underGraduate Homework In Mathematics

SetTheory 5

白永乐

202011150087

202011150087@mail.bnu.edu.cn

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General fire extinguisher

PROBLEM I Prove: $F \subset \mathcal{N}$ is closed set $\iff F = [T]$ for some $T \subset {}^{<\omega}\omega$.

SOLUTION.
- $\implies$: Let $T := T_F$, now we need to prove $F = [T]$. Form the defination of T_F and $[T]$ easily we get $F \subset [T]$. Now we prove $[T] \subset F$. For $f \in [T]$, we get $f \restriction n \in T$. i.e., $\forall n \in \mathbb{N}, f \restriction n = g \restriction n$ for some $g \in F$. So $d(f, F) \leq d(f, g) = \frac{1}{2^n}$. Since F is closed, we get $f \in F$.
- $\impliedby$: For any $[T] \in {}^{<\omega}\omega$, we need to prove $[T]$ is closed. Assume $f \in \overline{[T]}$, then $\forall n \in \mathbb{N}, \exists g \in [T], f \restriction n = g \restriction n$. Since $g \in [T]$ we get $f \restriction n = g \restriction n \in T$. So $f \in [T]$. So $[T]$ is closed.

$\square$

PROBLEM II Assume f is isolated point in closed set $F \subset \mathcal{N}$, then $\exists n \in \mathbb{N}, \forall g \in F, g \neq f \to g \restriction n \neq f \restriction n$.

SOLUTION. Since f is isolated, we get $\exists n \in \mathbb{N}, \forall g \in F \setminus \{f\}, d(f, g) > \frac{1}{2^n}$. Then $f \restriction n \neq g \restriction n$. $\square$

PROBLEM III A closed set $F \subset \mathcal{N}$ is perfect $\iff T_F$ is perfecr tree.

SOLUTION.
- $\implies$: For $t \in T_F$, by defination we know $\exists f \in F, n \in \mathbb{N}, t = f \restriction n$. Since f is perfect we know $\exists g \in F \wedge g \neq f, d(f, g) < \frac{1}{2^{n+1}}$. Then $t = f \restriction n \sqsubset g$. Since $f \neq g$, we get $\exists m \in \mathbb{N} \wedge m > n, f \restriction m \neq g \restriction m$. So $t \sqsubset f \restriction m, t \sqsubset g \restriction m$, and $f \restriction m, g \restriction m$ are incomparable.
- $\impliedby$: For $f \in F$, we need to prove f is limit point. $\forall n \in \mathbb{N}, t := f \restriction n \in T_F$. So $\exists s_1, s_2 \in T_F$ such that $t \sqsubset s_1, s_2$ and s_1, s_2 are incomparable. Then $s_1, s_2 \sqsubset f$ is impossible. Without loss of generality assume $s_1 \not\sqsubset f$. Then $s_1 = g \restriction m$ for some $g \in F, m \in \mathbb{N}$. So $d(f, g) \leq \frac{1}{2^n}$. So f is not isolated.

$\square$

PROBLEM IV For $\alpha < \omega_1$, we let $\Sigma_0 =$ the set of all open set in $\mathbb{R}$, and $\Pi_0 =$ the set of all closed set in $\mathbb{R}$. And $\Sigma_{\alpha+1} = \{\bigcup_{n \in \mathbb{N}} A(n) : A \in {}^{\mathbb{N}}\Pi_\alpha$. $\Pi_{\alpha+1} = \{\mathbb{R} \setminus A : A \in \Sigma_\alpha\}$. $\Sigma_\alpha = \bigcup_{\beta<\alpha} \Sigma_\beta, \Pi_\alpha = \bigcup_{\beta<\alpha} \Pi_\beta$ for limit ordinal α. Prove that $\mathcal{B}(\mathbb{R}) = \bigcup_{\alpha<\omega_1} \Sigma_\alpha$.

SOLUTION. Use MI easily we get $\bigcup_{\alpha<\omega_1} \Sigma_\alpha \subset \mathcal{B}(\mathbb{R})$. Now we prove $\mathcal{B}(\mathbb{R}) \subset \bigcup_{\alpha<\omega_1} \Sigma_\alpha$. Since open sets is subset of $\bigcup_{\alpha<\omega_1} \Sigma_\alpha$, we only need to prove $\bigcup_{\alpha<\omega_1} \Sigma_\alpha =: \mathcal{A}$ is σ-field. Easily we get $\Sigma_\alpha \subset \Sigma_{\alpha+2}$. Obviously $\mathbb{R} \in \mathcal{A}$. For $A \in \mathcal{A}$, assume $A \in \Sigma_\alpha$. Then $\mathbb{R} \setminus A \in \Pi_{\alpha+1} \subset \Sigma_{\alpha+1} \subset \mathcal{A}$. Assume $A \in {}^{\mathbb{N}}\mathcal{A}$, let $f \in {}^{\mathbb{N}}\omega_1, f(n) = \min\{\alpha \in \omega_1 : A(n) \in \Sigma_\alpha\}$. Consider $\sup \operatorname{ran} f =: \gamma$. Since $\forall \alpha \in \operatorname{ran} f, \alpha$ is countable. And $\operatorname{ran} f$ is countable. So $\sup \operatorname{ran} f$ is countable, thus $\sup \operatorname{ran} f < \omega_1$. Then $\operatorname{ran} A \subset \Pi_{\gamma+1}$. So we get $\bigcup_{n \in \mathbb{N}} A(n) \subset \Sigma_{\gamma+2} \subset \mathcal{A}$. So we get $\mathcal{A}$ is σ-field. So $\mathcal{B}(\mathbb{R}) \subset \mathcal{A}$, thus $\mathcal{A} = \mathcal{B}(\mathbb{R})$. $\square$

PROBLEM V Show that $\mathcal{M} := \{A \subset \mathbb{R} : A \text{ is measurable}\}$ is a σ-field.

Lemma 1. For $A \in {}^{\mathbb{N}}\mathcal{P}(\mathbb{R})$, we have $\mu^*(\bigcup_{n \in \mathbb{N}} A(n)) \leq \sum_{n \in \mathbb{N}} \mu^*(A(n))$.

证明. For any $\varepsilon > 0, n \in \mathbb{N}, \exists O(n) \in \mathcal{O}, A(n) \subset O(n) \wedge \mu^*(A(n)) \leq |O(n)| + \frac{\varepsilon}{2^{n+1}}$. Let $U := \bigcup_{n \in \mathbb{N}} O(n)$, then $\bigcup_{n \in \mathbb{N}} A(n) \subset U$. So $\mu^*(\bigcup_{n \in \mathbb{N}} A(n)) \leq |U| \leq \sum_{n \in \mathbb{N}} |O(n)| \leq \sum_{n \in \mathbb{N}} \mu^*(A(n)) + \varepsilon$. Since ε is arbitry, we get the lemma. $\square$

Lemma 2. If $G \in G_\delta$, then $\forall \varepsilon > 0, \exists O \in \mathcal{O}, G \subset O \wedge \mu^*(O \setminus G) \leq \varepsilon$.

证明. We first consider G is bonded. Assume $G \subset [-M, M], M > 0$. Assume $G = \bigcap_{n\in\mathbb{N}} O_n$, where $O_n \in \mathcal{O}$. Then $G = \bigcap_{n\in\mathbb{N}}(O_n \cap (-M-1, M+1))$. By convinence we assume $O_n \subset (-M-1, M+1)$. And $G = \bigcap_{n\in\mathbb{N}} \bigcap_{m=1}^{n} O_m$, by convinence we assume $O_n \supset O_{n+1}$. Use LCDT we easily get $\lim \mu(O_n) = \mu^*(G)$. So we easily get the lemma.

Now consider general G. We can write $G = \bigcup_{n\in\mathbb{N}} G_n$, where G_n is bounded G_δ set. So $\exists O_n, G_n \subset O_n, \mu^*(O_n \setminus G_n) < \frac{\varepsilon}{2^n}$. Let $O = \bigcup_{n\in\mathbb{N}} O_n$ is OK. $\square$

SOLUTION. First, for $A = \mathbb{R}$, easily we can let $F = G = \mathbb{R}$. Then F is F_σ and G is G_δ. Second, assume $A \in \mathcal{M}$, consider $B = \mathbb{R} \setminus A$. Assume $F \subset A \subset G$ and $\mu^*(G \setminus F) = 0$. Then $G^c \subset B \subset F^c$. And G^c is F_σ, F^c is G_δ. And $\mu^*(F^c \setminus G^c) = \mu^*(G \setminus F) = 0$. So $B \in \mathcal{M}$. Finally, assume $A \in {}^{\mathbb{N}}\mathcal{M}$, we need to prove $\bigcup_{n\in\mathbb{N}} A(n) =: A \in \mathcal{M}$. Use AC we can find $F \in {}^{\mathbb{N}}F_\sigma, G \in {}^{\mathbb{N}}G_\delta$ such that $F(n) \subset A(n) \subset G(n), \mu^*(G(n) - F(n)) = 0$. Let $T = \bigcup_{n\in\mathbb{N}} F(n)$. Since $F(n)$ is F_σ, we get $T \in F_\sigma$. And easily $T = \bigcup_{n\in\mathbb{N}} F(n) \subset \bigcup_{n\in\mathbb{N}} A(n) = A$. Now we let $O(n,m) \in \mathcal{O}, G(n) \subset O(n,m), \mu^*(O(n,m) \setminus G(n)) < \frac{1}{m2^n}$ by Lemma 2. Let $G := \bigcap_{m\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} O(n,m)$ is G_δ. Easily $\mu^*(G \setminus A) \leq \mu^*(G \setminus \bigcup_{n\in\mathbb{N}} G(n)) = 0$. And $A \subset G$. So we get $\mu^*(G \setminus F) \leq \mu^*(G \setminus A) + \mu^*(A \setminus F) = 0 + 0 = 0$. So $A \in \mathcal{M}$. $\square$

PROBLEM VI Show that $\mathcal{A} := \{A \subset \mathbb{R} : A \text{ has property of Baire}\}$ is σ-field.

SOLUTION. Easily $\mathbb{R}\Delta\mathbb{R}$ is meager, so $\mathbb{R} \in \mathcal{A}$.

If $A \in \mathcal{A}$, we need to prove $\mathbb{R} \setminus A \in \mathcal{A}$. Assume $G \in \mathcal{O}$ and $A\Delta G$ is meager, write $B = \mathbb{R} \setminus A$, only need to prove $\exists U \in \mathcal{O}$, such that $B \setminus U, U \setminus B$ are meager. Let $U = \mathbb{R} \setminus \overline{G}$. Then $B \setminus U = A \setminus \overline{G}$ is meager. Now only need to prove $U \setminus B = \overline{G} \setminus A$ is meager. Since $G \setminus A$ is meager, we only need to prove $\overline{G} \setminus G$ is meager. In fact, we can prove $\overline{G} \setminus G$ is nowhere dense. Consider $I \in \mathcal{O}$, we need to prove $\exists J \subset I, J \in \mathcal{O}, J \cap \partial G = \varnothing$. If $I \cap \partial G = \varnothing$, we can let $J = I$. Else, assume $a \in I \cap \partial G$. Form the defination of ∂G, we get $\exists b \in I \cap G$. Let $J = I \cap G \neq \varnothing$ is OK. So $B\Delta U$ is meager.

Assume $A \in {}^{\mathbb{N}}\mathcal{P}(\mathcal{A})$, we need to prove $\bigcup_{n\in\mathbb{N}} A(n) =: A \in \mathcal{A}$. Assume $G(n) \in \mathcal{O}$ and $A(n)\Delta G(n)$ is meager. Consider $G := \bigcup_{n\in\mathbb{N}} G(n)$. We only need to prove $G\Delta A$ is meager. Only need $G \setminus A, A \setminus G$ is meager. Since $G \setminus A \subset \bigcup_{n\in\mathbb{N}} G(n) \setminus A(n)$ and $G(n) \setminus A(n)$ is meager, we get $G \setminus A$ is meager. For the same reason, we get $A \setminus G \subset \bigcup_{n\in\mathbb{N}} A(n) \setminus G(n)$ is meager.

So finally we get $\mathcal{A}$ is σ-field. $\square$

PROBLEM VII Assume $A \subset {}^{\omega}\omega$ has the property of Baire, prove A is nonmerger $\iff \exists O \in \mathcal{O}({}^{\omega}\omega), O \neq \varnothing \wedge O \setminus A$ is meager.

SOLUTION. $\implies$: Since A has the property of Baire, we know $\exists O \in \mathcal{O}, O\Delta A$ is meager. Then $O \setminus A, A \setminus O$ are meager. Since A is nonmeager, $A \setminus O$ is meager, we get $O \neq \varnothing$.

$\impliedby$: Assume $O \in \mathcal{O}, O \neq \varnothing, O \setminus A$ is meager. Noting $O \subset O \setminus A \cup A$ and O is nonmeager, we get A is nonmeager. $\square$

PROBLEM VIII Let $C_A := \bigcup\{O_s : s \in {}^{<\omega}\omega, O_s \setminus A \text{ is meager}\}$. Prove that $C_A \setminus A$ is meager.

SOLUTION . We know $\mathbb{R}$ satisfy the second countable axiom, i.e., $\exists \mathcal{B} \subset \mathcal{O}({}^{\omega}\omega)$ such that $\forall O \in \mathcal{O}, \forall x \in O, \exists B \in \mathcal{B}, x \in B \subset O$. Now we consider $\mathcal{X} := \{X \in \mathcal{B} : \exists O_s, X \subset O_s \wedge O_s \setminus A$ is meager $\}$. Consider $Y = \bigcup \mathcal{X}$, we will prove $C_A = Y$.

On one hand, for $x \in Y$, we get $\exists X \in \mathcal{X}$ such that $x \in X$. So $\exists O_s$ such that $x \in X \subset O_s \wedge O_s \setminus A$ is meager. So $x \in C_A$.

On the other hand, for $x \in C_A$, we get $\exists O_s, x \in O_s, O_s \setminus A$ is meager. Since O_s is open, we get $\exists B \in \mathcal{B}, x \in B \subset O_s$. So $B \in \mathcal{X}$. Thus $x \in Y$.

So we get $Y = C_A$. So $C_A \setminus A = Y \setminus A = \bigcup_{X \in \mathcal{X}} X \setminus A$. From the defination of $\mathcal{X}$ we know $X \setminus A$ is meager, and Since $\mathcal{X} \subset \mathcal{B}$ we get $\mathcal{X}$ is countable. So finally we get $C_A \setminus A = \bigcup_{X \in \mathcal{X}} X \setminus A$ is meager. $\square$

PROBLEM IX Let $\pi : {}^{\omega}\omega \to {}^{\omega}2, \pi(x) = s_{x(0)}{}^\frown s_{x(1)}{}^\frown \cdots$. Where $s_{x(k)} = 11\cdots10$ for even k, there is k "1" in total, and $s_{x(k)} = 00\cdots01$ for odd k, there is k "0" in total. Prove that ${}^{\omega}2 \setminus \operatorname{ran}\pi$ is countable.

SOLUTION . As we all know, $\{f \in {}^{\omega}2 : \limsup f = \liminf f\}$ is countable. So we only need to prove $\forall f \in {}^{\omega}2 \setminus \operatorname{ran}\pi, \limsup f = \liminf f$. Consider $g \in {}^{\omega}2$ and $\liminf g = 0, \limsup g = 1$. We only need to prove $g \in \operatorname{ran}\pi$. Only need to prove $\exists h \in {}^{\omega}\omega, \pi(h) = g$. We construct h rescusively. Let $h(0) := \min\{n \in \omega : g(n) = 0\}$. Assume $h \restriction n$ is already defined. Let $M(n) = \sum_{k=0}^{n-1}(h(k)+1)$. Let $h(n) = \min\{k : g(M(n)+k) = a_n\}$, where $a_n = 0$ for even n and $a_n = 1$ for odd n. Since $\liminf g = 0 \wedge \limsup g = 1$, we know h is well-defined. Now we prove $\pi(h) = g$. For $k < h(0)$, form the defination of $h(0)$ we know $g(k) = 1 = \pi(h)(k)$. For $k = h(0)$ we get $g(k) = 0 = \pi(h)(k)$. Now assume $\sum_{i=0}^{n}(h(i)+1) < k \leq \sum_{i=0}^{n+1}(h(i)+1)$. Easily we know $\operatorname{len}(s_{h(k)} = h(k)+1)$, so we get $\pi(h)(k) = s_{h(n)}(k - M(n))$. So from the defination of $h(n)$ we easily get $\pi(h)(k) = g(k)$. $\square$

PROBLEM X Assume AD, then $\mathrm{AC}_{\omega}({}^{\omega}\omega)$. Consequently, ω_1 is regular.

SOLUTION . Assume $X : \omega \to \mathcal{P}({}^{\omega}\omega)$ and $\forall n \in \omega, X(n) \neq \varnothing$. Let $\theta : {}^{\omega}\omega \to {}^{\omega}\omega, \theta(f)(n) := f(2n+1)$. Consider $A := \{x \in {}^{\omega}\omega : \theta(x) \in X(x(0))\}$. Since I have no w.s because $\forall n \in \omega, X(n) \neq \varnothing$. By AD we get II has a w.s., write τ. Now consider $\gamma : \omega \to {}^{\omega}\omega, \gamma(n) := \theta((n,0,0,\cdots) * \tau)$. Since $\theta((n,0,\cdots) * \tau) \in X(n)$. So γ is the choose function.

Nov we prove ω_1 is regular. Only need to prove union countable many countable ordinal is countable. Assume $f : \omega \to \omega_1$, now we only need to prove $\bigcup \operatorname{ran} f \in \omega_1$. Consider $F : \operatorname{ran} f \to \mathcal{P}(\omega \times \omega)$, $F(\alpha) := \{R \subset \omega \times \omega : (\omega, R) \cong \alpha\}$. Since ${}^{\omega}\omega \approx \mathcal{P}(\omega \times \omega)$, we get $\mathrm{AC}_{\omega}(\mathcal{P}(\omega \times \omega))$. So $\exists \theta : \operatorname{ran} f \to \omega \times \omega, \theta(\alpha) \in F(\alpha), \forall \alpha \in \operatorname{ran} f$. Consider $G : \omega \to {}^{\omega}\omega_1, G(n)$ is the isomorphic from $(\omega, \theta(\omega))$ to $f(n)$. Let $h : \omega \times \omega \to \bigcup \operatorname{ran} f, h(n,m) := G(n)(m)$. Easily h is surjective. And since we have $\mathrm{AC}_{\omega}(\mathcal{P}(\omega \times \omega))$, we get $\bigcup \operatorname{ran} f \approx A$ for some $A \subset \omega \times \omega$. So we get $\bigcup \operatorname{ran} f$ is countable. So ω_1 is regular. $\square$