

under Graduate Homework In Mathematics

Set Theory 3

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General fire extinguisher

PROBLEM I Prove the following statements.

1. If $x \cap y = \emptyset$ and $x \cup y \preccurlyeq y$, then $\omega \times x \preccurlyeq y$.
2. If $x \cap y = \emptyset$ and $\omega \times x \preccurlyeq y$, then $x \cup y \approx y$.

SOLUTION. 1. Assume $f : x \cup y \rightarrow y$ is injective, define $f_1 = f, f_{n+1} = f_n \circ f$. Let $g : \omega \times x \rightarrow y, g(n, t) \mapsto f_{n+1}(t)$. We only need to prove g is injective. For $(n, u), (m, v) \in \omega \times x$, if $n = m$, then since f is injective we get f_n is injective, so $f_n(u) \neq f_n(v)$. Else, $m \neq n$, assume $n < m, m = n + k$. Obviously $f_{n+k}[x] = f_n[f_k[x]] \subset f_n[y]$. Since f_n is injective, we get $f_n[x] \cap f_n[y] = \emptyset$. So $f_n[x] \ni g(n, u) \neq g(m, v) \in f_n[y]$. So g is injective.

2. Assume $f : \omega \times x \rightarrow y$ is injective. Let $x_n := \{(n, t) : t \in x\}$. Then $\omega \times x = \bigcup_{n \in \omega} x_n$. Consider $g : x \cup y \rightarrow y$, for $t \in x$ let $g(t) := f(0, t)$, for $t \in f[x_n]$, let $g(t) = f(n + 1, t)$, for other t , let $g(t) = t$. Then we prove g is bijection.

First we prove g is injection. For $u, v \in x \cup y, u \neq v$, we will prove $g(u) \neq g(v)$.

- $u, v \in x$: Since f is injective, we have $g(u) = f(0, u) \neq f(0, v) = g(v)$.
- $u \in x, v \notin x$: From definition we obtain $f(u) \in f[x_0]$. If $v \in f[x_n]$ for some n , then $f(v) \in f[x_{n+1}]$. Since f is injective, $f[x_0] \cap f[x_{n+1}] = \emptyset$. So $f(u) \neq f(v)$. Else, we know $g(v) = v \notin f[x_0] \ni f(u)$.
- $u \in f[x_m], v \in f[x_n]$: If $m = n$ then $g(u) = f(m + 1, u) \neq f(n + 1, v) = g(v)$. Else $m \neq n$, then $g(u) \in f[x_{m+1}], g(v) \in f[x_{n+1}], f[x_{m+1}] \cap f[x_{n+1}] = \emptyset$. So $g(u) \neq g(v)$.
- $u \in x_n, v \notin x, \forall m, v \notin f[x_m]$: Easily $g(u) \in f[x_{n+1}]$ and $g(v) = v \notin f[x_{n+1}]$.
- $u, v \notin x, \forall n, u, v \notin f[x_n]$: Easily $g(u) = u \neq v = g(v)$.

Second we prove g is surjective. i.e., $\forall u \in y, \exists t \in x \cup y, g(t) = u$.

- $u \in f[x_n]$ for some n : If $n = 0$ then $y = f(0, t)$ for some $t \in x$. Then $g(t) = u$. Else $n \geq 1$, write $n = m + 1$. Then $y = f(m + 1, t)$ for some $t \in x$. So $g(t) = u$.
- $u \notin f[x_n], \forall n$: Easily we get $g(u) = u$.

So all in all g is bijective.

□

PROBLEM II

1. A subset of a finite set is finite.
2. The union of a finite set of finite sets is finite.
3. The power set of a finite set is finite.
4. The image of a finite set (under a mapping) is finite.

SKETCH. 1. Use MI to prove $\varphi(n) := \forall A \approx n, \forall B \subset A, \exists m \in \omega, B \approx m$ for $n \in \omega$. When $n = 0$, we know $A \approx 0 \rightarrow A = \emptyset$. So $B = \emptyset$ and thus $B \approx 0$. Now we prove $\varphi(n) \rightarrow \varphi(n+1)$. For $A \approx n+1$, if $B = A$ then $B \approx n+1$. Else, $\exists x \in A \setminus B$. Assume $f : A \rightarrow n+1$ is bijection.

$$\text{Consider } g : A \rightarrow n+1 \begin{cases} g(t) = f(t) & t \neq x \wedge g(t) \neq n \\ g(t) = n+1 & t = x \\ g(t) = f(x) & f(t) = n \end{cases} \quad \text{Easy to know } g \text{ is bijection, too.}$$

And since $x \notin B$ we get $B \subset g^{-1}[n] \approx n$, so by induction we get $\exists m \in \omega, B \approx m$.

2. First we prove union of two disjoint finite sets A and B is finite.

Use MI to the card of the second set B . For $B = \emptyset$ it's obvious. For $B \approx 1$, assume $A \approx n$, Thrn $B \approx \{n\}$, so $A \cup B \approx n \cup \{n\} = n+1$ is finite. Assume for certain $n, \forall B \approx n$ it's right, now we prove it's right for $n+1$. Assume $f : B \rightarrow n+1$ is bijection, then $f^{-1}[n] \approx n$, so $A \cup f^{-1}[n]$ is finite. Since $B = f^{-1}[n] \cup \{f^{-1}(n)\}$ we get $A \cup B = A \cup f^{-1}[n] \cup \{f^{-1}(n)\}$. Since $\{f^{-1}(n)\} \approx 1$, so the union is finite.

For general two finite sets A, B we have $A \cup B = A \cup (B \setminus A)$ and from II.1 we know $B \setminus A$ is finite, so $A \cup B$ is finite. Now we use MI to prove $\varphi(n) := \forall x \approx n, (\forall y \in x, \text{isFinite}(y)) \rightarrow \text{isFinite}(\bigcup x)$ for $n \in \omega$. When $n = 0, 1, 2$ it's obvious. Assume for certain $n \geq 2$ we have $\varphi(n)$, then we prove $\varphi(n+1)$. Assume $f : x \rightarrow n+1$ is bijective, let $y = f^{-1}[n] \subset x$. Then $y \approx n$, by induction we know $\bigcup y$ is finite. Since $x = y \cup \{f^{-1}(n)\}$ we get $\bigcup x = (\bigcup y) \cup f^{-1}(n)$. So $\bigcup x$ is finite, too.

3. Use MI of the card. For $x \approx 0$ we know $\mathcal{P}(x) = \{\emptyset\} \approx 1$. Assume for certain n we have $\forall x \approx n, \text{isFinite}(\mathcal{P}(x))$, then for $x \approx n+1$: Assume $f : x \rightarrow n+1$ is bijection. Let $y = f^{-1}[n]$ and $t = f^{-1}(n)$. Then $y \approx n$. Let $\theta : \mathcal{P}(x) \setminus \mathcal{P}(y) \rightarrow \mathcal{P}(y), \theta(a) := a \setminus \{t\}$. Easily θ is bijective, so $\mathcal{P}(x) \setminus \mathcal{P}(y) \approx \mathcal{P}(y)$ is finite. From II.2 we know $\mathcal{P}(x) = \mathcal{P}(y) \cup (\mathcal{P}(x) \setminus \mathcal{P}(y))$ is finite.

4. Use MI by card. For $A \approx 0$ it's obvious. Assume for $A \approx n$ it's right, now we prove for $A \approx n+1$ it's right, too. Let $f : A \rightarrow n+1$ is a bijection, and $g : A \rightarrow \text{Set}$ is a map on A . Let $B := f^{-1}[n] \subset A, t = f^{-1}(n) \in A$. Then $B \approx n$, so by induction we know $g[B]$ is finite. Since $A = B \cup \{t\}$ we get $g[A] = g[B] \cup g[\{t\}] = g[B] \cup \{g(t)\}$. Noting $\{g(t)\} \approx 1$ is finite, from II.2 we get $g[A]$ is finite, too.

□

PROBLEM III

1. A subset of a countable set is at most countable.
2. The union of a finite set of countable sets is countable.
3. The image of a countable set (under a mapping) is at most countable.

SKETCH. 1. Assume A is countable and $\theta : A \rightarrow \omega$ is bijection. For $B \subset A$, we have $B \approx \theta[B]$. So we only need to prove every subset of ω is at most countable. Let $x \subset \omega$. If x is finite,

then there is nothing to do. Now assume x is infinite. We define f on ω by induction. Let $f(0) = \min x$ and $f(n) = \min(x \setminus f[n])$. Since x is infinite, we know $f[n] \subsetneq x$, so f is well-defined. And easily to prove f is a bijection. So $x \approx \omega$ is countable.

2. Use MI to the number of sets to cup, write n . When $n = 1$ it's obvious. For $n = 2$, we should prove two countable sets u, v 's union $u \cup v$ is countable. Let $f : \omega \rightarrow u, g : \omega \rightarrow v$ is bijections, we need to find a bijection $h : \omega \rightarrow u \cup v$. We define h by induction. Let $h(n) = f(\min f^{-1}[u \setminus h[n]])$ for $2 \mid n$ and let $h(n) = g(\min g^{-1}[v \setminus h[n]])$ for $n \nmid 2$. Since u, v is infinite we obtain h is well-defined. Now we prove h is bijective. First we prove h is injective. For $m, n \in \omega, m \neq n$, assume $m < n$. If $x \mid n$, then $h(n) = f(\min f^{-1}[u \setminus h[n]]) \in f[f^{-1}[u \setminus h[n]]] = u \setminus h[n]$ and $h(m) \in h[n]$. So $h(m) \neq h(n)$. If $x \nmid n$ for the same reason we get $h(m) \neq h(n)$.

Second we prove h is surjective. Only need to prove $u, v \subset h[\omega]$. By symmetry we only need to prove $u \subset h[\omega]$. Since $u = f[\omega]$, we only need to prove $f[n] \subset h[2n - 1], \forall n \in \omega$. Use MI to prove it. For $n = 0$ it's obvious. Assume for certain n it's right, for $n + 1$, we only need to prove $a := f(n) \in h[2n + 1]$. If not, since $h(2n) = f(\min f^{-1}[u \setminus h[2n]])$ and $a \notin h[2n]$, we have $a \in u \setminus h[2n]$. Then $n = f^{-1}(a) \in f^{-1}[u \setminus h[2n]]$. For $m < n$, by induction we get $f(m) \in h[2m - 1] \subset h[2n]$, so $m \notin f^{-1}[u \setminus h[2n]]$, thus $n = \min f^{-1}[u \setminus h[2n]]$. So $h(2n) = a$, contradiction! So h is surjective.

Now we assume for certain $n \geq 2$ we have union of n countable sets is countable, we need to prove so do $n + 1$ sets. Assume $A \approx n + 1$ and $\forall x \in A, x \approx \omega$. Assume $f : A \rightarrow n + 1$ is bijection, and let $B := f^{-1}[n], t = f^{-1}(n)$, then $\bigcup A = (\bigcup B) \cup t$. By induction we know $\bigcup B$ is countable. And we have proved union of two countable sets is countable. So finally we get $\bigcup A$ is countable.

3. Only need to prove image of ω is at most countable. For $f : \omega \rightarrow \text{Set}$ is a map, we need to prove $\text{ran}(f)$ is at most countable. Let $h : \text{ran}(f) \rightarrow \omega, t \mapsto \min f^{-1}[\{t\}]$. Obviously h is a injective, so $\text{ran}(f)$ is at most countable.

□

PROBLEM IV $\mathbb{N} \times \mathbb{N}$ is countable.

SOLUTION. We will prove $f : \mathbb{N}^2 \rightarrow \mathbb{N}, (m, n) \mapsto 2^m(2n + 1) - 1$ is bijection. First we prove it's injection. Assume $f(a, b) = f(c, d)$, then $2^a(2b + 1) = 2^c(2d + 1)$. If $a \neq c$, assume $a < c$, then $2b + 1 = 2^{c-a}(2d + 1)$. But $2 \mid 2^{c-a}(2d + 1), 2 \nmid 2b + 1$, contradiction! So $a = c$. Then we get $2b + 1 = 2d + 1$, so $b = d$. So f is injective.

Second we prove f is surjective. For $t \in \mathbb{N}$, let $m := \sup\{k : 2^k \mid t + 1\}$. Since $0 < t + 1 < \omega$ and $2^k \mid t + 1 \rightarrow 2^k \leq t + 1$ we get $m < \omega$. Assume $t + 1 = 2^m \cdot l$, then easily $2 \nmid l$. So we can assume $l = 2n + 1$. Then $t = f(m, n)$. All in all, we get f is bijective. □

PROBLEM V Prove that $\kappa^\kappa \leq 2^{\kappa \times \kappa}$.

SOLUTION. Only need to find a injection $h : {}^\kappa\kappa \rightarrow {}^{\kappa \times \kappa}2$. For $f \in {}^\kappa\kappa$, let $h(f) \in {}^{\kappa \times \kappa}2$, and for $u, v \in \kappa$ let $h(f)(u, v) := \begin{cases} 1 & u = f(v) \\ 0 & u \neq f(v) \end{cases}$. Then we prove h is a injection. Assume $f, g \in {}^\kappa\kappa$ and $h(f) = h(g)$. Then $\forall v \in \kappa$, we have $h(g)(f(v), v) = h(f)(f(v), v) = 1$, so $f(v) = g(v)$. So h is injective. \square

PROBLEM VI If $A \preceq B$, then $A \preceq^* B$.

SOLUTION. If $A = \emptyset$ then it's obvious. Now assume $A \neq \emptyset$ and $a \in A$. Assume $f : A \rightarrow B$ is injection. Let $h : B \rightarrow A, h(y) := \begin{cases} f^{-1}(y) & y \in \text{ran}(f) \\ a & y \notin \text{ran}(f) \end{cases}$. Then $\forall x \in A, h(f(x)) = x$. So h is surjective. \square

PROBLEM VII If $A \preceq^* B$, then $\mathcal{P}(A) \preceq \mathcal{P}(B)$

SOLUTION. If $A = \emptyset$ then $\mathcal{P}(A) = 1$. Let $f : \mathcal{P}(A) \rightarrow \mathcal{P}(B), 0 \mapsto B$, then f is injective. Else we get $A \neq \emptyset$. Then assume $f : B \rightarrow A$ is surjective. Let $h : \mathcal{P}(A) \rightarrow \mathcal{P}(B), U \mapsto f^{-1}[U]$. Then we only need to prove h is injective. Assume $U, V \subset A$ and $h(U) = h(V)$. We get $f^{-1}[U] = f^{-1}[V]$. If $U \neq V$, assume $U \setminus V \neq \emptyset$ and $x \in U \setminus V$, then since f is surjective we get $\exists t \in A, f(t) = x$. So $t \in f^{-1}[U]$ but $t \notin f^{-1}[V]$, contradiction! So h is injective. Then $\mathcal{P}(A) \preceq \mathcal{P}(B)$. \square

PROBLEM VIII Let X be a set. If there is an injective function $f : X \rightarrow X$ such that $\text{ran}(f) \subsetneq X$, then X is infinite.

SOLUTION. Use MI to prove $\forall n \in \omega, X \not\approx n$. For $n = 0$, if $X \approx n$ then $X = 0$. So $X \subset \text{ran}(f)$, contradiction! Assume for certain $n \geq 1$ we get $\forall m < n, X \not\approx m$, then we need to prove $X \not\approx n$. If not, assume $h : X \rightarrow n$ is bijection. Consider $h[\text{ran}(f)] \subsetneq n$, we get $\exists m < n, h[\text{ran}(f)] \approx m$. Since f is injective, and h is bijection, we get $X \approx m$. Contradiction to the induction! So we finally proved $\forall n \in \omega, X \not\approx n$. \square