PROBLEM I Prove: If $m \in \mathbb{Z}^+, a \in \mathbb{Z}, \gcd(a, m) = 1$, A is reduced residue system of m, then

$$\sum_{i\in A} \left\{\frac{ai}{m}\right\} = \frac{1}{2}\varphi(m)$$

SOLUTION. Let $f : \mathbb{Z} \to \{0, \cdots, m-1\}$ and $f(x) \equiv x \mod m$. Then easily $\{\frac{x}{m}\} = \{\frac{f(x)}{m}\}$. So we get $\sum_{i\in A} \left\{\frac{ai}{m}\right\} = \sum_{i\in A} \left\{\frac{f(ai)}{m}\right\}$. Let $B = \{f(ai) : i \in A\}$, since $\gcd(m, a) = 1$, we obtain B is reduced residue system of m, too. Easily to know $\left\{\frac{f(ai)}{m}\right\} = \frac{f(ai)}{m}$, then $\sum_{i\in A} \left\{\frac{ai}{m}\right\} = \sum_{j\in B} \frac{j}{m}$. Noting that $g : B \to B, j \mapsto m - j$ is bijection, so $\sum_{j\in B} \frac{j}{m} = \sum_{j\in B} \frac{m-j}{m} = \frac{1}{2}\sum_{j\in B} \frac{j+m-j}{m} = \frac{1}{2}|B|$. Obvious since B is reduced residue system of m, we easily get $|B| = \varphi(m)$. So finally we get $\sum_{i\in A} \left\{\frac{i}{m}\right\} = \frac{1}{2}\varphi(m)$. □

PROBLEM II

1. Prove: $\sum_{i=0}^{a} \varphi(p^i) = p^a$, where p is prime.
2. Prove: $\sum_{d\in\mathbb{N}:d|a} \varphi(d) = a$.

SOLUTION.

1. Easily to know $\varphi(p^k) = p^k \times \frac{p-1}{p} = (p-1)p^{k-1})$. So we get $\sum_{i=1}^{a} \varphi(p^i) = \sum_{i=1}^{a}(p-1)p^{i-1} = (p-1)\frac{p^a-1}{p-1} = p^a - 1$. And $\varphi(1) = 1$, so finally we get $\sum_{i=0}^{a} \varphi(p^i) = p^a - 1 + 1 = p^a$.
2. Let $A := \{n \in \mathbb{N} : \sum_{d\in\mathbb{N}:d|n} \varphi(d) = n\}$, then from 1 we get $\{p^k : p \in \mathbb{P}, k \in \mathbb{N}\} \subset A$, where $\mathbb{P}$ is the set of primes. Now to prove $A = \mathbb{N}$, we only need to prove that for $m, n \in A \wedge \gcd(m, n) = 1$ we have $nm \in A$.

   Let $M := \{d \in \mathbb{N} : d \mid m\}, N := \{d \in \mathbb{N} : d \mid n\}, D := \{d \in \mathbb{N} : d \mid nm\}$, $f : M \times N \to D, f(x, y) := xy$, we will prove that f is bijection, and et $g : D \to M \times N, g(z) = (\gcd(z, m), \gcd(z, n))$, we will prove $g = f^{-1}$.

   For $(x, y) \in M \times N$, we need to prove $g \circ f(x, y) = (x, y)$. i.e., $\gcd(xy, m) = x, \gcd(xy, n) = y$. Since $y \mid n$ and $\gcd(n, m) = 1$, we easily get $\gcd(y, m) = 1$, so $\gcd(xy, m) = \gcd(x, m)$. Noting $x \mid m$, we get $\gcd(x, m) = x$. So $\gcd(xy, m) = x$. For the same reason we get $\gcd(xy, n) = y$.

   For $z \in D$, write $x = \gcd(z, m), y = \gcd(z, n)$, then $g(z) = (x, y)$. We need to prove $f(x, y) = z$, i.e., $xy = z$. Since $\gcd(m, n) = 1$, easily $z = \gcd(z, nm) = \gcd(z, m)\gcd(z, n) = xy$.

   So $g = f^{-1}$ and thus f is bijection. So we know $\sum_{d\in D} \varphi(d) = \sum_{(x,y)\in M\times N} \varphi(xy)$. Noting $\gcd(x, y) \mid \gcd(m, n) = 1$, we know $\varphi(x, y) = \varphi(x)\varphi(y)$. So $\sum_{d\in D} \varphi(d) = \sum_{x\in M} \sum_{y\in N} \varphi(x)\varphi(y) = \sum_{x\in M} \varphi(x) \sum_{y\in N} \varphi(y)$. Recalling $m, n \in A$, we know $\sum_{x\in M} \varphi(x) = m, \sum_{y\in N} \varphi(y) = n$, so finally $\sum_{d\in D} \varphi(d) = nm$. So $nm \in A$.

   Now for any $a \in \mathbb{N}^+$, we know $a = \prod_{k=1}^{t} p_k^{\alpha_k}$, where $p_k : k = 1, \cdots, t$ are different primes. Then $p_k^{\alpha_k} \in A$. So $\forall a \in \mathbb{N}^+, a \in A$, thus $\sum_{d\in\mathbb{N}:d|a} \varphi(d) = a$.

□

PROBLEM III If today is Monday, then what day is it $10^{10^{10}}$ days after today?

SOLUTION. Only need to find the remainder of $10^{10^{10}} \mod 7$. Noting that $\varphi(7) = 6$ and $\gcd(10, 7) = 1$, so $10^6 \equiv 1 \mod 7$. So we only need to find $10^{10} \mod 6$. Since $6 = 2 \times 3$, we only need to find $10^{10} \mod 2, 10^{10} \mod 3$. Easy to know $10^{10} \equiv 0 \mod 2$. Noting $10 \equiv 1 \mod 3$, we get $10^{10} \equiv 1 \mod 3$. So from the Chinese remainder theorem we get $10^{10} \equiv 4 \mod 6$. So $10^{10^{10}} \equiv 10^4 \equiv 3^4 \mod 7$. By calculation easy to know $3^4 \equiv 9^2 \equiv 2^2 \equiv 4 \mod 7$. So it is Friday $10^{10^{10}}$ days after today. $\square$

PROBLEM IV Find the remainder of $(12371^{56} + 34)^{28} \mod 111$.

SOLUTION. Easily $111 = 3 \times 37$, so we only need to find the remainder $\mod 3$, $\mod 37$ respectively. Easily $(12371^{56} + 34)^{28} \equiv ((-1)^{56} + 1)^{28} \equiv (-1)^{28} \equiv 1 \mod 3$. Easily $\varphi(37) = 36$, and $\gcd(12371, 37) = 1$, so $(12371^{56} + 34)^{28} \equiv (13^{20} - 3)^{28} \mod 37$. Noting $13^4 \equiv -3 \mod 37$, we get $13^{20} \equiv (-3)^5 \equiv -243 \equiv 16 \mod 37$. Thus, $(13^{20} - 3)^{28} \equiv (16 - 3)^{28} \equiv 13^{28} \equiv 13^{20} \times 13^8 \equiv 16 \times (-3)^2 \equiv 144 \equiv 33 \mod 37$.

Now by Chinese remainder theorem we get $(12371^{56} + 34)^{28} \equiv 70 \mod 111$. $\square$

PROBLEM V Assume $\frac{a}{b} \in \mathbb{Q}$, where $0 < a < b, \gcd(a, b) = 1$. Then $\frac{a}{b}$ is pure recurring decimal $\iff \exists t \in \mathbb{N}^+$ s.t. $10^t \equiv 1 \mod b$. Besides, $\min\{t \in \mathbb{N}^+ : 10^t \equiv 1 \mod b\}$ is the length of cycle section.

SOLUTION. Let l be the length of cycle section of $\frac{a}{b}$.

"$\implies$": Assume $\frac{a}{b} = \sum_{k=1}^{\infty} 10^{-kl} x$, where $x \in \mathbb{N}, 0 < x < 10^l$. Then we get $\frac{a}{b} = \frac{1}{10^l} \frac{1}{1-10^{-l}} = \frac{x}{10^l - 1}$. Then $a(10^l - 1) = bx$. Since $\gcd(a, b) = 1$, we get $b \mid 10^l - 1$. Let $t = l$ will work.

"$\impliedby$": Assume $10^t \equiv 1 \mod b$, where $t \in \mathbb{N}^+$. Let $10^t - 1 = bk$, where $k \in \mathbb{N}^+$. Let $x = ak$, we will prove $\frac{a}{b} = \sum_{k=1}^{\infty} 10^{-kt} x$. Easily $\sum_{k=1}^{\infty} 10^{-kt} x = \frac{x}{10^t - 1} = \frac{ak}{bk} = \frac{a}{b}$. So $\frac{a}{b}$ is pure recurring decimal and $l \mid t$.

In the first part we have proved that $l \in \{t \in \mathbb{N}^+ : 10^t \equiv 1 \mod b\}$. In the second part we have proved that $\forall t \in \mathbb{N}^+ \wedge 10^t \equiv 1 \mod b, l \mid t$. So obviously $l = \min\{t \in \mathbb{N}^+ : 10^t \equiv 1 \mod b\}$. $\square$