

**PROBLEM I** Assume  $N(t)$  is updating process.  $X$  is the time interval distrabution of  $N(t)$ . Assume  $\mathbb{D}(X) < \infty$ . Let  $R(t) := S_{N(t)+1} - t$ . Find:

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T R(t) dt$$

**SOLUTION**. Easily  $N(t) + 1 \geq T \geq N(t)$ . So  $\int_0^T R(t) dt \leq \sum_{i=1}^{N(t)+1} \int_{S_{i-1}}^{S_i} (S_i - t) dt = \frac{1}{2} \sum_{i=1}^{N(t)+1} (S_i - S_{i-1})^2 = \frac{1}{2} \sum_{i=1}^{N(t)+1} \xi_i^2$ . For the same reason, we get that  $\int_0^T R(t) dt \geq \frac{1}{2} \sum_{i=1}^{N(t)} \xi_i^2$ .

Easy to know that  $\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{i=1}^{N(t)} \xi_i^2 = \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{i=1}^{N(t)+1} \xi_i^2 = \frac{\mathbb{E}(X^2)}{\mathbb{E}(X)^2}$ . So finally we get that

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T R(t) dt = \frac{\mathbb{E}(X^2)}{2\mathbb{E}(X)^2}$$

□

**PROBLEM II** Assume the number of people arriving the cinema is distributed as a Possion process with parameter  $\lambda$ . Assume the film begin at a fixed time  $t \geq 0$ . Let  $A(t)$  be the sum of waiting time of all people arriving in  $(0, t]$ , find  $\mathbb{E}(A(t))$ .

**SOLUTION**. Let  $V_k$  be the arriving time of  $k$ -th people. Let  $N(t)$  be the number of people in  $(0, t]$ . Then  $A(t) = \sum_{k=1}^{N(t)} (t - V_k)$ . Let  $\xi_k := V_k - V_{k-1}$ . Then  $\sum_{k=1}^{N(t)} V_k = \sum_{k=1}^{N(t)} (N(t) - k) \xi_k = \sum_{k=0}^{N(t)-1} k \xi_{N(t)-k}$ . So  $\mathbb{E}(A(t)) = t\mathbb{E}(N(t)) - \mathbb{E}(\sum_{k=0}^{N(t)-1} k \xi_{N(t)-k})$ . Easy to get that  $\mathbb{E}(\sum_{k=0}^{N(t)-1} k \xi_{N(t)-k} \mid N(t) = n) = \frac{nt}{2}$ . So  $\mathbb{E}(A(t) \mid N(t) = n) = nt - \frac{nt}{2} = \frac{nt}{2}$ . So finally we have  $\mathbb{E}(A(t)) = \mathbb{E}(\mathbb{E}(A(t) \mid N(t))) = \mathbb{E}(\frac{N(t)t}{2}) = \frac{\lambda t^2}{2}$ . □

**PROBLEM III** Assume a machine has life distrabuted  $p$ . When machine is broken or has been used  $T$  years, we will change a new machine. The price of new machine is  $C_1$ , and if the machine is broken, it would cause loss  $C_2$ .

1. Give the long-time average fee of this machine.
2. Let  $C_1 = 10, C_2 = 0.5$ , and  $p(x) = \mathbb{1}_{(0,10)}(x) \frac{1}{10}$ . Which  $T$  can let the fee be minimum.

**SOLUTION**. 1. Let  $\xi$  be the time when the machine will broken. Let  $\gamma := \xi \wedge T$ . Then the machine will be changed at  $\gamma$ . Obviously  $\mathbb{E}(\gamma) = T\mathbb{P}(\xi > T) + \mathbb{E}(\xi \mathbb{1}(\xi \leq T)) = T \int_T^\infty p(x) dx + \int_0^T Txp(x) dx$ . Let  $\eta$  be the fee of this machine, then we have  $\eta = C_1 \mathbb{1}(\xi > T) + (C_1 + C_2) \mathbb{1}(\xi \leq T) = C_1 + C_2 \mathbb{1}(\xi \leq T)$ . So  $\mathbb{E}(\eta) = C_1 + C_2 \int_0^T p(x) dx$ . So the long-time average fee is

$$g(T) = \frac{C_1 + C_2 \int_0^T Txp(x) dx}{T \int_T^\infty p(x) dx + \int_0^T x p(x) dx}$$

2. Easy to get that  $g(T) = \frac{200+T}{20T-T^2}$  when  $T \in (0, 10)$ . And  $g'(T) = \frac{T^2+400T-4000}{(20T-T^2)^2}$ . Let  $g'(T) = 0$ , then  $T^2 + 400T - 4000 = 0$ , then  $T = 20\sqrt{110} - 200 \approx 9.76$ . So  $T = 9.76$  can make the fee get minimum.

□