underGraduate Homework In Mathematics

SetTheory 4

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General fire extinguisher

PROBLEM I Consider $\mathbb{Q} = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) / \sim$, where $(a, b) \sim (c, d) \iff ad = bc$. Define $+_{\mathbb{Q}}, \cdot_{\mathbb{Q}}$ and $<_{\mathbb{Q}}$ and verify that your definitions doesn't depend on the choice of representatives.

SOLUTION. Let $[(a, b)] +_{\mathbb{Q}} [(c, d)] = [(ad + bc, bd)], [(a, b)] \cdot_{\mathbb{Q}} [(c, d)] = [(ac, bd)]$, and $[(a, b)] <_{\mathbb{Q}} [(c, d)] \iff abd^2 < cdb^2$. Now we prove they are well-defined, i.e., doesn't depend on the choice of representatives.

For $+_{\mathbb{Q}}$, assume $(a, b) \sim (e, f)$, we need to prove $(ad + bc, bd) \sim (ed + fc, df)$. Since $af = be$, we have $(ad + bc)bf = ad^2 f + bdcf = bed^2 + bdcf = (ed + fc)bd$. So $+_{\mathbb{Q}}$ is well defined.

For $\cdot_{\mathbb{Q}}$, assume $(a, b) \sim (e, f)$, we need to prove $(ac, bd) \sim (ec, fd)$. Since $af = be$, we have $acfd = bced = bdec$. So $\cdot_{\mathbb{Q}}$ is well defined.

For $<_{\mathbb{Q}}$, assume $(a_1, b_1) \sim (a_2, b_2), (c_1, d_1) \sim (c_2, d_2)$ and $(a_1, b_1) < (c_1, d_1)$. Now we need to prove $(a_2, b_2) < (c_2, d_2)$. Since $a_1 b_2 = a_2 b_1, c_1 d_2 = c_2 d_1$ we get $a_1 b_1 d_2^2 < c_2 d_2 b_1^2$ □

PROBLEM II The set of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ has cardinality $\mathfrak{c}$ (while the set of all functions has cardinality $2^{\mathfrak{c}}$). [A continuous function on $\mathbb{R}$ is determined by its values at rational points.]

SOLUTION. Consider $\theta : {}^{\mathbb{R}}\mathbb{R} \to 2^{\mathbb{Q}}, f \mapsto \{(a, b) \in \mathbb{Q} : f(a) < b\}$. Now we prove f is a injection. Assume $\theta(f) = \theta(g)$, to prove $f = g$. First we prove for $x \in \mathbb{Q}$ we have $f(x) = g(x)$. We have $f(x) = \sup\{y \in \mathbb{Q} : y < f(x)\} = \sup\{y \in \mathbb{Q} : (x, y) \in \theta(f)\} = \sup\{y \in \mathbb{Q} : (x, y) \in \theta(g)\} = g(x)$. For $x \in \mathbb{R}$, choose a sequence $x_n \in \mathbb{Q}$ such that $x_n \to x$, then $f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = g(x)$. So we get $f = g$. So $\mathrm{card}{}^{\mathbb{R}}\mathbb{R} \le \mathrm{card} 2^{\mathbb{Q}} = 2^{\aleph_0}$. Obviously $\mathrm{card}{}^{\mathbb{R}}\mathbb{R} \ge 2^{\aleph_0}$, so we get they are equal. □

PROBLEM III There are at least $\mathfrak{c}$ countable order-types of linearly ordered sets.

SOLUTION. For every sequence $a = \langle a_n : n \in \mathbb{N} \rangle$ of natural numbers consider the ordertype

$$\tau_a = \{(x, y) \in \mathbb{Z} \times \mathbb{N} : 2 \nmid y \vee 0 < x < a_{\frac{y}{2}}\}$$

And for $(x, y), (z, w) \in \tau_a$ we define $(x, y) < (z, w) \iff y < w \vee y = w, x < z$. Now we will show that if $a \ne b$, then $\tau_a \ne \tau_b$. Assume $\tau_a \cong \tau_b$, we need to prove $a = b$. assume $\theta : \tau_a \to \tau_b$ is the isomorfism.

We know $(x, 0)$ can be defined as $\phi(p) = \exists_{k=1}^{x-1} t_k, \vee_{1 \le i < j \le x-1} t_i \ne t_j, \forall k = 1, \cdots x - 1, t_k < p$. And θ is isomorphism. So $\theta(x, 0) = (x, 0)$. For $(x, 1)$, we let b_0 satisfy $\theta(0, 1) = (b_0, m)$. Since the set $\{(x, y) : y = 1\}$ can be defined by $\psi(p) = \forall r, s(r, s < p \vee \tau(r) \vee \tau(s) \to \mathrm{card}[r, s] < \infty)$, where $\tau(r) := \{s : s < r\}$ and $[r, s] = \{y : r < y < s\}$. we get $\theta[\{(x, y) : y = 1\}] = \{(x, y) : y = 1\}$. So we can delete the element whose second coordinary is $0, 1$, and θ is isomorphism, too. Do this repeatedly, we get $\theta(x, 2n + 1) = (x, 2n + 1)$. So $a_n = \mathrm{card}\{(x, 2n + 1) \in \tau_a\} = \mathrm{card}\{(x, 2n + 1) \in \tau_b\} = b_n$ and thus $a = b$. □

PROBLEM IV The set of all algebraic reals is countable.

SOLUTION. Assume $\{f_n : n \in \mathbb{N}\}$ is the set of all integral coefficient polynomial. Consider $A_n := \{x \in \mathbb{C} : f(x) = 0\}$ is finite set. Then we get $\cup_{n\in\mathbb{N}} A_n$ is at most countable. Obviouly $\bigcup_{n\in\mathbb{N}} A_n$ is infinite, so it's countable. □

PROBLEM V If S is a countable set of reals, then $|\mathbb{R} - S| = \mathfrak{c}$. [Use $\mathbb{R} \times \mathbb{R}$ rather than $\mathbb{R}$ (because $|\mathbb{R} \times \mathbb{R}| = 2^{\aleph_0}$).]

SOLUTION. Assume $\theta : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ is bijection, and $T = \theta(S)$. Then T is countable. And $\mathrm{card}(\mathbb{R} \setminus S) = \mathrm{card}(\mathbb{R} \times \mathbb{R} \setminus T)$. So we only need to prove $\mathbb{R} \times \mathbb{R} \approx \mathbb{R} \times \mathbb{R} \setminus T$. Obviously $\mathrm{card}\mathbb{R} \times \mathbb{R} \setminus T \leq \mathrm{card}\mathbb{R} \times \mathbb{R}$, so we only need $\mathbb{R} \times \mathbb{R} \setminus T \geq \mathbb{R}$. Since T is countable, we get $\{x : \exists y, (x, y) \in T\}$ is countable. Choose $t \notin \{x : \exists y, (x, y) \in T\}$. Let $f : \mathbb{R} \to \mathbb{R} \times \mathbb{R} \setminus T, x \mapsto (t, x)$. Easily we get f is injection. So $\mathrm{card}\mathbb{R} \times \mathbb{R} \setminus T = \mathfrak{c}$. $\square$

PROBLEM VI Assume T is a tree.

1. If $s, t, u \in T$, then $R_{stu} := \{\delta_{st}, \delta_{tu}, \delta_{us}\}$ has at most 2 elements. And if $p, q \in R_{stu}$, then $p \subset q \land q \subset p$.

2. $\prec$ is a linear ordering of T which extends $\sqsubset$.