PROBLEM I Assume $(N_t : t \geq 0)$ is Possion process with parameter $\alpha$, and $\{\xi_n : n \in \mathbb{N}^+\}$ is a sequence of i.i.d random variable. More over, assume $(N_t : t \geq 0) \perp \{\xi_n : n \in \mathbb{N}^+\}$. Let $X_t = \sum_{k=1}^{N_t} \xi_k$. Let $r > 0$, prove that:

1. $(N_{t+r} - N_r : t \geq 0)$ is Possion process.
2. $\{\xi_{N_r+n} : n \in \mathbb{N}^+\}$ is also i.i.d sequence with the same distribution of $\{\xi_n : n \in \mathbb{N}^+\}$.
3. $(N_{t+r} - N_r : t \geq 0) \perp (\xi_{N_r+k} : k \in \mathbb{N}^+)$.
4. For $0 = t_0 < t_1 < \cdots < t_n$, we have $(X_{t_1}, X_{t_{k+1}} - X_{t_k} : k = 1, 2, \cdots, n-1)$ are independent.

SOLUTION. 1. Let $\mathcal{F}_t := \sigma(N_s : 0 \leq s \leq t)$ and $\mathcal{G}_t := \sigma(N_{r+s} - N_r : 0 \leq s \leq t)$. For $0 \leq s \leq t$, we have $N_{t+r} - N_r - (N_{s+r} - N_r) = N_{t+r} - N_{s+r} \sim Possion(\alpha(t-s))$. And easily to know $N_{r+s} - N_r \in \mathcal{F}_{r+s}$, so $\mathcal{G}_t \subset \mathcal{F}_{t+r}, \forall t \geq 0$. Since $(N_t : t \geq 0)$ is Possion process, easily $\mathcal{F}_{s+r} \perp N_{t+r} - N_{s+r}$. Since $\mathcal{G}_t \subset \mathcal{F}_{t+r}$, we obtain $\mathcal{G}_t \perp N_{t+r} - N_{s+r} = N_{t+r} - N_r - (N_{s+r} - N_r)$. Easily since $N_t$ is right-continuous we get $N_{t+r} - N_r$ is right-continuous. For the same reason, we know $\forall s \in [0, \infty), \lim_{t \to s-} N_{t+r} - N_r$ exists. So $(N_{t+r} : t \geq 0)$ is Possion process.

2. Only need to prove that for $m \in \mathbb{N}^+$, the distribution of $(\xi_{N_r+k} : 1 \leq k \leq m)$ is same as that of $(\xi_k : 1 \leq k \leq m)$. For $A_1, A_2, \cdots, A_m \in \mathcal{B}$, we have:

$$\begin{aligned}
&\mathbb{P}(\xi_{N_r+k} \in A_k, 1 \leq k \leq m) \\
&= \sum_{t=0}^{\infty} \mathbb{P}(\xi_{N_r+k} \in A_k, 1 \leq k \leq m, N_r = t) \\
&= \sum_{t=0}^{\infty} \mathbb{P}(\xi_{t+k} \in A_k, 1 \leq k \leq m, N_r = t) \\
(N_r \perp (\xi_n : n \in \mathbb{N}^+)) &= \sum_{t=0}^{\infty} \mathbb{P}(\xi_{t+k} \in A_k, 1 \leq k \leq m)\mathbb{P}(N_r = t) \\
&= \sum_{t=0}^{\infty} \prod_{k=1}^{m} \mathbb{P}(\xi_{k+t} \in A_k)\mathbb{P}(N_r = t) \\
&= \sum_{t=0}^{\infty} \prod_{k=1}^{m} \mu(A_k)\mathbb{P}(N_r = t) \\
&= \prod_{k=1}^{m} \mu(A_k)
\end{aligned} \tag{1}$$

where $\mu$ is the distribution of $\xi_1$. So we get for $m \in \mathbb{N}^+$, the distribution of $(\xi_{N_r+k} : 1 \leq k \leq m)$ is same as that of $(\xi_k : 1 \leq k \leq m)$.

3. We know that $\forall t \in \mathbb{N}^+, \xi_{N_r+t} \in \sigma(N_r, \xi_k : k \in \mathbb{N}^+)$. So $\sigma(\xi_{N_r+k} : k \in \mathbb{N}^+) \subset \sigma(N_r, \xi_k : k \in \mathbb{N}^+)$. Since $(N_t : t \geq 0)$ is Possion process, we get that $N_{t+r} - N_r \perp N_r, \forall t \geq 0$. So $\sigma(N_{t+r} - N_r : t \geq 0) \perp N_r$. Easily $\sigma(N_{t+r} - N_r : t \geq 0) \perp \sigma(\xi_k : k \in \mathbb{N}^+)$, so finally we get that $\sigma(N_{t+r} - N_r : t \geq 0) \perp \sigma(N_r, \xi_k : k \in \mathbb{N}^+) \supset \sigma(\xi_{N_r+k} : k \in \mathbb{N}^+)$.

4. $\forall 0=t_0<t_1<\cdots<t_n$, then $X_{t_1}=\sum_{i=1}^{N_{t_1}}\xi_i, X_{t_k}-X_{t_{k-1}}=\sum_{i=1}^{N_{t_k}-N_{t_{k-1}}}\xi_{N_{t_{k-1}}+i}\xi_i, k=2,\cdots,n$, then $\forall\{A_k\in\mathscr{E}:k=1,\cdots,n\}$,

$$
\begin{aligned}
&\mathbb{P}(\bigcap_{k=1}^{n}\sum_{i=1}^{N_{t_k}-N_{t_{k-1}}}\xi_{i+N_{t_{k-1}}}\in A_k)\\
=&\mathbb{P}(\bigcup_{0\le u_1\le\cdots\le u_n}\{\sum_{i=u_{k-1}+1}^{u_k}\xi_i\in A_k, N_{t_k}=u_k, k=1,\cdots,n\})\\
=&\sum_{0\le u_1\le\cdots\le u_n}\mathbb{P}(\sum_{i=u_{k-1}+1}^{u_k}\xi_i\in A_k, k=1,\cdots,n|N_{t_k}=u_k,k=1,\cdots,n\})\mathbb{P}(N_{t_k}=u_k,k=1,\cdots,n)\\
=&\sum_{0\le u_1\le\cdots\le u_n}\mathbb{P}(\sum_{i=u_{k-1}+1}^{u_k}\xi_i\in A_k, k=1,\cdots,n\})\mathbb{P}(N_{t_k}=u_k,k=1,\cdots,n)\\
=&\sum_{0\le u_1\le\cdots\le u_n}\prod_{k=1}^{n}\mathbb{P}(\sum_{i=u_{k-1}+1}^{u_k}\xi_i\in A_k)\prod_{j=1}^{n}\mathbb{P}(N_{t_j}=u_j)\\
=&\sum_{0\le u_1\le\cdots\le u_n}\prod_{k=1}^{n}\mathbb{P}(\sum_{i=1}^{u_k-u_{k-1}}\xi_{u_{k-1}+i}\in A_k)\prod_{j=1}^{n}\mathbb{P}(N_{t_j}-N_{t_{j-1}}=u_j-u_{j-1})\\
=&\sum_{0\le u_1\le\cdots\le u_n}\prod_{k=1}^{n}\mathbb{P}(\sum_{i=1}^{u_k-u_{k-1}}\xi_{u_{k-1}+i}\in A_k)\mathbb{P}(N_{t_k}-N_{t_{k-1}}=u_k-u_{k-1})\\
=&\sum_{u_1-u_0\in\mathbb{N}}\cdots\sum_{u_n-u_{n-1}\in\mathbb{N}}\prod_{k=1}^{n}\mathbb{P}(\sum_{i=1}^{u_k-u_{k-1}}\xi_{u_{k-1}+i}\in A_k, N_{t_k}-N_{t_{k-1}}=u_k-u_{k-1})\\
=&\prod_{k=1}^{n}\sum_{u_k-u_{k-1}\in\mathbb{N}}\mathbb{P}(\sum_{i=1}^{u_k-u_{k-1}}\xi_{u_{k-1}+i}\in A_k, N_{t_k}-N_{t_{k-1}}=u_k-u_{k-1})\\
=&\prod_{k=1}^{n}\mathbb{P}(\sum_{i=1}^{N_{t_k}-N_{t_{k-1}}}\xi_{i+N_{t_{k-1}}}\in A_k)
\end{aligned}
\tag{2}
$$

□

PROBLEM II Assume that $X$ is Possion random measure on $(E,\mathcal{E})$ with intensity $\mu$, which is a $\sigma$-finite measure. Assume $f:E\to\mathbb{R}$ is measurable and non-negative, prove that:

$$\mathbb{E}(\mathrm{e}^{-X(f)})=\exp\left\{-\int_E(1-\mathrm{e}^{-f(x)})\mu(\mathrm{d}x)\right\}$$

SOLUTION. Let $\mathcal{L}:=\{g\in\mathcal{M}(E,[0,\infty)):\mathbb{E}(\mathrm{e}^{-X(f)})=\exp\left(-\int_E(1-\mathrm{e}^{-f(x)})\mu(\mathrm{d}x)\right)\}$. First we prove that if $g$ is simple measurable function from $E$ to $[0,\infty)$, then $g\in\mathcal{L}$. Assume $g(x)=\sum_{k=1}^{n}a_k\mathbb{1}_{A_k}(x)$, where $A_k\in\mathcal{E}, a_k>0, A_i\cap A_j=\emptyset$. Then $\mathbb{E}(\exp(-X(g)))=\mathbb{E}(\prod_{k=1}^{n}\exp(-a_kX(A_k)))=$

$\prod_{k=1}^{n} \mathbb{E}(\exp(-a_k X(A_k)))$, since $X(A_k) : k = 1, \cdots, n$ are independent. Easily to know

$$\mathbb{E}(\exp(-a_k X(A_k))) = \sum_{i=0}^{\infty} \mathbb{P}(X(A_k) = i) \exp(-a_k i) = \sum_{i=0}^{\infty} \frac{\exp(-\mu(A_k))\mu(A_k)^i}{i!} \exp(-a_k i)$$

Noting that

$$\exp(-\int_E (1-\exp(-a_k \mathbb{1}_{A_k}(x)))\mu(\mathrm{d}x)) = \exp(\exp(-a_k)\mu(A_k)-\mu(A_k)) = \exp(-\mu(A_k)) \sum_{i=0}^{\infty} \frac{(\exp(-a_k)\mu(A_k))^i}{i!}$$

we get $\mathbb{E}(\exp(-a_k X(A_k))) = \exp(-\int_E (1-\exp(-a_k \mathbb{1}_{A_k}(x)))\mu(\mathrm{d}x))$. Noting $\int_E (1-\exp(-g(x)))\mu(\mathrm{d}x) = \sum_{k=1}^{n} \int_E (1-\exp(-a_k \mathbb{1}_{A_k}(x)))\mu(\mathrm{d}x)$, we get $\mathbb{E}(\exp(-X(g))) = \exp\left(-\int_E (1-\exp(-g(x)))\mu(\mathrm{d}x)\right)$.

Now for non-negative function $f$, consider $f_n$ satisfy that $\forall n, f_n$ is simple, and $f_n \nearrow f$ and $f_n \geq 0$. Then easily to know $\mathbb{E}(\exp(-X(f))) = \lim_{n\to\infty} \mathbb{E}(\exp(-X(f_n))) = \lim_{n\to\infty} \exp(-\int_E (1 - \exp(-f_n(x)))\mu(\mathrm{d}x)) = \exp(-\int_E (1-\exp(-f(x)))\mu(\mathrm{d}x))$. □

PROBLEM III Assume $\mu$ is finite measure on $(E, \mathcal{E})$, and $X$ is Possion random measure with intensity $\mu$. Assume $\phi : (E, \mathcal{E}) \to (F, \mathcal{F})$ is measurable, prove that $X \circ \phi^{-1}$ is Possion random measure with intensity $\mu \circ \phi^{-1}$.

SOLUTION. Assume $B_k \in \mathcal{F}, \forall k \in \mathbb{N}$ and $\forall i \neq j, B_i \cap B_j = \emptyset$. Then $X \circ \phi^{-1}(\bigcup_{k\in\mathbb{N}} B_k) = X(\bigcup_{k\in\mathbb{N}} \phi^{-1}(B_k)) = \sum_{k\in\mathbb{N}} X(\phi^{-1}(B_k))$. Since $X$ is possion random measure with intensity $\mu$, and for $B_1, \cdots, B_n \in \mathcal{F}$ and $B_i \cap B_j = \emptyset$, we have $\phi^{-1}(B_k)$ are disjoint set in $(E, \mathcal{E})$, so $\mathbb{E}(\exp(\mathrm{i} \sum_{k=1}^{n} \alpha_k X \circ \phi^{-1}(B_k))) = \exp(\sum_{k=1}^{n} (\exp(\mathrm{i}\alpha_k) - 1)\mu \circ \phi^{-1}(B_k))$. So $X \circ \phi^{-1}$ is Possion random measure on $(F, \mathcal{F})$ with intensity $\mu \circ \phi^{-1}$. □

PROBLEM IV Assume $\alpha \geq 0$, and $\mu$ is probability measure on $\mathbb{R}$ with $\mu(\{0\}) = 0$. Let $N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ is Possion random measure on $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+$ with intensity $\mathrm{d}s\mu(\mathrm{d}z)\,\mathrm{d}u$. Let $Y_t = Y_0 + \int_0^t \int_{\mathbb{R}} \int_0^{\alpha} z N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$, where $Y_0 \perp N$. Prove that $(Y_t : t \geq 0)$ is compound Possion process with rate $\alpha$ and jumping distribution $\mu$.

SOLUTION. We know that $\forall t \geq 0, \forall r : 0 \leq r \leq t, Y_r \in \sigma(N(B) : B \subset [0, r] \times \mathbb{R} \times [0, \alpha])$. And $\forall w \geq t$, $Y_w - Y_t \in \sigma(N(B) : B \subset (t, w] \times \mathbb{R} \times [0, \alpha])$. Easily $(t, w] \cap [0, r] = \emptyset$, so we get $Y_w - Y_t \perp (Y_r : 0 \leq r \leq t)$. Now we only need to check the distribution of $Y_t + w - Y_t$ for $t, w \geq 0$.

Easily to know that:

$$\begin{aligned}
\mathbb{E}\left(\mathrm{e}^{\mathrm{i}\theta(Y_{t+w}-Y_t)}\right) =& \mathbb{E}\exp\left(\int_s^{s+t} \int_{\mathbb{R}} \int_0^{\alpha} i\theta z N(\mathrm{d}s, \mathrm{d}u, \mathrm{d}z)\right) \\
=& \exp\left(t\alpha \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}\theta z} - 1)\mu(\mathrm{d}z)\right) \\
=& \exp(-t\alpha) \sum_{k=0}^{\infty} \frac{1}{k!} (t\alpha \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\theta z} \mu(\mathrm{d}z))^k \\
=& \mathrm{e}^{-t\alpha} \sum_{k=0}^{\infty} \frac{(\alpha k)^k}{k!} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\theta z} \mu^{*k}(\mathrm{d}z)
\end{aligned}$$

So we get the result. □

PROBLEM V Assume $X$ is Possion random measure on $(E,\mathcal{E})$ with intensity $\mu$, a finite measure. Assume $f,g$ are non-negative measure function on $E$. Prove that:

1. $\mathbb{E}(X(f)\mathrm{e}^{-X(g)}) = \mu(f\mathrm{e}^{-g})\mathbb{E}(\mathrm{e}^{-X(g)})$.
2. $\mathbb{E}(X(f)^2\mathrm{e}^{-X(g)}) = (\mu(f^2\mathrm{e}^{-g} + \mu(f\mathrm{e}^{-g})'2))\mathbb{E}(\mathrm{e}^{-X(g)})$.

SOLUTION.

1. Let $h(\theta) := \mathbb{E}(\mathrm{e}^{-X(\theta f+g)}) = \exp\left(-\int_E(1-\mathrm{e}^{-\theta f(x)-g(x)}\mu(\mathrm{d}x))\right)$. Then

$$h'(\theta) = \mathbb{E}(X(f)\mathrm{e}^{-X(\theta f+g)}) = \exp\left(-\int_E(1-\mathrm{e}^{-\theta f(x)-g(x)}\mu(\mathrm{d}x))\right)\cdot\int_E f(x)\mathrm{e}^{-\theta f(x)-g(x)}\mu(\mathrm{d}x)$$

   Since they are all non-negative, the differential is valid. Let $\theta=0$, we get $\mathbb{E}(X(f)\mathrm{e}^{-X(g)}) = \mu(f\mathrm{e}^{-g})\mathbb{E}(\mathrm{e}^{-X(g)})$.

2. Take $h$ as above, easily to get $h''(\theta) = \mathbb{E}(X(f)^2)\mathrm{e}^{-X(g)} = \exp\left(-\int_E(1-\mathrm{e}^{-\theta f(x)-g(x)}\mu(\mathrm{d}x))\right)\cdot\left(\int_E f(x)\mathrm{e}^{-\theta f(x)-g(x)}\mu(\mathrm{d}x)\right)^2 + \exp\left(-\int_E(1-\mathrm{e}^{-\theta f(x)-g(x)}\mu(\mathrm{d}x))\right)\cdot\int_E f(x)^2\mathrm{e}^{-\theta f(x)-g(x)}\mu(\mathrm{d}x)$. Let $\theta=0$, then easily $\mathbb{E}(X(f)^2\mathrm{e}^{-X(g)}) = (\mu(f^2\mathrm{e}^{-g} + \mu(f\mathrm{e}^{-g})'2))\mathbb{E}(\mathrm{e}^{-X(g)})$.

$\square$