underGraduate Homework In Mathematics

AlgebraicGeometry 9

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General fire extinguisher

PROBLEM I Assume V is irreducible algebraic set in $\mathbb{A}_k^n$. Assume $\theta : \mathbb{A}_k^n \to \mathbb{P}_k^n$ is the imbedding. Prove that $\overline{\theta(V)} \subset \mathbb{P}_k^n$ is irreducible.

SOLUTION. If not, assume $\overline{\theta(V)} = W_1 \cup W_2$, where $W_1, W_2 \subsetneq \theta(V)$ are algebraic set. First we prove $\overline{\theta(V)} \cap U_0 = \theta(V)$, where $U_0 = \{[x_0, \cdots, x_n] : x_0 \neq 0\}$. Obviously $\theta(V) \subset \overline{\theta(V)} \cap U_0$, so we only need to prove $\overline{\theta(V)} \cap U_0 \subset \theta(V)$. Consider $\Phi : k[x_0, \cdots, x_n] \to k[x_1, \cdots, x_n], f(x_0, \cdots, x_n) \mapsto f(1, x_1, \cdots, x_n)$. Then we have for any algebraic set W in $\mathbb{P}_k^n$, $\Phi(\mathbb{I}(W)) = \mathbb{I}(\theta^{-1}(W \cap U_0))$. Let $W = \overline{\theta(V)}$, we get $\mathbb{I}(\theta^{-1}(W \cap U_0)) = \Phi(\mathbb{I}(W)) = \mathbb{I}(V)$. So $\theta^{-1}(W \cap U_0) \subset V$.

Now consider $W_1 \cap U_0, W_2 \cap U_0$. Since $\overline{W_1 \cap U_0} \subset W_1 \subsetneq V = \overline{\theta(V)}$, we get $W_1 \cap U_0, W_2 \cap U_0 \subsetneq \theta(V)$, so $V = \theta^{-1}(W_1 \cap U_0) \cup \theta^{-1}(W_2 \cap U_0)$ is reducible, contradiction! So we get $\overline{(\theta(V))}$ is irreducible. $\square$

PROBLEM II Assume k is a algebraic closed field, $f \in k[x_1, \cdots, x_n]$ is irreducible, prove that $\mathbb{V}(f) \subset \mathbb{A}_k^n$ is irreducible algebraic set.

SOLUTION. Only need to prove (f) is prime ideal. i.e., $f \mid gh \to f \mid g \vee f \mid h$. Since $k[x_1, \cdots, x_n]$ is Unique factorization domain, we get f is prime element. So it's obvious. $\square$

PROBLEM III Assume V is irreducible algebraic set in $\mathbb{A}_k^n$, and $U \subset V$ is nonempty open set in V. Assume $f, g \in k(V)$ and $\forall p \in U, f(p) = g(p)$. Prove that $f = g$ in $k(V)$.

SOLUTION. Assume $f = \frac{f_1}{f_2}, g = \frac{g_1}{g_2}$, where $f_1, f_2, g_1, g_2 \in k[x_1, \cdots, x_n]$. And without loss of generality we assume $U \subset \mathrm{dom}(f_2), \mathrm{dom}(g_2)$, or we use $\mathrm{dom}(f_2) \cap \mathrm{dom}(g_2)$ replace U. To prove $f = g$ in $k(V)$, we only need to prove $f_1 g_2 - f_2 g_1 = 0$. Consider $h = f_1 g_2 - f_2 g_1 \in k[x_1, \cdots, x_n]$. We have $U \subset \mathbb{V}(h)$. So $(V \setminus U) \cup \mathbb{V}(h) = V$. Since V is irreducible, we get $V \setminus U = V \vee \mathbb{V}(h) = V$. Since $U \neq \varnothing$, we get $\mathbb{V}(h) = V$, so $h = 0$ in $k(V)$. $\square$