PROBLEM I

1. Assume $\{Y_1(n) : n \geq 0\}, \{Y_2(n) : n \geq 0\}$ are two independent migrating branch process with offspring distribution $(p(i) : i \in \mathbb{N})$ and the migrating probobility respectively are $(\gamma_1(i) : i \in \mathbb{Z}_+), (\gamma_2(i) : i \in \mathbb{N})$. Prove: $\{Y_1(n) + Y_2(n) : n \geq 0\}$ is migrating branching process with offspring distribution $p(i) : i \in \mathbb{N}$ and migrating probability $\gamma_1 * \gamma_2$.

2. Let $\{Y(n) : n \in \mathbb{N}\}$ be migrating branch process with offspring distribution $p(j) : j \in \mathbb{N}$ and the migrating distribution $\gamma(i) : i \in \mathbb{N}$. $P_n^\gamma = (p_n^\gamma(i,j); i,j \in \mathbb{N})$ is the n-th transition matrix. Prove: $\forall i, n \geq 1$
$$\sum_{j=0}^{\infty} p_n^\gamma(i,j) z^j = g_n(z)^i \prod_{k=1}^{n} h(g_{k-1}(z)), |z| \leq 1$$
where h is the generating function of $(\gamma(j) : j \in \mathbb{N})$. g is the generating function of $(p(j) : j \in \mathbb{N})$.

3. h, g are defined as above. Assume $m := g'(1-) < \infty, \mu := h'(1-) < \infty$. Prove: $\forall i, n \geq 1$,
$$\mathbb{P}(Y_n \mid Y_0 = i) = im^n + \mu \sum_{k=1}^{n} m^{k-1}$$

SOLUTION . 1. Since Y_1, Y_2 are independent Markov chain, we easily get $\mathbb{P}(Y_1(n+1)+Y_2(n+1) = i \mid \sigma(Y_1(j), Y_2(j) : 0 \leq j \leq n)) = \mathbb{P}(Y_1(n+1) + Y_2(n+1) = i \mid Y_1(n), Y_2(n))$. So to prove $Y_1 + Y_2$ is Markov chain, we only need to prove $\mathbb{P}(Y_1(n+1) + Y_2(n+1) = i \mid Y_1(n), Y_2(n))\mathbb{P}(Y_1(n+1) + Y_2(n+1) = i \mid Y_1(n) + Y_2(n))$.

$$\begin{aligned}
&\mathbb{P}(Y_1(n+1) + Y_2(n+1) = i \mid Y_1(n) = j, Y_2(n) = k) \\
=&\sum_{x+y=i} \mathbb{P}(Y_1(n+1) = x \mid Y_1(n) = j)\mathbb{P}(Y_2(n+1) = y \mid Y_2(n) = k) \\
=&\sum_{x+y=i} p^{*j} * \gamma_1(x) p^{*k} * \gamma_2(y) \\
=&p^{*j} * \gamma_1 * p^{*k} * \gamma_2(i) \\
=&p^{*(j+k)} * \gamma_1 * \gamma_2(i)
\end{aligned}$$

So $\mathbb{P}(Y_1(n+1)+Y_2(n+1) = i \mid Y_1(n), Y_2(n)) = p^{*(Y_1(n)+Y_2(n))} * (\gamma_1 * \gamma_2)(i) \in \sigma(Y_1(n)+Y_2(n)) \subset \sigma(Y_1(n), Y_2(n))$. So $Y_1 + Y_2$ is Markov chain. More over, we have obtained $\mathbb{P}(Y_1(n+1) + Y_2(n+1) = j \mid Y_1(n) + Y_2(n) = i) = p^{*i} * (\gamma_1 * \gamma_2)(j)$. So $\{Y_1(n) + Y_2(n) : n \geq 0\}$ is migrating branching process with offspring distribution $p(i) : i \in \mathbb{N}$ and migrating probability $\gamma_1 * \gamma_2$.

2. Use MI to prove it. Write $G_n(i,z) := \sum_{j=0}^{\infty} p_n^\gamma(i,j) z^j$. When $n = 0$, we have $p_0^\gamma(i,j) = \delta_{ij}$, so $G_0(i,z) = z^i = g_0(z)^i$. When $n = 1$, we have $p_1^\gamma(i,j) = p^{*i} * \gamma(j)$. So $G_1(i,z) = g(z)^i h(z)$. Assume for certain n we have proved that $G_n(i,z) = g_n(z)^i \prod_{k=1}^{n} h(g_{k-1}(z))$, Consider $n+1$.

Easily $p_{n+1}^{\gamma}(i,j) = \sum_{k\in\mathbb{N}} p_n^{\gamma}(k,j)p(i,\cdot) * \gamma(k)$. So

$$
\begin{aligned}
G_{n+1}(i,z) &= \sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{N}} p_1^{\gamma}(i,k)p_n^{\gamma}(k,j)z^j \\
&= \sum_{k\in\mathbb{N}} p_1^{\gamma}(i,k)G_n(k,z) \\
&= \prod_{k=1}^{n} h(g_{k-1}(z)) \sum_{k\in\mathbb{N}} p_1^{\gamma}(i,k)g_n(z)^k \\
&= \prod_{k=1}^{n} h(g_{k-1}(z))G_1(i,g_n(z)) \\
&= g_{n+1}(z)\prod_{k=1}^{n+1} h(g_{k-1}(z))
\end{aligned}
$$

3. Easily $\mathbb{P}(Y_n \mid Y_0 = i) = D_z G_n(i,z) \mid_{z\to 1-}$. Noting $g(1) = h(1) = 1$, easy to get that $\mathbb{P}(Y_n \mid Y_0 = i) = im^n + \mu\sum_{k=1}^{n} m^{k-1}$.

$\square$

PROBLEM II Assume $b \in (0,1), p \in (0,1)$. Let $\mu(0) = \frac{1-b-p}{1-p}, \mu(j) = bp^{j-1}, j \geq 1$. Prove:

1. $(\mu(j) : j \in \mathbb{N})$ is probability distribution and

$$
g(z) := \sum_{j=0}^{\infty} \mu(j)z^j = \frac{1-b-p}{1-p} + \frac{bz}{1-pz}.
$$

2. Let $b = (1-p)^2$. Then $g'(1) = 1$ and

$$
g(z) = p + \frac{(1-p)^2 z}{1-pz} = \frac{p-(2p-1)z}{1-pz}.
$$

Prove: $\forall n \geq 1$,

$$
g_n(z) = \frac{np - ((n+1)p-1)z}{1+(n-1)p-npz}.
$$

SOLUTION. 1. Easily $\sum_{j=1}^{\infty}\mu(j) = \frac{b}{1-p}$. So $\sum_{j=0}^{\infty}\mu(j) = 1$. Easily $\sum_{j=1}^{\infty}\mu(j)z^j = \frac{bz}{1-pz}$. So $g(z) = \mu(0) + \frac{bz}{1-pz} = \frac{1-b-p}{1-p} + \frac{bz}{1-pz}$.

2. $g_{n+1}(z) = g(g_n(z)) = \frac{p-(2p-1)g_n(z)}{1-pg_n(z)}$. So $g_{n+1}(z) - 1 = \frac{(g_n(z)-1)(1-p)}{1-pg_n(z)}$. Thus, we obtain $\frac{1}{g_{n+1}(z)-1} = \frac{1}{g_n(z)-1} - \frac{p}{1-p}$. So $\frac{1}{g_n(z)-1} = \frac{1}{z-1} - \frac{np}{1-p}$, and finally we get $g_n(z) = \frac{np-((n+1)p-1)z}{1+(n-1)p-npz}$.

$\square$

PROBLEM III Let $\{X(n) : n \in \mathbb{N}\}$ be branch process with offspring distribution $p(j) : j \in \mathbb{N}$. And g is the generating function. Let $m_2 := g'(1) + g''(1) < \infty$. Let $m = g'(1) < \infty$. $\forall k \geq 1$, $X_n^{(k)} = k^{-1}X_n$. Prove: $\forall \varepsilon > 0, i, n \geq 1$, $\mathbb{P}(|X_n^{(k)} - im^n| \geq \varepsilon \mid X_0^{(k)} = i) \to 0, k \to \infty$.

SOLUTION. In fact, we don't need $m_2 < \infty$. We let $(Y(k,n) : n \in \mathbb{N}), k \in \mathbb{N}$ are independent branch process with offspring distribution $p(j) : j \in \mathbb{N}$ and $Y(k,0) = i$. Then $\sum_{j=1}^{k} Y(j,n)$ is branch process with offspring distribution $p(j) : j \in \mathbb{N}$ and initial value ki. So $\sum_{j=1}^{k} Y(j,n) \overset{d}{=} X_n \mid X_0^{(k)} = i$. So $\mathbb{P}(|X_n^{(k)} - im^n| \geq \varepsilon \mid X_0^{(k)} = i) = \mathbb{P}(|\frac{\sum_{j=1}^{k} Y(j,n)}{k} - im^n| \geq \varepsilon)$. By LLN we obtain $\frac{1}{k}\sum_{j=1}^{k} Y(j,n) \overset{\text{a.s.}}{\to} im^n$. So finally we get $\lim_{k\to\infty} \mathbb{P}(|X_n^{(k)} - im^n| \geq \varepsilon \mid X_0^{(k)} = i) = \lim_{k\to\infty} \mathbb{P}(|\frac{\sum_{j=1}^{k} Y(j,n)}{k} - im^n| \geq \varepsilon) = 0$. □

PROBLEM IV Let $\{X(n) : n \in \mathbb{N}\}$ be branch process with offspring distribution $p(j) : j \in \mathbb{N}$. And g is the generating function, where $m := g'(1) \in (1,\infty), m_2 := g'(1) + g''(1) < \infty$. Let $\sigma^2 := m_2 - m^2 = \mathbb{D}(X(1))$. It is well known that $\exists W, \lim_{n\to\infty} \frac{X_n}{m^n} = W$. Prove:

$$\lim_{n\to\infty} \mathbb{E}_1[(m^{-n}X_n - W)^2] = 0, \mathbb{D}_1(W) = \sigma^2 m^{-1}(m-1)^{-1}$$

SOLUTION. For convinence we write $\mathbb{E}, \mathbb{D}$ instead of $\mathbb{E}_1, \mathbb{D}_1$. Easy to get that $\mathbb{E}(m^{-2n}X_n^2) = \frac{\sigma^2(1-m^{-n})}{m^2-m} + 1$. So by Fatou theorem we get that $\mathbb{E}(W^2) \leq \lim_{n\to\infty} \mathbb{E}(m^{-2n}X_n^2) = \frac{\sigma^2}{m^2-m} + 1 < \infty$. And by Doob Stochastic Processes p317 theorem 3.4 we get that $\mathbb{E}(\max_{n\in\mathbb{N}} m^{-2n}X_n^2) < \infty$. Thus, $m^{-2n}X_n^2$ are integrable uniformly, and so do $(m^{-n}X_n - W)^2$. So by LCDT we can get $\lim_{n\to\infty} \mathbb{E}((m^{-n}X_n - W)^2) = 0$. Noting that

$$\mathbb{E}(m^{-2n}X_n^2 - W^2) = \mathbb{E}((m^{-n}X_n + W)(m^{-n}X_n - W)) \leq \sqrt{\mathbb{E}((m^{-n}X_n + W)^2)\mathbb{E}((m^{-n}X_n - W)^2)} \to 0$$

, we get $\mathbb{E}(W^2) = \lim \mathbb{E}(m^{-n}X_n) = \frac{\sigma^2}{m^2-m} + 1$. Also, $\mathbb{E}(|m^{-n}X_n - W|)^2 \leq \mathbb{E}((m^{-n}X_n - W)^2)$, so $\mathbb{E}(W) = \lim \mathbb{E}(m^{-n}X_n) = 1$. So $\mathbb{D}(W) = \mathbb{E}(W^2) - \mathbb{E}(W)^2 = \frac{\sigma^2}{m(m-1)}$. □

PROBLEM V Let $\{Y(n) : n \in \mathbb{N}\}$ be branch process with offspring distribution $p(j) : j \in \mathbb{N}$. And g is the generating function, where $m := g'(1) \leq 1$. Prove $(p^\gamma(j) : j \in \mathbb{N})$ is the steady-state vector of transition matrix P_n^γ, that is $\sum_{i=0}^{\infty} p^\gamma(i)p_n^\gamma(i,j) = p^\gamma(j), i \geq 0$.

SOLUTION. Since $\lim_{m\to\infty} p_m^\gamma(i,j) = p^\gamma(j)$, and fix $k \in \mathbb{N}$, we have $\sum_{j=0}^{\infty} p_m^\gamma(k,i)p_n^\gamma(i,j) = p_{n+m}^\gamma(k,j)$, we only need to prove that $\lim_{m\to\infty} \sum_{i=0}^{\infty}(p_m^\gamma(k,i) - p^\gamma(i))p_n^\gamma(i,j) = 0$. Since $\lim_{m\to\infty} p_m^\gamma(k,i) = p^\gamma(i)$ and $\sum_{i\in\mathbb{N}} p_m^\gamma(k,i) = 1$, we can easily get that $\sum_{i\in\mathbb{N}} p^\gamma(i) = 1$. For $\varepsilon > 0$, we let N large enough such that $\sum_{k=N}^{\infty} p^\gamma(k) < \varepsilon$. Then we let M large enough such that $\forall i : 0 \leq i < N, \forall m \geq$

$M, |p_m^\gamma(k,i) - p^\gamma(k)| < \frac{\varepsilon}{N}$. Then

$$\begin{aligned}
&\left|\sum_{i=0}^{\infty}(p_m^\gamma(k,i) - p^\gamma(i))p_n^\gamma(i,j)\right| \\
\leq &\sum_{i=0}^{\infty}|p_m^\gamma(k,i) - p^\gamma(i)|p_n^\gamma(i,j) \\
\leq &\sum_{i=0}^{N-1}|p_m^\gamma(k,i) - p^\gamma(i)|p_n^\gamma(i,j) + \sum_{i=N}^{\infty}(p_m^\gamma(k,i) + p^\gamma(i))p_n^\gamma(i,j) \\
\leq &\sum_{i=0}^{N-1}\frac{\varepsilon}{N} + \sum_{i=N}^{\infty}p_m^\gamma(k,i) + p^\gamma(i) \\
\leq &\varepsilon + \sum_{i=N}^{\infty}p^\gamma(i) + 1 - \sum_{i=1}^{N-1}p_m^\gamma(k,i) \\
\leq &\varepsilon + \varepsilon + 1 - \sum_{i=1}^{N-1}p^\gamma(i) + \sum_{i=1}^{N-1}|p_m^\gamma(k,i) - p^\gamma(i)| \\
\leq &4\varepsilon
\end{aligned}$$

So finally we get $\lim_{m\to\infty}\sum_{i=0}^{\infty}(p_m^\gamma(k,i) - p^\gamma(i))p_n^\gamma(i,j) = 0$. Thus, $\sum_{i=0}^{\infty}p^\gamma(i)p_n^\gamma(i,j) = p^\gamma(j), i \geq 0$. □

PROBLEM VI Let $\{Y(n) : n \in \mathbb{N}\}$ be branch process with offspring distribution $p(j) : j \in \mathbb{N}$. And g is the generating function, where $m := g'(1) \leq 1$. Discuss $\lim_{n\to\infty}\mathbb{E}(Y_n \mid Y_0 = i)$.

SOLUTION. Easy to get that $\mathbb{E}(Y_n \mid Y_0 = i) = im^n + \mu\sum_{k=1}^{n}m^{k-1}$. When $m = 1$, we know $\mathbb{E}(Y_n \mid Y_0 = i) \to \infty$. When $m < 1$, we know $\mathbb{E}(Y_n \mid Y_0 = i) \to \frac{\mu}{1-m}$. □