

*Lemma 1.* Assume  $(B_t : t \geq 0)$  is a random process ranging in  $\mathbb{R}$ ,  $a \in \mathbb{R}^+$ , and  $\forall s, t : 0 \leq s \leq t, B_t - B_s \sim N(0, a(t-s))$ . Assume  $B_t$  is continuous about  $t$ , a.s. Let  $\mathcal{F}_t := \sigma(B_s : 0 \leq s \leq t)$ . Then  $(B_t : t \geq 0)$  is Brownian motion  $\iff \forall 0 \leq s \leq t, B_t - B_s \perp \mathcal{F}_s$ .

*证明.* “ $\implies$ ”: To prove  $B_t - B_s \perp \mathcal{F}_s$ , only need to prove for  $t_1 < t_2 < \dots < t_{n-1} = s < t = t_n$ , we have  $B_t - B_s \perp \sigma(B_{t_k} : k = 1, \dots, n-1)$ . Easily  $B_t - B_s \perp \sigma(B_{t_{k+1}} - B_{t_k}, B_{t_1} : k = 1, \dots, n-2) = \sigma(B_{t_k} : k = 1, \dots, n-1)$ , so  $B_t - B_s \perp \mathcal{F}_s$ .

“ $\impliedby$ ”: For  $t_1 < \dots < t_n$ , we need to prove  $B_{t_1}, B_{t_{k+1}} - B_{t_k} : 1 \leq k \leq n-1$  are independent. Use MI to  $n$ . When  $n = 1$  it's obvious. Assume we have proved it for certain  $n \geq 1$ , now consider  $n+1$ . Since  $B_{t_{k+1}} - B_{t_k} \in \mathcal{F}_{t_n}, k = 1, \dots, n-1$ , we have  $B_{t_{n+1}} - B_{t_n} \perp \sigma(B_{t_1}, B_{t_{k+1}} - B_{t_k} : k = 1, \dots, n-1)$ . So  $\mathbb{P}(B_{t_1} \in A_1, B_{t_{k+1}} - B_{t_k} \in A_{k+1}, k = 1, \dots, n) = \mathbb{P}(B_{t_1} \in A_1, B_{t_{k+1}} - B_{t_k} \in A_{k+1}, k = 1, \dots, n-1) \mathbb{P}(B_{t_{n+1}} - B_{t_n} \in A_{n+1})$ . By Induction assumption we get  $\mathbb{P}(B_{t_1} \in A_1, B_{t_{k+1}} - B_{t_k} \in A_{k+1}, k = 1, \dots, n) = \mathbb{P}(B_{t_1} \in A_1) \prod_{k=1}^n \mathbb{P}(B_{t_{k+1}} - B_{t_k} \in A_{k+1})$ . So finally we get  $B_{t_1}, B_{t_{k+1}} - B_{t_k} : 1 \leq k \leq n$  are independent.  $\square$

**PROBLEM I** Assume  $(B_t : t \geq 0)$  is Brownian motion, prove that for  $r > 0$ , we have  $(B_{t+r} - B_r : t \geq 0)$  is Brownian motion, too.

*SOLUTION.* Assume  $B_t - B_s \sim N(0, a(t-s)), a > 0$ . Let  $\mathcal{F}_t := \sigma(B_s : 0 \leq s \leq t)$  and  $\mathcal{G}_t := \sigma(B_{r+s} - B_r : 0 \leq s \leq t)$ . For  $0 \leq s \leq t$ , we have  $B_{t+r} - B_r - (B_{s+r} - B_r) = B_{t+r} - B_{s+r} \sim N(0, a(t-s))$ . And easily to know  $B_{r+s} - B_r \in \mathcal{F}_{r+s}$ , so  $\mathcal{G}_t \subset \mathcal{F}_{t+r}, \forall t \geq 0$ . Since  $(B_t : t \geq 0)$  is Brownian motion, easily  $\mathcal{F}_{s+r} \perp B_{t+r} - B_{s+r}$ . Since  $\mathcal{G}_t \subset \mathcal{F}_{t+r}$ , we obtain  $\mathcal{G}_t \perp B_{t+r} - B_{s+r} = B_{t+r} - B_r - (B_{s+r} - B_r)$ . Easily since  $B_t$  is continuous we get  $B_{t+r} - B_r$  is continuous. So  $(B_{t+r} : t \geq 0)$  is Brownian motion.  $\square$

**PROBLEM II** Assume  $(B_t : t \geq 0)$  is standard Brownian motion start at 0. Prove that  $\forall c > 0, (cB_{\frac{t}{c^2}} : t \geq 0)$  is standard Brownian motion start at 0, too.

*SOLUTION.* Since  $B_0 = 0$  we get  $cB_{\frac{0}{c^2}} = 0$ . Let  $\mathcal{F}_t := \sigma(B_s : 0 \leq s \leq t)$  and  $\mathcal{G}_t := \sigma(cB_{\frac{s}{c^2}} : 0 \leq s \leq t)$ . Easily to know  $\mathcal{G}_t = \mathcal{F}_{\frac{t}{c^2}}$ . For  $0 \leq s \leq t$ , we have  $cB_{\frac{t}{c^2}} - cB_{\frac{s}{c^2}} = c(B_{\frac{t}{c^2}} - B_{\frac{s}{c^2}}) \sim N(0, t-s)$ , because  $B_{\frac{t}{c^2}} - B_{\frac{s}{c^2}} \sim N(0, \frac{t-s}{c^2})$ . And since  $(B_t : t \geq 0)$  is Brownian motion, we get  $B_{\frac{t}{c^2}} - B_{\frac{s}{c^2}} \perp \mathcal{F}_{\frac{s}{c^2}} = \mathcal{G}_s$ . Easily since  $B_t$  is continuous we get  $cB_{\frac{t}{c^2}}$  is continuous. So  $(cB_{\frac{t}{c^2}} : t \geq 0)$  is standard Brownian motion starts at 0, too.  $\square$

**PROBLEM III** Assume  $(X_t : t \geq 0)$  and  $(Y_t : t \geq 0)$  are two independent standard Brownian motion,  $a, b \in \mathbb{R}$  and  $\sqrt{a^2 + b^2} > 0$ . Prove that  $(aX_t + bY_t : t \geq 0)$  is a Brownian motion with parameter  $c^2 = a^2 + b^2$ .

*SOLUTION.* Let  $\mathcal{F}_t := \sigma(X_s : 0 \leq s \leq t)$  and  $\mathcal{G}_t := \sigma(Y_s : 0 \leq s \leq t)$ . Let  $\mathcal{H}_t := \sigma(aX_s + bY_s : 0 \leq s \leq t)$ . Since  $(X_t : t \geq 0), (Y_t : t \geq 0)$  are two independent Brownian motion, we know  $\forall 0 \leq s \leq t, X_t - X_s \perp \mathcal{F}_s, \mathcal{G}_s; Y_t - Y_s \perp \mathcal{F}_s, \mathcal{G}_s$ . So we get  $aX_t + bY_t - aX_s - bY_s \perp \mathcal{F}_s, \mathcal{G}_s$ , thus  $aX_t + bY_t - aX_s - bY_s \perp \sigma(\mathcal{F}_s, \mathcal{G}_s)$ . Easily  $aX_s + bY_s \in \sigma(\mathcal{F}_s, \mathcal{G}_s)$ , so  $\mathcal{H}_t \subset \sigma(\mathcal{F}_t, \mathcal{G}_t), \forall t \geq 0$ . So  $aX_t + bY_t - aX_s - bY_s \perp \mathcal{H}_s$ . And easily  $a(X_t - X_s) \sim N(0, a^2(t-s)), b(Y_t - Y_s) \sim N(0, b^2(t-s))$ , and since  $\mathcal{F}_t \perp \mathcal{G}_t$  we get  $a(X_t - X_s) \perp b(Y_t - Y_s)$ , so  $aX_t + bY_t - aX_s - bY_s \sim N(0, (a^2 + b^2)(t-s))$ .

Easily since  $X_t, Y_t$  is continuous we get  $aX_t + bY_t$  is continuous. So  $(aX_t + bY_t : t \geq 0)$  is a Brownian motion with parameter  $a^2 + b^2 = c^2$ .  $\square$

**PROBLEM IV** Assume  $(B_t : t \geq 0)$  is standard Brownian motion start at 0. Let  $X_0 = 0$  and  $X_t := tB_{\frac{1}{t}}$ . Given

$$\limsup_{t \rightarrow \infty} \frac{B_t}{\sqrt{2t \log \log t}} = 1$$

Prove that  $(X_t : t \geq 0)$  is standard Brownian motion start at 0.

**SOLUTION.** First consider the distribution of  $X_t - X_s$  for  $s < t$ . If  $s = 0$  then  $X_t - X_s = tB_{\frac{1}{t}} \sim N(0, t^2 \times \frac{1}{t}) = N(0, t)$ . Else,  $s > 0$ , then  $X_t - X_s = s(B_{\frac{1}{t}} - B_{\frac{1}{s}}) + (t - s)B_{\frac{1}{t}}$ . Easily  $B_{\frac{1}{s}} - B_{\frac{1}{t}} \sim N(0, \frac{1}{s} - \frac{1}{t})$ , and  $B_{\frac{1}{t}} \sim N(0, \frac{1}{t})$ , and since  $\frac{1}{t} < \frac{1}{s}$  we know  $B_{\frac{1}{s}} - B_{\frac{1}{t}} \perp B_{\frac{1}{t}}$ . So  $X_t - X_s \sim N(0, s^2(\frac{1}{s} - \frac{1}{t}) + (t - s)^2 \frac{1}{t}) = N(0, t - s)$ .

Second let  $\mathcal{G}_t := \sigma(X_s : 0 \leq s \leq t)$ , we need to check  $X_t - X_s \perp \mathcal{G}_s, \forall 0 \leq s \leq t$ . For  $s = 0$  we get  $\mathcal{G}_s = \{\emptyset, \Omega\}$ , so it's obvious. Now assume  $s > 0$ . Then  $\mathcal{G}_s = \sigma(B_{\frac{1}{r}} : 0 \leq r \leq s)$ . Only need to prove for any finite set  $I \subset [0, s]$ , we have  $X_t - X_s \perp \sigma(B_{\frac{1}{r}} : r \in I)$ . Only need to check  $\mathbb{E}(B_{\frac{1}{r}}(X_t - X_s)) = 0$  because they are all normal distributed random variable. Easily  $\mathbb{E}(B_{\frac{1}{r}}X_t) = \mathbb{E}((B_{\frac{1}{r}} - B_{\frac{1}{t}})tB_{\frac{1}{t}}) + \mathbb{E}(tB_{\frac{1}{t}}^2) = 1$ , and  $\mathbb{E}(B_{\frac{1}{r}}X_s) = \mathbb{E}((B_{\frac{1}{r}} - B_{\frac{1}{s}})sB_{\frac{1}{s}}) + \mathbb{E}(sB_{\frac{1}{s}}^2) = 1$  So  $\mathbb{E}(B_{\frac{1}{r}}(X_t - X_s)) = 0$ .

Finally we need to check  $X_t$  is continuous a.s. Easily for  $t \neq 0$  we know  $X_t$  is continuous at  $t$ . Only need to check  $X_t$  is continuous at 0 with probability 1. Easily to know  $(-B_t : t \geq 0)$  is standard Brownian motion, too. So  $\limsup_{t \rightarrow \infty} \frac{-B_t}{\sqrt{2t \log \log t}} = 1$ . So  $\limsup_{t \rightarrow \infty} \frac{|B_t|}{\sqrt{2t \log \log t}} = 1$ . So  $\limsup_{t \rightarrow 0+} |X_t| = \lim_{t \rightarrow \infty} |\frac{1}{t}B_t| \leq \limsup_{t \rightarrow \infty} \frac{\sqrt{2t \log \log t}}{t} = 0$ . So  $\lim_{t \rightarrow 0+} |X_t| = 0$ . So  $X_t$  is continuous with probability 1.  $\square$