

Problem 1.

- (a) We assume the marble is a quantum particle with the given mass and energy in a well of the given length. From the energy levels of the infinite square well:

$$\begin{aligned}
 E_n &= \frac{\hbar^2 n^2 \pi^2}{2mL^2} \\
 \Rightarrow n &= \sqrt{\frac{2mL^2 E_n}{\hbar^2 \pi^2}} \\
 &= \frac{\sqrt{2mLE_n}}{\hbar \pi} \\
 &= \frac{\sqrt{2(1 \times 10^{-3} \text{kg})(1 \times 10^{-2} \text{m})(1 \times 10^{-3} \text{J})}}{(1.05 \times 10^{-34} \text{J} \cdot \text{s})\pi} \\
 &= 1.356 \times 10^{27}.
 \end{aligned}$$

- (b) The excitation energy is

$$\begin{aligned}
 E_{n+1} - E_n &= \frac{\hbar^2 \pi^2}{2mL^2} [(n+1)^2 - n^2] \\
 &= \frac{(2n+1)\hbar^2 \pi^2}{2mL^2} \\
 &= \frac{[2(1.356 \times 10^{27}) + 1](1.05 \times 10^{-34} \text{J} \cdot \text{s})^2 \pi^2}{2(1 \times 10^{-3} \text{kg})(1 \times 10^{-2} \text{m})^2} \\
 &= 0.141 \text{J}.
 \end{aligned}$$

Problem 2. [TODO]**Problem 3.**

- (a) We have that

$$\begin{aligned}
 \langle x \rangle &= \int_0^L dx \psi_n^* x \psi_n \\
 &= \frac{2}{L} \int_0^L dx x \sin\left(\frac{n\pi x}{L}\right)^2 \\
 &= \frac{1}{L} \int_0^L dx x \left[1 - \cos\left(\frac{2n\pi x}{L}\right)\right] \\
 &= \frac{1}{L} \cdot \frac{L^2}{2} - \int_0^L dx x \cos\left(\frac{2n\pi x}{L}\right) \\
 &= \frac{L}{2} - \left[x \frac{L}{2n\pi} \sin\left(\frac{2n\pi x}{L}\right) + \left(\frac{L}{2n\pi}\right)^2 \cos\left(\frac{2n\pi x}{L}\right) \right] \Big|_0^L \quad (\text{repeated IBP}) \\
 &= \frac{L}{2}.
 \end{aligned}$$

- (b) The momentum operator is $\hat{p} = (\hbar/i) \partial/\partial x$, thus

$$\begin{aligned}
 \langle p \rangle &= \int_0^L dx \psi_n^* \frac{\hbar}{i} \frac{\partial}{\partial x} \psi_n \\
 &= \frac{\hbar}{i} \frac{2}{L} \frac{n\pi}{L} \int_0^L dx \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right)
 \end{aligned}$$

$$\begin{aligned}
&= \frac{2n\pi\hbar}{iL^2} \int_0^L dx \sin\left(\frac{2n\pi x}{L}\right) \\
&= 0.
\end{aligned}$$

(c) We have that

$$\begin{aligned}
\langle x^2 \rangle &= \int_0^L dx \psi_n^* x^2 \psi_n \\
&= \frac{2}{L} \int_0^L dx x^2 \sin\left(\frac{n\pi x}{L}\right)^2 \\
&= \frac{1}{L} \int_0^L dx x^2 \left[1 - \cos\left(\frac{2n\pi x}{L}\right)\right] \\
&= \frac{1}{L} \cdot \frac{L^3}{3} - \frac{1}{L} \int_0^L dx x^2 \cos\left(\frac{2n\pi x}{L}\right) \\
&= \frac{L^2}{3} - \frac{1}{L} \left[x^2 \frac{L}{2n\pi} \sin\left(\frac{2n\pi x}{L}\right) + 2x \left(\frac{L}{2n\pi}\right)^2 \cos\left(\frac{2n\pi x}{L}\right) + 2 \left(\frac{L}{2n\pi}\right)^3 \sin\left(\frac{2n\pi x}{L}\right) \right] \Big|_0^L \\
&= \frac{L^2}{3} - 2 \left(\frac{L}{2n\pi}\right)^2 \\
&= \frac{(2n^2\pi^2 - 3)L^2}{6n^2\pi^2}.
\end{aligned}$$

(d) We have that

$$\begin{aligned}
\langle p^2 \rangle &= -\hbar^2 \int_0^L dx \psi_n^* \frac{\partial^2}{\partial x^2} \psi_n \\
&= \hbar^2 \frac{2}{L} \frac{n^2\pi^2}{L^2} \int_0^L dx \sin\left(\frac{n\pi x}{L}\right)^2 \\
&= \frac{n^2\pi^2\hbar^2}{L^3} \int_0^L dx \left[1 - \cos\left(\frac{2n\pi x}{L}\right)\right] \\
&= \frac{n^2\pi^2\hbar^2}{L^2}.
\end{aligned}$$

(e) By definition,

$$\begin{aligned}
\Delta x &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \\
&= \sqrt{\frac{(2n^2\pi^2 - 3)L^2}{6n^2\pi^2} - \frac{L^2}{4}} \\
&= L \sqrt{\frac{4(2n^2\pi^2 - 3) - 6n^2\pi^2}{24n^2\pi^2}} \\
&= \frac{L}{n\pi} \sqrt{\frac{2n^2\pi^2 - 3}{24}}.
\end{aligned}$$

(f) By definition,

$$\begin{aligned}
\Delta p &= \sqrt{\langle p^2 \rangle - \langle p \rangle^2} \\
&= \sqrt{\frac{n^2\pi^2\hbar^2}{L^2}} \\
&= \frac{n\pi\hbar}{L}.
\end{aligned}$$

(g) Thus

$$\begin{aligned}\Delta x \Delta p &= \hbar \sqrt{\frac{2n^2\pi^2 - 3}{24}} \\ &= \frac{\hbar}{2} \sqrt{\frac{2n^2\pi^2 - 3}{6}}.\end{aligned}$$

The lowest this value can possibly be is when $n = 1$, where

$$\Delta x \Delta p = \frac{\hbar}{2} \sqrt{\frac{2\pi^2 - 3}{6}} \approx 1.67 \cdot \frac{\hbar}{2}.$$

Problem 4.

(a) We have that

$$\begin{aligned}\langle x \rangle &= \int_{\mathbb{R}} dx \psi_0^* x \psi_0 \\ &= C_0^2 \int_{\mathbb{R}} dx x e^{-2ax^2} \\ &= 0 \quad (\text{odd function}).\end{aligned}$$

(b) We have that

$$\begin{aligned}\langle p \rangle &= \frac{\hbar}{i} \int_{\mathbb{R}} dx \psi_0^* \frac{\partial}{\partial x} \psi_0 \\ &= -\frac{2a\hbar}{i} \int_{\mathbb{R}} dx x e^{-2ax^2} \\ &= 0 \quad (\text{odd function}).\end{aligned}$$

(c) We have that

$$\begin{aligned}\langle x^2 \rangle &= \int_{\mathbb{R}} dx \psi_0^* x^2 \psi_0 \\ &= \int_{\mathbb{R}} dx x^2 e^{-2ax^2} \\ &= -\frac{1}{4a} x e^{-2ax^2} \Big|_{-\infty}^{\infty} + \frac{1}{4a} \int_{\mathbb{R}} dx e^{-2ax^2} \\ &= \frac{1}{4a} \sqrt{\frac{\pi}{2a}}.\end{aligned}$$

(d) We have that

$$\begin{aligned}\langle p^2 \rangle &= -\hbar^2 \int_{\mathbb{R}} dx \psi_0^* \frac{\partial^2}{\partial x^2} \psi_0 \\ &= -\hbar^2 \int_{\mathbb{R}} dx (4a^2 x^2 e^{-2ax^2} - 2a e^{-2ax^2}) \\ &= \sqrt{2\pi a} \hbar^2 - \frac{\sqrt{2\pi a}}{2} \hbar^2 \\ &= \frac{\sqrt{2\pi a}}{2} \hbar^2 \\ &= \sqrt{\frac{\pi a}{2}} \hbar^2\end{aligned}$$

(e) We have that

$$\begin{aligned}\Delta x &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \\ &= \sqrt{\frac{1}{4a} \sqrt{\frac{\pi}{2a}} - 0^2} \\ &= \frac{\pi^{1/4}}{2^{3/2} a^{3/4}}.\end{aligned}$$

(f) We have that

$$\begin{aligned}\Delta p &= \sqrt{\langle p^2 \rangle - \langle p \rangle^2} \\ &= \sqrt{\sqrt{\frac{\pi a}{2}} \hbar^2 - 0^2} \\ &= \left(\frac{\pi a}{2}\right)^{1/4} \hbar.\end{aligned}$$

(g) Therefore,

$$\begin{aligned}\Delta x \Delta p &= \frac{\pi^{1/4}}{2^{3/2} a^{3/4}} \left(\frac{\pi a}{2}\right)^{1/4} \hbar \\ &= \end{aligned}$$