GroupRepresentation 13

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PROBLEM I Compute the characters of $\mathrm{Sym}^k V$ and $\bigwedge^k V$.

SOLUTION. Assume $\{v_i : 1 \le i \le n\}$ is a basis of V, assume $\varphi(g)$ has characters $\{\lambda_i : 1 \le i \le n\}$. Then $\left\{\sum_{\sigma\in S_k} \bigotimes_{i=1}^k v_{\tau\sigma(i)} : \tau \in \mathcal{A}\right\}$ is a basis of $\mathrm{Sym}^k V$, where $\mathcal{A} = \left\{f \in \{1, \cdots, n\}^{\{1,2,\cdots,k\}} : f \text{ is injection}\right\}$. And $\left\{\sum_{\sigma\in S_k} \prod_{i=1}^k \lambda_{\tau\sigma(i)} : \tau \in \mathcal{A}\right\}$ are it's characters. For the same reason, we get $\left\{\sum_{\sigma\in S_k} \bigotimes_{i=1}^k (-1)^{\mathrm{sgn}\,\sigma} v_{\tau\sigma(i)} : \tau \in \mathcal{A}\right\}$ is a basis of $\bigwedge^k V$. And $\left\{\sum_{\sigma\in S_k} (-1)^{\mathrm{sgn}\,\sigma} \prod_{i=1}^k \lambda_{\tau\sigma(i)} : \tau \in \mathcal{A}\right\}$ are it's characters. □

PROBLEM II Find the decomposition of the reperesentation $V^{\otimes n}$ using character theory.

SOLUTION. Assume $V^{\otimes n} = U_1^{\oplus a_n} \oplus U_2^{\oplus b_n} \oplus V^{\oplus c_n}$. And $V \otimes V = U_1 \oplus U_2 \oplus V$. Now we try to caculate a_n, b_n, c_n. Since $U_1 \otimes V \cong V$ and $U_2 \otimes V \cong V$, we get

$$V^{\otimes n+1} = V^{\otimes n} \otimes V = (U_1^{\oplus a_n} \oplus U_2^{\oplus b_n} \oplus V^{\oplus c_n}) \otimes V \cong U_1^{\oplus c_n} \oplus U_2^{\oplus c_n} \oplus V^{a_n+b_n+c_n}$$

So we get

$$\begin{cases} a_{n+1} = c_n \\ b_{n+1} = c_n \\ c_{n+1} = a_n + b_n + c_n \end{cases}$$

Then $c_{n+2} = c_{n+1} + 2c_n$. Since $c_1 = c_2 = 1$, we get $c_n = \frac{2^n-(-1)^n}{3}$. Thus $a_n = b_n = \frac{2^{n-1}-(-1)^{n-1}}{3}$. □