underGraduate Homework In Mathematics

AlgebraicGeometry 8

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PROBLEM I Assume $V \subset \mathbb{P}_k^n$ is irreducible, let $U_i := \{[x_0, \cdots, x_n] : [x_0, \cdots, x_n] \in \mathbb{P}_k^n \wedge x_i \neq 0\}$, for $i = 0, 1, \cdots, n$. Let $\phi_i : U_i \to \mathbb{A}_k^n, [x_0, \cdots, x_n] \mapsto \frac{1}{x_i}(x_0, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$. Prove that $\phi_i(V \cap U_i) \subset \mathbb{A}_k^n$ is irreducible.

SOLUTION. If not, we assume $\phi_i(V \cap U_i) = W_1 \cup W_2$, where $W_1, W_2 \subsetneq \phi_i(V \cap U_i)$ are closed algebraic set in $\mathbb{A}_k^n$. Since ϕ_i is bijective, we can assume $T_1 = \phi_i^{-1}(W_1), T_2 = \phi_i^{-1}(W_2)$. Let $S_1 = \mathbb{V}(\mathbb{I}(T_1)), S_2 = \mathbb{V}(\mathbb{I}(T_2))$, then $S_1 = \mathbb{V}(\mathbb{I}(T_1)) \subset \mathbb{V}(\mathbb{I}(V)) = V$. So $S_1, S_2 \subset V$. And we know $V \cap U_i = T_1 \cup T_2 \subset S_1 \cup S_2$, so $V = (V \cap U_i^c) \cup S_1 \cup S_2$. Since V is irreducible, and $V \cap U_i \neq \varnothing$, we get $S_1 = V \vee S_2 = V$. Without loss of generality assume $S_1 = V$. Let $\Phi : k[x_0, \cdots, x_n] \to k[x_0, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n], f(x_0, \cdots, x_n) \mapsto f(x_0, \cdots, x_{i-1}, 1, x_{i+1}, \cdots, x_n)$. Now we prove $\Phi(\mathbb{I}(T_1)) = \mathbb{I}(W_1)$. For $f(x_0, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n) \in \mathbb{I}(W_1)$,

On one hand, assume $f(x_0, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n) = \sum_j a_j \prod_{k \neq i} x_k^{j_k}$. Consider $g(x_0, \cdots, x_n) = \sum_j a_j x_i^{d - \sum_{k \neq i} j_k} \prod_{k \neq i} x_k^{j_k} \in k[x_0, \cdots, x_n]$ is homogenous poly. Where $d \geq \deg f$ is a integer. Easily $\Phi(g) = f$. Now we prove $g \in \mathbb{I}(T_1)$. For $[x_0, \cdots, x_n] \in T_1$, we have $g(x_0, \cdots, x_n) = 0 \iff f(\phi_i(x_0, \cdots, x_n)) = 0$. Since $[x_0, \cdots, x_n] \in T_1$ we get $\phi_i(x_0, \cdots, x_n) \in W_1$, so $g(x_0, \cdots, x_n) = f(\phi_i(x_0, \cdots, x_n)) = 0$. So we get $\mathbb{I}(W_1) \subset \Phi(\mathbb{I}(T_1))$.

On the other hand, Assume $g(x_0, \cdots, x_n) \in \mathbb{I}(T_1)$ is homogenous poly. Consider

$$f(x_0, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n) = \Phi(g)(x_0, \cdots, x_n) = g(x_0, \cdots, x_{i-1}, 1, x_{i+1}, \cdots, x_n)$$

For $(x_0, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n) \in W_1$, we have

$$f(x_0, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n) = g(x_0, \cdots, x_{i-1}, 1, x_{i+1}, \cdots, x_n)$$

Easily $\phi_i(x_0, \cdots, x_{i-1}, 1, x_{i+1}, \cdots, x_n) = (x_0, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n) \in W_1$,
so $[x_0, \cdots, x_{i-1}, 1, x_{i+1}, \cdots, x_n] \in T_1$. So $g(x_0, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n) = 0$. So we get $f = \Phi(g) \in \mathbb{I}(W_1)$.

So we get $\Phi(\mathbb{I}(T_1)) = \mathbb{I}(W_1)$. Since $\mathbb{I}(V) = \mathbb{I}(S_1) = \mathbb{I}(T_1)$, we get $\Phi(\mathbb{I}(V)) = \mathbb{I}(W_1)$. Now we prove $W_1 = \phi_i(V \cap U_i)$.

For $(x_0, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n) \in \phi_i(V \cap U_i)$, we get $[x_0, \cdots, x_{i-1}, 1, x_{i+1}, \cdots, x_n] \in V$. For any $f(x_0, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n) \in \Phi(\mathbb{I}(V))$, there exists $g \in \mathbb{I}(V)$ such that $f = \Phi(g)$. So $f(x_0, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n) = g(x_0, \cdots, x_{i-1}, 1, x_{i+1}, \cdots, x_n) = 0$. So $(x_0, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n) \in \mathbb{V}(\Phi(\mathbb{I}(V))) = \mathbb{V}(\mathbb{I}(W_1)) = W_1$. So $W_1 = \phi_i(V \cap U_i)$.

It's contradiction with $W_1, W_2 \subsetneq \phi_i(V \cap U_i)$. So finally we get $\phi_i(V \cap U_i)$ is irreducible. $\square$