Problem 1.

(a) We assume the marble is a quantum particle with the given mass and energy in a well of the given length. From the energy levels of the infinite square well:

$$\begin{aligned} E_n &= \frac{\hbar^2 n^2 \pi^2}{2mL^2} \\ \implies n &= \sqrt{\frac{2mL^2 E_n^2}{\hbar^2 \pi^2}} \\ &= \frac{\sqrt{2m}LE_n}{\hbar \pi} \\ &= \frac{\sqrt{2(1\times 10^{-3}\text{kg})}(1\times 10^{-2}\text{m})(1\times 10^{-3}\text{J})}{(1.05\times 10^{-34}\text{J}\cdot\text{s})\pi} \\ &= 1.356 \times 10^{27}. \end{aligned}$$

(b) The excitation energy is

$$\begin{aligned} E_{n+1} - E_n &= \frac{\hbar^2 \pi^2}{2mL^2}[(n+1)^2 - n^2] \\ &= \frac{(2n+1)\hbar^2\pi^2}{2mL^2} \\ &= \frac{[2(1.356\times 10^{27})+1](1.05\times 10^{-34}\text{J}\cdot\text{s})^2\pi^2}{2(1\times 10^{-3}\text{kg})(1\times 10^{-2}\text{m})^2} \\ &= 0.141\text{J}. \end{aligned}$$

Problem 2.

(a) We solve the Schrödinger equation in each of the three regions. Factoring $\Psi(x,t) = \psi(x)T(t)$, we always have that $T = \exp(-iE_n t/\hbar)$. For the region in which $V = V_0$:

$$\begin{aligned} -\frac{\hbar^2}{2m}\psi''(x) + V_0\psi(x) &= E_n\psi(x) \\ \implies \psi'' &= \frac{2m(V_0 - E_n)}{\hbar^2}\psi. \end{aligned}$$

In the middle region, the coefficient of ψ is some positive value β^2. In the other two regions, the resulting equation for ψ is the same except $V_0 = 0$, yielding a negative coefficient k^2. We thus have that ψ is the piecewise function

$$\psi(x) = \begin{cases} \psi^1, & x \in (-\infty, -a] \\ \psi^2, & x \in [-a, a] \\ \psi^3, & x \in [a, \infty), \end{cases}$$

where

$$\begin{aligned} \psi^1(x) &= Ae^{ikx} + Be^{-ikx} \\ \psi^2(x) &= Ce^{\beta x} + De^{-\beta x} \\ \psi^3(x) &= Ee^{ikx} + Fe^{-ikx}. \end{aligned}$$

The coefficient E is not to be confused with the energy E_n. Since wavefunctions are invariant under scaling, we can set $A = 1$. And as there is no possible reflection in the region $[a, \infty)$, it must be that $F = 0$. Thus we have

$$\psi^1(x) = e^{ikx} + Be^{-ikx}$$

$$\psi^2(x) = Ce^{\beta x} + De^{-\beta x}$$
$$\psi^3(x) = Ee^{ikx}.$$

(b) Let $z = e^{ika}$, $w = e^{\beta a}$, $\bar{z} = 1/z$, and $\bar{w} = 1/w$. To ensure $C^0(\mathbb{R})$ continuity, we must have

$$\psi^1(-a) = \psi^2(-a) \implies \bar{z} + Bz = C\bar{w} + Dw$$
$$\psi^1(a) = \psi^2(a) \implies Cw + D\bar{w} = Ez$$

(c) To ensure $C^1(\mathbb{R})$ continuity, we must have

$$(\psi^1)'(-a) = (\psi^2)'(-a) \implies ik\bar{z} - ikBz = \beta C\bar{w} - \beta Dw$$
$$(\psi^2)'(a) = (\psi^3)'(a) \implies \beta Cw - \beta D\bar{w} = ikEz$$

Solving this 4×4 system of equations:

$$B = \frac{\bar{z}(-\bar{w} + w)(\bar{w} + w)(\beta^2 + k^2)}{z(-i\bar{w}\beta + \bar{w}k + i\beta w + kw)(i\bar{w}\beta - \bar{w}k + i\beta w + kw)}$$
$$C = -\frac{2\bar{w}\bar{z}k(-i\beta + k)}{(-i\bar{w}\beta + \bar{w}k + i\beta w + kw)(i\bar{w}\beta - \bar{w}k + i\beta w + kw)}$$
$$D = \frac{2\bar{z}kw(i\beta + k)}{(-i\bar{w}\beta + \bar{w}k + i\beta w + kw)(i\bar{w}\beta - \bar{w}k + i\beta w + kw)}$$
$$E = \frac{4i\bar{z}\beta k}{z(-i\bar{w}\beta + \bar{w}k + i\beta w + kw)(i\bar{w}\beta - \bar{w}k + i\beta w + kw)}$$

Substituting $w + \bar{w} = 2\cosh(\beta a)$ and $w - \bar{w} = 2\sinh(\beta a)$:

$$B = \frac{\bar{z}\cosh(\beta a)\sinh(\beta a)(\beta^2 + k^2)}{z(k\cosh(\beta a) + i\beta\sinh(\beta a))(k\sinh(\beta a) + i\beta\cosh(\beta a))}$$
$$C = -\frac{\bar{w}\bar{z}k(-i\beta + k)}{2(k\cosh(\beta a) + i\beta\sinh(\beta a))(k\sinh(\beta a) + i\beta\cosh(\beta a))}$$
$$D = \frac{\bar{z}kw(i\beta + k)}{2(k\cosh(\beta a) + i\beta\sinh(\beta a))(k\sinh(\beta a) + i\beta\cosh(\beta a))}$$
$$E = \frac{i\bar{z}\beta k}{z(k\cosh(\beta a) + i\beta\sinh(\beta a))(k\sinh(\beta a) + i\beta\cosh(\beta a))}$$

Expanding the denominator:

$$B = \frac{2\bar{z}^2\cosh(\beta a)\sinh(\beta a)(\beta^2 + k^2)}{(k^2 - \beta^2)\sinh(2\beta a) + 2\beta ki\cosh(2\beta a)}$$
$$C = -\frac{\bar{w}\bar{z}k(-i\beta + k)}{(k^2 - \beta^2)\sinh(2\beta a) + 2\beta ki\cosh(2\beta a)}$$
$$D = \frac{\bar{z}kw(i\beta + k)}{(k^2 - \beta^2)\sinh(2\beta a) + 2\beta ki\cosh(2\beta a)}$$
$$E = \frac{2i\bar{z}^2\beta k}{(k^2 - \beta^2)\sinh(2\beta a) + 2\beta ki\cosh(2\beta a)}$$

The maximum amplitude of ψ^1 is $1 + |B|$, and the maximum amplitude of ψ^3 is $|E|$. Thus, the transmission coefficient is

$$\begin{aligned}\tau &= \frac{|E|}{1 + |B|} \\ &= \frac{2\beta k}{2(\beta^2 + k^2)\cosh(\beta a)\sinh(\beta a) + \sqrt{(k^2 - \beta^2)^2\sinh^2(2\beta a) + 4\beta^2k^2\cosh^2(2\beta a)}}.\end{aligned}$$

Problem 3.

(a) We have that

$$
\begin{aligned}
\langle x\rangle &= \int_0^L \mathrm{d}x\, \psi_n^* x \psi_n \\
&= \frac{2}{L}\int_0^L \mathrm{d}x\, x \sin\left(\frac{n\pi x}{L}\right)^2 \\
&= \frac{1}{L}\int_0^L \mathrm{d}x\, x \left[1-\cos\left(\frac{2n\pi x}{L}\right)\right] \\
&= \frac{1}{L}\cdot\frac{L^2}{2} - \int_0^L \mathrm{d}x\, x\cos\left(\frac{2n\pi x}{L}\right) \\
&= \frac{L}{2} - \left[x\frac{L}{2n\pi}\sin\left(\frac{2n\pi x}{L}\right) + \left(\frac{L}{2n\pi}\right)^2\cos\left(\frac{2n\pi x}{L}\right)\right]\Bigg|_0^L \quad \text{(repeated IBP)} \\
&= \frac{L}{2}.
\end{aligned}
$$

(b) The momentum operator is $\hat{p} = (\hbar/i)\,\partial/\partial x$, thus

$$
\begin{aligned}
\langle p\rangle &= \int_0^L \mathrm{d}x\, \psi_n^* \frac{\hbar}{i}\frac{\partial}{\partial x}\psi_n \\
&= \frac{\hbar}{i}\frac{2}{L}\frac{n\pi}{L}\int_0^L \mathrm{d}x \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right) \\
&= \frac{2n\pi\hbar}{iL^2}\int_0^L \mathrm{d}x \sin\left(\frac{2n\pi x}{L}\right) \\
&= 0.
\end{aligned}
$$

(c) We have that

$$
\begin{aligned}
\langle x^2\rangle &= \int_0^L \mathrm{d}x\, \psi_n^* x^2 \psi_n \\
&= \frac{2}{L}\int_0^L \mathrm{d}x\, x^2 \sin\left(\frac{n\pi x}{L}\right)^2 \\
&= \frac{1}{L}\int_0^L \mathrm{d}x\, x^2 \left[1-\cos\left(\frac{2n\pi x}{L}\right)\right] \\
&= \frac{1}{L}\cdot\frac{L^3}{3} - \frac{1}{L}\int_0^L \mathrm{d}x\, x^2\cos\left(\frac{2n\pi x}{L}\right) \\
&= \frac{L^2}{3} - \frac{1}{L}\left[x^2\frac{L}{2n\pi}\sin\left(\frac{2n\pi x}{L}\right) + 2x\left(\frac{L}{2n\pi}\right)^2\cos\left(\frac{2n\pi x}{L}\right) + 2\left(\frac{L}{2n\pi}\right)^3\sin\left(\frac{2n\pi x}{L}\right)\right]\Bigg|_0^L \\
&= \frac{L^2}{3} - 2\left(\frac{L}{2n\pi}\right)^2 \\
&= \frac{(2n^2\pi^2-3)L^2}{6n^2\pi^2}.
\end{aligned}
$$

(d) We have that

$$
\langle p^2\rangle = -\hbar^2\int_0^L \mathrm{d}x\, \psi_n^* \frac{\partial^2}{\partial x^2}\psi_n
$$

$$
\begin{aligned}
&= \hbar^2 \frac{2}{L} \frac{n^2\pi^2}{L^2} \int_0^L \mathrm{d}x \sin\Big(\frac{n\pi x}{L}\Big)^2 \\
&= \frac{n^2\pi^2\hbar^2}{L^3} \int_0^L \mathrm{d}x \left[1 - \cos\Big(\frac{n\pi x}{L}\Big)\right] \\
&= \frac{n^2\pi^2\hbar^2}{L^2}.
\end{aligned}
$$

(e) By definition,

$$
\begin{aligned}
\Delta x &= \sqrt{\langle x^2\rangle - \langle x\rangle^2} \\
&= \sqrt{\frac{(2n^2\pi^2 - 3)L^2}{6n^2\pi^2} - \frac{L^2}{4}} \\
&= L\sqrt{\frac{4(2n^2\pi^2 - 3) - 6n^2\pi^2}{24n^2\pi^2}} \\
&= \frac{L}{n\pi}\sqrt{\frac{2n^2\pi^2 - 3}{24}}.
\end{aligned}
$$

(f) By definition,

$$
\begin{aligned}
\Delta p &= \sqrt{\langle p^2\rangle - \langle p\rangle^2} \\
&= \sqrt{\frac{n^2\pi^2\hbar^2}{L^2}} \\
&= \frac{n\pi\hbar}{L}.
\end{aligned}
$$

(g) Thus

$$
\begin{aligned}
\Delta x \Delta p &= \hbar\sqrt{\frac{2n^2\pi^2 - 3}{24}} \\
&= \frac{\hbar}{2}\sqrt{\frac{2n^2\pi^2 - 3}{6}}.
\end{aligned}
$$

The lowest this value can possibly be is when $n = 1$, where

$$
\Delta x \Delta p = \frac{\hbar}{2}\sqrt{\frac{2\pi^2 - 3}{6}} \approx 1.67 \cdot \frac{\hbar}{2}.
$$

Problem 4.

(a) We have that

$$
\begin{aligned}
\langle x\rangle &= \int_{\mathbb{R}} \mathrm{d}x\, \psi_0^* x \psi_0 \\
&= C_0^2 \int_{\mathbb{R}} \mathrm{d}x\, x e^{-2ax^2} \\
&= 0 \quad \text{(odd function)}.
\end{aligned}
$$

(b) We have that

$$
\begin{aligned}
\langle p\rangle &= \frac{\hbar}{i} \int_{\mathbb{R}} \mathrm{d}x\, \psi_0^* \frac{\partial}{\partial x} \psi_0 \\
&= -\frac{2a\hbar}{i} C_0^2 \int_{\mathbb{R}} \mathrm{d}x\, x e^{-2ax^2} \\
&= 0 \quad \text{(odd function)}.
\end{aligned}
$$

(c) We have that

$$\begin{aligned}\langle x^2\rangle &= \int_{\mathbb{R}} \mathrm{d}x\, \psi_0^* x^2 \psi_0 \\ &= C_0^2 \int_{\mathbb{R}} \mathrm{d}x\, x^2 e^{-2ax^2} \\ &= -\frac{C_0^2}{4a} x e^{-2ax^2}\bigg|_{-\infty}^{\infty} + \frac{C_0^2}{4a}\int_{\mathbb{R}} \mathrm{d}x\, e^{-2ax^2} \\ &= \frac{C_0^2}{8}\sqrt{\frac{2\pi}{a}}.\end{aligned}$$

(d) We have that

$$\begin{aligned}\langle p^2\rangle &= -\hbar^2 \int_{\mathbb{R}} \mathrm{d}x\, \psi_0^* \frac{\partial^2}{\partial x^2}\psi_0 \\ &= -\hbar^2 C_0^2 \int_{\mathbb{R}} \mathrm{d}x\, (4a^2x^2 e^{-2ax^2} - 2ae^{-2ax^2}) \\ &= \sqrt{2\pi a}\hbar^2 C_0^2 - \frac{\sqrt{2\pi a}}{2}\hbar^2 C_0^2 \\ &= \frac{\sqrt{2\pi a}}{2} C_0^2\hbar^2 \\ &= \sqrt{\frac{\pi a}{2}} C_0^2 \hbar^2\end{aligned}$$

(e) We have that

$$\begin{aligned}\Delta x &= \sqrt{\langle x^2\rangle - \langle x\rangle^2} \\ &= \sqrt{\frac{C_0^2}{8}\sqrt{\frac{2\pi}{a}} - 0^2} \\ &= \frac{2^{3/4}\pi^{1/4}C_0}{4a^{1/4}}\end{aligned}$$

(f) We have that

$$\begin{aligned}\Delta p &= \sqrt{\langle p^2\rangle - \langle p\rangle^2} \\ &= \sqrt{\sqrt{\frac{\pi a}{2}} C_0^2\hbar^2 - 0^2} \\ &= \left(\frac{\pi a}{2}\right)^{1/4} C_0\hbar.\end{aligned}$$

(g) Therefore,

$$\begin{aligned}\Delta x \Delta p &= \frac{2^{3/4}\pi^{1/4}C_0}{4a^{1/4}}\left(\frac{\pi a}{2}\right)^{1/4} C_0\hbar \\ &= \frac{\pi^{1/2}C_0^2}{2a^{1/2}}\frac{\hbar}{2} \\ &= \frac{\sqrt{2}}{4}\hbar \quad (????)\end{aligned}$$