PROBLEM I Assume $(X_n : n \geq 0)$ is an irreducible Markov chain on E. Prove that $(X_n : n \geq 0)$ is recurrent (or transient) $\iff$ $\forall i \in E$,

$$\mathbb{P}\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\{X_k = i\}\right) = 1(\text{ or } 0).$$

SOLUTION. Only need to prove " $\Longrightarrow$ ".

First we assume $(X_n : n \in \mathbb{N})$ is recurrent, we should prove $\mathbb{P}\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\{X_k = i\}\right) = 1$. Let $\tau_1 = \inf\{n > 0 : X_n = i\}$, and for $n \in \mathbb{N}^+$, we let $\tau_{n+1} = \inf\{n > \tau_n : X_n = i\}$. Since i is recurrent and (X_n) is irreducible, we know that $\tau_1 < \infty, a.s.$. Then $(X_{\tau_1+n} : n \in \mathbb{N})$ is a Markov chain with the same transition matrix as (X_n). So we get that $\tau_2 - \tau_1 < \infty, a.s.$. So $\tau_2 < \infty$, a.s.. Use MI, we can easily get that $\forall n \in \mathbb{N}^+, \tau_n < \infty, a.s.$. Easy to get that $\tau_{n+1} > \tau_n$ and $\tau_1 > 0$, so $\tau_n \geq n$. So $\tau_n < \infty \implies \exists k \geq n, X_k = i$. So $\forall n \in \mathbb{N}, \mathbb{P}\left(\bigcup_{k=n}^{\infty}\{X_k = i\}\right) = 1$. Thus, $\mathbb{P}\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\{X_k = i\}\right) = 1$.

Second we assume $(X_n : n \in \mathbb{N})$ is transient, we should prove that $\mathbb{P}\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\{X_k = i\}\right) = 0$. Write $A = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\{X_k = i\}$. We define τ_n as above. Easy to know $\forall \omega \in A, \forall n \in \mathbb{N}^+, \tau_n < \infty$. And easy to know that $\tau_{n+1} - \tau_n \mid_{\tau_n < \infty}$ has the same distribution for every n. And since (X_n) is transient, we know (X_{τ_k+n}) is transient for every $k \in \mathbb{N}^+$. So we know $\mathbb{P}(\tau_{n+1} - \tau_n < \infty \mid \tau_n < \infty) < 1$. Then $\mathbb{P}(A) = \mathbb{P}(\forall n, \tau_n < \infty) \leq \mathbb{P}(\forall n, \tau_{n+1} - \tau_n < \infty) \leq \prod_{n=1}^{\infty}\mathbb{P}(\tau_{n+1} - \tau_n < \infty \mid \tau_n < \infty) = \prod_{n=1}^{\infty}\mathbb{P}(\tau_2 - \tau_1 < \infty \mid \tau_1 < \infty) = 0$. $\square$

PROBLEM II Assume $(X_n : n \geq 0)$ is Markov chain on E, where E is finite. Prove that $\exists x \in E$, x is recurrent.

SOLUTION. Easily $\sum_{i \in E}\sum_{n=1}^{\infty} p_{ki}(n) = \sum_{n \in \mathbb{N}^+}\sum_{i \in E} p_{ki}(n) = \sum_{n \in \mathbb{N}^+} 1 = +\infty$. Since E is finite, we obtain that there is at least one i such that $\sum_{n \in \mathbb{N}^+} p_{ki}(n) = \infty$, then p^*_{ki}. Then i is recurrent. $\square$

PROBLEM III Assume $(X_n : n \geq 0)$ is Markov chain on $\mathbb{Z}$. Prove it is transient $\iff$ $\forall \mu_0$ is primitive distribution, $\lim_{n\to\infty} |X_n| \overset{\text{a.s.}}{=} \infty$.

SOLUTION. Only need to prove that $\forall k \in \mathbb{N}, \liminf_{n\to\infty} |X_n| > k, a.s.$. Consider the event $\liminf_{n\to\infty} |X_n| \leq k$, it means $\forall n \in \mathbb{N}, \exists t \geq n, X_t \in [-k, k]$. So we only need to prove $\mathbb{P}\left(\bigcap_{n=1}^{\infty}\bigcup_{t=n}^{\infty}\{X_t \in [-k,k]\}\right) = 0$. It is sufficient to prove that $\mathbb{P}(\bigcup_{u\in[-k,k]}\bigcap_{n=1}^{\infty}\bigcup_{t=n}^{\infty}\{X_t = u\}) = 0$. Since (X_n) is transient, it has been proved that $\mathbb{P}(\bigcap_{n=1}^{\infty}\bigcup_{t=n}^{\infty}\{X_t = u\}) = 0$. So $\mathbb{P}(\bigcup_{u\in[-k,k]}\bigcap_{n=1}^{\infty}\bigcup_{t=n}^{\infty}\{X_t = u\}) = 0 \leq \sum_{u\in[-k,k]}\mathbb{P}(\bigcap_{n=1}^{\infty}\bigcup_{t=n}^{\infty}\{X_t = u\}) = 0$. $\square$

PROBLEM IV Assume $\{a_i : i \geq 1\} \subset (0,1)$. Consider $E := \mathbb{N}$, P is a transition matrix on E, where $p_{ij} = a_i\mathbb{1}_{\{j=0\}} + (1 - a_i)\mathbb{1}_{\{j=i+1\}}$. Prove:

1. P is irreducible.
2. P is recurrent $\iff$ $\sum_i a_i = \infty$.
3. P is ergodic $\iff$ $\sum_{k=1}^{\infty}\prod_{i=1}^{k-1}(1 - a_i) < \infty$.

SOLUTION. 1. Easy to prove that $p_{i0}(1) > 0, \forall i \in \mathbb{N}$. And easily $p_{0i}(i) = \prod_{k=0}^{i-1}(1-a_k) > 0$. So P is irreducible.

2. Since P is irreducible, we only need to consider $X_0 = 0$. Then $\{T_0 > n\} \overset{\text{a.s.}}{=} \{X_k = k, k = 0, \cdots, n\}$. Then $\mathbb{P}_0(T_0 = \infty) = \mathbb{P}_0(\bigcap_n \{T_0 > n\}) = \lim_{n\to\infty} \mathbb{P}_0(X_k = k, k = 0, \cdots, n) = \lim_{n\to\infty} \prod_{k=0}^{n-1}(1-a_k) = \prod_{k=0}^{\infty}(1-a_k)$. Then $\mathbb{P}_0(T_0 = \infty) = 0 \iff \prod_{k=0}^{\infty}(1-a_k) = 0 \iff \sum_k a_k = \infty$.

3. Since $\mathbb{E}_0(T_0) = \sum_{n\in E} \mathbb{P}_0(T_0 > n) = \sum_{n=0}^{\infty} \prod_{k=0}^{n-1}(1-a_k)$, then P is ergodic $\iff \mathbb{E}_0(T_0) < \infty$ $\iff \sum_{n=0}^{\infty} \prod_{k=0}^{n-1}(1-a_k) < \infty$.

□

PROBLEM V Assume P is a transition matrix on E and P is irreducible, $j \in E$. Prove: P is recurrent $\iff$ 1 is the minimum non-negative solution of

$$y_i = \sum_{k\neq j} p_{ik} y_k + p_{ij}, i \in E \setminus \{j\} \tag{1}$$

SOLUTION. " $\Longrightarrow$ ": If P is recurrent, then the bounded solution of $z_i = \sum_{k\in E} p_{ik} z_k, i \in E \setminus \{j\}$ is constant. Easy to get that 1 is one of solution of Equation (1), now we will prove that 1 is the unique solution. Assume there is another solution, then $y_i - 1 = \sum_{k\neq j} p_{ik}(y_k - 1), \forall i \in E \setminus \{j\}$. Then we let $z_j = 0, z_i = y_i - 1, \forall i \neq j$, we find a non-constant solution, contradiction! So 1 is the unique solution and thus minimum solution.

" $\Longleftarrow$ ": If P is transient, then the bounded solution of $z_i = \sum_{k\in E} p_{ik} z_k, i \in E \setminus \{j\}$ has non-constant solution. Without loss of generality, we can assume $z_j = 0, \forall i \in E, |z_i| \leq 1, \exists i_0 \in E, z_{i_0} < 0$. Let $y_i = 1 + z_i, i \in E$, then $\{y_i : i \in E\}$ is the bounded solution of Equation (1). But $y_{i_0} < 1, y_i \geq 0, i \in E$. So 1 is not the minimum solution. □

PROBLEM VI Let $\{a_k : k \geq 0\}$ satisfies $\sum_{k\geq 0} a_k = 1, a_k \geq 1, a_0 > 0$, $\mu := \sum_{k=1}^{\infty} k a_k > 1$. Define

$$p_{ij} = \begin{cases} a_j & , i = 0 \\ a_{j-i+1} & , i \geq 1 \wedge j \geq i-1 \\ 0 & , \text{otherwise} \end{cases}$$

. Prove: P is transient.

SOLUTION. First, we prove that P is irreducible: Since $\sum_{k=1}^{\infty} k a_k > 1$, then $\exists m, a_m > 0$. And $\forall i \geq 1$, $p_{i-1,i} = a_0 > 0$. Then $\forall i, j$, if $i < j$, then $p_{ij}(j-i) = a_0^{j-i} > 0$. If $i \geq j$, let $t \equiv i - j \pmod{m}, 1 \leq t \leq m$, then $p_{ij}(t+1) = a_0^t a_m > 0$.

Let $\xi_n : n \in \mathbb{N}$ is a sequence of i.i.d r.v with $\mathbb{P}(\xi_0 = i) = a_i$. Since P is irreducible, we only consider the chain begin at 0. Let $X_0 = 0, X_{n+1} = X_n + \xi_n - \mathbb{1}_{X_n > 0}$. Then easily X_n is the Markov chain begin at 0 with transition matrix P. And $X_n = \sum_{k=0}^{n-1} \xi_k - \sum_{k=0}^{n-1} \mathbb{1}_{X_n > 0} \geq \sum_{k=0}^{n-1}(\xi_k - 1)$. So we obtain $\liminf_{n\to\infty} \frac{X_n}{n} \geq \liminf_{n\to\infty} \frac{\sum_{k=0}^{n-1} \xi_k - 1}{n} = \mu - 1 > 0$. So $\liminf_{n\to\infty} X_n = \infty$, so 0 is transient. □