# GroupRepresentation 3

白永乐

202011150087

202011150087@mail.bnu.edu.cn

2024 年 1 月 4 日

## 1 problem

PROBLEM I Let $\varphi$ is representation of $\mathrm{GL}_n(K)$ over $K^n$. And $\varphi(A)\alpha := A\alpha$. Prove:$\varphi$ is faithful and irreducible and $n-$dimentional.

SOLUTION . Obviously it's $n$-dimentional. If $A \neq B$, then exists $\alpha \in K^n$ s.t. $(A-B)\alpha \neq 0$. So $\varphi(A)\alpha \neq \varphi(B)\alpha$. So $\varphi(A) \neq \varphi(B)$, so $\varphi$ is faithful. To prove $\varphi$ is irreducible, we only need to prove there is no invariant subspace of $K^n$. Obviously for $\alpha, \beta \in K^n \setminus \{0\}$, obviously there exists $A \in \mathrm{GL}_n(K)$ such that $A\alpha = \beta$. So there is no nontrival invariant subspace of $K^n$. So it's irreducible. □

PROBLEM II For $A \in \mathrm{GL}_n(K)$, let $\psi(A)X = AX, \forall X \in M_n(K)$. Then:

1. $\psi$ is $n^2-$dimentional representation of $\mathrm{GL}_n(K)$ over $K$.
2. For $j : 1 \leq j \leq n$, let $M_n^{(j)}(K) := \{(a_{ik})_{n\times n} : a_{ik} \neq 0 \to k = j\}$. Prove $M_n^{(j)}$ is invariant subspace of $\mathrm{GL}_n(K)$. Let $\psi_j$ is subrepresentation of $\psi$ in $M_n^{(j)}$, prove $\psi_j$ is irreducible and $\psi = \bigoplus_{j=1}^n \psi_j$.
3. Prove $\psi_j \cong \varphi$, where $\varphi =$ (**??**).$\varphi$

SOLUTION .

1. Obviously $M_n(K)$ is $n^2$-dimentional, so $\psi$ is $n^2$-dimentional. Easily we have $\psi(AB)X = ABX = \psi(A)BX = \psi(A)\psi(B)X$, so $\psi(AB) = \psi(A)\psi(B)$. So $\psi$ is representation.
2. For $X \in M_n^{(j)}, A \in \mathrm{GL}_n(K)$, we have $(AX)_{ik} = \sum_t a_{it}x_{tk}$. So for $k \neq j$ we have $(AX)_{ik} = \sum_t a_{it} \cdot 0 = 0$. So $AX \in M_n^{(j)}$. So $M_n^{(j)}$ is invariant subspace. Easily $M_n(K) = \bigoplus_{j=1}^n M_n^{(j)}$, so $\psi = \bigoplus_{j=1}^n \psi_j$. From **??** we know $\psi_j \cong \varphi$, and from **??** we get *phi* is irreducible, so $\psi_j$ is irreducible.
3. Consider $\tau : M_n^{(j)} \to K^n, (\tau(X))_k := x_{jk}$, then $\tau$ is isomophism. Easily get $\tau$ is isomophism form $\psi_j$ to (**??**).$\varphi$. So $\psi_j \cong \varphi$.

□

PROBLEM III Let $K = \mathbb{C}$ and $n = 2$ in (Group representation second homework).(**??**), prove the subrepresentation of $\varphi$ over $M_2^0(\mathbb{C})$ is irreducible.

SOLUTION . Since every matrix in $M_2(\mathbb{C})$ can be diagonalized, so $\forall X \in M_2^0(\mathbb{C}), \exists A \in \mathrm{GL}_2(\mathbb{C}), \varphi(A)X = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} = \lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. So for a invariant subspace $V$, we have $X \in V \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in V \to \forall Y \in M_2^0(\mathbb{C}), Y \in V$. So $\varphi$ is irreducible. □

PROBLEM IV Assume $n \geq 3$ and $n \nmid \operatorname{char} K$, prove: the $n-$dimentional permutate representation of $S_n$ can be decomposed as the direct sum of a main representation and a $(n-1)-$dimentional irreducible subrepresentation

SOLUTION . In fact, **??** of second homework has given the decomposed. easily we get $\varphi|_{V_1}$ is a main representation. And since $\dim V_2 = n - 1$, we get $\varphi|_{V_2}$ is $(n-1)-$dimentional. So we only need to prove $\varphi|_{V_2}$ is irreducible. Assume $V \subset V_2$ is a invariant subspace and $V \neq \{0\}$, consider $x = \sum_{i=1}^n a_i x_i \in V \setminus \{0\}$. Obviously all of $a's$ can't be equal because $nk = 0 \to k = 0$ since $\operatorname{char} K \nmid n$. WLOG assume $a_1 \neq a_2$. Then $y = a_1 x_2 + a_2 x_1 + \sum_{i=3}^n a_i x_i = \varphi((1\ 2))x \in V$, and thus $x - y = (a_1 - a_2)(x_1 - x_2) \in V, x_1 - x_2 \in V$. So $x_1 - x_j = \varphi((2\ j))(x_1 - x_2) \in V, \forall j \geq 2$. Obviously these $n-1$ vecter is linearly independent, so $\dim V \geq n-1, V = V_2$. So $\varphi|_{V_2}$ is irreducible. □

PROBLEM V Caculate the $1-$ dimentional $\mathbb{C}$ representation:

1. $(2,4)-$type of $8-$order elementary Abel group.
2. the addition group of $\mathbb{Z}_p^n$

SOLUTION .
1. Assume this group is $\mathbb{Z}_2 \times \mathbb{Z}_4$. Then $\varphi(k, j) = \mathrm{e}^{\frac{(2k+j)\pi \mathrm{i}}{2}}$.
2. 
$$\varphi(a_1, a_2, \cdots, a_n) = \mathrm{e}^{\frac{(2\pi \mathrm{i} \sum_{k=1}^n a_k)}{p}}$$

□

# 2 appendix

PROBLEM I Group $G$ has an action on set $\Omega = \{x_1, x_2, \cdots, x_n\}$, let $(\varphi, V)$ be the $n-$ dimensional $K$ permutation representation of $G$, where $K$ is the field of vector space $V$, and

$$V = \left\{ \sum_{i=1}^n a_i x_i : a_i \in K, i = 1, 2, \cdots, n \right\}.$$

Let

$$V_1 = \left\langle \sum_{i=1}^n x_i \right\rangle,$$
$$V_2 = \left\{ \sum_{i=1}^n a_i x_i : \sum_{i=1}^n a_i = 0, a_i \in K \right\}.$$

Prove:

1. $V_1$ and $V_2$ are invariant subspaces of $G$ ;

2. If char $K \nmid n$, then $\varphi = \varphi_{V_1} \oplus \varphi_{V_2}$.

ℝOBEM III $\mathcal{O}(n) := \{A \in M_n(\mathbb{R}) : AA^T = I_n\}$ is the set of all $n$-dimensional otheretic matrix over $\mathbb{R}$. Let:

$$\begin{aligned} \varphi : \mathcal{O}(n) &\to \mathrm{GL}\left(M_n(\mathbb{R})\right) \\ A &\mapsto \varphi(A), \end{aligned} \tag{1}$$

Where,

$$\varphi(A)X := AXA^{-1} : \quad \forall X \in M_n(\mathbb{R}) \tag{2}$$

$M_n^+(\mathbb{R}) := \{A \in M_n^0(\mathbb{R}) : A = A^T\}$, $M_n^-(\mathrm{R}) := \{A \in M_n^0(\mathbb{R}) : A^T = -A\}$.

1. Proof: $M_n^+(\mathrm{R})$ and $M_n^-(\mathrm{R})$ are invariant spaces of $\varphi$;

2. Let the subrepresentation of $\varphi$ on $\langle I \rangle$,$M_n^+(\mathbb{R}), M_n^-(\mathbb{R})$ be $\varphi_0, \varphi_1, \varphi_2$. Proof:

$$\varphi = \varphi_0 \oplus \varphi_1 \oplus \varphi_2$$

3. calculate a $\frac{1}{2}n(n-1)-$ dimensional $\mathbb{R}$ representation of $\mathcal{O}(n)$.