

**PROBLEM I** Let  $(X_n : n \geq 0) \perp (Y_n : n \geq 0)$  are Markov chain on  $E$  with Transition matrix  $(p_{ij} : i, j \in E), (q_{ij} : i, j \in E)$  respectively. Prove:  $\{(X_n, Y_n) : n \geq 0\}$  are Markov chain on  $E \times E$ . And calculate the transition matrix of  $(X_n, Y_n) : n \geq 0$ .

*SOLUTION.*

$$\begin{aligned}
& \mathbb{P}(X_0 = i_0, \dots, X_{n+1} = i_{n+1}, Y_0 = j_0, \dots, Y_{n+1} = j_{n+1}) \\
&= \mathbb{P}(X_0 = i_0, \dots, X_{n+1} = i_{n+1}) \mathbb{P}(Y_0 = j_0, \dots, Y_{n+1} = j_{n+1}) \\
&= \mathbb{P}(X_0 = i_0) \prod_{k=0}^n p_{i_k i_{k+1}} \mathbb{P}(Y_0 = j_0) \prod_{k=0}^n q_{j_k j_{k+1}} \\
&= \mathbb{P}((X_0, Y_0) = (i_0, j_0)) \prod_{k=0}^n p_{i_k i_{k+1}} q_{j_k j_{k+1}} \\
&= \mathbb{P}((X_0, Y_0) = (i_0, j_0)) \prod_{k=0}^n \mathbb{P}(X_k = i_k, X_{k+1} = i_{k+1}) \mathbb{P}(Y_k = j_k, Y_{k+1} = j_{k+1}) \\
&= \mathbb{P}((X_0, Y_0) = (i_0, j_0)) \prod_{k=0}^n \mathbb{P}((X_k, Y_k) = (i_k, j_k), (X_{k+1}, Y_{k+1}) = (i_{k+1}, j_{k+1}))
\end{aligned}$$

So we get that  $((X_n, Y_n) : n \in \mathbb{N})$  is Markov chain with transition matrix  $r_{(i,j),(m,n)} = p_{im}q_{jn}$ .  $\square$

**PROBLEM II** Let  $S_n$  be 1-dimensional simple random walk,  $a \in \mathbb{Z}$ . Let  $\tau := \inf\{n \geq 0 : S_n = a\}$ . Prove:

1.  $(S_{\tau+n} : n \geq 0)$  is a one dimensional simple random walk.
2.  $(S_{n \wedge \tau} : n \geq 0)$  is a Markov chain on  $\mathbb{Z}$  and give its Transition matrix.
3.  $(S_{n \wedge \tau} : n \geq 0) \perp (S_{\tau+n} : n \geq 0)$ .

SOLUTION . 1.

$$\begin{aligned}
& \mathbb{P}(S_\tau = i_0, S_{\tau+1} = i_1, \dots, S_{\tau+n} = i_n \mid \tau < \infty) \\
&= \sum_{k \in \mathbb{N}} \mathbb{P}(\tau = k, S_\tau = i_0, S_{\tau+1} = i_1, \dots, S_{\tau+n} = i_n \mid \tau < \infty) \\
&= \sum_{k \in \mathbb{N}} \mathbb{P}(\tau = k, S_k = i_0, S_{k+1} = i_1, \dots, S_{k+n} = i_n \mid \tau < \infty) \\
&= \mathbb{1}(a = i_0) \sum_{k \in \mathbb{N}} \mathbb{P}(S_0 \neq a, \dots, S_{k-1} \neq a, S_k = a, S_{k+1} = i_1, \dots, S_{k+n} = i_n \mid \tau < \infty) \\
&= \frac{\mathbb{1}(a = i_0) \sum_{k \in \mathbb{N}} \mathbb{P}(S_0 \neq a, \dots, S_{k-1} \neq a, S_k = a, S_{k+1} = i_1, \dots, S_{k+n} = i_n)}{\mathbb{P}(\tau < \infty)} \\
&= \frac{\mathbb{1}(a = i_0)}{\mathbb{P}(\tau < \infty)} \sum_{k \in \mathbb{N}} \mathbb{P}(S_0 \neq a, \dots, S_{k-1} \neq a, S_k = a, S_{k+1} = i_1, \dots, S_{k+n} = i_n) \\
&= \frac{\mathbb{1}(a = i_0)}{\mathbb{P}(\tau < \infty)} \sum_{k \in \mathbb{N}} \mathbb{P}(S_{k+1} = i_1, \dots, S_{k+n} = i_n \mid S_0 \neq a, \dots, S_{k-1} \neq a, S_k = a) \\
&\quad \times \mathbb{P}(S_0 \neq a, \dots, S_{k-1} \neq a, S_k = a) \\
&= \frac{\mathbb{1}(a = i_0)}{\mathbb{P}(\tau < \infty)} \sum_{k \in \mathbb{N}} \mathbb{P}(S_{k+1} = i_1, \dots, S_{k+n} = i_n \mid S_k = a) \mathbb{P}(\tau = k) \\
&= \frac{\mathbb{1}(a = i_0)}{\mathbb{P}(\tau < \infty)} \sum_{k \in \mathbb{N}} \prod_{l=0}^{n-1} p_{i_l i_{l+1}} \mathbb{P}(\tau = k) = \mathbb{1}(a = i_0) \prod_{l=0}^{n-1} p_{i_l i_{l+1}}
\end{aligned}$$

Where  $p_{ij} : i, j \in \mathbb{Z}$  is the transition matrix of  $S_n : n \in \mathbb{N}$ . So  $(S_{\tau+n} : n \in \mathbb{N})$  is Markov chain with transition matrix same as  $S_n$ .

2.

$$\begin{aligned}
& \mathbb{P}(S_{\tau \wedge 0} = i_0, S_{\tau \wedge 1} = i_1, \dots, S_{\tau \wedge n} = i_n \mid \tau < \infty) \\
&= \sum_{k \in \mathbb{N}} \mathbb{P}(\tau = k, S_{\tau \wedge 0} = i_0, S_{\tau \wedge 1} = i_1, \dots, S_{\tau \wedge n} = i_n \mid \tau < \infty) \\
&= \sum_{k \in \mathbb{N}} \mathbb{P}(\tau = k, S_{k \wedge 0} = i_0, S_{k \wedge 1} = i_1, \dots, S_{k \wedge n} = i_n \mid \tau < \infty) \\
&= \sum_{k \geq n} \mathbb{P}(\tau = k, S_0 = i_0, \dots, S_n = i_n \mid \tau < \infty) \\
&\quad + \sum_{k < n} \mathbb{P}(\tau = k, S_0 = i_0, \dots, S_{k-1} = i_{k-1}, S_k = i_k = i_{k+1} = \dots = i_n \mid \tau < \infty) \\
&= \mathbb{1}(i_0, i_1, \dots, i_n \neq a) \prod_{k=0}^{n-1} p_{i_k i_{k+1}} + \sum_{k=0}^{n-1} \mathbb{1}(i_0, \dots, i_{k-1} \neq a, i_k = i_{k+1} = \dots = i_n = a) \prod_{l=0}^{k-1} p_{i_l i_{l+1}} \\
&= \prod_{k=0}^{n-1} (\mathbb{1}(i_k = i_{k+1} = a) + \mathbb{1}(i_k \neq a) p_{i_k, i_{k+1}})
\end{aligned}$$

So  $(S_{n \wedge \tau} : n \in \mathbb{N})$  is Markov chain with transition matrix  $q_{i,j} = \mathbb{1}(i = j = a) + \mathbb{1}(i \neq a) p_{i,j}$ .

3. By the corollary 3.2.11, we only need to proof  $\tau$  is stopping time on  $(\mathcal{F}_n : n \geq 0)$ , Where  $\mathcal{F}_n = \sigma(S_k : k \leq n)$ . So we only need to prove  $\forall n \in \mathbb{N}, \{\tau = n\} \in \mathcal{F}_n$ . Since  $\{\tau = n\} = \{\omega \in \omega : S_0, \dots, S_{n+1} \neq a, S_n = a\} = \bigcap_{0 \leq k \leq n} \{S_k \neq a\} \cap \{S_n = a\}$ , And  $\{S_k \neq a\} \in \sigma(S_k), \forall 0 \leq k \leq n, \{S_n = a\} \in \sigma(S_n)$ , Then  $\{\tau = n\} \in \mathcal{F}_n$ .

□

**PROBLEM III** Let  $S_n$  be 1-dimensional symmetry simple random walk starting from zero. Prove:  $(|S_n| : n \geq 0)$  is a Markov chain on  $\mathbb{Z}^+$  and give its transition matrix.

**SOLUTION**. Only need to solve problem IV. □

**PROBLEM IV** Let  $S_n$  be 1-dimensional simple random walk starting from zero. Prove:  $(|S_n| : n \geq 0)$  is a Markov chain on  $\mathbb{Z}^+$  and give its transition matrix.

**SOLUTION**. By the definition of  $|S_n|$ , we can easily get to know  $\forall (i_0, \dots, i_n) \in \mathbb{Z}^+, \mathbb{P}(|S_k| = i_k, k = 0, \dots, n) > 0 \iff i_0 = 0, |i_k - i_{k-1}| = 1, k = 1, \dots, n$ . Let  $S_n = \sum_{k=1}^n \xi_k$ , where  $(\xi_n : n \geq 1)$  are i.i.d. r.v. and  $\mathbb{P}(\xi_1 = 1) = p, \mathbb{P}(\xi_1 = -1) = q$ .  $A := \{(i_0, \dots, i_{n+1}) \in \mathbb{Z} : i_0 = 0, |i_k - i_{k-1}| = 1, k = 1, \dots, n+1\}$ .  $\forall (i_0, \dots, i_{n+1}) \in A$ , let  $r := \max\{k < n+1 : i_k = 0\}$ . Then  $i_r = 0, \forall k : n+1 > k \geq r+1, i_k \geq 1$ .

1.  $\forall (i_0, \dots, i_{n+1}) \notin A$ , then  $\mathbb{P}(|S_k| = i_k, k = 0, \dots, n) = 0$ , Then we have no need to calculate  $\mathbb{P}(|S_{n+1}| = i_{n+1} | |S_k| = i_k, k = 0, \dots, n)$ .
2. If  $(i_0, \dots, i_{n+1}) \in A \wedge r = n$ , then  $i_n = 0, i_{n+1} = 1$ . Then  $|S_n| = 0 \iff S_n = 0 \implies S_{n+1} = \pm 1 \iff |S_{n+1}| = 1$ . So we get that  $\mathbb{P}(|S_{n+1}| = i_{n+1} | |S_k| = i_k, k = 1, \dots, n) = 1 = \mathbb{P}(|S_{n+1}| = i_{n+1} | |S_n| = i_n)$ .
3.  $\forall (i_0, \dots, i_{n+1}) \in A, i_n \neq 0$ ,

$$\begin{aligned}
& \mathbb{P}(|S_k| = i_k, S_n = i_n, k = 0, \dots, n | |S_k| = i_k, k = 0, \dots, r) \\
&= \mathbb{P}(|S_k| = i_k, S_n = i_n, k = r+1, \dots, n | |S_k| = i_k, k = 0, \dots, r-1, S_r = 0) \\
&= \mathbb{P}(|S_k| = i_k, S_n = i_n, k = r+1, \dots, n | S_r = 0) \\
&= \mathbb{P}(S_k = i_k, S_n = i_n, k = r+1, \dots, n | S_r = 0) \\
&= p^{\frac{n-r+i_n}{2}} q^{\frac{n-r-i_n}{2}}
\end{aligned}$$

In the same way, we can get

$$\mathbb{P}(|S_k| = i_k, S_n = -i, k = 0, \dots, n | |S_k| = i_k, k = 0, \dots, r) = p^{\frac{n-r-i_n}{2}} q^{\frac{n-r+i}{2}}$$

So

$$\begin{aligned}
& \mathbb{P}(S_n = i_n | |S_k| = i_k, k = 0, \dots, n) \\
&= \frac{\mathbb{P}(S_n = i_n, |S_k| = i_k, k = 0, \dots, n)}{\mathbb{P}(|S_k| = i_k, k = 0, \dots, n)} \\
&= \frac{\mathbb{P}(S_n = i_n, |S_k| = i_k, k = 0, \dots, n | |S_k| = i_k, k = 0, \dots, r) \mathbb{P}(|S_k| = i_k, k = 0, \dots, r)}{\mathbb{P}(|S_k| = i_k, k = 0, \dots, n)} \\
&= \frac{\mathbb{P}(S_n = i_n, |S_k| = i_k, k = 0, \dots, n | |S_k| = i_k, k = 0, \dots, r)}{\frac{\mathbb{P}(|S_k| = i_k, k = 0, \dots, n)}{\mathbb{P}(|S_k| = i_k, k = 0, \dots, r)}} \\
&= \frac{\mathbb{P}(S_n = i_n, |S_k| = i_k, k = 0, \dots, n | |S_k| = i_k, k = 0, \dots, r)}{\mathbb{P}(|S_k| = i_k, k = 0, \dots, n | |S_k| = i_k, k = 0, \dots, r)} \\
&= \frac{\mathbb{P}(S_n = i_n, |S_k| = i_k, k = 0, \dots, n | |S_k| = i_k, k = 0, \dots, r)}{\mathbb{P}(|S_k| = i_k, k = r+1, \dots, n | |S_k| = i_k, k = 0, \dots, r)} \\
&= \frac{p^{n-r+\frac{i_n}{2}} q^{n-r-\frac{i_n}{2}}}{p^{n-r+\frac{i_n}{2}} q^{n-r-\frac{i_n}{2}} + p^{n-r-\frac{i_n}{2}} q^{n-r+\frac{i_n}{2}}} \\
&= p^{i_n} (p^{i_n} + q^{i_n})^{-1}
\end{aligned}$$

In the same way, we can get

$$\mathbb{P}(S_n = -i_n | |S_k| = i_k, k = 0, \dots, n) = q^{i_n} (p^{i_n} + q^{i_n})^{-1}$$

Then

$$\begin{aligned}
& \mathbb{P}(|S_{n+1}| = i_{n+1} | |S_k| = i_k, k = 0, \dots, n) \\
&= \mathbb{P}(|S_{n+1}| = i_{n+1} | S_n = i_n, |S_k| = i_k, k = 0, \dots, n) \mathbb{P}(S_n = i_n | |S_k| = i_k, k = 0, \dots, n) \\
&+ \mathbb{P}(|S_{n+1}| = i_{n+1} | S_n = -i_n, |S_k| = i_k, k = 0, \dots, n) \mathbb{P}(S_n = -i_n | |S_k| = i_k, k = 0, \dots, n) \\
&= \mathbb{P}(S_{n+1} = i_{n+1} | S_n = i_n) \mathbb{P}(S_n = i_n | |S_k| = i_k, k = 0, \dots, n) \\
&+ \mathbb{P}(S_{n+1} = -i_{n+1} | S_n = -i_n) \mathbb{P}(S_n = -i_n | |S_k| = i_k, k = 0, \dots, n) \\
&= \mathbb{1}(i_{n+1} = i_n + 1) (p^{i_n+1} + q^{i_n+1}) (p^{i_n} + q^{i_n})^{-1} + \mathbb{1}(i_{n+1} = i_n - 1) (p^{i_n} q + p q^{i_n}) (p^{i_n} + q^{i_n})^{-1}
\end{aligned}$$

Thus,  $(|S_n| : n \geq 0)$  is Markov chain on  $\mathbb{Z}^+$ , with transition matrix  $r_{ij} = \mathbb{1}(0 \neq i = j - 1) (p^{i+1} + q^{i+1}) (p^i + q^i)^{-1} + \mathbb{1}(0 \neq i = j + 1) (p^i q + p q^i) (p^i + q^i)^{-1} + \mathbb{1}(i = 0, j = 1)$ . When  $p = q = \frac{1}{2}$ , we get  $r_{ij} = \frac{1}{2} \mathbb{1}(i \neq 0, |j - i| = 1) + \mathbb{1}(i = 0, j = 1)$ .

□