underGraduate Homework In Mathematics

RiemannGeometry 1

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PROBLEM I Assume $\mathcal{A}_0 = \{(U_\alpha, \phi_\alpha) : \alpha \in I\}$ is a C^r-compatible coordinate covery of a m-dimensional manifold M, let

$$\mathcal{A} := \{(U, \phi) : (U, \phi) \text{ is chart of } M, \wedge \forall (V, \psi) \in \mathcal{A}_0, (U, \phi) \text{ is compatible with } (V, \psi)\}$$

. Then $\mathcal{A}$ is unique C^r-differential structure on M contains $\mathcal{A}_0$.

SOLUTION . First, easily $\mathcal{A}_0 \subset \mathcal{A}$ by definition of $\mathcal{A}_0$. Now we should prove $\mathcal{A}$ is differential structure on M. Let $(U, \phi), (V, \psi) \in \mathcal{A}$. If $(U, \phi) \in \mathcal{A}_0$, then by definition of $\mathcal{A}_0$ we know (U, ϕ) is compatible to (V, ψ). If $(U, \phi), (V, \psi) \notin \mathcal{A}_0$, then consider $U \cap V$. If $U \cap V = \varnothing$, then (U, ϕ) is compatible with (V, ψ). Now assume $W = U \cap V \neq \varnothing$. Consider $\gamma := \psi \circ \phi^{-1} : \phi(W) \to \psi(W)$. For any $x \in W$, since $\mathcal{A}_0$ is covery of M, we know $\exists (T, \tau) \in \mathcal{A}_0, x \in T$. Then by the definition of $\mathcal{A}$, we know (T, τ) is compatible locally on x with both (U, ϕ) and (V, ψ). So (U, ϕ) is locally compatible with (V, ψ) on x. Since x is arbitrary, we know (U, ϕ) is compatible with (V, ψ). So $\mathcal{A}$ is differential structure of M.

Now we assume $\mathcal{B}$ is another differential structure of M contains $\mathcal{A}_0$. Since $\mathcal{B}$ is compatible, we get $\mathcal{B} \subset \mathcal{A}$. Since $\mathcal{B}$ is maximal, we get $\mathcal{B} = \mathcal{A}$. So $\mathcal{A}$ is unique. $\square$

PROBLEM II Assume $(U, \phi; x^i), (V, \psi; y^i), (W, \chi; z^i)$ are three local coordinate on an m-dimensional smooth manifold M, and $W \cap V \cap U \neq \varnothing$. Prove that on $\phi(U \cap V \cap W)$, we have:

$$\left(\frac{\partial z^i}{\partial x^j}\right) = \left(\frac{\partial z^i}{\partial y^k}\right)\left(\frac{\partial y^k}{\partial x^j}\right)$$

.

SOLUTION . For fixed $1 \leq i, j \leq m$, we have

$$\frac{\partial z^i}{\partial x^j} = \sum_{k=1}^{m} \frac{\partial z^i}{\partial y^k} \frac{\partial y^k}{\partial x^j}$$

. So easily to get that

$$\left(\frac{\partial z^i}{\partial x^j}\right) = \left(\frac{\partial z^i}{\partial y^k}\right)\left(\frac{\partial y^k}{\partial x^j}\right)$$

.

We let $(W, \chi; z^i) = (U, \phi; x^i)$, then we get:

$$I_m = \left(\frac{\partial x^i}{\partial y^k}\right)\left(\frac{\partial y^k}{\partial x^j}\right)$$

. So both terms on the right side are invertible, thus non-singular. $\square$

PROBLEM III Assume M is orientable and connected, prove that M has exactly two different oriention.

SOLUTION . Since M is orientable, we can assume that $\mathcal{B} \subset \mathcal{A}$ is an oriention of M, where $\mathcal{A}$ is all local coordinate of M. Now consider $\mathcal{C} := \{(U; -x^i) : (U; x^i) \in \mathcal{B}\}$. Easily to check that $\mathcal{C}$ is an oriention of M, too. And obviously $\mathcal{B} \cap \mathcal{C} = \varnothing$, thus $\mathcal{B} \neq \mathcal{C}$. So there is two oriention. Now we need to prove there is no other oriention.

Assume $\mathcal{D}$ is an oriention of M. We will define a function $\mathrm{sgn}: M \to \{1,-1\}$ by

$$\mathrm{sgn}(p) = 1 \iff \exists (U_0, x_0^i) \in \mathcal{B}, \exists (V_0, y_0^i) \in \mathcal{D}, p \in U_0 \cap V_0, J_{x_0,y_0}(p) > 0$$

. We will prove that

$$\mathrm{sgn}(p) = 1 \implies \forall (U, x^i) \in \mathcal{B}, \forall (V; y^i) \in \mathcal{D}, p \in U \cap V \implies J_{x,y}(p) > 0$$

. It's easy because $J_{x,y} = J_{x,x_0} J_{x_0,y_0} J_{y_0,y}$, and $J_{x,x_0} > 0, J_{y,y_0} > 0$ by definition of oriention. So

$$\mathrm{sgn}(p) = -1 \iff \exists (U_0, x_0^i) \in \mathcal{C}, \exists (V_0, y_0^i) \in \mathcal{D}, p \in U_0 \cap V_0, J_{x_0,y_0}(p) > 0$$

$$\iff \forall (U, x^i) \in \mathcal{B}, \forall (V; y^i) \in \mathcal{D}, p \in U \cap V \implies J_{x,y}(p) > 0$$

. Noting that if $\mathrm{sgn}(p) = 1$, we have $\forall q \in U_0 \cap V_0, \mathrm{sgn}(q) = 1$, and so is $\mathrm{sgn}(p) = -1$. So sgn is continuous. So sgn is constant because M is connected. Easy to check that $\mathrm{sgn}(p) = 1 \iff \mathcal{B} = \mathcal{D}$, and $\mathrm{sgn}(p) = -1 \iff \mathcal{C} = \mathcal{D}$. So there is only two oriention of M. □

PROBLEM IV Let

$$S^n(a) := \{(x^{(1)}, \cdots, x^{(n+1)}) \in \mathbb{R}_1^{n+1} : \sum_{k=1}^{n+1} (x^{(k)})^2 = a^2\}$$

be the ball in $\mathbb{R}^{n+1}$ with radius $a > 0$. Let $S := (0, \cdots, -a), N := (0, \cdots, a)$ be the South Pole and North Pole respectively. Let $U_+ = S^n(a) \setminus \{S\}, U_- = S^n(a) \setminus \{N\}$. Let $\phi_+ : U_+ \to \mathbb{R}^n, \phi_- : U_- \to \mathbb{R}^n$.

$$(\xi^{(1)}, \cdots, \xi^{(n)}) = \phi_+(x^{(1)}, \cdots, x^{(n+1)}) := \left(\frac{ax^{(1)}}{a + x^{(n+1)}}, \cdots, \frac{ax^{(n)}}{a + x^{(n+1)}} \right)$$

.

$$(\eta^{(1)}, \cdots, \eta^{(n)}) = \phi_+(x^{(1)}, \cdots, x^{(n+1)}) := \left(\frac{ax^{(1)}}{a - x^{(n+1)}}, \cdots, \frac{ax^{(n)}}{a - x^{(n+1)}} \right)$$

. Calculate the inverses of ϕ_+ and ϕ_-, thus prove $\{(U_+, \phi_+), (U_-, \phi_-)\}$ gives a smooth structuction of $S^n(a)$.

SOLUTION. Noting

$$\xi^{(i)} = \frac{ax^{(i)}}{a + x^{(n+1)}}$$

, so

$$x^{(i)} = \frac{\xi^{(i)}(a + x^{(n+1)})}{a}$$

. And

$$\sum_{k=1}^{n+1} (x^{(k)})^2 = a^2$$

, so

$$\sum_{k=1}^{n} \frac{(\xi^{(i)})^2 (a + x^{(n+1)})^2}{a^2} + (x^{(n+1)})^2 = a^2$$

. □

PROBLEM V