underGraduate Homework In Mathematics

AlgebraicGeometry 7

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General fire extinguisher

PROBLEM I Assume k is an infinite field. $f \in k[x_0, \cdots, x_n]$. kdn $\forall t \in t, f(tx_0, \cdots, tx_n) = t^d f(x_0, \cdots, x_n)$, where $d \in \mathbb{N}$ is a constant. Prove f is homogeneous.

SOLUTION. Consider $g(t, x_0, \cdots, x_n) = f(tx_0, \cdots tx_n) - t^d f(x_0, \cdots x_n) \in k[t, x_0, \cdots, x_n]$. We get for $p \in \mathbb{A}_k^n, g(p) = 0$. Since k is infinite, we get $g = 0$. Assume $f(x_0, \cdots, x_n) = \sum_i a_i x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$. Then we get $g(t, x_0, \cdots x_n) = \sum_i (t^d - t^{|i|}) x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} = 0$. Where $|i| = \sum_{j=1}^n i_j$. So we get $\sum_{j=1}^n i_j = d$. So f is homogeneous with degree d. □

PROBLEM II For an ideal I of $k[x_0, \cdots, x_n]$, prove following to definition of homogeneous ideal is equivalent.

1. $\forall f \in I, f = \sum_t f_t$, where f_t are homogeneous poly with different degree. then $f_t \in I, \forall t$.
2. $\exists g_1, \cdots g_n$ are homogeneous such that $I = (g_1, \cdots, g_n)$.

SOLUTION.

1. $II.1 \implies II.2$: Assume $I = (f_1, f_2, \cdots, f_n)$ and $f_k = \sum_{t=1}^{a_k} g_{kt}$, where g_{kt} are homogeneous. Then $I = (g_{kt} : k = 1, 2, \cdots, n, t = 1, 2, \cdots, a_k)$.
2. $II.2 \implies II.1$: Assume $I = (g_1, \cdots, g_n)$, where g_k are homogeneous. Now consider $f \in I$. Assume $f = \sum_{j=1}^n h_j g_j$. Assume $h_j = \sum_{i=1}^{b_j} l_{ij}$, where l_{ij} are homogeneous. Then $f = \sum_{j=1}^n \sum_{i=1}^{b_j} l_{ij} g_j$. Assume $l_{ij} g_j$ has degree d_{ij}, then we have $f = \sum_d (\sum_{i,j:d_{ij}=d} l_{ij} g_j)$ is the homogeneous deconposition of f. Easily $\sum_{i,j:d_{ij}=d} l_{ij} g_j \in I$ since $g_j \in I$.

□

PROBLEM III

1. Assume J_λ are homogeneous ideal. Prove that $\mathbb{V}(\sum_\lambda J_\lambda) = \bigcap_\lambda \mathbb{V}(J_\lambda)$.
2. Assume J_1, J_2 are homogeneous ideal, then $\mathbb{V}(J_1 \cap J_2) = \mathbb{V}(J_1) \cup \mathbb{V}(J_2)$.

SOLUTION.

1. On one hand, assume $p \in \mathbb{V}(J_\lambda), \forall \lambda$, to prove $p \in \mathbb{V}(\sum_\lambda J_\lambda)$. Consider $f \in \sum_\lambda J_\lambda, f = \sum_{t=1}^n f_t$, where $f_t \in J_{\lambda_t}$ for some λ_t. Then $f(p) = \sum_{t=1}^n f_t(p) = 0$. So we get $p \in \mathbb{V}(\sum_\lambda J_\lambda)$.

   On the other hand, assume $p \in \mathbb{V}(\sum_\lambda J_\lambda)$, to prove $p \in \mathbb{V}(J_\lambda), \forall \lambda$. Since $J_\lambda \subset \sum_\lambda J_\lambda$, we get $\mathbb{V}(\sum_\lambda J_\lambda) \subset \mathbb{V}(J_\lambda)$. So $p \in \mathbb{V}(J_\lambda)$.
2. On one hand, assume $p \in \mathbb{V}(J_1 \cap J_2)$, to prove $p \in \mathbb{V}(J_1) \vee p \in \mathbb{V}(J_2)$. If not, assuem $f_1 \in J_1, f_2 \in J_2, f_1(p), f_2(p) \neq 0$, then we get $f_1 f_2)(p) \neq 0$. But $f_1 f_2 \in J_1 \cap J_2$, contradiction!

   On the other hand, assume $p \in \mathbb{V}(J_1) \cup \mathbb{V}(J_2)$, to prove $p \in \mathbb{V}(J_1 \cap J_2)$. Without loss of generality we assume $p \in \mathbb{V}(J_1)$, then since $J_1 \cap J_2 \subset J_1$ we get $p \in \mathbb{V}(J_1 \cap J_2)$.

□

PROBLEM IV Assume $C = \{(x, y) \in \mathbb{C}^2 : y^2 = x(x-1)(x-2)\}$ and $\hat{C} = \{[z, x, y] : y^2 z = x(x-z)(x-2z)\} \in \mathbb{P}_{\mathbb{C}}^2$. Prove $\hat{C}$ is one point compactification of C.

Solution. Write $C = \{[z, x, y] : y^2 z = x(x - z)(x - 2z), z \neq 0\} \subset \mathbb{P}^2_{\mathbb{C}}$. Then $\hat{C} \setminus C = \{[z, x, y] : y^2 z = x(x - z)(x - 2z), z = 0\} = \{[0, x, y] : x^3 = 0\} = \{[0, 0, y]\} = \{[0, 0, 1]\}$. So $\hat{C}$ is one point compactification of C. $\square$

Problem V Assume $X \subset \mathbb{P}^n_k$ is an algebraic set, then $\mathbb{V}(\mathbb{I}(X)) = X$.

Solution. Assume $X = \mathbb{V}(J)$ for some homogeneous ideal J. Easily we get $X \subset \mathbb{V}(\mathbb{I}(X))$ and $J \subset \mathbb{I}(\mathbb{V}(J))$. So we get $\mathbb{V}(J) \supset \mathbb{V}(\mathbb{I}(\mathbb{V}(J))) = \mathbb{V}(\mathbb{I}(X))$. So $X = \mathbb{V}(\mathbb{I}(X))$. $\square$

Problem VI Assume J is a homogeneous ideal, then $\sqrt{J}$ is homogeneous ideal, too.

Solution. Consider $g \in \sqrt{J}$ and $g = \sum_{t=1}^{n} g_t$ and g_t are homogeneous poly with different degree. Without loss of generality assume $d_1 < d_2 < \cdots < d_n$, where d_t is the degree of g_t. Now we need to prove $g_t \in \sqrt{J}$. If not, assume j is the least such that $g_j \notin \sqrt{J}$. Let $g' = g - \sum_{t=1}^{j-1} g_t$, then $g' \in \sqrt{J}$ because $g_j \in \sqrt{J}$ for $t < j$. Assume $g'^m \in J$, then $(\sum_{t=j}^{n} g_t)^m \in J$. We consider the homogeneous deconposition of g'^m with degree md_j. Since $d_j < d_{j+1} < \cdots < d_n$, we get the part is g_j^m. Since J is homogeneous, we get $g_j^m \in J$, i.e., $g_j \in \sqrt{J}$, contradiction! So finally we get $\forall t, g_t \in \sqrt{J}$ $\square$

Problem VII Assume J is a homogeneous ideal, then $\sqrt{J} \subset \mathbb{I}(\mathbb{V}(J))$.

Solution. Assume $f \in \sqrt{J}, f^n \in J$. From **??** we know $\sqrt{J}$ is homogeneous, so we only need to prove $\forall p \in \mathbb{V}(J), f(p) = 0$. Since $f^n \in J$ we get $f^n(p) = 0$. So $f(p) = 0$. So we obtain $\sqrt{J} \subset \mathbb{I}(\mathbb{V}(J))$. $\square$