GroupRepresentation 6

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PROBLEM I Assume H_1 and H_2 are sub module of left module M over ring R, prove that $H_1 + H_2$ is direct sum if and only if $H_1 \cap H_2 = \{0\}$.

SOLUTION. $\Rightarrow$: If $H_1 \cap H_2 \, neq\{0\}$, assume $a \in H_1 \cap H_2 \setminus \{0\}$. Then $a = a + 0 = 0 + a$ is two different method to compenent a into sum of elements in H_1, H_2. Contridiction!

$\Leftarrow$: Assume $a = x_1 + y_1 = x_2 + y_2, x_1, y_1 \in H_1, x_2, y_2 \in H_2, a \in H_1 + H_2$. Then $H_1 \ni x_1 - x_2 = y_1 - y_2 \in H_2$, so $x_1 - x_2 = y_1 - y_2 \in H_1 \cap H_2 = \{0\}$, so $x_1 = x_2, y_1 = y_2$. $\square$

PROBLEM II Assume M is R-module and $M = H_1 \oplus H_2$, prove $M/H_1 \cong H_2$ and $M/H_2 \cong H_1$.

SOLUTION. By symmetry we only need to prove $M/H_2 \cong H_1$. Consider $f : M \to H_2, h_1 + h_2 \mapsto h_2$, where $h_1 \in H_1, h_2 \in H_2$. Obviously f is surjective homomorism, so $M/\ker f \cong H_2$. So we only need to prove $\ker f = H_1$. On one hand, for $h_2 \in H_2$ we have $h_2 = 0 + h_2$, so $f(h_2) = 0, h_2 \in \ker f$. On the other hand, assume $h = h_1 + h_2 \in \ker f$, then $f(h) = h_1 = 0$, so $h = h_2 \in H_2$. So $\ker f = H_2$. So we get $M/H_2 \cong H_1$. $\square$

PROBLEM III Assume M is R-module and $M = \oplus_{k=1}^{s} H_k$. Prove: $M/(\oplus_{k=2}^{s} H_k) \cong H_1$.

SOLUTION. Noting PROBLEM II we only need to prove $M = (\oplus_{k=2}^{s} H_k) \oplus H_1$. Obviously $H_1 \oplus H_2 \oplus H_3 = H_1 \oplus (H_2 \oplus H_2)$, i.e., $\oplus$ has commulative law and associative law. So $M = \oplus_{k=1}^{s} H_k = (\oplus_{k=2}^{s} H_k) \oplus H_1$. So finally $M / \oplus_{k=2}^{s} H_k \cong H_1$. $\square$