GROUP REPRESENTATION

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PROBLEM I Group G has an action on set $\Omega = \{x_1, x_2, \cdots, x_n\}$, let (φ, V) be the $n-$ dimensional K permutation representation of G, where K is the field of vector space V, and

$$V = \left\{ \sum_{i=1}^{n} a_i x_i : a_i \in K, i = 1, 2, \cdots, n \right\}.$$

Let

$$V_1 = \left\langle \sum_{i=1}^{n} x_i \right\rangle,$$

$$V_2 = \left\{ \sum_{i=1}^{n} a_i x_i : \sum_{i=1}^{n} a_i = 0, a_i \in K \right\}.$$

Prove:

1. V_1 and V_2 are invariant subspaces of G ;
2. If $\operatorname{char} K \nmid n$, then $\varphi = \varphi_{V_1} \oplus \varphi_{V_2}$.

SOLUTION . 1. For $g \in G$, we have $g \sum_{k=1}^{n} x_k = \sum_{k=1}^{n} g x_k$. Assume $g x_k = x_{\sigma(k)}, \sigma \in S_n$, then $\sum_{k=1}^{n} g x_k = \sum_{k=1}^{n} x_k$, so V_1 is invariant subspace. Also, $g \sum_{k=1}^{n} a_k x_k = \sum_{k=1}^{n} a_k g x_k = \sum_{k=1}^{n} a_{\sigma^{-1}(k)} x_k \in V_2$. So V_2 is invariant subspace, too.

2. Since $\operatorname{char} K \nmid n$ we know $\sum_{k=1}^{n} x_k \notin V_2$, so $V_1 \cap V_2 = \{0\}$. Obviously $\dim V_1 = 1, \dim V_2 = n - 1$, so $V = V_1 \oplus V_2$. So $\varphi = \varphi_{V_1} \oplus \varphi_{V_2}$.

□

PROBLEM II Using exercise 1, calculate a 2−dimensional complex representation of S_3 and its matrix of the representation.

SOLUTION . In Problem I let $n = 2, K = \mathbb{C}, G = S_3$. Consider $\cdot : G \times \Omega \to \Omega$,

$$\sigma \cdot x_i = \begin{cases} x_i, \sigma \text{ is even} \\ x_{3-i}, \sigma \text{ is odd} \end{cases}$$

Then easily $\cdot$ is a group action. Consider $\varphi : G \to \mathrm{GL}(V), \varphi(g)(x) = g{\cdot}x$. We get $g(x_1+x_2) = x_1+x_2$, so $\Phi_{V_1} = I_1$. And $g(x_1 - x_2) = \begin{cases} x_1 - x_2, g \text{ is even} \\ x_2 - x_1, g \text{ is odd} \end{cases}$, so $\Phi_{V_2} = \pm I_1$. Finally we get the matrix representation Φ:

$$\Phi(g) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & g \text{ is even} \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & g \text{ is odd} \end{cases} \tag{1}$$

□

PROBLEM III $M_{\mathrm{n}}(K) := \{(a_{i,j})_{n\times n} : a_{ij} \in K, \forall 1 \le i, j \le n\}$. Let

$$\varphi : \mathrm{GL}_n(K) \to \mathrm{GL}\left(M_n(K)\right)$$
$$A \to \varphi(A),$$

where

$$\varphi(A)X := AXA^{-1}; \quad \forall X \in M_n(K).$$

1. Illustrate φ is the n^2-dimensional K representation of group $\mathrm{GL}_n(K)$;
2. $M_n^0(K) := \{A \in M_n(K) : \operatorname{tr} A = 0\}$. Illustrate $M_n^0(K)$ and $\langle I \rangle$ are invariant subspaces of φ;
3. Prove: If $\operatorname{char} K \nmid n$, then

$$\varphi = \varphi_{\langle I \rangle} \oplus \varphi_{M_n^0(K)}$$

SOLUTION. 1. Obviously $\dim M_n(K) = n^2$, so only need to prove φ is group homomorphism. We have $\varphi(AB)X = (AB)X(AB)^{-1} = A(BXB^{-1})A^{-1} = \varphi(A)(\varphi(B)X) = \varphi(A)\varphi(B)X$, so $\varphi(AB) = \varphi(A)\varphi(B)$.

2. Since $\varphi(A)$ is Similarity transformation over $M_n(K)$, so $\operatorname{tr}(\varphi(A)X) = \operatorname{tr} X$. So $M_n^0(K)$ is invariant subspace. Noting $\varphi(A)I = AIA^{-1} = I$, so $\langle I \rangle$ is invariant subspace, too.

3. Obviously $\dim M_n^0(K) = n^2 - 1, \dim\langle I \rangle = 1$. Since $\operatorname{char} K \nmid n$, we get $\operatorname{tr} I = n \neq 0$, so $M_n^0(K) \oplus \langle I \rangle = M_n(K)$. So $\varphi = \varphi_{\langle I \rangle} \oplus \varphi_{M_n^0(K)}$.

□

PROBLEM IV $\mathcal{O}(n) := \{A \in M_n(\mathbb{R}) : AA^T = I_n\}$ is the set of all n-dimensional otheretic matrix over $\mathbb{R}$. Let:

$$\begin{aligned} \varphi : \mathcal{O}(n) &\to \mathrm{GL}\left(M_n(\mathbb{R})\right) \\ A &\mapsto \varphi(A), \end{aligned} \tag{2}$$

Where,

$$\varphi(A)X := AXA^{-1} : \quad \forall X \in M_n(\mathbb{R}) \tag{3}$$

$M_n^+(\mathbb{R}) := \{A \in M_n^0(\mathbb{R}) : A = A^T\}$, $M_n^-(\mathrm{R}) := \{A \in M_n^0(\mathbb{R}) : A^T = -A\}$.

1. Proof: $M_n^+(\mathrm{R})$ and $M_n^-(\mathrm{R})$ are invariant spaces of φ;
2. Let the subrepresentation of φ on $\langle I\rangle$,$M_n^+(\mathbb{R}), M_n^-(\mathbb{R})$ be $\varphi_0, \varphi_1, \varphi_2$. Proof:

$$\varphi = \varphi_0 \oplus \varphi_1 \oplus \varphi_2$$

3. calculate a $\frac{1}{2}n(n-1)-$ dimensional $\mathbb{R}$ representation of $\mathcal{O}(n)$.

SOLUTION.

1. Since $(\varphi(A)X)^T = (A^{-1})^T X^T A^T = \varphi((A^{-1})^T)X^T = \varphi(A)X^T$, so M_n^+, M_n^- is invariant subspace.
2. Only need to prove $M_n(\mathbb{R}) = \langle I\rangle \oplus M_n^+(\mathbb{R}) \oplus M_n^-(\mathbb{R})$. From Problem III we know $M_n(\mathbb{R}) = \langle I\rangle \oplus M_n^0(\mathbb{R})$, so we only need to prove $M_n^0(\mathbb{R}) = M_n^+(\mathbb{R}) \oplus M_n^-(\mathbb{R})$. For $A \in M_n^0(\mathbb{R})$, we have $A = \frac{A+A^T}{2} + \frac{A-A^T}{2}$, where $\frac{A+A^T}{2} \in M_n^+(\mathbb{R})$ and $\frac{A-A^T}{2} \in M_n^-(\mathbb{R})$. So we only need to prove $M_n^+ \cap M_n^- = \{0\}$. If $A \in M_n^+ \cap M_n^-$, then $A = A^T = -A^T$, so $A = 0$. So $M_n^0(\mathbb{R}) = M_n^+(\mathbb{R}) \oplus M_n^-(\mathbb{R})$, thus $\varphi = \varphi_0 \oplus \varphi_1 \oplus \varphi_2$.
3. Obviously $\dim M_n^-(\mathbb{R}) = \frac{1}{2}n(n-1)$, so φ_2 satisfy the condition.

$\square$