PROBLEM I Let $S = (S_n : n \geq 0)$ be the one-dimensional symmetry simple random walk with $S_0 = c \geq 0$. Let $k \geq 1$ and τ be the time of the k-th downcrossing 0. X_b is the times of $(S_{n\wedge\tau} : n \geq 0)$ downcrossing b. Prove:

1. $(X_b : b \geq c-1)$ is branch process. And offspring distribution is $Geo(\frac{1}{2})$
2. $(X_{-a} : a \geq 1)$ is branch process. And offspring distribution is $Geo(\frac{1}{2})$
3. $(X_b : 0 \leq b \leq c-1)$ is migrating branch process. And offspring distribution is $Geo(\frac{1}{2})$ And the migrating distribution is concentrating on 1.

SOLUTION. For a random walk y, we let $D(n, y)$ be the number of downcrossings of y of n.

1. Fix $b \geq c-1$. Let ϕ_0 be the journey from start point to $b+1$. Let e_n be n-th journey from $b+1$ to b. Let ε_n be n-th journey after ϕ_0 from b to $b+1$. Then we know that e_n, ε_n are independent. Easy to get that $D(e_n, b) = 1$ and $D(\varepsilon_n, b) = 0, D(\varepsilon_n, b+1 = 0)$. Easy to get that $D((S_{n\wedge\tau} : n \in \mathbb{N}), b+1) = \sum_{t=1}^{D((S_{n\wedge\tau}:n\in\mathbb{N}),b)} D(e_t, b+1)$. Noting that $\forall d : c-1 \leq d \leq b, D((S_{n\wedge\tau} : n \in \mathbb{N}), d) \in \sigma(\varepsilon_n : n \in \mathbb{N})$. We easily get that $D(e_t, b+1) \perp \sigma(\varepsilon_n : n \in \mathbb{N})$. So X_b is Markov process. And to prove it's branch process, we only need to prove that $D(e_t, b+1)$ are i.i.d. It has been proved that $D(e_t, b+1)$ are i.i.d and $Geo(\frac{1}{2})$. So the offspring distribution is $Geo(\frac{1}{2})$.
2. Fix $a \geq 1$. Let ϕ_0 be the journey from start point to $-a$. Let e_n be n-th journey from $-a$ to $-a-1$, and ε_n be n-th journey from $-a-1$ to $-a$. Then easy to get that $X_{-a-1} = \sum_{t=1}^{X_{-a}} D(\varepsilon_t, -a-1)$. For the same reason we easily get that $D(\varepsilon_t, -a-1) \perp \sigma(e_n : n \in \mathbb{N})$. And by reflecting easy to get that $D(\varepsilon_t, -a-1) \sim Geo(\frac{1}{2})$, too. So $(X_{-a} : a \geq 1)$ is branch process and offspring distribution is $Geo(\frac{1}{2})$
3. Fix $b < c-1$. Let ϕ_0 be the journey from start point to $b+1$. Let e_n be the n-th journey from $b+1$ to b and ε_n be n-th journey from b to $b+1$. Then easy to prove that $X_{b+1} = D(\phi_0, b+1) + \sum_{t=1}^{X_b} D(e_n, b+1)$. Noting that $D(\phi_0, b+1) = 1$. So for the same reason, we get that $(X_b : 0 \leq b \leq c-1)$ is migrating branch process, with offspring distribution $Geo(\frac{1}{2})$ and migrating distribution δ_1.

□

PROBLEM II $c < b \in \mathbb{Z}_+$. Let $W = (W_n : n \geq 0)$ be the one-dimensional reflecting simple random walk with $W_0 = c \geq 0$ on $\mathbb{Z}^{0,b}$, whose transition matrix is $P^{0,b}$, where $a = 0, p, q > 0, p+q = 1$. Let $k \geq 1$ and τ be the time of the k-th downcrossing 0 of (W_n). $0 \leq a \leq b$, X_a is the times of $(S_{n\wedge\tau} : n \geq 0)$ downcrossing a. Prove:

1. $(X_a : c-1 \leq a \leq b-1)$ is branch process. And offspring distribution is $Geo(p)$.
2. $(X_a : 0 \leq a \leq c-1)$ is migrating branch process. And offspring distribution is $Geo(p)$. And the migrating distribution is concentrating on 1.

SOLUTION. For a random walk y, we let $D(n, y)$ be the number of downcrossings of y of n.

1. Fix a such that $c-1 \le a < b-1$. Let ϕ_0 be the journey from start point to a. Let e_n be the n-th journey from a to $a+1$, and ε_n be the n-th journey from $a+1$ to a. For reflecting simple random walk, we can also prove that e_n, ε_n are independent. Noting that $X_{a+1} = \sum_{t=1}^{X_a} D(\varepsilon_t, a+1)$, we easily get the conclusion.
2. Fix $a : 0 \le a < c-1$. Let ϕ_0 be the journey from start point to $a+1$. Let e_n be the n-th journey from $a+1$ to a and ε_n be n-th journey from a to $a+1$. Then easy to prove that $X_{a+1} = D(\phi_0, a+1) + \sum_{t=1}^{X_a} D(e_n, a+1)$. Noting that $D(\phi_0, a+1) = 1$. So for the same reason, we get that $(X_a : 0 \le a \le c-1)$ is migrating branch process, with offspring distribution $Geo(\frac{1}{2})$ and migrating distribution δ_1.

$\square$

PROBLEM III Let $W = (W_n : n \ge 0)$ be the one-dimensional simple random walk with $W_0 = 0$, whose transition matrix P given by equation (4.4.3) on textbook, $0 < p < q < 1$. X_a is the times of $(W_{n\wedge\tau} : n \ge 0)$ downcrossing a. $r = \frac{p}{q}$. Prove:

1. $\mathbb{P}(X_0 = i) = r^i(1-r), i \ge 0$;
2. $a \ge 0$, $\mathbb{P}(X_a = 0) = 1 - r^{a+1}$, $\mathbb{P}(X_a = i) = r^{a+1}(1-r), i \ge 1$.

SOLUTION. 1. Since $p < q$, then $W_n \to -\infty, n \to \infty$. Let $\tau_0 = 0, \forall k \ge 1$, $\sigma_k = \inf\{n \ge \tau_{k-1} : W_n = 1\}, \tau_k = \inf\{n \ge \sigma_k : W_n = 0\}$.

   (a) If $i = 0$, then $\{X_0 = i\} \overset{\text{a.s.}}{=} \{\sigma_1 = \infty\}$. Then $\mathbb{P}(X_0 = i) = \mathbb{P}(\sigma_1 = \infty) = r$.

   (b) If $i \ge 1$, then $\{X_0 = i\} \overset{\text{a.s.}}{=} \{\sigma_i < \infty, \sigma_{i+1} = \infty\}$. Since $\{\tau_i < \infty\} \subset \{\sigma_i \infty\}, \mathbb{P}(\sigma_i < \infty, \tau_i = \infty) = 0$, then by strong markov property,

$$\begin{aligned}\mathbb{P}(\sigma_{i+1} < \infty \mid \sigma_i < \infty) &= \mathbb{P}(\sigma_{i+1} < \infty \mid \sigma_i < \infty, \tau_i < \infty)\\ &= \mathbb{P}(\sigma_{i+1} < \infty \mid \tau_i < \infty)\\ &= \mathbb{P}(\sigma_{i+1} < \infty \mid \tau_i < \infty, W_{\tau_i} = 0)\\ &= \mathbb{P}(\sigma_1 < \infty) = r\end{aligned}$$

   Therefore,

$$\mathbb{P}(\sigma_{i+1} < \infty) = \mathbb{P}(\sigma_{i+1} < \infty \mid \sigma_i < \infty)\mathbb{P}(\sigma_i < \infty)$$

   Then $\mathbb{P}(\sigma_i < \infty) = r^i$. Therefore, $\mathbb{P}(X_0 = i) = \mathbb{P}(\sigma_i < \infty, \sigma_{i+1} = \infty) = \mathbb{P}(\sigma_i < \infty)\mathbb{P}(\sigma_{i+1} = \infty \mid \sigma_i < \infty) = r^i(1-r)$.

2. Let $D_a = \inf(n \ge 0 : W_n = a)$, then $\mathbb{P}(D_a < \infty) = r^a$. By strong markov property, $(W_{D_a+n-a:n\ge 0})$ is a random walk starting from 0 under $\mathbb{P}(\cdot \mid D_a < \infty) = \mathbb{P}(\cdot \mid D_a < \infty, W_{D_a} = a)$. By the conclusion in 1, $\mathbb{P}(X_a = i \mid D_a < \infty) = r^i(1-r), i \ge 0$. Then

$$\begin{aligned}\mathbb{P}(X_a = 0) &= \mathbb{P}(D_a = \infty) + \mathbb{P}(D_a < \infty, X_a = 0)\\ &= 1 - r^a + \mathbb{P}(D_a < \infty)\mathbb{P}(X_a = 0 \mid D_a < \infty)\\ &= 1 - r^a + r^a(1-r) = 1 - r^{a+1}\end{aligned}$$

$\forall i \geq 1$,

$$\begin{aligned}\mathbb{P}(X_a = i) &= \mathbb{P}(D_a < \infty, X_a = i)\\ &= \mathbb{P}(D_a < \infty)\mathbb{P}(X_a = i \mid D_a < \infty)\\ &= r^a r^i (1-r) = r^{a+i}(1-r)\end{aligned}$$

$\square$

PROBLEM IV Let $W = (W_n : n \geq 0)$ be the one-dimensional simple random walk with $W_0 = 0$, whose transition matrix P given by equation (4.4.3) on textbook, $0 < p < q < 1$. X_a is the times of $(W_{n\wedge\tau} : n \geq 0)$ downcrossing a. $r = \frac{p}{q}$. Prove: if $a \leq -1$, then $X_a - 1 \sim G(1-r)$, i.e. $\mathbb{P}(X_a = i) = r^{i-1}(1-r), i \geq 1$.

SOLUTION. Let $\tau = \inf\{n \in \mathbb{N} : W_n = a\}$. Then $\tau < \infty, a.s.$, then W_n downcross a at τ. And $(W_{\tau+n} - a : n \in \mathbb{N})$ is simple random walk start at 0. So by III we easily get $X_a - 1 \sim Geo(1-r)$. $\square$