underGraduate Homework In Mathematics

AlgebraicGeometry 10

白永乐

202011150087

202011150087@mail.bnu.edu.cn

2023 年 11 月 28 日

General fire extinguisher

PROBLEM I Assume $V \subset \mathbb{A}_k^n$ is irreducible, and $p \in V$. Let $m_p := \{f \in k[V] : f(p) = 0\}$, $M_p := \{f \in k[x_1, \cdots, x_n] : f(p) = 0\}$. Prove that $m_p \cong M_p/\mathbb{I}(V)$.

SOLUTION. Consider $\theta : M_p \to m_p, \theta(f) := f + \mathbb{I}(V)$. Since $p \in V$ we get θ is well-defined. And easily we get θ is homomorphism. Now consider $\ker\theta$. Obviously $\mathbb{I}(V) \subset \ker\theta$, now we prove $\ker\theta \subset \mathbb{I}(V)$. Assume $f \in \ker\theta$, to prove $f \in \mathbb{I}(V)$. Since $\theta(f) = \mathbb{I}(V)$, we get $f + \mathbb{I}(V) = \mathbb{I}(V)$, so $f \in \mathbb{I}(V)$. So $\ker\theta = \mathbb{I}(V)$. And easily θ is surjective, so we get $m_p = M_p/\ker\theta = M_p/\mathbb{I}(V)$. $\square$

PROBLEM II Prove that $M_p/M_p^2 + \mathbb{I}(V) \cong m_p/m_p^2$.

SOLUTION. Consider $\theta : M_p \to m_p/m_p^2, f \mapsto f|_V + m_p^2$. Easily θ is homomorphism and surjective. Now we prove $\ker\theta = M_p^2 + \mathbb{I}(V)$.

On one hand, assume $f \in \ker\theta$, i.e., $f|_V \in m_p^2$. Then $\exists g_1, \cdots, g_n, h_1, \cdots, h_n \in m_p$ such that $f|_V = \sum_{k=1}^n g_k h_k$. Assume $g_k = G_k|_V, h_k = H_k|_V$. Then $G_k, H_k \in M_p$. Consider $f - \sum_{k=1}^n G_k H_k =: h \in k[x_1, \cdots, x_n]$, easily to know $h(x) = 0, \forall x \in V$. So $h \in \mathbb{I}(V)$, thus $f \in M_p^2 + \mathbb{I}(V)$.

On the other hand, assume $f \in M_p^2 + \mathbb{I}(V)$, to prove $\theta f = 0$. Assume $f = \sum_{k=1}^n G_k H_k + h$, where $G_k, H_k \in M_p$ and $h \in \mathbb{I}(V)$. Then $\theta f = \sum_{k=1}^n g_k h_k + m_p^2$, where $g_k = G_k|_V, h_k = H_k|_V$. So we get $\theta f = m_p^2 = 0$.

Finally we get $m_p/m_p^2 = M_p/\ker\theta = M_p/M_p^2 + \mathbb{I}(V)$. $\square$

PROBLEM III Assume $V \subset \mathbb{A}_k^n$ is irreducible, and $p \in V, f \in k[V], f(p) \neq 0$. Consider $V_f := \{x \in V : f(x) \neq 0\}$. Let $\theta : V_f \to \mathbb{A}_k^{n+1}, x \mapsto (x, \frac{1}{f(x)})$. Let $U = \theta(V_f)$, prove that $T_pV \cong T_{\theta(p)}U$.

SOLUTION. Write $k[\mathbb{A}_k^n] = k[x_1, \cdots, x_n, y]$. Assume $V = \mathbb{V}(I) = \mathbb{V}(f_1, \cdots, f_m)$, where $f_i \in k[x_1, \cdots, x_n]$. Then $U = \mathbb{V}(f_1, \cdots, f_n, yf - 1)$. Now consider $\tau : T_p(V) \to \mathbb{A}_k^{n+1}, \tau(x) := (x, \frac{1}{f(p)} - \frac{f_p^{(1)}(x)}{f^2(p)})$. Now we prove $\tau(T_pV) = T_{\theta(p)}U$. Only need to prove $(yf-1)_{\theta(p)}^{(1)}(\tau(x)) = 0$. i.e., $\frac{f_p^{(1)}(x)}{f(p)} + f(p)(y - \frac{1}{f(p)}) = 0$, where $y = \frac{1}{f(p)} - \frac{f_p^{(1)}}{f^2(p)}$. Substitute y into the equation, we get is't obvious.

Obviously τ is injective, so it's isomorphic. $\square$