GROUP REPRESENTATION

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PROBLEM I Find an $n-$dimensional real matrix representation of $(\mathbb{R}, +)$.

SOLUTION. Let:

$$\phi : \mathbb{R} \to \mathrm{GL}_n(\mathbb{R}), x \mapsto e^x I_n \tag{1}$$

Then obviously $(M_n(\mathbb{R}), \phi)$ is an $n-$dimensional real matrix representation of $(\mathbb{R}, +)$. □

PROBLEM II Find an infinitely dimensional representation of $(\mathbb{R}, +)$.

SOLUTION. Let:

$$\phi : \mathbb{R} \to \mathrm{GL}(\mathbb{R}[x]), \phi(a)(f)(x) := f(a + x) \tag{2}$$

Then it's easy to know $(\phi, \mathbb{R}[x])$ is a representation of $(\mathbb{R}, +)$. □

PROBLEM III Determine whether the representation is faithful or not in Example 1 4 and **??**, **??**.

SOLUTION.

- Example 1: When $a = 0$ we have $f_0(x) = \mathrm{e}^0 = 1$, so it's obviously not faithful. When $a \neq 0$, we can easily get f_a is injective, so it's faithful.
- Example 2: When $a = 0$ we have $f_0(x) = \mathrm{e}^0 = 1$, so it's obviously not faithful. When $a \in \mathbb{R}^*$, let $x = y + \frac{2\pi}{a}$, then $f_a(x) = f_a(y)$, so f_a is not faithful. When $a \in \mathbb{C} \setminus \mathbb{R}$, we get $|f_a(x)| = \mathrm{e}^{-\mathrm{Im}(a)x}$ is injective, so f_a is injective, thus faithful.
- Example 3: Obviously $\phi(x + 2\pi) = \phi(x)$ so it's not faithful.
- Example 4: Consider $f \in \mathbb{R}_n[x], f(x) = x$, then for $a \neq b$, we have $\phi(a)(f) = x + a \neq x + b = \phi(b)(f)$, so $\phi(a) \neq \phi(b)$, thus ϕ is faithful.
- **??**: Obviously it's faithful.
- **??**: Obviously it's faithful.

□

PROBLEM IV Let $\lambda \in \mathbb{C}^*$, and:

$$\phi_\lambda : (\mathbb{Z}, +) \to \mathbb{C}^*, n \mapsto \lambda^n \tag{3}$$

Prove that ϕ_λ is an $1-$dimensional complex representation of $(\mathbb{Z}, +)$, and find when it's faithful.

SOLUTION . For $m, n \in \mathbb{Z}$, we have $\phi_\lambda(m+n) = \lambda^{m+n} = \lambda^m \lambda^n = \phi_\lambda(m)\phi_\lambda(n)$, so ϕ_λ is complex representation. Obviously $\mathbb{C}^* \cong \mathrm{GL}(\mathbb{C})$, so it has dimension one.

Noting ϕ_λ is faithful $\iff \ker(\phi) = \{0\} \iff \forall n \neq 0, \lambda^n \neq 1$. So ϕ_λ is not faithful if and only if $\lambda = \mathrm{e}^{q\pi}$ for some $q \in \mathbb{Q}$. □

PROBLEM V Let $V = \mathbb{R}[x]$. $\forall a \in \mathbb{R}$, let:

$$L_a(f(x)) := f(ax), \quad \forall f(x) \in \mathbb{R}[x],$$
$$S_a(f(x)) := f\left(\mathrm{e}^a x\right), \quad \forall f(x) \in \mathbb{R}[x].$$

and:

$$\varphi(a) = L_a, \quad \forall a \in \mathbb{R},$$
$$\psi(a) = S_a, \quad \forall a \in \mathbb{R}.$$

Question: Is φ and ψ is infinitely dimensional real representation of $(\mathbb{R}, +)$?

SOLUTION . $\phi(a+b)(f)(x) = f((a+b)x), \phi(a)\phi(b)(f)(x) = \phi(a)(f(bx)) = f(abx)$, let $f(x) = x, a = b = 1$ we get $\phi(a+b)(f)(x) = f((a+b)x) \neq f(abx)$, so ϕ is not representation of $\mathbb{R}$.

$\psi(a+b)(f)(x) = f(\mathrm{e}^{a+b}x), \psi(a)\psi(b)(f)(x) = \psi(a)(f(\mathrm{e}^b x)) = f(\mathrm{e}^a \mathrm{e}^b x) = \psi(a+b)(f)(x)$, so ψ is a representation of $(\mathbb{R}, +)$. Obviously $\mathbb{R}[x]$ is infinite-dimension, so ψ is infinitely dimensional. □

PROBLEM VI Give the matrix representation of $G = \langle a \rangle$ given by the regular representation, ρ, of field K, where $\mathrm{rank}(a) = 4$.

SOLUTION . $\{e, a, a^2, a^3\}$ is basis of $K[G]$. And:

$$\rho(a)e = a, \rho(a)a = a^2, \rho(a)a^2 = a^3, \rho(a)a^3 = e \tag{4}$$

So matrix of $\rho(a)$ is:

$$P(a) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{5}$$

And thus $P(a^n) = P(a)^n$. Then P is the matrix representation of G. □