GroupRepresentation 10

白永乐

202011150087

202011150087@mail.bnu.edu.cn

2023 年 12 月 22 日

PROBLEM I Assume $(\varphi, V), (\psi, W)$ are two finite-dim reperentation of group G, find the matrix of $\varphi \otimes \psi$.

SOLUTION . Assume $\{v_i : i = 1, \cdots, n\}, \{w_i : i = 1, \cdots, m\}$ are basis of V, W. Then we get $\{v_i \otimes w_j : 1 \leq i \leq n, 1 \leq j \leq m\}$ is a basis of $V \otimes W$. Assume Φ, Ψ, Γ is the matrix of $\varphi, \psi, \varphi \otimes \psi$. We use $\{1, \cdots, n\}^4$ as the dom of $\Gamma(g)$. Then we get $(\varphi \otimes \psi)(g)(v_i \otimes w_j) = \varphi(g)(v_i) \otimes \psi(g)(w_j)$. So $\Gamma(g)(e_{ij}) = (\sum_{k=1}^{n} \sum_{t=1}^{m} \Phi(g)_{ki} \Psi(g)_{tj} e_{kt})$. So finally we get $\Gamma(g)_{kt,ij} = \Phi(g)_{ki} \Psi(g)_{tj}$. $\square$

PROBLEM II Assume $\mathrm{Sym}^2 V := \mathrm{Span}\{v \otimes w + w \otimes v : v, w \in V\}$ and $\bigwedge^2 V := \mathrm{Span}\{v \otimes w - w \otimes v : v, w \in V\}$. Prove that $V \otimes V = \mathrm{Sym}^2 V \oplus \bigwedge^2 V$.

SOLUTION . First since $x \otimes y = \frac{x}{2} \otimes y + y \otimes \frac{x}{2} + \frac{x}{2} \otimes y - y \otimes \frac{x}{2}$ we get $x \otimes y \in \mathrm{Sym}^2 V + \bigwedge^2 V$. Since $\mathrm{Span}\{x \otimes y : x, y \in V\} = V \otimes V$, we get $V \otimes V = \mathrm{Sym}^2 V + \bigwedge^2 V$. Now assume $\dim V = n$, we only need to prove $\dim \mathrm{Sym}^2 V + \dim \bigwedge^2 V \leq n^2$. Assume $\{v_i : 1 \leq i \leq n\}$ is a basis of V, then $\{v_i \otimes v_j : 1 \leq i, j \leq n\}$ is basis of $V \otimes V$. Then easily $\mathrm{Span}\{v_i \otimes v_j + v_j \otimes v_i : 1 \leq i, j \leq n\} = \mathrm{Sym}^2 V, \mathrm{Span}\{v_i \otimes v_j - v_j \otimes v_i : 1 \leq i, j \leq n\} = \bigwedge^2 V$. Since for $i \neq j$ we get $v_i \otimes v_j + v_j \otimes v_i = v_j \otimes v_i + v_i \otimes v_j$ we get $\dim \mathrm{Sym}^2 V \leq n + \frac{n^2 - n}{2}$. Since for $i \neq j$ we have $v_i \otimes v_j - v_j \otimes v_i = -(v_j \otimes v_i - v_i \otimes v_j)$ and for $i = j$ we have $v_i \otimes v_j - v_j \otimes v_i = 0$ we get $\dim \bigwedge^2 V \leq \frac{n^2 - n}{2}$. So finally we get dim $\mathrm{Sym}^2 V + \dim \bigwedge^2 V \leq n + \frac{n^2 - n}{2} + \frac{n^2 - n}{2} = n^2$. So $V \otimes V = \mathrm{Sym}^2 V \oplus \bigwedge^2 V$. $\square$

PROBLEM III Find the complex character table of the group D_5.

SOLUTION . First we should find all of irreducible complex reperentation of D_5. Easily all of conjugate of D_5 are $\{e\}, \{\sigma, \sigma^4\}, \{\sigma^2, \sigma^3\}, \{\tau, \sigma\tau, \sigma^2\tau, \sigma^3\tau, \sigma^4\tau\}$. So there are four different irreducible complex reperentation of D_5. Now we try to find the one-dim irreducible complex reperentation. Easily we get $D_5' = \langle \sigma \rangle$. So $D_5 / D_5' \cong \mathbb{Z}_2$. So D_5 has two different irreducible complex reperentation, φ_0, φ_1. Where φ_0 is the main reperentation, and $\varphi_1(\sigma^i) = 1, \varphi_1(\sigma^i\tau) = -1$. Now we try to find other reperentation of D_5. Since $|D_5| = 10 = 1^2 + 1^2 + 2^2 + 2^2$, we get D_5 has two different two-dim irreducible reperentation. Consider $\varphi_\theta(\sigma) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ and $\varphi_\theta(\tau) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, let $\varphi_2 = \varphi_{\frac{2\pi}{5}}, \varphi_3 = \varphi_{\frac{4\pi}{5}}$. Easily φ_2, φ_3 are irreducible and different. So all of different irreducible complex reperentation of D_5 are $\varphi_0, \varphi_1, \varphi_2, \varphi_3$. Now we let $g_1 = e, g_2 = \sigma, g_3 = \sigma^2, g_4 = \tau$ and $W_{ij} = \chi_{i-1}(g_j)$, we have

$$W = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 2 & 2\cos\frac{2\pi}{5} & 2\cos\frac{4\pi}{5} & 0 \\ 2 & 2\cos\frac{4\pi}{5} & 2\cos\frac{2\pi}{5} & 0 \end{pmatrix}$$

□