Problem 1.

(a) We assume the marble is a quantum particle with the given mass and energy in a well of the given length. From the energy levels of the infinite square well:

$$\begin{aligned}
E_n &= \frac{\hbar^2 n^2 \pi^2}{2mL^2} \\
\implies n &= \sqrt{\frac{2mL^2 E_n^2}{\hbar^2\pi^2}} \\
&= \frac{\sqrt{2mLE_n}}{\hbar\pi} \\
&= \frac{\sqrt{2(1\times 10^{-3}\text{kg})}(1\times 10^{-2}\text{m})(1\times 10^{-3}\text{J})}{(1.05\times 10^{-34}\text{J}\cdot\text{s})\pi} \\
&= 1.356\times 10^{27}.
\end{aligned}$$

(b) The excitation energy is

$$\begin{aligned}
E_{n+1} - E_n &= \frac{\hbar^2\pi^2}{2mL^2}[(n+1)^2 - n^2] \\
&= \frac{(2n+1)\hbar^2\pi^2}{2mL^2} \\
&= \frac{[2(1.356\times 10^{27})+1](1.05\times 10^{-34}\text{J}\cdot\text{s})^2\pi^2}{2(1\times 10^{-3}\text{kg})(1\times 10^{-2}\text{m})^2} \\
&= 0.141\text{J}.
\end{aligned}$$

Problem 2. [TODO]

Problem 3.

(a) We have that

$$\begin{aligned}
\langle x\rangle &= \int_0^L \mathrm{d}x\, \psi_n^* x \psi_n \\
&= \frac{2}{L}\int_0^L \mathrm{d}x\, x\sin\left(\frac{n\pi x}{L}\right)^2 \\
&= \frac{1}{L}\int_0^L \mathrm{d}x\, x\left[1-\cos\left(\frac{2n\pi x}{L}\right)\right] \\
&= \frac{1}{L}\cdot\frac{L^2}{2} - \int_0^L \mathrm{d}x\, x\cos\left(\frac{2n\pi x}{L}\right) \\
&= \frac{L}{2} - \left[x\frac{L}{2n\pi}\sin\left(\frac{2n\pi x}{L}\right) + \left(\frac{L}{2n\pi}\right)^2\cos\left(\frac{2n\pi x}{L}\right)\right]\Bigg|_0^L \qquad \text{(repeated IBP)} \\
&= \frac{L}{2}.
\end{aligned}$$

(b) The momentum operator is $\hat{p} = (\hbar/i)\,\partial/\partial x$, thus

$$\begin{aligned}
\langle p\rangle &= \int_0^L \mathrm{d}x\, \psi_n^* \frac{\hbar}{i}\frac{\partial}{\partial x}\psi_n \\
&= \frac{\hbar}{i}\frac{2}{L}\frac{n\pi}{L}\int_0^L \mathrm{d}x \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2n\pi\hbar}{iL^2}\int_0^L \mathrm{d}x\,\sin\left(\frac{2n\pi x}{L}\right)\\
&= 0.
\end{aligned}
$$

(c) We have that

$$
\begin{aligned}
\langle x^2\rangle &= \int_0^L \mathrm{d}x\,\psi_n^* x^2 \psi_n\\
&= \frac{2}{L}\int_0^L \mathrm{d}x\,x^2\sin\left(\frac{n\pi x}{L}\right)^2\\
&= \frac{1}{L}\int_0^L \mathrm{d}x\,x^2\left[1-\cos\left(\frac{2n\pi x}{L}\right)\right]\\
&= \frac{1}{L}\cdot\frac{L^3}{3}-\frac{1}{L}\int_0^L \mathrm{d}x\,x^2\cos\left(\frac{2n\pi x}{L}\right)\\
&= \frac{L^2}{3}-\frac{1}{L}\left[x^2\frac{L}{2n\pi}\sin\left(\frac{2n\pi x}{L}\right)+2x\left(\frac{L}{2n\pi}\right)^2\cos\left(\frac{2n\pi x}{L}\right)+2\left(\frac{L}{2n\pi}\right)^3\sin\left(\frac{2n\pi x}{L}\right)\right]\Bigg|_0^L\\
&= \frac{L^2}{3}-2\left(\frac{L}{2n\pi}\right)^2\\
&= \frac{(2n^2\pi^2-3)L^2}{6n^2\pi^2}.
\end{aligned}
$$

(d) We have that

$$
\begin{aligned}
\langle p^2\rangle &= -\hbar^2\int_0^L \mathrm{d}x\,\psi_n^*\frac{\partial^2}{\partial x^2}\psi_n\\
&= \hbar^2\frac{2}{L}\frac{n^2\pi^2}{L^2}\int_0^L \mathrm{d}x\,\sin\left(\frac{n\pi x}{L}\right)^2\\
&= \frac{n^2\pi^2\hbar^2}{L^3}\int_0^L \mathrm{d}x\,\left[1-\cos\left(\frac{n\pi x}{L}\right)\right]\\
&= \frac{n^2\pi^2\hbar^2}{L^2}.
\end{aligned}
$$

(e) By definition,

$$
\begin{aligned}
\Delta x &= \sqrt{\langle x^2\rangle-\langle x\rangle^2}\\
&= \sqrt{\frac{(2n^2\pi^2-3)L^2}{6n^2\pi^2}-\frac{L^2}{4}}\\
&= L\sqrt{\frac{4(2n^2\pi^2-3)-6n^2\pi^2}{24n^2\pi^2}}\\
&= \frac{L}{n\pi}\sqrt{\frac{2n^2\pi^2-3}{24}}.
\end{aligned}
$$

(f) By definition,

$$
\begin{aligned}
\Delta p &= \sqrt{\langle p^2\rangle-\langle p\rangle^2}\\
&= \sqrt{\frac{n^2\pi^2\hbar^2}{L^2}}\\
&= \frac{n\pi\hbar}{L}.
\end{aligned}
$$

(g) Thus

$$\begin{aligned}\Delta x \Delta p &= \hbar\sqrt{\frac{2n^2\pi^2 - 3}{24}} \\ &= \frac{\hbar}{2}\sqrt{\frac{2n^2\pi^2 - 3}{6}}.\end{aligned}$$

The lowest this value can possibly be is when $n = 1$, where

$$\Delta x \Delta p = \frac{\hbar}{2}\sqrt{\frac{2\pi^2 - 3}{6}} \approx 1.67 \cdot \frac{\hbar}{2}.$$

Problem 4.

(a) We have that

$$\begin{aligned}\langle x \rangle &= \int_{\mathbb{R}} \mathrm{d}x\, \psi_0^* x \psi_0 \\ &= C_0^2 \int_{\mathbb{R}} \mathrm{d}x\, x e^{-2ax^2} \\ &= 0 \quad \text{(odd function).}\end{aligned}$$

(b) We have that

$$\begin{aligned}\langle p \rangle &= \frac{\hbar}{i}\int_{\mathbb{R}} \mathrm{d}x\, \psi_0^* \frac{\partial}{\partial x}\psi_0 \\ &= -\frac{2a\hbar}{i}\int_{\mathbb{R}} \mathrm{d}x\, x e^{-2ax^2} \\ &= 0 \quad \text{(odd function).}\end{aligned}$$

(c) We have that

$$\begin{aligned}\langle x^2 \rangle &= \int_{\mathbb{R}} \mathrm{d}x\, \psi_0^* x^2 \psi_0 \\ &= \int_{\mathbb{R}} \mathrm{d}x\, x^2 e^{-2ax^2} \\ &= -\frac{1}{4a} x e^{-2ax^2}\bigg|_{-\infty}^{\infty} + \frac{1}{4a}\int_{\mathbb{R}} \mathrm{d}x\, e^{-2ax^2} \\ &= \frac{1}{4a}\sqrt{\frac{\pi}{2a}}.\end{aligned}$$

(d) We have that

$$\begin{aligned}\langle p^2 \rangle &= -\hbar^2 \int_{\mathbb{R}} \mathrm{d}x\, \psi_0^* \frac{\partial^2}{\partial x^2}\psi_0 \\ &= -\hbar \int_{\mathbb{R}} \mathrm{d}x\, e^{-ax^2}\left(2ae^{-ax^2} - 4a^2x^2e^{-ax^2}\right) \\ &= -2a\hbar^2 \int_{\mathbb{R}} \mathrm{d}x\, e^{-2ax^2} + 4a^2\hbar^2 \int_{\mathbb{R}} \mathrm{d}x\, x^2 e^{-2ax^2} \\ &= -2a\hbar^2\sqrt{\frac{\pi}{2a}} + 4a^2\hbar^2 \frac{1}{4a}\sqrt{\frac{\pi}{2a}}\end{aligned}$$

$$= -\sqrt{\frac{\pi a}{2}}\hbar^2.$$

[TODO]

(e) We have that

$$\begin{aligned}\Delta x &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \\ &= \sqrt{\frac{1}{4a}\sqrt{\frac{\pi}{2a}} - 0^2} \\ &= \frac{\pi^{1/4}}{2^{3/2}a^{3/4}}.\end{aligned}$$