PROBLEM I Let $(X_n : n \geq 0) \perp (Y_n : n \geq 0)$ are Markov chain on E with Transition matrix $(p_{ij} : i, j \in E), (q_{ij} : i, j \in E)$ respectively. Prove: $\{(X_n, Y_n) : n \geq 0\}$ are Markov chain on $E \times E$. And calculate the transition matrix of $(X_n, Y_n) : n \geq 0$.

SOLUTION .

$$
\begin{aligned}
&\mathbb{P}(X_0 = i_0, \cdots, X_{n+1} = i_{n+1}, Y_0 = j_0, \cdots, Y_{n+1} = j_{n+1}) \\
=&\mathbb{P}(X_0 = i_0, \cdots, X_{n+1} = i_{n+1})\mathbb{P}(Y_0 = j_0, \cdots, Y_{n+1} = j_{n+1}) \\
=&\mathbb{P}(X_0 = i_0) \prod_{k=0}^{n} p_{i_k i_{k+1}} \mathbb{P}(Y_0 = j_0) \prod_{k=0}^{n} q_{j_k j_{k+1}} \\
=&\mathbb{P}((X_0, Y_0) = (i_0, j_0)) \prod_{k=0}^{n} p_{i_k i_{k+1}} q_{j_k j_{k+1}} \\
=&\mathbb{P}((X_0, Y_0) = (i_0, j_0)) \prod_{k=0}^{n} \mathbb{P}(X_k = i_k, X_{k+1} = i_{k+1})\mathbb{P}(Y_k = j_k, Y_{k+1} = j_{k+1}) \\
=&\mathbb{P}((X_0, Y_0) = (i_0, j_0)) \prod_{k=0}^{n} \mathbb{P}((X_k, Y_k) = (i_k, j_k), (X_{k+1}, Y_{k+1}) = (i_{k+1}, j_{k+1}))
\end{aligned}
$$

So we get that $((X_n, Y_n) : n \in \mathbb{N})$ is Markov chain with transition matrix $r_{(i,j),(m,n)} = p_{im} q_{jn}$. $\square$

PROBLEM II Let S_n be 1-dimensional simple random walk, $a \in \mathbb{Z}$. Let $\tau := \inf\{n \geq 0 : S_n = a\}$. Prove:

1. $(S_{\tau+n} : n \geq 0)$ is a one dimensional simple random walk.
2. $(S_{n\wedge\tau} : n \geq 0)$ is a Markov chain on $\mathbb{Z}$ and give its Transition matrix.
3. $(S_{n\wedge\tau} : n \geq 0) \perp (S_{\tau+n} : n \geq 0)$.

SOLUTION . 1.

$$
\begin{aligned}
&\mathbb{P}(S_\tau = i_0, S_{\tau+1} = i_1, \cdots, S_{\tau+n} = i_n \mid \tau < \infty) \\
=&\sum_{k\in\mathbb{N}} \mathbb{P}(\tau = k, S_\tau = i_0, S_{\tau+1} = i_1, \cdots, S_{\tau+n} = i_n \mid \tau < \infty) \\
=&\sum_{k\in\mathbb{N}} \mathbb{P}(\tau = k, S_k = i_0, S_{k+1} = i_1, \cdots, S_{k+n} = i_n \mid \tau < \infty) \\
=&\mathbb{1}(a = i_0) \sum_{k\in\mathbb{N}} \mathbb{P}(S_0 \neq a, \cdots, S_{k-1} \neq a, S_k = a, S_{k+1} = i_1, \cdots, S_{k+n} = i_n \mid \tau < \infty) \\
=&\frac{\mathbb{1}(a = i_0) \sum_{k\in\mathbb{N}} \mathbb{P}(S_0 \neq a, \cdots, S_{k-1} \neq a, S_k = a, S_{k+1} = i_1, \cdots, S_{k+n} = i_n)}{\mathbb{P}(\tau < \infty)} \\
=&\frac{\mathbb{1}(a = i_0)}{\mathbb{P}(\tau < \infty)} \sum_{k\in\mathbb{N}} \mathbb{P}(S_0 \neq a, \cdots, S_{k-1} \neq a, S_k = a, S_{k+1} = i_1, \cdots, S_{k+n} = i_n) \\
=&\frac{\mathbb{1}(a = i_0)}{\mathbb{P}(\tau < \infty)} \sum_{k\in\mathbb{N}} \mathbb{P}(S_{k+1} = i_1, \cdots, S_{k+n} = i_n \mid S_0 \neq a, \cdots, S_{k-1} \neq a, S_k = a) \\
\times&\mathbb{P}(S_0 \neq a, \cdots, S_{k-1} \neq a, S_k = a) \\
=&\frac{\mathbb{1}(a = i_0)}{\mathbb{P}(\tau < \infty)} \sum_{k\in\mathbb{N}} \mathbb{P}(S_{k+1} = i_1, \cdots, S_{k+n} = i_n \mid S_k = a)\mathbb{P}(\tau = k) \\
=&\frac{\mathbb{1}(a = i_0)}{\mathbb{P}(\tau < \infty)} \sum_{k\in\mathbb{N}} \prod_{l=0}^{n-1} p_{i_l i_{l+1}} \mathbb{P}(\tau = k) = \mathbb{1}(a = i_0) \prod_{l=0}^{n-1} p_{i_l i_{l+1}}
\end{aligned}
$$

Where $p_{ij} : i, j \in \mathbb{Z}$ is the transition matrix of $S_n : n \in \mathbb{N}$. So $(S_{\tau+n} : n \in \mathbb{N})$ is Markov chain with transition matrix same as S_n.

2.

$$
\begin{aligned}
&\mathbb{P}(S_{\tau\wedge 0} = i_0, S_{\tau\wedge 1} = i_1, \cdots, S_{\tau\wedge n} = i_n \mid \tau < \infty) \\
=&\sum_{k\in\mathbb{N}} \mathbb{P}(\tau = k, S_{\tau\wedge 0} = i_0, S_{\tau\wedge 1} = i_1, \cdots, S_{\tau\wedge n} = i_n \mid \tau < \infty) \\
=&\sum_{k\in\mathbb{N}} \mathbb{P}(\tau = k, S_{k\wedge 0} = i_0, S_{k\wedge 1} = i_1, \cdots, S_{k\wedge n} = i_n \mid \tau < \infty) \\
=&\sum_{k\geq n} \mathbb{P}(\tau = k, S_0 = i_0, \cdots, S_n = i_n \mid \tau < \infty) \\
+&\sum_{k<n} \mathbb{P}(\tau = k, S_0 = i_0, \cdots, S_{k-1} = i_{k-1}, S_k = i_k = i_{k+1} = \cdots = i_n \mid \tau < \infty) \\
=&\mathbb{1}(i_0, i_1, \cdots, i_n \neq a) \prod_{k=0}^{n-1} p_{i_k i_{k+1}} + \sum_{k=0}^{n-1} \mathbb{1}(i_0, \cdots, i_{k-1} \neq a, i_k = i_{k+1} = \cdots = i_n = a) \prod_{l=0}^{k-1} p_{i_l i_{l+1}} \\
=&\prod_{k=0}^{n-1} (\mathbb{1}(i_k = i_{k+1} = a) + \mathbb{1}(i_k \neq a) p_{i_k, i_{k+1}})
\end{aligned}
$$

So $(S_{n\wedge\tau} : n \in \mathbb{N})$ is Markov chain with transition matrix $q_{i,j} = \mathbb{1}(i = j = a) + \mathbb{1}(i \neq a)p_{i,j}$.

3. By the corollary 3.2.11, we only need to proof τ is stopping time on $(\mathcal{F}_n : n \geq 0)$, Where $\mathcal{F}_n = \sigma(S_k : k \leq n)$. So we only need to prove $\forall n \in \mathbb{N}$, $\{\tau = n\} \in \mathcal{F}_n$. Since $\{\tau = n\} = \{\omega \in \omega : S_0, \cdots, S_{n+1} \neq a, S_n = a\} = \bigcap_{0\leq k\leq n}\{S_k \neq a\} \cap \{S_n = a\}$, And $\{S_k \neq a\} \in \sigma(S_k), \forall 0 \leq k \leq n, \{S_n = a\} \in \sigma(S_n)$, Then $\{\tau = n\} \in \mathcal{F}_n$.

□

PROBLEM III Let S_n be 1-dimensional symmetry simple random walk starting from zero. Prove: $(|S_n| : n \geq 0)$ is a Markov chain on $\mathbb{Z}^+$ and give its transition matrix.

SOLUTION. Only need to solve problem IV. □

PROBLEM IV Let S_n be 1-dimensional simple random walk starting from zero. Prove: $(|S_n| : n \geq 0)$ is a Markov chain on $\mathbb{Z}^+$ and give its transition matrix.

SOLUTION. By the definition of $|S_n|$, we can easily get to know $\forall(i_0, \cdots, i_n) \in \mathbb{Z}^+$, $\mathbb{P}(|S_k| = i_k, k = 0, \cdots, n) > 0 \iff i_0 = 0, |i_k - i_{k-1}| = 1, k = 1, \cdots, n$. Let $S_n = \sum_{k=1}^n \xi_k$, where $(\xi_n : n \geq)$ are i.i.d. r.v. and $\mathbb{P}(\xi_1 = 1) = p, \mathbb{P}(\xi_1 = -1) = q$. $A := \{(i_0, \cdots, i_{n+1}) \in \mathbb{Z} : i_0 = 0, |i_k - i_{k-1}| = 1, k = 1, \cdots, n + 1\}$. $\forall(i_0, \cdots, i_{n+1}) \in A$, let $r := \max\{k < n + 1 : i_k = 0\}$. Then $i_r = 0, \forall k : n + 1 > k \geq r + 1, i_k \geq 1$.

1. $\forall(i_0, \cdots, i_{n+1}) \notin A$, then $\mathbb{P}(|S_k| = i_k, k = 0, \cdots, n) = 0$, Then we have no need to calculate $\mathbb{P}\Big(|S_{n+1}| = i_{n+1}\Big||S_k| = i_k, k = 0, \cdots, n\Big)$.
2. If $(i_0, \cdots, i_{n+1}) \in A \wedge r = n$, then $i_n = 0, i_{n+1} = 1$. Then $|S_n| = 0 \iff S_n = 0 \implies S_{n+1} = \pm 1 \iff |S_{n+1}| = 1$. So we get that $\mathbb{P}\Big(|S_{n+1}| = i_{n+1}\Big||S_k| = i_k, k = 1, \cdots, n\Big) = 1 = \mathbb{P}\Big(|S_{n+1}| = i_{n+1}\Big||S_n| = i_n\Big)$.
3. $\forall(i_0, \cdots, i_{n+1}) \in A, i_n \neq 0$,

$$\begin{aligned}
&\mathbb{P}\Big(|S_k| = i_k, S_n = i_n, k = 0, \cdots, n\Big||S_k| = i_k, k = 0, \cdots, r\Big) \\
&= \mathbb{P}\Big(|S_k| = i_k, S_n = i_n, k = r + 1, \cdots, n\Big||S_k| = i_k, k = 0, \cdots, r - 1, S_r = 0\Big) \\
&= \mathbb{P}\Big(|S_k| = i_k, S_n = i_n, k = r + 1, \cdots, n\Big|S_r = 0\Big) \\
&= \mathbb{P}\Big(S_k = i_k, S_n = i_n, k = r + 1, \cdots, n\Big|S_r = 0\Big) \\
&= p^{\frac{n-r+i_n}{2}} q^{\frac{n-r-i_n}{2}}
\end{aligned}$$

In the same way, we can get

$$\mathbb{P}\Big(|S_k| = i_k, S_n = -i, k = 0, \cdots, n\Big||S_k| = i_k, k = 0, \cdots, r\Big) = p^{\frac{n-r-i_n}{2}} q^{\frac{n-r+i}{2}}$$

So

$$
\begin{aligned}
&\mathbb{P}\Big(S_n = i_n \Big| |S_k| = i_k, k = 0, \cdots, n\Big) \\
&= \frac{\mathbb{P}(S_n = i_n, |S_k| = i_k, k = 0, \cdots, n)}{\mathbb{P}(|S_k| = i_k, k = 0, \cdots, n)} \\
&= \frac{\mathbb{P}\Big(S_n = i_n, |S_k| = i_k, k = 0, \cdots, n \Big| |S_k| = i_k, k = 0, \cdots, r\Big)\mathbb{P}(|S_k| = i_k, k = 0, \cdots, r)}{\mathbb{P}(|S_k| = i_k, k = 0, \cdots, n)} \\
&= \frac{\mathbb{P}\Big(S_n = i_n, |S_k| = i_k, k = 0, \cdots, n \Big| |S_k| = i_k, k = 0, \cdots, r\Big)}{\frac{\mathbb{P}(|S_k| = i_k, k=0, \cdots, n)}{\mathbb{P}(|S_k| = i_k, k=0, \cdots, r)}} \\
&= \frac{\mathbb{P}\Big(S_n = i_n, |S_k| = i_k, k = 0, \cdots, n \Big| |S_k| = i_k, k = 0, \cdots, r\Big)}{\mathbb{P}\Big(|S_k| = i_k, k = 0, \cdots, n \Big| |S_k| = i_k, k = 0, \cdots, r\Big)} \\
&= \frac{\mathbb{P}\Big(S_n = i_n, |S_k| = i_k, k = 0, \cdots, n \Big| |S_k| = i_k, k = 0, \cdots, r\Big)}{\mathbb{P}\Big(|S_k| = i_k, k = r+1, \cdots, n \Big| |S_k| = i_k, k = 0, \cdots, r\Big)} \\
&= \frac{p^{n-r+\frac{i_n}{2}} q^{n-r-\frac{i_n}{2}}}{p^{n-r+\frac{i_n}{2}} q^{n-r-\frac{i_n}{2}} + p^{n-r-\frac{i_n}{2}} q^{n-r+\frac{i_n}{2}}} \\
&= p^{i_n}(p^{i_n} + q^{i_n})^{-1}
\end{aligned}
$$

In the same way, we can get

$$
\mathbb{P}\Big(S_n = -i_n \Big| |S_k| = i_k, k = 0, \cdots, n\Big) = q^{i_n}(p^{i_n} + q^{i_n})^{-1}
$$

Then

$$
\begin{aligned}
&\mathbb{P}\Big(|S_{n+1}| = i_{n+1} \Big| |S_k| = i_k, k = 0, \cdots, n\Big) \\
&= \mathbb{P}\Big(|S_{n+1}| = i_{n+1} \mid S_n = i_n, |S_k| = i_k, k = 0, \cdots, n)\mathbb{P}(S_n = i_n \Big| |S_k| = i_k, k = 0, \cdots, n\Big) \\
&+ \mathbb{P}\Big(|S_{n+1}| = i_{n+1} \mid S_n = -i_n, |S_k| = i_k, k = 0, \cdots, n)\mathbb{P}(S_n = -i_n \Big| |S_k| = i_k, k = 0, \cdots, n\Big) \\
&= \mathbb{P}\Big(S_{n+1} = i_{n+1} \Big| S_n = i_n\Big)\mathbb{P}\Big(S_n = i_n \Big| |S_k| = i_k, k = 0, \cdots, n\Big) \\
&+ \mathbb{P}\Big(S_{n+1} = -i_{n+1} \Big| S_n = -i_n\Big)\mathbb{P}\Big(S_n = -i_n \Big| |S_k| = i_k, k = 0, \cdots, n\Big) \\
&= \mathbb{1}(i_{n+1} = i_n + 1)(p^{i_n+1} + q^{i_n+1})(p^{i_n} + q^{i_n})^{-1} + \mathbb{1}(i_{n+1} = i_n - 1)(p^{i_n}q + pq^{i_n})(p^{i_n} + q^{i_n})^{-1}
\end{aligned}
$$

Thus, $(|S_n| : n \geq 0)$ is Markov chain on $\mathbb{Z}^+$, with transition matrix $r_{ij} = \mathbb{1}(0 \neq i = j-1)(p^{i+1} + q^{i+1})(p^i + q^i)^{-1} + \mathbb{1}(0 \neq i = j+1)(p^i q + pq^i)(p^i + q^i)^{-1} + \mathbb{1}(i = 0, j = 1)$. When $p = q = \frac{1}{2}$, we get $r_{ij} = \frac{1}{2}\mathbb{1}(i \neq 0, |j - i| = 1) + \mathbb{1}(i = 0, j = 1)$.

$\square$