PROBLEM I let $(x_n : n \geq 0) \perp (y_n : n \geq 0)$ are markov chain on e with transition matrix $(p_{ij} : i, j \in e), (q_{ij} : i, j \in e)$ respectively. prove: $\{(x_n, y_n) : n \geq 0\}$ are markov chain on $e \times e$. and calculate the transition matrix of $(x_n, y_n) : n \geq 0$.

SOLUTION. easy to get that

$$
\begin{aligned}
&\mathbb{P}(x_0 = i_0, \cdots, x_{n+1} = i_{n+1}, y_0 = j_0, \cdots, y_{n+1} = j_{n+1}) \\
=&\mathbb{P}(x_0 = i_0, \cdots, x_{n+1} = i_{n+1})\mathbb{P}(y_0 = j_0, \cdots, y_{n+1} = j_{n+1}) \\
=&\mathbb{P}(x_0 = i_0) \prod_{k=0}^{n} p_{i_k i_{k+1}} \mathbb{P}(y_0 = j_0) \prod_{k=0}^{n} q_{j_k j_{k+1}} \\
=&\mathbb{P}((x_0, y_0) = (i_0, j_0)) \prod_{k=0}^{n} p_{i_k i_{k+1}} q_{j_k j_{k+1}} \\
=&\mathbb{P}((x_0, y_0) = (i_0, j_0)) \prod_{k=0}^{n} \mathbb{P}(x_k = i_k, x_{k+1} = i_{k+1})\mathbb{P}(y_k = j_k, y_{k+1} = j_{k+1}) \\
=&\mathbb{P}((x_0, y_0) = (i_0, j_0)) \prod_{k=0}^{n} \mathbb{P}((x_k, y_k) = (i_k, j_k), (x_{k+1}, y_{k+1}) = (i_{k+1}, j_{k+1}))
\end{aligned}
$$

so we get that $((x_n, y_n) : n \in \mathbb{N})$ is markov chain with transition matrix $r_{(i,j),(m,n)} = p_{im} q_{jn}$. $\square$

PROBLEM II let s_n is a one dimensional simple random walk. let $a \in \mathbb{Z}$. let $\tau := \inf\{n \geq 0 : s_n = a\}$. prove:

1. $(s_{\tau+n} : n \geq 0)$ is a one dimensional simple random walk.
2. $(s_{n \wedge \tau} : n \geq 0)$ is a markov chain on $\mathbb{Z}$ and give its transition matrix.
3. $(s_{n \wedge \tau} : n \geq 0) \perp (s_{\tau+n} : n \geq 0)$.

SOLUTION. 1. easy to know that

$$
\begin{aligned}
&\mathbb{P}(s_\tau = i_0, s_{\tau+1} = i_1, \cdots, s_{\tau+n} = i_n \mid \tau < \infty) \\
=&\sum_{k\in\mathbb{N}} \mathbb{P}(\tau = k, s_\tau = i_0, s_{\tau+1} = i_1, \cdots, s_{\tau+n} = i_n \mid \tau < \infty) \\
=&\sum_{k\in\mathbb{N}} \mathbb{P}(\tau = k, s_k = i_0, s_{k+1} = i_1, \cdots, s_{k+n} = i_n \mid \tau < \infty) \\
=&\mathbb{1}(a = i_0) \sum_{k\in\mathbb{N}} \mathbb{P}(s_0 \neq a, \cdots, s_{k-1} \neq a, s_k = a, s_{k+1} = i_1, \cdots, s_{k+n} = i_n \mid \tau < \infty) \\
=&\frac{\mathbb{1}(a = i_0) \sum_{k\in\mathbb{N}} \mathbb{P}(s_0 \neq a, \cdots, s_{k-1} \neq a, s_k = a, s_{k+1} = i_1, \cdots, s_{k+n} = i_n)}{\mathbb{P}(\tau < \infty)} \\
=&\frac{\mathbb{1}(a = i_0)}{\mathbb{P}(\tau < \infty)} \sum_{k\in\mathbb{N}} \mathbb{P}(s_0 \neq a, \cdots, s_{k-1} \neq a, s_k = a, s_{k+1} = i_1, \cdots, s_{k+n} = i_n) \\
=&\frac{\mathbb{1}(a = i_0)}{\mathbb{P}(\tau < \infty)} \sum_{k\in\mathbb{N}} \mathbb{P}(s_{k+1} = i_1, \cdots, s_{k+n} = i_n \mid s_0 \neq a, \cdots, s_{k-1} \neq a, s_k = a) \\
&\times \mathbb{P}(s_0 \neq a, \cdots, s_{k-1} \neq a, s_k = a) \\
=&\frac{\mathbb{1}(a = i_0)}{\mathbb{P}(\tau < \infty)} \sum_{k\in\mathbb{N}} \mathbb{P}(s_{k+1} = i_1, \cdots, s_{k+n} = i_n \mid s_k = a) \mathbb{P}(\tau = k) \\
=&\frac{\mathbb{1}(a = i_0)}{\mathbb{P}(\tau < \infty)} \sum_{k\in\mathbb{N}} \prod_{l=0}^{n-1} p_{i_l i_{l+1}} \mathbb{P}(\tau = k) = \mathbb{1}(a = i_0) \prod_{l=0}^{n-1} p_{i_l i_{l+1}}
\end{aligned}
$$

where $p_{ij} : i, j \in F$ is the transition matrix of $s_n : n \in \mathbb{N}$. so $(s_{\tau+n} : n \in \mathbb{N})$ is markov chain with transition matrix same as s_n.

2. easy to know that

$$
\begin{aligned}
&\mathbb{P}(s_{\tau\wedge 0} = i_0, s_{\tau\wedge 1} = i_1, \cdots, s_{\tau\wedge n} = i_n \mid \tau < \infty) \\
=&\sum_{k\in\mathbb{N}} \mathbb{P}(\tau = k, s_{\tau\wedge 0} = i_0, s_{\tau\wedge 1} = i_1, \cdots, s_{\tau\wedge n} = i_n \mid \tau < \infty) \\
=&\sum_{k\in\mathbb{N}} \mathbb{P}(\tau = k, s_{k\wedge 0} = i_0, s_{k\wedge 1} = i_1, \cdots, s_{k\wedge n} = i_n \mid \tau < \infty) \\
=&\sum_{k\geq n} \mathbb{P}(\tau = k, s_0 = i_0, \cdots, s_n = i_n \mid \tau < \infty) \\
&+ \sum_{k<n} \mathbb{P}(\tau = k, s_0 = i_0, \cdots, s_{k-1} = i_{k-1}, s_k = i_k = i_{k+1} = \cdots = i_n \mid \tau < \infty) \\
=&\mathbb{1}(i_0, i_1, \cdots, i_n \neq a) \prod_{k=0}^{n-1} p_{i_k i_{k+1}} + \sum_{k=0}^{n-1} \mathbb{1}(i_0, \cdots, i_{k-1} \neq a, i_k = i_{k+1} = \cdots = i_n = a) \prod_{l=0}^{k-1} p_{i_l i_{l+1}} \\
=&\prod_{k=0}^{n-1} (\mathbb{1}(i_k = i_{k+1} = a) + \mathbb{1}(i_k \neq a) p_{i_k, i_{k+1}})
\end{aligned}
$$

so $(s_{n\wedge\tau} : n \in \mathbb{N})$ is markov chain with transition matrix $q_{i,j} = \mathbb{1}(i = j = a) + \mathbb{1}(i \neq a) p_{i,j}$.

3. by the corollary 3.2.11, we only need to proof τ is stopping time on $({}_n : n \geq 0)$, where ${}_n = \sigma(s_k : k \leq n)$. so we only need to prove $\forall n \in \ltimes$, $\{\tau = n\} \in {}_n$. since $\{\tau = n\} = \{\omega \in \omega : s_0, \cdots, s_{n+1} \neq a, s_n = a\} = \bigcap_{0 \leq k \leq n} \{s_k \neq a\} \cap \{s_n = a\}$, and $\{s_k \neq a\} \in \sigma(s_k), \forall 0 \leq k \leq n, \{s_n = a\} \in \sigma(s_n)$, then $\{\tau = n\} \in {}_n$.

□

PROBLEM III let s_n is a one dimensional symmetry simple random walk starting from zero. prove: $(|s_n| : n \geq 0)$ is a markov chain on F^+ and give its transition matrix.

SOLUTION. only need to solve problem IV. □

PROBLEM IV let s_n is a one dimensional simple random walk starting from zero. prove: $(|s_n| : n \geq 0)$ is a markov chain on F^+ and give its transition matrix.

SOLUTION. by the definition of $|s_n|$, we can easily get to know $\forall (i_0, \cdots, i_n) \in F^+$, $\imath(|s_k| = i_k, k = 0, \cdots, n) > 0 \iff i_0 = 0, |i_k - i_{k-1}| = 1, k = 1, \cdots, n$. let $s_n = \sum_{k=1}^n \xi_k$, where $(\xi_n : n \geq)$ are i.i.d. r.v. and $\imath(\xi_1 = 1) = p, \imath(\xi_1 = -1) = q$. $a := \{(i_0, \cdots, i_{n+1}) \in F : i_0 = 0, |i_k - i_{k-1}| = 1, k = 1, \cdots, n+1\}$. $\forall (i_0, \cdots, i_{n+1}) \in a$, let $r := \max\{k : i_k = 0\}$. then $i_r = 0, \forall k \geq r+1, i_k \geq 1$.

1. $\forall (i_0, \cdots, i_{n+1}) \notin a$, then $\imath(|s_k| = i_k, k = 0, \cdots, n) = 0$, then we have no need to calculate $\imath(|s_{n+1}| = i_{n+1} | |s_k| = i_k, k = 0, \cdots, n)$.

2. $\forall (i_0, \cdots, i_{n+1}) \in a$,

$$\begin{aligned}
&\imath(|s_k| = i_k, s_n = i_n, k = 0, \cdots, n | |s_k| = i_k, k = 0, \cdots, r) \\
&= \imath(|s_k| = i_k, s_n = i_n, k = r+1, \cdots, n | |s_k| = i_k, k = 0, \cdots, r-1, s_r = 0) \\
&= \imath(|s_k| = i_k, s_n = i_n, k = r+1, \cdots, n | s_r = 0) \\
&= \imath(s_k = i_k, s_n = i_n, k = r+1, \cdots, n | s_r = 0) \\
&= p^{\frac{n-r+r_n}{2}} q^{\frac{n-r-r_n}{2}}
\end{aligned}$$

in the same way, we can get $\imath(|s_k| = i_k, s_n = -i_n, k = 0, \cdots, n | |s_k| = i_k, k = 0, \cdots, r) =$

$p^{\frac{n-r-r_n}{2}}q^{\frac{n-r+r_n}{2}}$ so

$$
\begin{aligned}
&\mathbb{P}(s_n = i_n \| s_k| = i_k, k = 0, \cdots, n) \\
&= \frac{\mathbb{P}(s_n = i_n, |s_k| = i_k, k = 0, \cdots, n)}{\mathbb{P}(|s_k| = i_k, k = 0, \cdots, n)} \\
&= \frac{\mathbb{P}(s_n = i_n, |s_k| = i_k, k = 0, \cdots, n \| s_k| = i_k, k = 0, \cdots, r)\mathbb{P}(|s_k| = r_k, k = 0, \cdots, r)}{\mathbb{P}(|s_k| = i_k, k = 0, \cdots, n)} \\
&= \frac{\mathbb{P}(s_n = i_n, |s_k| = i_k, k = 0, \cdots, n \| s_k| = i_k, k = 0, \cdots, r)}{\frac{\mathbb{P}(|s_k| = i_k, k = 0, \cdots, n)}{\mathbb{P}(|s_k| = r_k, k = 0, \cdots, r)}} \\
&= \frac{\mathbb{P}(s_n = i_n, |s_k| = i_k, k = 0, \cdots, n \| s_k| = i_k, k = 0, \cdots, r)}{\mathbb{P}(|s_k| = i_k, k = 0, \cdots, n \| s_k| = r_k, k = 0, \cdots, r)} \\
&= \frac{\mathbb{P}(s_n = i_n, |s_k| = i_k, k = 0, \cdots, n \| s_k| = i_k, k = 0, \cdots, r)}{\mathbb{P}(|s_k| = i_k, k = r+1, \cdots, n \| s_k| = r_k, k = 0, \cdots, r)} \\
&= \frac{p^{n-r+\frac{r_n}{2}}q^{n-r-\frac{r_n}{2}}}{p^{n-r+\frac{r_n}{2}}q^{n-r-\frac{r_n}{2}} + p^{n-r-\frac{r_n}{2}}q^{n-r+\frac{r_n}{2}}} \\
&= p^{i_n}(p^{i_n} + q^{i_n})^{-1}
\end{aligned}
$$

In the same way, we can get $\mathbb{P}(s_n = -i_n \| s_k| = i_k, k = 0, \cdots, n) = q^{r_n}(p^{r_n} + q^{r_n})^{-1}$. Then

$$
\begin{aligned}
&\mathbb{P}(|s_{n+1}| = i_{n+1} \| s_k| = i_k, k = 0, \cdots, n) \\
&= \mathbb{P}(|s_{n+1}| = i_{n+1} | s_n = i_n, |s_k| = i_k, k = 0, \cdots, n) \\
&\times \mathbb{P}(s_n = i_n \| s_k| = i_k, k = 0, \cdots, n) \\
&+ \mathbb{P}(|s_{n+1}| = i_{n+1} | s_n = -i_n, |s_k| = i_k, k = 0, \cdots, n) \\
&\times \mathbb{P}(s_n = -i_n \| s_k| = i_k, k = 0, \cdots, n) \\
&= \mathbb{P}(s_{n+1} = i_{n+1} | s_n = i_n) \\
&\times \mathbb{P}(s_n = i_n \| s_k| = i_k, k = 0, \cdots, n) \\
&+ \mathbb{P}(s_{n+1} = -i_{n+1} | s_n = -i_n) \\
&\times \mathbb{P}(s_n = -i_n \| s_k| = i_k, k = 0, \cdots, n) \\
&= \mathbb{1}(i_{n+1} = i_n + 1)(p^{i_n+1} + q^{i_n+1})(p^{i_n} + q^{i_n})^{-1} + \mathbb{1}(i_{n+1} = i_n - 1)(p^{r_n}q + pq^{r_n})(p^{r_n} + q^{r_n})^{-1}
\end{aligned}
$$

Thus, $(|S_n| : n \geq 0)$ is markov chain on $\mathbb{Z}^+$.

$\square$