AlgebraicGeometry 1

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PROBLEM I P is an ideal of a unitary commutative ring A, then P is prime ideal of A $\iff$ A/P is integral domain.

SOLUTION. $\Rightarrow$:

Since A is a unitary commutative ring, so A/P is unitary commutative ring, too. So we only need to prove $[ab] = [0] \Rightarrow [a] = [0] \vee [b] = [0]$. Obviously $[ab] = 0 \iff ab \in P \iff a \in P \vee b \in P \iff [a] = [0] \vee [b] = [0]$.

$\Leftarrow$:

As the same, $ab \in P \iff [ab] = [0] \Rightarrow [a] = [0] \vee [b] = [0] \iff a \in P \vee b \in P$, so P is prime ideal. $\square$

PROBLEM II M is an ideal of a unitary commutative ring A, then M is maximal ideal of A $\iff$ A/M is a field.

SOLUTION. $\Rightarrow$:

Consider $[a] \in A/M \setminus [0]$, we will prove it has a reverse. Consider $N := \{xm + ya : x, y \in A, m \in M\}$ is the minimum ideal of A contains M and a. Since $[a] \neq [0]$ we know $a \notin M$, so $M \subsetneqq N$. Noting M is maximal, so $N = A$. That means $\exists x, y \in A, m \in M, xm + ya = 1$. So $[xm + ya] = [1]$. Since $[xm] = [0]$ we get $[y][a] = 1$, i.e., $[y] = [a]^{-1}$.

$\Leftarrow$:

Consider $a \in A \setminus M, N := \{xp + ya : x, y \in A, p \in P\}$, we will prove $N = A$, which means M is maximal. Since A/M is field, $\exists y \in A, [y] = [a]^{-1}$. That's means $ay - 1 \in M \subset N$. Noting $ay \in N$, so $1 \in N$, thus $N = A$. $\square$

PROBLEM III A ring A is noetherian, $I \subset A$ is an ideal of A, then A/I is noetherian.

SOLUTION. Consider an ideal $J \subset A/I$, let $M := \{x \in A : [x] \in J\}$. Then $\forall a \in A, x \in M, [ax] = [a][x] \in J$, so $ax \in M$. $\forall a, b \in M, [a - b] = [a] - [b] \in J$, so $a - b \in M$. So M is an ideal of A. Since A is noetherian, we can assume $M = (f_i, i = 1, 2, \cdots n)$. Now we will prove $J = ([f_i], i = 1, 2, \cdots n)$. Consider $[f] \in J$, from definition of M we know $f \in M$, so $f = \sum_{i=1}^{n} a_i f_i, a_i \in A$, thus $[f] = [\sum_{i=1}^{n} a_i f_i] = \sum_{i=1}^{n} [a_i][f_i]$. So $J = ([f_i], i = 1, 2, \cdots n)$. $\square$

PROBLEM IV K is a field, $A = K[x_1, x_2, \cdots x_n], \mathbb{A}_K^n = K^n$. For ideal I of A let $V(I) := \{p \in \mathbb{A}_K^n : f(p) = 0, \forall f \in I\}$. Then $V(I_1) \cup V(I_2) = V(I_1 \cap I_2) = V(I_1 I_2)$

SOLUTION .

Lemma 1. $I \subset J \Rightarrow V(I) \supset V(J)$.

证明. Consider $p \in V(J)$, we get $\forall f \in J, f(p) = 0$. Since $I \subset J$, so $\forall f \in I, f(p) = 0$, i.e., $f \in V(I)$. □

From Lemma Lemma 1 we know $V(I_1 \cap I_2) \subset V(I_1 I_2)$, so we only need to prove $V(I_1) \cup V(I_2) \subset V(I_1 \cap I_2), V(I_1 I_2) \subset V(I_1) \cup V(I_2)$.

- $V(I_1) \cup V(I_2) \subset V(I_1 \cap I_2)$: From Lemma 1 It's obvious.
- $V(I_1 I_2) \subset V(I_1) \cup V(I_2)$: Consider $p \in V(I_1 I_2)$. If $p \notin V(I_1) \cup V(I_2)$, then $\exists f_1 \in I_1, f_2 \in I_2, f_1(p) \neq 0, f_2(p) \neq 0$. Now consider $f = f_1 f_2 \in I_1 I_2$, we get $f(p) = f_1(p) f_2(p) \neq 0$, so $p \notin V(I_1 I_2)$, it's a contradiction.

□

PROBLEM V K is an infinite field, then $\mathbb{A}_k^n$ is not Hausdorff.

SOLUTION .

Lemma 2. K is a infinite field, $f \in K[x_1, x_2, \cdots x_n] \setminus \{0\}$, then exists $p \in \mathbb{A}_K^n, f(p) \neq 0$.

证明. Use MI. When $n = 0, K[x_1, x_2, \cdots x_n] = K$, so it's obvious. Assume for $n = k$ it's right, when goes to $k + 1$:

Consider $h \in K(x_1, x_2, \cdots x_k)[x_{k+1}], h(x_{k+1}) := f(x_1, x_2, \cdots x_k, x_{k+1})$ is a non-zero polynomial so it has finite root. So exists $a \in K, h(a) \neq 0$. So $g := f(x_1, x_2, \cdots x_k, a) \in K[x_1, x_2, \cdots x_k] \neq 0$. By induction hypothesis we get $\exists b \in \mathbb{A}_K^k, g(b) \neq 0$. Let $p := (b, a) \in \mathbb{A}_K^{k+1}$, then $f(p) \neq 0$. □

In fact, it's not only not Hausdorff, it's kind of "absolutly not Hausdorff" because every pair of point can't be seperated. Consider two point $p \neq q, p, q \in \mathbb{A}_K^n$. Assume $p = (p_1, p_2, \cdots p_n), q = (q_1, q_2, \cdots q_n), p_1 \neq q_1$. Assume two open set $V(I_1)^c, V(I_2)^c$ can seperate p, q, then $V(I_1) \cup V(I_2) = \mathbb{A}_K^n$. From PROBLEM IV we know $V(I_1 I_2) = \mathbb{A}_K^n$. So $\forall f \in V(I_1 I_2), \forall p \in \mathbb{A}_K^n, f(p) = 0$. Then from LemmaLemma 2 we can get $f = 0$. So $I_1 I_2 = \{0\}$. Since $p \notin V(I_1), q \notin V(I_2)$, we know $I_1, I_2 \neq \{0\}$. So $\exists f_1 \in I_1 \neq 0, \exists f_2 \in I_2 \neq 0$, and thus $f = f_1 f_2 \in I_1 I_2 \neq 0$, contradiction! So these is not a pair of points can be seperated by two disjoint open set. □