# underGraduate Homework In Mathematics

## RiemannGeometry 1


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2024 年 11 月 28 日

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PROBLEM I Assume $\mathcal{A}_0 = \{(U_\alpha, \phi_\alpha) : \alpha \in I\}$ is a $C^r$-compatible coordinate covery of a $m$-dimensional manifold $M$, let

$$\mathcal{A} := \{(U,\phi) : (U,\phi) \text{ is chart of } M, \wedge \forall (V,\psi) \in \mathcal{A}_0, (U,\phi) \text{ is compatible with } (V,\psi)\}$$

. Then $\mathcal{A}$ is unique $C^r$-differential structure on $M$ contains $\mathcal{A}_0$.

SOLUTION . First, easily $\mathcal{A}_0 \subset \mathcal{A}$ by definition of $\mathcal{A}_0$. Now we should prove $\mathcal{A}$ is differential structure on $M$. Let $(U,\phi), (V,\psi) \in \mathcal{A}$. If $(U,\phi) \in \mathcal{A}_0$, then by definition of $\mathcal{A}_0$ we know $(U,\phi)$ is compatible to $(V,\psi)$. If $(U,\phi), (V,\psi) \notin \mathcal{A}_0$, then consider $U \cap V$. If $U \cap V = \varnothing$, then $(U,\phi)$ is compatible with $(V,\psi)$. Now assume $W = U \cap V \neq \varnothing$. Consider $\gamma := \psi \circ \phi^{-1} : \phi(W) \to \psi(W)$. For any $x \in W$, since $\mathcal{A}_0$ is covery of $M$, we know $\exists (T,\tau) \in \mathcal{A}_0, x \in T$. Then by the definition of $\mathcal{A}$, we know $(T,\tau)$ is compatible locally on $x$ with both $(U,\phi)$ and $(V,\psi)$. So $(U,\phi)$ is locally compatible with $(V,\psi)$ on $x$. Since $x$ is arbitrary, we know $(U,\phi)$ is compatible with $(V,\psi)$. So $\mathcal{A}$ is differential structure of $M$.

Now we assume $\mathcal{B}$ is another differential structure of $M$ contains $\mathcal{A}_0$. Since $\mathcal{B}$ is compatible, we get $\mathcal{B} \subset \mathcal{A}$. Since $\mathcal{B}$ is maximal, we get $\mathcal{B} = \mathcal{A}$. So $\mathcal{A}$ is unique. □

PROBLEM II Assume $(U,\phi; x^i), (V,\psi; y^i), (W,\chi; z^i)$ are three local coordinate on an $m$-dimensional smooth manifold $M$, and $W \cap V \cap U \neq \varnothing$. Prove that on $\phi(U \cap V \cap W)$, we have:

$$\left(\frac{\partial z^i}{\partial x^j}\right) = \left(\frac{\partial z^i}{\partial y^k}\right)\left(\frac{\partial y^k}{\partial x^j}\right)$$

.

SOLUTION . For fixed $1 \le i, j \le m$, we have

$$\frac{\partial z^i}{\partial x^j} = \sum_{k=1}^{m} \frac{\partial z^i}{\partial y^k}\frac{\partial y^k}{\partial x^j}$$

. So easily to get that

$$\left(\frac{\partial z^i}{\partial x^j}\right) = \left(\frac{\partial z^i}{\partial y^k}\right)\left(\frac{\partial y^k}{\partial x^j}\right)$$

.

We let $(W,\chi; z^i) = (U,\phi; x^i)$, then we get:

$$I_m = \left(\frac{\partial x^i}{\partial y^k}\right)\left(\frac{\partial y^k}{\partial x^j}\right)$$

. So both terms on the right side are invertible, thus non-singular. □

PROBLEM III Assume $M$ is orientable and connected, prove that $M$ has exactly two different oriention.

SOLUTION . Since $M$ is orientable, we can assume that $\mathcal{B} \subset \mathcal{A}$ is an oriention of $M$, where $\mathcal{A}$ is all local coordinate of $M$. Now consider $\mathcal{C} := \{(U; -x^i) : (U; x^i) \in \mathcal{B}\}$. Easily to check that $\mathcal{C}$ is an oriention of $M$, too. And obviously $\mathcal{B} \cap \mathcal{C} = \varnothing$, thus $\mathcal{B} \neq \mathcal{C}$. So there is two oriention. Now we need to prove there is no other oriention.