Lemma 1. Assume $(N_t : t \geq 0)$ is a random process, and $\mathbb{P}(\forall t \in [0,\infty), \lim_{s\to t+} N_s = N_t, \forall t \in (0,\infty), \lim_{s\to t-} N_s \in \mathbb{R}) = 1$. Assume $\alpha > 0$, and $\forall t, s \geq 0$, we have $N_{t+s} - N_s \sim \mathit{Possion}(\alpha t)$. Then $(N_t : t \geq 0)$ is a Possion process $\iff \forall 0 \leq s \leq t, N_t - N_s \perp \mathcal{F}_s$, where $\mathcal{F}_s := \sigma(N_x : x \leq s)$.

证明. "$\implies$": To prove $N_t - N_s \perp \mathcal{F}_s$, only need to prove for $t_1 < t_2 < \cdots < t_{n-1} = s < t = t_n$, we have $N_t - N_s \perp \sigma(N_{t_k} : k = 1, \cdots, n-1)$. Easily $N_t - N_s \perp \sigma(N_{t_{k+1}} - N_{t_k}, N_{t_1} : k = 1, \cdots, n-2) = \sigma(N_{t_k} : k = 1, \cdots, n-1)$, so $N_t - N_s \perp \mathcal{F}_s$.

"$\impliedby$": For $t_1 < \cdots < t_n$, we need to prove $N_{t_1}, N_{t_{k+1}} - N_{t_k} : 1 \leq k \leq n-1$ are independent. Use MI to n. When $n = 1$ it's obvious. Assume we have proved it for certain $n \geq 1$, now consider $n+1$. Since $N_{t_{k+1}} - N_{t_k} \in \mathcal{F}_{t_n}, k = 1, \cdots, n-1$, we have $N_{t_{n+1}} - N_{t_n} \perp \sigma(N_{t_1}, N_{t_{k+1}} - N_{t_k} : k = 1, \cdots, n-1)$. So $\mathbb{P}(N_{t_1} \in A_1, N_{t_{k+1}} - N_{t_k} \in A_{k+1}, k = 1, \cdots, n) = \mathbb{P}(N_{t_1} \in A_1, N_{t_{k+1}} - N_{t_k} \in A_{k+1}, k = 1, \cdots, n-1)\mathbb{P}(N_{t_{n+1}} - N_{t_n} \in A_{n+1})$. By Induction assumption we get $\mathbb{P}(N_{t_1} \in A_1, N_{t_{k+1}} - N_{t_k} \in A_{k+1}, k = 1, \cdots, n) = \mathbb{P}(N_{t_1} \in A_1)\prod_{k=1}^n \mathbb{P}(N_{t_{k+1}} - N_{t_k} \in A_{k+1})$. So finally we get $N_{t_1}, N_{t_{k+1}} - N_{t_k} : 1 \leq k \leq n$ are independent. $\square$

PROBLEM I Assume $(N_t : t \geq 0)$ is a Possion process with parameter α. Let $P(t) := \mathbb{P}(2 \nmid N_t), Q(t) := \mathbb{P}(2 \mid N_t)$. Prove that $P(t) = \mathrm{e}^{-\alpha t}\sinh(\alpha t), Q(t) = \mathrm{e}^{-\alpha t}\cosh(\alpha t)$.

SOLUTION. Easily to get

$$P(t) = \sum_{k=0}^{\infty} \mathbb{P}(N_t = 2k+1) = \sum_{k=0}^{\infty} \frac{(\alpha t)^{2k=1}}{(2k+1)!}\mathrm{e}^{-\alpha t}$$

Noting that $\sinh(\alpha t) = \frac{1}{2}\left(\sum_{k=0}^{\infty} \frac{(\alpha t)^n}{n!} - \sum_{k=0}^{\infty} \frac{(-\alpha t)^n}{n!}\right) = \sum_{k=0}^{\infty} \frac{(\alpha t)^{2k+1}}{(2k+1)!}$, we easily obtain $P(t) = \mathrm{e}^{-\alpha t}\sinh(\alpha t)$. Noting $P(t) + Q(t) = 1$, we easily get $Q(t) = 1 - P(t) = \mathrm{e}^{-\alpha t}\cosh(\alpha t)$. $\square$

PROBLEM II Assume $(N_t : t \geq 0)$ is a Possion process with parameter α. Prove that $\lim_{t\to\infty} \frac{N_t}{t} = \alpha, a.s.$.

Lemma 2. Assume $(N_t : t \geq 0)$ is a Possion process with parameter α. Then $\mathbb{P}(\forall 0 \leq s \leq t, N_s \leq N_t) = 1$.

证明. For $s, t \in \mathbb{Q} \wedge 0 \leq s \leq t$, we have $\mathbb{P}(N_s > N_t) = 0$ since $N_t - N_s \sim \mathit{Possion}(\alpha(t-s))$. So we get $\mathbb{P}(\exists s, t \in \mathbb{Q}, 0 \leq s \leq t, N_s > N_t) = 0$.

Now we will prove $\exists s, t \in \mathbb{R}, 0 \leq s \leq t, N_s > N_t \implies \exists a, b \in \mathbb{Q}, 0 \leq a \leq b, N_a > N_b$. Let $a_n = \frac{\lceil ns \rceil}{n}, b_n = \frac{\lceil nt \rceil}{n}$. Then $\lim a_n = s, \lim b_n = t$. Easily $a_n \geq s, b_n \geq t$. So since N is continuous we get $\lim N_{a_n} = N_s, \lim N_{b_n} = N_t$. Since $N_s > N_t$, we get $\exists n, N_{a_n} > N_{b_n}$. Let $a = a_n, b = b_n$ will work.

So $\mathbb{P}(\forall 0 \leq s \leq t, N_s \leq N_t) = 1 - \mathbb{P}(\exists 0 \leq s \leq t, N_s > N_t) = 1 - \mathbb{P}(\exists s, t \in \mathbb{Q}, 0 \leq s \leq t, N_s > N_t) = 1 - 0 = 1$. $\square$

SOLUTION. Consider $N_n : n \in \mathbb{N}$. Let $X_n := N_n - N_{n-1}, n \geq 1$. Then easily $(X_n : n \in \mathbb{N}^+)$ is i.i.d sequence and $X_1 \sim \mathit{Possion}(\alpha)$. So from the strong law of large numbers we get $\lim_{n\to\infty} \frac{N_n}{n} = \alpha$. From Lemma 2 we get $\frac{\lfloor t \rfloor}{t}\frac{N_{\lfloor t \rfloor}}{\lfloor t \rfloor} \leq \frac{N_t}{t} \leq \frac{N_{\lceil t \rceil}}{\lceil t \rceil}\frac{\lceil t \rceil}{t}, \forall t \in \mathbb{R}$, let $t \to \infty$ we get $\lfloor t \rfloor, \lceil t \rceil \to \infty$, and $\lfloor t \rfloor \sim t \sim \lceil t \rceil$. So finally we get $\lim_{t\to\infty} \frac{N_t}{t} = \alpha$. $\square$

PROBLEM III Assume $(N_t : t \geq 0)$ is a Possion process with parameter $\alpha > 0$. Prove that $\frac{N_t-\alpha t}{\sqrt{\alpha t}} \xrightarrow{d} N(0,1)$.

SOLUTION . Consider $N_n : n \in \mathbb{N}$. Let $X_n := N_n - N_{n-1}, n \geq 1$. Then easily $(X_n : n \in \mathbb{N}^+)$ is i.i.d sequence and $X_1 \sim Possion(\alpha)$. Easily $\mathbb{V}(X_n) = \alpha < \infty, \mathbb{E}(X_n) = \alpha$. So from the central limit theorem we get $\frac{N_n-\alpha n}{\sqrt{\alpha n}} \xrightarrow{d} N(0,1)$. Noting $\frac{N_t-\alpha t}{\sqrt{\alpha t}} = \frac{N_{\lfloor t \rfloor}-\alpha\lfloor t \rfloor}{\sqrt{\alpha\lfloor t \rfloor}}\frac{\sqrt{\lfloor t \rfloor}}{\sqrt{t}} + \frac{N_t-N_{\lfloor t \rfloor}-\alpha(t-\lfloor t \rfloor)}{\sqrt{\alpha t}}$. Let $t \to \infty$ we get $\lfloor t \rfloor, \to \infty$, and $\lfloor t \rfloor \sim t$. Noting $N_t - N_{\lfloor t \rfloor} \overset{d}{=} N_{t-\lfloor t \rfloor}$, and $t - \lfloor t \rfloor \leq 1$, we easily get $\frac{N_t-N_{\lfloor t \rfloor}}{\sqrt{\alpha t}} \xrightarrow{d} 0$. Easily $\frac{\alpha(t-\lfloor t \rfloor)}{\alpha t} \to 0$, so finally we get that $\frac{N_t-\alpha t}{\sqrt{\alpha t}} \xrightarrow{d} N(0,1)$ □

PROBLEM IV Assume $(X_t : t \geq 0), (Y_t : t \geq 0)$ are two independent Possion processes with parameter α, β respectively. Prove that $(X_t + Y_t : t \geq 0)$ is Possion process with parameter $\alpha + \beta$.

SOLUTION . Write $Z_t = X_t + Y_t$. First we prove $Z_{t+s} - Z_s \sim Possion((\alpha+\beta)t)$. Since $X_{t+s} - X_s \sim Possion(\alpha t), Y_{t+s} - Y_s \sim Possion(\beta t)$, and $X_{t+s} - X_s \perp Y_{s+t} - Y_s$, easily to get $Z_{t+s} - Z_s = X_{t+s} - X_s + Y_{s+t} - Y_s \sim Possion((\alpha+\beta)t)$.

Second we prove $\forall 0 \leq s \leq t, Z_t - Z_s \perp \mathcal{H}_s$, where $\mathcal{H}_s = \sigma(Z_x : 0 \leq x \leq s)$. Easily $Z_t - Z_s \in \sigma(X_t - X_s, Y_t - Y_s)$. Easily $X_t - X_s \perp \mathcal{F}_s := \sigma(X_x : 0 \leq x \leq s)$ since $(X_x : x \geq 0)$ is Possion process. Since $(X_x : x \geq 0) \perp (Y_x : x \geq 0)$, we get $X_t - X_s \perp \mathcal{G}_s := \sigma(Y_x : 0 \leq x \leq s)$. For the same reason we get $Y_t - Y_s \perp \mathcal{F}_s, \mathcal{G}_s$. So we get $Z_t - Z_s \perp \sigma(F_s, G_s) \supset \mathcal{H}_s$.

Finally, we prove that $\mathbb{P}(\forall t \in [0,\infty), \lim_{s\to t+} Z_s = Z_t, \forall t \in (0,\infty), \lim_{s\to t-} Z_s \in \mathbb{R}) = 1$ Since $Z_t = X_t + Y_t$, and $\mathbb{P}(\forall t \in [0,\infty), \lim_{s\to t+} Y_s = Y_t, \forall t \in (0,\infty), \lim_{s\to t-} Y_s \in \mathbb{R}) = 1$, $\mathbb{P}(\forall t \in [0,\infty), \lim_{s\to t+} X_s = X_t, \forall t \in (0,\infty), \lim_{s\to t-} X_s \in \mathbb{R}) = 1$, obvious we get the requirement.

All in all, $(X_t + Y_t : t \geq 0)$ is a Possion process with parameter $\alpha + \beta$. □

PROBLEM V Assume $(\xi_n : n \in \mathbb{N}^+)$ is a sequence of i.i.d. random variable ranging in $\mathbb{Z}^d$. Let $X_n = X_0 + \sum_{k=1}^n \xi_k$, and $X_0 \perp (\xi_n : n \in \mathbb{N}^+)$ ranging in $\mathbb{Z}^d$, too. Assume $(N_t : t \geq 0)$ is a Possion process with parameter $\alpha > 0$. Discuss $\frac{X_{N_t}}{t}$ when $t \to \infty$.

SOLUTION . First we prove that $\lim_{t\to\infty} N_t = \infty, a.s.$. We have $\mathbb{P}(\sup_t N_t \geq n) \geq \mathbb{P}(N_t \geq n), \forall t, \forall n \in \mathbb{N}$. Easily $\lim_{t\to\infty} \mathbb{P}(N_t \geq n) = 1$, so $\mathbb{P}(\sup_t N_t \geq n) = 1, \forall n \in \mathbb{N}$. So $\mathbb{P}(\sup_t N_t = \infty) = 1$. Noting Lemma 2 we easily get $\mathbb{P}(\lim_{t\to\infty} N_t = \infty) = 1$.

Now we can decompose $\frac{X_{N_t}}{t}$ into $\frac{X_{N_t}}{N_t}\frac{N_t}{t}$. We have proved that $\frac{N_t}{t} \to \alpha, a.s.$ in Problem II, so we only need to find $\frac{X_{N_t}}{N_t}$. Since $N_t \to \infty, a.s.$, we only need to find $\frac{X_n}{n}$ when $n \to \infty$.

If $\mathbb{E}(\xi_1)$ exists, then easily $\frac{X_n}{n} \to \mathbb{E}(\xi_1), a.s.$. Then we easily get $\frac{X_{N_t}}{t} \to \alpha\mathbb{E}(\xi_1), a.s.$. □