Problem 1.

(a) Assuming the ansatz $E(x,t) = X(x)T(t)$, the wave equation becomes

$$\begin{aligned}\frac{\partial^2 E}{\partial x^2} &= \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}\\ \implies X''(x)T(t) &= \frac{1}{c^2}X(x)T''(t)\\ \implies \frac{X''}{X} &= \frac{1}{c^2}\frac{T''}{T}.\end{aligned}$$

As the LHS is a function of only x and the RHS is a function of only t, both sides must be equal to some separation constant, which we define as $-k^2$. Thus

$$\begin{aligned}X'' &= -k^2 X\\ T'' &= -c^2k^2T\end{aligned}$$

The solutions to each of these equations are

$$\begin{aligned}X(x) &= A\sin(kx) + B\cos(kx)\\ T(t) &= C\sin(ckt) + D\cos(ckt).\end{aligned}$$

As we have the initial condition $E(x,0) = X(x)T(0) = 0$, a nontrivial solution must have that $T(0) = 0$, hence $D = 0$ by substitution into the above equation. Thus

$$T(t) = C\sin(ckt).$$

(b) We thus have that E is of the form (absorbing C into the other constants):

$$E(x,t) = (A\sin(kx) + B\cos(kx))\sin(ckt).$$

Applying the boundary condition $E(0,t) = 0$:

$$\begin{aligned}0 &= B\sin(ckt)\\ \implies 0 &= B.\end{aligned}$$

Applying the boundary condition $E(L,t) = 0$:

$$\begin{aligned}0 &= A\sin(kL)\sin(ckt)\\ \implies 0 &= \sin(kL)\\ \implies kL &= n\pi,\ n = 1,2,3,\dots\\ \implies k &= \frac{n\pi}{L}.\ n = 1,2,3,\dots \qquad (1)\end{aligned}$$

We thus have a family of solutions indexed by n:

$$E_n(x,t) = C_n \sin\Big(\frac{n\pi}{L}x\Big)\sin\Big(\frac{cn\pi}{L}t\Big).$$

(c) Defining $\omega = ck$, the allowed angular frequencies are

$$\omega_n = \frac{cn\pi}{L},\ n = 1,2,3,\dots$$

For each mode of the wave, we define N as its index in order to count the number of possible modes. We want to know how a change in N leads to a change in the wavenumber k. As the number of modes becomes very large, a small change is comparatively infinitesimal, and so we can approximately use the language of differentials. In 1D, from (1):

$$\mathrm{d}N = \frac{L}{\pi}\,\mathrm{d}k\,.$$

In 3D, we can choose modes along each of the three axes, yielding an $(L/\pi)^3$ term. However, we also must account for polarization, thus

$$\mathrm{d}N = 2\left(\frac{L}{\pi}\right)^3 \mathrm{d}^3k\,.$$

In abstract **k**-space, we treat the volume element d^3k as a spherical shell of radius k and width $\mathrm{d}k$ centered at the origin. We are only concerned with positive k values, so we pick up a factor of 1/8 as well:

$$\mathrm{d}^3k = \frac{2\pi k^2}{8}\,\mathrm{d}k\,.$$

As we are concerned with angular frequencies, we substitute $k = \omega/c$:

$$\mathrm{d}N = 2\left(\frac{L}{\pi}\right)^3 \cdot \frac{4\pi(\omega/c)^2}{8} \cdot \frac{\mathrm{d}\omega}{c} = \frac{L^3\omega^2}{\pi^2c^3}\,\mathrm{d}\omega\,. \tag{2}$$

(d) We will use Equation 2 to compute the energy density per unit ω, given as

$$\mathcal{U}(\omega) = \frac{1}{L^3}\frac{\partial E}{\partial \omega}.$$

We can compute $\mathrm{d}\omega$ from above, now we ust need $\mathrm{d}E$. From the Equipartition Theorem, each degree of freedom in a statistical system has the same amount of energy:

$$E = \underbrace{\frac{1}{2}k_\mathrm{B}T}_{\text{K.E.}} + \underbrace{\frac{1}{2}k_\mathrm{B}T}_{\text{P.E.}} = k_\mathrm{B}T,$$

where k_B is Boltzmann's constant. Thus

$$\begin{aligned} \mathrm{d}E &= k_\mathrm{B}\,\mathrm{d}T \\ &= k_\mathrm{B}T\,\mathrm{d}N \\ &= \frac{L^3\omega^2}{\pi^2c^3}\,\mathrm{d}\omega \\ \implies \frac{\partial E}{\partial\omega} &= \frac{L^3\omega^2}{\pi^2c^3}. \end{aligned}$$

Therefore, we obtain the Rayleigh-Jeans law:

$$\mathcal{U}(\omega) = \frac{1}{L^3}\frac{\partial E}{\partial\omega} = \frac{k_\mathrm{B}\omega^2T}{\pi^2c^3}.$$

(e) Suppose our black body is made of atoms with two energy levels, a ground state $|g\rangle$ and an excited state $|e\rangle$. Classically, the ratio between the number of systems in $|e\rangle$ vs the number in $|g\rangle$ is given by the Boltzmann distribution:

$$\frac{N_e}{N_\mathrm{g}} = e^{-E/(k_\mathrm{B}T)}.$$

Let $\bar{n}$ count the average number of photons in the black box and $p_{\mathrm{e}\leftrightarrow\mathrm{g}}$ the probability of transitioning between either state. Then the absorption rate is $N_\mathrm{g}\bar{n}p_{\mathrm{e}\leftrightarrow\mathrm{g}}$, and the emission rate is $N_\mathrm{e}(\bar{n}+1)p_{\mathrm{e}\leftrightarrow\mathrm{g}}$.

Assuming thermal equilibrium, the system must be in detailed balance, and so these rates must be the same:

$$\begin{aligned} N_\mathrm{g}\bar{n}p_{\mathrm{e}\leftrightarrow\mathrm{g}} &= N_\mathrm{e}(\bar{n}+1)p_{\mathrm{e}\leftrightarrow\mathrm{g}} \\ \implies \frac{\bar{n}}{\bar{n}+1} &= \frac{N_\mathrm{e}}{N_\mathrm{g}} = e^{-E/(k_\mathrm{B}T)}. \end{aligned}$$

Taking Planck's assumption that the atoms can only absorb or emit photons of discrete chunks $E = \hbar\omega$, we get

$$\begin{aligned} \frac{\bar{n}}{\bar{n}+1} &= e^{-E/(k_\mathrm{B}T)} \\ \implies \bar{n} &= \frac{1}{e^{\hbar\omega/(k_\mathrm{B}T)}-1}. \end{aligned}$$

The total average energy of photons in the box is $\bar{E} = \bar{n}\hbar\omega$, thus

$$\bar{E} = \frac{\hbar\omega}{e^{\hbar\omega/(k_\mathrm{B}T)}-1}.$$

Counting photon states, we have that

$$\begin{aligned} \mathrm{d}E &= \bar{n}\hbar\omega\,\mathrm{d}N \\ &= \bar{n}\hbar\omega\frac{L^3\omega^2}{\pi^2c^3}\,\mathrm{d}\omega \\ \implies \frac{\partial E}{\partial\omega} &= \frac{L^3\omega^2}{\pi^2c^3}\frac{\hbar\omega}{e^{\hbar\omega/(k_\mathrm{B}T)}-1}. \end{aligned}$$

Calculating the energy density as before, we obtain Planck's law:

$$\mathcal{U}(\omega) = \frac{\hbar\omega^3}{\pi^2c^3}\frac{1}{e^{\hbar\omega/(k_\mathrm{B}T)}-1}.$$

Problem 2.

(a) We consider the 1D infinite square well and solve the 1D Schrödinger equation

$$\left[-\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\psi(x,t) = i\hbar\frac{\partial}{\partial t}\psi(x,t),$$

where the potential energy function is

$$V(x) = \begin{cases} 0, & x \in [0, L] \\ \infty, & \text{otherwise.} \end{cases}$$

In effect, the form of V enforces that $\psi = 0$ outside of $[0, L]$ and the boundary conditions $\psi(0,t) = \psi(L,t) = 0$, thus we only need to solve for the wavefunction within the well. Taking the ansatz $\psi(x,t) = X(x)T(t)$, we get

$$\begin{aligned} &-\frac{\hbar^2}{2m}X''(x)T(t) = i\hbar X(x)T'(t) \\ \implies &-\frac{\hbar^2}{2m}\frac{X''(x)}{X(x)} = i\hbar\frac{T'(t)}{T(t)} := E, \end{aligned}$$

where E is a constant. Thus

$$X''(x) = -\frac{2mE}{\hbar^2}X(x),$$

$$T'(t) = -\frac{iE}{\hbar} T(t).$$

Solving the time component:

$$T(t) = C \exp\left(-\frac{iE}{\hbar} t\right).$$

Solving the spatial component:

$$X(x) = A \sin\left(\frac{\sqrt{2mE}}{\hbar} x\right) + B \cos\left(\frac{\sqrt{2mE}}{\hbar} x\right).$$

Therefore (absorbing C) into the other integration factors

$$\psi(x,t) = \exp\left(-\frac{iE}{\hbar} t\right) \left[A \sin\left(\frac{\sqrt{2mE}}{\hbar} x\right) + B \cos\left(\frac{\sqrt{2mE}}{\hbar} x\right)\right].$$

Applying $\psi(0,t) = 0$:

$$\begin{aligned} 0 &= \exp\left(-\frac{iE}{\hbar} t\right) \cdot B \\ \implies B &= 0. \end{aligned}$$

Then, applying $\psi(L,t) = 0$:

$$\begin{aligned} 0 &= A \exp\left(-\frac{iE}{\hbar} t\right) \sin\left(\frac{\sqrt{2mE}}{\hbar} L\right) \\ \implies 0 &= \sin\left(\frac{\sqrt{2mE}}{\hbar} L\right) \\ \implies \frac{\sqrt{2mE}}{\hbar} L &= n\pi, \ n = 1, 2, 3, \ldots \\ \implies E_n &= \frac{n^2 \pi^2 \hbar^2}{2mL^2}, \ n = 1, 2, 3, \ldots \end{aligned}$$

Thus the non-normalized solutions are of the form

$$\psi_n(x,t) = A_n \exp\left(-\frac{iE_n}{\hbar} t\right) \sin\left(\frac{n\pi}{L} x\right).$$

Normalization entails that $\|\psi_n\|_{L^2} = 1$, where $\|\cdot\|_{L^2}$ is the $L^2(\mathbb{R})$-norm. Thus

$$\begin{aligned} 1 &= \|\psi_n\|_{L^2}^2 \\ &= \int_{\mathbb{R}} \mathrm{d}x \, \psi_n^* \psi_n \\ &= \int_{[0,L]} \mathrm{d}x \, \psi_n^* \psi_n \quad (\mathrm{supp}(\psi_n) = [0, L]) \\ &= \int_0^L \mathrm{d}x \, A_n^* \exp\left(\frac{iE_n}{\hbar} t\right) \sin\left(\frac{n\pi}{L} x\right) A_n \exp\left(-\frac{iE_n}{\hbar} t\right) \sin\left(\frac{n\pi}{L} x\right) \\ &= |A_n|^2 \int_0^L \mathrm{d}x \sin^2\left(\frac{n\pi}{L} x\right) \\ &= |A_n|^2 \int_0^L \mathrm{d}x \, \frac{1 - \cos\left(\frac{2n\pi}{L} x\right)}{2} \end{aligned}$$

$$
\begin{aligned}
&= \frac{L}{2}|A_n|^2 \\
\implies |A_n| &= \sqrt{\frac{2}{L}}.
\end{aligned}
$$

By linearity of the Schrödinger equation, a general solution ψ can be built via a series expansion

$$\psi = \sum_n c_n \psi_n.$$

Without loss of generality we can move any complex phase factor of A_n to the coefficient c_n and assume that A_n is real and nonnegative, hence $A_n := \sqrt{2/L}$. Thus the basis eigenfunctions and their corresponding energies are

$$
\begin{aligned}
\psi_n(x,t) &= \sqrt{\frac{2}{L}}\exp\left(-\frac{iE_n}{\hbar}t\right)\sin\left(\frac{n\pi}{L}x\right), \\
E_n &= \frac{n^2\pi^2\hbar^2}{2mL^2},\ n = 1, 2, 3, \ldots
\end{aligned}
$$

Orthogonality of the basis $\{\psi_n\}$ follows from the orthogonality of the subset $\{\sin(\pi n x/L)\}$ of the Fourier basis. Thus for the general solution to be normalized:

$$
\begin{aligned}
1 &= \|\psi\|_{L^2}^2 \\
&= \langle \psi, \psi \rangle_{L^2} \\
&= \left\langle \sum_m c_m \psi_m, \sum_n c_n \psi_n \right\rangle_{L^2} \\
&= \sum_{m,n} c_m^* c_n \langle \psi_m, \psi_n \rangle \\
&= \sum_{m,n} c_m^* c_n \delta_{m,n} \\
&= \sum_m |c_m|^2 \\
&= \|c\|_{\ell^2}^2 \\
\implies \|c\|_{\ell^2} &= 1,
\end{aligned}
$$

where $\langle \cdot, \cdot \rangle_{L^2}$ is the L^2-inner product, $\|\cdot\|_{\ell^2}$ is the ℓ^2-norm, and c is the (possibly finite or infinite) coefficient sequence $c = (c_1, c_2, c_3, \ldots)$.

(b) (A) As per the conditions above, we require $1 = \sqrt{A^2 + A^2}$, thus $A = 1/\sqrt{2}$, and so

$$\Psi(x,0) = \frac{1}{\sqrt{2}}\psi_1(x) + \frac{1}{\sqrt{2}}\psi_2(x).$$

(B) The coefficients in the series expansion do not change with time, so

$$
\begin{aligned}
\Psi(x,t) &= \frac{1}{\sqrt{2}}\psi_1(x,t) + \frac{1}{\sqrt{2}}\psi_2(x,t) \\
&= \frac{1}{\sqrt{L}}\exp\left(-\frac{iE_1}{\hbar}t\right)\sin\left(\frac{\pi}{L}x\right) + \frac{1}{\sqrt{L}}\exp\left(-\frac{iE_2}{\hbar}t\right)\sin\left(\frac{2\pi}{L}x\right)
\end{aligned}
$$

and

$$|\Psi(x,t)|^2 = \frac{1}{L}\sin^2\left(\frac{\pi}{L}x\right)$$

$$
\begin{aligned}
&+ \frac{1}{L} \exp\left(\frac{i(E_1 - E_2)t}{\hbar}\right) \sin\left(\frac{\pi}{L}x\right) \sin\left(\frac{2\pi}{L}x\right) \\
&+ \frac{1}{L} \exp\left(\frac{i(E_2 - E_1)t}{\hbar}\right) \sin\left(\frac{2\pi}{L}x\right) \sin\left(\frac{\pi}{L}x\right) \\
&+ \frac{1}{L} \sin^2\left(\frac{2\pi}{L}x\right) \\
= {}& \frac{1}{L}\left[\sin^2\left(\frac{\pi}{L}x\right) + \sin^2\left(\frac{2\pi}{L}x\right) + 2\cos\left(\frac{E_1 - E_2}{\hbar}t\right) \sin\left(\frac{\pi}{L}x\right) \sin\left(\frac{2\pi}{L}x\right)\right] \\
= {}& \frac{1}{L}\left[\sin^2\left(\frac{\pi}{L}x\right) + \sin^2\left(\frac{2\pi}{L}x\right) + 2\cos\big((1^2\omega - 2^2\omega)t\big) \sin\left(\frac{\pi}{L}x\right) \sin\left(\frac{2\pi}{L}x\right)\right] \\
= {}& \frac{1}{L}\left[\sin^2\left(\frac{\pi}{L}x\right) + \sin^2\left(\frac{2\pi}{L}x\right) + 2\cos(3\omega t) \sin\left(\frac{\pi}{L}x\right) \sin\left(\frac{2\pi}{L}x\right)\right].
\end{aligned}
$$

(C) We have that

$$
\begin{aligned}
\langle x\rangle &= \int_{\mathbb{R}} \mathrm{d}x\, \psi^* x \psi \\
&= \frac{1}{L}\int_0^L \mathrm{d}x\, x\left[\sin^2\left(\frac{\pi}{L}x\right) + \sin^2\left(\frac{2\pi}{L}x\right) + 2\cos(3\omega t)\sin\left(\frac{\pi}{L}x\right)\sin\left(\frac{2\pi}{L}x\right)\right] \\
&= \frac{1}{L}\int_0^L \mathrm{d}x\, x\left[\frac{1-\cos\left(\frac{2\pi}{L}x\right)}{2} + \frac{1-\cos\left(\frac{4\pi}{L}x\right)}{2} + \cos(3\omega t)\left(\cos\left(\frac{\pi}{L}x\right) - \cos\left(\frac{3\pi}{L}x\right)\right)\right] \\
&= 1 - \frac{1}{2L}\int_0^L \mathrm{d}x\left(x\left[\cos\left(\frac{2\pi}{L}x\right) + \cos\left(\frac{4\pi}{L}x\right)\right]\right) \\
&\quad + \frac{\cos(3\omega t)}{L}\int_0^L \mathrm{d}x\, x\left(\cos\left(\frac{\pi}{L}x\right) - \cos\left(\frac{3\pi}{L}x\right)\right) \\
&= 1 - \frac{1}{2L}\left[\frac{\cos\left(\frac{2\pi}{L}x\right)}{(2\pi/L)^2} + \frac{x\sin\left(\frac{2\pi}{L}x\right)}{(2\pi/L)} + \frac{\cos\left(\frac{4\pi}{L}x\right)}{(4\pi/L)^2} + \frac{x\sin\left(\frac{4\pi}{L}x\right)}{(4\pi/L)}\right]\Bigg|_0^a \\
&\quad + \frac{\cos(3\omega t)}{L}\left[\frac{\cos\left(\frac{\pi}{L}x\right)}{(\pi/L)^2} + \frac{x\sin\left(\frac{\pi}{L}x\right)}{(\pi/L)} + \frac{\cos\left(\frac{3\pi}{L}x\right)}{(3\pi/L)^2} + \frac{x\sin\left(\frac{3\pi}{L}x\right)}{(3\pi/L)}\right]\Bigg|_{x=0}^a \\
&= 1 - \frac{1}{2L}\left[\frac{\cos\left(\frac{2\pi a}{L}\right)-1}{(2\pi/L)^2} + \frac{a\sin\left(\frac{2\pi a}{L}\right)}{(2\pi/L)} + \frac{\cos\left(\frac{4\pi a}{L}\right)-1}{(4\pi/L)^2} + \frac{a\sin\left(\frac{4\pi a}{L}\right)}{(4\pi/L)}\right] \\
&\quad + \frac{\cos(3\omega t)}{L}\left[\frac{\cos\left(\frac{\pi a}{L}\right)-1}{(\pi/L)^2} + \frac{a\sin\left(\frac{\pi a}{L}\right)}{(\pi/L)} + \frac{\cos\left(\frac{3\pi a}{L}-1\right)}{(3\pi/L)^2} + \frac{\sin\left(\frac{3\pi a}{L}\right)}{(3\pi/L)}\right]
\end{aligned}
$$

The frequency of oscillation is 3ω. The amplitude of oscillation is

$$
\frac{1}{L}\left[\frac{\cos\left(\frac{\pi a}{L}\right)-1}{(\pi/L)^2} + \frac{a\sin\left(\frac{\pi a}{L}\right)}{(\pi/L)} + \frac{\cos\left(\frac{3\pi a}{L}-1\right)}{(3\pi/L)^2} + \frac{\sin\left(\frac{3\pi a}{L}\right)}{(3\pi/L)}\right] \quad (????).
$$

(D) The expected value of momentum can be computed as

$$
\begin{aligned}
\langle p\rangle &= \frac{\mathrm{d}}{\mathrm{d}t}\langle x\rangle \\
&= \frac{-3\omega\sin(3\omega t)}{L}\left[\frac{\cos\left(\frac{\pi a}{L}\right)-1}{(\pi/L)^2} + \frac{a\sin\left(\frac{\pi a}{L}\right)}{(\pi/L)} + \frac{\cos\left(\frac{3\pi a}{L}-1\right)}{(3\pi/L)^2} + \frac{\sin\left(\frac{3\pi a}{L}\right)}{(3\pi/L)}\right].
\end{aligned}
$$

(E) You might get E_1 or E_2. The expectation value is

$$
\langle H\rangle = \left(\frac{1}{\sqrt{2}}\right)^2 E_1 + \left(\frac{1}{\sqrt{2}}\right)^2 E_2
$$

$$= \frac{E_1 + E_2}{2},$$

which is their mean.

(c) The probability of finding the particle in $[0, L/a]$ is

$$\begin{aligned}
P_n([0, L/a]) &= \int_0^a \mathrm{d}x \, \psi_n^* \psi_n \\
&= \frac{2}{L} \int_0^a \sin^2\left(\frac{n\pi}{L} x\right) \\
&= \frac{2}{L} \int_0^a \frac{1 - \cos\left(\frac{2n\pi}{L} x\right)}{2} \\
&= \frac{1}{L} \left[x - \frac{L}{2n\pi} \sin\left(\frac{2n\pi}{L} x\right) \right] \Bigg|_0^a \\
&= \frac{1}{L} \left[a - \frac{L}{2n\pi} \sin\left(\frac{2n\pi a}{L}\right) \right].
\end{aligned}$$

(d) As $n \to \infty$, the second term vanishes, leaving $P_n = a/L$. If the particle were a classical one, bouncing around randomly in the square well, one would also expect it to be in $[0, a]$ around a proportion a/L of the time. This suggests that quantum mechanical objects begin to behave classically when highly excited.