

# Algebraic Geometry 1

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2023 年 11 月 6 日

**PROBLEM I**  $P$  is an ideal of a unitary commutative ring  $A$ , then  $P$  is prime ideal of  $A \iff A/P$  is integral domain.

**SOLUTION.**  $\Rightarrow$ :

Since  $A$  is a unitary commutative ring, so  $A/P$  is unitary commutative ring, too. So we only need to prove  $[ab] = [0] \Rightarrow [a] = [0] \vee [b] = [0]$ . Obviously  $[ab] = [0] \iff ab \in P \iff a \in P \vee b \in P \iff [a] = [0] \vee [b] = [0]$ .

$\Leftarrow$ :

As the same,  $ab \in P \iff [ab] = [0] \Rightarrow [a] = [0] \vee [b] = [0] \iff a \in P \vee b \in P$ , so  $P$  is prime ideal.  $\square$

**PROBLEM II**  $M$  is an ideal of a unitary commutative ring  $A$ , then  $M$  is maximal ideal of  $A \iff A/M$  is a field.

**SOLUTION.**  $\Rightarrow$ :

Consider  $[a] \in A/M \setminus [0]$ , we will prove it has a reverse. Consider  $N := \{xm + ya : x, y \in A, m \in M\}$  is the minimum ideal of  $A$  contains  $M$  and  $a$ . Since  $[a] \neq [0]$  we know  $a \notin M$ , so  $M \subsetneq N$ . Noting  $M$  is maximal, so  $N = A$ . That means  $\exists x, y \in A, m \in M, xm + ya = 1$ . So  $[xm + ya] = [1]$ . Since  $[xm] = [0]$  we get  $[y][a] = 1$ , i.e.,  $[y] = [a]^{-1}$ .

$\Leftarrow$ :

Consider  $a \in A \setminus M, N := \{xp + ya : x, y \in A, p \in P\}$ , we will prove  $N = A$ , which means  $M$  is maximal. Since  $A/M$  is field,  $\exists y \in A, [y] = [a]^{-1}$ . That's means  $ay - 1 \in M \subset N$ . Noting  $ay \in N$ , so  $1 \in N$ , thus  $N = A$ .  $\square$

**PROBLEM III** A ring  $A$  is noetherian,  $I \subset A$  is an ideal of  $A$ , then  $A/I$  is noetherian.

**SOLUTION.** Consider an ideal  $J \subset A/I$ , let  $M := \{x \in A : [x] \in J\}$ . Then  $\forall a \in A, x \in M, [ax] = [a][x] \in J$ , so  $ax \in M$ .  $\forall a, b \in M, [a - b] = [a] - [b] \in J$ , so  $a - b \in M$ . So  $M$  is an ideal of  $A$ . Since  $A$  is noetherian, we can assume  $M = (f_i, i = 1, 2, \dots, n)$ . Now we will prove  $J = ([f_i], i = 1, 2, \dots, n)$ . Consider  $[f] \in J$ , from definition of  $M$  we know  $f \in M$ , so  $f = \sum_{i=1}^n a_i f_i, a_i \in A$ , thus  $[f] = [\sum_{i=1}^n a_i f_i] = \sum_{i=1}^n [a_i][f_i]$ . So  $J = ([f_i], i = 1, 2, \dots, n)$ .  $\square$

**PROBLEM IV**  $K$  is a field,  $A = K[x_1, x_2, \dots, x_n]$ ,  $\mathbb{A}_K^n = K^n$ . For ideal  $I$  of  $A$  let  $V(I) := \{p \in \mathbb{A}_K^n : f(p) = 0, \forall f \in I\}$ . Then  $V(I_1) \cup V(I_2) = V(I_1 \cap I_2) = V(I_1 I_2)$

**SOLUTION.**

*Lemma 1.*  $I \subset J \Rightarrow V(I) \supset V(J)$ .

**证明.** Consider  $p \in V(J)$ , we get  $\forall f \in J, f(p) = 0$ . Since  $I \subset J$ , so  $\forall f \in I, f(p) = 0$ , i.e.,  $p \in V(I)$ .  $\square$

From Lemma 1 we know  $V(I_1 \cap I_2) \subset V(I_1 I_2)$ , so we only need to prove  $V(I_1) \cup V(I_2) \subset V(I_1 \cap I_2), V(I_1 I_2) \subset V(I_1) \cup V(I_2)$ .

- $V(I_1) \cup V(I_2) \subset V(I_1 \cap I_2)$ : From Lemma 1 It's obvious.
- $V(I_1 I_2) \subset V(I_1) \cup V(I_2)$ : Consider  $p \in V(I_1 I_2)$ . If  $p \notin V(I_1) \cup V(I_2)$ , then  $\exists f_1 \in I_1, f_2 \in I_2, f_1(p) \neq 0, f_2(p) \neq 0$ . Now consider  $f = f_1 f_2 \in I_1 I_2$ , we get  $f(p) = f_1(p) f_2(p) \neq 0$ , so  $p \notin V(I_1 I_2)$ , it's a contradiction.

$\square$

**PROBLEM V**  $K$  is an infinite field, then  $\mathbb{A}_K^n$  is not Hausdorff.

**SOLUTION.**

*Lemma 2.*  $K$  is a infinite field,  $f \in K[x_1, x_2, \dots, x_n] \setminus \{0\}$ , then exists  $p \in \mathbb{A}_K^n, f(p) \neq 0$ .

**证明.** Use MI. When  $n = 0, K[x_1, x_2, \dots, x_n] = K$ , so it's obvious. Assume for  $n = k$  it's right, when goes to  $k + 1$ :

Consider  $h \in K[x_1, x_2, \dots, x_k][x_{k+1}], h(x_{k+1}) := f(x_1, x_2, \dots, x_k, x_{k+1})$  is a non-zero polynomial so it has finite root. So exists  $a \in K, h(a) \neq 0$ . So  $g := f(x_1, x_2, \dots, x_k, a) \in K[x_1, x_2, \dots, x_k] \neq 0$ . By induction hypothesis we get  $\exists b \in \mathbb{A}_K^k, g(b) \neq 0$ . Let  $p := (b, a) \in \mathbb{A}_K^{k+1}$ , then  $f(p) \neq 0$ .  $\square$

In fact, it's not only not Hausdorff, it's kind of "absolutely not Hausdorff" because every pair of point can't be separated. Consider two point  $p \neq q, p, q \in \mathbb{A}_K^n$ . Assume  $p = (p_1, p_2, \dots, p_n), q = (q_1, q_2, \dots, q_n), p_1 \neq q_1$ . Assume two open set  $V(I_1)^c, V(I_2)^c$  can separate  $p, q$ , then  $V(I_1) \cup V(I_2) = \mathbb{A}_K^n$ . From **PROBLEM IV** we know  $V(I_1 I_2) = \mathbb{A}_K^n$ . So  $\forall f \in V(I_1 I_2), \forall p \in \mathbb{A}_K^n, f(p) = 0$ . Then from Lemma 2 we can get  $f = 0$ . So  $I_1 I_2 = \{0\}$ . Since  $p \notin V(I_1), q \notin V(I_2)$ , we know  $I_1, I_2 \neq \{0\}$ . So  $\exists f_1 \in I_1 \neq 0, \exists f_2 \in I_2 \neq 0$ , and thus  $f = f_1 f_2 \in I_1 I_2 \neq 0$ , contradiction! So these is not a pair of points can be separated by two disjoint open set.  $\square$