ALGEBRAIC GEOMETRY

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PROBLEM I X is a topology space. Prove that X is Noetherian $\iff$ every subspace of X is compact.

SOLUTION. $\Rightarrow$: Obviously subspace of Noetherian space is Noetherian space, so we only need to prove Noetherian space is compact. If X is not compact, then $\exists v_n \subset X$ such that v_n is closed and $\bigcap_{k=1}^{n} v_k \neq \emptyset$ and $\bigcap_{k=1}^{\infty} v_k = \emptyset$. Then $u_n := \bigcap_{k=1}^{n} v_n$ is infinitly descending chain of closed sets, and since $\bigcap_{k=1}^{\infty} u_k = \bigcap_{k=1}^{\infty} v_k = \emptyset$ so it's not finally stable.

$\Leftarrow$: If X is not Noetherian, assume $\{u_n\} \subset \mathscr{P}(x)$ is a chain of closed sets such that $u_{n+1} \subsetneqq u_n$. Let $u := \bigcap_{k=1}^{\infty} u_k$, consider $X \setminus u \subset X$. $v_n := u_n \setminus u = u_n \cap (X \setminus u)$ is closed set in $X \setminus u$, but $\bigcap_{k=1}^{\infty} v_n = (\bigcap_{k=1}^{\infty} u_n) \setminus u = u \setminus u = \emptyset$. So $X \setminus u$ is not compact, contradiction! □

PROBLEM II Given X Noetherian, $A \subset X$, $A = \bigcup_{k=1}^{n} u_k = \bigcup_{k=1}^{m} v_k$ and u_k, v_k is irreducible non-empty closed set, $u_i \not\subset u_j, v_i \not\subset v_j$. Prove that $m = n$, and $\exists \sigma \in S_n, \forall k \in \{1, 2, \cdots n\}, u_k = v_{\sigma(k)}$.

Lemma 1. If u is irreducible, $u \subset \bigcap_{k=1}^{n} v_k$, where v_k is closed, then if $u \cap v_i \neq \emptyset$, then $u \subset v_i$. And $\exists k, u \in v_k$.

证明. Assume $u \cap v_1 \neq \emptyset$. If $u \not\subset v_1$, then $u = (u \cap v_1) \cup (u \cap \bigcup_{k=2}^{n} v_k)$, contradiction with u is irreducible. So $u \subset v_1$.

If $u = \emptyset$ it's obvious $u \subset v_1$. If not, then $u \cap v_k, k = 1, 2 \cdots n$ can't be all empty. So $u \subset v_k$ for some k. □

SOLUTION. From Lemma 1 we know $\forall i, \exists j, u_i = v_j$. If $u_i = v_j = v_k$ then $v_j \subset v_k$ thus $j = k$. So $\forall i, \exists! j, u_i = v_j$. Let $\phi : \{1, 2, \cdots n\} \to \{1, 2, \cdots m\}, i \mapsto j$. Then ϕ is a map. If $\phi(i) = \phi(j)$ then $u_i = u_j$, thus $i = j$. So ϕ is injection. Consider v_j, since $v_j \subset \bigcup_{k=1}^{n} u_k$, so from Lemma 1 we know $\exists i, u_i = v_j$. So $\phi(i) = j$. Thus ϕ is bijection. So $m = n, \phi \in S_n$. Let $\sigma = \phi \in S_n$ satisfy the condition. □

PROBLEM III K is a field, prove that $(xy - 1)$ is prime ideal of $K[x, y]$.

SOLUTION. Only need to prove $K[x,y]/(xy-1)$ is integral domain. Consider homeomorphism $\phi : K[x,y] \to K[x,x^{-1}], f(x,y) \mapsto f(x,x^{-1})$. Obviously $\ker(\phi) \ni xy-1$, now we prove $\ker(\phi) = (xy-1)$. $\forall f(x,y) \in \ker(\phi)$, we have $f(x,x^{-1}) = 0$. Obviously we can get $f(x,y) = (xy-1)g(x,y) + l(x) + m(y), g(x,y) \in K[x,y], l(x) \in K[x], m(y) \in K[y]$. So $f(x,x^{-1}) = l(x) + m(x^{-1}) = 0$. Since $x^n, n \in \mathbb{Z}$ is linear independent, so $l(x), m(x^{-1}) \in K, l+m = 0$. So $l(x) + m(y) = l+m = 0$. So $f(x,y) \in (xy-1)$.

So $\ker(\phi) = (xy-1)$, and thus $K(x,y)/(xy-1) \cong K[x,x^{-1}]$. So $K[x,y]/(xy-1)$ is integral domain and $(xy-1)$ is prime. □

PROBLEM IV Let k be a field, $I = (xy-1)$, $C = V(I) \in \mathbb{A}_k^2$. Show that $I(C) = (xy-1)$.

SOLUTION. Obviously $(xy-1) \subset I(C)$, so we only need to prove $I(C) \subset (xy-1)$. Consider $f(x,y) \in I(C)$, we get $f(t,t^{-1}) = 0, \forall x \in \mathbb{R} \setminus \{0\}$. For n large enough we know $x^n f(x,x^{-1}) \in \mathbb{R}[x]$, and it has infinite roots, so $x^n f(x,x^{-1}) = 0$. Thus $f(x,x^{-1}) = 0$. From Problem III we know $f \in \ker(\phi) = (xy-1)$. So $(xy-1) = I(C)$. □

PROBLEM V If X is Noetherian space, $Y \subset X$, then $\dim Y \le \dim X$.

Lemma 2. If X is Noetherian, $Y \subset X$, and $u \subset Y$ is irreducible closed set in Y, then exists least closed set $v \subset X$ such that $u \subset v$, and v is irreducible in X.

证明. Let $\mathcal{W} := \{w \subset X : u \subset w, w \text{ is irreducible closed set in } X\}$, and $v := \bigcap_{w \in \mathcal{W}} w$, we will prove v can satisfy the given condition. Obviously v is closed. Assuming $v = \bigcup_{k=1}^n v_k$, where v_k's are irreducible, then $u = \bigcup_{k=1}^n (v_k \cap Y)$. Since u is irreducible in Y, we get $u \subset v_k \cap Y$ for some k. So $v_k \in \mathcal{W}$, and then $v \subset v_k$. So $v = v_k$ and thus irreducible. Forall closed set $t \subset X$ such that $u \subset t$, there exists a irreducible closed set $s \subset X$ such that $u \subset s \subset t$. So $s \in \mathcal{W}$ and thus $v \subset s \subset t$. So v is the least. □

SOLUTION. Consider irreducible ascending chain $Y_1 \subsetneqq Y_2 \subsetneqq \cdots Y_n$ in Y. From Lemma 2 we know exists least irreducible closed $X_k \in X$ such that $Y_k \subset X_k$. Form the minimality of X_k we easily get $X_k \subset X_{k+1}$. Since Y_k is closed in Y we get $\exists V_k \subset X$ and V_k is closed and $Y_k = Y \cap V_k$. From the minimality of X_k we get $X_k \subset V_k$ and thus $X_k \cap Y = Y_k$. So $X_k \cap Y \subsetneqq X_{k+1} \cap Y$, thus $X_k \subsetneqq X_{k+1}$. So $\dim X \ge \dim Y$. □