L of this curve over $[a, b]$ is given as

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx. \tag{5.20}$$

:::

Example 5.18. Find the exact arc length of the curve $y = \frac{x^6+8}{16x^2}$ over the interval $x = 2$ to $x = 3$.

Solution. Notice that $f'(x) = \frac{1}{4}x^3 - x^{-3}$ and so

$$L = \int_2^3 \sqrt{1 + \left(\frac{1}{16}x^6 - \frac{1}{2} + x^{-6}\right)}\, dx$$
$$= \int_2^3 \sqrt{\left(\frac{1}{4}x^3 + x^{-3}\right)^2}\, dx = \int_2^3 \left(\frac{1}{4}x^3 + x^{-3}\right)\, dx = \frac{595}{144}.$$

as desired.

Example 5.19. Find the arc length of the graph of $y = \ln\cos x$ on $[0, \frac{\pi}{4}]$.

Solution. First we find that $y' = -\tan x$. Now we apply the Arc Length Formula to find

$$L = \int_0^{\pi/4} \sqrt{1 + (-\tan x)^2}\, dx = \int_0^{\pi/4} \sec x\, dx$$
$$= \int_0^{\pi/4} \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x}\right)\, dx.$$

Now to continue we let $u = \sec x + \tan x$. Then we find $du = (\sec x \tan x + \sec^2 x)\, dx$ and so we find the arc length as

$$L = \int_1^{1+\sqrt{2}} \left(\frac{1}{u}\right)\, du = \ln(1 + \sqrt{2})$$

as desired.

Switching the roles of x and y, we obtain the arc length for a smooth curve having the form $x = g(y)$.

::: {#thm- } If $x = f(y)$ is a smooth curve on the interval $[c, d]$, then the arc length L of this curve over $[c, d]$ is given as

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy. \tag{5.21}$$

:::

Example 5.20. Find the exact arc length of the curve $x = \frac{1}{8}y^4 + \frac{1}{4}y^{-2}$ over the interval $y = 1$ to $y = 4$.

Solution. Let $x = g(y) = \frac{1}{8}y^4 + \frac{1}{4}y^{-2}$ and notice that $g'(y) = \frac{1}{8}y^2 - 2y^{-2}$ and so

$$L = \int_1^4 \sqrt{1 + \left(\frac{1}{4}y^6 - \frac{1}{2} + \frac{1}{4}y^{-6}\right)}\, dy$$
$$= \int_1^4 \left(\frac{1}{2}y^3 + \frac{1}{2}y^{-3}\right)\, dy = \frac{2055}{64}$$

as desired.

Example 5.21. Find the arc length of the curve given by $y = \ln(x - \sqrt{x^2-1})$, for $1 \leq x \leq \sqrt{2}$ by integrating wth respect to y. :::

Solution. Notice that $e^y = x - \sqrt{x^2-1}$. Also notice that

$$e^{-y} = \frac{1}{x-\sqrt{x^2-1}} = \frac{1}{x-\sqrt{x^2-1}}\left(\frac{x+\sqrt{x^2-1}}{x+\sqrt{x^2-1}}\right) \tag{5.22}$$

We obtain $e^{-y} = x + \sqrt{x^2-1}$. Now by adding e^y and e^{-y}, we obtain $x = \frac{1}{2}(e^y + e^{-y})$. Therefore, $\frac{dx}{dy} = \frac{e^y - e^{-y}}{2}$ and so the arc length is

$$L = \int_{\ln(\sqrt{2}-1)}^{0} \sqrt{1 + \left(\frac{e^y - e^{-y}}{2}\right)^2}\, dy = \left(\frac{e^y - e^{-y}}{2}\right)\Big|_{\ln(\sqrt{2}-1)}^{0} = 1$$

as desired.

Surfaces of Revolution

::: {#thm- } Let f be a nonnegative smooth function on $[a, b]$. The **surface area** of the surface obtained by revolving the graph of f about the x-axis is

$$S = 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2}\, dx. \tag{5.23}$$

Example 5.22. Find the surface area generated by revolving the curve $y = \sqrt{2x - x^2}$ on $0.5 \leq x \leq 1.5$.

Solution. We have $\frac{dy}{dx} = \frac{1-x}{\sqrt{2x-x^2}}$ and so using the Surface Area Theorem we have

$$\begin{aligned} S &= \int_{0.5}^{1.5} 2\pi\sqrt{2x-x^2}\sqrt{1 + \frac{(1-x)^2}{2x-x^2}}\, dx \\ &= 2\pi \int_{0.5}^{1.5} \sqrt{2x-x^2}\left(\frac{\sqrt{2x-x^2+1-2x+x^2}}{\sqrt{2x-x^2}}\right) dx \\ &= 2\pi \int_{0.5}^{1.5} dx = 2\pi \end{aligned}$$

as desired.

Example 5.23. Verify that the surface area of a spher of radius r is $S = 4\pi r^2$ by evaluating a definite integral.

Solution. Let $y = \sqrt{r^2 - x^2}$. Then $y' = -\frac{x}{\sqrt{r^2-x^2}}$ and so

$$1 + (y')^2 = \frac{r}{r^2 - x^2}. \tag{5.24}$$

Therefore we find that

$$S = 2\pi \int_{-r}^{r} y \left(\frac{r}{\sqrt{r^2 - x^2}} \right) dx = 2\pi r \int_{-r}^{r} dx = 4\pi r^2 \tag{5.25}$$

as desired.

::: {#thm- } Let g be a nonnegative smooth function on $[c, d]$. The **surface area** of the surface obtained by revolving the graph of g about the y-axis is

$$S = 2\pi \int_{c}^{d} f(y) \sqrt{1 + [g'(y)]^2}\, dy. \tag{5.26}$$

:::

Example 5.24. Find the area of the surface obtained by revolving the curve

$$x = \frac{1}{6} y^3 + \frac{1}{2y} \tag{5.27}$$

for $1 \leq y \leq 2$ about the y-axis.

Solution. Notice that $x' = \frac{y^4 - 1}{2y^2}$ and so

$$1 + (x')^2 = 1 + \frac{(y^4 - 1)^2}{4y^4} = \frac{(y^4 + 1)^2}{4y^4} \tag{5.28}$$

Using the Surface Area Theorem we find the surface area to be

$$\begin{aligned} S &= \pi \int_{1}^{2} \left(\frac{y^3}{6} + \frac{1}{2y} \right) \left(y^2 + \frac{1}{y^2} \right) dy \\ &= \pi \int_{1}^{2} \left(\frac{y^5}{6} + \frac{2y}{3} + \frac{1}{2y^3} \right) dy = \frac{47\pi}{16} \end{aligned}$$

as desired.

5.12 Exercises

Exercise 5.33. Find the length of the curves. - $y = (1/3)(x^2+2)^{3/2}$ from $x = 0$ to $x = 3$ - $y = (x^3/3) + x^2 + x + 1/(4x + 4)$ from $x = 0$ to $x = 2$ - $y = (2/3)(x^2 + 1)^{3/2}$, from $x = 1$ to $x = 4$ - $y = x^2 \cos x$ from $x = 0$ to $x = \pi$ - $y = x^3 - x^2$ from $x = -1$ to $x = 1$ - $y = (1/2)(e^x + e^{-x})$ from $x = 0$ to $x = \ln 2$ - $x = \sqrt{36 - y^2}$, $0 \le y \le 3$ - $x = (1/3)\sqrt{y}(y - 3)$, $1 \le y \le 4$ - $x = 2e^{\sqrt{2}y} + (1/16)e^{-\sqrt{2}y}$, $0 \le y \le (\ln 2)/\sqrt{2}$ - $x = y^4/4 + 1/(8y^2)$, $1 \le y \le 2$

Exercise 5.34. Use integration to find the lateral surface area of the cone generated by revolving the line segment $y = x/2$, $0 \le x \le 4$ about the y-axis.

Exercise 5.35. Find the areas of the surfaces generate by revolving the curves about the indicated axis. - $y = x^3/9$, $0 \le x \le 2$, x-axis - $y = \sqrt{x}$ on $[4, 9]$, x-axis - $x = y^3/3$, $0 \le y \le 1$, y-axis - $y = \sqrt{4x + 6}$, $0 \le x \le 5$, x-axis - $y = (1/3)\sqrt{y(3 - y)^2}$ on $0 \le y \le 3$, y-axis - $x = 2\sqrt{4 - y}$, $0 \le y \le 15/4$, y-axis - $y = x^2/4$, $2 \le x \le 4$, y-axis

Exercise 5.36. Find the area of the surface generated by revolving about the x-axis the portion of the astroid $x^{2/3} + y^{2/3} = 1$, $0 \le x \le 1$.

Exercise 5.37. Find the area of the surface obtained by revolving the graph of $y = \sqrt{4 - x^2}$ on $[0, 1]$ about the x-axis

Exercise 5.38. Which curve has greater length on the interval $[-1, 1]$, $y = 1 - x^2$ or $y = \cos(\pi x/2)$?

Exercise 5.39. Find the surface area of an ellipsoid. Use the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

to represent the curve that is rotated about an axis to generate the solid, under a suitable restriction.

Chapter 6

Techniques of Integration

Techniques of Integration (A Student's Handbook) provides a comprehensive overview of the most common techniques for solving integrals. The book includes substitution, integration by parts, partial fractions, and trigonometric substitutions. Each technique is clearly explained with step-by-step examples, and practice problems are provided to help you hone your skills.

Integration by parts is a technique of integration that can be used to simplify the process of integration. The basic idea is to express a given function as the product of two simpler functions. The key to using this technique effectively is to choose the two functions wisely so that the resulting product is easier to integrate than the original function. With a little practice, integration by parts can be mastered and used to tackle even the most difficult integrals.

Integrals involving trigonometric functions can be tricky to solve. However, there are a few techniques that can be used to tackle these integrals. First, it can be helpful to rewrite the function in terms of another variable.

For example, if you're dealing with a sine function, you could try rewriting it in terms of cosine. This can sometimes make the integral easier to work with. Another technique is to use trigonometric identities.

In this chapter, you'll learn how to integrate powers of sine and cosine, and also powers of secant and tangent. With a bit of practice, trigonometric integrals can be conquered!

If you've ever been stuck trying to integrate a tricky function, then trigonometric substitution might be just the technique you need. This technique is based on the fact that many difficult integrals can be simplified by making a substitution that involves a trigonometric function

In this chapter you'll learn how to work with integrals involving sum and differences of squares. Trigonometric substitution can be a powerful tool for simplifying complex integrals of this form, and it's definitely worth learning if you're interested in techniques of integration. The idea is to replace a complicated integral involving a rational power of a trigonometric function with a simpler one. This can be achieved by making a suitable substitution in the integrand.

These techniques of integration have been developed over many centuries, with each new generation of mathematicians building on the work of their predecessors. One important area of research is the theory of partial fractions.

Partial fractions is a mathematical technique used to integrate rational functions. In the simplest case, it involves breaking up a fraction into a sum of simpler fractions, each of which can be integrated more easily. However, partial fractions can also be used to integrate more complex rational functions. The key is to identify the structure of the function and then to apply the appropriate techniques of integration.

Partial fractions are used to decompose a rational function into a sum of simpler functions, which makes it possible to integrate the function more easily. The first step in the process is to find the roots of the denominator polynomial. These roots are then used to split the denominator into a product of linear factors. The next step is to determine the coefficients of the partial fractions. This can be done by solving a system of linear equations, or by using a method known as undetermined coefficients. Once the coefficients are known, the final step is to integrate each partial fraction separately.

Although partial fractions may seem like a daunting topic at first, with a little practice it is possible to master this important technique. With a little practice, partial fractions can be used to solve a wide variety of integrals.

Integrals are one of the most fundamental concepts in mathematics, and improper integrals are a natural extension of this concept. However, improper integrals can be tricky to compute, due to the fact that they often involve infinity. As a result, there are a variety of techniques that can be used to tackle these integrals.

The most important thing is to make sure that the limits of integration are well-defined. Once this is done, you can use various techniques of integration to compute the improper integral. These techniques include integration by parts, partial fractions, and change of variables. With practice, you will be able to master these techniques and compute any improper integral.

But what exactly is an improper integral? Well, there are two types:

type I and type II. A type I improper integral is one where the function diverges at one or both of the limits of integration. A type II improper integral is one where the function converges, but the limit of integration is infinite. In both cases, the integral just doesn't exist in the usual sense.

So how do you deal with an improper integral? Well, intis chapter you'll learn a few techniques that can be used, depending on the situation. So if you ever find yourself faced with an improper integral, don't despair! There are techniques that can be used to deal with them. It just takes a little bit of creativity and thinking outside the box.

This book is not for everyone. In fact, it's quite possible that you will finish reading this sentence and immediately close the book, never to look at it again. But if you're still reading, then there's a good chance that this book is for you.

Specifically, it's for people who want to learn techniques of integration. Whether you're an experienced student or someone who just wants to brush up on your calculus skills, this book will provide you with the tools you need to succeed. So if you're looking for a book that can help you integrate techniques into your mathematical repertoire, then this is the book for you. By the time you finish this book, you should be well on your way to mastering integration techniques.

This book is specifically designed for students who are having trouble with integration techniques. In it, we'll go over all the major techniques and show you how to apply them to a variety of problems. We'll also provide plenty of worked examples so that you can see how the techniques are used in practice. By the time you finish this book, you should be well on your way to mastering integration techniques.

How I teach in this book: 1) I use clear and concise language 2) I provide plenty of worked examples 3) I focus on the major integrative techniques 4) I show you how to apply these techniques to a variety of problems 5) I provide plenty of opportunity for practice. So what are you waiting for? Let's get started!

6.1 Integration By Parts

Here we motivate and elaborate on an integration technique known as integration by parts. We also demonstrate repeated application of this formula to evaluate a single integral. The reduction formula for integral powers of the cosine function and an example on its use is also presented.

6.2 Introduction

There are many techniques of integration. Consider for example, the following three integrals.

$$\int \frac{4}{x^2+9}\,dx \qquad \int \frac{4x}{x^2+9}\,dx \qquad \int \frac{4x^2}{x^2+9}\,dx \tag{6.1}$$

How would you integrate these? The first uses the arctangent function, the second uses logarithms, and the third uses division. Can you integrate these three integrals?

6.3 The Integration by Parts Formula

Let f and g be differentiable functions. Recall the product rules implies that fg is a differentiable function and that

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x). \tag{6.2}$$

If we integrate both sides we obtain

$$\int [f(x)g(x)]'dx = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx. \tag{6.3}$$

It may happen that one of the integrals on the right-hand side is easier to integrate than the other.

::: {#thm- } Let $u = f(x)$ and $v = g(x)$ be differentiable functions. The differentials are $du = f'(x)\,dx$ and $dv = g'(x)\,dx$ and the formula

$$\int u\,dv = uv - \int v\,du \tag{6.4}$$

is called **integration by parts** . :::

Example 6.1. Find $\int xe^x\,dx$.

Solution. Let $u = x$ and $dv = e^x\,dx$. Then $du = dx$ and $v = e^x$, we have

$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C \tag{6.5}$$

where C is an arbitrary constant.

Example 6.2. Find $\int x \sin x\,dx$.

Solution. Let $u = x$ and $dv = \sin x\,dx$. Then $du = dx$ and $v = -\cos x$, we have

$$\int x\sin x\,dx = -x\cos x - \int(-\cos x)\,dx = -x\cos x + \sin x + C \tag{6.6}$$

where C is an arbitrary constant.

Theorem 6.1. *Let $f(x)$ and $g(x)$ be differentiable functions. Then*

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\big|_a^b - \int_a^b g(x)f'(x)\,dx. \tag{6.7}$$

Example 6.3. Find $\int_e^{e^2} \ln x\,dx$.

Solution. Let $u = \ln x$ and $dv = dx$. Then $du = (1/x)dx$ and $v = x$, we have

$$\int_e^{e^2} \ln x\,dx = x\ln x\big|_e^{e^2} - \int_e^{e^2} x\left(\frac{1}{x}\right)\,dx = (x\ln x - x)\big|_e^{e^2} = e^2 \tag{6.8}$$

or approximately 7.3891.

6.4 Solving for Unknown Integral

Example 6.4. Find $\int \sec^3 x\,dx$.

Solution. Let $u = \sec x$ and $dv = \sec^2 x$. Then $du = \sec x\tan x\,dx$ and $v = \int \sec^2 x\,dx$. We have

$$\begin{aligned}\int \sec^3 x\,dx &= \sec x\tan x - \int \tan^2 x\sec x\,dx\\ &= \sec x\tan x - \int(\sec^2 x - 1)\sec x\,dx\\ &= \sec x\tan x - \int \sec^3 x\,dx + \int \sec x\,dx\\ &= \sec x\tan x - \int \sec^3 x\,dx + \ln|\sec x + \tan x|.\end{aligned}$$

Combining these together we have

$$2\int \sec^3 x\,dx = \sec x\tan x + \ln|\sec x + \tan x| + C \tag{6.9}$$

where C is an arbitrary constant. Therefore we finally have

$$\int \sec^3 x\,dx = \frac{1}{2}\sec x \tan x + \ln\sqrt{\sec x + \tan x} + C \tag{6.10}$$

as desired.

6.5 Repeated Use of Integration by Parts

Example 6.5. Find $\int e^x \cos x\,dx$.

Solution. Let $u = e^x$ and $dv = \cos x\,dx$. Then $du = e^x$ and $v = \sin x$. We find that

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx. \tag{6.11}$$

Let $u = e^x$ and $dv = \sin x\,dx$. Then $du = e^x$ and $v = -\cos x$. We find that

$$\begin{aligned}\int e^x \cos x\,dx &= e^x \sin x - \left(-e^x \cos x - \int e^x(-\cos x)\,dx\right)\\ &= e^x \sin x + e^x \cos x - \int e^x \cos x\,dx.\end{aligned}$$

Combining these together we have

$$2\int e^x \cos x\,dx = e^2 \sin x + e^x \cos x + C \tag{6.12}$$

or in other words

$$\int e^x \cos x\,dx = \frac{e^x \sin x + e^x \cos x}{2} + C \tag{6.13}$$

where C is an arbitrary constant.

6.6 Reduction Formulas

Example 6.6. Demonstrate the repeated use of integration by parts by proving the reduction formula

$$\int \cos^n x\,dx = \frac{1}{n}\sin x \cos^{n-1} x + \frac{n-1}{n}\int \cos^{n-2} x\,dx \tag{6.14}$$

where n is an integer greater than 2.

Solution. Let $u = \cos^{n-1} x$ and $dv = \cos x \, dx$. Then we have

$$du = (n-1)\cos^{n-2} x(-\sin x)\, dx$$

and $v = \sin x$. Therefore

$$\begin{aligned}
\int \cos^n \, dx &= \cos^{n-1} \sin x + (n-1) \int \sin^2 x \cos^{n-2} x \, dx \\
&= \cos^{n-1} x \sin x + (n-1) \int (1 - \cos^2 x) \cos^{n-2} x \, dx \\
&= \cos^{n-1} \sin x + (n-1) \int \cos^{n-2} x \, dx - (n-1) \int \cos^n x \, dx
\end{aligned}$$

Combining these together we have

$$n \int \cos^n x \, dx = \cos^{n-1} \sin x + (n-1) \int \cos^{n-2} x \, dx. \tag{6.15}$$

The final result is

$$\int \cos^n x \, dx = \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \, dx \tag{6.16}$$

where C is an arbitrary constant.

Example 6.7. Find $\int \cos^5 x \, dx$.

Solution. We will use the reduction formula with $n = 5$ and $n = 3$. We have

$$\begin{aligned}
\int \cos^5 \, dx &= \frac{\cos^4 \sin x}{5} + \frac{4}{5} \int \cos^3 \, dx \\
&= \frac{\cos^4 \sin x}{5} + \frac{4}{5} \left(\frac{1}{3} \cos^2 \sin x + \frac{2}{3} \sin x \right) + C \\
&= \frac{1}{5} \cos^4 \sin x + \frac{4}{15} \cos^2 \sin x + \frac{8}{15} \sin x + C
\end{aligned}$$

where C is an arbitrary constant.

6.7 Exercises

Exercise 6.1. Evaluate the following integrals.

- $\int x \sin 3x \, dx$
- $\int (\ln x)^2 \, dx$
- $\int 2x e^{3x} \, dx$

- $\int \tan^{-1} x\, dx$
- $\int \cos^{-1}(2x)\, dx$
- $\int x \sec^2 x\, dx$
- $\int \frac{xe^x}{(x+1)^2}\, dx$
- $\int x \sin x \cos x\, dx$
- $\int x^2 e^{4x}\, dx$

Exercise 6.2. Evaluate the following definite integrals.

- $ _0^1 x e^{-5x}, dx $
- $\int_0^{\ln 2} \ln 2x\, dx$
- $\int_1^{e^2} x^2 \ln x\, dx$
- $\int_0^{1/\sqrt{2}} y \tan^{-1}\, dy$
- $\int_0^{\pi/8} x \sec^2 2x\, dx$
- $\int_0^1 x \arcsin^2\, dx$

Exercise 6.3. Evaluate the integral by making a u-substitution and then integration by parts.

- $\int e^{\sqrt{x}}\, dx$
- $\int x^5 e^{x^2}\, dx$
- $\int \cos \sqrt{x}\, dx$

Exercise 6.4. Use the reduction formulas to evaluate the integral. \begin{multicols}{4} - $\int \sec^4 x\, dx$ - $\int \tan^4 x\, dx$ - $\int_0^{\pi/2} \sin^5 x\, dx$ - $\int \cos^5 x\, dx$

Exercise 6.5. Find the area of the region between $y = x \sin x$ and $y = x$ for $0 \leq x \leq \pi/2$.

Exercise 6.6. Find the area of the region under the graph of $y = (\ln x)^2$ n the interval $[1, e]$.

Exercise 6.7. Find the area of the region under the graph of $y = \frac{xe^x}{(1+x)^2}$ on the interval $[0, 1]$.

Exercise 6.8. Given the region bounded by the graph of $y = x\cos x$, $y = 0$, $x = -1$, and $x = 1$. Find the area of the region. Find the volume of the solid generated by revolving abut the x-axis.

Exercise 6.9. The region bounded by the graph pf $y = e^{x/2}\cos x$, $x = 0$, $y = 0$, and $x = \pi/2$ is revolved about eh x-axis. Find the volume if the solid generated.

Exercise 6.10. If $f(0) = g(0)$ and f'' and g'' are continuous, prove that

$$\int_0^a f(x)g''(x)\,dx = f(a)g'(a) - f'(a)g(a) + \int_0^a f''(x)g(x)\,dx. \tag{6.17}$$

Exercise 6.11. The region bounded by the graphs of $y = \sin x$, $y = 0$, $x = 0$, and $x = \pi$ is revolved about the line $x = -1$. Find the volume if the solid generated.

Exercise 6.12. Suppose that $f(1) = 2$, $f(4) = 7$, $f'(1) = 5$, $f'(4) = 3$, and f'' is continuous. Find the value of $\int_1^4 xf''(x)\,dx$.

Exercise 6.13. Use integration by parts to show that

$$\int f(x)\,dx = xf(x) - \int xf'(x)\,dx.$$

Exercise 6.14. Use the method of cylindrical shells to find the volume generated by rotating the region bounded by $y = e^{-x}$, $y = 0$, $x = -1$, $x = 0$, about $x = 1$.

Exercise 6.15. Use integration by parts to establish the reduction formula - $\int \sin^n x\,dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$ - $\int x^n \sin x\,dx = -x^n \cos x + n\int x^{n-1}\cos x\,dx$ - $\int x^n \cos x\,dx = -x^n \sin x + n\int x^{n-1}\sin x\,dx$

6.8 Trigonometric Integrals

Integrals involving powers of sine and cosine and integrals involving powers of secant and tangent are studied. We also discuss integrals of products of sine and cosine functions involving different angles. Proficiency using trigonometric identities is assumed.

6.9 Integrals Involving Powers of Sine and Cosine

The goal now is to evaluate integrals involving trigonometric functions. These techniques will be indispensable we using trigonometric substitutions.
We begin with a simple example that demonstrates some of the ideas in this sections.

Example 6.8. Evaluate $\int \sin^3 x\,dx$.

Solution. We write $\sin^3 x = \sin^2 x \sin x = (1-\cos^2 x)\sin x$ and use this with the substitution $u = \cos x$ as follows

$$\begin{aligned}\int \sin^3 x\,dx &= \int (1-\cos^2 x)\sin x\,dx = -\int (1-u^2)\,du \\ &= -u + \frac{1}{3}u^3 + C = -\cos x + \frac{1}{3}\cos^3 x + C\end{aligned}$$

where C is an arbitrary constant.

The identity $\sin^2 x + \cos^2 x = 1$ allows use to convert back and forth between powers of sine and cosine as needed.

Theorem 6.2. *If m and n are positive integers, then the integral*

$$\int \sin^m x \cos^n x\,dx \tag{6.18}$$

can be evaluated by one of the three methods. - If m is odd, then split off a factor of $\sin x$, apply the identity $\sin^2 x = 1-\cos^2 x$, and make the substitution $u = \cos x$. - If n is odd, then split off a factor of $\cos x$, apply the identity $\cos^2 x = 1-\sin^2 x$, and make the substitution $u = \sin x$ - If m and n are both even, use the identities

$$\sin^2 x = \frac{1}{2}(1-\cos 2x) \qquad \cos^2 x = \frac{1}{2}(1+\cos 2x)$$

to reduce the powers on $\sin x$ and $\cos x$.

Example 6.9. Evaluate $\int_0^{\pi/3} \sin^4 3x \cos^3 3x\,dx$.

Solution. Notice that $n = 3$ is odd, so we have

$$\begin{aligned}\int_0^{\pi/3} \sin^4 3x \cos^3 3x\,dx &= \int_0^{\pi/3} \sin^4 3x(1-\sin^2 3x)\cos 3x\,dx \\ &= \left[\frac{1}{15}\sin^5 3x - \frac{1}{21}\sin^7 3x\right]_0^{\pi/3} = 0\end{aligned}$$

as desired.

Example 6.10. Evaluate $\int \sin^3 x \cos^2 x \, dx$.

Solution. Notice that $m = 3$ is odd, so we have

$$\int \sin^3 x \cos^2 x \, dx = \int (1 - \cos^2 x) \cos^2 x \sin x \, dx$$
$$= \int (\cos^2 x - \cos^4 x) \sin x \, dx = -\frac{1}{3} \cos^3 x + \frac{1}{5} \cos^5 x + C$$

where C is an arbitrary constant.

Example 6.11. Evaluate $\int \sin^2 x \cos^4 x \, dx$.

Solution. Notice that both m and n are even, so we have

$$\begin{aligned}
\int \sin^2 x \cos^4 x \, dx &= \frac{1}{8} \int (1 - \cos 2x)(1 + \cos 2x)^2 \, dx \\
&= \frac{1}{8} \int (1 - \cos^2 2x)(1 + \cos 2x) \, dx \\
&= \frac{1}{8} \int \sin^2 2x \, dx + \frac{1}{8} \int \sin^2 2x \cos 2x \, dx \\
&= \frac{1}{16} \int (1 - \cos 4x) \, dx + \frac{1}{48} \sin^3 2x \\
&= \frac{1}{16} x - \frac{1}{64} \sin 4x + \frac{1}{48} \sin^3 2x + C
\end{aligned}$$

where C is an arbitrary constant.

6.10 Integrals Involving Powers of Secant and Tangent

Recall the basic formulas from before:

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \int \tan x \, dx = \ln|\sec x| + C \tag{6.19}$$

and for the secant function

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \int \sec x \, dx = \ln|\sec x + \tan x| + C. \tag{6.20}$$

Example 6.12. Evaluate $\int \tan^3 x \, dx$.

Solution. We use the identity $\tan^2 x = \sec^2 x - 1$ and we have

$$\begin{aligned}\int \tan^3 x\, dx &= \int \tan x \tan^2 x\, dx = \int \tan x(\sec^2 x - 1)\, dx \\ &= \int \tan x \sec^2 x\, dx - \int \tan x\, dx = \frac{\tan^2 x}{2} - \ln|\sec x| + C\end{aligned}$$

as desired.

Theorem 6.3. *If m and n are positive integers, then the integral*

$$\int \sec^m x \tan^n x\, dx \tag{6.21}$$

can be evaluated by one of the three methods. - If m is even, then split off a factor of $\sec x$, apply the identity $\sec^2 x = 1 + \tan^2 x$, and make the substitution $u = \tan x$. - If n is odd, then split off a factor of $\sec^2 x$, apply the identity $\tan^2 x = \sec^2 x - 1$, and make the substitution $u = \sec x$. - If m is odd and n is even, use the identity $\tan^2 x = \sec^2 x - 1$ and the reduction formula for powers of $\sec x$, namely

$$\int \sec^n x\, dx = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} \int \sec^{n-2} x\, dx \tag{6.22}$$

for a positive integer n great than 1.

Example 6.13. Evaluate $\int \sec^4 x \tan^9 x\, dx$.

Solution. Notice that $m = 4$ is even. We let $u = \tan x$ so that $du = \sec^2 x$. We have

$$\begin{aligned}\int \sec^4 x \tan^9 x\, dx &= \int u^9(u^2+1)\, du = \int (u^11 + u^9)\, du \\ &= \frac{u^{12}}{12} + \frac{u^{10}}{10} + C = \frac{1}{12}\tan^{12} x + \frac{1}{10}\tan^{10} x + C\end{aligned}$$

where C is an arbitrary constant.

Example 6.14. Evaluate $\int \sec^7 x \tan^5 x\, dx$.

Solution. Notice that $n = 5$ is odd. We let $u = \sec x$ so that $du =$

$\sec x \tan x$. We have

$$\begin{aligned}
\int \sec^7 x \tan^5 x\, dx &= \int \tan^4 x \sec^6 x \sec x \tan x\, dx \\
&= \int (\sec^2 x - 1)^2 \sec^6 x \sec x \tan x\, dx \\
&= \int (u^2-1)^2 u^6\, du = \int (u^{10} - 2u^8 + u^6)\, du \\
&= \frac{u^{11}}{11} - 2\left(\frac{u^9}{9}\right) + \frac{u^7}{7} + C \\
&= \frac{1}{11}\sec^{11} x - \frac{2}{9}\sec^9 x + \frac{1}{17}\sec x + C
\end{aligned}$$

where C is an arbitrary constant.

6.11 Integrals Involving Products of Different Angles

Integral involving product of sine and cosines functions with different angle we will make use the trigonometric formulas:

$$\sin mx\, \sin nx = \frac{1}{2}\left(\cos[(m-n)x] - \cos[(m+n)x]\right) \tag{6.23}$$

$$\sin mx\, \cos nx = \frac{1}{2}\left(\sin[(m-n)x] + \sin[(m+n)x]\right) \tag{6.24}$$

$$\cos mx\, \cos nx = \frac{1}{2}\left(\cos[(m-n)x] + \cos[(m+n)x]\right) \tag{6.25}$$

Example 6.15. Evaluate $\int \sin 5x \sin 7x\, dx$.

Solution. We have

$$\begin{aligned}
\int \sin 5x \sin 7x\, dx &= \frac{1}{2}\left(\cos(-2x)\, dx - \int \cos 12x\, dx\right) \\
&= \frac{1}{2}\left(\cos 2x\, dx - \int \cos 12x\, dx\right) \\
&= \frac{1}{2}\left(\frac{\sin 2x}{2} - \frac{\sin 12x}{12}\right) + C \\
&= \frac{\sin 2x}{4} - \frac{\sin 12x}{24} + C
\end{aligned}$$

where C is an arbitrary constant.

Example 6.16. Evaluate $\int \sin 3x \cos 7x\, dx$.

Solution. We have

$$\begin{aligned}\int \sin 3x \cos 7x \, dx &= \frac{1}{2}\left(\sin(-4x)\, dx + \int \sin 10x \, dx\right) \\ &= \frac{1}{2}\left(\frac{\cos(-4x)}{4} - \frac{\cos 10x}{10}\right) + C \\ &= \frac{\cos 4x}{8} - \frac{\cos 10x}{20} + C\end{aligned}$$

where C is an arbitrary constant.

Example 6.17. Evaluate $\int \cos x \cos 2x \, dx$.

Solution. We have

$$\begin{aligned}\int \cos x \cos 2x \, dx &= \frac{1}{2}\left(\int \cos(-x)\, dx + \int \cos 3x \, dx\right) \\ &= \frac{\sin x}{2} + \frac{\sin 3x}{6} + C\end{aligned}$$

where C is an arbitrary constant.

6.12 Exercises

Exercise 6.16. Evaluate the integrals.

- $\displaystyle\int_0^{\pi} \sin^5\left(\frac{x}{2}\right) dx$
- $\displaystyle\int \sin^6 x \cos^3 x \, dx$
- $\displaystyle\int \sin^3 x \cos x \, dx$
- $\displaystyle\int_{-\pi/2}^{\pi/2} \cos^3 x \, dx$
- $\displaystyle\int_0^{\pi/2} \cos^5 x \, dx$
- $\displaystyle\int \cos^4 x \, dx$
- $\displaystyle\int_0^{\pi} \sin^4(3t)\, dt$
- $\displaystyle\int_0^{\pi} 8 \sin^4 y \cos^2 y \, dy$
- $\displaystyle\int_0^{\pi} \cos^6 \theta \, d\theta$

- $\displaystyle\int_0^{\pi} \sin^2 x \cos^4 x \, dx$
- $\displaystyle\int_0^{\pi/4} 8\cos^3 2\theta \sin 2\theta \, d\theta$
- $\displaystyle\int_0^{\pi} \sin^2 t \cos^4 t \, dt$
- $\displaystyle\int x\cos^2 x \, dx$
- $\displaystyle\int e^x \sec^3 e^x \, dx$
- $\displaystyle\int_{\pi/4}^{\pi/2} \cot^3 x \, dx$

Exercise 6.17. Evaluate the integrals.

- $\displaystyle\int_0^{\pi/2} \sin x \cos x \, dx$
- $\displaystyle\int \sin 3\theta \sin 4\theta \, d\theta$
- $\displaystyle\int \cot^3 x \csc^3 x \, dx$
- $\displaystyle\int_0^{\pi} \cos 3x \cos 4x \, dx$
- $\displaystyle\int \frac{\sin^3 x}{\sec^2 x} \, dx$
- $\displaystyle\int (\tan^2 + \tan^4 x) \, dx$
- $\displaystyle\int \frac{\cos x + \sin x}{\sin 2x} \, dx$
- $\displaystyle\int_{-\pi/2}^{\pi/2} \cos x \cos 7x \, dx$
- $\displaystyle\int \frac{1}{\csc x \cot^2 x} \, dx$
- $\displaystyle\int \frac{1}{\cos x - 1} \, dx$
- $\displaystyle\int \frac{1 - tan^x}{\sec^2 x} \, dx$
- $\displaystyle\int \sin 3x \sin 6x \, dx$

Exercise 6.18. Find the length of the curve $y = \ln(\cos x)$ on the interval $[0, \frac{\pi}{3}]$.

Exercise 6.19. Find the are under the graph of $y = \sin^2 \pi x$ on the interval $[0, 1]$.

Exercise 6.20. Find the area of the region bounded by the curves $y = \sin^2 x$ and $y = \cos^2 x$ on $-\pi/4 \le x \le \pi/4$.

Exercise 6.21. Find the area enclosed by the graphs of $y = \sin^4 x$, $y = \cos^4 x$, $x = 0$, and $x = \pi/4$.

Exercise 6.22. Find the volume of the solid generated by revolving the region under the graph of

$$f(x) = \frac{\sin x}{\cos^3 x}$$

on $\left[0, \frac{\pi}{4}\right]$ about the x-axis.

Exercise 6.23. Evaluate $\int \sin x \cos x \, dx$ four different ways. Two of them using u-substitutions, a third using integration by parts and a fourth using a trigonometric identity.

Exercise 6.24. Prove that

$$\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0$$

where m and n are positive integers.

Exercise 6.25. Prove that

$$\int \cos^m x \sin^n x \, dx = -\frac{\cos^{m+1} x \sin^{n-1} x}{m+n} + \frac{n-1}{m+n} \int \cos^m x \sin^{n-2} x \, dx$$

where m and n are positive integers.

Exercise 6.26. Prove that

$$\int \sec^n x \, dx = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx$$

where n is a positive integers.

6.13 Trigonometric Substitution

Trigonometric substitution refers to an integration technique that uses trigonometric functions (mostly tangent, sine, and secant) to reduce an integrand to another expression so that one may utilize another known technique of integration. Here we study these three main forms and also give examples where we can use complete the square to reduce to one of these three methods.

Trigonometric substitution are intended to transform integrals containing the expressions

$$a^2 + x^2 \qquad a^2 - x^2 \qquad x^2 - a^2 \tag{6.26}$$

into trigonometric integrals that can be evaluated using previously discussed methods.

6.14 Integrals Involving $a^2 + x^2$

Here we intend to integrate functions containing the expression $a^2 + x^2$ using a tangent substitution.
With $x = a\tan\theta$, we have

$$a^2 + x^2 = a^2 + a^2\tan^2\theta = a^2(1 + \tan^2\theta) = a^2\sec^2\theta. \tag{6.27}$$

This indicates that an integral containing the expression $a^2 + x^2$ may be evaluated by using an integral containing powers of secant. We would like the substitution to be reversible so that we can change back to the original variable when finished. This requires us to know that $x = a\tan\theta$ is solvable for θ. Therefore we require

$$\theta = \tan^{-1}\left(\frac{x}{a}\right) \qquad \text{with} \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}. \tag{6.28}$$

Example 6.18. Find $\displaystyle\int \frac{1}{\sqrt{4+x^2}}\,dx$.

Solution. Let $x = 2\tan\theta$ so that $dx = 2\sec^2\theta\,d\theta$ and

$$\begin{aligned}\int \frac{1}{\sqrt{4+x^2}}\,dx &= \int \frac{2\sec^2\theta}{2\sec\theta}\,d\theta = \int \sec\theta\,d\theta \\ &= \ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{x+\sqrt{a+x^2}}{2}\right| + C\end{aligned}$$

where C is an arbitrary constant.

Example 6.19. Find $\displaystyle\int \frac{1}{(1+x^2)^2}\,dx$.

Solution. Let $x = \tan\theta$ so that $dx = \sec^2\theta\,d\theta$ and

$$\int \frac{1}{(1+x^2)^2}\,dx = \int \frac{\sec^2\theta}{\sec^4\theta}\,d\theta = \int \cos^2\theta d\theta = \frac{\theta}{2} + \frac{\sin 2\theta}{4} + C$$

where C is an arbitrary constant. Using the reference triangle The integration can be completed:

$$\int \frac{1}{(1+x^2)^2}\,dx = \frac{\theta}{2} + \frac{\sin 2\theta}{4} + C = \frac{1}{2}\tan^{-1}x + \frac{x}{2(1+x^2)} + C$$

where $\sin 2\theta$ is found using the triangle.

6.15 Integrals Involving $a^2 - x^2$

Now we intend to integrate functions containing the expression $a^2 - x^2$ using a sine substitution.
With $x = a\sin\theta$, we have

$$a^2 - x^2 = a^2 - a^2\sin^2\theta = a^2(1-\sin^2\theta) = a^2\cos^2\theta. \tag{6.29}$$

This indicates that an integral containing the expression $a^2 + x^2$ may be evaluated by using an integral containing powers of cosine. We would like the substitution to be reversible so that we can change back to the original variable when finished. This requires us to know that $x = a\sin\theta$ is solvable for θ. Therefore we require

$$\theta = \sin^{-1}\left(\frac{x}{a}\right) \qquad \text{with} \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}. \tag{6.30}$$

Example 6.20. Find $\displaystyle\int \frac{1}{x^2\sqrt{4-x^2}}\,dx$.

Solution. Let $x = 2\sin\theta$ so that $dx = 2\cos\theta\,d\theta$. This yields

$$\begin{aligned}\int \frac{1}{x^2\sqrt{4-x^2}}\,dx &= \int \frac{2\cos\theta}{(2\sin\theta)^2\sqrt{4-4\sin^2\theta}}\,d\theta \\ &= \frac{1}{4}\int \frac{1}{\sin^2\theta}\,d\theta = \frac{1}{4}\int \csc^2\theta\,d\theta = -\frac{1}{4}\cot\theta + C\end{aligned}$$

Using the reference triangle

we find that $\cot\theta = \frac{\sqrt{4-x^2}}{x}$. The integration can be completed:

$$\int \frac{1}{x^2\sqrt{4-x^2}}\,dx = -\frac{1}{4}\frac{\sqrt{4-x^2}}{x} + C$$

where C is an arbitrary constant.

Example 6.21. Find $\displaystyle\int \frac{\sqrt{9-x^2}}{x^2}\,dx$.

Solution. Let $x = 3\sin\theta$ so that $dx = 3\cos\theta\,d\theta$. This yields

$$\begin{aligned}\int \frac{\sqrt{9-x^2}}{x^2}\,dx &= \int \frac{3\cos\theta}{9\sin^2\theta}(3\cos\theta)\,d\theta \\ &= \int \cot^2\theta\,d\theta = \int(\csc^2\theta - 1)\,d\theta = -\cot\theta - \theta + C\end{aligned}$$

Using the reference triangle

we find that $\cot\theta = \frac{\sqrt{9-x^2}}{x}$. The integration can be completed:

$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx = -\frac{\sqrt{9-x^2}}{x} - \sin^{1}\left(\frac{x}{3}\right) + C$$

where C is an arbitrary constant.

6.16 Integrals Involving $x^2 - a^2$

Next, we intend to integrate functions containing the expression $x^2 - a^2$ using a secant substitution.
With $x = a\sec\theta$, we have

$$x^2 - a^2 = a^2\sec^2 a - a^2 = a^2(\sec^2\theta - 1) = a^2\tan^2\theta \tag{6.31}$$

This indicates that an integral containing the expression $x^2 - a^2$ may be evaluated by using an integral containing powers of tangent. We would like the substitution to be reversible so that we can change back to the original variable when finished. This requires us to know that $x = a\sec\theta$ is solvable for θ. Therefore we require

$$\theta = \sec^{-1}\left(\frac{x}{a}\right) \qquad \text{with} \qquad \begin{cases} 0 \le \theta < \pi/2 & \text{if } x/a \ge 1 \\ \pi/2 < \theta \le \pi & \text{if } x/a \le -1. \end{cases} \tag{6.32}$$

Example 6.22. Find $\displaystyle\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x}\,dx$.

Solution. Let $x = \sqrt{3}\sec\theta$ so that $dx = \sqrt{3}\sec\theta\tan\theta\,d\theta$. This yields

$$\begin{aligned}
\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x}\,dx &= \int_0^{\pi/6} \frac{(\sqrt{3}\tan\theta)(\sqrt{3}\sec\theta\tan\theta)}{\sqrt{3}\sec\theta}\,d\theta \\
&= \int_0^{\pi/6} \sqrt{3}\tan^2\theta\,d\theta \\
&= \sqrt{3}\int_0^{\pi/6} (\sec^2\theta - 1)\,d\theta = 1 - \frac{\sqrt{3}\pi}{6}
\end{aligned}$$

as desired.

Example 6.23. Find $\displaystyle\int_1^4 \frac{\sqrt{x^2+4x-5}}{x+2}\,dx$.

Solution. Noting that $x^2+4x-5 = (x+2)^2-9)$ we let $u = x+2$. Then $dx = du$ and we have

$$\int_1^4 \frac{\sqrt{x^2+4x-5}}{x+2}\,dx = \int_3^6 \frac{\sqrt{u^2-9}}{u}\,du.$$

Let $u = 3\sec\theta$ (where $0 \le \theta < \pi/2$), so that $du = 3\sec\theta\tan\theta\,d\theta$. This yields

$$\begin{aligned}
\int_1^4 \frac{\sqrt{x^2+4x-5}}{x+2}\,dx &= \int_3^6 \frac{\sqrt{u^2-9}}{u}\,du \\
&= \int_0^{\pi/3} \frac{3\tan\theta}{3\sec\theta}(3\sec\theta\tan\theta\,d\theta) \\
&= 3\int_0^{\pi/3} (\sec^2\theta - 1)\,d\theta \\
&= 3\sqrt{3} - \pi.
\end{aligned}$$

as desired.

6.17 Exercises

Exercise 6.27. Evaluate the following integrals.

- $\displaystyle\int \sqrt{1-9t^2}\,dt$
- $\displaystyle\int \frac{1}{x^3\sqrt{x^2-4}}\,dx$
- $\displaystyle\int \frac{5}{\sqrt{25x^2-9}}\,dx$, $x > 3/5$

- $\displaystyle\int x^3\sqrt{4-x^2}\,dx$
- $\displaystyle\int \sqrt{25-t^2}\,dt$
- $\displaystyle\int (4-x^2)^{3/2}\,dx$
- $\displaystyle\int \frac{\sqrt{y^2-25}}{y^3}\,dy,\ y>5$
- $\displaystyle\int e^x\sqrt{4-e^{2x}}\,dx$
- $\displaystyle\int \frac{1}{(1+x^2)^{3/2}}\,dx$
- $\displaystyle\int \frac{1}{\sqrt{16+4x^2}}\,dx$

Exercise 6.28. Evaluate the following integrals.

- $\displaystyle\int_1^2 \frac{\sqrt{x^2-1}}{x}\,dx$
- $\displaystyle\int_4^6 \frac{x^2}{\sqrt{x^2-9}}\,dx$
- $\displaystyle\int_0^1 x\sqrt{x^2+4}\,dx$
- $\displaystyle\int_0^{\sqrt{3}/2} \frac{1}{(1-t^2)^{5/2}}\,dt$
- $\displaystyle\int_0^a x^2\sqrt{a^2-x^2}\,dx$
- $\displaystyle\int_0^{3/5} \sqrt{9-25x^2}\,dx$
- $\displaystyle\int_{\sqrt{2}/3}^{2/3} \frac{1}{x^5\sqrt{9x^2-1}}\,dx$
- $\displaystyle\int_4^8 \frac{\sqrt{x^2-16}}{x^2}\,dx$

Exercise 6.29. Find the area of the region bounded by the hyperbola $9x^2-4y^2=36$ and the line $x=3$.

Exercise 6.30. Find the arc length of the curve over $y=\frac{1}{2}x^2$ on the interval $[0,4]$.

Exercise 6.31. Evaluate the following integrals.

- $\displaystyle\int \frac{1}{2u^2 - 12u + 36}\, dx$
- $\displaystyle\int \frac{1}{\sqrt{3 + 2x - x^2}}\, dx$
- $\displaystyle\int \frac{x^2 - 2x + 1}{\sqrt{x^2 - 2x + 10}}\, dx$
- $\displaystyle\int \frac{e^x}{\sqrt{1 + e^x + e^{2x}}}\, dx$
- $\displaystyle\int_1^2 \frac{1}{\sqrt{4x - x^2}}\, dx$
- $\displaystyle\int_0^4 \sqrt{x(4-x)}\, dx$

Exercise 6.32. Verify the special integration formulas ($a > 0$).

- $\displaystyle\int \sqrt{a^2 - u^2}\, du = \frac{1}{2}\left(a^2 \arcsin\frac{u}{a} + u\sqrt{a^2 - u^2}\right) + C$
- $\displaystyle\int \sqrt{u^2 - a^2}\, du = \frac{1}{2}\left(u\sqrt{u^2 - a^2} - a^2 \ln|u + \sqrt{u^2 - a^2}|\right) + C,\ u > a$
- $\displaystyle\int \sqrt{u^2 + a^2}\, du = \frac{1}{2}\left(u\sqrt{u^2 + a^2} + a^2 \ln|u + \sqrt{u^2 + a^2}|\right) + C.$

6.18 Partial Fractions

The decomposition of a rational function into a sum of simpler rational functions is an important integration technique, known as the method of partial fractions. We illustrate this method using several examples. We concentrate on linear and quadratic factors.

6.19 The Method of Partial Fractions

We now concentrate on integrals of the form

$$\int \frac{P(x)}{Q(x)}\, dx$$

where $P(x)$ and $Q(x)$ are polynomial functions. In the case when $Q(x)$ is a constant, then the integration is accomplished using the power rule form before.
In the case when the degree of $P(x)$ is larger than the degree of $Q(x)$ we can use long division. For example, consider the integral

$$\int \frac{x^2}{x+1}\, dx.$$

By long division, we have

$$\frac{x^2}{x+1} = x - 1 + \frac{1}{x+1}$$

Using the power rule and the natural logarithmic function we can easily integrate,

$$\int \frac{x^2}{x+1}\, dx = \frac{1}{2}x - x + \ln|x+1| + C$$

where C is an arbitrary constant. The method of partial fractions assumes that the fraction in the integrand is always irreducible, meaning the degree of the numerator is less than the degree of the denominator.

The method of partial fractions takes advantages of the fact that certain fractions are easy to integrate using the logarithmic function. For example, in (6.19), the integral

$$\int \frac{1}{x+1}\, dx$$

is immediate. So our goal is to be able to decomposition a complex fraction into simpler fractions which can be integrated easier. For example, notice that

$$\frac{3x}{x^2+2x-8} = \frac{1}{x-2} + \frac{2}{x+4}.$$

We ask, which is easier to integrate: the left or the right hand side?

To begin the process of partial fractions we factor $Q(x)$ into factors of the form

$$(ax+b)^n \qquad \text{and} \qquad (ax^2+bx+c)^n$$

where ax^2+bx+c are irreducible. We then apply algebraic techniques to solve for the missing coefficients that decomposition $P(x)/Q(x)$ into its partial fractions.

6.20 Linear Factors

Theorem 6.4. *Suppose that $f(x) = P(x)/Q(x)$, where $P(x)$ and Q are polynomials with no common factors and with the degree of P less than the degree of Q. If Q is the product of simple linear factors, then for each factor of the form $(ax+b)^n$, the partial fraction decomposition is the following sum of n partial fractions:*

$$\frac{P(x)}{Q(x)} = \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n} \tag{6.33}$$

where A_i, for $i = 1, 2, \ldots, n$ are constants to be determined.

Example 6.24. Evaluate $\displaystyle\int \frac{1}{x(x-4)}\,dx$.

Solution. We use the method of partial fractions and write

$$\frac{1}{x(x-4)} = \frac{A}{x} + \frac{B}{x-4} = \frac{(A+B)x - 4A}{x(x-4)}.$$

This yields $A = -1/4$ and $B = 1/4$, so we have

$$\begin{aligned}\int \frac{1}{x(x-4)}\,dx &= -\frac{1}{4}\int \frac{1}{x}\,dx + \frac{1}{4}\int \frac{1}{x-4}\,dx \\ &= -\frac{1}{4}\ln|x| + \frac{1}{4}\ln|x-4| + C\end{aligned}$$

where C is an arbitrary constant

Example 6.25. Evaluate $\displaystyle\int \frac{x^2+2x+8}{x^3-4x}\,dx$.

Solution. We use the method of partial fractions and write

$$\begin{aligned}\frac{x^2+2x+8}{x^3-4x} &= \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-2} \\ &= \frac{A(x^2-4) + B(x^2-2x) + C(x^2+2x)}{x(x+2)(x-2)}\end{aligned}$$

This yields $A = -2$, $B = 1$ and $C = 2$, so we have

$$\begin{aligned}\int \frac{x^2+2x+8}{x^3-4x}\,dx &= -2\int \frac{1}{x}\,dx + \int \frac{1}{x+2}\,dx + 2\frac{1}{x-2}\,dx \\ &= -2\ln|x| + \ln|x+2| + 2\ln|x-2| + C\end{aligned}$$

where C is an arbitrary constant

Example 6.26. Evaluate $\displaystyle\int_2^4 \frac{3x-5}{(x-1)^2}\,dx$.

Solution. We use the method of partial fractions and write

$$\frac{3x-5}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} = \frac{Ax + (B-A)}{(x-1)^2}$$

This yields $A = 3$ and $B = -2$, so we have

$$\begin{aligned}\int_2^4 \frac{3x-5}{(x-1)^2}\,dx &= \int_2^4 \left[\frac{3}{x-1} - \frac{2}{(x-1)^2}\right] dx \\ &= \left(3\ln|x-1| + \frac{2}{x-1}\right)\Bigg|_2^4 = 3\ln 3 - \frac{4}{3}\end{aligned}$$

as desired.

Example 6.27. Evaluate $\displaystyle\int \frac{x^2+10x-36}{x(x-3)^2}\,dx$.

Solution. We use the method of partial fractions and write

$$\begin{aligned}\frac{x^2+10x-36}{x(x-3)^2} &= \frac{A}{x}+\frac{B}{x-3}+\frac{C}{(x-3)^2}\\ &= \frac{(A+B)x^2-(6A+3B-C)x+9A}{x(x-3)^2}\end{aligned}$$

This yields $A=-4$, $B=5$ and $C=1$, so we have

$$\begin{aligned}\int \frac{x^2+10x-36}{x(x-3)^2}\,dx &= \int_1^2 \left[\frac{-4}{x}+\frac{5}{x-3}+\frac{1}{(x-3)^2}\right]\\ &= \left(-4\ln|x|+\ln|x-3|-\frac{1}{x-3}\right)\Bigg|_1^2 = \frac{1}{2}-9\ln 2\end{aligned}$$

as desired.

6.21 Quadratic Factors

Theorem 6.5. *Suppose that $f(x)=P(x)/Q(x)$, where $P(x)$ and Q are polynomials with no common factors and with the degree of P less than the degree of Q. If Q is the product of irreducible quadratic factors, then for each factor of the form $(ax^2+bx+c)^n$, the partial fraction decomposition is the following sum of n partial fractions:*

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_nx+B_n}{(ax^2+bx+c)^n} \tag{6.34}$$

where A_i, B_i for $i=1,2,\dots,n$ are constants to be determined.

Example 6.28. Evaluate $\displaystyle\int \frac{8(x^2+4)}{x(x^2+8)}\,dx$.

Solution. We use the method of partial fractions and write

$$\frac{8(x^2+4)}{x(x^2+8)} = \frac{Ax+B}{x^2+8}+\frac{C}{x}$$

After equating coefficients, we obtain $A=4$, $B=0$ and $C=4$, so we have

$$\int \frac{8(x^2+4)}{x(x^2+8)}\,dx = \int\left(\frac{4x}{x^2+8}+\frac{4}{x}\right)dx = \ln[(x^2+8)^2x^4]+C$$

as desired.

Example 6.29. Evaluate $\displaystyle\int \frac{20x}{(x-1)(x^2+4x+5)}\,dx$.

Solution. We use the method of partial fractions and write

$$\frac{20x}{(x-1)(x^2+4x+5)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4x+5}$$

After equating coefficients, we obtain $A = 2$, $B = -2$ and $C = 10$, so we have

$$\begin{aligned}\int \frac{20x}{(x-1)(x^2+4x+5)}\,dx &= \int \left(\frac{2}{x-1} + \frac{-2x+10}{x^2+4x+5}\right) dx \\ &= \ln\left|\frac{(x-1)^2}{x^2+4x+5}\right| + 14\tan^{-1}(x+2) + C\end{aligned}$$

where C is an arbitrary constant.

Example 6.30. Evaluate $\displaystyle\int \frac{2}{x(x^2+1)^2}\,dx$.

Solution. We use the method of partial fractions and write

$$\frac{2}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

After equating coefficients, we obtain $A = 2$, $B = -2$ and $C = 0$, $D = -2$, and $E = 0$ so we have

$$\begin{aligned}\int \frac{2}{x(x^2+1)^2}\,dx &= \int \left(\frac{-2}{x} + \frac{-2x}{x^2+1} + \frac{-2x}{(x^2+1)^2}\right) dx \\ &= 2\ln|x| - \ln(x^2+1) + \frac{1}{x^2+1} + C\end{aligned}$$

where C is an arbitrary constant.

Example 6.31. Evaluate $\displaystyle\int \frac{3x^4+4x^3+16x^2+20x+9}{(x+2)(x^2+3)^2}\,dx$.

Solution. We use the method of partial fractions and write

$$\frac{3x^4+4x^3+16x^2+20x+9}{(x+2)(x^2+3)^2} = \frac{A}{x+2} + \frac{Bx+C}{x^2+3} + \frac{Dx+E}{(x^2+3)^2}$$

By equating coefficients, we obtain the linear system of equations

$$\begin{aligned} A+B &= 3 \\ 2B+C &= 4 \\ 6A+3B+2C+D &= 16 \\ 6B+3C+2D+E &= 20 \\ 9A+6C+2E &= 9. \end{aligned}$$

Using a computer algebra system we obtain $A = 1$, $B = 2$, $C = 0$, $D = 4$, and $E = 0$ so we have

$$\begin{aligned}
&\int \frac{3x^4 + 4x^3 + 16x^2 + 20x + 9}{(x+2)(x^2+3)^2}\,dx \\
&= \int \frac{1}{x+2} + \frac{2x}{x^2+3} + \frac{4x}{(x^2+3)^2}\,dx \\
&= \ln|x+2| + \ln(x^2+3) - \frac{2}{x^2+3} + C
\end{aligned}$$

where C is an arbitrary constant.

6.22 Exercises

Exercise 6.33. Evaluate the following integrals.

- $\displaystyle\int \frac{1}{4x^2 - 9}\,dx$
- $\displaystyle\int \frac{x^2 + 12x + 12}{x^3 - 4x}\,dx$
- $\displaystyle\int \frac{3x^2 + x + 4}{x^4 + 3x^2 + 2}\,dx$
- $\displaystyle\int \frac{2x^3 - 4x^2 - 15x + 5}{x^2 - 2x - 8}\,dx$
- $\displaystyle\int_0^1 \frac{2u + 3}{u^2 + 4u + 3}\,dx$
- $\displaystyle\int \frac{4x^2 + 2x - 1}{x^3 + x^2}\,dx$
- $\displaystyle\int_0^1 \frac{x}{x^2 + 4x + 13}\,dx$
- $\displaystyle\int \frac{x^2 + 3x + 2}{x(x^2 + 2x + 2)}\,dx$
- $\displaystyle\int \frac{x^2 + 3x - 4}{x^3 - 4x^2 + 4x}\,dx$
- $\displaystyle\int_2^3 \frac{x^3 - 2x + 7}{x^2 + x - 2}\,dx$
- $\displaystyle\int \frac{x^2}{x^3 - x^2 + 4x - 4}\,dx$
- $\displaystyle\int \frac{1}{(y^2+1)(y^2+2)}\,dy$
- $\displaystyle\int \frac{x^2 - 1}{x^3 + x}\,dx$
- $\displaystyle\int \frac{x^4 + 3x^2 + 1}{x^5 + 5x^3 + 5x}\,dx$

- $\displaystyle\int \frac{x^2}{x^4 - 2x^2 - 8}\,dx$
- $\displaystyle\int \frac{x^4 - 3x^2 - 3x - 2}{x^3 - x^2 - 2x}\,dx$
- $\displaystyle\int \frac{x^2 + 5}{x^3 - x^2 + x - 3}\,dx$
- $\displaystyle\int \frac{x^3}{x^3 + 1}\,dx$
- $\displaystyle\int \frac{x}{16x^4 - 1}\,dx$
- $\displaystyle\int_1^2 \frac{x^2 + 10x - 36}{x(x-3)^2}\,dx$
- $\displaystyle\int \frac{x^2 + 6x + 4}{x^4 + 8x^2 + 16}\,dx$
- $\displaystyle\int \frac{x^3 + 2x^3 + 3x - 2}{(x^2 + 2x + 2)^2}\,dx$
- $\displaystyle\int \frac{1}{(x+1)(x^2 + 2x + 2)^2}\,dx$
- $\displaystyle\int \frac{x^3 + 1}{x(x^2 + x + 1)^2}\,dx$

Exercise 6.34. Find the arc length of the curve $y = \ln x$ from $x = 1$ to $x = 2$.

Exercise 6.35. Find hte area of the region bounded by the graphs of $y = 12/(x^2 + 5x + 6)$, $y = 0$, $x = 0$, and $x = 1$.

Exercise 6.36. Find the ares of the surface generated when the curve $y = \ln x$ is revolved about the x-axis.

Exercise 6.37. Find the area of the region bounded by the curve $y = x/(1 + x)$, the x-axis, and the line $x = 4$.

Exercise 6.38. Find the volume of the solid generated when the region enclosed by $x = y(1 - y^2)^{1/4}$, $y = 0$, $y = 1$, and $x = 0$ is revolved about the y-axis.

Exercise 6.39. Find the volume of the solid obtained by revolving the region bounded by $y = 1/(x + 1)$, $y = 0$, $x = 0$, and $x = 2$ about the y-axis.

Exercise 6.40. Find the volume of the solid obtained by revolving the region bounded by $y = 1/(x+2)$, $y = 0$, $x = 0$, and $x = 3$ about the line $x = -1$.

Exercise 6.41. Consider the region bounded by the graphs of

$$y^2 = \frac{(2-x)^2}{(1+x)^2}$$

on the interval $[0, 1]$. Find the volume of the solid generated by revolving this region about the x-axis.

6.23 Improper Integrals

We integrate continuous functions over unbounded intervals. We also integrate functions which are continuous except for a possibly infinite discontinuity in its domain which have unbounded range.

6.24 Infinite Limits of integration

Until now, our regions of integration have been bounded regions. By this we mean that when evaluating

$$\int_a^b f(x)\,dx$$

we are integrating f over a bounded interval $[a, b]$ and the range of f is also a bounded region. We will now discuss more general regions of integration such as $(a, +\infty)$, $(-\infty, b)$, and $(-\infty, +\infty)$. Consider for example the following diagram. We will use this to define $\int_a^{+\infty} f(x)\,dx$ as a limit. Similarly we wil define $\int_{-\infty}^b f(x)\,dx$ as a limit by considering the diagram. In either case these integrals are said to be *improper*. There are other cases to consider, but in all cases, we will define an *improper integral* using a limit of an already known integral.

Definition 6.1. Suppose that f is a continuous function.

- The improper integral of f over the interval $[a, +\infty)$ is defined to be

$$\int_a^{+\infty} f(x)\,dx = \lim_{b\to+\infty} \int_a^b f(x)\,dx. \tag{6.35}$$

- The improper integral of f over the interval $[-\infty, b)$ is defined to be

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty}\int_{a}^{b} f(x)\,dx. \tag{6.36}$$

- The improper integral of f over the interval $(-\infty, +\infty)$ is defined to be

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{+\infty} f(x)\,dx. \tag{6.37}$$

Whenever the limit exists the integral is said to **converge**, and otherwise **diverge**.

Example 6.32. Determine whether the improper integral $\int_e^\infty \frac{1}{x\ln^2 x}\,dx$ converges or diverges.

Solution. We find that the integral converges to 1 since

$$\begin{aligned}\int_e^\infty \frac{1}{x\ln^2 x}\,dx &= \lim_{b\to\infty}\int_e^b \frac{(\ln x)^{-2}}{x}\,dx \\ &= \lim_{b\to\infty}\left[-\frac{1}{\ln x}\right]_e^b = \lim_{b\to\infty}\left(-\frac{1}{\ln b}+1\right) = 1\end{aligned}$$

as desired.

Example 6.33. Determine whether the improper integral $\int_0^\infty \sin x\,dx$ converges or diverges.

Solution. We find the integral diverges because

$$\int_0^\infty \sin x\,dx = \lim_{b\to\infty}\int_0^b \sin\,dx = \lim_{b\to\infty}[-\cos x]_0^b = \lim_{b\to\infty}(-\cos b+1) \tag{6.38}$$

which is a limit that does not exist.

Example 6.34. Determine whether the improper integral $\int_{-\infty}^0 \frac{1}{x^2+2x+5}\,dx$ converges or diverges.

Solution. We find the integral converges since

$$\begin{aligned}\int_{-\infty}^0 \frac{1}{x^2+2x+5}\,dx &= \lim_{a\to-\infty}\int_a^0 \frac{1}{x^2+2x+5}\,dx \\ &= \lim_{a\to-\infty}\left[\frac{1}{2}\tan^{-1}\left(\frac{1}{2}(x+1)\right)\right]_a^0 \\ &= \lim_{a\to-\infty}\left[\frac{1}{2}\tan^{-1}\frac{1}{2}-\frac{1}{2}\tan^{-1}\left(\frac{1}{2}(a+1)\right)\right] = \frac{\pi}{4}\end{aligned}$$

as desired.

Example 6.35. Determine whether the improper integral $\int_{-\infty}^{+\infty} \frac{e^x}{1+e^{2x}}\, dx$ converges or diverges.

Solution. We consider the following two limits

$$\int_{-\infty}^{+\infty} \frac{e^x}{1+e^{2x}}\, dx = \lim_{a\to-\infty} \int_a^0 \frac{e^x}{1+e^{2x}}\, dx + \lim_{b\to+\infty} \int_0^b \frac{e^x}{1+e^{2x}}\, dx.$$

To do so we integrate $I = \int \frac{e^x}{1+e^{2x}}\, dx$. Let $u = e^x$, so that $du = e^x dx$. Then we find

$$I = \int \frac{1}{1+u^2}\, du = \tan^{-1} u + C. \tag{6.39}$$

Using this result we have

$$\begin{aligned}
\int_{-\infty}^{+\infty} \frac{e^x}{1+e^{2x}}\, dx &= \lim_{a\to-\infty} \left[\tan^{-1} e^x\right]_a^0 + \lim_{b\to+\infty} \left[\tan^{-1} e^x\right]_0^b \\
&= \lim_{a\to-\infty} \left(\frac{\pi}{4} - \tan^{-1} e^a\right) + \lim_{b\to+\infty} \left[\tan^{-1} e^b - \frac{\pi}{4}\right] = \frac{\pi}{2}
\end{aligned}$$

as desired.

Example 6.36. Find the area of the region bounded by the graph of

$$y = \frac{2}{x^2 - 2x + 2}$$

and the x-axis.

Solution. If the area exists, it is found by evaluating an improper integral. We find

$$\begin{aligned}
A = \int_{-\infty}^{+\infty} y\, dx &= \int_{-\infty}^{0} y\, dx + \int_0^{+\infty} y\, dx \\
&= \lim_{a\to-\infty} \int_a^0 \frac{2}{x^2-2x+2}\, dx + \lim_{b\to+\infty} \int_0^b \frac{2}{x^2-2x+2}\, dx \\
&= 2 \lim_{a\to-\infty} \int_a^0 \frac{1}{(x-1)^2+1}\, dx + 2 \lim_{b\to+\infty} \int_0^b \frac{1}{(x-1)^2+1}\, dx \\
&= 2 \lim_{a\to-\infty} \left[\tan^{-1}(x-1)\right]_a^0 + \lim_{b\to+\infty} \left[\tan^{-1}(x-1)\right]_0^b = 2\pi
\end{aligned}$$

as desired.

6.25 Infinite Discontinuities

Definition 6.2. Let f be a function.

- If f is a continuous on the interval $[a, b]$, except for an infinite discontinuity at b, then theimproper integral of f over the interval $[a, b]$ is defined as
$$\int_a^b f(x)\,dx = \lim_{k\to b^-} \int_a^k f(x)\,dx \tag{6.40}$$
- If f is a continuous on the interval $[a, b]$, except for an infinite discontinuity at a, then the improper integral of f over the interval $[a, b]$ is defined as
$$\int_a^b f(x)\,dx = \lim_{k\to a^+} \int_k^a f(x)\,dx \tag{6.41}$$
- If f is a continuous on the interval (a, b), except for some c in (a, b) at which f has an infinite discontinuity, then the improper integral of f over the interval $[a, b]$ is defined as
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx. \tag{6.42}$$

Whenever the limit exists the integral is said to **converge**, and otherwise **diverge**.

Example 6.37. Evaluate $\int_0^1 \frac{x^3}{x^4-1}\,dx$ or state that it diverges.

Solution. We find

$$\begin{aligned}\int_0^1 \frac{x^3}{x^4-1}\,dx &= \lim_{c\to 1^-} \int_0^c \frac{x^3}{x^4-1}\,dx = \lim_{c\to 1^-} \left[\frac{1}{4}\ln|x^4-1|\right]_0^c \\ &= \frac{1}{4}\lim_{c\to 1^-} \ln|c^4-1| = -\infty\end{aligned}$$

and so the integral diverges.

Example 6.38. Evaluate $\int_1^{11} \frac{1}{(x-3)^{2/3}}\,dx$ or state that it diverges.

Solution. This integral is improper at $x = 3$ and so

$$\int_1^{11} \frac{1}{(x-3)^{2/3}}\,dx = \int_1^3 \frac{1}{(x-3)^{2/3}}\,dx + \int_3^{11} \frac{1}{(x-3)^{2/3}}\,dx \tag{6.43}$$

and we evaluate each one separately. We find

$$\begin{aligned}\int_1^3 \frac{1}{(x-3)^{2/3}}\,dx &= \lim_{c\to 3^-}\int_1^c (x-3)^{-2/3}\,dx = \lim_{c\to 3^-}\left[3(x-3)^{1/3}\right]_1^c \\ &= 3\lim_{c\to 3^-}\left(2^{1/3}-(3-c)^{1/3}\right) = (3)2^{1/3}\end{aligned}$$

and

$$\begin{aligned}\int_3^{11} \frac{1}{(x-3)^{2/3}}\,dx &= \lim_{c\to 3^+}\int_c^{11} \frac{1}{(x-3)^{2/3}}\,dx = \lim_{c\to 3^+}\left[3(x-3)^{1/3}\right]_c^{11} \\ &= 3\lim_{c\to 3^+}\left(8^{1/3}-(c-3)^{1/3}\right) = 6\end{aligned}$$

Therefore we have $\int_1^{11} \frac{1}{(x-3)^{2/3}}\,dx = 6 + (3)2^{1/3}$.

Example 6.39. The region bounded by the graph of $f(x) = -\ln x$ and the x-axis on the interval $(0, 1]$ is revolved about the x-axis. Find the volume of the solid of revolution or state that it does not exist.

Solution. The volume is

$$V = \pi\int_0^1 (-\ln x)^2\,dx = \lim_{a\to 0^+}\pi\int_a^1 \ln^2 x\,dx.$$

We use integration by parts with $u = \ln x$ and $dv = \ln x\,dx$. Then $du = (1/x)\,dx$ and $v = x\ln x - x$ and so

$$\begin{aligned}\int \ln^2 x\,dx &= x\ln^2 x - x\ln x - \int (lnx - 1)\,dx \\ &= x\ln^2 x - x\ln x - (x\ln x - x - x) + C \\ &= x\ln^2 x - 2x\ln x - 2x\ln x + 2x + C\end{aligned}$$

where C is an arbitrary constant. We now continuing finding the volume

$$\begin{aligned}V &= \lim_{a\to 0^+}\pi\left[x\ln^2 x - 2x\ln x + 2x\right]_a^1 \\ &= \lim_{a\to 0^+}\pi(2 - (a\ln^2 a - 2a\ln a + 2a)) = 2\pi\end{aligned}$$

where the last equality uses l'Hopital's rule.

6.26 Exercises

Exercise 6.42. Decide if the integral converges or not. Explain your reasoning.

- $\int_1^2 \frac{1}{x^3}\, dx$
- $\int_0^{\pi/4} \csc x\, dx$
- $\int_{-\infty}^{+\infty} \frac{\sin x}{4+x^2}\, dx$

Exercise 6.43. Evaluate the following integrals that converge.

- $\int_{-1}^{+\infty} \frac{x}{1+x^2}\, dx$
- $\int_1^{+\infty} \frac{6}{x^4}\, dx$
- $\int_1^{+\infty} 2^{-x}\, dx$
- $\int_1^{+\infty} \frac{4}{\sqrt[4]{x}}\, dx$
- $\int_0^{+\infty} xe^{-x^2}\, dx$
- $\int_0^{+\infty} xe^{-x/3}\, dx$
- $\int_0^{+\infty} \frac{x}{\sqrt[5]{x^2+1}}\, dx$
- $\int_0^{+\infty} e^{-x} \cos x\, dx$
- $\int_2^{+\infty} \frac{1}{x\sqrt{\ln x}}\, dx$
- $\int_1^{+\infty} \frac{\ln x}{x}\, dx$
- $\int_{-\infty}^{0} \frac{1}{\sqrt[3]{2-x}}\, dx$
- $\int_0^{+\infty} \frac{x^3}{(x^2+1)^2}\, dx$
- $\int_{-\infty}^{3} \frac{1}{x^2+9}\, dx$
- $\int_0^{+\infty} \frac{e^x}{1+e^x}\, dx$
- $\int_{-\infty}^{a} \sqrt{e^x}\, dx$
- $\int_0^{\infty} \sin \frac{x}{2}\, dx$
- $\int_{-\infty}^{+\infty} \frac{x}{\sqrt{x^2+2}}\, dx$
- $\int_0^{+\infty} \frac{1}{e^x+e^{-x}}\, dx$
- $\int_1^{+\infty} \frac{1}{x(x+1)}\, dx$
- $\int_0^{+\infty} \cos \pi x\, dx$

Exercise 6.44. Evaluate the following integrals that converge.

- $\int_0^{\pi/4} \frac{\sec^2 x}{1-\tan x}\, dx$
- $\int_0^5 \frac{10}{x}\, dx$
- $\int_0^{\pi/2} \sec x \tan x\, dx$
- $\int_0^8 \frac{3}{\sqrt{8-x}}\, dx$
- $\int_{-2}^2 \frac{1}{x^2}\, dx$
- $\int_0^e \ln x^2\, dx$

- $\int_0^{\pi/2} \tan x\, dx$
- $\int_0^{\pi/2} \sec x\, dx$
- $\int_0^{1} \frac{1}{(x-1)^{2/3}}\, dx$
- $\int_3^{6} \frac{1}{\sqrt{36-x^2}}\, dx$
- $\int_3^{4} \frac{1}{(x-3)^{3/2}}\, dx$
- $\int_0^{5} \frac{1}{25-x^2}\, dx$
- $\int_0^{1} \frac{1}{\sqrt{x}(x+1)}\, dx$
- $\int_4^{+\infty} \frac{\sqrt{x^2-16}}{x^2}\, dx$
- $\int_0^{\ln 3} \frac{e^x}{(e^x-1)^{2/3}}\, dx$
- $\int_0^{8} \frac{1}{\sqrt[3]{x}}\, dx$
- $\int_0^{9} \frac{1}{(x-1)^{1/3}}\, dx$

Exercise 6.45. Find the area of the region between the x-axis and the curve $y = e^{-3x}$ for $x \geq 0$.

Exercise 6.46. Let R be the region bounded by the graphs of $y = e^{-ax}$ and $y = e^{-bx}$, for $x \geq 0$, where $a > b > 0$. Find the area of R.

Exercise 6.47. Suppose that the region between the x-axis and the curve $y = e^{-x}$ for $x \geq 0$ is revolved about he x-axis. Find the volume and surface area for the solid generated.

Exercise 6.48. Use integration by parts to evaluate the integral.

- $\int_0^{+\infty} xe^{-}2x\, dx$
- $\int_0^{1} 2x \ln x\, dx$
- $\int_1^{\infty} \frac{\ln x}{2x^2}\, dx$

Exercise 6.49. Let R be the region bounded by the graph of $f(x) = 1/x^p$, where p is a real number. For what values of p does the integral $\int_0^1 f(x)\, dx$ exist? When the integral exists what is its value?

Exercise 6.50. Find the arc length of the graph of $y = \sqrt{16 - x^2}$ over the interval $[0, 4]$.

Exercise 6.51. Find the surface area of the surface formed by revolving the graph of $y = 2e^{-x}$ on the interval $[0, +\infty)$ about the x-axis.

Chapter 7

Infinite Series

Infinite series is one of the most fundamental concepts in calculus. This book will teach you everything you need to know about them, from their basic properties to more advanced concepts. With this knowledge, you'll be able to tackle any calculus problem that comes your way!

In short, infinite series are sums of infinitely many terms. For example, the infinite series $1+2+3+4+...$ describes the sum of all natural numbers. So why are infinite series important? Well, they pop up in a lot of places in mathematics, physics, and computer science.

In fact, infinite series are so important that they have their own branch of mathematics devoted to them. So the next time you're stuck trying to solve an infinite series, just remember that you're working on something that has puzzled mathematicians for centuries. And if that doesn't make you feel better, at least you're not alone.

You might be wondering, why would anyone want to add up an infinite number of things? Well, in some cases, the answer is that we don't actually have to add up all the terms in order to get a meaningful result.

While infinite series may seem like a theoretical concept, they actually have a wide range of applications in the real world. In fact, infinite series are so important that they were once described as "one of the most useful and powerful tools ever invented by mathematicians." So what makes infinite series so special? Let's take a closer look.

In mathematics, a sequence is an ordered set of terms. Sequences can be infinite, in which case they are infinitely long, or they can be finite, in which case they end after a certain number of terms. All the terms in a sequence are connected to each other and in order; there is no "jumping around" allowed!

You can think of a sequence as a list of numbers that might follow some kind of pattern. For example: 1, 2, 4, 8, 16 is a sequence where each term is double the previous term. Another example is 3, 6, 9, 12, ... , where each term is 3 more than the previous term.

There are many other types of patterns that sequences can follow. Some patterns are very simple and some are very complicated. You can even have sequences that go infinitely long without repeating (like the decimal representation of $1/3 = 0.3333...$). And some sequences repeat over and over again in a cycle (like 0, 1, 1, 0, 0, 1, 1,...). As you can see, there's a lot to explore with sequences!

Sequences can be found in nature, in art, and in many other places. The most famous sequence is probably the Fibonacci sequence, in which each number is the sum of the previous two numbers in the series. However, there are many other interesting sequences out there.

Sequences can be used to model real-world situations, such as population growth or the spread of disease. They can also be used to solve problems, such as finding the area of a circle. In short, sequences are a powerful tool for understanding and solving problems in mathematics.

An infinite series is a sum of an infinite sequence of numbers. In calculus, when studying infinite series, one must first understand whether an infinite series will converge or diverge. To determine whether an infinite series converges or diverges, one can use several methods such as the ratio test, the root test, or various comparison tests.

In general, if an infinite series converges, then it is said to have a sum, whereas if an infinite series diverges, then it is said to be unbound.

For example, the infinite series $1/2 + 1/4 + 1/8 + ...$ converges because each term in the sequence approaches 0 as n approaches infinity. However, the infinite series $1 + 2 + 3 + ...$ diverges because it does not have a sum. In conclusion, understanding infinite series and their convergence is important in calculus and other mathematical fields.

There are many different types of infinite series, and the study of infinite series is an active area of research in mathematics. In recent years, researchers have made significant progress in understanding the behavior of infinite series and their applications to other areas of mathematics.

The Integral Test is a way to determine if an infinite series converges or diverges. If the infinite series converges, then the corresponding infinite integral must also converge; if the infinite series diverges, then the corresponding infinite integral must also diverge.

The integral test is a test for infinite series that uses integration to determine whether a given series converges or diverges. The test is based on the fact that if a function f is positive and increasing on the interval

$[1, \infty)$, then the infinite series $\sum f(n)$ converges if and only if the infinite integral $\int f(x)dx$ converges.

In other words, the integral test can be used to determine whether an infinite series converges or diverges by first finding the corresponding integral and then checking to see if that integral converges or diverges.

The Integral Test is especially useful for infinite series with non-negative terms, such as the p-Series.

In general, the p-Series is a series where each term is equal to $1/p^n$, where p is a positive integer. The p-Series converges if $p > 1$ and diverges if $p < 1$. Thus, by the Integral Test, we can see that the p-Series will converge if and only if the corresponding infinite integral converges.

We all know the old saying, "comparing apples and oranges." But what about comparing series? In mathematics, a series is an infinite sequence of terms, often resulting from the repeated addition of finite sequences. And just like apples and oranges, not all series are created equal.

In fact, there are an infinite number of ways to compare two infinite series. So which one is better? It all depends on your perspective. Here's a look at some of the most popular methods of comparing series:

- The ratio test: This method compares the infinite sequences term by term. If the limit of the ratio between two terms is less than 1, then the infinite series will converge; if the limit is greater than 1, then the infinite series will diverge.

- The root test: This method looks at the infinite sequence of square roots. If the limit of the square root of a term is less than 1, then the infinite series will converge; if the limit is greater than 1, then the infinite series will diverge.

- The comparison test: This method compares an infinite series with another infinite series that we already know converges or diverges. If the infinite series being compared converges or diverges using the same criteria, then we can say that it has the same behavior as the other infinite series.

So which method is best? That's up for debate. Each method has its own strengths and weaknesses, so it really depends on your particular needs. In any case, there's no need to worry about choosing the wrong method - with so many ways to compare infinite series, you're sure to find a technique that works for you!

A Taylor polynomial is an infinite series that is used to approximate a function. The first term in the series is the function itself, and each subsequent term is derived from the derivatives of the function at a certain point.

The degree of the polynomial corresponds to the number of derivatives that are used. For example, a first-degree polynomial uses one derivative, a second-degree polynomial uses two derivatives, and so on.

Taylor polynomials are extremely useful in mathematics and physics because they allow us to approximate complicated functions using only a few terms. However, they are only approximations, so they are not always 100% accurate.

In general, higher-degree polynomials will be more accurate than lower-degree polynomials, but they will also be more difficult to work with. Ultimately, it's up to you to decide which degree of approximation is appropriate for your needs.

Who would have thought that infinite series could be so interesting? Me! Especially, when I learned about power series, Taylor series, and Maclaurin series in my calculus class. These mathematical concepts are used to approximate functions that can be difficult to work with directly. And while they may seem like a mouthful, they're actually not that complicated.

A power series is an infinite series in which the terms are powers of a variable. For example, the series $x+x^2+x^3+\cdots$ is a power series in which the terms are successive powers of x. A Taylor series is a power series that is used to approximate a function around a point. The Maclaurin series is just a special case of the Taylor series in which the point is 0. These concepts may seem like a lot to take in at first, but once you get the hang of them they're actually quite simple. So if you ever find yourself struggling with infinite series, remember: there's always a power in numbers.

Applications of infinite series abound in both pure mathematics and applied mathematics. In pure mathematics, infinite series appear in the study of asymptotic behavior, where they are used to describe the behavior of functions at infinity.

In applied mathematics, infinite series are used extensively in physics and engineering, particularly in electrical circuits and wave propagation. They also appear in many other areas of applied mathematics, such as statistics and numerical analysis.

In this book, I teach infinite series in a way that is rigorous but still accessible to the average reader. First, I introduce the concept of a limit and explain how it can be used to calculate the sum of an infinite series. Next, I discuss the different types of infinite series, including convergent and divergent series. Finally, I provide worked examples to illustrate each point. By the end of the book, readers will have a solid understanding of infinite series and how to apply them.

7.1 Sequences

Sequences are introduced. The limit of a sequence and various limits rules are studied, including the squeeze theorem. The convergence of a bounded monotonic sequence is explained.

7.2 Introduction to Sequences

Basically, a *sequence* is a function whose domain is a set of integers.

Definition 7.1. A **sequence** is a function whose domain is the set of positive integers. The functional values a_1, a_2, ..., a_n are the **terms** of the sequence, and the term a_n is called the nth term of the sequence.

Example 7.1. List the terms of the sequence.

- $a_n = 1/n$
- $a_n = n/(1+n)$

Solution. The first 5 terms in the first sequence are $1, 1/2, 1/3, 1/4$ and $1/5$. The first 5 terms in the second sequence are

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}.$$

where the index in both examples is n and $n = 1, 2, 3, \dots$.

Example 7.2. Find an expression for the nth term of the sequence.

- $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots$
- $\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \cdots$

Solution. For the first sequence, the denominators of the four known terms have been expressed as powers of 2. This suggests that the denominator of the nth term is 2^n. Thus, the sequence can be expressed as

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots, \frac{1}{2^n}, \cdots$$

Notice that the second sequence has alternating signs and that notice that the nth term in the sequence can be obtained by multiplying the nth term by $(-1)^{n+1}$. Thus, the sequence can be expressed as

$$\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \cdots, (-1)^{n+1}\frac{n}{n+1}, \cdots$$

as desired.

Example 7.3. List the first five terms of the recursively defined sequence $a_1 = 1$, $a_2 = 1$, $a_{n+1} = a_n + a_{n-1}$ for $n > 2$.

Solution. The sequence is called the Fibonacci sequence. The first 11 terms are $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89$.

7.3 The Limit of a Sequence

A sequence $\{a_n\}$ has a **limit**, written

$$\lim_{n\to\infty} a_n = L$$

if a_n can be made as close to L as we please by taking n sufficiently large. If the limit in (7.3) exists the sequence is said to **converge**. Otherwise we say that the sequence **diverges**.

Definition 7.2. A sequence $\{a_n\}$ **converges** to L, written

$$\lim_{n\to\infty} a_n = L$$

if for every $\epsilon > 0$ there exists a positive integer N such that $|a_n - L < \epsilon$ whenever $n > N$.

Theorem 7.1. *If* $\lim_{x\to\infty} f(x) = L$ *and* $\{a_n\}$ *is a sequence defined by* $a_n = f(n)$ *where* n *is a positive integer, then* $\lim_{n\to\infty} a_n = L$.

Theorem 7.2. *If* $\{a_n\}$ *and* $\{b_n\}$ *are convergent sequences and* c *is a constant, then* - $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$ - $\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$ - $\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n$ - $\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n$ - $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ *whenever* $\lim_{n\to\infty} b_n \neq 0$ - $\lim_{n\to\infty} a_n^p = [\lim_{n\to\infty} a_n]^p$, *if* $p > 0$ *and* $a_n > 0$.

Example 7.4. Determine if the sequence $\left\{\frac{n}{2n+1}\right\}$ converges or diverges.

Solution. We find that

$$\begin{aligned}
\lim_{n\to\infty}\frac{n}{2n+1} &= \lim_{n\to\infty}\frac{1}{2+1/n} \\
&= \frac{\lim_{n\to\infty}1}{\lim_{n\to\infty}(2+1/n)} \\
&= \frac{\lim_{n\to\infty}1}{\lim_{n\to\infty}2+\lim_{n\to\infty}1/n} \\
&= \frac{1}{2+0}=\frac{1}{2}
\end{aligned}$$

Example 7.5. Determine if the sequence $\left\{(-1)^{n+1}\frac{n}{2n+1}\right\}$ converges or diverges.

Solution. The sequence diverges as it oscillates between $1/2$ and $-1/2$.

Example 7.6. Determine if the sequence $\left\{(-1)^{n+1}\frac{1}{n}\right\}$ converges or diverges.

Solution. Notice that $1/n \to -$ as $n \to \infty$. Thus the product $(-1)^{n+1}(1/n)$ oscillates between positive and negative values. Since the odd numbered terms are approaching 0 and the even numbered terms also approaching zero, thus

$$\lim_{n\to\infty}(-1)^{n+1}\frac{1}{n}=0$$

so the sequence converges to 0.

Example 7.7. Determine if the sequence $\{3-2n\}$ converges or diverges.

Solution. Since $\lim_{n\to\infty}(3-2n)=-\infty$, the sequence diverges.

Example 7.8. Find the limit of the sequence $\left\{\frac{n}{2^n}\right\}$.

Solution. The expression

$$\lim_{n\to\infty}\frac{n}{2^n}$$

is an indeterminate type of the form ∞/∞, so l'Hopital's rule is applied by using the function $f(x)=x$ and $g(x)=2^x$. We have

$$\lim_{x\to\infty}\frac{x}{2^x}=\lim_{x\to\infty}\frac{1}{2^x\ln 2}=0$$

from which we conclude that

$$\lim_{n\to\infty} \frac{n}{2^n} = 0$$

as desired.

7.4 The Squeeze Theorem for Limits

If we are able to compare a sequence with two other familiar convergent sequences, then we can apply the squeeze theorem.

::: {#thm- } If $a_n \leq b_n \leq c_n$, for $n \geq n_0$ and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L,$$

then $\lim_{n\to\infty} b_n = L$. :::

Example 7.9. Find the limit of the sequence

$$b_n = \frac{\cos n}{n^2+1}.$$

Solution. Notice that $-1 \leq \cos n \leq 1$, for all n, and so

$$-\frac{1}{n^2+1} \leq \frac{\cos n}{n^2+1} \leq \frac{1}{n^2+1}.$$

Letting

$$a_n = -\frac{1}{n^2+1} \qquad \text{and} \qquad c_n = \frac{1}{n^2+1},$$

we have $a_n \leq b_n \leq c_n$ for $n \geq 1$. Therefore, by the Squeeze Theorem we have

$$\lim_{n\to\infty} b_n = 0.$$

as desired.

Example 7.10. Show that if $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Solution. Notice that $-|a_n| \leq a_n \leq |a_n|$. However, since

$$\lim_{a_n\to\infty} -|a_n| = \lim_{a_n\to\infty} |a_n| = 0$$

by the squeeze theorem, $\lim_{n\to\infty} a_n = 0$ follows at once.

Theorem 7.3. *If* $\lim_{n\to\infty} a_n = L$ *and the function* f *is continuous at* L*, then*

$$\lim_{n\to\infty} f(a_n) = f(L).$$

Example 7.11. Show that if $\lim_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} |a_n| = 0$.

Solution. Assume that $\lim_{n\to\infty} a_n = 0$ and recall that $f(x) = |x|$ is continuous at 0. Therefore,

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} |a_n| = |0| = 0$$

as desired.

7.5 Monotonic Sequences and Bounded Sequences

The sequence of natural numbers $1, 2, 3, 4, ...$ is strictly increasing sequence. The sequence

$$\frac{1}{2}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$$

is also a strictly increasing sequence. The constant sequence $\{4\}$ is a nondecreasing sequence.

In general, a sequence $\{a_n\}$ is called **strictly increasing** whenever

$$a_1 < a_2 < a_3 < \cdots < a_n < \cdots$$

and is called **increasing** whenever

$$a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n \leq \cdots$$

Similarly, a sequence $\{a_n\}$ is called **strictly decreasing** whenever

$$a_1 > a_2 > a_3 > \cdots > a_n > \cdots$$

and is called **decreasing** whenever

$$a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq \cdots$$

Definition 7.3. A sequence $\{a_n\}$ is **monotonic** when its terms are nondecreasing

$$a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n \leq \cdots$$

or when its terms are nonincreasing

$$a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq \cdots.$$

Example 7.12. Show that the sequence

$$\frac{1}{2}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$$

is a strictly increasing sequence.

Solution. Let $a_n = \frac{n}{n+1}$. Then

$$a_{n+1} = \frac{n+1}{n+2}$$

and so

$$a_{n+1} - a_n = \frac{n+1}{n+2} = \frac{n}{n+1} = \frac{1}{(n+1)(n+2)} > 0.$$

Whence $a_{n+1} > a_n$ for all $n \geq 1$. By definition, the given sequence is a strictly increasing sequence. Another approach is to use the function $f(x) = x/(x+1)$ and the first derivative test.

The sequence of natural numbers $1, 2, 3, 4, \ldots$ has no upper bound. The sequence

$$\frac{1}{2}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$$

is bounded above by 1. In fact, 1 is the least upper bound. The constant sequence $\{4\}$ is bounded above and below.

Definition 7.4. Let $\{a_n\}$ be a sequence. - A sequence $\{a_n\}$ is **bounded above** when there is a real number M such that $a_n \leq M$ for all n. The number M is called an **upper bound** of the sequence. - A sequence $\{a_n\}$ is **bounded below** when there is a real number N such that $N \leq a_n$ for all n. The number N is called a lower bound of the sequence. - A sequence $\{a_n\}$ is **bounded** when it is bounded above and bounded below.

::: {#thm- } If a sequence $\{a_n\}$ is bounded and monotonic, then it converges. :::

Example 7.13. Show that the sequence $\left\{\frac{e^n}{n!}\right\}$ converges and find its limit.

Solution. Fist we show that the sequence is strictly decreasing. Let $a_n = \frac{e^n}{n!}$. We have

$$\frac{a_{n+1}}{a_n} = \frac{\frac{e^{n+1}}{(n+1)!}}{\frac{e^n}{n!}} = \frac{e^{n+1}n!}{e^n(n+1)!} = \frac{e}{n+1}. \tag{7.1}$$

Whence $a_{n+1} < a_n$ for all $n \geq 1$. Since every term in the sequence is positive, it is bounded below by 0. By the Bounded Convergence Theorem,

we know that the sequence converges. By (7.1), the sequence is given by the recursion formula
$$a_{n+1} = \frac{e}{n+1} a_n.$$
Passing the limit on both sides we obtain
$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \left(\frac{e}{n+1} a_n\right) = \lim_{n\to\infty} \left(\frac{e}{n+1}\right) \lim_{n\to\infty} (a_n) = 0.$$
Therefore, the sequence converges to 0.

7.6 Exercises

Exercise 7.1. Write the first five terms of the sequence.

- $a_n = \frac{(-1)^{n+1} 2^n}{n+1}$
- $a_n = \frac{n+1}{3n-1}$
- $a_n = \sin \frac{n\pi}{2}$
- $a_1 = 4, a_{n+1} = \frac{a_n}{a_n - 1}$
- $a_n = \frac{2^n}{(2n)!}$
- $a_n = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!}$
- $a_1 = 3,\ a_{n+1} = 2a_n - 1$
- $a_1 = 2, a_{n+1} = 3a_n + 1$

Exercise 7.2. Find an expression for the nth term of the sequence.

- $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots\right\}$
- $\left\{\frac{1}{1\cdot 2}, \frac{2}{2\cdot 3}, \frac{3}{3\cdot 4}, \frac{4}{4\cdot 5}, \ldots\right\}$
- $\left\{-\frac{1}{4}, \frac{2}{9}, -\frac{3}{16}, \frac{4}{25}, \ldots\right\}$
- $\left\{1, -\frac{2}{3}, \frac{4}{9}, -\frac{8}{27}, \ldots\right\}$

Exercise 7.3. Which of the sequences converge and which diverge? Find the limit of each convergent sequence.

- $a_n = \frac{n+(-1)^n}{n}$
- $a_n = \sqrt{n+1}$
- $a_n = \frac{2n+1}{1-2\sqrt{n}}$
- $a_n = \frac{n^2-1}{2n^2+1}$
- $a_n = \frac{n+3}{n^2+5n+6}$
- $a_n = (-1)^n \frac{n+2}{3n+1}$
- $a_n = \left(1 + \frac{2}{n}\right)^n$
- $a_n = \frac{1-n^3}{70-4n^2}$

- $a_n = 1 + \left(\frac{-2}{e}\right)^n$
- $a_n = \frac{3^n}{n^3}$
- $a_n = \sqrt[n]{2^{1-3n}}$
- $a_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!}$
- $a_n = \sqrt[n]{n^2 + n}$
- $a_n = \frac{1}{n^2} + \frac{2}{n^2} + \frac{3}{n^2} + \cdots + \frac{n}{n^2}$
- $a_n = \frac{1}{n} \int_1^n \frac{1}{x}\, dx$
- $a_n = \frac{1+2+3+\cdots+n}{n+2} - \frac{n}{2}$

Exercise 7.4. Determine if the sequence $a_n = \frac{\ln(n+2)}{n+2}$ is increasing, decreasing or bounded.

Exercise 7.5. Which of the following properties are satisfied by the sequence $\left\{\frac{(-1)^n}{n}\right\}$?

Exercise 7.6. Compute the limit of the convergent sequence $\{\sqrt[n]{n}\}$.

Exercise 7.7. Find the 5th term of the recursively defined sequence $a_1 = 2$, $a_2 = 4$, and $a_{n+1} = 2a_n - a_{n-1}$ for $n \geq 2$.

Exercise 7.8. Find $\lim_{n\to\infty} \frac{(-1)^n}{n}$.

Exercise 7.9. Determine whether the sequence $\left\{\frac{2^n+1}{e^n}\right\}$ converges or diverges. If it converges, find its limit.

Exercise 7.10. Identify the sequence below for which $\{|a_n|\}$ converges but $\{a_n\}$ does not.

Exercise 7.11. Which is the largest set of r such that the sequence $\{r^n\}$ converge?

Exercise 7.12. Write the first five terms of the sequence $\{a_n\}$ where $a_1 = 1$ and $a_{n+1} = 3a_n - 2$.

Exercise 7.13. Determine if the sequence $\{a_n\}$ given by $a_n = \frac{2n^2+1}{n}$ converges or diverges. If it converges, find its limit.

Exercise 7.14. Find the limit of the following sequences or state that they diverge.

- $a_n = \frac{\cos n}{n}$
- $a_n = \frac{2\tan^{-1} n}{n^3+4}$
- $a_n = \frac{n\sin^3(n\pi/2)}{n+1}$

Exercise 7.15. Determine whether the sequence is increasing, decreasing, or not monotonic. Is the sequence bounded?

- $a_n = \frac{1}{2n+3}$
- $a_n = 4 - \frac{1}{n}$
- $a_n = \frac{2n-3}{3n+4}$
- $a_n = \left(\frac{2}{3}\right)^n$
- $\sin \frac{n\pi}{6}$
- $a_n = \frac{\ln(1/n)}{n}$

Exercise 7.16. Suppose that $f(x)$ is a differentiable for all x in $[0,1]$ and that $f(0) = 0$. Define the sequence $\{a_n\}$ by the rule $a_n = nf(1/n)$. Show that $lim_{n\to\infty} a_n = f'(0)$.

- $a_n = n\tan^{-1}\left(\frac{1}{n}\right)$
- $a_n = n\left(e^{1/n} - 1\right)$
- $a_n = n\ln\left(1 + \frac{2}{n}\right)$

7.7 Infinite Series and Convergence

We introduce infinite series and their basic properties such as the Divergence Test and elementary convergence rules. We also discuss the Harmonic Series and Geometric Series.

7.8 Infinite Series

Infinite sums occur naturally, for example, when we consider decimal representations of real numbers. Consider the representation of 1/7:

$$\frac{1}{7} = 0.\overline{142857} = 0.142857 + 0.000000142857 + 0.000000000000142857 + \cdots$$

this suggests that the decimal representation can be viewed as a sum of infinitely many real numbers.

Definition 7.5. An expression of the form

$$a_1 + a_2 + a_3 + \cdots a_n + \cdots$$

is called an **infinite series** or, more simply, a **series**. The numbers a_1, a_2, a_3, are called the **terms** of the series; a_n is called the nth term, or **general term**, of the series; and the series itself is denoted by the symbol

$$\sum_{n=1}^{\infty} a_n$$

or sometimes as $\sum a_n$.

Definition 7.6. Given an infinite series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

the **nth partial** sum of the series is

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n$$

If the sequence of partial sums $\{S_n\}$ converges to the number S, that is, if $\lim_{n\to\infty} S_n = S$, then the series $\sum a_n$ **converges** and has sum S, written

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots = S.$$

If $\{S_n\}$ diverges, then the series $\sum a_n$ **diverges** .

For example, the real number $1/7$ can be approximated using a sequence of partial sums

$$\begin{aligned}
s_1 &= 0.142857 = \frac{142857}{10^6} \\
s_2 &= s_1 + 0.000000142857 = \frac{142857}{10^6} + \frac{142857}{10^{12}} \\
s_3 &= s_1 + s_2 + 0.000000000000142857 = \frac{142857}{10^6} + \frac{142857}{10^{12}} + \frac{142857}{10^{18}} \\
&\vdots
\end{aligned}$$

Notice that the nth partial sum is

$$s_n = \frac{142857}{10^6} + \frac{142857}{10^{12}} + \frac{142857}{10^{18}} + \cdots + \frac{142857}{10^{6n}}. \tag{7.2}$$

Therefore we find

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(\frac{142857}{10^6} + \frac{142857}{10^{12}} + \frac{142857}{10^{18}} + \cdots + \frac{142857}{10^{6n}} \right).$$

To calculate this limit we need a closed form, so we multiple by sides of (7.2) by $1/10^6$ and obtain

$$\frac{1}{10^6} s_n = \frac{142857}{10^{12}} + \frac{142857}{10^{18}} + \frac{142857}{10^{24}} + \cdots + \frac{142857}{10^{6(n+1)}}. \tag{7.3}$$

Now we subtract (7.3) from (7.2) and obtain

$$s_n = \frac{142857}{10^6} - \frac{142857}{10^{6(n+1)}} = \frac{142857}{10^6} \left(1 - \frac{1}{10^{6n}}\right). \tag{7.4}$$

Whence

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{142857}{10^6} \left(1 - \frac{1}{10^{6n}}\right) = \frac{1}{7}$$

as desired.

Example 7.14. Show that the series $\sum_{n=1}^{\infty} \frac{4}{4n^2-1}$ is convergent, and find its sum.

Solution. Use partial fractions to write the general term as

$$a_n = \frac{2}{2n-1} - \frac{2}{2n+1}$$

Then we write the nth partial sum of the series as

$$S_n = 2 - \frac{2}{2n+1}$$

Therefore the sum is 2.

7.9 Geometric Series

Definition 7.7. Let a be a nonzero real number. A series of the form

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots + ar^n + \cdots$$

is called a **geometric series** with common ratio r.

Theorem 7.4. *If $|r| < 1$, then the geometric series*

$$\sum_{n=1}^{\infty} ar^{n-1}$$

converges, and its sum is $\frac{a}{1-r}$. The seres diverges if $|r| \geq 1$.

Proof. Multiply both sides of the nth partial sum of the series by r and then subtract from the original partial sums equation to obtain

$$(1-r)S_n = a(1-r^n)$$

Now consider each case.

□

Example 7.15. Evaluate the geometric series $\sum_{n=0}^{\infty} 1.1^n$ or state that the series diverges.

Solution. Notice that $a = 1$ and $r = 1.1$, and since $|1.1| \geq 1$, the series diverges.

Example 7.16. Evaluate the geometric series $\sum_{n=0}^{\infty} e^{-n}$ or state that the series diverges.

Solution. Notice that $a = 1$ and $r = 1/e$, and since $|1/e| < 1$, the series converges with

$$\sum_{n=0}^{\infty} e^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{e}\right) = \frac{1}{1-1/e} = \frac{e}{e-1}$$

as desired.

7.10 Harmonic Series

Definition 7.8. The series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is called the **harmonic series** .

Theorem 7.5. *The harmonic series is divergent.*

Proof. Write out the partial sums, for example,

$$\begin{aligned} S_8 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \\ &> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + 3\left(\frac{1}{2}\right) \end{aligned}$$

In general $S_{2^n} > 1 + n(1/2)$. Therefore, $\lim_{n\to\infty} S_{2^n} = \infty$ and so $\{S_n\}$ is divergent.

□

7.11 The Divergence Test

The Divergence Test is a convenient way to determine whether an infinite series diverges. For example the series $\sum \sin n$ diverges and which is easily seen from the Divergence Test. The following observation leads us to the Divergence Test.

::: {#lem- } If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. ::: ::: {.proof } Notice that $a_n = S_n - S_{n-1}$. Hence

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0$$

where S is the sum of the series. :::

::: {#thm- } If $\lim_{n\to\infty} a_n$ does not exist or $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges. :::

Example 7.17. Explain why the series $\sum_{n=1}^{\infty} n^2$ diverges.

Solution. If $\sum_{n=1}^{\infty} n^2$ converges, then we have $\lim_{n\to\infty} n^2 = 0$ by the Divergence Lemma. Therefore, $\sum_{n=1}^{\infty} n^2$ must diverge.

7.12 Convergence Rules

Often times it is easier to determine the convergence of a simpler series.

::: {#thm- } If $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$ are convergent and c is any real number, then $\sum_{n=1}^{\infty} ca_n$ and $\sum_{n=1}^{\infty} cb_n$ are also convergent, and

- $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n = cA$ and
- $\sum_{n=1}^{\infty} c(a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n = A \pm B$. :::

Example 7.18. Evaluate the geometric series

$$\sum_{n=2}^{\infty} 3(-0.75)^n$$

or state that the series diverges.

Solution. First we consider the geometric series,

$$\sum_{n=2}^{\infty} (-0.75)^n.$$

Notice that $r = -0.75$ and $|r| < 1$ and so the series converges. The first term is $a = (-0.75)^2$ and so the sum is

$$\sum_{n=2}^{\infty} (-0.75)^n = \frac{(-0.75)^2}{1-(-0.75)} = \frac{9}{28}.$$

Therefore we find that

$$\sum_{n=2}^{\infty} 3(-0.75)^n = 3\left(\frac{9}{28}\right) = \frac{27}{28}$$

using the Convergence Rules.

7.13 Exercises

Exercise 7.17. Determine whether the series converges. If the series converges, find its sum.

- $\sum_{n=1}^{\infty} n$
- $\sum_{n=1}^{\infty} \left(\frac{1}{2n+3} - \frac{1}{2n+1}\right)$
- $\sum_{n=1}^{\infty} \frac{n+1}{2n-3}$
- $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$.
- $\sum_{n=1}^{\infty} \frac{2}{\sqrt{n+1}+\sqrt{n}}$
- $\sum_{n=1}^{\infty} \frac{n(n+2)}{(n+3)^2}$
- $\sum_{n=1}^{\infty} \frac{4}{(2n+3)(2n+5)}$
- $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$
- $\sum_{n=1}^{\infty} \frac{1+3^n}{2^n}$

Exercise 7.18. Determine whether the series converges or diverges. If it converges, find its sum.

- $\sum_{n=1}^{\infty} 3\left(-\frac{1}{2}\right)^{n-1}$

- $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$
- $\sum_{n=1}^{\infty} \frac{2n-1}{3n+1}$
- $\sum_{n=1}^{\infty} 5\left(\frac{4}{3}\right)^{n-1}$
- $\sum_{n=1}^{\infty} \frac{3^n-1}{3^{n+1}}$
- $\sum_{n=1}^{\infty} \left[\cos\left(\frac{1}{n}\right) - \cos\left(\frac{1}{n+1}\right)\right]$

Exercise 7.19. Show that the following series are divergent.

- $\sum_{n=1}^{\infty}(-1)^{n-1}$
- $\sum_{n=1}^{\infty} \frac{n}{n+1}$
- $\sum_{n=1}^{\infty} \frac{2n^2+1}{3n^2-1}$
- $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$
- $\sum_{n=1}^{\infty} \frac{n!}{2^n}$
- $\sum_{n=1}^{\infty} \frac{2^n+1}{2^{n+1}}$

Exercise 7.20. Show that the series $\sum_{n=1}^{\infty} \left[\frac{2}{n(n+1)} - \frac{4}{3^n}\right]$ is convergent, and find its sum.

Exercise 7.21. Find a formula for the nth partial sum of each series and use it to find the series sum of the series converges. - $\frac{1}{2}+\frac{1}{5}+\frac{1}{10}+\frac{1}{17}+\frac{1}{26}+\cdots$ - $\frac{1}{3}+\frac{1}{7}+\frac{1}{11}+\frac{1}{15}+\frac{1}{19}+\cdots$ - $\frac{9}{100}+\frac{9}{100^2}+\frac{9}{100^3}+\cdots+\frac{9}{100^n}+\cdots$ - $\frac{5}{3}-\frac{5}{9}+\frac{5}{27}-\frac{5}{81}+\cdots$ - $1-2+4-8+\cdots+(-1)^{n-1}2^{n-1}+\cdots$ - $1+\frac{4}{3}+\frac{16}{9}+\frac{64}{27}+\cdots$

Exercise 7.22. Write out the first few terms of each series to show how the series starts. Then find the sum of the series. - $\sum_{k=0}^{\infty} \frac{(-1)^k}{4^k}$ - $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k+4}}$ - $\sum_{k=0}^{\infty} \left(\frac{2^{k+1}}{5^k}\right)$ - $\sum_{k=1}^{\infty} \frac{n}{2^k}$ - $\sum_{k=1}^{\infty} \frac{1}{e^{-k}+1}$ - $\sum_{n=1}^{\infty} \frac{1}{2n^2+7n+3}$

Exercise 7.23. Use partial fractions to find the sum of each series.

- $\sum_{n=1}^{\infty} \frac{6}{(2n-1)(2n+1)}$
- $\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$

Exercise 7.24. Which series converge, and which diverge?
Give reasons for your answers. If a series converges, find its sum.

- $\sum_{n=0}^{\infty}(\sqrt{2})^n$
- $\sum_{k=0}^{\infty} \frac{1}{(3k+1)(3k+4)}$
- $\sum_{n=0}^{\infty}(-1)^{n+1}n$

- $\sum_{n=1}^{\infty} \arctan n$
- $\sum_{n=1}^{\infty} (\sqrt{n+1} - \sqrt{n})$
- $\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^n$
- $\sum_{n=1}^{\infty} \ln\left(\frac{n}{2n+1}\right)$
- $\sum_{n=0}^{\infty} \frac{3}{5^n}$

Exercise 7.25. Let f_n denote the Fibonacci sequence. Show that each of the following statements is true.

- $\frac{1}{f_{n-1}f_{n+1}} = \frac{1}{f_{n-1}f_n} - \frac{1}{f_n f_{n+1}}$
- $\sum_{n=2}^{\infty} \frac{1}{f_{n-1}f_{n+1}} = 1$
- $\sum_{n=2}^{\infty} \frac{1}{f_{n-1}f_{n+1}} = 2$

7.14 Integral Test and P-Series

We motivate and discuss the Integral Test for convergence of an infinite series. We demonstrate how the convergence (or divergence) of a p-series can be determined using the Integral Test.

7.15 The Integral Test

Recall that the Harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \tag{7.5}$$

diverges. What about the following series?

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \tag{7.6}$$

The Integral Test compares an infinite series to an improper integral in order to determine convergence or divergence. For example, to determine the convergence or divergence of (7.6) we will determine the convergence or divergence of

$$\int_1^{\infty} \frac{1}{x^2}\, dx. \tag{7.7}$$

To do so, let $f(x) = 1/x^2$ and consider that (7.7) represents the area under the graph of f from $[1, +\infty)$. Thus we find that

$$\int_1^{\infty} \frac{1}{x^2}\, dx = \lim_{b \to +\infty} -\frac{1}{x}\bigg|_1^b = \lim_{b \to +\infty} \left(-\frac{1}{b} + 1\right) = 1.$$

Consider the following figure. Notice that the sum of the areas of the rectangles under the graph of $f(x) = 1/x^2$ is less than the area under the graph. Therefore, we have

$$\begin{aligned} s_n &= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{n^2} \\ &= f(1) + f(2) + f(3) + f(4) + \cdots + f(n) \\ &< f(1) + \int_1^\infty \frac{1}{x^2}\, dx \\ &= 1 + 1 = 2. \end{aligned}$$

This show that the sequence of partial sums of $\sum_{n=1}^\infty \frac{1}{n^2}$ are bounded above and the series converges. In fact it is well-known that

$$\sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6} \tag{7.8}$$

as was famously proven by Euler. Our argument here, for the convergence of $\sum_{n=1}^\infty \frac{1}{n^2}$ relies heavily on the fact that $f(x) = 1/x^2$ is a continuous, positive, and decreasing function on $[1, \infty)$.

::: {#thm- } Suppose that f is a continuous, positive, and decreasing function on $[1, \infty)$. If $f(n) = a_n$, for $n \geq 1$, then

$$\sum_{n=1}^\infty a_n \qquad \text{and} \qquad \int_1^\infty f(x)\, dx$$

either both converge or both diverge. :::

Example 7.19. Determine if the series $\displaystyle\sum_{n=1}^\infty \frac{8\tan^{-1} n}{1+n^2}$ converges or diverges.

Solution. Let

$$f(x) = \frac{8\tan^{-1} x}{1+x^2}$$

and notice that f is a continuous, positive, and decreasing function on $[1, \infty)$. Using $u = tan^{-1}x$ we see that the improper integral converges

$$\int_1^\infty \frac{8\tan^{-1} x}{1+x^2}\, dx = \int_{\pi/4}^{\pi/2} 8u\, du = 4\left(\frac{\pi^2}{4} - \frac{\pi^2}{16}\right) = \frac{3\pi^2}{4}.$$

Therefore, the series converges by the Integral Test.

Example 7.20. Determine if the series $\displaystyle\sum_{n=1}^\infty \frac{8n}{1+n^2}$ converges or diverges.

Solution. Let
$$f(x) = \frac{8x}{1+x^2}$$
and notice that f is a continuous, positive, and decreasing function on $[1,\infty)$. Using $u = 1+x^2$ we see that the improper integral diverges
$$\int_1^{\infty} \frac{8x}{1+x^2}\,dx = 4\int_2^{+\infty} \frac{1}{u}\,du = 4\lim_{b\to\infty} \ln u\Big|_2^b = +\infty.$$
Therefore, the series diverges by the Integral Test.

7.16 p-Series

Definition 7.9. A **p-series** is an infinite series of the form
$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$
where p is a constant.

What is the difference between a geometric series and a p-series? In a geometric series the exponent is a variable, i.e $\sum(1/2)^n$ is a geometric series. In a p-series the variable is in the base, i.e $\sum(1/n)^2$ is a p-series.

Example 7.21. Let $p > 1$ be a real number. Determine if the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges or diverges.

Solution. Let $f(x) = 1/x^p$ and notice that f is a continuous, positive, and decreasing function on $[1,\infty)$. We see that the improper integral converges
$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{b\to+\infty} \frac{x^{1-p}}{1-p}\Big|_1^b = \lim_{b\to+\infty}\left(\frac{1}{1-p}(b^{1-p}-1)\right) = \frac{1}{1-p}.$$
Therefore, the series converges by the Integral Test.

Example 7.22. Let $p \leq 1$ be a real number. Determine if the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges or diverges.

Solution. If $p = 1$, we have the harmonic series which diverges, and in fact

$$\int_1^\infty \frac{1}{x}\,dx$$

diverges also. Now assume that $p < 1$. Again we let $f(x) = 1/x^p$ and notice that f is a continuous, positive, and decreasing function on $[1, \infty)$. We see that the improper integral diverges

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to+\infty} \left.\frac{x^{1-p}}{1-p}\right|_1^b = \lim_{b\to+\infty} \left(\frac{1}{1-p}(b^{1-p} - 1)\right) = +\infty.$$

Therefore, the series diverges by the Integral Test.

Combining the two previous examples together we have the following theorem.

Theorem 7.6. *The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.*

Unfortunately, there is no theorem to give us the sum of a p-series

7.17 Exercises

Exercise 7.26. Use the Integral Test to determine whether the series diverges or converges.

- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{n+2}{n+1}$

Exercise 7.27. Determine whether the series diverges or converges.

- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{10^n}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}}$

- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$
- $\displaystyle\sum_{n=1}^{\infty} n^{-1.001}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{3n+2}{n(n+1)}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{e^n}$
- $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n}$
- $\displaystyle\sum_{n=3}^{\infty} \frac{(1/n)}{(\ln n)\sqrt{\ln^2 n - 1}}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(1+\ln^2 n)}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^4+1}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3+n}$

Exercise 7.28. Are there any values of x for which $\displaystyle\sum_{n=1}^{\infty}(1/(nx))$ converges?

Exercise 7.29. Explain why the Integral Test does not apply to the series

$$\sum_{n=1} \frac{(-1)^n}{n}.$$

Exercise 7.30. Find the values of p for which the series is convergent.

- $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^p}$
- $\displaystyle\sum_{n=1}^{\infty} n(1+n^2)^p$

Exercise 7.31. Find the value(s) of a for which the series converges. Justify your answer.

- $\displaystyle\sum_{n=1}^{\infty}\left(\frac{a}{n}-\frac{1}{n+1}\right)$
- $\displaystyle\sum_{n=1}^{\infty}\left(\frac{a}{n+1}-\frac{1}{n+2}\right)$

Exercise 7.32. Use the Integral Test to determine the convergence or divergence of the p-series.

- $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{1/2}}$
- $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{5}}$
- $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt[5]{n}}$
- $\displaystyle\sum_{n=3}^{\infty}\frac{1}{(n-2)^{4}}$
- $\displaystyle\sum_{n=1}^{\infty}\frac{3}{n^{5/3}}$
- $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{27k^{2}}}$

7.18 Comparisons of Series

The Direct Comparison Test and the Limit Comparison Test are discussed. We work through several examples for each case and provide many exercises. Convergence tests for infinite series are only mastered through practice.

7.19 Direct Comparison Test

The idea behind the comparison tests is to determine whether a series converges or diverges by comparing a given series to an already familiar series. For example, the series

$$\sum_{n=0}^{\infty}\frac{1}{e^{n}}$$

is known to converge. Do you think the similar series

$$\sum_{n=1}^{\infty}\frac{1}{e^{n}+1}$$

also converges? The idea is compare each term, so that the sequence of partial sums can be compared also. Notice that

$$\frac{1}{e^n+1} < \frac{1}{e^n}.$$

This means that its partial sums must form a bounded increasing sequence, which as we know, is convergent.

::: {#thm- } Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.
- If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.
- If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then a_n is also divergent. :::

Proof. For the first part, the partial sums of $\sum a_n$ are bounded above by

$$M = a_1 + a_2 + \cdots + a_N + \sum_{n=N+1}^{\infty} c_n.$$

Therefore, the partial sums form a nondecreasing sequence with a limit L with $L \leq M$. For the second part, the partial sums of $\sum a_n$ are not bounded from above. Consider that if the partial sums of $\sum a_n$ are bounded above then the partial sums for $\sum b_n$ would be bounded by

$$M^* = b_1 + b_2 + \cdots b_N + \sum_{n=N+1}^{\infty} a_n$$

and $\sum b_n$ would have to converge instead of diverge.

□

Example 7.23. Determine if the series $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n-1}}$ is convergent or divergent.

Solution. Let $a_n = \frac{1}{\sqrt{n-1}}$ and $b_n = \frac{1}{\sqrt{n}}$. Recall the series $\sum_{n=2}^{\infty} b_n$ is a divergent p-series. So by the Direct Comparison Test, with

$$a_n = \frac{1}{\sqrt{n}-1} > \frac{1}{\sqrt{n}} = b_n$$

we see that the given series is divergent.

Example 7.24. Determine if the series $\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$ is convergent or divergent.

Solution. Notice that for large n the term $2n^2$ is dominate in the denominator. Observe that

$$\frac{5}{2n^2+4n+3} < \frac{5}{2n^2}$$

for all $n \geq 1$. Since we know that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, so is the series $\frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$. Therefore the given series is convergent.

7.20 Limit Comparison Test

Theorem 7.7. *Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$. - If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converge of diverge. - If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, them $\sum a_n$ converges. - If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.*

Example 7.25. Determine if the series $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}+\sqrt[3]{n}}$ converges or diverges.

Solution. By the Limit Comparison Test with

$$\lim_{n\to\infty} \frac{\frac{1}{2\sqrt{n}+\sqrt[3]{n}}}{1/\sqrt{n}} = \lim_{n\to\infty} \frac{1}{2+n^{-1/6}} = \frac{1}{2},$$

we see that the series diverges since the p-series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges.

Example 7.26. Determine if the series $\sum_{n=1}^{\infty} \frac{n+1}{n^2\sqrt{n}}$ converges or diverges.

Solution. By the Limit Comparison Test with

$$\lim_{n\to\infty} \frac{\frac{n+1}{n^2\sqrt{n}}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{n+1}{n} = 1,$$

we see that the series converges since the p-series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges.

Example 7.27. Determine if the series $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^2}$ converges or diverges.

Solution. By the Limit Comparison Test with

$$\lim_{n\to\infty} \frac{1/(\ln n)^2}{1/n} = \lim_{n\to\infty} \frac{n}{(\ln n)^2} = \lim_{n\to\infty} n = \infty,$$

we see that the series diverges since the p-series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Example 7.28. Determine if the series $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ converges or diverges.

Solution. Recall that the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges but that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. However the Limit Comparison test with these two series, is inconclusive. Consider instead the series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$. We find that

$$\lim_{n\to\infty} \frac{\ln n/n^2}{1/n^{3/2}} = \lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} = 0$$

Therefore, by the Limit Comparison Test the given series converges.

7.21 Exercises

Exercise 7.33. Determine whether the series converges or diverges.

- $\sum_{n=1}^{\infty} \frac{1}{n^2+2}$
- $\sum_{n=3}^{\infty} \frac{1}{n-2}$
- $\sum_{n=1}^{\infty} \frac{1}{3+2^n}$
- $\sum_{n=2}^{\infty} \frac{1}{n^{2/3}-1}$
- $\sum_{n=1}^{\infty} \frac{1+4^n}{1+3^n}$
- $\sum_{n=1}^{\infty} \frac{1}{3n^2+2}$
- $\sum_{n=1}^{\infty} \frac{n+4^n}{n+6^n}$
- $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}-1}$
- $\sum_{n=1}^{\infty} \frac{1}{4\sqrt[3]{n}-1}$
- $\sum_{n=3}^{\infty} \frac{3^n}{2^n-4}$
- $\sum_{n=1}^{\infty} \frac{2+\sin n}{3^n}$
- $\sum_{n=1}^{\infty} e^{-n^2}$

Exercise 7.34. Determine whether the series converges or diverges.

- $\sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n}$
- $\sum_{n=1}^{\infty} \frac{1+\cos n}{n^2}$
- $\sum_{n=1}^{\infty} \frac{1}{n!}$
- $\sum_{n=1}^{\infty} \frac{2n}{3n-1}$
- $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1}$
- $\sum_{n=1}^{\infty} \frac{n^2}{n!}$
- $\sum_{n=1}^{\infty} \frac{2n^2+n}{\sqrt{4n^7+3}}$
- $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^{3/2}}$
- $\sum_{n=1}^{\infty} \frac{1}{(1+\ln n)^2}$
- $\sum_{n=1}^{\infty} \frac{\sqrt{n}+\ln n}{n^2+1}$
- $\sum_{n=1}^{\infty} \frac{\sqrt[n]{n}}{n^2}$
- $\sum_{n=1}^{\infty} \frac{1}{n\sqrt[n]{n}}$

Exercise 7.35. Show that if $a_n > 0$ and $\lim_{n\to\infty} na_n \neq 0$, then $\sum a_n$ is divergent.

Exercise 7.36. If $\sum_{n=1}^{\infty} a_n$ is a convergent series of nonnegative numbers, can anything be said about $\sum_{n=1}^{\infty}(a_n/n)$? Justify your conclusion.

Exercise 7.37. Show that if $\sum a_n$ is a convergent series of nonnegative terms, then $\sum a_n^2$ converges.

Exercise 7.38. Show that if $a_n > 0$ and $\sum a_n$ is convergent, then$\sum \ln(1+a_n)$ is convergent.

7.22 Alternating Series

Infinite series whose terms alternate in sign are called alternating series. We motivate and prove the Alternating Series Test and we also discuss absolute convergence and conditional convergence. Alternating p-series are detailed at the end.

7.23 Alternating Series Test

Infinite series whose terms alternate in sign are called alternating series.

Definition 7.10. An **alternating series** has one of the following forms

$$\sum_{n=1}^{\infty}(-1)^n a_n \qquad \text{or} \qquad \sum_{n=1}^{\infty}(-1)^{n+1} a_n$$

where the a_n's are positive.

For example, the two series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

$$\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots$$

are alternating series.

The next theorem gives sufficient conditions under which an alternating series must converge. The **Alternating Series Test** is sometimes called **Leibniz's Theorem**.

::: {#thm- } An alternating series converges whenever the following conditions are satisfied

- $a_1 \geq a_2 \geq a_3 \geq a_4 \geq \cdots \geq a_n \geq \cdots$
- $\lim_{n\to+\infty} a_n = 0$ :::

Proof. Consider the following diagram. First we plot $s_1 = a_1$. To find s_2 we subtract a_2, so s_2 is to the left of s_1. Then to find s_3 we add a_3, so s_3 is to the right of s_2. Since $a_3 < a_2$, s_3 is to the left of s_1. Continuing in this pattern, we see that the partial sums oscillate back and forth. However, by the assumption $\lim_{n\to+\infty} a_n = 0$, the successive steps are becoming smaller and smaller.

Notice that the odd partial sums $s_1, s_3, s_5, \ldots$ are decreasing and that the even partial sums $s_2, s_4, s_6, \ldots$ are increasing. We now consider the two cases.

If n is even, say $n = 2m$, then the sum of the first n terms is

$$\begin{aligned} s_{2m} &= (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2m-1} - a_{2m}) \\ &= a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2m-2} - a_{2m-1}) - a_{2m} \end{aligned}$$

Since each term in parentheses is positive or zero, we notice that s_{2m} is the sum of m nonnegative terms. Thus $s_{2m+2} \geq s_{2m}$, and the sequence $\{s_{2m}\}$ is nondecreasing. Also notice that $s_{2m} \leq a_1$ and so the sequence $\{s_{2m}\}$ is nondecreasing and bounded from above. Therefore, it has a limit, say

$$\lim_{m\to\infty} s_{2m} = L.$$

Next we consider the subsequence of odd terms $\{s_{2m+1}\}$ of $\{s_n\}$. Since

$$s_{2m+1} = s_{2m} + a_{2m+1}$$

and $\lim_{m\to\infty} a_{2m+1} = 0$ by hypothesis, we have

$$\lim_{m\to\infty} s_{2m+1} = \lim_{m\to\infty} s_{2m} + a_{2m+1} = \lim_{m\to\infty} s_{2m} + \lim_{m\to\infty} a_{2m+1} = L$$

Since the subsequence $\{s_{2m+1}\}$ and $\{s_{2m}\}$ of the sequence of partial sums $\{s_n\}$ both converge to L, we have $\lim_{n\to\infty} s_n = L$ as needed.

$\square$

Example 7.29. Show that the alternating harmonic series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ converges or diverges.

Solution. Notice that

$$a_n = \frac{1}{n} > \frac{1}{n+1} = a_{n+1}$$

and that

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

By the Alternating Series Test, the series must converge.

Example 7.30. Show that the alternating harmonic series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n+1}{n}$ converges or diverges.

Solution. Notice that $a_{n+1} \leq a_n$, for all n. However, the Alternating Series Test does not apply because

$$\lim_{n\to\infty} \frac{\ln n}{n} \neq 0.$$

In fact the series diverges.

Example 7.31. Show that the series $\sum_{n=1}^{\infty}(-1)^{n}\frac{\ln n}{n}$ converges or diverges.

Solution. Let $f(x) = (\ln x)/x$. By the Quotient Rule, we have $f'(x) = (1 - \ln x)/x^2$. The fact that $f'(x) < 0$, for $x > e$, implies the terms decrease, whenever $n \geq 3$. Also notice that

$$\lim_{n\to\infty} \frac{\ln n}{n} = 0,$$

and therefore, the condition of the Alternating Series Test are satisifed. By the Alternating Series Test, the series must converge.

7.24 Approximating Sums of Alternating Series

Theorem 7.8. *If an alternating series satisfies the hypothesis of the alternating series test, and if S is the sum of the series, then - either $s_n \leq S \leq s_{n+1}$ or $s_{n+1} \leq S \leq s_n$, and - if S is approximated by s_n, then the absolute error $|S - s_n|$ satisfies*

$$|S - s_n| \leq a_{n+1}.$$

Moreover, the sign of the error $S - s_n$ is the same as a_{n+1}.

Proof. From the proof of the Alternating Series Test we know that S lies between any two consecutive partial sums s_n and s_{n+1}. It follows that

$$|S - s_n| \leq |s_{n+1} - s_n| \leq a_{n+1}$$

as desired.

□

7.25 Absolute and Conditional Convergence

Definition 7.11. A series $\sum a_n$ is called **absolutely convergent** if the series $\sum |a_n|$ converges.

::: {#thm- } If a series $\sum a_n$ is absolutely convergent, then it is convergent. :::

Proof. Notice that for each n, $-|a_n| \leq a_n \leq |a_n|$ and so $0 a_n +|a_n| 2 a_n $. If the series $\sum |a_n|$ converges, then the series $\sum 2|a_n|$ converges, and so the nonnegative series $\sum (a_n + |a_n|)$ converges. The equality $a_n = (a_n + |a_n|) - |a_n|$ now lets us express the series $\sum a_n$ as the difference of two convergent series:

$$\sum a_n = \sum (a_n + |a_n|) - |a_n| = \sum (a_n + |a_n|) - \sum |a_n|.$$

Therefore $\sum a_n$ converges.

□

Notice that the alternating harmonic series converges but is not absolutely convergent.

Definition 7.12. A series $\sum a_n$ is **conditionally convergent** if it is convergent but not absolutely convergent.

Example 7.32. Determine whether the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}}$ diverges, converges absolutely, or converges conditionally.

Solution. Consider the series

$$\sum_{n=1}^{\infty} \left| (-1)^{n+1} \frac{1}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}.$$

This is a divergent p-series with $p = 1/2 < 1$. Therefore the series is not absolutely convergent.
Now notice that

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

and that the series is decreasing. By the Alternating Series Test, the given series converges. Therefore the given series converges conditionally.

Example 7.33. Determine whether the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n^3}}$ diverges, converges absolutely, or converges conditionally.

Solution. Consider the series

$$\sum_{n=1}^{\infty}\left|(-1)^{n+1}\frac{1}{\sqrt{n^3}}\right| = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n^3}}.$$

This is a convergent p-series with $p = 3/2 > 1$. Therefore the series is absolutely convergent.

Example 7.34. Determine whether the series $\sum_{n=1}^{\infty}\frac{\sin n}{n^2}$ diverges, converges absolutely, or converges conditionally.

Solution. Notice that the Alternating Series Test does not apply. Consider the series

$$\sum_{n=1}^{\infty}\left|\frac{\sin n}{n^2}\right| \le \sum_{n=1}^{\infty}\frac{1}{n^2}.$$

Recall that the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a convergent p-series with $p = 2 > 1$. By the Direct Comparison Test, the series $\sum_{n=1}^{\infty}\left|\frac{\sin n}{n^2}\right|$ converges. Therefore, the given series is absolutely convergent.

Example 7.35. Determine whether the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n+3}{n(n+1)}$ diverges, converges absolutely, or converges conditionally.

Solution. Consider the series

$$\sum_{n=1}^{\infty}\left|(-1)^{n+1}\frac{n+3}{n(n+1)}\right| = \sum_{n=1}^{\infty}\frac{n+3}{n(n+1)}. \tag{7.9}$$

We use the Limit Comparison Test with

$$a_n = \frac{n+3}{n(n+1)} \qquad \text{and} \qquad b_n = \frac{1}{n}.$$

We have

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n+3}{n+1} = 1.$$

It follows from the Limit Comparison Test that the series in (7.9) diverges. Therefore the given series converges conditionally.

Example 7.36. Determine whether the **alternating p-series**

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^p}$$

diverges, converges absolutely, or converges conditionally.

Solution. Consider the series

$$\sum_{n=1}^{\infty}\left|(-1)^{n+1}\frac{1}{n^p}\right| = \sum_{n=1}^{\infty}\frac{1}{n^p}. \tag{7.10}$$

This is a convergent p-series whenever $p > 1$. Therefore the alternating p-series is absolutely convergent if $p > 1$. Otherwise the series in (7.10) is divergent, and in this case, when $0 < p \leq 1$, the alternating p-series is conditionally convergent.

7.26 Exercises

Exercise 7.39. Which of the alternating series converge, and which diverge? Give reasons for your answers.

- $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^{3/2}}$
- $\sum_{n=1}^{\infty}(-1)^{n}\frac{\sqrt{n}}{n+5}$
- $\sum_{n=1}^{\infty}(-1)^{n}\frac{n}{3n-1}$
- $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{10^n}{n^{10}}$
- $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{2n^2-1}$
- $\sum_{n=1}^{\infty}(-1)^{n}\sin\left(\frac{\pi}{n}\right)$
- $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln n}{n}$
- $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{\sqrt{n^2+1}}$
- $\sum_{n=1}^{\infty}(-1)^{n}\ln\left(1+\frac{1}{n}\right)$
- $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{e^{1/n}}{n}$
- $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{\ln n}$
- $\sum_{n=1}^{\infty}(-1)^{n}\frac{\sqrt{n}}{1+2\sqrt{n}}$
- $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3\sqrt{n+1}}{\sqrt{n}+1}$
- $\sum_{n=1}^{\infty}(-1)^{n}\frac{\ln(n+1)}{n+2}$

Exercise 7.40. Which of the series converge absolutely, which converge, and which diverge? Give reasons for your answers.

- $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}$
- $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n+2}$
- $\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{n\ln n}$
- $\sum_{n=1}^{\infty}(-1)^{n}\frac{n}{n^3-5}$
- $\sum_{n=0}^{\infty}\frac{\cos n\pi}{n+1}$
- $\sum_{n=1}^{\infty}(-1)^{n+1}\arctan n$

Exercise 7.41. Estimate the magnitude of the error involved in using the sum of the first four terms to approximate the sum of the entire series.

- $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{10^n}$
- $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^6}$
- $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{\sqrt{n}}$
- $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(0.01)^n}{n}$
- $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^5}$
- $\sum_{n=2}^{\infty}(-1)^{n-1}\frac{1}{n\ln n}$
- $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{10^n}$
- $\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{n!}$
- $\sum_{n=1}^{\infty}(-1)^{n}\frac{3^n}{n!}$

Exercise 7.42. Show that if $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} |a_n|$ diverges.

Exercise 7.43. Show that if $\sum_{n=1}^{\infty} a_n$ converges absolutely, then

$$\left|\sum_{n=1}^{\infty} a_n\right| \leq \sum_{n=1}^{\infty} |a_n|.$$

Exercise 7.44. Give an example of an alternating p-series that converges, but whose corresponding p-series diverges.

Exercise 7.45. Find all values of s for which the series $\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{n^s}$ converges.

Exercise 7.46. For what values of p is the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(\ln n)^p}{n}$ convergent?

Exercise 7.47. Show that if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge absolutely, then so do $\sum_{n=1}^{\infty} a_n + b_n$, $\sum_{n=1}^{\infty} a_n - b_n$, and $\sum_{n=1}^{\infty} ka_n$ where k is a real number.

Exercise 7.48. Find all values of x for which the infinite series

$$\sum_{n=1}^{\infty} \frac{x^n}{n}$$

(a) converges absolutely and (b) converges conditionally.

7.27 Ratio and Root Tests

The Ratio Test and the Root Tests are criterions for the convergence of an infinite series. We provide several examples using these convergence tests and several exercises.

7.28 The Ratio Test

The Ratio Test is a criterion for the convergence (a convergence test) of an infinite series $\sum a_n$. It depends on the quantity

$$\lim_{n \to +\infty} \frac{a_{n+1}}{a_n}. \tag{7.11}$$

to determine convergence or diverges. The Ratio Test fails if the limit in (7.11) is 1.

Theorem 7.9. *Let $\sum a_n$ be a series with positive terms and suppose that*

$$\rho = \lim_{n \to +\infty} \frac{a_{n+1}}{a_n}.$$

- If $\rho < 1$, the series converges. - If $\rho > 1$ or $\rho = +\infty$, the series diverges. - If $\rho = 1$, the series may converge or diverge.

Example 7.37. Use the Ratio Test to determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges or diverges.

Solution. We find

$$\rho = \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{1/(n+1)!}{1/n!} = \lim_{n \to \infty} \frac{1}{n+1} = 0.$$

Since $\rho < 1$ the series converges by the Ratio Test.

Example 7.38. Use the Ratio Test to determine whether the series $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converges or diverges.

Solution. We find

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)/2^{n+1}}{n/2^n} = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2}.$$

Since $\rho < 1$ the series converges by the Ratio Test.

Example 7.39. Use the Ratio Test to determine whether the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ converges or diverges.

Solution. We find

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^{n+1}/(n+1)!}{n^n/n!} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e.$$

Since $\rho > 1$ the series diverges by the Ratio Test.

Example 7.40. Use the Ratio Test to determine whether the series $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ converges or diverges.

Solution. We find

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1/(2(n+1)-1)}{1/(2n-1)} = \lim_{n\to\infty} \frac{2n-1}{2n+1} = 1.$$

Since $\rho = 1$ the Ratio Test fails. However, the Integral Test proves that the series diverges since

$$\int_1^{+\infty} \frac{1}{2x-1}\,dx = \lim_{b\to+\infty} \int_1^b \frac{1}{2x-1}\,dx = \lim_{b\to+\infty} \frac{1}{2} \ln(2x-1)\Big|_1^b = +\infty$$

as desired.

7.29 The Root Test

The Root Test is a criterion for the convergence (a convergence test) of an infinite series $\sum a_n$. It depends on the quantity

$$\lim_{n\to+\infty} \sqrt[n]{a_n}. \tag{7.12}$$

to determine convergence or diverges. The Root Test fails if the limit in (7.12) is 1.

Theorem 7.10. *Let $\sum a_n$ be a series with positive terms and suppose that*

$$\rho = \lim_{n\to+\infty} \sqrt[n]{a_n}.$$

- *If $\rho < 1$, the series converges.* - *If $\rho > 1$ or $\rho = +\infty$, the series diverges.*
- *If $\rho = 1$, the series may converge or diverge.*

Example 7.41. Use the root test to determine whether the series

$$\sum_{n=1}^{\infty} \left(\frac{4n^2-3}{7n^2+6}\right)^n$$

converges or diverges.

Solution. We find

$$\rho = \lim_{n\to\infty} \sqrt[n]{\left(\frac{4n^2-3}{7n^2+6}\right)^n} = \lim_{n\to\infty} \frac{4n^2-3}{7n^2+6} = \frac{4}{7}$$

Since $\rho < 1$ the series converges.

Example 7.42. Use the root test to determine whether the series $\sum_{n=1}^{\infty} \frac{2^n}{n^{10}}$ converges or diverges by the Root Test.

Solution. We find

$$\rho = \lim_{n\to\infty} \sqrt[n]{\frac{2^n}{n^{10}}} = \lim_{n\to\infty} \frac{2}{(n^{1/n})^{10}} = 2$$

Since $\rho > 1$ the series diverges by the Root Test.

Example 7.43. Use the root test to determine whether the series $\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$ converges or diverges.

Solution. We find

$$\rho = \lim_{n\to\infty} \sqrt[n]{\frac{e^{2n}}{n^n}} = \lim_{n\to\infty} \frac{e^{2n/n}}{n^{n/n}} = \lim_{n\to\infty} \frac{2^2}{n} = 0$$

Since $\rho < 1$ the series converges by the Root Test.

Example 7.44. Determine whether the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^{n+3}}{(n+1)^n}$$

is absolutely convergent, conditionally convergent, or divergent. :::

Solution. We find

$$\rho = \lim_{n\to\infty} \sqrt[n]{\left|(-1)^{n-1}\frac{2^{n+3}}{(n+1)^n}\right|} = \lim_{n\to\infty}\left|\frac{2^{n+3}}{(n+1)^n}\right|^{1/n} = \lim_{n\to\infty}\frac{2^{1+3/n}}{n+1} = 0$$

Since $\rho < 1$ the series is absolutely convergent by the Root Test.

Exercises

Exercise 7.49. Use the Ratio Test to determine the convergence or divergence of the series.

- $\displaystyle\sum_{n=0}^{\infty}\frac{n!}{3^n}$
- $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n!}$
- $\displaystyle\sum_{n=1}^{\infty}\frac{5^n}{n^4}$
- $\displaystyle\sum_{n=1}^{\infty}\frac{n^2}{4^n}$
- $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{n3^n}$
- $\displaystyle\sum_{n=1}^{\infty}\frac{n^n}{2^n}$
- $\displaystyle\sum_{n=1}^{\infty}\frac{(n!)^2}{(3n)!}$
- $\displaystyle\sum_{n=1}^{\infty}\frac{n^6}{n!}$
- $\displaystyle\sum_{n=1}^{\infty}n\left(\frac{1}{2}\right)^n$

Exercise 7.50. Use the Root Test to determine the convergence or divergence of the series.

- $\displaystyle\sum_{n=1}^{\infty}\frac{1}{4^n}$
- $\displaystyle\sum_{n=1}^{\infty}\left(\frac{12n_n^3}{9n^2+n+1}\right)^n$
- $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n-2}{5n+1}\right)^n$

- $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{2n^2}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3^n}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{n}{e^n}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^n}{(n^n)^2}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$
- $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n!}{n^3}\right)$

Exercise 7.51. Which of the series converge, and which diverge? Give reasons for your answers.

- $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n}$
- $\displaystyle\sum_{n=1}^{\infty} (1-e^{-n})^n$
- $\displaystyle\sum_{n=1}^{\infty} \frac{100}{n}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{10^n}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{10}{2\sqrt{n^3}}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{3^n}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4+2^{-n}}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{4n^2-1}$
- $\displaystyle\sum_{n=1}^{\infty} \left(1-\frac{1}{3n}\right)^n$
- $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n 2^n}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{n \ln n}{2^n}$

- $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n}{3 \cdot 5 \cdot 7 \cdots (2n+1)}$

Exercise 7.52. Determine whether the series is convergent, absolutely convergent, conditionally convergent, or divergent.

- $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^n}{n\sqrt{\ln n}}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{10^n}{(n+1)4^{2n+1}}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n+1)}{n\sqrt{n}}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\tan^{-1} n}{n^2}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{100^n}$
- $\displaystyle\sum_{n=2}^{\infty} \frac{(-10^n}{(\ln n)^n}$
- $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{\sqrt{n^3+2}}$
- $\displaystyle\sum_{n=1}^{\infty} (-1)^n \tan\left(\frac{1}{n}\right)$
- $\displaystyle\sum_{n=1}^{\infty} \frac{\sin 4n}{4^n}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{(-n)^n}{[(n+1)\tan^{-1} n]^n}$
- $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2 2^n}{n!}$

Exercise 7.53. Show that the ratio and root tests fail on p-series.

Exercise 7.54. Show that the ratio and root tests fail on series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(\ln n)^p}$.

Exercise 7.55. The terms of a series are de?ned recursively by the equations $a_1 = 2$,

$$a_{n+1} = \frac{2 + \cos n}{\sqrt{n}} a_n.$$

Determine whether $\sum a_n$ converges or diverges.

Exercise 7.56. Around 1910, the mathematician Srinivasa Ramanujan discovered the formula

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!(1103 + 26390n)}{(n!)396^{4n}}.$$

Verify that the series is convergent.

Exercise 7.57. Find the values of the parameter $p > 0$ for which the series

$$\sum_{n=1}^{\infty} \left(1 - \frac{p}{n}\right)^n$$

converges.

Taylor Polynomials and Approximations

Recall a tangent line approximation of a function is used to obtain a local linear approximation of the function near the point of tangency. We consider how to improve on the accuracy of tangent linear approximations by using higher-order polynomials as approximating functions. We also discuss the error associated with such approximations.

Polynomial Approximations

The goal is to show how polynomial functions can be used to approximate other elementary functions. First we investigate local approximations around $x = 0$.

Maclaurin Polynomials

Definition 7.13. If f can be differentiated n times at 0, then the polynomial

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n \quad (7.13)$$

is called the nth **Maclaurin polynomial** for f.

Note that a Maclaurin polynomial has the property that its value and the values of its first n derivatives match the values of f and its first n derivatives at $x = 0$.

Example 7.45. Determine the nth Maclaurin polynomial for each of the following functions: $\sin x$, $\cos x$, $\ln x$, and e^x.

Solution. Respectively we have,

$$\begin{array}{ll}
\sin x & p_n(x) = x - \dfrac{x^3}{6} + \dfrac{x^5}{120} - \dfrac{x^7}{5040} + \cdots + \dfrac{(-1)^{(n-1)}}{(2n-1)!}x^{2n-1} \\
\cos x & p_n(x) = 1 - \dfrac{x^2}{2} + \dfrac{x^4}{24} - \dfrac{x^6}{720} + \cdots + \dfrac{(-1)^{(n-1)}}{(2n-2)!}x^{2n-2} \\
e^x & p_n(x) = 1 + x + \dfrac{1}{2!}x^2 + \dfrac{1}{3!}x^3 + \cdots + \dfrac{1}{n!}x^n
\end{array}$$

using (7.13) directly.

Taylor Polynomials

We want to find an nth-degree polynomial p with the property that its value and the values of its first n derivatives match the values of f and its first n derivatives at $x = x_0$. However, rather than expressing $p(x)$ in powers of x, it will simplify the computations if we express it in powers of $x - x_0$; that is, we what a polynomial

$$p(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + \cdots + c_n(x - x_0)^n \tag{7.14}$$

such that $f(x_0) = p(x_0)$ and

$$f'(x_0) = p'(x_0), \quad f''(x_0) = p''(x_0), \quad \ldots, \quad f^{(n)}(x_0) = p^{(n)}(x_0) \tag{7.15}$$

However notice that

$$\begin{aligned}
p(x) &= c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + c_3(x - x_0)^3 + \cdots + c_n(x - x_0)^n \\
p'(x) &= c_1 + 2c_2(x - x_0) + 3c_3(x - x_0)^2 + \cdots + nc_n(x - x_0)^{n-1} \\
p''(x) &= 2c_2 + 3c_3(x - x_0) + \cdots + n(n-1)c_n(x - x_0)^{n-2} \\
p'''(x) &= 3c_3 + \cdots + n(n-1)(n-2)c_n(x - x_0)^{n-3} \\
&\vdots \\
p^{(n)}(x) &= n(n-1)(n-2)\cdots(1)c_n
\end{aligned}$$

Therefore to satisfy (7.15) we must have

$$\begin{aligned}
f(x_0) &= p(x_0) = c_0 \\
f'(x_0) &= p(x_0) = c_1 \\
f''(x_0) &= p(x_0) = 2c_2 = 2!\, c_2 \\
f'''(x_0) &= p(x_0) = 6c_3 = 3!\, c_3 \\
&\vdots \\
f^{(n)}(x_0) &= p^{(n)}(x_0) = n(n-1)(n-2)\cdots(1)c_n = n!c_n.
\end{aligned}$$

Solving for the coefficients c_i for (7.14) yields $c_0 = f(x_0)$ and

$$c_1 = f'(x_0), \quad c_2 = \frac{f''(x_0)}{2!}, \quad c_3 = \frac{f'''(x_0)}{3!}, \quad \dots, \quad c_n = \frac{f^{(n)}(x_0)}{n!},$$

In other words we have just proven the following theorem.

Theorem 7.11. *If a function f has a power series expansion at c, that is, if*

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n, \qquad |c-a| < R$$

then its coefficients are given by $c_n = \frac{f^{(n)}(x_0)}{n!}$.

Definition 7.14. If f can be differentiated n times at x_0, then the polynomial

$$\begin{aligned}
p_n(x) = f(x_0) + f'(x_0)(x-x_0) &+ \frac{f''(x_0)}{2!}(x-x_0)^2 \\
&+ \frac{f'''(x_0)}{3!}(x-x_0)^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n
\end{aligned}$$

is called the nth **Taylor polynomial** for f.

Note that a Taylor polynomial has the property that its value and the values of its first n derivatives match the values of f and its first n derivatives at $x = x_0$.

Whenever $n = 1$ we say that the Taylor Polynomial $p_1(x)$ for a function f is a **local linear approximation** for f around $x = x_0$. Similarly, whenever $n = 2$ we say that the Taylor Polynomial $p_2(x)$ for a function f is a **local quadratic approximation** for f around $x = x_0$.

Example 7.46. Find the linear and quadratic approximations to $f(x) = \ln x$ at $x = 1$. Use these approximations to estimate $\ln 1.05$.

Solution. Notice that $f'(x) = 1/x$ and $f''(x) = -1/x^2$. Thus $f'(1) = 1$ and $f''(1) = -1$. So the 1st degree Taylor polynomial is

$$p_1(x) = f(1) + f'(1)(x-1) = x - 1.$$

This is the linear (tangent line) approximation of $\ln x$ at $x = 1$. Further the 2nd degree Taylor polynomial is

$$p_2(x) = p_1(x) + \frac{1}{2}f''(1)(x-1)^2 = (x-1) - \frac{1}{2}(x-1)^2.$$

This is the local quadratic approximation of $\ln x$ at $x = 1$. Using

$$p_1(1.05) = 1.05 - 1 = 0.05$$
$$p_2(1.05) = (1.05 - 1) - \frac{1}{2}(1.05-1)^2 = 0.04875.$$

The value of $\ln 1.05$ given by calculator is 0.04879.

Exercises

Exercise 7.58. For each of the following find the Taylor polynomial of orders $0, 1, 2,$ and 3 generated by f at c.

- $f(x) = 1/x$, $c = 2$
- $f(x) = 1/(x+2)$, $c = 0$
- $f(x) = \sqrt{x}$, $c = 4$
- $f(x) = \sin x$, $c = \pi/6$

Exercise 7.59. For each of the following find the linearization and quadratic approximation of f at $x = 0$.

- $f(x) = \ln(\sin x)$
- $f(x) = 1/\sqrt{1-x^2}$
- $f(x) = e^{\cos x}$

Exercise 7.60. Find the nth-order Taylor polynomial $p_n(x)$ at c for the function f and the values of n. Plot the graphs of f and the approximating polynomials on the same set of axes. - $f(x) = \cos x$, $c = \pi/4$, $n = 0, 1, 2, 3, 4$ - $f(x) = e^x x$, $c = 1$, $n = 0, 1, 2, 3, 4$

Exercise 7.61. Use a Taylor polynomial to approximate the given number to within the indicated accuracy.

- $e^{0.2}$, 0.0001
- $-\frac{1}{2.1}$, 0.0005
- $\sin 69^\circ$, 0.0001

Exercise 7.62. Find the Taylor polynomial $p_n(x)$ and the Taylor remainder $r_n(x)$ for the function f and the values of c and n. - $f(x) = x^4 + 3x^2 + 2x + 3$, $c = -1$, $n = 4$ - $f(x) = \tan x$, $c = \pi/4$, $n = 2$ - $f(x) = \sqrt[3]{x}$, $c = -8$, $n = 3$ - $f(x) = x \sin x$, $n = 2$, $c = \pi/4$ - $f(x) = \frac{2}{x}$, $n = 3$, $c = 1$ - $f(x) = x^2 \cos x$, $n = 2$, $c = \pi$

Power Series

We introduce power series and discuss convergence of power series. Finding the interval of convergence and finding the radius of convergence is explained through several examples. We also discuss term-by-term differentiation and integration power series.

What are Power Series?

We begin with the definition of power series.

Definition 7.15. Let x be a variable. An infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

is called a **power series** . More generally, an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n + \cdots$$

is called a **power series centered at** c.

For example,

$$\sum_{n=0}^{\infty} \frac{(x-1)^n}{n+1} = 1 + \frac{x-1}{2} + \frac{(x-1)^2}{3} + \frac{(x-1)^3}{4} + \cdots$$

is a power series in $x-1$. The series

$$\sum_{n=0}^{\infty} \frac{(-1)^n (x+4)^n}{n!} = 1 - (x+4) + \frac{(x+4)^2}{2!} - \frac{(x+4)^3}{3!} + \cdots$$

is a power series centered -4. More generally, the Taylor series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

is a power series in $x - x_0$.

For each x, the power series is an infinite series of constants that we can test for convergence or divergence. A power series may diverge for some values of x and converge for other values. We emphasize that a power series is a function

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

whose domain is the set of all x for which the power series converges.

Convergence of Power Series

The set of values of x for which the series is convergent is always an interval, either a finite interval, the infinite interval $(-\infty, +\infty)$, or a collapsed interval such as $\{0\}$. The following theorem guarantees this is always the case.

Theorem 7.12. *For any power series centered at c*

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

there are only three possibilities:

- *The series converges only when $x = c$.*
- *The series converges for all x.*
- *There is a positive real number R such that the power series converges whenever $|x-c| < R$ and diverges whenever $|x-c| > R$.*

The number R is called the ***radius of convergence*** *.*

In other words every power series has a radius of convergence, where in case 1 the radius of convergence is 0, in case 2 the radius of convergence is ∞, and in case 3 the radius of convergence is R.

In case three, there are four possibilities that may occur for a given series to converge. The **interval of convergence** of a series can be any of the possibilities:

$$(c-R, c+R) \qquad (c-R, c+R] \qquad [c-R, c+R) \qquad [c-R, c+R]$$

In this case we have $|x-c|<R$ which means that

$$c-R<x<c+R.$$

However, the endpoints of this interval must be check individually, that is, if $x=c\pm R$ the series might converge at one or both endpoints or it might diverge at both endpoints.

Example 7.47. For which values of x is the power series

$$\sum_{n=0}^{\infty} n!\, x^n$$

convergent?

Solution. Notice that the series converges when $x=0$. Assume $x\neq 0$. Let $a_n=n!x^n$ and we use the Ratio Test. We have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\, x^{n+1}}{n!\, x^n}\right| = \lim_{n\to\infty}(n+1)|x| = \infty.$$

By the Ratio Test the series diverges whenever $x\neq 0$.

Example 7.48. Find the interval of convergence and radius of convergence for the power series

$$\sum_{n=0}^{\infty}\frac{x^n}{n!}.$$

Solution. We apply the Ratio Test for absolute convergence. We have

$$\begin{aligned}\rho &= \lim_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to+\infty}\left|\frac{x^{n+1}}{(n+1)!}\frac{n!}{x^n}\right| \\ &= \lim_{n\to+\infty}\left|\frac{x}{n+1}\right| = 0\end{aligned}$$

By the Ratio Test for absolute convergence, the series converges absolutely for all values of x since $\rho<1$. The interval of convergence is $(-\infty,+\infty)$ and the radius of convergence is $R=+\infty$.

Example 7.49. Find the interval of convergence and radius of convergence for the power series

$$\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{3^n(n+1)}.$$

Solution. We apply the Ratio Test for absolute convergence. We have

$$\begin{aligned}\rho &= \lim_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to+\infty}\left|\frac{x^{n+1}}{3^{n+1}(n+2)}\,\frac{3^n(n+1)}{x^n}\right| \\ &= \lim_{n\to+\infty}\frac{|x|}{3}\,\frac{k+1}{k+2} = \frac{|x|}{3}\lim_{n\to+\infty}\frac{k+1}{k+2} = \frac{|x|}{3}\end{aligned}$$

By the Ratio Test for absolute convergence, the series converges absolutely if $|x| < 3$ and diverges if $|x| > 3$. The Ratio Test is inconclusive whenever $|x| = 3$. These cases must be considering separately. If $x = 3$, then the series is a conditionally convergent harmonic series:

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}.$$

If $x = -3$, then the series is a divergent harmonic series:

$$\sum_{n=0}^{\infty}\frac{1}{n+1}.$$

Thus, the interval of convergence for the given power series is $(-3, 3]$ and the radius of convergence is $R = 3$.

Example 7.50. Find the interval of convergence and radius of convergence for the power series

$$\sum_{n=1}^{\infty}\frac{(x-2)^n}{\sqrt{n}}.$$

Solution. We apply the Ratio Test for absolute convergence. We have

$$\begin{aligned}\rho &= \lim_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to+\infty}\left|\frac{(x-2)^{n+1}}{\sqrt{n+1}}\,\frac{\sqrt{n}}{(x-2)^n}\right| \\ &= |x-2|\lim_{n\to+\infty}\sqrt{\frac{k}{k+1}} = |x-2|\end{aligned}$$

By the Ratio Test for absolute convergence, the series converges absolutely if $|x-2| < 1$ and diverges if $|x-2| > 1$. The Ratio Test is inconclusive whenever $|x-2| = 1$. These cases must be considering separately. If $x = 3$, then the series is a divergent p-series with $p = 1/2$:

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}.$$

If $x = 1$, then the series is a convergent alternating series:

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}.$$

Thus, the interval of convergence for the given power series is $[1, 3)$ and the radius of convergence is $R = 1$.

Differentiating and Integrating Power Series

In advanced calculus it is proven that a power series can be differentiated (integrated) term by term at each interior point of its interval of convergence.

Theorem 7.13. *Suppose that the power series $\sum a_n(x-c)^n$ converges for $|x-c| < R$ and defines a function f on that interval. - Then f is differentiable for $|x-c| < R$, and f' is found by differentiating the power series for f term by term; that is*

$$f'(x) = \sum n a_n (x-c)^{n-1}.$$

for $|x-c| < R$. - The integral of f is found by integrating the power series for f term by term; that is

$$\int f(x)\,dx = \sum a_n \frac{(x-c)^{n+1}}{n+1} + C.$$

for $|x-c| < R$, where C is an arbitrary constant.

This theorem makes no conclusion about convergence of the differentiated (integrated) series to at the endpoints.

Example 7.51. Consider the power series

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots, \quad |x| < 1.$$

- Differentiate this power series term by term to find the power series for f' and identify the function it represents. - Integrate this power series term by term and identify the function it represents.

Solution. Differentiating this power series we find that

$$\begin{aligned} f'(x) &= 1 + 2x + \cdots + nx^{n-1} + \cdots \\ &= \sum_{n=0}^{\infty} nx^n. \end{aligned}$$

Therefore, whenever $|x| < 1$ we have

$$f'(x) = \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n.$$

as desired. In this case, substituting $x = \pm 1$ into the power series for f' reveals that the series diverges at both endpoints.

Integrating this power series we find that

$$\begin{aligned}\int f(x)\,dx &= x + \frac{x^2}{2} + \cdots + \frac{x^n}{n} + \cdots \\ &= \sum_{n=0}^{\infty}(n+1)x^n\end{aligned}$$

Therefore, whenever $|x| < 1$ we have

$$\int f(x)\,dx = -\ln|1-x| = \sum_{n=0}^{\infty}\frac{x^n}{n}.$$

as desired. Multiplying both sides by 1 we obtain a power series representation for $\ln(1-x)$:

$$\ln(1-x) = -\sum_{n=0}^{\infty}\frac{x^n}{n}.$$

When $x = 1$ the series is divergent by comparing to the divergent harmonic series. When $x = -1$ the series converges to $\ln 2$.

Functions Represented by Power Series

Example 7.52. Find a power series representation for $\tan^{-1}(x)$ by integrating a power series representation of $f(x) = 1/(1+x^2)$.

Solution. The series

$$f(t) = \frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + \cdots$$

converges on the open interval $-1 < t < 1$. Therefore,

$$\begin{aligned}\tan^{-1} x &= \int_0^x \frac{1}{1+t^2}\,dt = t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \cdots \Bigg|_0^x \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}\end{aligned}$$

for $-1 < x < 1$.

Example 7.53. Find a power series representation for $\ln(1+x)$ by integrating a power series representation of $f(x) = 1/(1+x)$.

Solution. The series

$$f(t) = \frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots$$

converges on the open interval $-1 < t < 1$. Therefore,

$$\begin{aligned} \ln(1+x) &= \int_0^x \frac{1}{1+t}\,dt = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots \Big|_0^x \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^{n-1} x^{n-1}}{n} \end{aligned}$$

for $-1 < x < 1$.

Combining Power Series

A power series defines a function in its interval of convergence. When power series are combined algebraically, new functions are defined.

::: {#thm- } Suppose that $\sum a_n x^n$ and $\sum b_n x^n$ converge to $f(x)$ and $g(x)$, respectively, on an interval I. - The power series $\sum (a_n \pm b_n) x^n$ converges to $f(x) \pm g(x)$ on I. - The power series $\sum a_n x^{n+m}$ converges to $f(x)x^m$ for all $x \neq 0$ in I provided that m is an integer such that $n + m \geq 0$ for all terms of the power series $\sum a_n x^{n+m}$. If $x = 0$, the series converges to $\lim_{x\to 0} x^m f(x)$. - If $h(x) = bx^m$, where m is a positive integer and b is a nonzero real number, the power series $\sum a_n (h(x))^n$ converges to the composite function $f(h(x))$, for all x such that $h(x)$ is in I.

Example 7.54. Consider the geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots, \qquad |x| < 1$$

find the power series and interval of convergence for the functions

$$f(x) = \frac{x^5}{1-x} \qquad \text{and} \qquad g(x) = \frac{1}{1-2x}.$$

Solution. We notice that

$$f(x) = \frac{x^5}{1-x} = x^5(1 + x + x^2 + x^3 + \cdots) = \sum_{n=0}^{\infty} x^{n+5}.$$

This series has a ratio $r = x$ and converges when $|r| = |x| < 1$. The interval of convergence is $|x| < 1$. For the function g, we use $2x$ for x in the power series for $1/(1-x)$ and find that

$$g(x) = \frac{1}{1-2x} = 1 + (2x) + (2x)^2 + \cdots = \sum_{n=0}^{\infty} (2x)^n.$$

This geometric series has a ratio $r = 2x$ and converges if $|r| = |2x| < 1$. In other words, the interval of convergence is $|x| < 1/2$.

Example 7.55. Find a power series for $f(x) = \frac{3x-1}{x^2-1}$.

Solution. Using partial fraction we have

$$\frac{3x-1}{x^2-1} = \frac{2}{x+1} + \frac{1}{x-1}.$$

By adding the two geometric power series

$$\frac{2}{x+1} = \frac{2}{1-(-x)} = \sum_{n=0}^{\infty} 2(-1)^n x^n, \quad |x| < 1$$

and

$$\frac{1}{x-1} = \frac{-1}{1-x} = -\sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

we obtain

$$\frac{3x-1}{x^2-1} = \sum_{n=0}^{\infty} 2(-1)^n x^n - \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} (2(-1)^n - 1) x^n$$

with an interval of convergence of $(-1, 1)$.

7.30 Exercises

Exercise 7.63. Find the radius and interval of convergence for each power series. For what values of x does the series converge absolutely or conditionally?

- $\displaystyle\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n}$
- $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} n x^n$
- $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (x+2)^n}{n}$

- $\displaystyle\sum_{n=1}^{\infty} \frac{n!x^n}{(2n)!}$
- $\displaystyle\sum_{n=0}^{\infty} \frac{nx^n}{4^n(n^2+1)}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{(3x-1)^n}{n^3+n}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x+2)^n}{n2^n}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n(\ln n)^2}$
- $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n^2+3}}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{n^n(3x+5)^n}{(2n)!}$
- $\displaystyle\sum_{n=0}^{\infty} (\ln n)x^n$
- $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-2)^n}{n3^n}$

Exercise 7.64. Show that the series

$$\sum_{n=1}^{\infty} \frac{\sin(n^3x)}{n^2}$$

converges for all values of x, but that

$$\sum_{n=1}^{\infty} \frac{d}{dx}\left(\frac{\sin(n^3x)}{n^2}\right)$$

diverges for all values of x. Why does this not contradict 7.13?

Exercise 7.65. Find the radius and interval of convergence for each power series. For what values of x does the series converge absolutely or conditionally?

- $\displaystyle\sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)!}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n!}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-4)^n}{n9^n}$

- $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^{n-1}}{3^{n-1}}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{n!x^n}{(2n)!}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^{n+1}}{(n+1)4^{n=1}}$

Exercise 7.66. Find a power series representation for the function and determine the interval of convergence.

- $f(x) = \dfrac{1}{1+x}$
- $f(x) = \dfrac{1}{9+x^2}$
- $f(x) = \dfrac{1+x}{1-x}$
- $f(x) = \dfrac{3}{1-x^4}$
- $f(x) = \dfrac{x^2}{(1-2x)^2}$
- $f(x) = \arctan(x/3)$

Exercise 7.67.

(a) Use differentiation to find a power series representation for

$$f(x) = \frac{1}{(1+x)^2}$$

Find the radius of convergence. Use part (a) to find a power series for

$$f(x) = \frac{1}{(1+x)^3}.$$

Use part (b) to find a power series for

$$f(x) = \frac{x^2}{(1+x)^3}.$$

Exercise 7.68. Use the power series for $\tan^{-1} x$ to prove the following expression for as the sum of an in?nite series:

$$\pi = 2\sqrt{3}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}.$$

Exercise 7.69. Suppose that f and g are functions defined by

$$f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad \text{and} \qquad g(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}.$$

- Find the intervals of convergence of f and g. - Show that $f'(x)=g(x)$ and $g'(x)=-f(x)$. - Identify the functions f and g.

Exercise 7.70. Find a power series representation for the function, centered at c, and determine the interval of convergence.

- $f(x)=\dfrac{1}{3-x}$, $c=1$
- $f(x)=\dfrac{5}{2x-3}$, $c=-3$
- $f(x)=\dfrac{3}{3x+4}$, $c=0$
- $f(x)=\dfrac{4x}{x^2+2x-3}$, $c=0$
- $f(x)=\dfrac{5}{5+x^2}$, $c=\pi/4$
- $f(x)=\dfrac{2}{6-x}$, $c=-2$

Exercise 7.71. Use the power series $\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^n x^n$ to determine a power series, centered at 0, for the function. Identitfy the interval of convergence.

- $h(x)=\dfrac{x}{x^2-1}=\dfrac{1}{2(1+x)}-\dfrac{1}{2(1-x)}$
- $h(x)=\dfrac{2}{(x+1)^3}\dfrac{d^2}{dx^2}\left(\dfrac{1}{x+1}\right)$
- $h(x)=\ln(1-x^2)\displaystyle\int\frac{1}{1+x}\,dx-\int\frac{1}{1-x}\,dx$

7.31 Taylor and Maclaurin Series

The Taylor series of a function is a representation as power series whose terms are calculated from the values of the function's derivatives at a single point (the center). If the Taylor series is centered at zero, then that series is also called a Maclaurin series. We discuss the Maclaurin series of the sine and cosine functions and examine precisely when the Maclaurin series for thses functions converges.

7.32 Taylor Series and Maclaurin Series

It is not a big step to extend the notions of Maclaurin and Taylor polynomials to series by not stopping the summation index at n. Thus, we have the following definition.

Definition 7.16. If f is represented by a power series $f(x) = \sum a_n (x-c)^n$ for all x in an open interval I containing c, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n \tag{7.16}$$

where $f^{(0)}$ denotes f.

Example 7.56. Find the Maclaurin series for $f(x) = e^x$.

Solution. The nth Maclaurin polynomial for e^x is

$$p_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}.$$

Thus the Maclaurin series for e^x is

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

as needed.

Example 7.57. Find the Maclaurin series for $f(x) = \sin x$.

Solution. The nth Maclaurin polynomial for $\sin x$ is

$$p_n(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!}.$$

Thus the Maclaurin series for $\sin x$ is

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

as needed.

Example 7.58. Find the Maclaurin series for $f(x) = \cos x$.

Solution. The nth Maclaurin polynomial for $\cos x$ is

$$p_n(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!},$$

Thus the Maclaurin series for $\cos x$ is

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

as needed.

Example 7.59. Find the Maclaurin series for $f(x) = 1/(1-x)$.

Solution. The nth Maclaurin polynomial for $1/(1-x)$ is

$$p_n(x) = \sum_{k=0}^{n} x^k.$$

Thus the Maclaurin series for $\cos x$ is

$$\sum_{n=0}^{\infty} x^n$$

as needed.

7.33 Convergence of Taylor Series

Our next result is often called **Taylor's Theorem** and the remainder given in the theorem is called the **Lagrange form of the remainder**.

Theorem 7.14. *Let f have derivatives of order through $n+1$ in an open interval I centered at c, then, for each x in I, there exists z between x and c such that*

$$f(x) = \sum_{n=0}^{n} \frac{f^{(n)}(c)}{n!}(x-c)^n + R_n(x)$$

where

$$R_n(x) = \frac{f^{n+1}(z)}{(n+1)!}(x-c)^{n+1}.$$

Given a function f, we know how to write its Taylor series centered at a point c, and we know how to find its interval of convergence. How do we know that the series actually converges to f? The remaining task is to determine when the Taylor series for f actually converges to f on its interval of convergence.

Theorem 7.15. *If* $\lim_{n\to\infty} R_n = 0$ *for all* x *in the interval* I*, then the Taylor series for* f *converges and equals to* $f(x)$*,*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

Example 7.60. Show that the Maclaurin series for $f(x) = e^x$ converges to $f(x)$ for all x.

Solution. The Maclaurin series for e^x is

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

which converges for all real numbers. The Taylor remainder is

$$R_n(x) = \frac{e^z}{(n+1)!}x^{n+1}$$

for some z between x and 0. Thus

$$|R_n(x)| \le \frac{e^{|x|}}{(n+1)!}|x|^{n+1}.$$

Therefore, for arbitrary x fixed, we find that

$$\lim_{n\to\infty} |R_n(x)| = \lim_{n\to\infty} \left| \frac{e^c}{(n+1)!}x^{n+1} \right| = 0$$

By 7.15, the Taylor series for e^x converges to e^x.

Example 7.61. Show that the Maclaurin series for $f(x) = \sin x$ converges to $f(x)$ for all x.

Solution. The Maclaurin series for $\sin x$ is

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

which converges for all real numbers. Notice that $f^{(n+1)}(z) = \pm \sin c$ or $f^{(n+1)}(z) = \pm \cos z$. In any case we have that $|f^{n+1}(z)| \le 1$ for all real numbers z. Thus

$$|R_n(x)| = \frac{f^{(n+1)}(z)}{(n+1)!}|x|^{n+1} \le \frac{|x|^{n+1}}{(n+1)!}.$$

Therefore, for arbitrary x fixed, we find that

$$\lim_{n\to\infty} |R_n(x)| = \lim_{n\to\infty} \left| \frac{|x|^{n+1}}{(n+1)!} \right| = 0$$

By 7.15, the Taylor series for $\sin x$ converges to $\sin x$.

7.34 Exercises

Exercise 7.72. Find the Taylor polynomial $p_n(x)$ and the Taylor remainder $R_n(x)$ for the function f and the values of c and n.

- $f(x) = x^4 + 3x^3 + 2x + 3,\ c = -1,\ n = 4$
- $f(x) = \sqrt{x},\ c = 4,\ n = 3$
- $f(x) = \cos x,\ c = \pi/4,\ n = 3$
- $f(x) = \cos 2x,\ c = \pi/6,\ n = 3$

Exercise 7.73. Use the definition of Taylor series to find the Taylor series, centered at c, for the function.

- $f(x) = \dfrac{1}{x},\ c = 1$
- $f(x) = xe^x,\ c = 1$
- $f(x) = \sin 3x,\ c = 0$
- $f(x) = e^x,\ c = 1$
- $f(x) = \tan x,\ c = \pi/4$
- $\tan 2x,\ c = 0$

Exercise 7.74. Prove that the Maclaurin series for the function converges to the function for all x.

- $f(x) = \sin x$
- $f(x) = e^{-x}$
- $f(x) = e^{-2x}$
- $f(x) = \sin + \cos x$

Exercise 7.75. Use the binomial series to find the Maclaurin series for the function.

- $f(x)\dfrac{1}{\sqrt{1-x}}$
- $f(x) = \dfrac{1}{4+x^2}$
- $f(x) = \dfrac{1}{(1+x)^4}$
- $f(x) = \sqrt[4]{1+x}$

Exercise 7.76. Find the Maclaurin series for the function.

- $f(x) = e^{x^2/3}$
- $f(x) = x^2 \sin x$

- $g(x) = e^{-4x}$
- $h(x) = \cos x^{3/2}$
- $f(x) = \cos^2 x$
- $h(x) = x^2 \cos x^2$

References

Index