## under Graduate Homework In Mathematics

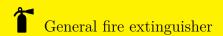
## RiemannGeometry 1

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 $(T,\tau)$  is compatible locally on x with both  $(U,\phi)$  and  $(V,\psi)$ . So  $(U,\phi)$  is locally compatible with  $(V,\psi)$  on x. Since x is arbitrary, we know  $(U,\phi)$  is compatible with  $(V,\psi)$ . So  $\mathcal A$  is differential structure of M. Now we assume  $\mathcal{B}$  is another differential structure of M contains  $\mathcal{A}_0$ . Since  $\mathcal{B}$  is compatible,

 $\mathbb{R}^{OBEM}$  I Assume  $\mathcal{A}_0 = \{(U_{\alpha}, \phi_{\alpha}) : \alpha \in I\}$  is a  $C^r$ -compatible coordinate covery of a m-

 $\mathcal{A} := \{(U, \phi) : (U, \phi) \text{ is chart of } M, \land \forall (V, \psi) \in \mathcal{A}_0, (U, \phi) \text{ is compatible with } (V, \psi)\}$ 

SOLION. First, easily  $A_0 \subset A$  by definition of  $A_0$ . Now we should prove A is differential structure on M. Let  $(U, \phi), (V, \psi) \in \mathcal{A}$ . If  $(U, \phi) \in \mathcal{A}_0$ , then by definition of  $\mathcal{A}_0$  we know  $(U, \phi)$  is compatible to  $(V, \psi)$ . If  $(U, \phi), (V, \psi) \notin \mathcal{A}_0$ , then consider  $U \cap V$ . If  $U \cap V = \emptyset$ , then  $(U, \phi)$  is compatible with  $(V,\psi)$ . Now assume  $W=U\cap V\neq\varnothing$ . Consider  $\gamma:=\psi\circ\phi^{-1}:\phi(W)\to\psi(W)$ . For any  $x\in W$ , since  $\mathcal{A}_0$  is covery of M, we know  $\exists (T,\tau) \in \mathcal{A}_0, x \in T$ . Then by the definition of  $\mathcal{A}$ , we know

. Then  $\mathcal{A}$  is unique  $C^r$ -differential structure on M contains  $\mathcal{A}_0$ .

we get  $\mathcal{B} \subset \mathcal{A}$ . Since  $\mathcal{B}$  is maximal, we get  $\mathcal{B} = \mathcal{A}$ . So  $\mathcal{A}$  is unique.

dimensional manifold M, let

. So easily to get that

 $\left(\frac{\partial z^i}{\partial x^j}\right) = \left(\frac{\partial z^i}{\partial y^k}\right) \left(\frac{\partial y^k}{\partial x^j}\right)$ SOLUTION. For fixed  $1 \leq i, j \leq m$ , we have

POBEM II Assume  $(U, \phi; x^i), (V, \psi; y^i), (W, \chi; z^i)$  are three local coordinate on an m-dimensional

smooth manifold M, and  $W \cap V \cap U \neq \emptyset$ . Prove that on  $\phi(U \cap V \cap W)$ , we have:

$$\frac{\partial z^{i}}{\partial x^{j}} = \sum_{k=1}^{m} \frac{\partial z^{i}}{\partial y^{k}} \frac{\partial y^{k}}{\partial x^{j}}$$
$$\left(\frac{\partial z^{i}}{\partial x^{j}}\right) = \left(\frac{\partial z^{i}}{\partial y^{k}}\right) \left(\frac{\partial y^{k}}{\partial x^{j}}\right)$$

We let  $(W, \chi; z^i) = (U, \phi; x^i)$ , then we get:

 $I_m = \left(\frac{\partial x^i}{\partial y^k}\right) \left(\frac{\partial y^k}{\partial x^j}\right)$ 

. So both terms on the right side are invertible, thus non-singular.  $\mathbb{R}^{OBEM}$  III Assume M is orientable and connected, prove that M has exactly two different orien-

tion.

SOLTION. Since M is orientable, we can assume that  $\mathcal{B} \subset \mathcal{A}$  is an oriention of M, where  $\mathcal{A}$  is all local coordinate of M. Now consider  $\mathcal{C} := \{(U; -x^i) : (U; x^i) \in \mathcal{B}\}$ . Easily to check that  $\mathcal{C}$  is an

oriention of M, too. And obviously  $\mathcal{B} \cap \mathcal{C} = \emptyset$ , thus  $\mathcal{B} \neq \mathcal{C}$ . So there is two oriention. Now we need to prove there is no other oriention.