Problem 1. Consider $\mathbb{S}^n$ as a subset of $\mathbb{R}^{n+1}$. Denote the north pole by $\mathbf{N} = (0, \dots, 0, 1)$ and the south pole by $\mathbf{S} = (0, \dots, 0, -1)$. For any point $\mathbf{p} = (x_1, \dots, x_{n+1}) \in \mathbb{S}^2$, we compute its north-pole and south-pole stereographic projections, ϕ_N and ϕ_S respectively, onto the equatorial hyperplane E of points $\mathbf{q} = (X_1, \dots, X_n, 0)$, and their associated inverses and transition functions.

We compute the north-pole projection $\phi_N : \mathbb{S}^2 \to E$. The line intersecting $\mathbf{N}$ and $\mathbf{p}$ is parameterized by

$$\mathbf{L}(t) = \mathbf{N} + t(\mathbf{p} - \mathbf{N}) = (tx_1, \dots, tx_n, 1 + t(x_{n+1} - 1)).$$

Solving for when this line intersects E:

$$0 = 1 + t(x_{n+1} - 1) \implies t = \frac{1}{1 - x_{n+1}}.$$

Hence

$$\phi_N(\mathbf{p}) = (X_1, \dots, X_n, 0) = \left(\frac{x_1}{1 - x_{n+1}}, \dots, \frac{x_n}{1 - x_{n+1}}, 0 \right).$$

We now compute ϕ_N^{-1}. For $\mathbf{q} = (X_1, \dots, X_n, 0) \in E$, the line intersecting $\mathbf{q}$ and $\mathbf{N}$ is parameterized by

$$\begin{aligned} \tilde{\mathbf{L}}(t) &= \mathbf{q} + t(\mathbf{N} - \mathbf{q}) \\ &= (X_1 - tX_1, \dots, X_n - tX_n, t) \\ &= ((1-t)X_1, \dots, (1-t)X_n, t). \end{aligned}$$

Solving for $\|\tilde{\mathbf{L}}\| = 1$:

$$\begin{aligned} 1 &= (1-t)^2 X_1^2 + \cdots + (1-t)^2 X_n^2 + t^2 \\ &= (1-t)^2 \|\mathbf{q}\|^2 + t^2 \\ \implies 0 &= (1-t)^2 \|\mathbf{q}\|^2 - (1 - t^2) \\ &= (1-t)((1-t)\|\mathbf{q}\|^2 - (1+t)). \end{aligned}$$

The solution $t = 1$ gives $\tilde{\mathbf{L}} = \mathbf{N}$. Hence, $\mathbf{p}$ occurs when

$$t = \frac{\|\mathbf{q}\|^2 - 1}{\|\mathbf{q}\|^2 + 1}.$$

Thus

$$\begin{aligned} \phi_N^{-1}(\mathbf{q}) &= (x_1, \dots, x_n, x_{n+1}) \\ &= \left(\left(1 - \frac{\|\mathbf{q}\|^2 - 1}{\|\mathbf{q}\|^2 + 1}\right) X_1, \dots, \left(1 - \frac{\|\mathbf{q}\|^2 - 1}{\|\mathbf{q}\|^2 + 1}\right) X_n, \frac{\|\mathbf{q}\|^2 - 1}{\|\mathbf{q}\|^2 + 1} \right) \\ &= \left(\frac{2}{\|\mathbf{q}\|^2 + 1} X_1, \dots, \frac{2}{\|\mathbf{q}\|^2 + 1} X_n, \frac{\|\mathbf{q}\|^2 - 1}{\|\mathbf{q}\|^2 + 1} \right). \end{aligned}$$

By symmetry, ϕ_S and ϕ_S^{-1} are given by the above formulas with $x_{n+1} \mapsto -x_{n+1}$:

$$\begin{aligned} \phi_S(\mathbf{p}) &= \left(\frac{x_1}{1 + x_{n+1}}, \dots, \frac{x_n}{1 + x_{n+1}}, 0 \right) \\ \phi_S^{-1}(\mathbf{q}) &= \left(\frac{2}{\|\mathbf{q}\|^2 + 1} X_1, \dots, \frac{2}{\|\mathbf{q}\|^2 + 1} X_n, -\frac{\|\mathbf{q}\|^2 - 1}{\|\mathbf{q}\|^2 + 1} \right). \end{aligned}$$

The transition functions are thus given by

$$\phi_S \circ \phi_N^{-1}(\mathbf{q}) = \phi_S \left(\frac{2}{\|\mathbf{q}\|^2 + 1} X_1, \dots, \frac{2}{\|\mathbf{q}\|^2 + 1} X_n, \frac{\|\mathbf{q}\|^2 - 1}{\|\mathbf{q}\|^2 + 1} \right)$$

$$= \left(\frac{\frac{2}{\|\mathbf{q}\|^2+1} X_1}{1 + \frac{\|\mathbf{q}\|^2-1}{\|\mathbf{q}\|^2+1}}, \ldots, \frac{\frac{2}{\|\mathbf{q}\|^2+1} X_n}{1 + \frac{\|\mathbf{q}\|^2-1}{\|\mathbf{q}\|^2+1}}, 0 \right)$$
$$= \left(\frac{X_1}{\|\mathbf{q}\|^2}, \ldots, \frac{X_n}{\|\mathbf{q}\|^2}, 0 \right)$$

and

$$\begin{aligned}
\phi_N \circ \phi_S^{-1}(\mathbf{q}) &= \phi_N \left(\frac{2}{\|\mathbf{q}\|^2+1} X_1, \ldots, \frac{2}{\|\mathbf{q}\|^2+1} X_n, -\frac{\|\mathbf{q}\|^2-1}{\|\mathbf{q}\|^2+1} \right) \\
&= \left(\frac{\frac{2}{\|\mathbf{q}\|^2+1} X_1}{1 + \frac{\|\mathbf{q}\|^2-1}{\|\mathbf{q}\|^2+1}}, \ldots, \frac{\frac{2}{\|\mathbf{q}\|^2+1} X_n}{1 + \frac{\|\mathbf{q}\|^2-1}{\|\mathbf{q}\|^2+1}}, 0 \right) \\
&= \left(\frac{X_1}{\|\mathbf{q}\|^2}, \ldots, \frac{X_n}{\|\mathbf{q}\|^2}, 0 \right).
\end{aligned}$$

Problem 2. Represent $\mathbb{T}^2$ by its fundamental polygon, fit to $[0,1]^2 \subset \mathbb{R}^2$. We choose for a chart the following open sets in $\mathbb{T}^2$:

$$\begin{aligned}
U_1 &= (0,1)^2 \\
U_2 &= ([0, \tfrac{1}{2}) \cup (\tfrac{1}{2}, 1]) \times (0,1) \\
U_3 &= (0,1) \times ([0, \tfrac{1}{2}) \cup (\tfrac{1}{2}, 1]) \\
U_4 &= ([0, \tfrac{1}{2}) \cup (\tfrac{1}{2}, 1])^2
\end{aligned}$$

together with the respective homeomorphisms $\phi_i : \mathbb{T}^2 \to \mathbb{R}^2$, $i = 1, \ldots, 4$:

$$\begin{array}{ll}
\phi_1(x,y) = (x,y), & \phi_1^{-1}(x,y) = (x,y) \\
\phi_2(x,y) = (x + \tfrac{1}{2} \mod 1, y) & \phi_2^{-1}(x,y) = (x - \tfrac{1}{2} \mod 1, y) \\
\phi_3(x,y) = (x, y + \tfrac{1}{2} \mod 1) & \phi_3^{-1}(x,y) = (x, y - \tfrac{1}{2} \mod 1) \\
\phi_4(x,y) = (x + \tfrac{1}{2} \mod 1, y + \tfrac{1}{2} \mod 1) & \phi_4^{-1}(x,y) = (x - \tfrac{1}{2} \mod 1, y - \tfrac{1}{2} \mod 1).
\end{array}$$

We thus have the following transition maps:

$$\begin{array}{ll}
\phi_1 \circ \phi_2^{-1}(x,y) = (x - \tfrac{1}{2} \mod 1, y), & \phi_2 \circ \phi_1^{-1}(x,y) = (x + \tfrac{1}{2} \mod 1, y) \\
\phi_1 \circ \phi_3^{-1}(x,y) = (x, y - \tfrac{1}{2} \mod 1), & \phi_3 \circ \phi_1^{-1}(x,y) = (x, y + \tfrac{1}{2} \mod 1) \\
\phi_1 \circ \phi_4^{-1}(x,y) = (x - \tfrac{1}{2} \mod 1, y - \tfrac{1}{2} \mod 1), & \phi_4 \circ \phi_1^{-1}(x,y) = (x + \tfrac{1}{2} \mod 1, y + \tfrac{1}{2} \mod 1) \\
\phi_2 \circ \phi_3^{-1}(x,y) = (x + \tfrac{1}{2} \mod 1, y - \tfrac{1}{2} \mod 1), & \phi_3 \circ \phi_2^{-1}(x,y) = (x - \tfrac{1}{2} \mod 1, y + \tfrac{1}{2} \mod 1) \\
\phi_2 \circ \phi_4^{-1}(x,y) = (x, y - \tfrac{1}{2} \mod 1), & \phi_4 \circ \phi_2^{-1}(x,y) = (x, y + \tfrac{1}{2} \mod 1) \\
\phi_3 \circ \phi_4^{-1}(x,y) = (x - \tfrac{1}{2} \mod 1, y), & \phi_4 \circ \phi_3^{-1}(x,y) = (x + \tfrac{1}{2} \mod 1, y).
\end{array}$$

Problem 3. We have $\mathbb{RP}^n = \{[x_0 : \cdots : x_n]\}$ with the atlas $\{(U_i, \phi_i)\}_{i=0}^n$ given by

$$\begin{aligned}
&U_i = \{[x_0 : \cdots : x_n] \mid x_i \neq 0\} \\
&\phi_i : U_i \to \mathbb{R}^n,\ [x_0 : \cdots : x_n] \mapsto \left(\frac{x_0}{x_i}, \ldots, \frac{\widehat{x_i}}{x_i}, \ldots, \frac{x_n}{x_i} \right).
\end{aligned}$$

We compute the inverse of the chart maps. For a fixed index $1 \leq i \leq n$, we label our coordinates in $\mathbb{R}^n$ by $(X_0, \ldots, \widehat{X_i}, \ldots, X_n)$. Then

$$\phi_i^{-1}(X_0, \ldots, \widehat{X_i}, \ldots, X_n) = [X_0 : \cdots : X_{i-1} : 1 : X_{i+1} : \cdots : X_n]$$

because

$$\phi_i([X_0 : \cdots : X_{i-1} : 1 : X_{i+1} : \cdots : X_n]) = \left(\frac{X_0}{1}, \ldots, \frac{X_{i-1}}{1}, \frac{X_{i+1}}{1}, \ldots, \frac{X_n}{1} \right).$$

Therefore, for $i \neq j$, the transition function $\phi_i \circ \phi_j^{-1}$ is given by

$$\begin{aligned}\phi_i \circ \phi_j^{-1}(X_0, \dots, \widehat{X_j}, \dots, X_n) &= \phi_i([X_0 : \cdots : X_{j-1} : 1 : X_{j+1} : X_n]) \\ &= \left(\frac{X_0}{X_i}, \dots, \frac{\widehat{X_i}}{X_i}, \dots, \frac{X_{j-1}}{X_i}, \frac{1}{X_i}, \frac{X_{j+1}}{X_i}, \dots, \frac{X_n}{X_i}\right).\end{aligned}$$

Problem 4. e choose for an atlas for $\mathbb{CP}^n$ one analogous to the atlas above for $\mathbb{RP}^n$. Consider $\{(U_i, \phi_i)\}_{i=0}^n$ given by

$$\begin{aligned}&U_i = \{[z_0 : \cdots : z_n] \mid z_i \neq 0\} \\ &\phi_i : U_i \to \mathbb{C}^n,\ [z_0 : \cdots : z_n] \mapsto \left(\frac{z_0}{z_i}, \dots, \frac{\widehat{z_i}}{z_i}, \dots, \frac{z_n}{z_i}\right).\end{aligned}$$

The inverses and transition functions are exactly as above, substituting $x_i \mapsto z_i$ and $X_i \mapsto Z_i$. For $i \neq j$, define the component functions $(f_1, \dots, \widehat{f_i}, \dots, f_n)$ of $\phi_i \circ \phi_j^{-1}$ by

$$\begin{aligned}f_k(Z_0, \dots, \widehat{Z_j}, \dots, Z_n) &= \frac{Z_k}{Z_i},\ k \neq i,\ k \neq j \\ f_j(Z-0, \dots, \widehat{Z_j}, \dots, Z_n) &= \frac{1}{Z_i}.\end{aligned}$$

In the first case, f_k is independent of and thus holomorphic with respect to all Z_r, where $r \neq k$ and $r \neq i$. Letting $Z_k = X_k + IY_j$ and $Z_i = X_i + IY_i$, where I is the imaginary unit, we get that

$$\begin{aligned}f_k &= \frac{X_k + IY_k}{X_i + IY_i} \\ &= \frac{X_kX_i + Y_kY_i}{X_i^2 + Y_i^2} + I\frac{Y_kX_i - X_kY_i}{X_i^2 + Y_i^2} \\ &:= u_k + Iv_k.\end{aligned}$$

Then we have that

$$\begin{aligned}\frac{\partial u_k}{\partial X_k} &= \frac{X_i}{X_i^2 + Y_i^2} \\ \frac{\partial v_k}{\partial Y_k} &= \frac{X_i}{X_i^2 + Y_i^2} \\ \frac{\partial u_k}{\partial Y_k} &= \frac{Y_i}{X_i^2 + Y_i^2} \\ \frac{\partial v_k}{\partial X_k} &= -\frac{Y_i}{X_i^2 + Y_i^2},\end{aligned}$$

satisfying the Cauchy-Riemann equations in Z_k, and

$$\begin{aligned}\frac{\partial u_k}{\partial X_i} &= \frac{X_k(Y_i^2 - X_i^2)}{(X_i^2 + Y_i^2)^2} \\ \frac{\partial v_k}{\partial Y_i} &= -\frac{X_k(X_i^2 - Y_i^2)}{(X_i^2 + Y_i^2)^2} = \frac{X_k(Y_i^2 - X_i^2)}{(X_i^2 + Y_i^2)^2} \\ \frac{\partial u_k}{\partial Y_i} &= \frac{Y_k(X_i^2 - Y_i^2)}{(X_i^2 + Y_i^2)^2} \\ \frac{\partial v_k}{\partial X_i} &= \frac{Y_k(Y_i^2 - X_i^2)}{(X_i^2 + Y_i^2)^2} = -\frac{Y_k(X_i^2 - Y_i^2)}{(X_i^2 + Y_i^2)^2}.\end{aligned}$$

Hence f_k is holomorphic in all coordinates. In the second case, f_j is only dependent on Z_i. We have

$$\begin{aligned} f_j &= \frac{1}{X_i + IY_i} \\ &= \frac{X_i}{X_i^2 + Y_i^2} - I\frac{Y_i}{X_i^2 + Y_i^2} \\ &:= u_j + Iv_j. \end{aligned}$$

We have that

$$\begin{aligned} \frac{\partial u_j}{\partial X_i} &= \frac{Y_i^2 - X_i^2}{(X_i^2 + Y_i^2)^2} \\ \frac{\partial v_j}{\partial Y_i} &= -\frac{X_i^2 - Y_i^2}{(X_i^2 + Y_i^2)^2} = \frac{Y_k(X_i^2 - Y_i^2)}{(X_i^2 + Y_i^2)^2} \\ \frac{\partial u_j}{\partial Y_i} &= -\frac{2X_iY_i}{(X_i^2 + Y_i^2)} \\ \frac{\partial v_j}{\partial X_i} &= \frac{2X_iY_i}{(X_i^2 + Y_i^2)^2}. \end{aligned}$$

Therefore, the transition functions are all holomorphic, and $\mathbb{CP}^n$ is a complex manifold.

Problem 5. Define $O_+ = (0,1) \in X$ and $O_- = (0,-1) \in X$. Let $q : X \to M$ be the quotient map. For any $p \in X \setminus \{O_+, O_-\}$ with x-coordinate p_x, $q(p) = [p] = \{(p_x, 1), (p_x, -1)\}$. Meanwhile, $q(O_+) = [O_+] = \{O_+\}$ and $q(O_-) = [O_-] = \{O_-\}$.

We claim that M is locally Euclidean. For $[p] \in M$ with $p \notin \{O_+, O_-\}$, define the neighborhood

$$N_\varepsilon([p]) := \{[p'] \in M \mid |p_x - p'_x| < \varepsilon\},$$

where $\varepsilon < |p_x|$. The preimage of this set under q is $(p_x - \varepsilon, p_x + \varepsilon) \times \{1, -1\}$, which is open in $X \subset \mathbb{R}^2$, thus $N_\varepsilon([p])$ is open. The neighborhood is homeomorphic to $(p_x - \varepsilon, p_x + \varepsilon)$ by the map $\phi : [p] \mapsto p_x$.

For $[O_+]$, define $N_\varepsilon([O_+])$ as above, with only that $\varepsilon > 0$. The preimage of this set is

$$q^{-1}(N_\varepsilon([O_+])) = [((-\varepsilon, 0) \cup (0, \varepsilon)) \times \{1, -1\}] \cup \{O_+\},$$

which is open, hence $N_\varepsilon([O_+])$ is open. This neighborhood is homeomorphic to $(-\varepsilon, \varepsilon)$ by the map ϕ. An analogous construction holds for O_-. Thus, M is locally Euclidean.

We now claim that M is second-countable. The rays $\{[p] \in M \mid p_x < 0 \text{ or } p_x > 0\}$ have the usual countable basis of rational-radius open balls centered at rational points. We additionally add to this basis the collection of open balls $N_\varepsilon([O_+])$ and $N_\varepsilon([O_-])$ where $\varepsilon \in \mathbb{Q}$. Hence M is second-countable.

Finally, we claim that M is not Haussdorff. For every open set U_1 containing $[O_+]$ and U_2 containing $[O_-]$, there exist neighborhoods $N_{\varepsilon_1}([O_+]) \subseteq U_1$ and $N_{\varepsilon_2}([O_-]) \subseteq U_2$ for some $\varepsilon_1, \varepsilon_2 > 0$. Therefore, if $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, then we have the neighborhoods $N_\varepsilon([O_+]) \subseteq U_1$ and $N_\varepsilon([O_-]) \subseteq U_2$. But these neighborhoods have nonempty intersection $N_{\varepsilon/2}([-\varepsilon/2]) \cup N_{\varepsilon/2}([\varepsilon/2])$, therefore M is not Haussdorff.

Problem 6. Let M, N be smooth manifolds, and let $F : M \to N$.

$\boxed{\implies}$ Suppose F satisfies Definition 1. Let $p \in M$ be arbitrary. Since M is a manifold, there exists a chart (U, ϕ) with $p \in U \subseteq M$. Similarly, since $F(p) \in N$ and N is a manifold, there exists a chart (V, ψ) with $F(p) \in V \subseteq N$. By Definition 1, the map

$$\psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \to \psi(V)$$

is smooth. Hence F satisfies Definition 2.

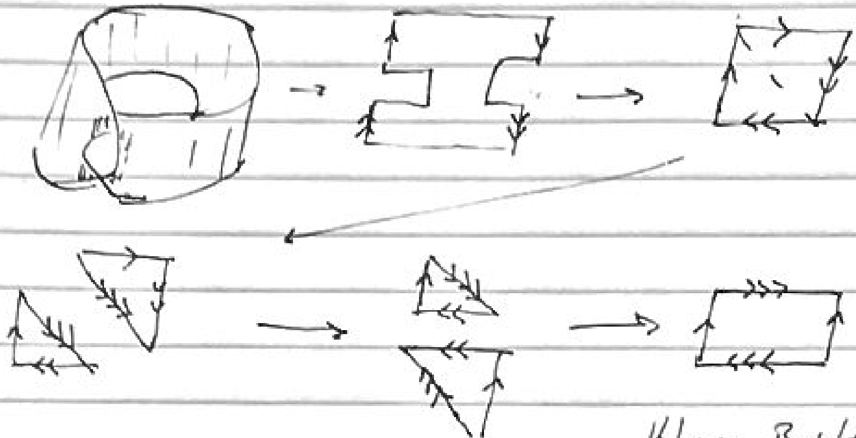
$\boxed{\Longleftarrow}$ Suppose F satisfies Definition 2 using the charts (U, ϕ) with $p \in U \subseteq M$ and (V, ψ) with $F(p) \in V \subseteq N$. Consider any charts (U', ϕ') with $p \in U' \subseteq M$ and (V', ψ') with $F(p) \in V' \subseteq N$. Then

$$\psi' \circ F \circ (\phi')^{-1} = (\psi' \circ \psi^{-1}) \circ (\psi \circ F \circ \phi^{-1}) \circ (\phi \circ (\phi')^{-1}) : \phi'(U' \cap F^{-1}(V')) \to \psi(V')$$

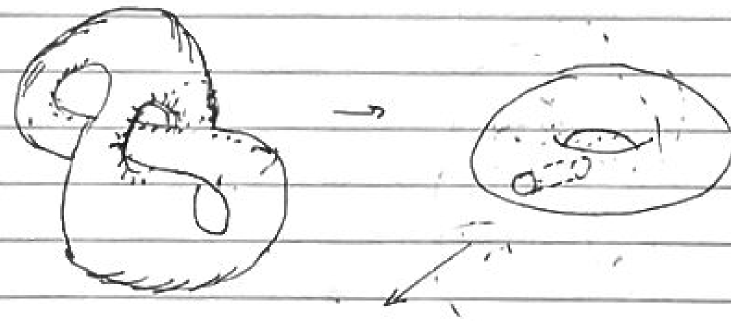
is smooth, since the transition functions of smooth manifolds are smooth and the composition of smooth maps is smooth. Hence F satisfies Definition 1.

Therefore, the two definitions are equivalent.

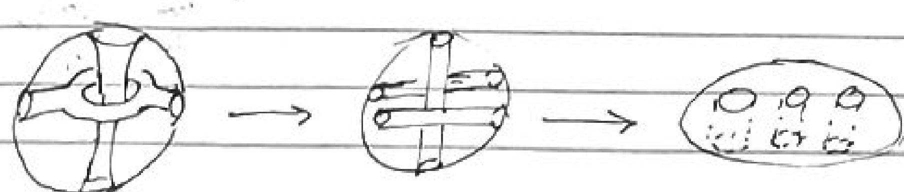
Problem 7. See the following figures.



Klein Bottle



Two tori glued together.



3 tori glued together