## under Graduate Homework In Mathematics

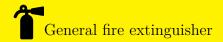
SetTheory 4

白永乐

202011150087

202011150087@mail.bnu.edu.cn

2023年11月5日



ROBEM I Consider  $\mathbb{Q} = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) / \sim$ , where  $(a, b) \sim (c, d) \iff ad = bc$ . Define  $+_{\mathbb{Q}}, \cdot_{\mathbb{Q}}$  and verify that your definitions doesn't depend on the choice of representatives.

SPETION. Let  $[(a,b)] +_{\mathbb{Q}} [(c,d)] = [(ad+bc,bd)], [(a,b)] \cdot_{\mathbb{Q}} [(c,d)] = [(ac,bd)],$  and  $[(a,b)] <_{\mathbb{Q}} [(c,d)] \iff abd^2 < cdb^2$ . Now we prove they are well-defined, i.e., doesn't depend on the choice of representatives.

For  $+_{\mathbb{Q}}$ , assume  $(a,b) \sim (e,f)$ , we need to prove  $(ad+bc,bd) \sim (ed+fc,df)$ . Since af=be, we have  $(ad+bc)bf=ad^2f+bdcf=bed^2+bdcf=(ed+fc)bd$ . So  $+_{\mathbb{Q}}$  is well defined.

For  $\cdot_{\mathbb{Q}}$ , assume  $(a,b) \sim (e,f)$ , we need to prove  $(ac,bd) \sim (ec,fd)$ . Since af = be, we have acfd = bced = bdec. So  $\cdot_{\mathbb{Q}}$  is well defined.

For  $<_{\mathbb{Q}}$ , assume  $(a_1, b_1) \sim (a_2, b_2), (c_1, d_1) \sim (c_2, d_2)$  and  $(a_1, b_1) < (c_1, d_1)$ . Now we need to prove  $(a_2, b_2) < (c_2, d_2)$ . Since  $a_1b_2 = a_2b_1, c_1d_2 = c_2d_1$  we get  $a_1b_1d_2^2 < c_2d_2b_1^2$ 

 $\mathbb{R}^{OBEM}$  II The set of all continuous functions  $f: \mathbb{R} \to \mathbb{R}$  has cardinality  $\mathfrak{c}$  (while the set of all functions has cardinality  $2^{\mathfrak{c}}$ ). [A continuous function on  $\mathbb{R}$  is determined by its values at rational points.]

SOUTON. Consider  $\theta: \mathbb{R} \to 2^{\mathbb{Q}}, f \mapsto \{(a,b) \in \mathbb{Q} : f(a) < b\}$ . Now we prove f is a injection. Assume  $\theta(f) = \theta(g)$ , to prove f = g. First we prove for  $x \in \mathbb{Q}$  we have f(x) = g(x). We have  $f(x) = \sup\{y \in \mathbb{Q} : y < f(x)\} = \sup\{y \in \mathbb{Q} : (x,y) \in \theta(f)\} = \sup\{y \in \mathbb{Q} : (x,y) \in \theta(g)\} = g(x)$ . For  $x \in \mathbb{R}$ , choose a sequence  $x_n \in \mathbb{Q}$  such that  $x_n \to x$ , then  $f(x) = \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} g(x_n) = g(x)$ . So we get f = g. So  $\operatorname{card}^{\mathbb{R}} \mathbb{R} \leq \operatorname{card} 2^{\mathbb{Q}} = 2^{\aleph_0}$ . Obviously  $\operatorname{card}^{\mathbb{R}} \mathbb{R} \geq 2^{\aleph_0}$ , so we get they are equal.

ROBEM III There are at least  $\mathfrak{c}$  countable order-types of linearly ordered sets. [For every sequence  $a = \langle a_n : n \in \mathbb{N} \rangle$  of natural numbers consider the ordertype

$$\tau_a = a_0 + \xi + a_1 + \xi + a_2 + \dots$$

where  $\xi$  is the order-type of the integers. Show that if  $a \neq b$ , then  $\tau_a \neq \tau_b$ .] A real number is algebraic if it is a root of a polynomial whose coefficients are integers. Otherwise, it is transcendental.

ROBEM IV The set of all algebraic reals is countable.

ROBEM V If S is a countable set of reals, then  $|\mathbb{R} - S| = \mathfrak{c}$ . [Use  $\mathbb{R} \times \mathbb{R}$  rather than  $\mathbb{R}$  (because  $|\mathbb{R} \times \mathbb{R}| = 2^{\aleph_0}$ ).]