PROBLEM I Assume $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, and $C \in \mathcal{F}$ satisfy $\mathbb{P}(C) > 0$. Let $\mathbb{P}_C : \mathcal{F} \to \mathbb{R}, \mathbb{P}_C(X) = \frac{\mathbb{P}(C \cap X)}{\mathbb{P}(C)}$. Assume $A, B \in \mathcal{F}$, and $\mathbb{P}(B \cap C) > 0$, prove that $\mathbb{P}_C(A \mid B) = \mathbb{P}(A \mid B \cap C)$.

SOLUTION. Easily $\mathbb{P}_C(B) = \frac{\mathbb{P}(B \cap C)}{\mathbb{P}(C)} > 0$, so $\mathbb{P}_C(A \mid B)$ is well-defined. Easily to get that

$$\mathbb{P}_C(A \mid B) = \frac{\mathbb{P}_C(A \cap B)}{\mathbb{P}_C(B)} = \frac{\frac{\mathbb{P}(A \cap B \cap C)}{\mathbb{P}(C)}}{\frac{\mathbb{P}(B \cap C)}{\mathbb{P}(C)}} = \frac{\mathbb{P}(A \cap B \cap C)}{\mathbb{P}(B \cap C)} = \mathbb{P}(A \mid B \cap C)$$

$\square$

PROBLEM II Assume that $(X_n : n \geq 0)$ is 1-dimentional simple symetry random walk, prove that $(|X_n| : n \geq 0)$ is a Markov chain ranges in $\mathbb{N}$.

SOLUTION. Easy to know that $(X_n : n \geq 0)$ is a Markov chain in $\mathbb{Z}$. Let $\mathcal{F}_n := \sigma(X_1, \cdots, X_n), \mathcal{G}_n := \sigma(|X_1|, \cdots, |X_n|)$, then easily $\mathcal{G}_n \subset \mathcal{F}_n$. Then we get that $\mathbb{P}(|X_{n+1}| = i \mid \mathcal{F}_n) = \mathbb{P}(X_{n+1} = i \mid \mathcal{F}_n) + \mathbb{P}(X_{n+1} = -i \mid \mathcal{F}_n) = \mathbb{P}(X_{n+1} = i \mid X_n) + \mathbb{P}(X_{n+1} = -i \mid X_n) = \mathbb{P}(|X_{n+1}| = i \mid X_n)$.

$\square$