GroupRepresentation 7

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PROBLEM I Find all of irreducible reperentation of $C_4 = \{e, a, a^2, a^3\}$ over $\mathbb{C}$ by give the irreducible decomposation of it's regular reperentation.

SOLUTION . Assume $\varphi : C_4 \to M_4(\mathbb{C})$ is the regular reperentation, and

$$\varphi(a) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Let $V_1 = \{x \in \mathbb{C}^4 : x_1 = x_2 = x_3 = x_4\}, V_2 = \{x \in \mathbb{C}^4 : x_1 = x_3 = -x_2 = -x_4\}, V_3 = \{x \in \mathbb{C}^4 : x_1 = -x_3, x_2 = -x_4\}$. Easily we get V_1, V_2, V_3 are invariant subspace over φ. Now we prove thry are irreducible. Obviously $\dim V_1 = \dim V_2 = 1$, so they are irreducible. Only need to prove V_3 is irreducible. Consider $W \subset V_3$ is a subspace and $W \neq \{0\}$, to prove $W = V_3$. Let $x \in W$ and $x \neq 0$. Then $\varphi(a)x = (x_2, x_3, x_4, x_1) \in W$. Consider the equation $\begin{cases} ax_1 + bx_2 = 1 \\ ax_2 - bx_1 = 0 \end{cases}$, Since x_1, x_2 can't be all 0, we know this eauqtion has a solution (a, b). Then $(1, 0, -1, 0) = ax + b\varphi(x) \in W$. For the same reason we get $(0, 1, 0, -1) \in W$, too. So $W = V_3$. So V_3 is irreducible. Easily we find $\varphi|_{V_1}$ is ordinary reperentation, $\varphi|_{V_2}$ is isomorphic to $\psi : C_4 \to \mathbb{C}, a \mapsto -1$, and $\varphi|_{V_3}$ is isomorphic to $\tau : C_4 \to M_2(\mathbb{C}), a \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. They are all of irreducible reperentation of C_4 over $\mathbb{C}$. $\square$