underGraduate Homework In Mathematics

AlgebraicGeometry 13

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PROBLEM I Assume $\Omega \subset \mathbb{C}$ is a domain. Prove that f is meromorphic map over Ω is equiv for Ω as opensubset of $\mathbb{C}$ and as Remian surface.

SOLUTION. First we assume $f : \Omega \to \mathbb{C}_\infty$ is holomorphic. Let $T := \{x \in \Omega : f(x) = \infty\}$. Now we prove f is meromorphic over Ω. Since $\forall x \in T, f(x) = \infty$ and f is continous, we get $\forall x \in T, \lim_{y\to x} f(y) = \infty$. Now we only need to prove every $x \in T$ is isolated point. If not, we will prove $f \equiv \infty$. Let $V := \{x \in \Omega : f(x) = \infty \wedge \exists x_n \in \Omega, x_n \neq x, x_n \to x, f(x_n) = \infty\} \neq \varnothing$. Easily V is closed in Ω, now we prove it's open, too. Assume $x \in V$, and $x_n \in \Omega, x_n \neq x, x_n \to x, f(x_n) = \infty$. Since f is holomorphic, we get $\exists V : \infty \in V \subset \mathbb{C}_\infty$ is open, $\exists U : x \in U \subset \Omega$ is open, such that $g := \phi \circ f|_U \circ \mathrm{id} : U \to \mathbb{C}$ is holomorphic, where $\phi(x) = \frac{1}{x}$. Then $g(x) = g(x_n) = 0$. So $g|_U \equiv 0$. So we get $f|_U \equiv 0$. So $U \subset V$. So V is open in Ω. Since Ω is connected, we get $V = \Omega$. So $f \equiv \infty$.

Second we assume $f : W \to \mathbb{C}$ is holomorphic, where $W \subset \Omega$ is open, and $\forall x \in T := \Omega \setminus W, \lim_{y\to x} f(y) = \infty$, and x is isolated point. Now let $h : \Omega \to \mathbb{C}_\infty, h|_W = f, h|_T \equiv \infty$. We only need to prove h is holomorphic from Ω(as Remian surface) to $\mathbb{C}_\infty$. Only need to prove h is holomorphic on $x \in T$. Let $\phi : \mathbb{C}\setminus\{0\} \to \mathbb{C}, x \mapsto \frac{1}{x}$. Let $U \subset h^{-1}(\mathbb{C}\setminus\{0\}) \subset \Omega$ is a neibor of x and $\forall y \in U\setminus\{x\}, h(y) \neq \infty$. Now we prove $g := \phi\circ h\circ \mathrm{id} : U \to \mathbb{C}$ is holomorphic. Easily $h(z) = \frac{\psi(z)}{(z-x)^n}$ for some $n \in \mathbb{N}^+$ and holomorphic map ψ where $0 \notin \psi(U)$. So we get $g(z) = \frac{1}{h(z)} = \frac{(z-x)^n}{\psi(z)}$ is holomorphic over U. $\square$

PROBLEM II Let M is a Remian surface, $f : M \to \mathbb{C}_\infty$ is holomorphic. Assume $p \in M$. Prove that $\mathrm{ord}_p(f)$ is well-defined.

SOLUTION. Assume $U \subset M$ is openset and $p \in U$, and $\phi, \psi : U \to \mathbb{C}_\infty$ are holomorphic. Consider $f_\phi := f \circ \phi^{-1} : \phi(U) \to \mathbb{C}$ and $f_\psi := f \circ \psi^{-1} : \psi(U) \to \mathbb{C}$ are meromorphic. We only need prove f_ϕ at $\phi(p)$ and f_ψ at $\psi(p)$ have the same ord. Since M is Remian surface we know $\phi \circ \psi^{-1} : \psi(U) \to \phi(U)$ is holomorphic. Without loss of generality we can assume $\phi(p) = \psi(p) = 0$. Assume $f_\phi(z) = z^n h(z)$, where $h(z)$ is holomorphic near 0 and $h(0) \neq 0$. Then $f_\psi(z) = f_\phi \circ (\phi \circ \psi^{-1})(z) = \phi \circ \psi^{-1}(z)^n h(\phi \circ \psi^{-1}(z))$ Since $\phi(p) = \psi(p) = 0$ we get $\phi \circ \psi^{-1}(0) = 0$. So $h(\phi \circ \psi^{-1}(0)) \neq 0$. And $\lim_{z\to 0} \frac{1}{z}\phi \circ \psi^{-1}(z)$ exists. For the same reason we get $\lim_{z\to 0} \frac{1}{z}\psi \circ \phi^{-1}(z)$ exists. So $\lim_{z\to 0} \phi \circ \psi^{-1}(z) \neq 0$. So finally we get $\mathrm{ord}_p(f)$ is well-defined. $\square$

PROBLEM III Assume $p(z) \in \mathbb{C}[z]$ is a poly and not const. Consider $f : \mathbb{C}_\infty \to \mathbb{C}_\infty, f(z) = \begin{cases} p(z) & z \in \mathbb{C} \\ \infty & z = \infty \end{cases}$. Prove that f is holomorphic.

SOLUTION. Obviously for $z \in \mathbb{C}$ we have f is holomorphic near z. So we only need to prove f is holomorphic near ∞. Let $V \subset \mathbb{C}_\infty$ is a neibor of ∞ and $0 \notin V$. Since $\deg p > 0$, we get $\lim_{z\to\infty} p(z) = \infty$. So $\exists U \subset \mathbb{C}_\infty$ such that $f(U) \subset V$. without loss of generality assume $0 \notin U$. Now we only need to prove $\phi\circ f\circ\phi$ is holomorphic at 0, where $\phi(z) = \frac{1}{z}$. Assume $f(z) = \sum_{i=0}^n a_i z^i$ and $a_n \neq 0$, then $\frac{1}{f(\frac{1}{z})} = \frac{z^n}{\sum_{i=0}^n a_i z^{n-i}}$. Since $a_n \neq 0$ we get $\frac{1}{f(\frac{1}{z})}$ is holomorphic near 0. $\square$

PROBLEM IV Let $\omega_1, \omega_1 \in \mathbb{C}^*$ and are $\mathbb{R}$-linear independent. Let Λ is the addtive group generated by ω_1, ω_2 and $k \in \mathbb{N}^+, k \geq 3$. Prove that $g(z) = \sum_{\omega \in \Lambda} \frac{1}{(z-\omega)^k}$ locally uniformly converge in $\mathbb{C} \setminus \Lambda$. And $g(z) = g(z+\omega), \forall \omega \in \Lambda$.

SOLUTION . Write $\Lambda = \{z_n : n \in \mathbb{N}^+\}$, then we only need to prove $\sum_{n=1}^{\infty} \frac{1}{(z-z_n)^k}$ locally uniformly converge in $\mathbb{C} \setminus \Lambda$. Without loss of generality we can assume $|z_n| \leq |z_{n+1}|, \forall n \in \mathbb{N}^+$. Easily we know $\mathrm{card}(B(0,n) \cap \Lambda) = O(n^2), n \to \infty$. So we get $|z_n| = O(\sqrt{n}), n \to \infty$. Assume $M \subset \mathbb{C} \setminus L$ is cpt, now we need to prove the series converge uniformly in M. Let $\lambda := d(M, \Lambda) > 0$, then we get $|z - z_n| > \lambda$. And $|z_n - z| \geq |z_n| - |z| \geq |z_n| - \max\{|z| : z \in M\} = O(\sqrt{n}), n \to \infty$. So $\exists N \in \mathbb{N}^+, t \in \mathbb{R}^+, \forall n > N, |z_n - z| \geq t\sqrt{n}$. For $n \leq N$, we have $|z_n - z| \geq \lambda$, without loss of generality assume $t < \frac{\lambda}{\sqrt{N}}$, then $|z_n - z| > t\sqrt{n}$, too. So finally we get $\left|\frac{1}{(z_n-z)^k}\right| \leq \frac{t}{n^{\frac{k}{2}}}, \forall z \in M$. Since $\sum_{n \in \mathbb{N}^+} \frac{t}{n^{\frac{k}{2}}} < \infty$, we get the series converge uniformly in M.

Now we prove $g(z) = g(z+\omega)$. If $z \in \Lambda$, then easily $g(z) = \infty = g(z+\omega)$. Now we assume $z \notin \Lambda$. Then we easily get $|z - z_n| = O(\sqrt{n})$, so $\sum_{n=1}^{\infty} \frac{1}{(z-z_n)^k}$ converge absolutely. So we can change the order of sum. Since $\Lambda \to \Lambda, x \mapsto x + \omega$ is bijection, we get $\exists \sigma : \mathbb{N}^+ \to \mathbb{N}^+$ is bijection and $z_{\sigma(n)} = z_n - \omega$. So we get $g(z+\omega) = \sum_{n=1}^{\infty} \frac{1}{(z-z_n+\omega)^k} = \sum_{n=1}^{\infty} \frac{1}{(z-z_{\sigma(n)})^k} = \sum_{n=1}^{\infty} \frac{1}{(z-z_n)^k} = g(z)$. □

PROBLEM V Let Λ be same as above, let $\wp(z) := \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right) + \frac{1}{z^2}$. Prove this series converge locally uniformly in $\mathbb{C} \setminus \Lambda$. And $\wp(z) = \wp(z+\omega), \forall \omega \in \Lambda$.

SOLUTION . First we prove $\wp(z) = \wp(z+\omega)$. If $\omega = 0$ or $z \in \Lambda$ then it's obvious. Now we assume $\omega \neq 0 \wedge z \notin \Lambda$. We write $\Lambda \setminus \{0\} = \{z_n : n \in \mathbb{N}^+\}$ as above, then $|z_n| = O(\sqrt{n})$. we first prove $\sum_{n \in \mathbb{N}^+} \left(\frac{1}{(z-z_n)^2} - \frac{1}{z_n^2}\right)$ converge absolutely for $z \notin \Lambda$. Since $\frac{1}{(z-z_n)^2} - \frac{1}{z_n^2} = \frac{2zz_n - z^2}{(z-z_n)^2 z_n^2}$ and $|z_n| = O(\sqrt{n})$, we get $\left|\frac{1}{(z-z_n)^2} - \frac{1}{z_n^2}\right| = O(\frac{1}{n^{\frac{3}{2}}})$. So we get the series is absolutely converge. So we can change the order to sum. Write $f(z,t) = \frac{1}{(z-t)^2}$ and $g(t) = \begin{cases} \frac{1}{t^2} & t \neq 0 \\ 0 & t = 0 \end{cases}$ for $z \in \mathbb{C}$ and $t \in \Lambda$. Then $\wp(z) = \sum_{t \in \Lambda} f(z,t) - g(t)$. For $t_1, t_2 \in \Lambda$, we define $t_1 \sim t_2 \iff \frac{t_1 - t_2}{\omega} \in \mathbb{Z}$. Then $\sim$ is a equivalence relation of Λ. We choose a class of reperentation element, write T. Then $\wp(z) = \sum_{t \in T} \sum_{n \in \mathbb{Z}} f(z, t+n\omega) - g(t+n\omega)$. And $\wp(z+\omega) = \sum_{t \in T} \sum_{n \in \mathbb{Z}} f(z+\omega, t+n\omega) - g(t+n\omega)$. So to prove $\wp(z) = \wp(z+\omega)$, we only need to prove $\forall t \in T, \sum_{n \in \mathbb{Z}} f(z, t+n\omega) - g(t+n\omega) = \sum_{n \in \mathbb{Z}} f(z+\omega, t+n\omega) - g(t+n\omega)$. We can easily get $|f(z, t+n\omega)| = O(frac1n^2), |g(t+n\omega)| = O(frac1n^2)$, so we get $\sum_{ninZ} f(z, t+n\omega)$ and $\sum_{n \in \mathbb{Z}} g(t+n\omega)$ converge absolutely. So we get $\sum_{n \in \mathbb{Z}} f(z, t+n\omega) - g(t+n\omega) = \sum_{n \in \mathbb{Z}} f(z, t+n\omega) - \sum_{n \in \mathbb{Z}} g(t+n\omega)$. For the same reason we get $\sum_{n \in \mathbb{Z}} f(z+\omega, t+n\omega) - g(t+n\omega) = \sum_{n \in \mathbb{Z}} f(z+\omega, t+n\omega) - \sum_{n \in \mathbb{Z}} g(t+n\omega)$. So we only need to prvoe $\sum_{n \in \mathbb{Z}} f(z, t+n\omega) = \sum_{n \in \mathbb{Z}} f(z+\omega, t+n\omega)$. Since $f(z+\omega, t+n\omega) = f(z, t+(n-1)\omega)$ and $n \mapsto n-1$ is bijection on $\mathbb{Z}$, we get $\sum_{n \in \mathbb{Z}} f(z+\omega, t+n\omega) = \sum_{n \in \mathbb{Z}} f(z, t+(n-1)\omega) = \sum_{n \in \mathbb{Z}} f(z, t+n\omega)$. So finally we get $\wp(z) = \wp(z+\omega), \forall \omega \in \Lambda$.

Now we prove the series converge locally uniformly in $\mathbb{C} \setminus \Lambda$. Let $M \subset \mathbb{C} \setminus \Lambda$ is cpt. Now we prove the series converge uniformly in M. Only need to prove $\sum_{n \in \mathbb{N}^+} \frac{2zz_n - z^2}{(z-z_n)^2 z_n^2}$ converge uniformly. Since $|z|$ is bounded and $|z_n| = O(\sqrt{n})$, we get $\exists s \in \mathbb{R}^+, |2zz_n - z^2| \leq s\sqrt{n}$. Since M is cpt we get $d(M, \Lambda) > 0$, so $|z - z_n|$ has positive inf. So as above we get $\exists t \in \mathbb{R}^+, |z - z_n| \geq t\sqrt{n}$.

Without loss of generality we can assume t is little enough such that $|z_n| \geq t\sqrt{n}$. So finally we get $\left|\frac{2zz_n - z^2}{(z-z_n)^2 z_n^2}\right| \leq \frac{s\sqrt{n}}{t^4 n^2} = \frac{s}{t^4} n^{-\frac{3}{2}}$. Since $\sum_{n\in\mathbb{N}^+} n^{-\frac{3}{2}} < \infty$, we get the series converge uniformly in M.

$\square$