DIFFERENTIAL TOPOLOGY
WINTER 2026
HOMEWORK #2

Due on Gradescope anytime before 10 p.m. EDT on Wednesday, January 21.

Problem 1. Let M be a smooth manifold, let $p \in M$, and let $X \in T_pM$ be a derivation. Prove the following two facts:

(1) If $f \equiv C$ is a constant function, then $Xf = 0$.
(2) If $f(p) = 0 = g(p)$, then $X(fg) = 0$.

(This was a lemma we stated in class.)

Problem 2. Let $F \colon M \to N$ and $G \colon N \to P$ be smooth maps between smooth manifolds, and let $p \in M$. Prove the following facts:

(1) The map $F_* \colon T_pM \to T_{F(p)}N$ is linear.
(2) $(G \circ F)_* = G_* \circ F_*$.
(3) $(\mathrm{Id}_M)_* = \mathrm{Id}_{T_pM}$.
(4) If F is a diffeomorphism, then F_* is an isomorphism.

(This was also a lemma we stated in class.)

Problem 3. Prove that the tangent bundle of S^3 is trivial.
Hint: Consider S^3 embedded in $\mathbb{R}^4$, identify $\mathbb{R}^4$ with the quaternions, and consider the vector fields $X_i \colon p \to ip$, $X_j \colon p \to jp$, and $X_k \colon p \to kp$ (you should prove that these maps are indeed sections of TS^3).

Problem 4. The *tautological line bundle on* $\mathbb{R}P^n$ has total space

$$E = \{(l, v) \mid l \text{ is a line in } \mathbb{R}^{n+1}, v \in l \subset \mathbb{R}^{n+1}\}$$

and projection map $\pi(l, v) = l$. Prove that this is indeed a vector bundle, and compute its transition functions (for your choice of trivializations).

Problem 5. *Lee, Problem 3-7.* Let M be a smooth manifold with or without boundary and p be a point of M. Let $C_p^\infty(M)$ denote the algebra of germs of smooth real-valued functions at p, and let $\mathcal{D}_pM$ denote the vector space of derivations of $C_p^\infty(M)$. Define a map $\Phi \colon \to T_pM$ by $(\Phi v)f = v([f]_p)$. Show that Φ is an isomorphism.
[Read the relevant discussion on p.71 of Lee's book.]

Problem 6. *Lee, Problem 10-3.* Let VB denote the category whose objects are smooth vector bundles and whose morphisms are smooth bundle homomorphisms, and let Diff denote the category whose objects are smooth manifolds and whose morphisms are smooth maps. Show

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that the assignment $M \mapsto TM$ and $F \mapsto dF$ defines a covariant functor from Diff to VB, called the **tangent functor**.
[This essientially follows from facts you proved earlier, but it's worth writing it out explicitly in this language; you don't need to reprove anything you would like to use. Read the relevant discussion on p.73-75 of Lee's book.]