PROBLEM I Assume $(B_t : t \geq 0)$ is Brownian motion, prove that for $r > 0$, we have $(B_{t+r} - B_r : t \geq 0)$ is Brownian motion, too.

SOLUTION . Assume $B_t - B_s \sim N(0, a(t-s)), a > 0$. Let $\mathcal{F}_t := \sigma(B_s : 0 \leq s \leq t)$ and $\mathcal{G}_t := \sigma(B_{r+s} - B_r : 0 \leq s \leq t)$. For $0 \leq s \leq t$, we have $B_{t+r} - B_r - (B_{s+r} - B_r) = B_{t+r} - B_{s+r} \sim N(0, a(t-s))$. And easily to know $B_{r+s} - B_r \in \mathcal{F}_{r+s}$, so $\mathcal{G}_t \subset \mathcal{F}_{t+r}, \forall t \geq 0$. Since $(B_t : t \geq 0)$ is Brownian motion, easily $\mathcal{F}_{s+r} \perp B_{t+r} - B_{s+r}$. Since $\mathcal{G}_t \subset \mathcal{F}_{t+r}$, we obtain $\mathcal{G}_t \perp B_{t+r} - B_{s+r} = B_{t+r} - B_r - (B_{s+r} - B_r)$. So $(B_{t+r} : t \geq 0)$ is Brownian motion. $\square$

PROBLEM II Assume $(B_t : t \geq 0)$ is standrad Brownian motion start at 0. Prove that $\forall c > 0, (cB_{\frac{t}{c^2}} : t \geq 0)$ is standrad Brownian motion start at 0, too.

SOLUTION . Since $B_0 = 0$ we get $cB_{\frac{0}{c^2}} = 0$. Let $\mathcal{F}_t := \sigma(B_s : 0 \leq s \leq t)$ and $\mathcal{G}_t := \sigma(cB_{\frac{s}{c^2}} : 0 \leq s \leq t)$. Easily to know $\mathcal{G}_t = \mathcal{F}_{\frac{t}{c^2}}$. For $0 \leq s \leq t$, we have $cB_{\frac{t}{c^2}} - cB_{\frac{s}{c^2}} = c(B_{\frac{t}{c^2}} - B_{\frac{s}{c^2}}) \sim N(0, t-s)$, because $B_{\frac{t}{c^2}} - B_{\frac{s}{c^2}} \sim N(0, \frac{t-s}{c^2})$. And since $(B_t : t \geq 0)$ is Brownian motion, we get $B_{\frac{t}{c^2}} - B_{\frac{s}{c^2}} \perp \mathcal{F}_{\frac{s}{c^2}} = \mathcal{G}_s$. So $(cB_{\frac{t}{c^2}} : t \geq 0)$ is standrad Brownian motion starts at 0, too. $\square$

PROBLEM III Assume $(X_t : t \geq 0)$ and $(Y_t : t \geq 0)$ are two independent standrad Brownian motion, $a, b \in \mathbb{R}$ and $\sqrt{a^2 + b^2} > 0$. Prove that $(aX_t + bY_t : t \geq 0)$ is a Brownian motion with parameter $c = \sqrt{a^2 + b^2}$.

SOLUTION . Let $\mathcal{F}_t := \sigma(X_s : 0 \leq s \leq t)$ and $\mathcal{G}_t := \sigma(Y_s : 0 \leq s \leq t)$. Let $\mathcal{H}_t := \sigma(aX_s + bY_s : 0 \leq s \leq t)$. Since $(X_t : t \geq 0), (Y_t : t \geq 0)$ are two independent Brownian motion, we know $\forall 0 \leq s \leq t, X_t - X_s \perp \mathcal{F}_s, \mathcal{G}_s; Y_t - Y_s \perp \mathcal{F}_s, \mathcal{G}_s$. So we get $aX_t + bY_t - aX_s - bY_s \perp \mathcal{F}_s, \mathcal{G}_s$, thus $aX_t + bY_t - aX_s - bY_s \perp \sigma(\mathcal{F}_s, \mathcal{G}_s)$ Easily $aX_s + bY_s \in \sigma(\mathcal{F}_s, \mathcal{G}_s)$, so $\mathcal{H}_t \subset \sigma(\mathcal{F}_t, \mathcal{G}, t), \forall t \geq 0$. So $aX_t + bY_t - aX_s - bY_s \perp \mathcal{H}_s$. And easily $a(X_t - X_s) \sim N(0, a^2(t-s)), b(Y_t - Y_s) \sin N(0, b^2(t-s))$, and since $\mathcal{F}_t \perp \mathcal{G}_t$ we get $a(X_t - X_s) \perp b(Y_t - Y_s)$, so $aX_t + bY_t - aX_s - bY_s \sim N(0, (a^2 + b^2)(t-s))$. So $(aX_t + bY_t : t \geq 0)$ is a Brownian motion with parameter $a^2 + b^2 = c^2$. $\square$

PROBLEM IV Assume $(B_t : t \geq 0)$ is standrad Brownian motion start at 0. Let $X_0 = 0$ and $X_t := tB_{\frac{1}{t}}$. Given

$$\limsup_{t \to \infty} \frac{B_t}{\sqrt{2t \log \log t}} = 1$$

Prove that $(X_t : t \geq 0)$ is standrad Brownian motion start at 0.

SOLUTION . $\square$