underGraduate Homework In Mathematics

AlgebraicGeometry 6

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General fire extinguisher

PROBLEM I Assume V is a set and k is a field. Assume $k[V] \subset k^V$ is f.g. $k-$algebra, and exists a class of generators $x_1, \cdots x_n$, s.t. $\phi : V \to \mathbb{A}_k^n, p \mapsto (x_1(p), \cdots x_n(p))$ embeds V as an irreducible algebra set in $\mathbb{A}_k^n$. Prove: $k[V] \cong k[y_1, \cdots, y_n]/\mathbb{I}(\mathrm{Im}(\phi))$.

SOLUTION. Consider $\psi : k[y_1, \cdots y_n] \to k[V], y_i \mapsto x_i$. Obviously ψ is homomorphism, so we only need to prove $\ker \psi = \mathbb{I}(\mathrm{Im}(\phi))$.

$\forall f(y_1, \cdots, y_n) \in \ker \psi, \psi(f(y_1, \cdots, y_n)) = f(x_1, \cdots x_n) = 0$. So for $(x_1(p), \cdots, x_n(p)) \in \mathrm{Im}(\phi)$, we have $f(x_1(p), x_2(p), \cdots x_n(p)) = f(x_1, \cdots, x_n)(p) = 0$. So $f(y_1, \cdots, y_n) \in \mathbb{I}(\mathrm{Im}(\phi))$.

On the other hand, consider $f(y_1, \cdots, y_n) \in \mathbb{I}(\mathrm{Im}(\phi))$, we have $\forall p \in \mathbb{A}_k^n, f(x_1(p), \cdots, x_n(p)) = 0$. So $f(x_1, x_2, \cdots x_n) = 0$, i.e., $f \in \ker \psi$.

So finally we get $k[V] \cong k[y_1, \cdots, y_n]/\mathbb{I}(\mathrm{Im}(\phi))$. $\square$

PROBLEM II Assume V is irreducible algebra set in $\mathbb{A}_k^n$ and $h \in k[V]$. Assume $H \in k[x_1 \cdots x_n], h(p) = H(p), \forall p \in V$. Let $\phi : V_h \to \mathbb{A}_k^{n+1}, p \mapsto (p, \frac{1}{H(p)})$. Let $J = \mathbb{I}(V) \subset k[x_1, \cdots x_n] \subset k[x_1, \cdots x_n, y]$, and $J' := (J, yH - 1)$. Prove: $\mathrm{Im}(\phi) = \mathbb{V}(J')$.

SOLUTION. First we prove $\mathrm{Im}(\phi) \subset \mathbb{V}(J')$. Consider $(x_1, \cdots, x_n, y) \in \mathrm{Im}(\phi)$, we need to prove $p = (x_1, \cdots, x_n) \in \mathbb{V}(J)$ and $yH(p) - 1 = 0$. It's obvious from the definition of V_h and ϕ.

Second we prove $\mathbb{V}(J') \subset \mathrm{Im}(\phi)$. Assume $(p, y) \in \mathbb{V}(J')$, then $p \in \mathbb{V}(J)$ and $H(p)y = 1$. So we get $H(p) \neq 0$. Since $p \in \mathbb{V}(J) = V$, we get $p \in V_h$. And $\phi(p) = (p, \frac{1}{H(p)}) = (p, y)$. So $(p, y) \in J'$. $\square$

PROBLEM III Let $u = \mathbb{A}_k^2 \setminus \{(0,0)\} \subseteq \mathbb{A}_k^n$, then u is not affine variety.

SOLUTION. Since $k[u] = k[x_1, x_2, x_1^{-1}] \cap k[x_1, x_2, x_2^{-1}]$ we know $k[u] = k[x_1, x_2]$. Now assume u is affine variety, assume $\phi : u \to \mathbb{A}_k^n$ embeds u to an irreducible algebra set. And assume $\{f_1, f_2, \ldots f_n\}$ is the generators of $k[u]$. Then $\phi(p) = (f_1(p), f_2(p), \ldots, f_n(p))$. Assume $\phi(u) = \mathbb{V}(I)$ for some prime ideal $I = (g_1, g_2, \ldots, g_n)$. Then $g_i(\phi(p)) = 0, \forall p \in u, i = 1, \ldots, n$. Now we prove $g_i(\phi(0,0)) = 0, \forall i = 1, \ldots, n$. If not, assume $h(x_1, x_2) = g_i(\phi(x_1, x_2)) \neq 0$. Consider $h(x_1, x_2) \in k[x_1][x_2]$, we get $x_2 - 1 \mid h(x_1, x_2)$. Assume $h(x_1, x_2) = (x_2 - 1)^n l(x_1, x_2), x_2 - 1 \nmid l(x_1, x_2)$. Obviously $x_2 - 1 \notin I$ because $(x_2 - 1)(0,0) = -1 \neq 0$. So $l(x_1, x_2) \in I$. Since $x_2 - 1 \nmid l(x_1, x_2)$ we get $l(x_1, 1) \neq 0$. So $\exists a \in k, l(a, 2) \neq 0$. Then $(a, 1) \notin \mathbb{V}(I) = \mathrm{Im}(\phi)$, contradiction! So $g_i(\phi) = 0 \in k[x_1, x_2]$. Then $\phi(0) \in \mathbb{V}(I) = \phi(u)$. So $\exists p \in u, \phi(p) = \phi(0)$. So u is not affine variety. $\square$

PROBLEM IV