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In Mathematic

SetTheory 7

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PROBLEM I Prove that there are arbitrarily large singular cardinals

SOLUTION. For cardinal λ, we cansider $\aleph_{\lambda+\omega}$. Easily $\mathrm{cf}(\aleph_{\lambda+\omega}) = \mathrm{cf}(\lambda+\omega) = \mathrm{cf}(\omega) = \aleph_0 < \aleph_{\lambda+\omega}$, and $\aleph_{\lambda+\omega} \geq \lambda+\omega \geq \lambda$. $\square$

PROBLEM II There are arbitrarily large singular cardinals $\aleph_\alpha$ such that $\aleph_\alpha = \alpha$.

SOLUTION. For cardinal λ, we let $x_0 = \lambda, x_{n+1} = \aleph_{x_n}$. Now consider $\kappa = \sup_{n\in\omega} x_n$. Easily κ is limit ordinal, so $\aleph_\kappa = \sup_{\alpha<\kappa} \aleph_\alpha = \sup_{n\in\omega} \aleph_{x_n} = \sup_{n\in\omega} \aleph_{x_{n+1}} = \kappa$. And since $\kappa = \bigcup_{n\in\omega} x_n$, we get $\mathrm{cf}(\kappa) \leq \omega$. Easily $\kappa \geq x_2 = \aleph_{\aleph_\lambda} \geq \aleph_{\aleph_0} > \omega$. So we get $\kappa > \mathrm{cf}(\kappa)$. So κ is singular. $\square$

PROBLEM III

1. $\mathrm{cf}(\aleph + \beta) = \mathrm{cf}(\beta)$.
2. $\mathrm{cf}(\aleph_\alpha) = \mathrm{cf}(\alpha)$ for limit ordinal α.
3. $\mathrm{cf}(\aleph_{\alpha+1}) = \aleph_{\alpha+1}$.

SOLUTION.

1. First we prove $\mathrm{cf}(\alpha+\beta) \leq \mathrm{cf}(\beta)$. Consider $\theta : \mathrm{cf}(\beta) \to \beta$ is unbound. Then we let $\tau : \mathrm{cf}(\beta) \to \alpha+\beta, x \mapsto \alpha+\theta(x)$. Easily we get τ is unbound. So we get $\mathrm{cf}(\alpha+\beta) \leq \mathrm{cf}(\beta)$.

   Second we prove $\mathrm{cf}(\alpha+\beta) \geq \mathrm{cf}(\beta)$. Consider $\theta : \mathrm{cf}(\alpha+\beta) \to \alpha+\beta$ is unbound. Now we consider $B := \{x \in \alpha+\beta : x \geq \alpha\}$ and $A = \theta_{-1}[B]$. Easily we get $B \cong \beta$, and $\mathrm{ordertype}(A) \leq \mathrm{cf}(\alpha+\beta)$. And $\theta \restriction A : A \to B$ is unbounded, so easily we get $\mathrm{cf}(\beta) \leq \mathrm{cf}(\alpha+\beta)$.

   Finally we get $\mathrm{cf}(\alpha+\beta) = \mathrm{cf}(\beta)$.

2. First we prove $\mathrm{cf}(\aleph_\alpha) \leq \mathrm{cf}(\alpha)$. Assume $\theta : \mathrm{cf}(\alpha) \to \alpha$ is unbound. Consider $\tau : \mathrm{cf}(\alpha) \to \aleph_\alpha, x \mapsto \aleph_{\theta(x)}$. Since α is limit ordinal, we get $\aleph_\alpha = \sup_{\beta<\alpha} \aleph_\beta$. So we get τ is unbounded. So we get $\mathrm{cf}(\aleph_\alpha) \leq \alpha$.

   Second we prove $\mathrm{cf}(\alpha) \leq \mathrm{cf}(\aleph_\alpha)$. Assume $\theta : \mathrm{cf}(\aleph_\alpha) \to \aleph_\alpha$ is unbounded. Let $f : \mathbb{O}\mathrm{rd} \to \mathbb{O}\mathrm{rd}, f(x) := \min\{y \in \mathbb{O}\mathrm{rd} : \aleph_y \geq x\}$. Let $\tau : \mathrm{cf}(\aleph_\alpha) \to \alpha, x \mapsto f(\theta(x))$. Since $\theta(x) < \aleph_\alpha = \sup_{\beta<\alpha} \aleph_\beta$, we get $\exists \beta < \alpha, \theta(x) < \aleph_\beta$. So we get $f(\theta(x)) \leq \beta < \alpha$. So τ is well-defined. Easily to get τ is unbounded. So we get $\mathrm{cf}(\aleph_\alpha) \leq \mathrm{cf}(\alpha)$.

   Finally we get $\mathrm{cf}(\aleph_\alpha) = \mathrm{cf}(\alpha)$.

$\square$

PROBLEM IV Assume GCH, prove that for cardinal $\lambda, \kappa > \omega$, we have:

$$\kappa^\lambda = \begin{cases} \kappa & \lambda < \mathrm{cf}(\kappa) \\ \kappa^+ & \mathrm{cf}(\kappa) \leq \lambda \leq \kappa \\ \lambda^+ & \kappa < \lambda \end{cases}$$

SOLUTION . Use MI to κ. For $\kappa = \omega$, when $\lambda = \omega$ we get $\kappa^\lambda = 2^\omega = \omega^+$. When $\lambda > \omega$, we get $\kappa^\lambda \geq 2^\lambda = \lambda^+$. And $\kappa^\lambda \leq (2^\lambda)^\lambda = 2^{\lambda\times\lambda} = 2^\lambda = \lambda^+$. Now assume for $\alpha : \omega \leq \alpha < \kappa$ it's right, consider κ.

- $\lambda < \mathrm{cf}(\kappa)$.

  Since $\lambda < \mathrm{cf}(\kappa)$, we get every $f : \lambda \to \kappa$ is bounded. So we get ${}^\lambda\kappa = \bigcup_{\alpha<\kappa} {}^\lambda\alpha$. So we get $\kappa^\lambda \leq \sum_{\alpha<\kappa} \alpha^\lambda = \kappa \sup_{\alpha<\kappa} \alpha^\lambda$. Since $\alpha < \kappa$, we get $\alpha^\lambda = \begin{cases} \alpha & \lambda < \mathrm{cf}(\alpha) \\ \alpha^+ & \mathrm{cf}(\alpha) \leq \lambda \leq \alpha \\ \lambda^+ & \alpha < \lambda \end{cases}$. Anyway, since $\alpha, \lambda < \kappa$, we get $\alpha^\lambda \leq \kappa$. So we get $\kappa^\lambda \leq \kappa \sup_{\alpha<\kappa} \alpha^\lambda \leq \kappa\kappa = \kappa$.

- $\mathrm{cf}(\kappa) \leq \lambda \leq \kappa$.

  Easily $\kappa^\lambda \leq \kappa^\kappa \leq 2^{\kappa\kappa} = 2^\kappa = \kappa^+$. Now we only need $\kappa^+ \leq \kappa^\lambda$. Only need to prove $\kappa^{\mathrm{cf}(\kappa)} > \kappa$. If not, assume $f : \kappa \to {}^{\mathrm{cf}(\kappa)}\kappa$ is bijection. Assume $\theta : \mathrm{cf}(\kappa) \to \kappa$ is unbounded. Without loss of generality assume θ is injective. Let $\tau : \kappa \to \mathrm{cf}(\kappa), x \mapsto \min\{y \in \mathrm{cf}(\kappa) : \theta(y) \geq x\}$. Now consider $A_\alpha := \tau_{-1}[\alpha]$ for $\alpha < \mathrm{cf}(\kappa)$. Easily we get $\forall y \in \tau_{-1}[\alpha], \alpha > \theta(y)$. Since θ is injective, we get $\mathrm{card}(A_\alpha) \leq \mathrm{card}(\alpha) < \kappa$. Let $B_\alpha := \{f(x)(\alpha) : x \in A_\alpha\}$, then easily $\mathrm{card}(B_\alpha) \leq \mathrm{card}(A_\alpha) < \kappa$. Now consider $g : \mathrm{cf}(\kappa) \to \kappa, g(\alpha) := \min(\kappa \setminus B_\alpha)$. Since f is bijection, we get $\exists x \in \kappa, g = f(x)$. But $f(x)(\tau(x)) \in B_{\tau(x)}$, and $g(\tau(x)) = \min(\kappa \setminus B_{\tau(x)}) \notin B_{\tau(x)}$, contradiction! So we get $\kappa^\lambda \geq \kappa^{\mathrm{cf}(\kappa)} > \kappa$, then $\kappa^\lambda \geq \kappa^+$.

- $\kappa < \lambda$.

  We get $\lambda^+ = 2^\lambda \leq \kappa^\lambda \leq 2^{\lambda\lambda} = 2^\lambda = \lambda^+$. So $\kappa^\lambda = \lambda^+$.

$\square$

PROBLEM V Assume a linearly ordered set P has a countable dense subset, then $\mathrm{card}P \leq 2^{\aleph_0}$.

SOLUTION. Assume $A \subset P$ is a countable dense subset. Now consider $f : P \to \mathcal{P}(A), x \mapsto \{y \in A : y < x\}$. Easily $\text{card}\mathcal{P}(A) = 2^{\aleph_0}$, so we only need to prove f is injection. Assume $f(x) = f(y)$. Without loss of generality assume $x \leq y$. If $x \neq y$, then we get $x < y$. Since A is dense, we get $\exists z, w \in A$ such that $x \leq z < w \leq y$. So we get $z \in f(y)$ but $z \notin f(x)$, contradiction! So we get $x = y$. So f is injective, so $\text{card}P \leq 2^{\aleph_0}$. $\square$

PROBLEM VI Find the cardinal of all null sets of reals.

SOLUTION. Let $\mathcal{A} \subset \mathcal{P}(\mathbb{R})$ is the set of all of null sets. Then we get $\text{card}\mathcal{A} \leq \text{card}\mathcal{P}(\mathbb{R}) = 2^{\mathfrak{c}}$. Now we prove $\text{card}\mathcal{A} \geq 2^{\mathfrak{c}}$. Consider $C \subset \mathbb{R}$ is the Canter set. We have C is null and $\text{card}C = \mathfrak{c}$. So we get $\mathcal{P}(C) \subset \mathcal{A}$, then $\text{card}\mathcal{A} \geq \text{card}\mathcal{P}(C) = 2^{\mathfrak{c}}$. $\square$

PROBLEM VII Prove that ${}^{\mathbb{N}}\mathbb{N}$ is uncountable.

SOLUTION. Easily we have $\text{card}^{\mathbb{N}}\mathbb{N} = \aleph_0^{\aleph_0} \geq 2^{\aleph_0} > \aleph_0$ is uncountable. $\square$