underGraduate Homework In Mathematics

SetTheory 4

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General fire extinguisher

PROBLEM I Consider $\mathbb{Q} = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})/\sim$, where $(a, b) \sim (c, d) \iff ad = bc$.j Define $+_{\mathbb{Q}}, \cdot_{\mathbb{Q}}$ and $<_{\mathbb{Q}}$ and verify that your definitions doesn't depend on the choice of representatives.

SOLUTION. Let $[(a, b)] +_{\mathbb{Q}} [(c, d)] = [(ad + bc, bd)], [(a, b)] \cdot_{\mathbb{Q}} [(c, d)] = [(ac, bd)]$, and $[(a, b)] <_{\mathbb{Q}} [(c, d)] \iff abd^2 < cdb^2$. Now we prove they are well-defined, i.e., doesn't depend on the choice of representatives.

For $+_{\mathbb{Q}}$, assume $(a, b) \sim (e, f)$, we need to prove $(ad + bc, bd) \sim (ed + fc, df)$. Since $af = be$, we have $(ad + bc)bf = ad^2 f + bdcf = bed^2 + bdcf = (ed + fc)bd$. So $+_{\mathbb{Q}}$ is well defined.

For $\cdot_{\mathbb{Q}}$, assume $(a, b) \sim (e, f)$, we need to prove $(ac, bd) \sim (ec, fd)$. Since $af = be$, we have $acfd = bced = bdec$. So $\cdot_{\mathbb{Q}}$ is well defined.

For $<_{\mathbb{Q}}$, assume $(a_1, b_1) \sim (a_2, b_2), (c_1, d_1) \sim (c_2, d_2)$ and $(a_1, b_1) < (c_1, d_1)$. Now we need to prove $(a_2, b_2) < (c_2, d_2)$. Since $a_1 b_2 = a_2 b_1, c_1 d_2 = c_2 d_1$ we get $a_1 b_1 d_2^2 < c_2 d_2 b_1^2$ $\square$

PROBLEM II The set of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ has cardinality $\mathfrak{c}$ (while the set of all functions has cardinality $2^{\mathfrak{c}}$). [A continuous function on $\mathbb{R}$ is determined by its values at rational points.]

SOLUTION. Consider $\theta : {}^{\mathbb{R}}\mathbb{R} \to 2^{\mathbb{Q}}, f \mapsto \{(a, b) \in \mathbb{Q} : f(a) < b\}$. Now we prove f is a injection. Assume $\theta(f) = \theta(g)$, to prove $f = g$. First we prove for $x \in \mathbb{Q}$ we have $f(x) = g(x)$. We have $f(x) = \sup\{y \in \mathbb{Q} : y < f(x)\} = \sup\{y \in \mathbb{Q} : (x, y) \in \theta(f)\} = \sup\{y \in \mathbb{Q} : (x, y) \in \theta(g)\} = g(x)$. For $x \in \mathbb{R}$, choose a sequence $x_n \in \mathbb{Q}$ such that $x_n \to x$, then $f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = g(x)$. So we get $f = g$. So $\mathrm{card}^{\mathbb{R}}\mathbb{R} \leq \mathrm{card} 2^{\mathbb{Q}} = 2^{\aleph_0}$. Obviously $\mathrm{card}^{\mathbb{R}}\mathbb{R} \geq 2^{\aleph_0}$, so we get they are equal. $\square$

PROBLEM III There are at least $\mathfrak{c}$ countable order-types of linearly ordered sets.

SOLUTION. For every sequence $a = \langle a_n : n \in \mathbb{N} \rangle$ of natural numbers consider the ordertype

$$\tau_a = \{(x, y) \in \mathbb{Z} \times \mathbb{N} : 2 \nmid y \wedge 0 < x < a_{\frac{y}{2}}\}$$

And for $(x, y), (z, w) \in \tau_a$ we define $(x, y) < (z, w) \iff y < w \wedge y = w, x < z$. Now we will show that if $a \neq b$, then $\tau_a \neq \tau_b$. Assume $\tau_a \cong \tau_b$, we need to prove $a = b$. assume $\theta : \tau_a \to \tau_b$ is the isomorfism.

We know $(x, 0)$ can be defined as $\phi(p) = \exists_{k=1}^{x-1} t_k, \wedge_{1 \leq i < j \leq x-1} t_i \neq t_j, \forall k = 1, \cdots x - 1, t_k < p$. And θ is isomorphism. So $\theta(x, 0) = (x, 0)$. For $(x, 1)$, we let b_0 satisfy $\theta(0, 1) = (b_0, m)$. Since the set $\{(x, y) : y = 1\}$ can be defined by $\psi(p) = \forall r, s(r, s < p \wedge \tau(r) \wedge \tau(s) \to \mathrm{card}[r, s] < \infty)$, where $\tau(r) := \{s : s < r\}$ and $[r, s] = \{y : r < y < s\}$. we get $\theta[\{(x, y) : y = 1\}] = \{(x, y) : y = 1\}$. So we can delete the element whose second coordinary is $0, 1$, and θ is isomorphism, too. Do this repeatedly, we get $\theta(x, 2n + 1) = (x, 2n + 1)$. So $a_n = \mathrm{card}\{(x, 2n + 1) \in \tau_a\} = \mathrm{card}\{(x, 2n + 1) \in \tau_b\} = b_n$ and thus $a = b$. $\square$

PROBLEM IV The set of all algebraic reals is countable.

SOLUTION. Assume $\{f_n : n \in \mathbb{N}\}$ is the set of all integral coefficient polynomial. Consider $A_n := \{x \in \mathbb{C} : f(x) = 0\}$ is finite set. Then we get $\cup_{n\in\mathbb{N}} A_n$ is at most countable. Obviouly $\bigcup_{n\in\mathbb{N}} A_n$ is infinite, so it's countable. $\square$

PROBLEM V If S is a countable set of reals, then $|\mathbb{R} - S| = \mathfrak{c}$. [Use $\mathbb{R} \times \mathbb{R}$ rather than $\mathbb{R}$ (because $|\mathbb{R} \times \mathbb{R}| = 2^{\aleph_0}$).]

SOLUTION . Assume $\theta : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ is bijection, and $T = \theta(S)$. Then T is countable. And $\text{card}(\mathbb{R} \setminus S) = \text{card}(\mathbb{R} \times \mathbb{R} \setminus T)$. So we only need to prove $\mathbb{R} \times \mathbb{R} \approx \mathbb{R} \times \mathbb{R} \setminus T$. Obviously $\text{card}\mathbb{R} \times \mathbb{R} \setminus T \leq \text{card}\mathbb{R} \times \mathbb{R}$, so we only need $\mathbb{R} \times \mathbb{R} \setminus T \geq \mathbb{R}$. Since T is countable, we get $\{x : \exists y, (x, y) \in T\}$ is countable. Choose $t \notin \{x : \exists y, (x, y) \in T\}$. Let $f : \mathbb{R} \to \mathbb{R} \times \mathbb{R} \setminus T, x \mapsto (t, x)$. Easily we get f is injection. So $\text{card}\mathbb{R} \times \mathbb{R} \setminus T = \mathfrak{c}$. □

PROBLEM VI

1. Prove that the set of all irrational number in $\mathbb{R}$ has cardinality $\mathfrak{c}$.
2. Prove that the set of all transcendental numbers in $\mathbb{R}$ has cardinality $\mathfrak{c}$.

SOLUTION .

1. Since $\text{card}\mathbb{Q}$ is countable, from PROBLEM V we get $\text{card}(\mathbb{R} \setminus \mathbb{Q}) = \mathfrak{c}$.
2. Since the set of all algebraic nbumber is countable(PROBLEM IV), for the same reason we get the set of transcendental number has cardinality $\mathfrak{c}$.

□

PROBLEM VII Assume T is a tree.

1. If $s, t, u \in T$, then $R_{stu} := \{\delta_{st}, \delta_{tu}, \delta_{us}\}$ has at most 2 elements. And if $p, q \in R_{stu}$, then $p \subset q \vee q \subset p$.
2. $\prec$ is a linear ordering of T which extends $\sqsubset$.
3. For every $t \in T$, Prove $T^t := \{s \in T : t \sqsubset s\}$ is an interval in $(T, \prec)$.

SOLUTION .

1. First we prove for $p, q \in R_{stu}$ we have $p \subset q \vee q \subset p$. Without loss of generality assume $p = \delta_{st}, q = \delta_{tu}$. We have $p, q \subset (\cdot, t)$. Since $(\cdot, t)$ is well ordered, and easily p, q are initial segment, so $p \subset q \vee q \subset p$.

   Now we prove there are at most two elements. From above we know $(R_{stu}, \subset)$ is linear order set, and it's finite. Without loss of generality we assume $\delta_{st} \subset \delta_{tu} \subset \delta_{us}$. Then we get $\delta_{tu} = \delta_{tu} \cap \delta_{us} = (\cdot, t) \cap (\cdot, u) \cap (\cdot, s) \subset \delta_{st}$. That means $\delta_{st} = \delta_{tu}$, so there is at most two elements.

2. Easily to prove $\sqsubset \subset \prec$. Now we prove $\prec$ is linear ordered. Consider a bigger linear ordered set Y is obtained by adding a minimum, $-\infty$, in X. Consider the tree $U := {}^{<\alpha}Y$. We try to make a map from T to B_U. Let $\theta : T \to B_U, \theta(f)(\beta) := \begin{cases} f(\beta), \beta \in \text{dom } f \\ -\infty, \beta \notin \text{dom } f \end{cases} \quad \forall \beta \in \alpha, f \in T.$
   Then we it's easily to prove θ is injective and $f \prec g \iff \theta(f)(\beta) < \theta(g)(\beta)$, where $\beta = \min\{t \in \alpha : \theta(f)(t) \neq \theta(g)(t)\}$. We define $f, g \in B_U, f < g \iff f(\beta) < g(\beta)$, where $\beta = \min\{t \in \alpha : f(t) \neq g(t)\}$. Now we only need to prove $(B_U, <)$ is linear ordered. Easily $f \not< f, \forall f \in B_U$. And for $f \neq g, f < g \vee g < f$. Assume $f < g < h$, to prove $f < h$.

If $n_{fg} < n_{gh}$ then we get $f(n_{fg}) < g(n_{fg}) = h(n_{fg})$. So $n_{fh} \le n_{fg}$. From VII.1 we get $n_{fh} = n_{fg} \vee n_{fh} = n_{gh}$. So $n_{fh} = n_{fg}$, and thus $f < h$.

If $n_{fg} > n_{gh}$, then we get $h(n_{gh}) > g(n_{gh}) = f(n_{gh})$. Same as above we get $n_{fh} = n_{gh}$, so $f < h$.

If $n_{fg} = n_{gh}$, it's obvious $f < h$.

So we have proved B_U is linear ordered, and thus $(T, \prec)$ is linear orderd.

3. Only need to prove if $t \sqsubset u, t \sqsubset v, u \prec v$, then $\forall s : u \prec s \prec v, t \sqsubset s$. If $u \sqsubseteq s$ then $t \sqsubset u \sqsubseteq s$. Else we get $u \not\sqsubseteq s$. So we get $s \not\sqsubseteq u \wedge s(n_{su}) > u(n_{su})$. From VII.2 we get $t \prec s$. So if $t \not\sqsubset s$ then $s \not\sqsubseteq t \wedge s(n_{st}) > t(n_{st})$. Since $t \sqsubseteq v$ we get $s(n_{st}) > t(n_{st}) = v(n_{st})$. Since $t \sqsubset v$ we get $n_{st} = n_{sv}$, so $v \prec s$, contradiction! So $t \sqsubset s$.

□

PROBLEM VIII

1. Prove that $\prec$ is linear ordered on $T \cup B_T$.
2. For every $t \in T$, prove that $B_t = \{f \in B_T : t \in f\} \cup \{f \in T : t \sqsubset f\}$ is interval in $(T \cup B_T, \prec)$.

SOLUTION. 1. consider a bigger tree U. For $f \in T$ let $\theta(f) = f$, for $f \in B_T$ we let $\theta(f)$ is a map from *ordertype*$(\mathrm{dom}(f))$ to X, and $\theta(f)(\beta) := g(\beta)$, where $g \in f$ and $\beta \in \mathrm{dom}(g)$. Let $U = \theta(T \cup B_T)$. Then easily $T \subset U$. Now we prove θ is isomorphic from $(T \cup B_T, \prec)$ to $(U, \prec)$. Easily for $f, g \in T$ we have $f \sqsubset g \iff \theta(f) \sqsubset \theta(g)$.

And for $f \in T, g \in B_T$ we have $f \in g \iff \theta(f) \sqsubset \theta(g)$. So from the defination of $\prec$ we get θ is isomorphic. Since we have proved $(U, \prec)$ is linear order(VII.2), we get $(T \cup B_T, \prec)$ is linear order, too.

2. Since $\theta(B_t) = U^{\theta(t)}$ is an interval(VII.3), we get B_t is interval, too.

□

PROBLEM IX Prove that if X is Suslinnnn line, then X^2 is not c.c.c.

SOLUTION. Let $\mathcal{A} := \{Y \subset X : \mathrm{card}(X \setminus Y) \le \omega\}$. For every $Y \in \mathcal{A}$, let $\theta(Y) := \{(a,c) \subset X : (a,c) \cap Y = \varnothing \wedge a < c\}$. Since X is not separable, we know Y is not dense in X, so $\theta(Y) \neq \varnothing$. So we can use AC to get a map $\tau : \mathcal{A} \to X \times X \times X, Y \mapsto (a,b,c)$, where $(a,c) \cap Y = \varnothing$ and $a < b < c$. Now we give a map from ω_1 to $X \times X \times X$ recursively. Assume for some cardinary α we have defined $f(\beta)$ for every $\beta < \alpha$, now we give $f(\alpha)$. Let $f(\alpha) := \tau(\{b : \exists a, c \in X, (a,b,c) \in f[\alpha]\})$. Since $\alpha < \omega_1$ is countable, we get the set $\{b : \exists a, c \in X, (a,b,c) \in f[\alpha]\}$ is countalbe, so it's in $\mathrm{dom}(\tau)$. So the defination of f is well.

Now consider $\{(a,b) \times (b,c) : (a,b,c) \in \mathrm{ran}(f)\}$. From the defination of f we easily get they are disjoint open interval. And easily we get f is injective, so $\mathrm{ran}(f)$ is uncountable. So we get a class of uncountable disjoint open interval in $X \times X$. So we obtain $X \times X$ is not c.c.c. □

PROBLEM X Assume P is a perfect set and $P \cap (a,b) \neq \varnothing$. Prove that $\mathrm{card}(P \cap (a,b)) = \mathfrak{c}$.

Lemma 1. If P is a perfect set and $P\cap(a,b)\neq\varnothing$, then $\exists c\in(a,b)\cap P$ such that $P\cap(a,c), P\cap(c,b)\neq\varnothing$.

证明. Sicne $P\cap(a,b)\neq\varnothing$, take $x\in P\cap(a,b)$. Since P is perfect set, and (a,b) is open, we get $\exists y\neq x, y\in P\cap(a,b)$. without loss of generality we assume $x<y$. Then $y\in(x,c)\cap P$, so $\exists z\neq y, z\in(x,c)\cap P$ for the same reason. Without loss of generality we assume $y<z$. Now let $c=y$, we get $c\in P\cap(a,b)$ and $x\in(a,c)\cap P, z\in(c,b)\cap P$. So the lemma have been proved. □

SOLUTION . To prove $\mathrm{card}(P\cap(a,b))=\mathfrak{c}$, we only need to find a injective from $(0,1)$ to $(P\cap(a,b))$. First we consider the tree $T:={}^{<\omega}2$. We define $\tau:T\to\{(m,n)\in X\times X: m<n, \exists x\in P, m<x<n\}$ recursively. Let $\tau(\varnothing)=(a,b)$. Assume for $x\in T$ we have defined τ on $(\cdot,x)_T$. Now we need to give $\tau(x)$. Assume *ordertype*$(x)=n+1$ for some $n\in\mathbb{N}$. Then $\mathrm{dom}(x)=n+1$. Consider $y=x|_n<_T x$, assume $\tau(y)=(m,n)$. From Lemma 1 we know $\exists l\in P\cap(m,n)$ such that $P\cap(m,l), P\cap(l,n)\neq\varnothing$. If $x(n)=0$ we let $\tau(x)=(m,l)$; else we get $x(n)=1$, we let $\tau(x)=(l,n)$.

From the defination of τ we can easily get $f\sqsubset g\to\tau(f)_1\leq\tau(g)_1\leq\tau(g)_2\leq\tau(f)_2$. Now we construct $\theta:(0,1)\to P\cap(a,b)$. For $x=\sum_{k=1}^{\infty}\frac{a_k}{2^k}\in(0,1)$, let $f:\omega\to 2, f(k)=a_k$. Consider $f|_n\in T$, assume $\theta(f|_n)=(m_k,n_k)$. Then we get $\bigcap_{k\in\mathbb{N}}[m_k,n_k]\cap P\neq\varnothing$ since they are descending compect set. Let $\theta(x):=\min\bigcap_{k\in\mathbb{N}}[m_k,n_k]\cap P$.

First we prove $\theta(x)\in(a,b)\cap P$. Obviously $\theta(x)\in P\cap[a,b]$, since $[m_k,n_k]\subset[a,b]$. Since $x\in(0,1), \exists k, a_k=0$. Then $[m_k,n_k]=[m_{k-1},l_{k-1}]$, where $l_{k-1}<m_{k-1}\leq b$. So we get $\theta(x)<b$. For the same reason we get $\theta(x)>a$.

Now we only need to prove θ is injective. Assume $x\neq y$. Assume $x=\sum_{k=1}^{\infty}\frac{a_k}{2^k}, y=\sum_{k=1}^{\infty}\frac{b_k}{2^k}$. Without loss of generality we assume $x<y$. And we can assume $\lim a_k, \lim b_k\neq 1$. Assume $t=\min\{k: a_k<b_k\}$. Let $f(k)=a_k, g(k)=b_k$, and $\tau(f|_{t+1})=(m,n), \tau(g|_{t+1})=(n,l)$. Then $\theta(x)\leq n\leq\theta(y)$. Now we only need to prove $\theta(x)<n$. Since $\lim a_k neq 1$, we obtain $\exists i>t, a_i=0$. Then $\tau(f|_{k+1})_2<n$, and thus $\theta(x)<n\leq\theta(y)$.

So we have found a injective from $(0,1)$ to $Pcap(a,b)$. And obviously $(a,b)\cap P\subset\mathbb{R}$, we get $\mathrm{card}((a,b)\cap P)=\mathfrak{c}$. □

PROBLEM XI If P_1, P_2 are perfect set and $P_1\subsetneq P_2$, then $\mathrm{card}(P_2\setminus P_1)=\mathfrak{c}$.

SOLUTION . Take $x\in P_2\backslash P_1$. Since P_1 is closed and $x\notin P_1$, we know $\exists a,b\in\mathbb{R}, a<x<b, (a,b)\cap P_1=\varnothing$. Now consider $(a,b)\cap P_2$, since $x\in(a,b)\cap P_2\neq\varnothing$, from PROBLEM X we get $\mathrm{card}((a,b)\cap P_2)=\mathfrak{c}$. Noting $(a,b)\cap P_2\subset P_1^c\cap P_2=P_2\setminus P_1\subset\mathbb{R}$, we finally get $\mathrm{card}(P_2\setminus P_1)=\mathfrak{c}$. □

PROBLEM XII If P is perfect set, then $P^*=P$.

SOLUTION . First we prove $P\subset P^*$. Take $a\in P$, consider a neibor of a, named I. Since $a\in I\cap P\neq\varnothing$, from PROBLEM X we get $\mathrm{card}(P\cap I)=\mathfrak{c}$. So $a\in P^*$.

Second we prove $P^*\subset P$. From the defination of P^* we know $P^*\subset\overline{P}=P$. So we get $P=P^*$. □

PROBLEM XIII If F is colsed and $P\subset F$ is a perfect set, then $P\subset F^*$.

SOLUTION. From the defination of * we easily get $A \subset B \to A^* \subset B^*$. So we get $P = P^* \subset F^*$. $\square$

PROBLEM XIV Assume $F = P \cup S$ is closed set, where P is perfect and S is at most countable. Prove that $F^* = P$.

SOLUTION. From PROBLEM XIII we get $F^* \supset P$, now we only need to prove $F^* \subset P$. Assume $a \in F^*$, then for any neibor of a, named I, we get $I \cap F$ is uncountable. Since $F = P \cup S$ and S is countable, we easily get $I \cap P \neq \varnothing$. So we get $a \in \overline{P} = P$. So $F^* \subset P$. $\square$

PROBLEM XV Assume F is uncountable closed set, prove that $F = F^* \cup (F \setminus F^*)$ is the unique way to part F into a perfect set and an at most countable set.

SOLUTION. First we prove $F^* \neq \varnothing$. If $F^* = \varnothing$, then $\forall a \in F^*, \exists I_a$ is neibor of a such that $I_a \cap F$ is at most countable. Since $F = \bigcup_{n \in \mathbb{N}} F \cap [-n, n]$ is uncountable, we know at least one of $F \cap [-n, n]$ is uncountable, name it E. Then E is compect. Notin $\{I_a : a \in E\}$ is a covery of E, so there exists finite subcovery. Then we get $E = \bigcap_{k=1}^{n} I_{a_k}$ is countable, contradiction! So $F^* \neq \varnothing$.

Next we prove F^* is colsed. From the defination of * it's obvious.

Then we prove $F \setminus F^*$ is at most countable. Consider $F_n := \{x \in F \setminus F^* : d(x, F^*) \geq \frac{1}{n}\}$. We get $F \setminus F^* = \bigcup_{n \in \mathbb{N}^+} F_n$. So we only need to prove F_n is countable. If not, since F_n is uncountable colsed set, from above we easily know $F_n^* \neq \varnothing$. But $F_n^* \subset F_n \subset F \setminus F^*$ and since $F_n \subset F$ we know $F_n^* \subset F^*$, contradiction! So we get $F \setminus F^*$ is at most countable.

Now we prove F^* is perfect set. Only need to prove every $a \in F^*$ is limit point. For any neibor I of a, we easily know $I \cap F$ is uncountable. Sicne $F \setminus F^*$ is countable, we get $I \cap F^* \neq \varnothing$. So a is limit point.

Finally we prove it's unique. Assume $F = P_1 \cup S_1 = P_2 \cup S_2$, where P_1, P_2 are perfect sets and S_1, S_2 are atmost countable. Consider $S := S_1 \cup S_2, P := P_1 \cap P_2 = F \setminus S$. Since S is atmost countable, we get $P \neq \varnothing$. So P is perfect, too. If $P \subsetneq P_1$, then from PROBLEM XI we get $\mathrm{card}(P_1 \setminus P) = \mathfrak{c}$. But $P_1 \setminus P \subset S_2$ is at most countable, contradiction! So $P = P_1$. For the same reason we get $P = P_2$. So $P_1 = P_2$ and thus $S_1 = S_2$. $\square$

PROBLEM XVI If B is Borel and f is continuous, then $f_{-1}(B)$ is Borel.

SOLUTION. Let $\mathcal{A} := \{A \in \mathcal{B} : f_{-1}(A) \in \mathcal{B}\}$. Since f is continuous we get for open set $I \subset \mathbb{R}, f_{-1}(I)$ is open, thus Borel. So $I \in \mathcal{A}$. Now we need to prove $\mathcal{A} = \mathcal{B}$, only need to prove $\mathcal{A}$ is σ-field.

First, for $A \in \mathcal{A}$, we have $f_{-1}(A^c) = (f_{-1}(A))^c \in \mathcal{B}$.

Second, for $A_n \in \mathcal{A}, \forall n \in \mathbb{N}$, we have $f_{-1}(\bigcup_{n \in \mathbb{N}} A_n) = \bigcup_{n \in \mathbb{N}} f_{-1}(A_n) \in \mathcal{B}$.

Finally, it's obviously that $\varnothing, \mathbb{R} \in \mathcal{A}$.

So we get $\mathcal{A}$ is σ-field, thus $\mathcal{B} = \mathcal{A}$. So $B \in \mathcal{A}$, thus $f_{-1}(B) \in \mathcal{B}$. $\square$

PROBLEM XVII For closed set $F \subset \mathcal{N}$, let $T_F := \{s \in \mathrm{Seq} : \exists f \in F, s \subset f\}$. Prove that T_F has no maximal node. And prove $F \mapsto T_F$ is bijection from closed set in $\mathcal{N}$ and sequnetial trees with no maximal node.

SOLUTION. For $s \in T_F$, we assume $s \subset f \in F$. Since s is finite, we can assume $s = f|_n$. Then $s \subsetneq f|_{n+1} \in T_F$, so s is not maximal node. So T_F has no maximal node.

Now we prove $F \mapsto T_F$ is injective. Assume $T_F = T_G$, where F, G are closed subset of $\mathcal{N}$. Only need to prove $F = G$. By symetry we only need to prove $F \subset G$. For $f \in F$, from the defination of T_F we know $\forall n \in \mathbb{N}, f_n := f|_n \in T_F = T_G$. From defination of T_G we get $\exists g_n \in G, g_n|_n = f|_n$. Then $d(g_n, f) \leq \frac{1}{2^n}$. So $g_n \to f$. Since G is closed, we get $f \in G$.

Finally we prove $F \to T_F$ is surjective. Assume $T \subset \mathrm{Seq}$ has no maximal node. Let $F := \{f \in \mathcal{N} : \forall n, f|_n \in T\}$. Now we prove $T_F = T$. Easily we get $T_F \subset T$, only need $T \subset T_F$. For $s \in T$, we need to prove $s \in T_F$, i.e., $\exists f \in F, s \subset f$. Now we construct f recursively. For $n \in \mathrm{dom}(s)$ we let $f(n) = s(n)$. Now assume $n \notin \mathrm{dom}(s)$ and for $m < n$ we have defined $f(m)$ and $f|_n \in T$. Since T has no maximal, we get $f|_n$ is not maximal. So $\{t \in \mathbb{N} : f|_n \cup \{(n,t)\} \in T\} \neq \varnothing$. Let $f(n) := \min\{t \in \mathbb{N} : f|_n \cup \{(n,t)\} \in T\}$ is OK. Easily we get $f|_n \in T, \forall n \in \mathbb{N}$. So $f \in F$, and $s \subset f$. $\square$