undergraduate Homework In Mathematics

AlgebraicGeometry 4

白永乐

202011150087

202011150087@mail.bnu.edu.cn

2023 年 10 月 15 日

General fire extinguisher

PROBLEM I Prove that $\mathbb{C}$ and $\mathbb{C} \setminus \{0\}$ are Homeomorphic by Zariski topology.

SOLUTION. Since $\mathrm{card}\mathbb{C} = \mathrm{card}\mathbb{C} \setminus \{0\}$, there is a bijection $f : \mathbb{C} \to \mathbb{C} \setminus \{0\}$. Then we prove f is homeomorphism.

First, we prove a set is closed iff it's finite or it's Universe. For a finite set A, obviously $V((\prod_{t \in A}(x-t))) = A$ is closed. For closed set $A = V(I), I \neq (0)$, consider $f \in I \setminus \{0\}$, we know $\forall a \in A, f(a) = 0$, so A is finite.

Since f is bijection, we know f preserve the cardinality, so preserve finite set. And $f(\mathbb{C}) = \mathbb{C} \setminus \{0\}$, so f preserve closed set. For the same reason f^{-1} preserve closed set. So f is homeomorphism. $\square$

PROBLEM II Assume V is irreducible algebric set and f is a fraction function on V. Let $D_f := \{h \in \mathcal{O}_v(V) : \exists g \in \mathcal{O}_v(V) \ s.t. \ f = \frac{g}{h}\} \cup \{0\}$. Prove that D_f is ideal of $\mathcal{O}_v(V)$.

SOLUTION. For $h_1, h_2 \in D_f$, if $h_1 = h_2$ then $h_1 - h_2 = 0 \in D_f$. Else, assume $f = \frac{g_1}{h_1} = \frac{g_2}{h_2}$, so $f = \frac{g_1 - g_2}{h_1 - h_2}$. So $h_1 - h_2 \in D_f$.

For $h \in D_f$ and $j \in \mathcal{O}_v(V)$, if $hj = 0$ then $hj \in D_f$. Else we get $hj \neq 0$, then $f = \frac{g}{h} = \frac{gj}{hj}$. So $hj \in D_f$.

Above all, we get D_f is a ideal. $\square$