underGraduate Homework In Mathematics

AlgebraicGeometry 12

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General fire extinguisher

PROBLEM I Assume $x, y \in \mathbb{C}$ and $\mathrm{Im}x, \mathrm{Im}y > 0$. Assume $\begin{pmatrix} b & a \\ d & c \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z})$, and $x = \frac{a+by}{c+dy}$. Prove that $\begin{vmatrix} b & a \\ d & c \end{vmatrix} = 1$.

SOLUTION. Assume $y = m+n\mathrm{i}, m, n \in \mathbb{R}, n > 0$. Then we get $\mathrm{Im}x = \mathrm{Im}\frac{a+bm+bn\mathrm{i}}{c+dm+dn\mathrm{i}} = \frac{(c+dm)bn-(a+bm)dn}{(c+dm)^2+(dn)^2} > 0$. So we get $cbn + dmbn - adn - bmdn = n(cb - ad) > 0$. Since $n > 0$ we get $cb - ad > 0$, so $\begin{vmatrix} b & a \\ d & c \end{vmatrix} > 0$.

Since $\begin{pmatrix} b & a \\ d & c \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z})$, we get $\begin{vmatrix} b & a \\ d & c \end{vmatrix} = \pm 1$, so finally we get $\begin{vmatrix} b & a \\ d & c \end{vmatrix} = 1$. $\square$

PROBLEM II Assume $\tau, \rho \in \mathbb{C}, \tau^2 = \rho^3 = 1$. Prove that $\exists \sigma \in \mathbb{C}$ such that $\rho = \sigma^4, \tau = \sigma^6$.

SOLUTION. Let $\sigma \in \sqrt{\tau\rho^2}$, i.e., $\sigma^2 = \tau\rho^2$. Then $\sigma^4 = \tau^2\rho^4 = \rho$ and $\sigma^6 = \tau^3\rho^6 = \tau$. $\square$