underGraduate Homework In Mathematics

SetTheory 4

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General fire extinguisher

PROBLEM I Consider $\mathbb{Q} = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})/\sim$, where $(a,b) \sim (c,d) \iff ad = bc$. Define $+_{\mathbb{Q}}, \cdot_{\mathbb{Q}}$ and $<_{\mathbb{Q}}$ and verify that your definitions doesn't depend on the choice of representatives.

SOLUTION . Let $[(a,b)] +_{\mathbb{Q}} [(c,d)] = [(ad+bc, bd)], [(a,b)] \cdot_{\mathbb{Q}} [(c,d)] = [(ac,bd)]$, and $[(a,b)] <_{\mathbb{Q}} [(c,d)] \iff abd^2 < cdb^2$. Now we prove they are well-defined, i.e., doesn't depend on the choice of representatives.

For $+_{\mathbb{Q}}$, assume $(a,b) \sim (e,f)$, we need to prove $(ad+bc, bd) \sim (ed+fc, df)$. Since $af = be$, we have $(ad+bc)bf = ad^2f + bdcf = bed^2 + bdcf = (ed+fc)bd$. So $+_{\mathbb{Q}}$ is well defined.

For $\cdot_{\mathbb{Q}}$, assume $(a,b) \sim (e,f)$, we need to prove $(ac,bd) \sim (ec,fd)$. Since $af = be$, we have $acfd = bced = bdec$. So $\cdot_{\mathbb{Q}}$ is well defined.

For $<_{\mathbb{Q}}$, assume $(a_1,b_1) \sim (a_2,b_2), (c_1,d_1) \sim (c_2,d_2)$ and $(a_1,b_1) < (c_1,d_1)$. Now we need to prove $(a_2,b_2) < (c_2,d_2)$. Since $a_1b_2 = a_2b_1, c_1d_2 = c_2d_1$ we get $a_1b_1d_2^2 < c_2d_2b_1^2$ □

PROBLEM II The set of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ has cardinality $\mathfrak{c}$ (while the set of all functions has cardinality $2^{\mathfrak{c}}$). [A continuous function on $\mathbb{R}$ is determined by its values at rational points.]

SOLUTION . Consider $\theta : {}^{\mathbb{R}}\mathbb{R} \to 2^{\mathbb{Q}}, f \mapsto \{(a,b) \in \mathbb{Q} : f(a) < b\}$. Now we prove f is a injection. Assume $\theta(f) = \theta(g)$, to prove $f = g$. First we prove for $x \in \mathbb{Q}$ we have $f(x) = g(x)$. We have $f(x) = \sup\{y \in \mathbb{Q} : y < f(x)\} = \sup\{y \in \mathbb{Q} : (x,y) \in \theta(f)\} = \sup\{y \in \mathbb{Q} : (x,y) \in \theta(g)\} = g(x)$. For $x \in \mathbb{R}$, choose a sequence $x_n \in \mathbb{Q}$ such that $x_n \to x$, then $f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = g(x)$. So we get $f = g$. So $\mathrm{card}^{\mathbb{R}}\mathbb{R} \le \mathrm{card} 2^{\mathbb{Q}} = 2^{\aleph_0}$. Obviously $\mathrm{card}^{\mathbb{R}}\mathbb{R} \ge 2^{\aleph_0}$, so we get they are equal. □

PROBLEM III There are at least $\mathfrak{c}$ countable order-types of linearly ordered sets.

SOLUTION . For every sequence $a = \langle a_n : n \in \mathbb{N}\rangle$ of natural numbers consider the ordertype

$$\tau_a = \{(x,y) \in \mathbb{Z} \times \mathbb{N} : 2 \nmid y \vee 0 < x < a_{\frac{y}{2}}\}$$

And for $(x,y),(z,w) \in \tau_a$ we define $(x,y) < (z,w) \iff y < w \vee y = w, x < z$. Now we will show that if $a \neq b$, then $\tau_a \neq \tau_b$. Assume $\tau_a \cong \tau_b$, we need to prove $a = b$. assume $\theta : \tau_a \to \tau_b$ is the isomorfism.

We know $(x,0)$ can be defined as $\phi(p) = \exists_{k=1}^{x-1} t_k, \vee_{1\le i<j\le x-1} t_i \neq t_j, \forall k = 1, \cdots x-1, t_k < p$. And θ is isomorphism. So $\theta(x,0) = (x,0)$. For $(x,1)$, we let b_0 satisfy $\theta(0,1) = (b_0, m)$. Since the set $\{(x,y) : y = 1\}$ can be defined by $\psi(p) = \forall r, s(r,s < p \vee \tau(r) \vee \tau(s) \to \mathrm{card}[r,s] < \infty)$, where $\tau(r) := \{s : s < r\}$ and $[r,s] = \{y : r < y < s\}$. we get $\theta[\{(x,y) : y = 1\}] = \{(x,y) : y = 1\}$. So we can delete the element whose second coordinary is $0, 1$, and θ is isomorphism, too. Do this repeatedly, we get $\theta(x, 2n+1) = (x, 2n+1)$. So $a_n = \mathrm{card}\{(x,2n+1) \in \tau_a\} = \mathrm{card}\{(x,2n+1) \in \tau_b\} = b_n$ and thus $a = b$. □

PROBLEM IV The set of all algebraic reals is countable.

SOLUTION . □

PROBLEM V If S is a countable set of reals, then $|\mathbb{R} - S| = \mathfrak{c}$. [Use $\mathbb{R} \times \mathbb{R}$ rather than $\mathbb{R}$ (because $|\mathbb{R} \times \mathbb{R}| = 2^{\aleph_0}$).]