PROBLEM I Solve this equation: $1215x \equiv 560 \mod 2755$.

SOLUTION. Easily $1215x \equiv 560 \mod 2755 \iff 243x \equiv 112 \mod 551$. Obviously $x = 200$ is a solution, and $\gcd(243, 551) = 1$, so all the solutions are $x = 200 + 551t, t \in \mathbb{Z}$. □

PROBLEM II Find the solution of $\begin{cases} x + 4y - 29 \equiv 0 \mod 143 \\ 2x - 9y + 84 \equiv 0 \mod 143 \end{cases}$

SOLUTION. Double the first equation then minus the second, we get $17y - 142 \equiv 0 \mod 143$. Then $y \equiv 42 \mod 143$. Subsititute it in the first equation, we get $x \equiv 4 \mod 143$. □

PROBLEM III

1. Assume $m \in \mathbb{N}^+, \gcd(a, m) = 1$, prove that $x \equiv ba^{\phi(n)-1} \mod m$ is the solution of $ax \equiv b \mod m$.
2. Assume p is prime and $0 < a < p$. Prove that $x = b(-1)^{a-1}\frac{\binom{p}{a}}{p} \mod p$ is solution of $ax \equiv b \mod p$.

SOLUTION.

1. Only need to check $aba^{\phi(m)-1} \equiv b \mod m$. Since $\gcd(a, m) = 1$, easily $a^{\phi(m)} \equiv 1 \mod m$, so it's obvious.
2. We multiply $a!$ to the equation, we get $a!x \equiv b(-1)^{a-1} \prod_{k=1}^{a-1}(-k) \equiv b(a-1)! \mod p$. Since $0 < a < p$, we get $\gcd((a-1)!, p) = 1$, so $ax \equiv b \mod p$.

□

PROBLEM IV Solve the equation:

$$\begin{cases} x \equiv 1 \mod 2 \\ x \equiv 2 \mod 5 \\ x \equiv 3 \mod 7 \\ x \equiv 4 \mod 9 \end{cases}$$

SOLUTION. Let $m_1 = 2, m_2 = 5, m_3 = 7, m_4 = 9$, and $M_1 = 315, M_2 = 126, M_3 = 90, M_4 = 70$. Then $M_1' = 1, M_2' = 1, M_3' = -1, M_4' = 4$. So $x \equiv 1 \times 315 \times 1 + 1 \times 126 \times 2 - 1 \times 90 \times 3 + 4 \times 70 \times 4 \equiv 1417 \equiv 157 \mod 630$ □

I think this question should be as follows, but may be I make a mistake. PROBLEM V

1. Assume $m_1, \cdots, m_k \in \mathbb{N}^+, b_1, \cdots, b_k \in \mathbb{Z}$, and $\forall i, j, \gcd(m_i, m_j) \mid b_i - b_j$. Let $m_i' := \prod_{p \in \mathbb{P}, \forall j < i, v_p(m_j) < v_p(m_i) \wedge \forall j, v_p(m_j) \le v_p(m_i)} p^{v_p(m_i)}$, where $\mathbb{P}$ is the set of primes, and $v_p(x)$ is the biggest integer t such that $p^t \mid x$. Then following two equation has same solution:

$$x \equiv b_i \mod m_i, \forall i \tag{1}$$

$$x \equiv b_i \mod m_i', \forall i \tag{2}$$

2. find the solution of

$$\begin{cases} x \equiv 0 \mod 5 \\ x \equiv 10 \mod 715 \\ x \equiv 140 \mod 247 \\ x \equiv 245 \mod 391 \\ x \equiv 109 \mod 187 \end{cases}$$

SOLUTION .

1. Easily solution of Equation (1) must be solution of Equation (2), now we will prove the reverse. Assume $x \equiv b_i \mod m_i', \forall i$. Now we will prove $x \equiv b_i \mod m_i$. Only need to prove $\forall p \in \mathbb{P}, x \equiv b_i \mod p^{v_p(m_i)}$. Assume $j = \min\{t : v_p(m_t) = \max_r v_p(m_r)\}$, then by the defination of m_i', we know that $p^{v_p(m_j)} \mid m_j'$. And easily $p^{v_p(m_i)} \mid p^{v_p(m_j)}$, so we get $x \equiv b_j \mod p^{v_p(m_i)}$. More over, easily to know $p^{v_p(m_i)} \mid \gcd(m_i, m_j)$, so $b_j \equiv b_i \mod p^{v_p(m_i)}$. So finally we get the result.

   It is easy to prove that $\forall i \neq j, \gcd(m_i', m_j') = 1$, so we can solve the second equation.

2. From above we get the given equation is equvilate to

$$\begin{cases} x \equiv 0 \mod 5 \\ x \equiv 10 \mod 143 \\ x \equiv 7 \mod 19 \\ x \equiv 245 \mod 391 \\ x \equiv 0 \mod 1 \end{cases}$$

   Assume $x = 5y$, then we get $\begin{cases} y \equiv 2 \mod 143 \\ y \equiv 9 \mod 19 \\ y \equiv 49 \mod 391 \end{cases}$. Solve this equation, we get $y \equiv 2004 \mod 1062347$. So finally we get $x \equiv 10020 \mod 5311735$.

$\square$