PROBLEM I Assume $(X_n : n \geq 0)$ is an irreducible Markov chain on E. Prove that $(X_n : n \geq 0)$ is recurrent (or transient) $\iff \forall i \in E$,

$$\mathbb{P}\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\{X_k = i\}\right) = 1(\text{or } 0).$$

SOLUTION. Only need to prove " $\implies$ ".

First we assume $(X_n : n \in \mathbb{N})$ is recurrent, we should prove $\mathbb{P}\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\{X_k = i\}\right) = 1$. Let $\tau_1 = \inf\{n > 0 : X_n = i\}$, and for $n \in \mathbb{N}^+$, we let $\tau_{n+1} = \inf\{n > \tau_n : X_n = i\}$. Since i is recurrent and (X_n) is irreducible, we know that $\tau_1 < \infty, a.s.$. Then $(X_{\tau_1+n} : n \in \mathbb{N})$ is a Markov chain with the same transition matrix as (X_n). So we get that $\tau_2 - \tau_1 < \infty, a.s.$. So $\tau_2 < \infty$, a.s.. Use MI, we can easily get that $\forall n \in \mathbb{N}^+, \tau_n < \infty, a.s.$. Easy to get that $\tau_{n+1} > \tau_n$ and $\tau_1 > 0$, so $\tau_n \geq n$. So $\tau_n < \infty \implies \exists k \geq n, X_k = i$. So $\forall n \in \mathbb{N}, \mathbb{P}\left(\bigcup_{k=n}^{\infty}\{X_k = i\}\right) = 1$. Thus $\mathbb{P}\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\{X_k = i\}\right) = 1$.

Second we assume $(X_n : n \in \mathbb{N})$ is transient, we should prove that $\mathbb{P}\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\{X_k = i\}\right) = 0$. Write $A = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\{X_k = i\}$. If not, we consider $(A, \mathcal{F} \cap A, \mathbb{P}_A := \frac{\mathbb{P}}{\mathbb{P}(A)})$. We define τ_n as above. Easy to know $\forall \omega \in A, \forall n \in \mathbb{N}^+, \tau_n < \infty$. And easy to know that $\tau_{n+1} - \tau_n \mid_{\tau_n < \infty}$ has the same distribution for every n. And since (X_n) is transient, we know (X_{τ_k+n}) is transient for every $k \in \mathbb{N}^+$. So we know $\mathbb{P}(\tau_{n+1} - \tau_n < \infty \mid \tau_n < \infty) < 1$. Then $\mathbb{P}(A) = \mathbb{P}(\forall n, \tau < \infty) \leq \mathbb{P}(\forall n, \tau_{n+1} - \tau_n < \infty) \leq \prod_{n=1}^{\infty}\mathbb{P}(\tau_{n+1} - \tau_n < \infty \mid \tau_n < \infty) = \prod_{n=1}^{\infty}\mathbb{P}(\tau_2 - \tau_1 < \infty \mid \tau_1 < \infty) = 0$. $\square$

PROBLEM II Let $(X_n : n \geq 0)$ is a one dimension simple random walk, and P is it's transition matrix. Let $a \leq b \in \mathbb{Z}$ satisfies $\mathbb{P}(a \leq X_0 \leq b) = 1$. Define $\tau = \inf\{n \geq 0 : X_n = a \text{or } b\}, Y_n = X_{n \wedge \tau}$. Prove: $(Y_n : n \geq 0)$ is Markov chain on $[a, b] \cap \mathbb{Z}$, and give its transition matrix and the classification.

PROBLEM III Prove: $(X_n : n \geq 0)$ is Markov chain on E, where E is finite. Then $\exists x \in E$, x is recurrent. PROBLEM IV Assume $(X_n : n \geq 0)$ is Markov chain on $\mathbb{Z}$. Prove it is transient $\iff \forall \mu_0$ is primitive distribution, $\lim_{n\to\infty} |X_n| \overset{\text{a.s.}}{=} \infty$. PROBLEM V Assume P is a transition matrix on $\mathbb{Z}^+$, which has a first line $\{a_0, a_1, \cdots\}$, $\forall i \geq 1$, $p_{i,i-1} = 1$, and $\forall j \neq i-1$, $p_{i,j} = 0$. Discuss the irreducibility, recurrence, ergodicity and periodicity of 0. PROBLEM VI Assume P is a transition matrix on E. Prove: $\forall i \in E$, $\lim_{n\to\infty} p_{ii}(n)$ exists, and $\lim_{n\to\infty} p_{ii}(n) = \frac{1}{F'_{ii}(1)} = \frac{1}{\mathbb{P}_i(T_i)}$.

PROBLEM VII Assume P is a transition matrix on E and P is irreducible, $j \in E$. Prove: P is recurrent $\iff$ 1 is the minimum non negtive solution of

$$y_i = \sum_{k \neq j} p_{ik} y_k + p_{ij}, i \in E$$

PROBLEM VIII Let $\{a_k : k \geq 0\}$ satisfies $\sum_{k\geq 0} a_k = 1, a_k \geq 1, a_0 > 0$, $\mu := \sum_{k=1}^{\infty} k a_k > 1$. Define

$$p_{ij} = \begin{cases} a_j & , i = 0 \\ a_{j-i+1} & , i \geq 1 \wedge j \geq i-1 \\ 0 & , \text{otherwise} \end{cases}.$$ Prove: P is transient.