GroupRepresentation 8

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PROBLEM I R is a ring with identity element. If every non zero element in R is inversible, we call R is division ring. Prove: if D is a division ring, then $M_n(D)$ is a monocycle.

SOLUTION . Assume I is a non-zero two-sided ideal of $M_n(D)$, now we only need to prove $I = M_n(D)$. Only need to prove $E_{ij} \in I, \forall i, j$. Since $I \neq \{0\}$, assume $A \in I$ and $A_{st} \neq 0$. Then $\forall i, j, E_{is}AE_{tj} \in I$. i.e., $a_{st}E_{ij} \in I$. So $a_{st}^{-1}I_n a_{st}E_{ij} \in I$, i.e., $E_{ij} \in I$. So $I = M_n(D)$. □

PROBLEM II V is right module of division ring D, let $\mathrm{Hom}_D(V, V)$ is the set of all module isomorphic of V. Given $\dim_D V = n$, prove that $\mathrm{Hom}_D(V, V) \cong M_n(D)$.

SOLUTION . Assume $\{a_1, \cdots, a_n\}$ is a maximal linearly indenpendent set of V. First we prove $\forall x \in V, \exists! d_1, \cdots, d_n \in D$ such that $x = \sum_{k=1}^n d_k a_k$.

Existence: Since A is maximal linearly indenpendent set, we have $A \cup \{x\}$ is not linearly indenpendent. So $\exists t_1, \cdots, t_n, t \in D$ such that $\sum_{k=1}^n t_k a_k + tx = 0$ and $t_1, \cdots, t_n, t$ are not all 0. If $t = 0$, then we get $a_1, \cdots, a_n$ are not linearly indenpendent, contradiction! So $t \neq 0$. Since D is division ring, we get $\exists w \in D$ such that $tw = wt = 1$. Let $d_k = -wt_k, k = 1, \cdots, n$, then $\sum_{k=1}^n d_k a_k = \sum_{k=1}^n -wd_k a_k = wtx = x$.

Uniqueness: If $x = \sum_{k=1}^n d_k a_k = \sum_{k=1}^n t_k a_k$, then $\sum_{k=1}^n (d_k - t_k)a_k = 0$. Since $a_k, k = 1, \cdots, n$ is linearly indenpendent, we get $d_k - t_k = 0, k = 1, \cdots, n$. So the form is unique.

Now let $f_j : V \to D, \sum_{k=1}^n d_k a_k \mapsto d_j$, for $j = 1, \cdots, n$. Then f_k is well-defined. Let $F : \mathrm{Hom}_D(V, V) \to M_n(D), F(\varphi)_{ij} := f_i(\varphi(a_j))$. Now we prove F is isomorphic.

Since $\varphi(\sum_{k=1}^n d_k a_k) = \sum_{k=1}^n d_k \varphi(a_k)$, so $\varphi(\sum_{k=1}^n d_k a_k)_i = \sum_{k=1}^n d_k F(\varphi)_{ik}$. So φ can be reperesented by $F(\varphi)$, so F is injection.

For $A \in M_n(D)$, assume $A = (a_{ij})$. Let $\varphi(\sum_{k=1}^n d_k a_k)_i := \sum_{k=1}^n d_k a_{ik}$. Easily we get $\varphi \in \mathrm{Hom}_D(V, V)$. And $F(\varphi) = A$. So F is surjective.

Easily $F(\varphi\psi)_{ij} = f_i(\varphi\psi a_j) = f_i(\varphi \sum_{k=1}^n f_k(\psi a_j)a_k) = \sum_{k=1}^n F(\varphi)_{ik} F(\psi)_{kj}$. So $F(\varphi\psi) = F(\varphi)F(\psi)$. And obviously $F(a\varphi + b\psi) = aF(\varphi) + bF(\psi)$.

So all in all we get F is a isomorphic. □