

under Graduate Homework In Mathematics

MarkovProcess

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General fire extinguisher

PROBLEM I Assume $(\mathcal{F}_t : t \geq 0, t \in \mathbb{R})$ is a filtration. For $t \geq 0$ we let $\mathcal{F}_{t+} := \bigcap_{s>t} \mathcal{F}_s$. Prove that $\mathcal{F}_t \subset \mathcal{F}_{t+}$ and $(\mathcal{F}_{t+} : t \geq 0)$ is a filtration.

SOLUTION. To prove $\mathcal{F}_t \subset \mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_s$, we only need to prove $\forall s > t, \mathcal{F}_t \subset \mathcal{F}_s$. By the definition of filtration it's obvious. Now we will prove $(\mathcal{F}_{t+} : t \geq 0)$ is a filtration. Only need to prove $\forall t, s \in \mathbb{R} \wedge t \leq s, \mathcal{F}_{t+} \subset \mathcal{F}_{s+}$. By the definition of \mathcal{F}_{t+} we know that $\mathcal{F}_{t+} = \bigcap_{x>t} \mathcal{F}_x = \bigcap_{x>s} \mathcal{F}_x \cap \bigcap_{x:t<x\leq s} \mathcal{F}_x \subset \bigcap_{x>s} \mathcal{F}_x = \mathcal{F}_{s+}$. So $(\mathcal{F}_{t+} : t \geq 0)$ is a filtration. \square

PROBLEM II Assume $(X_t : t \geq 0, t \in \mathbb{R})$ is a stochastic process on probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Prove that $\forall s, t \geq 0, \varepsilon > 0, \{\rho(X_s, X_t) \geq \varepsilon\} \in \mathcal{F}$.

SOLUTION. Easily $\{\rho(X_s, X_t) \geq \varepsilon\} = \bigcup_{k=1}^{\infty} \{\rho(X_s, X_t) > \varepsilon - \frac{1}{k}\varepsilon\}$. So we only need to prove $\forall k \in \mathbb{N}^+, \{\rho(X_s, X_t) > \varepsilon(1 - \frac{1}{k})\} \in \mathcal{F}$. Take $\delta = \varepsilon(1 - \frac{1}{k})$, only need to prove $\forall \delta > 0, \{\rho(X_s, X_t) > \delta\} \in \mathcal{F}$.

$\forall t \geq 0, X_t : \Omega \rightarrow E$ is measurable, where $E \subset \mathbb{R}^d$. So we can find a countable dense set in \mathbb{R}^d , write D . We will prove that $\{\rho(X_s, X_t) > \delta\} = \bigcup_{q \in D} \{\rho(X_s, q) - \rho(X_t, q) > \delta\}$. On one hand, easily $\rho(X_s, q) - \rho(X_t, q) > \delta \implies \rho(X_s, X_t) > \delta$ from triangle inequality. So we easily get $\{\rho(X_s, X_t) > \delta\} \supset \bigcup_{q \in D} \{\rho(X_s, q) - \rho(X_t, q) > \delta\}$. On the other hand, assume for certain $\omega \in \Omega$ we have $\rho(X_s(\omega), X_t(\omega)) > \delta$, we will prove $\exists q \in D, \rho(X_s(\omega), q) - \rho(X_t(\omega), q) > \delta$. For convenience, we omit (ω) from now on to the end of this paragraph. Since $\rho(X_s, X_t) > \delta$, we know $\gamma := \frac{\rho(X_s, X_t) - \delta}{2} > 0$. Since D is dense, we obtain $\exists q \in D, \rho(X_t, q) < \gamma$. So from triangle inequality we get $\rho(X_s, q) \geq \rho(X_s, X_t) - \rho(X_t, q) > 2\gamma + \delta - \gamma = \gamma + \delta$. So we get $\rho(X_s, q) - \rho(X_t, q) > \gamma + \delta - \gamma = \delta$. Finally, we get $\{\rho(X_s, X_t) > \delta\} = \bigcup_{q \in D} \{\rho(X_s, q) - \rho(X_t, q) > \delta\}$.

Noting $\bigcup_{q \in D} \{\rho(X_s, q) - \rho(X_t, q) > \delta\} = \bigcup_{q \in D} \bigcup_{p \in \mathbb{Q}^+} \{\rho(X_s, q) > \delta + p, \rho(X_t, q) < p\}$, and D, \mathbb{Q}^+ are countable, so we only need to check $\{\rho(X_s, q) > \delta + p, \rho(X_t, q) < p\} \in \mathcal{F}, \forall q \in D, p \in \mathbb{Q}^+$. Noting $\{\rho(X_s, q) > \delta + p, \rho(X_t, q) < p\} = \{\rho(X_s, q) > \delta + p\} \cap \{\rho(X_t, q) < p\}$, and X_s, X_t are measurable from Ω to E , we obtain $\{\rho(X_s, q) > \delta + p\}, \{\rho(X_t, q) < p\} \in \mathcal{F}$. So we proved $\{\rho(X_s, X_t) > \delta\} \in \mathcal{F}, \forall s, t \geq 0, \forall \delta > 0$.

Finally, we obtain $\{\rho(X_s, X_t) \geq \varepsilon\} \in \mathcal{F}, \forall s, t \geq 0, \varepsilon > 0$. \square

PROBLEM III Let $\mathcal{D}_X := \{\mu_J^X : J \in S(I)\}$ be the family of finite-dimensional distributions of a stochastic process $(X_t : t \geq 0, t \in \mathbb{R})$. $\forall (s_1, s_2) \in S(I)$ and $J = (t_1, \dots, t_n) \in S(I)$, write $K_1 := (s_1, s_2, t_1, \dots, t_n) \in S(I), K_2 := (s_2, s_1, t_1, \dots, t_n) \in S(I)$. Take $A_1, A_2 \in \mathcal{E}, B \in \mathcal{E}^n$, prove that

$$\mu_{K_1}^X(A_1 \times A_2 \times B) = \mu_{K_2}^X(A_2 \times A_1 \times B)$$

and

$$\mu_{K_1}^X(E \times E \times B) = \mu_{K_2}^X(E \times E \times B) = \mu_J^X(B)$$

SOLUTION. By the definition of finite-dimensional distributions we get

$$\begin{aligned} \mu_{K_1}^X(A_1 \times A_2 \times B) &= \mathbb{P}((X_{s_1}, X_{s_2}, X_{t_1}, \dots, X_{t_n}) \in A_1 \times A_2 \times B) \\ &= \mathbb{P}(X_{s_1} \in A_1) \mathbb{P}(X_{s_2} \in A_2) \mathbb{P}((X_{t_1}, X_{t_2}, \dots, X_{t_n}) \in B) \end{aligned}$$

For the same reason, we obtain

$$\begin{aligned} \mu_{K_2}^X(A_2 \times A_1 \times B) &= \mathbb{P}((X_{s_2}, X_{s_1}, X_{t_1}, \dots, X_{t_n}) \in A_2 \times A_1 \times B) \\ &= \mathbb{P}(X_{s_2} \in A_2) \mathbb{P}(X_{s_1} \in A_1) \mathbb{P}((X_{t_1}, X_{t_2}, \dots, X_{t_n}) \in B) \end{aligned}$$

So we get

$$\mu_{K_1}^X(A_1 \times A_2 \times B) = \mu_{K_2}^X(A_2 \times A_1 \times B)$$

Also, let $A_1 = A_2 = E$, we get

$$\begin{aligned} \mu_{K_1}^X(E \times E \times B) &= \mu_{K_2}^X(E \times E \times B) \\ &= \mathbb{P}(X_{s_1} \in E) \mathbb{P}(X_{s_2} \in E) \mathbb{P}((X_{t_1}, \dots, X_{t_n}) \in B) \\ &= \mathbb{P}((X_{t_1}, \dots, X_{t_n}) \in B) \end{aligned}$$

By the definition of finite-dimensional distributions we get

$$\mu_J^X(B) = \mathbb{P}((X_{t_1}, \dots, X_{t_n}) \in B)$$

So finally we get

$$\mu_{K_1}^X(E \times E \times B) = \mu_{K_2}^X(E \times E \times B) = \mu_J^X(B)$$

□

PROBLEM IV Assume $(\tau_k : k \in \mathbb{N}^+)$ is an i.i.d sequence of r.v. with exponential distribution with parameter $\alpha > 0$. Let $S_n := \sum_{k=1}^n \tau_k$. For $t \geq 0, t \in \mathbb{R}$, let:

$$N_t := \sum_{n=1}^{\infty} \mathbb{1}_{\{S_n \leq t\}}, X_t := \sum_{n=1}^{\infty} \mathbb{1}_{\{S_n < t\}}$$

Prove that N and X are modifications of each other, but they are not indistinguishable.

SOLUTION. First we prove N and X are modifications of each other. Fix $t \in [0, \infty)$, we need to prove $\mathbb{P}(N_t = X_t) = 1$. Noting $N_t - X_t = \sum_{n=1}^{\infty} \mathbb{1}_{S_n \leq t} - \mathbb{1}_{S_n < t} = \sum_{n=1}^{\infty} \mathbb{1}_{S_n = t}$, we get $\mathbb{P}(N_t =$

$X_t) = \mathbb{P}(\sum_{n=1}^{\infty} \mathbb{1}_{S_n=t}) = \mathbb{P}(\forall n \in \mathbb{N}^+, S_n \neq t)$. So we only need to prove $\mathbb{P}(S_n = t) = 0, \forall n \in \mathbb{N}^+$.

Since $\tau_k, k \in \mathbb{N}^+$ are continuous-distributed, we know $S_n = \sum_{k=1}^n \tau_k$ is continuous-distributed, so $\mathbb{P}(S_n = t) = 0$. So we proved N and X are modifications of each other.

Next we will prove they are not indistinguishable. Only need to prove $\mathbb{P}(\forall t \in [0, \infty), X_t = N_t) = 0 \neq 1$. Since $N_t - X_t = \sum_{n=1}^{\infty} \mathbb{1}_{S_n \leq t} - \mathbb{1}_{S_n < 1} = \sum_{n=1}^{\infty} \mathbb{1}_{S_n=t}$, we know $\forall t, N_t = X_t \iff \forall t, \forall n \in \mathbb{N}^+, S_n \neq t$. But S_n is ranged in $[0, \infty)$, so R.H.S is an impossible event. So we finally get $\mathbb{P}(\forall t \in [0, \infty), X_t = N_t) = 0$ and thus X and N are not indistinguishable. \square

PROBLEM V Assume T is non-negative r.v. with distribution function F continuous on \mathbb{R} . Let $X_t = \mathbb{1}_{\{T \leq t\}}$. Prove that X is stochastically continuous.

SOLUTION. Only need to check $\forall t \geq 0, X_s \xrightarrow{\mathbb{P}} X_t, s \rightarrow t$. Take $\varepsilon > 0$, we need to prove $\lim_{s \rightarrow t} \mathbb{P}(\rho(X_s, X_t) > \varepsilon) = 0$. For $u > v \geq 0$, we have $X_u - X_v = \mathbb{1}_{v < T \leq u}$. So $\mathbb{P}(\rho(X_u, X_v) > \varepsilon) \leq \mathbb{P}(X_u \neq X_v) = \mathbb{P}(v < T \leq u) \leq \mathbb{P}(T \in [v, u])$. So we easily get $\lim_{s \rightarrow t+} \mathbb{P}(\rho(X_s, X_t) > \varepsilon) \leq \lim_{s \rightarrow t+} \mathbb{P}(T \in [t, s]) = 0$ and $\lim_{s \rightarrow t-} \mathbb{P}(\rho(X_s, X_t) > \varepsilon) \leq \lim_{s \rightarrow t-} \mathbb{P}(T \in [s, t]) = 0$. So $\lim_{s \rightarrow t} \mathbb{P}(\rho(X_s, X_t) > \varepsilon) = 0$. \square

PROBLEM VI Assume $I = \mathbb{Z}^+$, then the stochastic process $X = (X_0, X_1, \dots)$ is a r.v. from Ω to E^∞ . Define the distribution of X , μ_X , as follows:

$$\mu_X(A) = \mathbb{P}(X \in A), A \in \mathcal{E}^\infty$$

Then stochastic process X, Y are equivalent $\iff \mu_X = \mu_Y$.

SOLUTION. “ \implies ”: Assume X, Y are equivalent, now we will prove $\mu_X = \mu_Y$. Let $\mathcal{A} := \{A \in \mathcal{P}(E^\infty) : \exists n \in \mathbb{N}^+, A = A_1 \times A_2 \times \dots \times A_n \times E \times E \times \dots\}$. Then we can get $\mu_X(A) = \mu_{(1,2,\dots,n)}^X(A_1 \times \dots \times A_n)$. So for $A \in \mathcal{A}$ we know $\mu_X(A) = \mu_Y(A)$. By the definition of $\mathcal{E}^\infty = \sigma(\mathcal{A})$, and noting \mathcal{A} is a Semiset algebra, by the Measure extension theorem we get $\mu_X = \mu_Y$.

“ \impliedby ”: Assume $\mu_X = \mu_Y$, then easily $\mu_{(s_1,\dots,s_n)}^X(A_{s_1} \times \dots \times A_{s_n}) = \mu_X(\prod_{k \in \mathbb{N}^+} B_k)$, where $B_k = A_{s_t}$ for $k = s_t$ and $B_k = E$ for $k \neq s_t, \forall t = 1, \dots, n$. So easily $\mu_J^X = \mu_J^Y, \forall J \subset I \wedge |J| < \infty$. \square