Lemma 1. Assume $(B_t : t \geq 0)$ is a random process ranging in $\mathbb{R}$, $a \in \mathbb{R}^+$, and $\forall s, t : 0 \leq s \leq t, B_t - B_s \sim N(0, a(t-s))$. Assume B_t is continuous about t, a.s. Let $\mathcal{F}_t := \sigma(B_s : 0 \leq s \leq t)$. Then $(B_t : t \geq 0)$ is Brownian motion $\iff \forall 0 \leq s \leq t, B_t - B_s \perp \mathcal{F}_s$.

证明. "$\Longrightarrow$": To prove $B_t - B_s \perp \mathcal{F}_s$, only need to prove for $t_1 < t_2 < \cdots < t_{n-1} = s < t = t_n$, we have $B_t - B_s \perp \sigma(B_{t_k} : k = 1, \cdots, n-1)$. Easily $B_t - B_s \perp \sigma(B_{t_{k+1}} - B_{t_k}, B_{t_1} : k = 1, \cdots, n-2) = \sigma(B_{t_k} : k = 1, \cdots, n-1)$, so $B_t - B_s \perp \mathcal{F}_s$.

"$\Longleftarrow$": For $t_1 < \cdots < t_n$, we need to prove $B_{t_1}, B_{t_{k+1}} - B_{t_k} : 1 \leq k \leq n-1$ are independent. Use MI to n. When $n = 1$ it's obvious. Assume we have proved it for certain $n \geq 1$, now consider $n+1$. Since $B_{t_{k+1}} - B_{t_k} \in \mathcal{F}_{t_n}, k = 1, \cdots, n-1$, we have $B_{t_{n+1}} - B_{t_n} \perp \sigma(B_{t_1}, B_{t_{k+1}} - B_{t_k} : k = 1, \cdots, n-1)$. So $\mathbb{P}(B_{t_1} \in A_1, B_{t_{k+1}} - B_{t_k} \in A_{k+1}, k = 1, \cdots, n) = \mathbb{P}(B_{t_1} \in A_1, B_{t_{k+1}} - B_{t_k} \in A_{k+1}, k = 1, \cdots, n-1)\mathbb{P}(B_{t_{n+1}} - B_{t_n} \in A_{n+1})$. By Induction assumption we get $\mathbb{P}(B_{t_1} \in A_1, B_{t_{k+1}} - B_{t_k} \in A_{k+1}, k = 1, \cdots, n) = \mathbb{P}(B_{t_1} \in A_1) \prod_{k=1}^{n} \mathbb{P}(B_{t_{k+1}} - B_{t_k} \in A_{k+1})$. So finally we get $B_{t_1}, B_{t_{k+1}} - B_{t_k} : 1 \leq k \leq n$ are independent. □

PROBLEM I Assume $(B_t : t \geq 0)$ is Brownian motion, prove that for $r > 0$, we have $(B_{t+r} - B_r : t \geq 0)$ is Brownian motion, too.

SOLUTION. Assume $B_t - B_s \sim N(0, a(t-s)), a > 0$. Let $\mathcal{F}_t := \sigma(B_s : 0 \leq s \leq t)$ and $\mathcal{G}_t := \sigma(B_{r+s} - B_r : 0 \leq s \leq t)$. For $0 \leq s \leq t$, we have $B_{t+r} - B_r - (B_{s+r} - B_r) = B_{t+r} - B_{s+r} \sim N(0, a(t-s))$. And easily to know $B_{r+s} - B_r \in \mathcal{F}_{r+s}$, so $\mathcal{G}_t \subset \mathcal{F}_{t+r}, \forall t \geq 0$. Since $(B_t : t \geq 0)$ is Brownian motion, easily $\mathcal{F}_{s+r} \perp B_{t+r} - B_{s+r}$. Since $\mathcal{G}_t \subset \mathcal{F}_{t+r}$, we obtain $\mathcal{G}_t \perp B_{t+r} - B_{s+r} = B_{t+r} - B_r - (B_{s+r} - B_r)$. Easily since B_t is continuous we get $B_{t+r} - B_r$ is continuous. So $(B_{t+r} : t \geq 0)$ is Brownian motion. □

PROBLEM II Assume $(B_t : t \geq 0)$ is standard Brownian motion start at 0. Prove that $\forall c > 0, (cB_{\frac{t}{c^2}} : t \geq 0)$ is standard Brownian motion start at 0, too.

SOLUTION. Since $B_0 = 0$ we get $cB_{\frac{0}{c^2}} = 0$. Let $\mathcal{F}_t := \sigma(B_s : 0 \leq s \leq t)$ and $\mathcal{G}_t := \sigma(cB_{\frac{s}{c^2}} : 0 \leq s \leq t)$. Easily to know $\mathcal{G}_t = \mathcal{F}_{\frac{t}{c^2}}$. For $0 \leq s \leq t$, we have $cB_{\frac{t}{c^2}} - cB_{\frac{s}{c^2}} = c(B_{\frac{t}{c^2}} - B_{\frac{s}{c^2}}) \sim N(0, t-s)$, because $B_{\frac{t}{c^2}} - B_{\frac{s}{c^2}} \sim N(0, \frac{t-s}{c^2})$. And since $(B_t : t \geq 0)$ is Brownian motion, we get $B_{\frac{t}{c^2}} - B_{\frac{s}{c^2}} \perp \mathcal{F}_{\frac{s}{c^2}} = \mathcal{G}_s$. Easily since B_t is continuous we get $cB_{t^2_c}$ is continuous. So $(cB_{\frac{t}{c^2}} : t \geq 0)$ is standard Brownian motion starts at 0, too. □

PROBLEM III Assume $(X_t : t \geq 0)$ and $(Y_t : t \geq 0)$ are two independent standard Brownian motion, $a, b \in \mathbb{R}$ and $\sqrt{a^2 + b^2} > 0$. Prove that $(aX_t + bY_t : t \geq 0)$ is a Brownian motion with parameter $c^2 = a^2 + b^2$.

SOLUTION. Let $\mathcal{F}_t := \sigma(X_s : 0 \leq s \leq t)$ and $\mathcal{G}_t := \sigma(Y_s : 0 \leq s \leq t)$. Let $\mathcal{H}_t := \sigma(aX_s + bY_s : 0 \leq s \leq t)$. Since $(X_t : t \geq 0), (Y_t : t \geq 0)$ are two independent Brownian motion, we know $\forall 0 \leq s \leq t, X_t - X_s \perp \mathcal{F}_s, \mathcal{G}_s; Y_t - Y_s \perp \mathcal{F}_s, \mathcal{G}_s$. So we get $aX_t + bY_t - aX_s - bY_s \perp \mathcal{F}_s, \mathcal{G}_s$, thus $aX_t + bY_t - aX_s - bY_s \perp \sigma(\mathcal{F}_s, \mathcal{G}_s)$ Easily $aX_s + bY_s \in \sigma(\mathcal{F}_s, \mathcal{G}_s)$, so $\mathcal{H}_t \subset \sigma(\mathcal{F}_t, \mathcal{G}, t), \forall t \geq 0$. So $aX_t + bY_t - aX_s - bY_s \perp \mathcal{H}_s$. And easily $a(X_t - X_s) \sim N(0, a^2(t-s)), b(Y_t - Y_s)$ sin $N(0, b^2(t-s))$, and since $\mathcal{F}_t \perp \mathcal{G}_t$ we get $a(X_t - X_s) \perp b(Y_t - Y_s)$, so $aX_t + bY_t - aX_s - bY_s \sim N(0, (a^2+b^2)(t-s))$.

Easily since X_t, Y_t is continuous we get aX_t+bY_t is continuous. So $(aX_t+bY_t : t \geq 0)$ is a Brownian motion with parameter $a^2+b^2=c^2$. $\square$

PROBLEM IV Assume $(B_t : t \geq 0)$ is standard Brownian motion start at 0. Let $X_0 = 0$ and $X_t := tB_{\frac{1}{t}}$. Given

$$\limsup_{t\to\infty} \frac{B_t}{\sqrt{2t\log\log t}} = 1$$

Prove that $(X_t : t \geq 0)$ is standard Brownian motion start at 0.

SOLUTION. First consider the distribution of $X_t - X_s$ for $s < t$. If $s = 0$ then $X_t - X_s = tB_{\frac{1}{t}} \sim N(0, t^2 \times \frac{1}{t}) = N(0,t)$. Else, $s > 0$, then $X_t - X_s = s(B_{\frac{1}{t}} - B_{\frac{1}{s}}) + (t-s)B_{\frac{1}{t}}$. Easily $B_{\frac{1}{s}} - B_{\frac{1}{t}} \sim N(0, \frac{1}{s}) - \frac{1}{t})$, and $B_{\frac{1}{t}} \sim N(0, \frac{1}{t})$, and since $\frac{1}{t} < \frac{1}{s}$ we know $B_{\frac{1}{s}} - B_{\frac{1}{t}} \perp B_{\frac{1}{t}}$. So $X_t - X_s \sim N(0, s^2(\frac{1}{s} - \frac{1}{t}) + (t-s)^2\frac{1}{t}) = N(0, t-s)$.

Second let $\mathcal{G}_t := \sigma(X_s : 0 \leq s \leq t)$, we need to check $X_t - X_s \perp \mathcal{G}_s, \forall 0 \leq s \leq t$. For $s = 0$ we get $\mathcal{G}_s = \{\emptyset, \Omega\}$, so it's obvious. Now assume $s > 0$. Then $\mathcal{G}_s = \sigma(B_{\frac{1}{r}} : 0 \leq r \leq s)$. Only need to prove for any finite set $I \subset [0,s]$, we have $X_t - X_s \perp \sigma(B_{\frac{1}{r}} : r \in I)$. Only need to check $\mathbb{E}(B_{\frac{1}{r}}(X_t - X_s)) = 0$ because they are all normal distributed random variable. Easily $\mathbb{E}(B_{\frac{1}{r}}X_t) = \mathbb{E}((B_{\frac{1}{r}} - B_{\frac{1}{t}})tB_{\frac{1}{t}}) + \mathbb{E}(tB_{\frac{1}{t}}^2) = 1$, and $\mathbb{E}(B_{\frac{1}{r}}X_s) = \mathbb{E}((B_{\frac{1}{r}} - B_{\frac{1}{s}})sB_{\frac{1}{s}}) + \mathbb{E}(sB_{\frac{1}{s}}^2) = 1$ So $\mathbb{E}(B_{\frac{1}{r}}(X_t - X_s)) = 0$.

Finally we need to check X_t is continuous a.s. Easily for $t \neq 0$ we know X_t is continuous at t. Only need to check X_t is continuous at 0 with probability 1. Easily to know $(-B_t : t \geq 0)$ is standard Brownian motion, too. So $\limsup_{t\to\infty} \frac{-B_t}{\sqrt{2t\log\log t}} = 1$. So $\limsup_{t\to\infty} \frac{|B_t|}{2t\log\log t} = 1$. So $\limsup_{t\to 0+} |X_t| = \lim_{t\to\infty} |\frac{1}{t}B_t| \leq \limsup_{t\to\infty} \frac{\sqrt{2t\log\log t}}{t} = 0$. So $\lim_{t\to 0+} |X_t| = 0$. So X_t is continuous with probability 1. $\square$