PROBLEM I Assume $n \in \mathbb{N}^+$ and $2^n + 1$ is prime. Prove that $\exists k \in \mathbb{N}, n = 2^k$.

Lemma 1. Assume $b = ka$ and k is odd, then for $x, y \in \mathbb{N}$, we have $x^a + y^a \mid x^b + y^b$.

证明. Easily $x^b + y^b \equiv (x^a)^k + y^b \equiv (x^a + y^a - y^a)^k + y^b \equiv (-y^a)^k + y^b \equiv 0 \mod x^a + y^a$. So $x^a + y^a \mid x^b + y^b$. □

SOLUTION. Assume n is not power of 2, then $\exists p > 2$ is prime such that $p \mid n$. Let $a = \frac{n}{p}$, then from Lemma 1 we have $2^a + 1^a \mid 2^n + 1^n$. Easily $a = \frac{n}{p} < n$, so $2^a + 1 < 2^n + 1$. And easily $1 < 2^a + 1$. So $2^n + 1$ is not prime, contradiction! So $\exists k \in \mathbb{N}, n = 2^k$. □

PROBLEM II Find the standard decomposition of 30!.

SOLUTION. There are $2, 3, 5, 7, 11, 13, 17, 19, 23, 29$, 10 primes, below 30. So we know 30! can be broken down into power and product of them. By calculation, we can get that:

$$\begin{aligned}
v_2(30!) &= \sum_{k=1}^{\infty}\left[\frac{30}{2^k}\right] &= 15 + 7 + 3 + 1 = 26 \\
v_3(30!) &= \sum_{k=1}^{\infty}\left[\frac{30}{3^k}\right] &= 10 + 3 + 1 = 14 \\
v_5(30!) &= \sum_{k=1}^{\infty}\left[\frac{30}{5^k}\right] &= 6 + 1 = 7 \\
v_7(30!) &= \sum_{k=1}^{\infty}\left[\frac{30}{7^k}\right] &= 4 \\
v_{11}(30!) &= \sum_{k=1}^{\infty}\left[\frac{30}{11^k}\right] &= 2 \\
v_{13}(30!) &= \sum_{k=1}^{\infty}\left[\frac{30}{13^k}\right] &= 2 \\
v_{17}(30!) &= \sum_{k=1}^{\infty}\left[\frac{30}{17^k}\right] &= 1 \\
v_{19}(30!) &= \sum_{k=1}^{\infty}\left[\frac{30}{19^k}\right] &= 1 \\
v_{23}(30!) &= \sum_{k=1}^{\infty}\left[\frac{30}{23^k}\right] &= 1 \\
v_{29}(30!) &= \sum_{k=1}^{\infty}\left[\frac{30}{29^k}\right] &= 1
\end{aligned} \tag{1}$$

So finally we get $30! = 2^{26}3^{14}5^7 7^4 11^2 13^2 17^1 19^1 23^1 29^1$. □

PROBLEM III Assume $n \in \mathbb{N}^+$ and $\alpha \in \mathbb{R}$, prove that:

1. $\left[\frac{[n\alpha]}{n}\right] = [\alpha]$.

2. $\sum_{k=0}^{n-1}[\alpha+\frac{k}{n}]=[n\alpha]$.

SOLUTION. 1. Easily $\left[\frac{[n\alpha]}{n}\right]\leq\left[\frac{n\alpha}{n}\right]\leq[\alpha]$. Now we will prove $\left[\frac{[n\alpha]}{n}\right]\geq[\alpha]$. By the definition of $[\cdot]$ we only need to prove $\frac{[n\alpha]}{n}\geq[\alpha]$. So we only need $[n\alpha]\geq n[\alpha]$. By the definition of $[\cdot]$ it is sufficient to show $n\alpha\geq n[\alpha]$, which is obvious.

2. By 1 easily to know $[\alpha+\frac{k}{n}]=\left[\frac{[n(\alpha+\frac{k}{n})]}{n}\right]=\left[\frac{[n\alpha]+k}{n}\right]$. Let $f:\mathbb{Z}\to\{0,\cdots,n-1\}$ and $f(x)\equiv x \mod n$. Then easily $[\frac{x}{n}]=\frac{x}{n}-\frac{f(x)}{n}$. So we know $\sum_{k=0}^{n-1}[\alpha+\frac{k}{n}]=\sum_{k=0}^{n-1}[\frac{[n\alpha]+k}{n}]=\sum_{k=0}^{n-1}\frac{[n\alpha]+k}{n}-\sum_{k=0}^{n-1}\frac{f([n\alpha]+k)}{n}$. Easily to know $(f([n\alpha]+k):k=1,\cdots,n-1)$ is a replacement of $(k:k=0,\cdots,n-1)$. So finally we get $\sum_{k=0}^{n-1}\frac{[n\alpha]+k}{n}-\sum_{k=0}^{n-1}\frac{f([n\alpha]+k)}{n}=\sum_{k=0}^{n-1}\frac{[n\alpha]+k}{n}-\sum_{k=0}^{n-1}\frac{k}{n}=\sum_{k=0}^{n-1}\frac{[n\alpha]}{n}=[n\alpha]$.

$\square$

PROBLEM IV Assume $r>0, r\in\mathbb{R}$. Let T be the number of integer point in $x^2+y^2\leq r^2$. Prove that

$$T=1+4[r]+8\sum_{0<x\leq\frac{r}{\sqrt{2}}}[\sqrt{r^2-x^2}]-4\left[\frac{r}{\sqrt{2}}\right]^2$$

SOLUTION.

$$T=\sum_{x,y\in\mathbb{Z},x^2+y^2\leq r^2}1=\sum_{x^2+y^2\leq r^2,xy=0}1+\sum_{x^2+y^2\leq r^2,xy\neq0}1=1+\sum_{0<x^2+y^2\leq r^2,xy=0}1+4\sum_{x^2+y^2\leq r^2,x>0,y>0}1$$

By symmetry, we know

$$\sum_{0<x^2+y^2\leq r^2,xy=0}1=4\sum_{x^2+y^2\leq r^2,x>0,y=0}1=4[r]$$

And

$$\sum_{x^2+y^2\leq r^2,x,,y>0}1=\sum_{x^2+y^2\leq r^2,0<x\leq\frac{r}{\sqrt{2}},0<y}1+\sum_{x^2+y^2\leq r^2,0<y\leq\frac{r}{\sqrt{2}},0<x}1-\sum_{x^2+y^2\leq r^2,0<x\leq\frac{r}{\sqrt{2}},0<y\leq\frac{r}{\sqrt{2}}}1$$

Since for $y:0<y\leq\frac{r}{\sqrt{2}}$ we have $0<x<\sqrt{r^2-y^2}$, so there are $[\sqrt{r^2-y^2}]$ different x for each y. So easily

$$\sum_{x^2+y^2\leq r^2,0<y\leq\frac{r}{\sqrt{2}},0<x}1=\sum_{x^2+y^2\leq r^2,0<x\leq\frac{r}{\sqrt{2}},0<y}1=\sum_{0<x\leq\frac{r}{\sqrt{2}}}[\sqrt{r^2-x^2}]$$

For $x,y:0<x\leq\frac{r}{\sqrt{2}}$ we have $x^2+y^2\leq r^2$, so:

$$\sum_{x^2+y^2\leq r^2,0<x\leq\frac{r}{\sqrt{2}},0<y\leq\frac{r}{\sqrt{2}}}1=\left[\frac{r}{\sqrt{2}}\right]^2$$

So finally we get

$$T=1+4[r]+8\sum_{0<x\leq\frac{r}{\sqrt{2}}}[\sqrt{r^2-x^2}]-4\left[\frac{r}{\sqrt{2}}\right]^2$$

$\square$

PROBLEM V Find all integer solution of $306x - 360y = 630$.

SOLUTION. The origin equation is equivalent to $17x - 20y = 35$. Consider $\mod 5$, we get $5 \mid 17x$. So $5 \mid x$. Assume $x = 5k$, then $17k - 4y = 7$. Then $17(k+1) - 4y = 24$, consider $\mod 4$, we get $4 \mid k+1$, so $k+1 = 4s$ and $17s - y = 6$. So $y = 17s - 6$ and easily $x = 5s = 5(4s-1) = 20s - 5$. So $\begin{cases} x = 20s - 5 \\ y = 17s - 6 \end{cases}$ is all solutions of the equation. $\square$

PROBLEM VI Assume $N, a, b \in \mathbb{N}, a, b > 0, \gcd(a, b) = 1$. Prove that the number of positive integer solutions of the equation $ax + by = N$ is $\left[\frac{N}{ab}\right]$ or $\left[\frac{N}{ab}\right] + 1$.

SOLUTION. Since $\gcd(a, b) = 1$, we know $\exists s, t \in \mathbb{Z}, as + bt = N$. So we know $x = s + kb, y = t - ka$. Let $x, y > 0$, we get $k > -\frac{s}{b}, k < \frac{t}{a}$. So we know the number of solution is $\left[\frac{t}{a}\right] + \left[\frac{s}{b}\right] + 1$. Now we only need $\left[\frac{N}{ab}\right] \le \left[\frac{t}{a}\right] + \left[\frac{s}{b}\right] + 1 \le \left[\frac{N}{ab}\right] + 1$.

To prove $\left[\frac{N}{ab}\right] \le \left[\frac{t}{a}\right] + \left[\frac{s}{b}\right] + 1$, it is sufficient to show $\left[\frac{N}{ab}\right] \le \left[\frac{t}{a}\right] + \frac{s}{b} + 1$. Only need to show $\left[\frac{N}{ab}\right] \le \frac{t}{a} + \frac{s}{b}$. Noting $ab\left[\frac{N}{ab}\right] \le ab\frac{N}{ab} = N = as + bt = ab(\frac{t}{a} + \frac{s}{b})$ it's obvious.

To prove $\left[\frac{t}{a}\right] + \left[\frac{s}{b}\right] + 1 \le \left[\frac{N}{ab}\right] + 1$, we only need $\left[\frac{t}{a}\right] + \left[\frac{s}{b}\right] \le \frac{N}{ab}$. Noting $ab(\left[\frac{t}{a}\right] + \left[\frac{s}{b}\right]) \le ab(\frac{t}{a} + \frac{s}{b}) = as + bt = N = ab(\frac{N}{ab})$ it's obvious. $\square$

PROBLEM VII Write $\frac{17}{60}$ as sum of three reduced fraction whose denominators are coprime to each other.

SOLUTION. Consider $\frac{17}{60} = \frac{x}{4} + \frac{y}{3} + \frac{z}{5}$, i.e., $17 = 15x + 20y + 12z$. Since $\gcd(15, 20, 12) = 1$, we know this equation has some solution. Easy to know $x = -1, y = 1, z = 1$ is a solution. So $\frac{17}{60} = -\frac{1}{4} + \frac{1}{3} + \frac{1}{5}$ satisfy the condition. $\square$