GroupRepresentation 11

白永乐

202011150087

202011150087@mail.bnu.edu.cn

2023 年 12 月 21 日

PROBLEM I Find the complex character table of following group.

1. D_4.
2. $Q = \langle j, i : i^4 = j^4 = 1, jij^{-1} = i^{-1} \rangle$.
3. A_4.
4. D_6.

SOLUTION. 1. Write $D_4 = \langle \sigma, \tau : \sigma^4 = \tau^2 = 1, \tau\sigma\tau = \sigma^{-1} \rangle$. First we find all of conjugate class of D_4, they are $C_1 = \{1\}, C_2 = \{\sigma, \sigma^3\}, C_3 = \{\sigma^2\}, C_4 = \{\tau, \sigma^2\tau\}, C_5 = \{\sigma\tau, \sigma^3\tau\}$. Second we find all of 1-dimetional reperesentation of D_4. Only need to find all 1-dimetional reperesentation of D_4/D_4'. Easily we get $D_4' = \{\sigma^2, 1\}$, so $D_4/D_4' = \{D_4', D_4'\sigma, D_4'\tau, D_4'\sigma\tau\}$. So D_4/D_4' has 4 different reperesentation, write $\overline{\varphi_0}, \overline{\varphi_1}, \overline{\varphi_2}, \overline{\varphi_3}$, where $\overline{\varphi_0}$ is main reperesentation. And let $\overline{\varphi_1}(D_4'\sigma) = -1, \overline{\varphi_1}(D_4'\tau) = 1, \overline{\varphi_2}(D_4'\sigma) = 1, \overline{\varphi_2}(D_4'\tau) = -1, \overline{\varphi_3}(D_4'\sigma) = -1, \overline{\varphi_3}(D_4'\tau) = -1$. Then improve them to D_4, we get $\varphi_0, \varphi_1, \varphi_2, \varphi_3$, where φ_0 is main reperesentation, and $\varphi_i(x) = \overline{\varphi_i}(D_4'x)$. They are all of 1-dimetional reperesentation of D_4. Now we find other irreducible reperesentation of D_4. Since $|D_4| = 8 = 1^2 + 1^2 + 1^2 + 1^2 + 2^2$ we get D_4 has a 2-dimetional irreducible reperesentation. Consider $\varphi_4 : D_4 \to M_2(\mathbb{C}), \sigma \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tau \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Obviously it's irreducible reperesentation of D_4. So all of irreducible reperesentation are $\varphi_0, \varphi_1, \varphi_2, \varphi_3, \varphi_4$. Now let $g_1 = 1, g_2 = \sigma, g_3 = \sigma^2, g_4 = \tau, g_5 = \sigma\tau$, and let $W_{ij} = \chi_{i-1}(g_j)$. Then we get

$$W = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 \\ 2 & 0 & -2 & 0 & 0 \end{pmatrix}$$

2. Write $Q = \{\pm 1, \pm i, \pm j, \pm k\}$. Easily we get $C_0 = \{1\}, C_1 = \{-1\}, C_2 = \{\pm i\}, C_3 = \{\pm j\}, C_4 = \{\pm k\}$ are conjugate class of Q. So Q has 5 different irreducible reperesentation. Easily we know $Q' = \{\pm 1\}$ and $Q/Q' = \{Q', Q'i, Q'j, Q'k = Q'ij\}$. Easily Q/Q' has 4 different 1-dimetional reperesentation, write $\overline{\varphi_0}, \overline{\varphi_1}, \overline{\varphi_2}, \overline{\varphi_3}$, where $\overline{\varphi_0}$ is main reperesentation. And $\overline{\varphi_1}(Q'i) = -1, \overline{\varphi_1}(Q'j) = 1; \overline{\varphi_2}(Q'i) = 1, \overline{\varphi_2}(Q'j) = -1; \overline{\varphi_3}(Q'i) = \overline{\varphi_3}(Q'j) = -1$. Improve them we get $\varphi_0, \varphi_1, \varphi_2, \varphi_3$, and φ_0 is main reperesentation, and $\varphi_t(x) = \overline{\varphi_t}(Q'x)$. Since $|Q| = 8 = 1^2 + 1^2 + 1^2 + 1^2 + 2^2$, we get the last reperesentation is 2-dimetional. Consider $\varphi_4 : Q \to M_2(\mathbb{C}), i \mapsto \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, j \mapsto \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$. Easily we get φ_4 is irreducible, so $\varphi_t, t = 0, \cdots, 4$ are all irreducible reperesentation of Q.

   Now let $g_1 = 1, g_2 = -1, g_3 = i, g_4 = j, g_5 = k$ and $W_{ij} = \chi_{i-1}(g_j)$. Then we have

$$W = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 \\ 2 & -2 & 0 & 0 & 0 \end{pmatrix}$$

3. Obviously $A_4' = K_4 = \{(12)(34), (13)(24), (14)(23), (1)\}$. And $C_1 = \{(1)\}, C_2 = K_4 \setminus C_1, C_3 = \{(123), (243), (134), (142)\}, C_4 = \{(132), (124), (143), (234)\}$ are all of conjugate class of A_4. Easily $A_4/K_4 = \{(123)K_4, (132)K_4, K_4\}$. So it has 3 different irreducible 1-dimetional reperesentation. Write $\overline{\varphi_0}, \overline{\varphi_1}, \overline{\varphi_2}$, where $\overline{\varphi_0}$ is main reperesentation. And $\overline{\varphi_1}((123)K_4) = \omega, \overline{\varphi_2}((123)K_4) = \omega^2$. Now improve then to A_4, we get $\varphi_0, \varphi_1, \varphi_2$, where φ_0 is main reperesentation, and $\varphi_t(x) = \overline{\varphi_t}(xK4)$. Sicne $|A_4| = 1^2 + 1^2 + 1^2 + 1^2 + 3^2$, we know the last irreducible reperesentation is 3-dimetional. Consider $\varphi_3 : A_4 \to M_3(\mathbb{C})$, and

$$\varphi_3((123)) = \begin{pmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \varphi_3((124)) = \begin{pmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

   Easily we get φ_3 is irreducible. So all irreducible reperesentation of A_4 are $\varphi_0, \varphi_1, \varphi_2, \varphi_3$.

   Now let $g_1 = (1), g_2 = (12)(34), g_3 = (123), g_4 = (132)$ and $W_{ij} = \chi_{i-1}(g_j)$, then we have

$$W = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & \omega & \omega^2 \\ 1 & 1 & \omega^2 & \omega \\ 3 & -1 & 0 & 0 \end{pmatrix}$$

4. First easily we get $C_1 = \{e\}, C_2 = \{\sigma^3\}, C_3 = \{\sigma, \sigma^5\}, C_4 = \{\sigma^2, \sigma^4\}, C_5 = \{\tau, \sigma^2\tau, \sigma^4\tau\}, C_6 = \{\sigma\tau, \sigma^3\tau, \sigma^5\tau\}$ are conjugate classes of D_6. So there are 6 different irreducible reperesentation of D_6. Second we should find all of 1-dim reperesentation of D_6. Easily we get $D_6' = \{e, \sigma^2, \sigma^4\}$, so we get $D_6/D_6' \cong K_4$, where K_4 is the Klein group. So we get $\overline{\varphi_0}, \overline{\varphi_1}, \overline{\varphi_2}, \overline{\varphi_3}$

are 1-dim irreducible reperesentation of D_6/D_6', where $\overline{\varphi_0}$ is the main reperesentation, and $\overline{\varphi_1}(\sigma D_6') = -1, \overline{\varphi_1}(\tau D_6') = 1, \overline{\varphi_2}(\sigma D_6') = 1, \overline{\varphi_2}(\tau D_6') = -1, \overline{\varphi_3}(\sigma D_6') = \overline{\varphi_3}(\tau D_6') = -1$. Improve them by $\varphi_i(x) = \overline{\varphi_i}(xD_6')$, we get $\varphi_0, \varphi_1, \varphi_2, \varphi_3$ are all of 1-dim reperesentation of D_6. Now we should find other reperesentation. Since $|D_6| = 12 = 1^2 + 1^2 + 1^2 + 1^2 + 2^2 + 2^2$, we get D_6 has two 2-dim irreducible reperesentation. Consider $\varphi_\theta(\sigma) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ and $\varphi_\theta(\tau) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, let $\varphi_4 = \varphi_{\frac{\pi}{3}}$ and $\varphi_5 = \varphi_{\frac{2\pi}{3}}$. then easily we get φ_5, φ_6 are irreducible. So all of irreducible reperesentation of D_6 are $\varphi_i, i = 0, \cdots, 5$. Let $g_1 = e, g_2 = \sigma^3, g_3 = \sigma, g_4 = \sigma^2, g_5 = \tau, g_6 = \sigma\tau$ and $W_{ij} = \chi_{i-1}(g_j)$, then we get

$$W = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 \\ 2 & -2 & 1 & -1 & 0 & 0 \\ 2 & 2 & -1 & -1 & 0 & 0 \end{pmatrix}$$

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