PROBLEM I Let $X = \{X(n) : n \geq 0\}$ be Markov chain defined on probability space $(\Omega, \mathscr{F}, \mathbb{P})$, with state space E and transition probability matrix $P = (p(i,j) : i, j \in E)$. Let $a, b \in E$, $\tau_0 = 0$, $\sigma_k = \inf\{n \geq \tau_{k-1} : X(n) = b\}, \tau_k = \inf\{n \geq \sigma_{k-1} : X(n) = a\}$. Prove: $\tau_n, \sigma_n, n \geq 1$ are all stopping time on $(\mathscr{F}_n : n \geq 0)$.

SOLUTION. We use MI to prove it. Easily $\sigma_1 = \inf\{n \geq \tau_0 : X(n) = b\} = \inf\{n \geq 0 : X(n) = b\}$ is stopping time. Assume for certain $n \geq 1$, we have proved that σ_n, τ_{n-1} are stopping times, now we need to prove σ_{n+1}, τ_n are stopping times. Since σ_n is stopping time, we know $\forall k \leq m, \{k \geq \sigma_n\} \in \mathcal{F}_m$. And obviously $\forall k \leq m, \{X(k) = a\} \in \mathcal{F}_m$. So we obtain that $\{\sigma_n \leq m\} = \bigcup_{k=1}^m \{k \geq \sigma_n, X(k) = a\} \in \mathcal{F}_m$. So τ_n is stopping time. Since τ_n is stopping time, we know $\forall k \leq m, \{k \geq \tau_n\} \in \mathcal{F}_m$. And obviously $\forall k \leq m, \{X(k) = b\} \in \mathcal{F}_m$. So we obtain that $\{\tau_n \leq m\} = \bigcup_{k=1}^m \{k \geq \tau_n, X(k) = b\} \in \mathcal{F}_m$. So σ_{n+1} is stopping time. So we finally obtain that $\forall n \in \mathbb{N}^+, \sigma_n, \tau_n$ are stopping times. □

PROBLEM II Let $(X_n : n \geq 0)$ be a one-dimension simple random walk starting at 1. Let $e(n) = \{X_{n \wedge \tau_1} : n \geq 0\}$, where $\tau_1 = \inf\{n \geq 0 : X_n = 0\}$. Find the distribution of $\sup_{n \geq 0} e(n)$.

SOLUTION. Assume $\mathbb{P}(X_{n+1} - X_n = 1) = p, \mathbb{P}(X_{n+1} - X_n = -1) = q$, where $p + q = 1$. Let $E := \sup_{n \geq 0} e(n)$. Let $m \in \mathbb{N}^+$. Let $\sigma := \inf\{n \geq 0 : X_n = 0 \text{ or } X_n = m\}$. Then σ is a stopping time.

First we assume $p \neq q$. Let $Y_n := \left(\frac{q}{p}\right)^{X_n}$, then it's easy to check that Y_n is martingale. So we know that $Y_{n \wedge \sigma}$ is martingale, too. Easy to get that $\sigma < \infty, a.s.$, so $Y_{n \wedge \sigma} \xrightarrow{\text{a.s.}} Y_\sigma$. And $0 \leq Y_{n \wedge \sigma} \leq m$, so $\mathbb{E}(Y_\sigma) = \mathbb{E}(Y_{n \wedge \sigma}) = \mathbb{E}(Y_0)$. Noting that $\{X_\sigma = 0\} \overset{\text{a.s.}}{=} \{E < m\}$ and $\{X_\sigma = m\} \overset{\text{a.s.}}{=} \{E \geq m\}$, we get two equations:

$$\begin{cases} \mathbb{P}(E < m) + \mathbb{P}(E \geq m) = 1 \\ \mathbb{P}(E < m) + \mathbb{P}(E \geq m) \left(\frac{q}{p}\right)^m = \frac{q}{p} \end{cases}.$$

Solve this equation, we get $\mathbb{P}(E \geq m) = \frac{\frac{q}{p} - 1}{(\frac{q}{p})^m - 1}$. Then $\mathbb{P}(E = m) = \frac{(\frac{p}{q})^m (\frac{p}{q} - 1)}{((\frac{p}{q})^m - 1)((\frac{p}{q})^{m+1} - 1)}$. Furthermore, easily $\mathbb{P}(E = \infty) = \lim_{m \to \infty} \mathbb{P}(E \geq m) = \begin{cases} 0 & \frac{q}{p} > 1 \\ 1 - \frac{q}{p} & \frac{q}{p} < 1 \end{cases}$.

Second, we consider $p = q = \frac{1}{2}$. Then easily X_n is martingale. So we know that $X_{n \wedge \sigma}$ is martingale, too. Easy to get that $\sigma < \infty, a.s.$, so $X_{n \wedge \sigma} \xrightarrow{\text{a.s.}} X_\sigma$. And $0 \leq X_{n \wedge \sigma} \leq m$, so $\mathbb{E}(X_\sigma) = \mathbb{E}(X_{n \wedge \sigma}) = \mathbb{E}(X_0)$. Noting that $\{X_\sigma = 0\} \overset{\text{a.s.}}{=} \{E < m\}$ and $\{X_\sigma = m\} \overset{\text{a.s.}}{=} \{E \geq m\}$, we get two equations:

$$\begin{cases} \mathbb{P}(E < m) + \mathbb{P}(E \geq m) = 1 \\ 0\mathbb{P}(E < m) + m\mathbb{P}(E \geq m) = 1 \end{cases}.$$

Solve this equation, we get $\mathbb{P}(E \geq m) = \frac{1}{m}$. So $\mathbb{P}(E = m) = \frac{1}{m(m+1)}$, and easily $\mathbb{P}(E = \infty) = 0$. □

PROBLEM III Prove:

1. When $0 < p \leq q$, the reflecting random walk with transition matrix Q_+^a is recurrent.

2. When $0 < q \le p$, the reflecting random walk with transition matrix Q^a_- is recurrent.

SOLUTION. By symmetry, only need to prove 1. By shifting, without loss of generality we can assume $a = 0$. We consider the equation

$$y_0 = y_1, \forall i \ge 1, y_i = q y_{i-1} + p y_{i+1}$$

Only need to prove its all bounded solution are all constant. Easy to get $y_{i+2} = \frac{1}{p} y_{i+1} - \frac{q}{p} y_i$. Consider the charasteristic equation of this sequence, $x^2 - \frac{x}{p} + \frac{q}{p} = 0$. We get $x_1 = 1, x_2 = \frac{q}{p} \ge 1$. If $x_2 > 1$, then $y_n = c_1 x_1^n + c_2 x_2^n$ is bounded $\iff$ $c_2 = 0$, so $y_n = c_1 x_1^n = c_1$ is constant. Else, $x_2 = x_1 = 1$, then $y_n = (an+b) x_1^n = an + b$ is bounded $\iff$ $a = 0$, so $y_n = b$ is constant. So the Markov chain is recurrent. □

PROBLEM IV Prove colloary 4.4.3. i.e., let $\phi_0(n : n \in \mathbb{N}^+)$ be simple random walk begin at $\phi_0(0) \ge a+1$, let $\zeta_0 := \inf\{m : \phi_0(m) = a+1\}$, let $Y_n : n \in \mathbb{N}$ be reflecting simple random walk on $\mathbb{Z}^a_+$, starting at $a+1$, indenpendent with ϕ_0. Let $X_n := \begin{cases} \phi_0(n) & n \le \zeta_0 \\ Y_n - \zeta_0 & n \ge \zeta_0 \end{cases}$. Prove that $X_n : n \in \mathbb{N}$ is reflecting random walk on $\mathbb{Z}^a_+$ begin at $\phi_0(0)$.

SOLUTION. Now we consider $n \in \mathbb{N}^+$ and $i_0, i_1, i_2, \cdots, i_{n+1} \in \mathbb{Z}^a_+$.

1. If $\forall k : 1 \le k \le n, i_k \ne a+1$, then we have

$$\begin{aligned} \mathbb{P}(X_0 = i_0, \cdots, X_{n+1} = i_{n+1}) &= \mathbb{P}(\phi_0(0) = i_0, \cdots, \phi_0(n+1) = i_{n+1}) \\ &= \mathbb{P}(\phi_0(0) = i_0, \cdots, \phi_0(n) = i_n)\mathbb{P}(\phi_0(n+1) = i_{n+1} \mid \phi_0(n) = i_n) \\ &= \mathbb{P}(X_0 = i_0, \cdots, X_n = i_n) q^a_+(i_n, i_{n+1}) \end{aligned}$$

2. Else, we let $k := \inf\{m : 1 \le m \le n, i_m = a+1\}$. Then we have

$$\begin{aligned} \mathbb{P}(X_0 = i_0, \cdots, X_{n+1} = i_{n+1}) &= \mathbb{P}(\phi_0(0) = i_0, \cdots, \phi_0(k) = i_k, Y_0 = a+1, Y_1 = i_{k+1}, \cdots, Y_{n-k} = i_n, Y_{n-k+} \\ &= \mathbb{P}(\phi_0(0) = i_0, \cdots, \phi_0(k) = i_k)\mathbb{P}(Y_0 = a+1, Y_1 = i_{k+1}, \cdots, Y_{n-k} = i_n, Y_{n-} \\ &= \mathbb{P}(\phi_0(0) = i_0, \cdots, \phi_0(k) = i_k)\mathbb{P}(Y_0 = a+1, Y_1 = i_{k+1}, \cdots, Y_{n-k} = i_n) q^a_+(\\ &= \mathbb{P}(X_0 = i_0, \cdots, X_n = i_n) q^a_+(i_n, i_{n+1}) \end{aligned}$$

So we get $(X_n : n \ge 0)$ is reflecting simple random walk on $\mathbb{Z}^a_+$. □