SET THEORY

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1 Question

PROBLEM I. Using only $\hat{\in}$ and $\hat{=}$ to express the following formulas

1. $z \hat{=} ((x, y), (u, v))$
2. $\forall x[\neg(x \hat{=} \varnothing) \to (\exists y \hat{\in} x)(x \cap y \hat{=} \varnothing)]$
3. $\forall u[\forall x \exists y (x, y) \hat{\in} u \to \exists f \forall x (x, f(x)) \hat{\in} u]$

SOLUTION.

1. $\forall a(a \in z \leftrightarrow ((\forall b(b \in a \leftrightarrow (\forall c(c \in b \leftrightarrow (\forall d(d \in c \leftrightarrow d = x) \vee (\forall d(d \in c \leftrightarrow (d = x \vee d = y)))))))) \vee (\forall b(b \in a \leftrightarrow ((\forall c(c \in b \leftrightarrow (\forall d(d \in c \leftrightarrow d = x) \vee (\forall d(d \in c \leftrightarrow (d = x \vee d = y)))))) \vee (\forall c(c \in b \leftrightarrow (\forall d(d \in c \leftrightarrow d = u) \vee (\forall d(d \in c \leftrightarrow (d = u \vee d = v)))))))))))$
2. $\forall x(\neg(\forall u \neg(u \in x)) \to \neg(\forall y \neg(y \in x \wedge \forall u(u \in x \to \neq (u \in y)))))$
3. $\forall u(\forall x \exists y (x, y) \in u \to \exists f((\forall x \exists y((x, y) \in f \wedge \forall z((x, z) \in f \to z = y)))\forall x \forall y((x, y) \in f \to (x, y) \in u)))$

$\square$

PROBLEM II. Suppose that R, S are two binary relations (as sets). Show that R_{-1} and $S \circ R$ exist, where

$$S \circ R = \{(x, z) \mid \exists y((x, y) \in R \wedge (y, z) \in S)\}$$

SOLUTION. since $A = \mathrm{dom}(R), B = \mathrm{ran}(R)$ exists, we can get $(x, y) \in R_{-1} \iff (y, x) \in R \Rightarrow y \in \mathrm{ran}(R) \wedge x \in \mathrm{dom}(R) \iff (x, y) \in \mathrm{ran}(R) \times \mathrm{dom}(R)$, so we get $R_{-1} \subset \mathrm{ran}(R) \times \mathrm{dom}(R)$.

For the same reason we can easily get $S \circ R \subset \mathrm{dom}(R) \times \mathrm{ran}(S)$.

So from axiom 2 we finially get $R_{-1}, R \circ S$ are sets. $\square$

PROBLEM III. There is no set X such that $\mathscr{P}(X) \subseteq X$.

SOLUTION. If there is such X, we consider the set $Y := \{x \in X : x \notin x\} \subset X$. If $Y \in Y$, then we get $Y \notin Y$. If $Y \notin Y$, since $Y \in \mathscr{P}(X) \subset X$ we get $Y \in Y$. So it's a contradiction. So there is no such X. $\square$

PROBLEM IV. If X is inductive, then the set

$$\{x \in X \mid x \subseteq X\}$$

is inductive. Hence N is transitive, and for each n, $n = \{m \in N \mid m < n\}$.

SOLUTION. Let $x \in X \wedge x \subset X$. Since X is inductive, we get $x \cup \{x\} \in X$. Since $x \in X$ we get $\{x\} \subset X$. So $x \cup \{x\} \subset X$. So $\{x \in X : x \subset X\}$ is inductive. Obviously $\emptyset \in \{x \in X : x \subset X\}$, so by MI we can get $\mathbb{N} \subset \{x \in \mathbb{N} : x \subset \mathbb{N}\}$, so $\mathbb{N} = \{x \in \mathbb{N} : x \subset \mathbb{N}\}$. i.e., $\mathbb{N}$ is transitive.

Since $\forall m, n \in \mathbb{N}(m < n \leftrightarrow m \in n)$, so $\{m \in \mathbb{N} : m < n\} \subset n$. Since $\mathbb{N}$ is transitive, so $m \in n \to m \in \mathbb{N}$, so $n \subset \{m \in \mathbb{N} : m < n\}$. So finally we get $n = \{m \in \mathbb{N} : m < n\}$. $\square$

PROBLEM V. If X is inductive, then the set

$$\{x \in X \mid x \text{ is transitive }\}$$

is inductive. Hence every $n \in N$ is transitive.

SOLUTION. Use $\tau(x)$ to reperesent x is transitive. Let $x \in X \wedge \tau(x)$, consider $x \cup \{x\}$. Let $y \in x \cup \{x\}$, if $y = x$, then $y \subset x \cup \{x\}$; else, $y \in x$, since x is transitive, so $y \subset x \subset x \cup \{x\}$. Then we get $\tau(x \cup \{x\})$. So $\{x \in X : \tau(x)\}$ is inductive.

For $\mathbb{N}$ we have $\{n \in \mathbb{N} : \tau(n)\}$ is inductive, and 0 is transitive, so $\forall n \in \mathbb{N}, \tau(n)$. $\square$

PROBLEM VI. If X is inductive, then the set

$$\{x \in X \mid x \text{ is transitive and } x \notin x\}$$

is inductive. Hence $n \notin n$ and $n \neq n + 1$ for each $n \in N$

SOLUTION. Use $\tau(x)$ to reperesent x is transitive. Let $x \in X \wedge \tau(x) \wedge x \notin x$, consider $x \cup \{x\}$. Let $y \in x \cup \{x\}$, if $y = x$, then $y \subset x \cup \{x\}$; else, $y \in x$, since x is transitive, so $y \subset x \subset x \cup \{x\}$. Then we get $\tau(x \cup \{x\})$. If $x \cup \{x\} \in x \cup \{x\}$, then $x \cup \{x\} \in x \vee x \cup \{x\} = x$. Since $x \notin x$ so $x \cup \{x\} \neq x$, so $x \cup \{x\} \in x$. Since x is transitive, so $x \cup \{x\} \subset x$. But $x \notin x$, so it's impossible. So $\{x \in X : \tau(x) \wedge x \notin x\}$ is inductive.

For $\mathbb{N}$, we get $\{n \in \mathbb{N} : \tau(n) \wedge x \notin x\}$ is inductive, so $\mathbb{N} = \{n \in \mathbb{N} : \tau(n) \wedge x \notin x\}$, so $n \in \mathbb{N} \to n \notin n$. And since $n + 1 = n \cup \{n\}$, we get $n + 1 \neq n$. $\square$

PROBLEM VII. If X is inductive, then the set $\{x \in X \mid x$ is transitive and every nonempty

$$z \subseteq x \text{ has an } \in\text{-minimal element } \}$$

is inductive. (t is $\in$-minimal in z if there is no $s \in z$ such that $s \in t$.)

SOLUTION. Use $\tau(x)$ to reperesent x is transitive, use $\phi(x)$ to reperesent every nonempty $z \subset x$ has an $\in -$minimal element. Let $x \in X \wedge \tau(x) \wedge \phi(x)$, consider $x \cup \{x\}$. If $x \in x$ then $x \cup \{x\} = x$ and thus $x \cup \{x\} \in \{y \in X : \tau(y) \wedge \phi(y)\}$. Now we assume $x \notin x$.

Let $y \in x \cup \{x\}$, if $y = x$, then $y \subset x \cup \{x\}$; else, $y \in x$, since x is transitive, so $y \subset x \subset x \cup \{x\}$. Then we get $\tau(x \cup \{x\})$. Consider $z \subset x \cup \{x\} \wedge z \neq \emptyset$. If $z = \{x\}$, since $x \notin x$ so x is the $\in -$minimal element of z. Else, consider $t = z \setminus \{x\} \subset x$, and $u \in t$ is $\in -$minimal element of t. Since $u \in x \wedge \tau(x)$ we get $u \subset x$. And $x \notin x \to x \notin u$. So u is $\in -$minimal element of $t \cup \{x\}$, too. So u is $\in -$minimal element of z. So $\{x \in X : \tau(x) \wedge \phi(x)\}$ is inductive. □

PROBLEM VIII. Every nonempty $X \subseteq \mathbb{N}$ has an $\in$-minimal element.

SOLUTION. Consider $X \subset \mathbb{N} \wedge X \neq \emptyset$. Assume $n \in X$. Consider $Y := n \cap X \subset n$. If $Y = \emptyset$, then n is the $\in -$minimal element of X. Else, from Problem VII we know Y has $\in -$minimal element u. Now we just need to prove u is $\in -$minimal element of X. If not, $\exists m \in X, m \in u$, then since $u \in n$ we get $m \in u \subset n$, so $m \in n$ and thus $m \in Y$, contradiction with u is $\in -$minimal element of Y. □

PROBLEM IX. If X is inductive then so is

$$\{x \in X \mid x = \varnothing \vee x = y \cup \{y\} \text{ for some } y\}.$$

Hence each $n \neq \varnothing$ is $m + 1$ for some m.

SOLUTION. Consider $x \in X \wedge (x = \emptyset \vee (\exists z x = z \cup \{z\}))$, then for $y = x \cup \{x\}$ we have $y \in X \wedge y = x \cup \{x\}$, thus $y \in \{x \in X : x = \emptyset \vee (\exists y \in X) x = y \cup \{y\}\}$. So the set is inductive, too.

For $\mathbb{N}$ we get $\forall n \in \mathbb{N}, n = \emptyset \vee \exists m \in \mathbb{N}, n = m \cup \{m\}$. If $n = \emptyset$ then $n = 0$. If $n = m \cup \{m\}$ then $m \in n$. Since $\mathbb{N}$ is transitive, we get $m \in n \subset \mathbb{N}$. So $m \in \mathbb{N}$. So $n = m \cup \{m\} = m + 1$. □

PROBLEM X. Let A be a subset of N such that $0 \in A$, and if $n \in A$ then $n + 1 \in A$. Then $A = N$.

SOLUTION. Obviously A is inductive, and $\mathbb{N}$ is the least inductive set, so $\mathbb{N} \subset A$. Noting $A \subset \mathbb{N}$, so $A = \mathbb{N}$. □