underGraduate Homework In Mathematics

AlgebraicGeometry 5

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General fire extinguisher

PROBLEM I If $f : V \to W, g : W \to U$ is two poly maps, then $(g \circ f)^* = f^* \circ g^*$.

SOLUTION. For $u \in k[U]$, we have $(g \circ f)^* u = u \circ g \circ f = g^*(u) \circ f = f^*(g^*(u)) = (f^* \circ g^*)(u)$, so $(g \circ f)^* = (f^* \circ g^*)$. $\square$

PROBLEM II $\mathcal{O}_{V,p}$ is local ring.

SOLUTION. To prove $\mathcal{O}_{V,p}$ is local ring, we only need to prove $\forall f \in \mathcal{O}_{V,p}$, one of f and $1 - f$ is unit.

First we prove f is unit iff $f(p) \neq 0$. If f is unit then exists $g \in \mathcal{O}_{V,p}$ s.t. $fg = 1$, so $f(p)g(p) = 1$. Then we get $f(p) \neq 0$. If $f(p) \neq 0$, then we assume $f = \frac{g}{h}, h(p) \neq 0$. Since $f(p) \neq 0$ we get $g(p) \neq 0$, so $\frac{h}{g} \in \mathcal{O}_{V,p}$, then f is a unit.

Now we prove f or $1 - f$ is a unit. Obviously $f(p) \neq 0$ or $1 - f(p) \neq 0$, so one of them is unit. $\square$

PROBLEM III Prove: $\{V_h : h \in k[V]\}$ is topological basis of V.

SOLUTION. Only need to prove for any open set $U \subset V$, we can find a subclass of $\{V_h : h \in k[V]\}$ such that U is union of the class. Obviously from PROBLEM IV we get exists a finite subclass satisfy the requirement, it's even stronger! $\square$

PROBLEM IV Prove: For open set $U \subset V, \exists h_1, h_2, \cdots h_n$ s.t. $U = \cup_{k=1}^n V_{h_k}$.

SOLUTION. Since U is open set in V, so $\exists I$ is ideal in $k[x_1, \cdots x_n]$ such that $\mathbb{V}I = U^c$. Since $k[x_1, \cdots x_n]$ is Noetherian, we obtain $\exists f_1, \cdots f_k, I = (f_1, \cdots f_n)$. Let $h_k := f_k|_V$, then $U^c = \{p \in V : \forall k, h_k(p) = 0\}$, so $U^c = \cap_{k=1}^n (V_{h_k}^c)$, i.e., $U = \cup_{k=1}^n V_{h_k}$. $\square$