underGraduate Homework In Mathematics

SetTheory 4

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General fire extinguisher

PROBLEM I Consider $\mathbb{Q} = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})/\sim$, where $(a,b) \sim (c,d) \iff ad = bc$. Define $+_{\mathbb{Q}}, \cdot_{\mathbb{Q}}$ and $<_{\mathbb{Q}}$ and verify that your definitions doesn't depend on the choice of representatives.

SOLUTION. Let $[(a,b)] +_{\mathbb{Q}} [(c,d)] = [(ad+bc, bd)], [(a,b)] \cdot_{\mathbb{Q}} [(c,d)] = [(ac, bd)]$, and $[(a,b)] <_{\mathbb{Q}} [(c,d)] \iff abd^2 < cdb^2$. Now we prove they are well-defined, i.e., doesn't depend on the choice of representatives.

For $+_{\mathbb{Q}}$, assume $(a,b) \sim (e,f)$, we need to prove $(ad+bc, bd) \sim (ed+fc, df)$. Since $af = be$, we have $(ad+bc)bf = ad^2f + bdcf = bed^2 + bdcf = (ed+fc)bd$. So $+_{\mathbb{Q}}$ is well defined.

For $\cdot_{\mathbb{Q}}$, assume $(a,b) \sim (e,f)$, we need to prove $(ac, bd) \sim (ec, fd)$. Since $af = be$, we have $acfd = bced = bdec$. So $\cdot_{\mathbb{Q}}$ is well defined.

For $<_{\mathbb{Q}}$, assume $(a_1,b_1) \sim (a_2,b_2), (c_1,d_1) \sim (c_2,d_2)$ and $(a_1,b_1) < (c_1,d_1)$. Now we need to prove $(a_2,b_2) < (c_2,d_2)$. Since $a_1b_2 = a_2b_1, c_1d_2 = c_2d_1$ we get $a_1b_1d_2^2 < c_2d_2b_1^2$ □

PROBLEM II The set of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ has cardinality $\mathfrak{c}$ (while the set of all functions has cardinality $2^{\mathfrak{c}}$). [A continuous function on $\mathbb{R}$ is determined by its values at rational points.]

SOLUTION. Consider $\theta : {}^{\mathbb{R}}\mathbb{R} \to 2^{\mathbb{Q}}, f \mapsto \{(a,b) \in \mathbb{Q} : f(a) < b\}$. Now we prove f is a injection. Assume $\theta(f) = \theta(g)$, to prove $f = g$. First we prove for $x \in \mathbb{Q}$ we have $f(x) = g(x)$. We have $f(x) = \sup\{y \in \mathbb{Q} : y < f(x)\} = \sup\{y \in \mathbb{Q} : (x,y) \in \theta(f)\} = \sup\{y \in \mathbb{Q} : (x,y) \in \theta(g)\} = g(x)$. For $x \in \mathbb{R}$, choose a sequence $x_n \in \mathbb{Q}$ such that $x_n \to x$, then $f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = g(x)$. So we get $f = g$. So $\operatorname{card}{}^{\mathbb{R}}\mathbb{R} \leq \operatorname{card} 2^{\mathbb{Q}} = 2^{\aleph_0}$. Obviously $\operatorname{card}{}^{\mathbb{R}}\mathbb{R} \geq 2^{\aleph_0}$, so we get they are equal. □

PROBLEM III There are at least $\mathfrak{c}$ countable order-types of linearly ordered sets. [For every sequence $a = \langle a_n : n \in \mathbb{N} \rangle$ of natural numbers consider the ordertype

$$\tau_a = a_0 + \xi + a_1 + \xi + a_2 + \dots$$

where ξ is the order-type of the integers. Show that if $a \neq b$, then $\tau_a \neq \tau_b$.] A real number is algebraic if it is a root of a polynomial whose coefficients are integers. Otherwise, it is transcendental.

PROBLEM IV The set of all algebraic reals is countable.

PROBLEM V If S is a countable set of reals, then $|\mathbb{R} - S| = \mathfrak{c}$. [Use $\mathbb{R} \times \mathbb{R}$ rather than $\mathbb{R}$ (because $|\mathbb{R} \times \mathbb{R}| = 2^{\aleph_0}$).]