underGraduate Homework In Mathematics

AlgebraicGeometry 11

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PROBLEM I Let $E := \{(x,y) \in \mathbb{A}^2_{\mathbb{C}} : y^2 = x^3+ax+b\}$. Prove that E is nonsingular $\iff$ $4a^3+27b^2 \neq 0$. i.e., x^3+ax+b has no repeated roots.

SOLUTION. Let $f(x,y) := y^2 - x^3 - ax - b \in \mathbb{C}[x,y]$. Then we have $f_x(x,y) = 3x^2 - a$, $f_y(x,y) = 2y$. Now consider the equation $f(x,y) = f_x(x,y) = f_y(x,y) = 0$. We know E is nonsigular iff this equation has solutions. From $f_y = 0$ we get $y = 0$, then $x^3 - ax - b = 3x^2 - a = 0$. So the equation has solutions iff $x^3 - ax - b$ has repeated roots. Since $3x^2 - a = 0$ we get $0 = x^3 - ax - b = 3x^3 - 3ax - 3b = ax - 3ax - 3b$, so $2ax = -3b$, then $0 = 3x^2 - a = 12a^2x^2 - 4a^3 = 27b^2 - 4a^3$. So we get $x^3 - ax - b$ has repeated roots iff $27b^2 - 4a^3 = 0$.

So finally we get E is nonsingular iff $27b^2 - 4a^3 = 0$ iff $x^3 - ax - b$ has no repeated roots. $\square$

PROBLEM II Assume $E := \{[x,y,z] \in \mathbb{P}^2_{\mathbb{C}} : y^2z = x^3 - axz^2 - bz^3\}$. Prove that E is regular at $[0,1,0]$.

SOLUTION. Consider $\theta : E \setminus \mathbb{V}(y) \to \mathbb{A}^2_{\mathbb{C}}, [x,y,z] \mapsto (\frac{x}{y}, \frac{z}{y})$. Noting $\theta([0,1,0]) = (0,0)$, so we only need to prove $(0,0)$ is nonsigular in $F = \{(x,z) \in \mathbb{A}^2_{\mathbb{C}} : z = x^3 - axz^2 - bz^3\}$. Let $f(x,z) = x^3 - axz^2 - bz^3 - z$, then $f_x(x,z) = 3x^2 - az^2$, $f_z(x,z) = 2axz - 3bz^2 - 1$. Then $f_z(0,0) = -1 \neq 0$. So $(0,0)$ is nonsigular. $\square$

PROBLEM III Let $\omega_1, \omega_2 \in \mathbb{C}$ are $\mathbb{R}$-linear independent and $\Lambda := \{m\omega_1 + n\omega_2 : m,n \in \mathbb{Z}\} \subset \mathbb{C}$. Then $\mathbb{C}/\Lambda$ is Riemann surface. Now consider $T_w : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda, [z] \mapsto [z+w]$ for some $w \in \mathbb{C}$. Prove that T_w is homomorphism.

SOLUTION. For $p = [z] \in \mathbb{C}/\Lambda$, let $q = [z+w] = T_w([p])$. Let ϕ_p, ϕ_q is homomorphism from neibor of p, q to $\mathbb{C}$. Then we easily get $\phi_q \circ T_w - \phi_p = w + r$ for some $r \in \Lambda$. Let $f : \mathbb{C} \to \mathbb{C}, x \mapsto x + w + r$. Then $f + \phi_q \circ T_w \circ \phi_p^{-1}$ is homomorphism. So we get T_w is homomorphism. Easily we get $T_{-w} = T_w^{-1}$, so T_w is isomorphic. $\square$