underGraduate Homework In Mathematics

SetTheory 3

白永乐

202011150087

202011150087@mail.bnu.edu.cn

2023 年 10 月 25 日

General fire extinguisher

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PROBLEM I Prove the following statements.

1. If $x \cap y = \varnothing$ and $x \cup y \preccurlyeq y$, then $\omega \times x \preccurlyeq y$.

2. If $x \cap y = \varnothing$ and $\omega \times x \preccurlyeq y$, then $x \cup y \approx y$.

SOLUTION. 1. Assume $f : x \cup y \to y$ is injective, define $f_1 = f, f_{n+1} = f_n \circ f$. Let $g : \omega \times x \to y, g(n,t) \mapsto f_{n+1}(t)$. We only need to prove g is injective. For $(n,u),(m,v) \in \omega \times x$, if $n = m$, then since f is injective we get f_n is injective, so $f_n(u) \neq f_n(v)$. Else, $m \neq n$, assume $n < m, m = n + k$. Obviously $f_{n+k}[x] = f_n[f_k[x]] \subset f_n[y]$. Since f_n is injective, we get $f_n[x] \cap f_n[y] = \varnothing$. So $f_n[x] \ni g(n,u) \neq g(m,v) \in f_n[y]$. So g is injective.

2. Assume $f : \omega \times x \to y$ is injective. Let $x_n := \{(n,t) : t \in x\}$. Then $\omega \times x = \cup_{n \in \omega} x_n$. Consider $g : x \cup y \to y$, for $t \in x$ let $g(t) := f(0,t)$, for $t \in f[x_n]$, let $g(t) = f(n+1,t)$, for other t, let $g(t) = t$. Then we prove g is bijection.

   First we prove g is injection. For $u, v \in x \cup y, u \neq v$, we will prove $g(u) \neq g(v)$.

   - $u, v \in x$: Since f is injective, we have $g(u) = f(0,u) \neq f(0,v) = g(v)$.
   - $u \in x, v \notin x$: From definition we obtain $f(u) \in f[x_0]$. If $v \in f[x_n]$ for some n, then $f(v) \in f[x_{n+1}]$. Since f is injective, $f[x_0] \cap f[x_{n+1}] = \varnothing$. So $f(u) \neq f(v)$. Else, we know $g(v) = v \notin f[x_0] \ni f(u)$.
   - $u \in f[x_m], v \in f[x_n]$: If $m = n$ then $g(u) = f(m+1,u) \neq f(n+1,v) = g(v)$. Else $m \neq n$, then $g(u) \in f[x_{m+1}], g(v) \in f[x_{n+1}], f[x_{m+1}] \cap f[x_{n+1}] = \varnothing$. So $g(u) \neq g(v)$.
   - $u \in x_n, v \notin x, \forall m, v \notin f[x_m]$: Easily $g(u) \in f[x_{n+1}]$ and $g(v) = v \notin f[x_{n+1}]$.
   - $u, v \notin x, \forall n, u, v \notin f[x_n]$: Easily $g(u) = u \neq v = g(v)$.

   Second we prove g is surjective. i.e., $\forall u \in y, \exists t \in x \cup y, g(t) = u$.

   - $u \in f[x_n]$ for some n: If $n = 0$ then $y = f(0,t)$ for some $t \in x$. Then $g(t) = u$. Else $n \geq 1$, write $n = m + 1$. Then $y = f(m+1,t)$ for some $t \in x$. So $g(t) = u$.
   - $u \notin f[x_n], \forall n$: Easily we get $g(u) = u$.

   So all in all g is bijective.

$\square$

PROBLEM II

1. A subset of a finite set is finite.

2. The union of a finite set of finite sets is finite.

3. The power set of a finite set is finite.

4. The image of a finite set (under a mapping) is finite.

PROBLEM I Prove the following statements.

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2. If $x \cap y = \varnothing$ and $\omega \times x \preccurlyeq y$, then $x \cup y \approx y$.

SOLUTION.
1. Assume $f : x \cup y \to y$ is injective, define $f_1 = f, f_{n+1} = f_n \circ f$. Let $g : \omega \times x \to y, g(n,t) \mapsto f_{n+1}(t)$. We only need to prove g is injective. For $(n,u),(m,v) \in \omega \times x$, if $n = m$, then since f is injective we get f_n is injective, so $f_n(u) \neq f_n(v)$. Else, $m \neq n$, assume $n < m, m = n + k$. Obviously $f_{n+k}[x] = f_n[f_k[x]] \subset f_n[y]$. Since f_n is injective, we get $f_n[x] \cap f_n[y] = \varnothing$. So $f_n[x] \ni g(n,u) \neq g(m,v) \in f_n[y]$. So g is injective.

2. Assume $f : \omega \times x \to y$ is injective. Let $x_n := \{(n,t) : t \in x\}$. Then $\omega \times x = \cup_{n \in \omega} x_n$. Consider $g : x \cup y \to y$, for $t \in x$ let $g(t) := f(0,t)$, for $t \in f[x_n]$, let $g(t) = f(n+1,t)$, for other t, let $g(t) = t$. Then we prove g is bijection.

   First we prove g is injection. For $u, v \in x \cup y, u \neq v$, we will prove $g(u) \neq g(v)$.

   - $u, v \in x$: Since f is injective, we have $g(u) = f(0,u) \neq f(0,v) = g(v)$.
   - $u \in x, v \notin x$: From definition we obtain $f(u) \in f[x_0]$. If $v \in f[x_n]$ for some n, then $f(v) \in f[x_{n+1}]$. Since f is injective, $f[x_0] \cap f[x_{n+1}] = \varnothing$. So $f(u) \neq f(v)$. Else, we know $g(v) = v \notin f[x_0] \ni f(u)$.
   - $u \in f[x_m], v \in f[x_n]$: If $m = n$ then $g(u) = f(m+1,u) \neq f(n+1,v) = g(v)$. Else $m \neq n$, then $g(u) \in f[x_{m+1}], g(v) \in f[x_{n+1}], f[x_{m+1}] \cap f[x_{n+1}] = \varnothing$. So $g(u) \neq g(v)$.
   - $u \in x_n, v \notin x, \forall m, v \notin f[x_m]$: Easily $g(u) \in f[x_{n+1}]$ and $g(v) = v \notin f[x_{n+1}]$.
   - $u, v \notin x, \forall n, u, v \notin f[x_n]$: Easily $g(u) = u \neq v = g(v)$.

   Second we prove g is surjective. i.e., $\forall u \in y, \exists t \in x \cup y, g(t) = u$.

   - $u \in f[x_n]$ for some n: If $n = 0$ then $y = f(0,t)$ for some $t \in x$. Then $g(t) = u$. Else $n \geq 1$, write $n = m + 1$. Then $y = f(m+1,t)$ for some $t \in x$. So $g(t) = u$.
   - $u \notin f[x_n], \forall n$: Easily we get $g(u) = u$.

   So all in all g is bijective.

□

PROBLEM II

1. A subset of a finite set is finite.
2. The union of a finite set of finite sets is finite.
3. The power set of a finite set is finite.
4. The image of a finite set (under a mapping) is finite.

SOLUTION. 1. Use MI to prove $\varphi(n) := \forall A \approx n, \forall B \subset A, \exists m \in \omega, B \approx m$ for $n \in \omega$. When $n = 0$, we know $A \approx 0 \to A = \varnothing$. So $B = \varnothing$ and thus $B \approx 0$. Now we prove $\varphi(n) \to \varphi(n+1)$. For $A \approx n+1$, if $B = A$ then $B \approx n+1$. Else, $\exists x \in A \setminus B$. Assume $f : A \to n+1$ is bijection. Consider $g : A \to n+1 \begin{cases} g(t) = f(t) & t \neq x \wedge g(t) \neq n \\ g(t) = n+1 & t = x \\ g(t) = f(x) & f(t) = n \end{cases}$ Easy to know g is bijection, too. And since $x \notin B$ we get $B \subset g^{-1}[n] \approx n$, so by induction we get $\exists m \in \omega, B \approx m$.

2. First we prove union of two disjoint finite sets A and B is finite.

   Use MI to the card of the second set B. For $B = \varnothing$ it's obvious. For $B \approx 1$, assume $A \approx n$, Thrn $B \approx \{n\}$, so $A \cup B \approx n \cup \{n\} = n+1$ is finite. Assume for certain $n, \forall B \approx n$ it's right, now we prove it's right for $n+1$. Assume $f : B \to n+1$ is bijection, then $f^{-1}[n] \approx n$, so $A \cup f^{-1}[n]$ is finite. Since $B = f^{-1}[n] \cup \{f^{-1}(n)\}$ we get $A \cup B = A \cup f^{-1}[n] \cup \{f^{-1}(n)\}$. Since $\{f^{-1}(n)\} \approx 1$, so the union is finite.

   For general two finite sets A, B we have $A \cup B = A \cup (B \setminus A)$ and from II.1 we know $B \setminus A$ is finite, so $A \cup B$ is finite. Now we use MI to prove $\varphi(n) := \forall x \approx n((\forall y \in x, isFinite(y)) \to isFinite(\bigcup x))$ for $n \in \omega$. When $n = 0, 1, 2$ it's obvious. Assume for certain $n \geq 2$ we have $\varphi(n)$, then we prove $\varphi(n+1)$. Assume $f : x \to n+1$ is bijective, let $y = f^{-1}[n] \subset x$. Then $y \approx n$, by induction we know $\bigcup y$ is finite. Since $x = y \cup \{f^{-1}(n)\}$ we get $\bigcup x = (\bigcup y) \cup f^{-1}(n)$. So $\bigcup x$ is finite, too.

3. Use MI of the card. For $x \approx 0$ we know $\mathscr{P}(x) = \{\varnothing\} \approx 1$. Assume for certain n we have $\forall x \approx n, isFinite(\mathscr{P}(x))$, then for $x \approx n+1$: Assume $f : x \to n+1$ is bijection. Let $y = f^{-1}[n]$ and $t = f^{-1}(n)$. Then $y \approx n$. Let $\theta : \mathscr{P}(x) \setminus \mathscr{P}(y) \to \mathscr{P}(y), \theta(a) := a \setminus \{t\}$. Easily θ is bijective, so $\mathscr{P}(x) \setminus \mathscr{P}(y) \approx \mathscr{P}(y)$ is finite. From II.2 we know $\mathscr{P}(x) = \mathscr{P}(y) \cup (\mathscr{P}(x) \setminus \mathscr{P}(y))$ is finite.

4. Use MI by card. For $A \approx 0$ it's obvious. Assume for $A \approx n$ it's right, now we prove for $A \approx n+1$ it's right, too. Let $f : A \to n+1$ is a bijection, and $g : A \to \mathbb{S}\mathrm{et}$ is a map on A. Let $B := f^{-1}[n] \subset A, t = f^{-1}(n) \in A$. Then $B \approx n$, so by induction we know $g[B]$ is finite. Since $A = B \cup \{t\}$ we get $g[A] = g[B] \cup g[\{t\}] = g[B] \cup \{g(t)\}$. Noting $\{g(t)\} \approx 1$ is finite, from II.2 we get $g[A]$ is finite, too.

$\square$

PROBLEM III

1. A subset of a countable set is at most countable.
2. The union of a finite set of countable sets is countable.
3. The image of a countable set (under a mapping) is at most countable.

SOLUTION. 1. Assume A is countable and $\theta : A \to \omega$ is bijection. For $B \subset A$, we have $B \approx \theta[B]$. So we only need to prove every subset of ω is at most countable. Let $x \subset \omega$. If x is finite, SOLUTION. 1. Use MI to prove $\varphi(n) := \forall A \approx n, \forall B \subset A, \exists m \in \omega, B \approx m$ for $n \in \omega$. When $n = 0$, we know $A \approx 0 \to A = \varnothing$. So $B = \varnothing$ and thus $B \approx 0$. Now we prove $\varphi(n) \to \varphi(n+1)$. For $A \approx n+1$, if $B = A$ then $B \approx n+1$. Else, $\exists x \in A \setminus B$. Assume $f : A \to n+1$ is bijection. Consider $g : A \to n+1 \begin{cases} g(t) = f(t) & t \neq x \wedge g(t) \neq n \\ g(t) = n+1 & t = x \\ g(t) = f(x) & f(t) = n \end{cases}$ Easy to know g is bijection, too. And since $x \notin B$ we get $B \subset g^{-1}[n] \approx n$, so by induction we get $\exists m \in \omega, B \approx m$.

2. First we prove union of two disjoint finite sets A and B is finite.

   Use MI to the card of the second set B. For $B = \varnothing$ it's obvious. For $B \approx 1$, assume $A \approx n$, Thrn $B \approx \{n\}$, so $A \cup B \approx n \cup \{n\} = n+1$ is finite. Assume for certain $n, \forall B \approx n$ it's right, now we prove it's right for $n+1$. Assume $f : B \to n+1$ is bijection, then $f^{-1}[n] \approx n$, so $A \cup f^{-1}[n]$ is finite. Since $B = f^{-1}[n] \cup \{f^{-1}(n)\}$ we get $A \cup B = A \cup f^{-1}[n] \cup \{f^{-1}(n)\}$. Since $\{f^{-1}(n)\} \approx 1$, so the union is finite.

   For general two finite sets A, B we have $A \cup B = A \cup (B \setminus A)$ and from II.1 we know $B \setminus A$ is finite, so $A \cup B$ is finite. Now we use MI to prove $\varphi(n) := \forall x \approx n((\forall y \in x, isFinite(y)) \to isFinite(\bigcup x))$ for $n \in \omega$. When $n = 0, 1, 2$ it's obvious. Assume for certain $n \geq 2$ we have $\varphi(n)$, then we prove $\varphi(n+1)$. Assume $f : x \to n+1$ is bijective, let $y = f^{-1}[n] \subset x$. Then $y \approx n$, by induction we know $\bigcup y$ is finite. Since $x = y \cup \{f^{-1}(n)\}$ we get $\bigcup x = (\bigcup y) \cup f^{-1}(n)$. So $\bigcup x$ is finite, too.

3. Use MI of the card. For $x \approx 0$ we know $\mathscr{P}(x) = \{\varnothing\} \approx 1$. Assume for certain n we have $\forall x \approx n, isFinite(\mathscr{P}(x))$, then for $x \approx n+1$: Assume $f : x \to n+1$ is bijection. Let $y = f^{-1}[n]$ and $t = f^{-1}(n)$. Then $y \approx n$. Let $\theta : \mathscr{P}(x) \setminus \mathscr{P}(y) \to \mathscr{P}(y), \theta(a) := a \setminus \{t\}$. Easily θ is bijective, so $\mathscr{P}(x) \setminus \mathscr{P}(y) \approx \mathscr{P}(y)$ is finite. From II.2 we know $\mathscr{P}(x) = \mathscr{P}(y) \cup (\mathscr{P}(x) \setminus \mathscr{P}(y))$ is finite.

4. Use MI by card. For $A \approx 0$ it's obvious. Assume for $A \approx n$ it's right, now we prove for $A \approx n+1$ it's right, too. Let $f : A \to n+1$ is a bijection, and $g : A \to \mathbb{S}\mathrm{et}$ is a map on A. Let $B := f^{-1}[n] \subset A, t = f^{-1}(n) \in A$. Then $B \approx n$, so by induction we know $g[B]$ is finite. Since $A = B \cup \{t\}$ we get $g[A] = g[B] \cup g[\{t\}] = g[B] \cup \{g(t)\}$. Noting $\{g(t)\} \approx 1$ is finite, from II.2 we get $g[A]$ is finite, too.

$\square$

PROBLEM III

1. A subset of a countable set is at most countable.
2. The union of a finite set of countable sets is countable.
3. The image of a countable set (under a mapping) is at most countable.

SOLUTION. 1. Assume A is countable and $\theta : A \to \omega$ is bijection. For $B \subset A$, we have $B \approx \theta[B]$. So we only need to prove every subset of ω is at most countable. Let $x \subset \omega$. If x is finite, then there is nothing to do. Now assume x is infinite. We define f on ω by induction. Let $f(0) = \min x$ and $f(n) = \min(x \setminus f[n])$. Since x is infinite, we know $f[n] \subsetneqq x$, so f is well-defined. And easily to prove f is a bijection. So $x \approx \omega$ is countable.

2. Use MI to the number of sets to cup, write n. When $n = 1$ it's obvious. For $n = 2$, we should prove two countable sets u, v's union $u \cup v$ is countable. Let $f : \omega \to u, g : \omega \to v$ is bijections, we need to find a bijection $h : \omega \to u \cup v$. We define h by induction. Let $h(n) = f(\min f^{-1}[u \setminus h[n]])$ for $2 \mid n$ and let $h(n) = g(\min g^{-1}[v \setminus h[n]])$ for $n \nmid 2$. Since u, v is infinite we obtain h is well-defined. Now we prove h is bijective. First we prove h is injective. For $m, n \in \omega, m \neq n$, assume $m < n$. If $x \mid n$, then $h(n) = f(\min f^{-1}[u \setminus h[n]]) \in f[f^{-1}[u \setminus h[n]]] = u \setminus h[n]$ and $h(m) \in h[n]$. So $h(m) \neq h(n)$. If $x \nmid n$ for the same reason we get $h(m) \neq h(n)$.

   Second we prove h is surjective. Only need to prove $u, v \subset h[\omega]$. By symmetry we only need to peove $u \subset h[\omega]$. Since $u = f[\omega]$, we only need to prove $f[n] \subset h[2n-1], \forall n \in \omega$. Use MI to prove it. For $n = 0$ it's obvious. Assume for certain n it's right, for $n+1$, we only need to prove $a := f(n) \in h[2n+1]$. If not, since $h(2n) = f(\min f^{-1}[u \setminus h[2n]])$ and $a \notin h[2n]$, we have $a \in u \setminus h[2n]$. Then $n = f^{-1}(a) \in f^{-1}[u \setminus h[2n]]$. For $m < n$, by induction we get $f(m) \in h[2m-1] \subset h[2n]$, so $m \notin f^{-1}[u \setminus h[2n]]$, thus $n = \min f^{-1}[u \setminus h[2n]]$. So $h(2n) = a$, contridiction! So h is surjective.

   Now we assume for certain $n \geq 2$ we have union of n countable sets is countable, we need to prove so do $n+1$ sets. Assume $A \approx n+1$ and $\forall x \in A, x \approx \omega$. Assume $f : A \to n+1$ is bijection, and let $B := f^{-1}[n], t = f^{-1}(n)$, then $\bigcup A = (\bigcup B) \cup t$. By induction we know $\bigcup B$ is countable. And we have proved union of two countable sets is countable. So finally we get $\bigcup A$ is countable.

3. Only need to prove image of ω is at most countable. For $f : \omega \to \mathbb{S}\mathrm{et}$ is a map, we need to prove $\mathrm{ran}(f)$ is at most countable. Let $h : \mathrm{ran}(f) \to \omega, t \mapsto \min f^{-1}[\{t\}]$. Obviously h is a injective, so $\mathrm{ran}(f)$ is at most countable.

$\square$

PROBLEM IV $\mathbb{N} \times \mathbb{N}$ is countable.

SOLUTION. We will prove $f : \mathbb{N}^2 \to \mathbb{N}, (m, n) \mapsto 2^m(2n+1) - 1$ is bijection. First we prove it's injection. Assume $f(a, b) = f(c, d)$, then $2^a(2b+1) = 2^c(2d+1)$. If $a \neq c$, assume $a < c$, then $2b+1 = x^{c-a}(2d+1)$. But $2 \mid x^{c-a}(2d+1), 2 \nmid 2b+1$, contridiction! So $a = c$. Then we get $2b+1 = 2d+1$, so $b = d$. So f is injective.

Second we prove f is surjective. For $t \in \mathbb{N}$, let $m := \sup\{k : 2^k \mid t+1\}$. Since $0 < t+1 < \omega$ and $2^k \mid t+1 \to 2^k \leq t+1$ we get $m < \omega$. Assume $t+1 = 2^m \cdot l$, then easily $2 \nmid l$. So we can assume $l = 2n+1$. Then $t = f(m, n)$. All in all, we get f is bijective. $\square$

PROBLEM V Prove that $\kappa^\kappa \leq 2^{\kappa \times \kappa}$.

then there is nothing to do. Now assume x is infinite. We define f on ω by induction. Let $f(0) = \min x$ and $f(n) = \min(x \setminus f[n])$. Since x is infinite, we know $f[n] \subsetneqq x$, so f is well-defined. And easily to prove f is a bijection. So $x \approx \omega$ is countable.

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   Second we prove h is surjective. Only need to prove $u, v \subset h[\omega]$. By symmetry we only need to peove $u \subset h[\omega]$. Since $u = f[\omega]$, we only need to prove $f[n] \subset h[2n-1], \forall n \in \omega$. Use MI to prove it. For $n = 0$ it's obvious. Assume for certain n it's right, for $n+1$, we only need to prove $a := f(n) \in h[2n+1]$. If not, since $h(2n) = f(\min f^{-1}[u \setminus h[2n]])$ and $a \notin h[2n]$, we have $a \in u \setminus h[2n]$. Then $n = f^{-1}(a) \in f^{-1}[u \setminus h[2n]]$. For $m < n$, by induction we get $f(m) \in h[2m-1] \subset h[2n]$, so $m \notin f^{-1}[u \setminus h[2n]]$, thus $n = \min f^{-1}[u \setminus h[2n]]$. So $h(2n) = a$, contridiction! So h is surjective.

   Now we assume for certain $n \geq 2$ we have union of n countable sets is countable, we need to prove so do $n+1$ sets. Assume $A \approx n+1$ and $\forall x \in A, x \approx \omega$. Assume $f : A \to n+1$ is bijection, and let $B := f^{-1}[n], t = f^{-1}(n)$, then $\bigcup A = (\bigcup B) \cup t$. By induction we know $\bigcup B$ is countable. And we have proved union of two countable sets is countable. So finally we get $\bigcup A$ is countable.

3. Only need to prove image of ω is at most countable. For $f : \omega \to \mathbb{S}\mathrm{et}$ is a map, we need to prove $\mathrm{ran}(f)$ is at most countable. Let $h : \mathrm{ran}(f) \to \omega, t \mapsto \min f^{-1}[\{t\}]$. Obviously h is a injective, so $\mathrm{ran}(f)$ is at most countable.

$\square$

PROBLEM IV $\mathbb{N} \times \mathbb{N}$ is countable.

SOLUTION. We will prove $f : \mathbb{N}^2 \to \mathbb{N}, (m, n) \mapsto 2^m(2n+1) - 1$ is bijection. First we prove it's injection. Assume $f(a, b) = f(c, d)$, then $2^a(2b+1) = 2^c(2d+1)$. If $a \neq c$, assume $a < c$, then $2b + 1 = x^{c-a}(2d+1)$. But $2 \mid x^{c-a}(2d+1), 2 \nmid 2b+1$, contridiction! So $a = c$. Then we get $2b+1 = 2d+1$, so $b = d$. So f is injective.

Second we prove f is surjective. For $t \in \mathbb{N}$, let $m := \sup\{k : 2^k \mid t+1\}$. Since $0 < t+1 < \omega$ and $2^k \mid t+1 \to 2^k \leq t+1$ we get $m < \omega$. Assume $t + 1 = 2^m \cdot l$, then easily $2 \nmid l$. So we can assume $l = 2n+1$. Then $t = f(m, n)$. All in all, we get f is bijective. $\square$

PROBLEM V Prove that $\kappa^\kappa \leq 2^{\kappa \times \kappa}$.

SOLUTION. Only need to find a injection $h : {}^{\kappa}\kappa \to {}^{\kappa\times\kappa}2$. For $f \in {}^{\kappa}\kappa$, let $h(f) \in {}^{\kappa\times\kappa}2$, and for $u, v \in \kappa$ let $h(f)(u,v) := \begin{cases} 1 & u = f(v) \\ 0 & u \neq f(v) \end{cases}$. Then we prove h is a injection. Assume $f, g \in {}^{\kappa}\kappa$ and $h(f) = h(g)$. Then $\forall v \in \kappa$, we have $h(g)(f(v), v) = h(f)(f(v), v) = 1$, so $f(v) = g(v)$. So h is injective. $\square$

PROBLEM VI If $A \preccurlyeq B$, then $A \preccurlyeq^{*} B$.

SOLUTION. If $A = \varnothing$ then it's obvious. Now assume $A \neq \varnothing$ and $a \in A$. Assume $f : A \to B$ is injection. Let $h : B \to A, h(y) := \begin{cases} f^{-1}(y) & y \in \operatorname{ran}(f) \\ a & y \notin \operatorname{ran}(f) \end{cases}$ Then $\forall x \in A, h(f(x)) = x$. So h is surjective. $\square$

PROBLEM VII If $A \preccurlyeq^{*} B$, then $\mathscr{P}(A) \preccurlyeq \mathscr{P}(B)$

SOLUTION. If $A = \varnothing$ then $\mathscr{P}(A) = 1$. Let $f : \mathscr{P}(A) \to \mathscr{P}(B), 0 \mapsto B$, then f is injective. Else we get $A \neq \varnothing$. Then assume $f : B \to A$ is surjective. Let $h : \mathscr{P}(A) \to \mathscr{P}(B), U \mapsto f^{-1}[U]$. Then we only need to prove h is injective. Assume $U, V \subset A$ and $h(U) = h(V)$. We get $f^{-1}[U] = f^{-1}[V]$. If $U \neq V$, assume $U \setminus V \neq \varnothing$ and $x \in U \setminus V$, then since f is surjective we get $\exists t \in A, f(t) = x$. So $t \in f^{-1}[U]$ but $t \notin f^{-1}[V]$, contridiction! So h is injective. Then $\mathscr{P}(A) \preccurlyeq \mathscr{P}(B)$. $\square$

PROBLEM VIII Let X be a set. If there is an injective function $f : X \to X$ such that $\operatorname{ran}(f) \subsetneq X$, then X is infinite.

SOLUTION. Use MI to prove $\forall n \in \omega, X \not\approx n$. For $n = 0$, if $X \approx n$ then $X = 0$. So $X \subset \operatorname{ran}(f)$, contridiction! Assume for certain $n \geq 1$ we get $\forall m < n, X \not\approx m$, then we need to prove $X \not\approx n$. If not, assume $h : X \to n$ is bijection. Consider $h[\operatorname{ran}(f)] \subsetneq n$, we get $\exists m < n, h[\operatorname{ran}(f)] \approx m$. Since f is injective, and h is bijection, we get $X \approx m$. Contridiction to the induction! So we finally proved $\forall n \in \omega, X \not\approx n$. $\square$

Solution. Only need to find a injection $h : {}^{\kappa}\kappa \to {}^{\kappa\times\kappa}2$. For $f \in {}^{\kappa}\kappa$, let $h(f) \in {}^{\kappa\times\kappa}2$, and for $u, v \in \kappa$ let $h(f)(u,v) := \begin{cases} 1 & u = f(v) \\ 0 & u \neq f(v) \end{cases}$. Then we prove h is a injection. Assume $f, g \in {}^{\kappa}\kappa$ and $h(f) = h(g)$. Then $\forall v \in \kappa$, we have $h(g)(f(v), v) = h(f)(f(v), v) = 1$, so $f(v) = g(v)$. So h is injective. $\square$

Problem VI If $A \preccurlyeq B$, then $A \preccurlyeq^* B$.

Solution. If $A = \varnothing$ then it's obvious. Now assume $A \neq \varnothing$ and $a \in A$. Assume $f : A \to B$ is injection. Let $h : B \to A, h(y) := \begin{cases} f^{-1}(y) & y \in \operatorname{ran}(f) \\ a & y \notin \operatorname{ran}(f) \end{cases}$ Then $\forall x \in A, h(f(x)) = x$. So h is surjective. $\square$

Problem VII If $A \preccurlyeq^* B$, then $\mathscr{P}(A) \preccurlyeq \mathscr{P}(B)$

Solution. If $A = \varnothing$ then $\mathscr{P}(A) = 1$. Let $f : \mathscr{P}(A) \to \mathscr{P}(B), 0 \mapsto B$, then f is injective. Else we get $A \neq \varnothing$. Then assume $f : B \to A$ is surjective. Let $h : \mathscr{P}(A) \to \mathscr{P}(B), U \mapsto f^{-1}[U]$. Then we only need to prove h is injective. Assume $U, V \subset A$ and $h(U) = h(V)$. We get $f^{-1}[U] = f^{-1}[V]$. If $U \neq V$, assume $U \setminus V \neq \varnothing$ and $x \in U \setminus V$, then since f is surjective we get $\exists t \in A, f(t) = x$. So $t \in f^{-1}[U]$ but $t \notin f^{-1}[V]$, contridiction! So h is injective. Then $\mathscr{P}(A) \preccurlyeq \mathscr{P}(B)$. $\square$

Problem VIII Let X be a set. If there is an injective function $f : X \to X$ such that $\operatorname{ran}(f) \subsetneq X$, then X is infinite.

Solution. Use MI to prove $\forall n \in \omega, X \not\approx n$. For $n = 0$, if $X \approx n$ then $X = 0$. So $X \subset \operatorname{ran}(f)$, contridiction! Assume for certain $n \geq 1$ we get $\forall m < n, X \not\approx m$, then we need to prove $X \not\approx n$. If not, assume $h : X \to n$ is bijection. Consider $h[\operatorname{ran}(f)] \subsetneq n$, we get $\exists m < n, h[\operatorname{ran}(f)] \approx m$. Since f is injective, and h is bijection, we get $X \approx m$. Contridiction to the induction! So we finally proved $\forall n \in \omega, X \not\approx n$. $\square$