under**Graduate Homework**

In Mathematics

GroupRepresentation 4

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General fire extinguisher

PROBLEM I Find all of 1−dimentional complex representation of the alternating group A_4.

SOLUTION. Consider the conjugacy classes of A_4. They are: $T_1 = \{(1)\}, T_2 = \{(1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\}, T_3 =$ $\{(1\ 2\ 3), (1\ 4\ 2), (1\ 3\ 4), (2\ 4\ 3)\}, T_4 = \{(1\ 3\ 2), (1\ 2\ 4), (1\ 4\ 3), (2\ 3\ 4)\}$. Assume φ is the representation, then for $a \sim b$ we obtain $\varphi(a) = \varphi(g^{-1}bg) = \varphi(g)^{-1}\varphi(b)\varphi(g) = \varphi(b)$. So $\tau : G/\sim \to \mathbb{C}, [a] \mapsto \varphi(a)$ is well-defined. Since $T_2 \subset A_4'$ we get $\tau(T_2) = 1$. And easily $\tau(T_3)\tau(T_4) = 1, \tau(T_3)^3 = 1$. So $\tau(T_3) = 1, \omega, \bar{\omega}$, where $\omega = -\frac{1}{2} + \frac{\sqrt{3}i}{2}$.

1. $\tau(T_3) = 1$, we get $\varphi(a) = 1, \forall a \in A_4$.
2. $\tau(T_3) = \omega$, we get $\varphi(a) = \begin{cases} \omega & a \in T_3 \\ \bar{\omega} & a \in T_4 \\ 1 & otherwise \end{cases}$.
3. $\tau(T_3) = \bar{\omega}$, we get $\varphi(a) = \begin{cases} \bar{\omega} & a \in T_3 \\ \omega & a \in T_4 \\ 1 & otherwise \end{cases}$.

□

PROBLEM II Consider $N \trianglelefteq S_4$ and $N = \{(1), (12)(34), (13)(24), (14)(23)\}$.

1. Prove: $S_4/N \cong S_3$.
2. Find a 2−dimentional irreducible complex matrix representation of S_4.

SOLUTION.

1. Since $|S_4/N| = 6$ and obviously $S_4/N \ncong C_6$, because $[(1\ 2)][(1\ 3)] \neq [(1\ 3)][(1\ 2)]$, we get $S_4/N \cong S_3$.
2. Consider $\varphi : S_3 \to \mathrm{GL}_2(\mathbb{C}), (2\ 3) \mapsto \bar{\cdot}, (1\ 2\ 3) \mapsto A$, where A is the rotation of $\frac{2\pi}{3}$. Then easily φ is a group representation. Obviously φ is irreducible, so $\bar{\varphi}$ is irreducible. So $\bar{\varphi}$ satisfy the requirement.

□

PROBLEM III Assume K is a field and $m \in \mathbb{N}^*$. Let $\varphi_m(t) := t^m, \forall t \in K^*$, then φ_m is a 1−dimentional K−representation of $(K^*, \cdot)$. Use φ_m to find a 1− dimentional K− representation of $\mathrm{GL}_n(K)$.

SOLUTION. Consider $f : \mathrm{GL}_n(K) \to K, f(A) = |A|$. Since φ_m is group representation, $\varphi_m \circ f$ is group representation of $\mathrm{GL}_n(K)$. So $\bar{\varphi_m} : \mathrm{GL}_n(K) \to K^*, A \mapsto |A|^m$ satisfy the requirement. □

PROBLEM IV Prove that if φ is 1−dimentional complex representation of finite group G, then $G/\ker\varphi$ is a cyclic group.

Solution. Let $\varphi(G) =: T \subset \mathbb{C}$. Since G is finite we get $\forall x \in T, |x| = 1$. Let $a \in T$ and $\arg a \in [0, 2\pi)$ is minimum. For $b \in T$, if $\arg a \nmid \arg b$, then assume $\arg b = \arg a \cdot n + \theta$, where $\theta \in (0, \arg a)$. Then we get $\mathrm{e}^{\theta} = ba^{-n} \in T$ since T is subgroup. Contridiction to $\arg a$ is minimum. So $\forall b \in T, \arg a \mid \arg b$. That means $\exists n \in \mathbb{N}, b = a^n$. So T is cyclic group. Noting $G/\ker f \cong \mathrm{ran}(f) = T$, we get $G/\ker f$ is cyclic group. $\square$

Problem V Prove: If G is a non-cyclic finite group, then there is no faithful $1-$dimentional complex representation of G.

Solution. Assume there is a faithful φ.

If G is not Abel, then exists $a, b \in G$ such that $aba^{-1}b^{-1} \neq e$.
But $\varphi(aba^{-1}b^{-1}) = \varphi(a)\varphi(b)\varphi(a)^{-1}\varphi(b)^{-1} = 1 = \varphi(e)$, contridiction!

If G is Abel, then assume $G = \oplus_{k=1}^{n} G_k$, where G_k is cyclic, and $|G_k| = p_k^{\alpha_k}$. If $\forall i \neq j, p_i \neq p_j$, then G is cyclic, contridiction! So exists $i \neq j$ such that $p_i = p_j$. Assume $p_1 = p_2$. Let $f_i : G \to G_i$ is projection, then $\varphi_i := \varphi \circ f_i$ is group representation of G_i. Assume $G_1 = \langle x \rangle, G_2 = \langle y \rangle$, then $o(\varphi_1(x^{\alpha_1 - 1})) = p_1 = p_2$. So $\exists z \in G_2$ such that $\varphi_2(z) = \varphi_1(\varphi_1(x^{\alpha_1 - 1}))$. Contridiction to φ is faithful! $\square$

Problem VI Assume (φ, V) and (ψ, W) are two $K-$representation of group G. Prove: $(\varphi \dot{+} \psi)^* \approx \varphi^* \dot{+} \psi^*$.

Solution. First we prove $V^* \oplus W^* \cong (V \oplus W)^*$. Consider $\theta : V^* \oplus W^* \to (V \oplus W)^*, \theta(f, g)(u, v) := (f(u), g(v))$. Then obviously θ is a bijection. And $\theta(a(f, g) + b(h, l))(u, v) = \theta(af + bh, ag + bl)(u, v) = ((af + bh)(u), (ag + bl)(v)) = (af(u) + bh(u), ag(u) + bl(u)) = a\theta(f, g)(u, v) + b\theta(h, l)(u, v)$, so θ is isomorphism.

Now we only need to prove $(\varphi \dot{+} \psi)^*(a)\theta = \theta(\varphi^* \dot{+} \psi^*)(a), \forall a \in G$. Forall $f \in V^*, g \in W^*$, we have $(\varphi \dot{+} \psi)^*(a)\theta(f, g) = \theta(f, g)(\varphi \dot{+} \psi)(a)$. And $\theta(\varphi^* \dot{+} \psi^*)(a)(f, g) = \theta(\varphi^*(a)(f), \psi^*(a)(g)) = \theta(f\varphi(a), g\psi(a))$. Easily $\theta(f\varphi(a), g\psi(a)) = \theta(f, g)(\varphi \dot{+} \psi)(a)$, so θ is isomorphism of $(\varphi \dot{+} \psi)^*$ and $\varphi^* \dot{+} \psi^*$. $\square$