DIFFERENTIAL TOPOLOGY
WINTER 2026
HOMEWORK #1

Due on Gradescope anytime before 10 p.m. EDT on Wednesday, January 14.

Problem 1. Consider the atlas for S^n whose two charts are given by the two stereographic projections from the north pole $(0,\dots,0,1)$ and the south pole $(0,\dots,0,-1)$, respectively. Calculate the transition functions.

Problem 2. Find an atlas for the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$, and calculate the transition functions.

Problem 3. Consider the atlas on $\mathbb{RP}^n$ with charts (U_i, ϕ_i) where

$$U_i = \{[x_0 \colon x_1 \colon \cdots \colon x_n] \mid x_i \neq 0\}$$

and

$$\begin{aligned} \phi_i \colon U_i &\to \mathbb{R}^n \\ [x_0 \colon x_1 \colon \cdots \colon x_n] &\mapsto \left(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{\widehat{x_i}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}\right). \end{aligned}$$

Calculate the transition functions.

Problem 4. A function $f : \mathbb{C}^m \to \mathbb{C}^n$ is holomorphic if, holding all but one variables fixed, each of the resulting component functions satisfies the Cauchy-Riemann equations.

Prove that $\mathbb{CP}^n$ is a complex manifold.

Problem 5. *Lee, Problem 1.1.* Let X be the set of all points $(x, y) \in \mathbb{R}^2$ such that $y = \pm 1$, and let M be the quotient of X by the equivalence relation generated by $(x,-1) \sim (x, 1)$ for all $x \neq 0$. Show that M is locally Euclidean and second-countable, but not Hausdorff. (This space is called *the line with two origins.*)

Problem 6. In class we defined smooth maps on manifolds as follows:

Definition 1. *A map $F \colon M \to N$ between smooth manifolds is* smooth *if for every $p \in M$ and for every chart (U, ϕ) with $p \in U$ and every chart (V, ψ) with $F(p) \in V$, the map*

$$\psi \circ F \circ \phi^{-1} \colon \phi(U \cap F^{-1}(V)) \to \psi(V)$$

is smooth.

An alternative definition is:

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Definition 2. *A map* $F \colon M \to N$ *between smooth manifolds is* smooth *if for every* $p \in M$ *there exist charts* (U, ϕ) *with* $p \in U$ *and* (V, ψ) *with* $F(p) \in V$ *such that the map*

$$\psi \circ F \circ \phi^{-1} \colon \phi(U \cap F^{-1}(V)) \to \psi(V)$$

is smooth.

Prove that the two definitions agree.
Note: The map $\psi \circ F \circ \phi^{-1}$ is called the *coordinate representation of* F.

Problem 7. *Spivak, Problem 1.16.* To which "standard" surfaces are the three surfaces homeomorphic? (See the full question and corresponding figure in Spivak's book.)