underGraduate Homework In Mathematics

PROBLEM I Assume $(\mathscr{F}_t : t \geq 0, t \in \mathbb{R})$ is a filtration. For $t \geq 0$ we let $\mathscr{F}_{t+} := \bigcap_{s>t} \mathscr{F}_s$. Prove that $\mathscr{F}_t \subset \mathscr{F}_{t+}$ and $(\mathscr{F}_{t+} : t \geq 0)$ is a filtration.

SOLUTION. To prove $\mathscr{F}_t \subset \mathscr{F}_{t+} = \bigcap_{s>t} \mathscr{F}_s$, we only need to prove $\forall s > t, \mathscr{F}_t \subset \mathscr{F}_s$. By the definition of filtration it's obvious. Now we will prove $(\mathscr{F}_{t+} : t \geq 0)$ is a filtration. Only need to prove $\forall t, s \in \mathbb{R} \wedge t \leq s, \mathscr{F}_{t+} \subset \mathscr{F}_{s+}$. By the definition of $\mathscr{F}_{\cdot+}$ we know that $\mathscr{F}_{t+} = \bigcap_{x>t} \mathscr{F}_x = \bigcap_{x>s} \mathscr{F}_x \cap \bigcap_{x:t<x\leq s} \mathscr{F}_x \subset \bigcap_{x>s} \mathscr{F}_x = \mathscr{F}_{s+}$. So $(\mathscr{F}_{t+} : t \geq 0)$ is a filtration. $\square$

PROBLEM II Assume $(X_t : t \geq 0, t \in \mathbb{R})$ is a stochastic process on probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Prove that $\forall s, t \geq 0, \varepsilon > 0, \{\rho(X_s, X_t) \geq \varepsilon\} \in \mathscr{F}$.

SOLUTION. Easily $\{\rho(X_s, X_t) \geq \varepsilon\} = \bigcup_{k=1}^{\infty} \{\rho(X_s, X_t) > \varepsilon - \frac{1}{k}\varepsilon\}$. So we only need to prove $\forall k \in \mathbb{N}^+, \{\rho(X_s, X_t) > \varepsilon(1 - \frac{1}{k})\} \in \mathscr{F}$. Take $\delta = \varepsilon(1 - \frac{1}{k})$, only need to prove $\forall \delta > 0, \{\rho(X_s, X_t) > \delta\} \in \mathscr{F}$.

$\forall t \geq 0, X_t : \Omega \to E$ is measurable, where $E \subset \mathbb{R}^d$. So we can find a countable dense set in $\mathbb{R}^d$, write D. We will prove that $\{\rho(X_s, X_t) > \delta\} = \bigcup_{q\in D} \{\rho(X_s, q) - \rho(X_t, q) > \delta\}$. On one hand, easily $\rho(X_s, q) - \rho(X_t, q) > \delta \implies \rho(X_s, X_t) > \delta$ from triangle inequality. So we easily get $\{\rho(X_s, X_t) > \delta\} \supset \bigcup_{q\in D} \{\rho(X_s, q) - \rho(X_t, q) > \delta\}$. On the other hand, assume for certain $\omega \in \Omega$ we have $\rho(X_s(\omega), X_t(\omega)) > \delta$, we will prove $\exists q \in D, \rho(X_s(\omega), q) - \rho(X_t(\omega), q) > \delta$. For convenience, we omit (ω) from now on to the end of this paragraph. Since $\rho(X_s, X_t) > \delta$, we know $\gamma := \dfrac{\rho(X_s, X_t) - \delta}{2} > 0$. Since D is dense, we obtain $\exists q \in D, \rho(X_t, q) < \gamma$. So from triangle inequality we get $\rho(X_s, q) \geq \rho(X_s, X_t) - \rho(X_t, q) > 2\gamma + \delta - \gamma = \gamma + \delta$. So we get $\rho(X_s, q) - \rho(X_t, q) > \gamma + \delta - \gamma = \delta$. Finally, we get $\{\rho(X_s, X_t) > \delta\} = \bigcup_{q\in D} \{\rho(X_s, q) - \rho(X_t, q) > \delta\}$.

Noting $\bigcup_{q\in D} \{\rho(X_s, q) - \rho(X_t, q) > \delta\} = \bigcup_{q\in D} \bigcup_{p\in\mathbb{Q}^+} \{\rho(X_s, q) > \delta + p, \rho(X_t, q) < p\}$, and $D, \mathbb{Q}^+$ are countable, so we only need to check $\{\rho(X_s, q) > \delta + p, \rho(X_t, q) < p\} \in \mathscr{F}, \forall q \in D, p \in \mathbb{Q}^+$. Noting $\{\rho(X_s, q) > \delta + p, \rho(X_t, q) < p\} = \{\rho(X_s, q) > \delta + p\} \cap \{\rho(X_t, q) < p\}$, and X_s, X_t are measurable from Ω to E, we obtain $\{\rho(X_s, q) > \delta + p\}, \{\rho(X_t, q) < p\} \in \mathscr{F}$. So we proved $\{\rho(X_s, X_t) > \delta\} \in \mathscr{F}, \forall s, t \geq 0, \forall \delta > 0$.

Finally, we obtain $\{\rho(X_s, X_t) \geq \varepsilon\} \in \mathscr{F}, \forall s, t \geq 0, \varepsilon > 0$. $\square$

PROBLEM III Let $\mathscr{D}_X := \{\mu_J^X : J \in S(I)\}$ be the family of finite-dimentional distributions of a stochastic process $(X_t : t \geq 0, t \in \mathbb{R})$. $\forall (s_1, s_2) \in S(I)$ and $J = (t_1, \cdots, t_n) \in S(I)$, write $K_1 := (s_1, s_2, t_1, \cdots, t_n) \in S(I), K_2 := (s_2, s_1, t_1, \cdots, t_n) \in S(I)$. Take $A_1, A_2 \in \mathscr{E}, B \in \mathscr{E}^n$, prove that

$$\mu_{K_1}^X(A_1 \times A_2 \times B) = \mu_{K_2}^X(A_2 \times A_1 \times B)$$

and

$$\mu^X_{K_1}(E \times E \times B) = \mu^X_{K_2}(E \times E \times B) = \mu^X_J(B)$$

SOLUTION. By the definition of finite-dimentional distributions we get

$$\begin{aligned}\mu^X_{K_1}(A_1 \times A_2 \times B) &= \mathbb{P}((X_{s_1}, X_{s_2}, X_{t_1}, \cdots, X_{t_n}) \in A_1 \times A_2 \times B) \\ &= \mathbb{P}(X_{s_1} \in A_1)\mathbb{P}(X_{s_2} \in A_2)\mathbb{P}((X_{t_1}, X_{t_2}, \cdots, X_{t_n}) \in B)\end{aligned}$$

For the same reason, we obtain

$$\begin{aligned}&\mu^X_{K_2}(A_2 \times A_1 \times B) \\ &\qquad = \mathbb{P}((X_{s_2}, X_{s_1}, X_{t_1}, \cdots, X_{t_n}) \in A_2 \times A_1 \times B) \\ &\qquad = \mathbb{P}(X_{s_2} \in A_2)\mathbb{P}(X_{s_1} \in A_1)\mathbb{P}((X_{t_1}, X_{t_2}, \cdots, X_{t_n}) \in B)\end{aligned}$$

So we get

$$\mu^X_{K_1}(A_1 \times A_2 \times B) = \mu^X_{K_2}(A_2 \times A_1 \times B)$$

Also, let $A_1 = A_2 = E$, we get

$$\begin{aligned}\mu^X_{K_1}(E \times E \times B) &= \mu^X_{K_2}(E \times E \times B) \\ &= \mathbb{P}(X_{s_1} \in E)\mathbb{P}(X_{s_2} \in E)\mathbb{P}((X_{t_1}, \cdots, X_{t_n}) \in B) \\ &= \mathbb{P}((X_{t_1}, \cdots, X_{t_n}) \in B)\end{aligned}$$

By the definition of finite-dimentional distributions we get

$$\mu^X_J(B) = \mathbb{P}((X_{t_1}, \cdots, X_{t_n}) \in B)$$

So finally we get

$$\mu^X_{K_1}(E \times E \times B) = \mu^X_{K_2}(E \times E \times B) = \mu^X_J(B)$$

□

PROBLEM IV Assume $(\tau_k : k \in \mathbb{N}^+)$ is an i.i.d sequence of r.v. with exponential distribution with parameter $\alpha > 0$. Let $S_n := \sum_{k=1}^{n} \tau_k$. For $t \geq 0, t \in \mathbb{R}$, let:

$$N_t := \sum_{n=1}^{\infty} \mathbb{1}_{\{S_n \leq t\}}, X_t := \sum_{n=1}^{\infty} \mathbb{1}_{\{S_n < t\}}$$

Prove that N and X are modifications of each other, but they are not indistinguishable.

SOLUTION. First we prove N and X are modifications of each other. Fix $t \in [0, \infty)$, we need to prove $\mathbb{P}(N_t = X_t) = 1$. Noting $N_t - X_t = \sum_{n=1}^{\infty} \mathbb{1}_{S_n \leq t} - \mathbb{1}_{S_n < 1} = \sum_{n=1}^{\infty} \mathbb{1}_{S_n = t}$, we get $\mathbb{P}(N_t =$

$X_t) = \mathbb{P}(\sum_{n=1}^{\infty} \mathbb{1}_{S_n=t}) = \mathbb{P}(\forall n \in \mathbb{N}^+, S_n \neq t)$. So we only need to prove $\mathbb{P}(S_n = t) = 0, \forall n \in \mathbb{N}^+$. Since $\tau_k, k \in \mathbb{N}^+$ are continuous-distributed, we know $S_n = \sum_{k=1}^{n} \tau_k$ is continuous-distributed, so $\mathbb{P}(S_n = t) = 0$. So we proved N and X are modifications of each other.

Next we will prove they are not indistinguishable. Only need to prove $\mathbb{P}(\forall t \in [0, \infty), X_t = N_t) = 0 \neq 1$. Since $N_t - X_t = \sum_{n=1}^{\infty} \mathbb{1}_{S_n \le t} - \mathbb{1}_{S_n < 1} = \sum_{n=1}^{\infty} \mathbb{1}_{S_n = t}$, we know $\forall t, N_t = X_t \iff \forall t, \forall n \in \mathbb{N}^+, S_n \neq t$. But S_n is ranged in $[0, \infty)$, so R.H.S is an impossible event. So we finally get $\mathbb{P}(\forall t \in [0, \infty), X_t = N_t) = 0$ and thus X and N are not indistinguishable. $\square$

PROBLEM V Assume T is non-negetive r.v. with distribution function F continuous on $\mathbb{R}$. Let $X_t = \mathbb{1}_{\{T \le t\}}$. Prove that X is stochastically continuous.

SOLUTION. Only need to check $\forall t \ge 0, X_s \xrightarrow{\mathbb{P}} X_t, s \to t$. Take $\varepsilon > 0$, we need to prove $\lim_{s \to t} \mathbb{P}(\rho(X_s, X_t) > \varepsilon) = 0$. For $u > v \ge 0$, we have $X_u - X_v = \mathbb{1}_{v < T \le u}$. So $\mathbb{P}(\rho(X_u - X_v) > \varepsilon) \le \mathbb{P}(X_u \neq X_v) = \mathbb{P}(v < T \le u) \le \mathbb{P}(T \in [v, u])$. So we easily get $\lim_{s \to t+} \mathbb{P}(\rho(X_s, X_t) > \varepsilon) \le \lim_{s \to t+} \mathbb{P}(T \in [t, s]) = 0$ and $\lim_{s \to t-} \mathbb{P}(\rho(X_s, X_t) > \varepsilon) \le \lim_{s \to t-} \mathbb{P}(T \in [s, t]) = 0$. So $\lim_{s \to t} \mathbb{P}(\rho(X_s, X_t) > \varepsilon) = 0$. $\square$

PROBLEM VI Assume $I = \mathbb{Z}^+$, then the stochastic process $X = (X_0, X_1, \cdots)$ is a r.v. from Ω to E^{∞}. Define the distribution of X, μ_X, as follows:

$$\mu_X(A) = \mathbb{P}(X \in A), A \in \mathscr{E}^{\infty}$$

Then stochastic process X, Y are equivalent $\iff \mu_X = \mu_Y$.

SOLUTION. " $\Longrightarrow$ ":Assume X, Y are equivalent, now we will prove $\mu_X = \mu_Y$. Let $\mathscr{A} := \{A \in \mathscr{P}(E^{\infty}) : \exists n \in \mathbb{N}^+, A = A_1 \times A_2 \times \cdots \times A_n \times E \times E \times \cdots\}$. Then we can get $\mu_X(A) = \mu^X_{(1,2,\cdots,n)}(A_1 \times \cdots \times A_n)$. So for $A \in \mathscr{A}$ we know $\mu_X(A) = \mu_Y(A)$. By the definition of $\mathscr{E}^{\infty} = \sigma(\mathscr{A})$, and noting $\mathscr{A}$ is a Semiset algebra, by the Measure extension theorem we get $\mu_X = \mu_Y$.

" $\Longleftarrow$ ": Assume $\mu_X = \mu_Y$, then easily $\mu^X_{(s_1,\cdots,s_n)}(A_{s_1} \times \cdots \times A_{s_n}) = \mu_X(\prod_{k \in \mathbb{N}^+} B_k)$, where $B_k = A_{s_t}$ for $k = s_t$ and $B_k = E$ for $k \neq s_t, \forall t = 1, \cdots, n$. So easily $\mu^X_J = \mu^Y_J, \forall J \subset I \wedge |J| < \infty$. $\square$