# GroupRepresentation 5

白永乐

202011150087

202011150087@mail.bnu.edu.cn

2024 年 1 月 4 日

PROBLEM I $K$ is a field, $A$ is algebra on $K$, $\emptyset \neq A_1 \subset A$, we call $A_1$ is a subalgebra of $A$, if $A_1$ is a subring of $A$ which contains 1 of $A$ and $A_1$ is a subspace of $A$ on K and is also an algebra on $K$. Let $Z(A) := \{c \in A : ca = ac, \forall a \in A\}$. Prove: $Z(A)$ is a subalgebra of $A$, we call $Z(A)$ is the center of algebra $A$.

SOLUTION. 1. $Z(A)$ is a subring of $A$ which contains 1 of $A$: Since $\forall a \in A$, A is a ring, then $1a = a1$. So $1 \in Z(A)$. $\forall c_1, c_2 \in Z(A)$, $\forall a \in A$, $(c_1 - c_2)a = (c_1 + (-c_2))a = c_1 a + (-1)c_2 a = ac_1 + (-1)ac_2 = ac_1 + a(-1)c_2 = a(c_1 + (-1)c_2) = a(c_1 - c_2)$, then $c_1 - c_2 \in Z(A)$. $(c_1c_2)a = c_1(c_2a) = c_1(ac_2) = (c_1a)c_2 = (ac_1)c_2 = a(c_1c_2)$, then $c_1c_2 \in A$.

2. $Z(A)$ is a subspace of $A$ on K:$\forall k \in K$, $\forall c, c_1, c_2 \in Z(A)$, $\forall a \in A$, since $A$ is an algebra on $K$, then $(kc)a = k(ca) = k(ac) = a(kc)$, then $kc \in Z(A)$. And by **??**, we get $c_1 + c_2 \in Z(A)$.

3. By **??**, **??**, we get $Z(A)$ is also an algebra on $K$.
   So $Z(A)$ is a subalgebra of $A$. □

PROBLEM II Let $G$ is infinite group, $K$ is a field. Prove:

1. $\sum_{g \in G} g \in Z(K[G])$;

2. $C_a := \{gag^{-1} : g \in G\}$, $\sum_{x \in C_a} x \in Z(K[G])$.

SOLUTION. 1. $\forall \sum_{h \in G} a_h h \in K([G])$, then $\sum_{g \in G} g \sum_{h \in G} a_h h = \sum_{g \in G} \sum_{x \in G} x a_{x^{-1}g} x^{-1} g = \sum_{g \in G} \sum_{x \in G} a_{x^{-1}g} g = \sum_{g \in G} \sum_{x \in G} a_x g = \sum_{g \in G} \sum_{x \in G} a_x x x^{-1} g = \sum_{x \in G} \sum_{g \in G} a_x x x^{-1} g = \sum_{h \in G} a_h h \sum_{g \in G} g$.

2. $\forall \sum_{h \in G} a_h h \in K([G])$, then $\sum_{x \in C_a} x \sum_{g \in G} a_g g = \sum_{x \in C_a} \sum_{g \in G} x a_g g = \sum_{x \in C_a} \sum_{g \in G} a_g x g = \sum_{g \in G} a_g \sum_{x \in C_a} x g$, $\sum_{g \in G} a_g g \sum_{x \in C_a} x = \sum_{g \in G} \sum_{x \in C_a} a_g g x = \sum_{g \in G} \sum_{x \in C_a} a_g g x = \sum_{g \in G} a_g \sum_{x \in C_a} g x$. Since $\forall g \in G$, then $\{hah^{-1}g : h \in G\} = \{g(g^{-1}h)a(h^{-1}g) : h \in G\} = \{ghah^{-1} : h \in G\}$, then, $\sum_{x \in C_a} x \sum_{g \in G} a_g g = \sum_{g \in G} a_g g \sum_{x \in C_a} x$.

□