PROBLEM I Prove that if $(X_n : n \geq 0)$ is a simple random walk, then so is $(-X_n : n \geq 0)$.

SOLUTION. Let $\xi_n := X_n - X_{n-1}$ for $n \in \mathbb{N}^+$. Then Since $(X_n : n \in \mathbb{N})$ is simple random walk we have $X_0, \xi_1, \xi_2, \cdots$ are independent r.v. ranges in $\mathbb{Z}$, and $\xi_i, i = 1, 2\cdots$ are i.i.d., and $\mathbb{P}(|\xi_i| = 1) = 1$. So we easily get $-X_0, -\xi_1, -\xi_2, \cdots$ are independent r.v. ranges in $\mathbb{Z}$, and $-\xi_i, i = 1, 2, \cdots$ are i.i.d., and $\mathbb{P}(|-\xi_i| = 1) = 1$. Since $-X_n = X_0 + \sum_{k=1}^{n} \xi_k$, by the definition of simple random walk we obtain $(-X_n : n \in \mathbb{N})$ is a simple random walk. $\square$

PROBLEM II Let $(X_n : n \geq 0)$ be a d-dimentional random walk with $\mathbb{P}(|\xi_i| \geq 1) > 0$, prove that $\mathbb{P}(\sup_n |X_n| = \infty) = 1$.

SOLUTION. Let $t \in \mathbb{Z}^d, t \neq 0$ and $\mathbb{P}(\xi_i = t) > 0$. Since $\mathbb{P}(\sup_n |X_n| = \infty) = \inf_{k\in\mathbb{N}} \mathbb{P}(\sup_n |X_n| \geq k)$, we only need to prove $\mathbb{P}(\sup_n |X_n| \geq k) = 1$ for every $k \in \mathbb{N}$. Take $K > 3k, K \in \mathbb{N}$. Let $A_s := \{\xi_i = t : i = sK+1, sK+2, \cdots, sK+K-1\}$. Then for $\omega \in A_s$, we have $|X_{sK+K} - X_{sK}| = |\sum_{u=1}^{K-1} t| = K|t| \geq K \geq 3k$. Then $\sup_n |X_n| \geq \max\{|X_{sK+K}|, |X_{sK}|\} \geq \frac{1}{2}|X_{sK+K} - X_{sK}| \geq k$. So we get $\forall s, A_s \subset \{\sup_n |X_n| \geq k\}$. Since ξ_i are independent, easily get $A_s, s = 1, 2, \cdots$ are independent. Noting $\mathbb{P}(A_s) = \mathbb{P}(\xi_i = t)^K > 0$, we get $\sum_{s\in\mathbb{N}} \mathbb{P}(A_s) = \infty$. So from BC-theorem we get $\mathbb{P}(A_s, i.o.) = 1$, thus $\mathbb{P}(\bigcup_{s\in\mathbb{N}} A_s) = 1$. Thus, $\mathbb{P}(\sup_n |X_n| \geq k) = 1$, for every $k \in \mathbb{N}$. Thus, $\mathbb{P}(\sup_n |X_n| = \infty) = \mathbb{P}(\bigcap_{k\in\mathbb{N}} \{\sup_n |X_n| \geq k\}) = 1$. $\square$

PROBLEM III Let $(S_n : n \geq 0)$ be a symmetry simple random walk with $S_0 = 0$, for $d = 2$, prove that

$$\mathbb{P}(S_{2n} = 0) = \frac{1}{4^{2n}} \left(\frac{(2n)!}{(n!)^2} \right)^2$$

For $d = 3$, prove that

$$\mathbb{P}(S_{2n} = 0) = \frac{1}{2^{2n}} \frac{(2n)!}{(n!)^2} \sum_{i+j+k=n} \left(\frac{1}{3^n} \frac{n!}{i!j!k!} \right)^2$$

SOLUTION. First we consider $d = 2$. Write $\xi_i = S_i - S_{i-1}$. Then we know S_{2n} occur $\iff$ the number of $(1,0)$ and $(-1,0)$ in $\{\xi_i : i = 1, \cdots, 2n\}$, and the number of $(0,1)$ and $(0,-1)$ in $\{\xi_i : i = 1, \cdots, 2n\}$. We assume there is k pairs of $(1,0), (-1,0)$, then easily there is $n-k$ pairs of $(0,1), (0,-1)$. The probability is $\binom{2n}{k}\binom{2n-k}{k}\binom{2n-2k}{n-k}\frac{1}{4^{2n}}$. So the total probability is $\mathbb{P}(S_{2n} = 0) = \sum_{k=0}^{n} \binom{2n}{k}\binom{2n-k}{k}\binom{2n-2k}{n-k}\frac{1}{4^{2n}} = \sum_{k=0}^{n} \frac{(2n)!}{k!k!(n-k)!(n-k)!4^{2n}}$. Noting that $\sum_{k=0}^{n} \frac{(n!)^2}{k!k!(n-k)!(n-k)!} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \binom{2n}{n} = \frac{(2n)!}{n!n!}$, we finally get $\mathbb{P}(S_{2n} = 0) = \frac{1}{4^{2n}} \left(\frac{(2n)!}{n!n!} \right)^2$.

Use the same method, consider $d = 3$, we have

$$\mathbb{P}(S_{2n} = 0) = \sum_{i+j+k=n} \binom{2n}{i}\binom{2n-i}{i}\binom{2n-2i}{j}\binom{2n-2i-j}{j}\binom{2n-2i-2j-k}{k}\frac{1}{6^{2n}}$$

So easily to get $\mathbb{P}(S_{2n} = 0) = \frac{1}{2^{2n}} \frac{(2n)!}{(n!)^2} \sum_{i+j+k=n} \left(\frac{1}{3^n} \frac{n!}{i!j!k!} \right)^2$. $\square$

PROBLEM IV Assume $(S_n : n \geq 0)$ is a symmetry simple random walk with $S_0 = i \in \mathbb{Z}$. Prove that $\forall a \in \mathbb{Z}$, let $\tau_a := \min\{n \in \mathbb{N} : S_n = a\}$, then $\mathbb{P}(\tau_a < \infty) = 1$.

Solution. Without loss of generality assume $a < 0, i = 0$. Take $N \in \mathbb{N}^+$. Consider $\tau := \min\{n \in \mathbb{N} : S_n = a \vee S_n = N\}$. From Problem II we can easily know $\mathbb{P}(\tau < \infty) = 1$ because $\{\sup_n |S_n| = \infty\} \subset \{\tau < \infty\}$, a.s. So we get $\{\tau_a = \tau\} \subset \{\tau_a < \infty\}$, a.s. Let $Y_n := S_{n \wedge \tau} := S_{\min\{n,\tau\}}$. Easily $(S_n : n \in \mathbb{N})$ is a martingale, and τ is a stopping time, so we get $(Y_n : n \in \mathbb{N})$ is a martingale, too. And easily $Y_n \in [a, N]$, so Y_n is bounded. So we get $\mathbb{E}(S_\tau) = \lim_{n\to\infty} \mathbb{E}(Y_n) = \mathbb{E}(Y_0) = 0$. Easily to know $\mathbb{E}(S_\tau) = \mathbb{P}(\tau = \tau_a)a + \mathbb{P}(\tau \neq \tau_a)N = 0$. And $\mathbb{P}(\tau = \tau_a) + \mathbb{P}(\tau \neq \tau_a) = 1$, so easily $\mathbb{P}(\tau = \tau_a) = \frac{N}{N-a}$. So $\mathbb{P}(\tau_a < \infty) \geq \frac{N}{N-a}$. Let $N \to \infty$, we get $\mathbb{P}(\tau_a < \infty) = 1$. $\square$