

# under Graduate Homework In Mathematics

## Set Theory 3

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**PROBLEM I** Prove the following statements.

1. If  $x \cap y = \emptyset$  and  $x \cup y \preccurlyeq y$ , then  $\omega \times x \preccurlyeq y$ .
2. If  $x \cap y = \emptyset$  and  $\omega \times x \preccurlyeq y$ , then  $x \cup y \approx y$ .

**SOLUTION.** 1. Assume  $f : x \cup y \rightarrow y$  is injective, define  $f_1 = f, f_{n+1} = f_n \circ f$ . Let  $g : \omega \times x \rightarrow y, g(n, t) \mapsto f_{n+1}(t)$ . We only need to prove  $g$  is injective. For  $(n, u), (m, v) \in \omega \times x$ , if  $n = m$ , then since  $f$  is injective we get  $f_n$  is injective, so  $f_n(u) \neq f_n(v)$ . Else,  $m \neq n$ , assume  $n < m, m = n + k$ . Obviously  $f_{n+k}[x] = f_n[f_k[x]] \subset f_n[y]$ . Since  $f_n$  is injective, we get  $f_n[x] \cap f_n[y] = \emptyset$ . So  $f_n[x] \ni g(n, u) \neq g(m, v) \in f_n[y]$ . So  $g$  is injective.

2. Assume  $f : \omega \times x \rightarrow y$  is injective. Let  $x_n := \{(n, t) : t \in x\}$ . Then  $\omega \times x = \bigcup_{n \in \omega} x_n$ . Consider  $g : x \cup y \rightarrow y$ , for  $t \in x$  let  $g(t) := f(0, t)$ , for  $t \in f[x_n]$ , let  $g(t) = f(n + 1, t)$ , for other  $t$ , let  $g(t) = t$ . Then we prove  $g$  is bijection.

First we prove  $g$  is injection. For  $u, v \in x \cup y, u \neq v$ , we will prove  $g(u) \neq g(v)$ .

- $u, v \in x$ : Since  $f$  is injective, we have  $g(u) = f(0, u) \neq f(0, v) = g(v)$ .
- $u \in x, v \notin x$ : From definition we obtain  $f(u) \in f[x_0]$ . If  $v \in f[x_n]$  for some  $n$ , then  $f(v) \in f[x_{n+1}]$ . Since  $f$  is injective,  $f[x_0] \cap f[x_{n+1}] = \emptyset$ . So  $f(u) \neq f(v)$ . Else, we know  $g(v) = v \notin f[x_0] \ni f(u)$ .
- $u \in f[x_m], v \in f[x_n]$ : If  $m = n$  then  $g(u) = f(m + 1, u) \neq f(n + 1, v) = g(v)$ . Else  $m \neq n$ , then  $g(u) \in f[x_{m+1}], g(v) \in f[x_{n+1}], f[x_{m+1}] \cap f[x_{n+1}] = \emptyset$ . So  $g(u) \neq g(v)$ .
- $u \in x_n, v \notin x, \forall m, v \notin f[x_m]$ : Easily  $g(u) \in f[x_{n+1}]$  and  $g(v) = v \notin f[x_{n+1}]$ .
- $u, v \notin x, \forall n, u, v \notin f[x_n]$ : Easily  $g(u) = u \neq v = g(v)$ .

Second we prove  $g$  is surjective. i.e.,  $\forall u \in y, \exists t \in x \cup y, g(t) = u$ .

- $u \in f[x_n]$  for some  $n$ : If  $n = 0$  then  $y = f(0, t)$  for some  $t \in x$ . Then  $g(t) = u$ . Else  $n \geq 1$ , write  $n = m + 1$ . Then  $y = f(m + 1, t)$  for some  $t \in x$ . So  $g(t) = u$ .
- $u \notin f[x_n], \forall n$ : Easily we get  $g(u) = u$ .

So all in all  $g$  is bijective.

□

**PROBLEM II**

1. A subset of a finite set is finite.
2. The union of a finite set of finite sets is finite.
3. The power set of a finite set is finite.
4. The image of a finite set (under a mapping) is finite.

**SKETCH.** 1. Use MI to prove  $\varphi(n) := \forall A \approx n, \forall B \subset A, \exists m \in \omega, B \approx m$  for  $n \in \omega$ . When  $n = 0$ , we know  $A \approx 0 \rightarrow A = \emptyset$ . So  $B = \emptyset$  and thus  $B \approx 0$ . Now we prove  $\varphi(n) \rightarrow \varphi(n+1)$ . For  $A \approx n+1$ , if  $B = A$  then  $B \approx n+1$ . Else,  $\exists x \in A \setminus B$ . Assume  $f : A \rightarrow n+1$  is bijection.

$$\text{Consider } g : A \rightarrow n+1 \begin{cases} g(t) = f(t) & t \neq x \wedge g(t) \neq n \\ g(t) = n+1 & t = x \\ g(t) = f(x) & f(t) = n \end{cases} \quad \text{Easy to know } g \text{ is bijection, too.}$$

And since  $x \notin B$  we get  $B \subset g^{-1}[n] \approx n$ , so by induction we get  $\exists m \in \omega, B \approx m$ .

2. First we prove union of two disjoint finite sets  $A$  and  $B$  is finite.

Use MI to the card of the second set  $B$ . For  $B = \emptyset$  it's obvious. For  $B \approx 1$ , assume  $A \approx n$ , Thrn  $B \approx \{n\}$ , so  $A \cup B \approx n \cup \{n\} = n+1$  is finite. Assume for certain  $n, \forall B \approx n$  it's right, now we prove it's right for  $n+1$ . Assume  $f : B \rightarrow n+1$  is bijection, then  $f^{-1}[n] \approx n$ , so  $A \cup f^{-1}[n]$  is finite. Since  $B = f^{-1}[n] \cup \{f^{-1}(n)\}$  we get  $A \cup B = A \cup f^{-1}[n] \cup \{f^{-1}(n)\}$ . Since  $\{f^{-1}(n)\} \approx 1$ , so the union is finite.

For general two finite sets  $A, B$  we have  $A \cup B = A \cup (B \setminus A)$  and from II.1 we know  $B \setminus A$  is finite, so  $A \cup B$  is finite. Now we use MI to prove  $\varphi(n) := \forall x \approx n, (\forall y \in x, \text{isFinite}(y)) \rightarrow \text{isFinite}(\bigcup x)$  for  $n \in \omega$ . When  $n = 0, 1, 2$  it's obvious. Assume for certain  $n \geq 2$  we have  $\varphi(n)$ , then we prove  $\varphi(n+1)$ . Assume  $f : x \rightarrow n+1$  is bijective, let  $y = f^{-1}[n] \subset x$ . Then  $y \approx n$ , by induction we know  $\bigcup y$  is finite. Since  $x = y \cup \{f^{-1}(n)\}$  we get  $\bigcup x = (\bigcup y) \cup f^{-1}(n)$ . So  $\bigcup x$  is finite, too.

3. Use MI of the card. For  $x \approx 0$  we know  $\mathcal{P}(x) = \{\emptyset\} \approx 1$ . Assume for certain  $n$  we have  $\forall x \approx n, \text{isFinite}(\mathcal{P}(x))$ , then for  $x \approx n+1$ : Assume  $f : x \rightarrow n+1$  is bijection. Let  $y = f^{-1}[n]$  and  $t = f^{-1}(n)$ . Then  $y \approx n$ . Let  $\theta : \mathcal{P}(x) \setminus \mathcal{P}(y) \rightarrow \mathcal{P}(y), \theta(a) := a \setminus \{t\}$ . Easily  $\theta$  is bijective, so  $\mathcal{P}(x) \setminus \mathcal{P}(y) \approx \mathcal{P}(y)$  is finite. From II.2 we know  $\mathcal{P}(x) = \mathcal{P}(y) \cup (\mathcal{P}(x) \setminus \mathcal{P}(y))$  is finite.
4. Use MI by card. For  $A \approx 0$  it's obvious. Assume for  $A \approx n$  it's right, now we prove for  $A \approx n+1$  it's right, too. Let  $f : A \rightarrow n+1$  is a bijection, and  $g : A \rightarrow \text{Set}$  is a map on  $A$ . Let  $B := f^{-1}[n] \subset A, t = f^{-1}(n) \in A$ . Then  $B \approx n$ , so by induction we know  $g[B]$  is finite. Since  $A = B \cup \{t\}$  we get  $g[A] = g[B] \cup g[\{t\}] = g[B] \cup \{g(t)\}$ . Noting  $\{g(t)\} \approx 1$  is finite, from II.2 we get  $g[A]$  is finite, too.

□

### PROBLEM III

1. A subset of a countable set is at most countable.
2. The union of a finite set of countable sets is countable.
3. The image of a countable set (under a mapping) is at most countable.

**SKETCH.** 1. Assume  $A$  is countable and  $\theta : A \rightarrow \omega$  is bijection. For  $B \subset A$ , we have  $B \approx \theta[B]$ . So we only need to prove every subset of  $\omega$  is at most countable. Let  $x \subset \omega$ . If  $x$  is finite,

then there is nothing to do. Now assume  $x$  is infinite. We define  $f$  on  $\omega$  by induction. Let  $f(0) = \min x$  and  $f(n) = \min(x \setminus f[n])$ . Since  $x$  is infinite, we know  $f[n] \subsetneq x$ , so  $f$  is well-defined. And easily to prove  $f$  is a bijection. So  $x \approx \omega$  is countable.

2. Use MI to the number of sets to cup, write  $n$ . When  $n = 1$  it's obvious. For  $n = 2$ , we should prove two countable sets  $u, v$ 's union  $u \cup v$  is countable. Let  $f : \omega \rightarrow u, g : \omega \rightarrow v$  is bijections, we need to find a bijection  $h : \omega \rightarrow u \cup v$ . We define  $h$  by induction. Let  $h(n) = f(\min f^{-1}[u \setminus h[n]])$  for  $2 \mid n$  and let  $h(n) = g(\min g^{-1}[v \setminus h[n]])$  for  $n \nmid 2$ . Since  $u, v$  is infinite we obtain  $h$  is well-defined. Now we prove  $h$  is bijective. First we prove  $h$  is injective. For  $m, n \in \omega, m \neq n$ , assume  $m < n$ . If  $x \mid n$ , then  $h(n) = f(\min f^{-1}[u \setminus h[n]]) \in f[f^{-1}[u \setminus h[n]]] = u \setminus h[n]$  and  $h(m) \in h[n]$ . So  $h(m) \neq h(n)$ . If  $x \nmid n$  for the same reason we get  $h(m) \neq h(n)$ .

Second we prove  $h$  is surjective. Only need to prove  $u, v \subset h[\omega]$ . By symmetry we only need to prove  $u \subset h[\omega]$ . Since  $u = f[\omega]$ , we only need to prove  $f[n] \subset h[2n - 1], \forall n \in \omega$ . Use MI to prove it. For  $n = 0$  it's obvious. Assume for certain  $n$  it's right, for  $n + 1$ , we only need to prove  $a := f(n) \in h[2n + 1]$ . If not, since  $h(2n) = f(\min f^{-1}[u \setminus h[2n]])$  and  $a \notin h[2n]$ , we have  $a \in u \setminus h[2n]$ . Then  $n = f^{-1}(a) \in f^{-1}[u \setminus h[2n]]$ . For  $m < n$ , by induction we get  $f(m) \in h[2m - 1] \subset h[2n]$ , so  $m \notin f^{-1}[u \setminus h[2n]]$ , thus  $n = \min f^{-1}[u \setminus h[2n]]$ . So  $h(2n) = a$ , contradiction! So  $h$  is surjective.

Now we assume for certain  $n \geq 2$  we have union of  $n$  countable sets is countable, we need to prove so do  $n + 1$  sets. Assume  $A \approx n + 1$  and  $\forall x \in A, x \approx \omega$ . Assume  $f : A \rightarrow n + 1$  is bijection, and let  $B := f^{-1}[n], t = f^{-1}(n)$ , then  $\bigcup A = (\bigcup B) \cup t$ . By induction we know  $\bigcup B$  is countable. And we have proved union of two countable sets is countable. So finally we get  $\bigcup A$  is countable.

3. Only need to prove image of  $\omega$  is at most countable. For  $f : \omega \rightarrow \text{Set}$  is a map, we need to prove  $\text{ran}(f)$  is at most countable. Let  $h : \text{ran}(f) \rightarrow \omega, t \mapsto \min f^{-1}[\{t\}]$ . Obviously  $h$  is a injective, so  $\text{ran}(f)$  is at most countable.

□

#### PROBLEM IV $\mathbb{N} \times \mathbb{N}$ is countable.

*SKETCH.* We will prove  $f : \mathbb{N}^2 \rightarrow \mathbb{N}, (m, n) \mapsto 2^m(2n + 1) - 1$  is bijection. First we prove it's injection. Assume  $f(a, b) = f(c, d)$ , then  $2^a(2b + 1) = 2^c(2d + 1)$ . If  $a \neq c$ , assume  $a < c$ , then  $2b + 1 = 2^{c-a}(2d + 1)$ . But  $2 \mid 2^{c-a}(2d + 1), 2 \nmid 2b + 1$ , contradiction! So  $a = c$ . Then we get  $2b + 1 = 2d + 1$ , so  $b = d$ . So  $f$  is injective.

Second we prove  $f$  is surjective. For  $t \in \mathbb{N}$ , let  $m := \sup\{k : 2^k \mid t + 1\}$ . Since  $0 < t + 1 < \omega$  and  $2^k \mid t + 1 \rightarrow 2^k \leq t + 1$  we get  $m < \omega$ . Assume  $t + 1 = 2^m \cdot l$ , then easily  $2 \nmid l$ . So we can assume  $l = 2n + 1$ . Then  $t = f(m, n)$ . All in all, we get  $f$  is bijective. □

#### PROBLEM V Prove that $\kappa^\kappa \leq 2^{\kappa \times \kappa}$ .

**SOLUTION.** Only need to find a injection  $h : {}^\kappa\kappa \rightarrow {}^{\kappa \times \kappa}2$ . For  $f \in {}^\kappa\kappa$ , let  $h(f) \in {}^{\kappa \times \kappa}2$ , and for  $u, v \in \kappa$  let  $h(f)(u, v) := \begin{cases} 1 & u = f(v) \\ 0 & u \neq f(v) \end{cases}$ . Then we prove  $h$  is a injection. Assume  $f, g \in {}^\kappa\kappa$  and  $h(f) = h(g)$ . Then  $\forall v \in \kappa$ , we have  $h(g)(f(v), v) = h(f)(f(v), v) = 1$ , so  $f(v) = g(v)$ . So  $h$  is injective.  $\square$

**PROBLEM VI** If  $A \preceq B$ , then  $A \preceq^* B$ .

**SOLUTION.** If  $A = \emptyset$  then it's obvious. Now assume  $A \neq \emptyset$  and  $a \in A$ . Assume  $f : A \rightarrow B$  is injection. Let  $h : B \rightarrow A, h(y) := \begin{cases} f^{-1}(y) & y \in \text{ran}(f) \\ a & y \notin \text{ran}(f) \end{cases}$ . Then  $\forall x \in A, h(f(x)) = x$ . So  $h$  is surjective.  $\square$

**PROBLEM VII** If  $A \preceq^* B$ , then  $\mathcal{P}(A) \preceq \mathcal{P}(B)$

**SOLUTION.** If  $A = \emptyset$  then  $\mathcal{P}(A) = 1$ . Let  $f : \mathcal{P}(A) \rightarrow \mathcal{P}(B), 0 \mapsto B$ , then  $f$  is injective. Else we get  $A \neq \emptyset$ . Then assume  $f : B \rightarrow A$  is surjective. Let  $h : \mathcal{P}(A) \rightarrow \mathcal{P}(B), U \mapsto f^{-1}[U]$ . Then we only need to prove  $h$  is injective. Assume  $U, V \subset A$  and  $h(U) = h(V)$ . We get  $f^{-1}[U] = f^{-1}[V]$ . If  $U \neq V$ , assume  $U \setminus V \neq \emptyset$  and  $x \in U \setminus V$ , then since  $f$  is surjective we get  $\exists t \in A, f(t) = x$ . So  $t \in f^{-1}[U]$  but  $t \notin f^{-1}[V]$ , contradiction! So  $h$  is injective. Then  $\mathcal{P}(A) \preceq \mathcal{P}(B)$ .  $\square$

**PROBLEM VIII** Let  $X$  be a set. If there is an injective function  $f : X \rightarrow X$  such that  $\text{ran}(f) \subsetneq X$ , then  $X$  is infinite.

**SOLUTION.** Use MI to prove  $\forall n \in \omega, X \not\approx n$ . For  $n = 0$ , if  $X \approx n$  then  $X = 0$ . So  $X \subset \text{ran}(f)$ , contradiction! Assume for certain  $n \geq 1$  we get  $\forall m < n, X \not\approx m$ , then we need to prove  $X \not\approx n$ . If not, assume  $h : X \rightarrow n$  is bijection. Consider  $h[\text{ran}(f)] \subsetneq n$ , we get  $\exists m < n, h[\text{ran}(f)] \approx m$ . Since  $f$  is injective, and  $h$  is bijection, we get  $X \approx m$ . Contradiction to the induction! So we finally proved  $\forall n \in \omega, X \not\approx n$ .  $\square$