

PROBLEM I Assume $n \in \mathbb{N}^+$ and $2^n + 1$ is prime. Prove that $\exists k \in \mathbb{N}, n = 2^k$.

Lemma 1. Assume $b = ka$ and k is odd, then for $x, y \in \mathbb{N}$, we have $x^a + y^a \mid x^b + y^b$.

证明. Easily $x^b + y^b \equiv (x^a)^k + y^b \equiv (x^a + y^a - y^a)^k + y^b \equiv (-y^a)^k + y^b \equiv 0 \pmod{x^a + y^a}$. So $x^a + y^a \mid x^b + y^b$. \square

SECTION. Assume n is not power of 2, then $\exists p > 2$ is prime such that $p \mid n$. Let $a = \frac{n}{p}$, then from Lemma 1 we have $2^a + 1^a \mid 2^n + 1^n$. Easily $a = \frac{n}{p} < n$, so $2^a + 1 < 2^n + 1$. And easily $1 < 2^a + 1$. So $2^n + 1$ is not prime, contradiction! So $\exists k \in \mathbb{N}, n = 2^k$. \square

PROBLEM II Find the standard decomposition of $30!$.

SECTION. There are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 10 primes, below 30. So we know $30!$ can be break down into power and product of them. By calculation we can get that:

$$\begin{aligned}
 v_2(30!) &= \sum_{k=1}^{\infty} \left[\frac{30}{2^k} \right] = 15 + 7 + 3 + 1 = 26 \\
 v_3(30!) &= \sum_{k=1}^{\infty} \left[\frac{30}{3^k} \right] = 10 + 3 + 1 = 14 \\
 v_5(30!) &= \sum_{k=1}^{\infty} \left[\frac{30}{5^k} \right] = 6 + 1 = 7 \\
 v_7(30!) &= \sum_{k=1}^{\infty} \left[\frac{30}{7^k} \right] = 4 \\
 v_{11}(30!) &= \sum_{k=1}^{\infty} \left[\frac{30}{11^k} \right] = 2 \\
 v_{13}(30!) &= \sum_{k=1}^{\infty} \left[\frac{30}{13^k} \right] = 2 \\
 v_{17}(30!) &= \sum_{k=1}^{\infty} \left[\frac{30}{17^k} \right] = 1 \\
 v_{19}(30!) &= \sum_{k=1}^{\infty} \left[\frac{30}{19^k} \right] = 1 \\
 v_{23}(30!) &= \sum_{k=1}^{\infty} \left[\frac{30}{23^k} \right] = 1 \\
 v_{29}(30!) &= \sum_{k=1}^{\infty} \left[\frac{30}{29^k} \right] = 1
 \end{aligned} \tag{1}$$

So finally we get $30! = 2^{26} 3^{14} 5^7 7^4 11^2 13^2 17^1 19^1 23^1 29^1$. \square

PROBLEM III Assume $n \in \mathbb{N}^+$ and $\alpha \in \mathbb{R}$, prove that:

$$1. \left[\frac{[n\alpha]}{n} \right] = [\alpha].$$

2. $\sum_{k=0}^{n-1} [\alpha + \frac{k}{n}] = [n\alpha]$.

SOLUTION. 1. Easily $\left\lfloor \frac{[n\alpha]}{n} \right\rfloor \leq \left\lfloor \frac{n\alpha}{n} \right\rfloor \leq [\alpha]$. Now we will prove $\left\lfloor \frac{[n\alpha]}{n} \right\rfloor \geq [\alpha]$. By the definition of $[\cdot]$ we only need to prove $\frac{[n\alpha]}{n} \geq [\alpha]$. So we only need $[n\alpha] \geq n[\alpha]$. By the definition of $[\cdot]$ we only need $n\alpha \geq n[\alpha]$, which is obvious.

2. By 1 easily to know $[\alpha + \frac{k}{n}] = \left\lfloor \frac{[n(\alpha + \frac{k}{n})]}{n} \right\rfloor = \left\lfloor \frac{[n\alpha] + k}{n} \right\rfloor$. Let $f : \mathbb{Z} \rightarrow \{0, \dots, n-1\}$ and $f(x) \equiv x \pmod{n}$. Then easily $\left\lfloor \frac{x}{n} \right\rfloor = \frac{x}{n} - \frac{f(x)}{n}$. So we know $\sum_{k=0}^{n-1} [\alpha + \frac{k}{n}] = \sum_{k=0}^{n-1} \left\lfloor \frac{[n\alpha] + k}{n} \right\rfloor = \sum_{k=0}^{n-1} \frac{[n\alpha] + k}{n} - \sum_{k=0}^{n-1} \frac{f([n\alpha] + k)}{n}$. Easily to know $(f([n\alpha] + k) : k = 1, \dots, n-1)$ is a replacement of $(k : k = 0, \dots, n-1)$. So finally we get $\sum_{k=0}^{n-1} \frac{[n\alpha] + k}{n} - \sum_{k=0}^{n-1} \frac{f([n\alpha] + k)}{n} = \sum_{k=0}^{n-1} \frac{[n\alpha] + k}{n} - \sum_{k=0}^{n-1} \frac{k}{n} = \sum_{k=0}^{n-1} \frac{[n\alpha]}{n} = [n\alpha]$. \square

PROBLEM IV Assume $r > 0, r \in \mathbb{R}$. Let T be the number of integer point in $x^2 + y^2 \leq r^2$. Prove that $T = 1 + 4[r] + 8 \sum_{0 < x \leq \frac{r}{\sqrt{2}}} [\sqrt{r^2 - x^2}] - 4 \left\lfloor \frac{r}{\sqrt{2}} \right\rfloor^2$.

SOLUTION. $T = \sum_{x, y \in \mathbb{Z}, x^2 + y^2 \leq r^2} 1 = \sum_{x^2 + y^2 \leq r^2, xy=0} 1 + \sum_{x^2 + y^2 \leq r^2, xy \neq 0} 1 = 1 + \sum_{0 < x^2 + y^2 \leq r^2, xy=0} 1 + 4 \sum_{x^2 + y^2 \leq r^2, x > 0, y > 0} 1$. By symmetry, we know $\sum_{0 < x^2 + y^2 \leq r^2, xy=0} 1 = 4 \sum_{x^2 + y^2 \leq r^2, x > 0, y=0} 1 = 4[r]$. And $\sum_{x^2 + y^2 \leq r^2, x, y > 0} 1 = \sum_{x^2 + y^2 \leq r^2, 0 < x \leq \frac{r}{\sqrt{2}}, 0 < y} 1 + \sum_{x^2 + y^2 \leq r^2, 0 < y \leq \frac{r}{\sqrt{2}}, 0 < x} 1 - \sum_{x^2 + y^2 \leq r^2, 0 < x \leq \frac{r}{\sqrt{2}}, 0 < y \leq \frac{r}{\sqrt{2}}} 1$. Easily to know $\sum_{x^2 + y^2 \leq r^2, 0 < y \leq \frac{r}{\sqrt{2}}, 0 < x} 1 = \sum_{x^2 + y^2 \leq r^2, 0 < x \leq \frac{r}{\sqrt{2}}, 0 < y} 1 = \sum_{0 < x \leq \frac{r}{\sqrt{2}}} [\sqrt{r^2 - x^2}]$, and $\sum_{x^2 + y^2 \leq r^2, 0 < x \leq \frac{r}{\sqrt{2}}, 0 < y \leq \frac{r}{\sqrt{2}}} 1 = \left\lfloor \frac{r}{\sqrt{2}} \right\rfloor^2$. So finally we get $T = 1 + 4[r] + 8 \sum_{0 < x \leq \frac{r}{\sqrt{2}}} [\sqrt{r^2 - x^2}] - 4 \left\lfloor \frac{r}{\sqrt{2}} \right\rfloor^2$. \square

PROBLEM V Find all integer solution of $306x - 360y = 630$.

SOLUTION. The origin equation is equivalent to $17x - 20y = 35$. Consider $\pmod{5}$, we get $5 \mid 17x$. So $5 \mid x$. Assume $x = 5k$, then $17k - 4y = 7$. Then $17(k+1) - 4y = 24$, consider $\pmod{4}$, we get $4 \mid k+1$, so $k+1 = 4s$ and $17s - y = 6$. So $y = 17s - 6$ and easily $x = 5s = 5(4s - 1) = 20s - 5$. So $\begin{cases} x = 20s - 5 \\ y = 17s - 6 \end{cases}$ is all of solutions of the equation. \square

PROBLEM VI Assume $N, a, b \in \mathbb{N}, a, b > 0, \gcd(a, b) = 1$. Prove that the number of positive integer solutions of the equation $ax + by = N$ is $\left\lfloor \frac{N}{ab} \right\rfloor$ or $\left\lfloor \frac{N}{ab} \right\rfloor + 1$.

SOLUTION. Since $\gcd(a, b) = 1$, we know $\exists s, t \in \mathbb{Z}, as + bt = N$. So we know $x = s + kb, y = t - ka$. Let $x, y > 0$, we get $k > -\frac{s}{b}, k < \frac{t}{a}$. So we know the number of solution is $\left\lfloor \frac{t}{a} \right\rfloor + \left\lfloor \frac{s}{b} \right\rfloor + 1$. Now we only need $\left\lfloor \frac{N}{ab} \right\rfloor \leq \left\lfloor \frac{t}{a} \right\rfloor + \left\lfloor \frac{s}{b} \right\rfloor + 1 \leq \left\lfloor \frac{N}{ab} \right\rfloor + 1$.

To prove $\left\lfloor \frac{N}{ab} \right\rfloor \leq \left\lfloor \frac{t}{a} \right\rfloor + \left\lfloor \frac{s}{b} \right\rfloor + 1$ we only need $\left\lfloor \frac{N}{ab} \right\rfloor \leq \left\lfloor \frac{t}{a} \right\rfloor + \frac{s}{b} + 1$. Only need $\left\lfloor \frac{N}{ab} \right\rfloor \leq \frac{t}{a} + \frac{s}{b}$. Noting $ab \left\lfloor \frac{N}{ab} \right\rfloor \leq ab \frac{N}{ab} = N = as + bt = ab(\frac{t}{a} + \frac{s}{b})$ it's obvious.

To prove $\left\lfloor \frac{t}{a} \right\rfloor + \left\lfloor \frac{s}{b} \right\rfloor + 1 \leq \left\lfloor \frac{N}{ab} \right\rfloor + 1$, we only need $\left\lfloor \frac{t}{a} \right\rfloor + \left\lfloor \frac{s}{b} \right\rfloor \leq \frac{N}{ab}$. Noting $ab(\left\lfloor \frac{t}{a} \right\rfloor + \left\lfloor \frac{s}{b} \right\rfloor) \leq ab(\frac{t}{a} + \frac{s}{b}) = as + bt = N = ab(\frac{N}{ab})$ it's obvious. \square

PROBLEM VII Write $\frac{17}{60}$ as sum of three reduced fraction whose denominators are coprime to each other.

SOLUTION. Consider $\frac{17}{60} = \frac{x}{4} + \frac{y}{3} + \frac{z}{5}$, i.e., $17 = 15x + 20y + 12z$. Since $\gcd(15, 20, 12) = 1$, we know this equation has some solution. Easy to know $x = -1, y = 1, z = 1$ is a solution. So $\frac{17}{60} = -\frac{1}{4} + \frac{1}{3} + \frac{1}{5}$ satisfy the condition. \square