## Graduate Homework In Mathematics

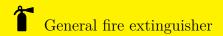
AlgebraicGeometry 10

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ROBEM I Assume  $V \subset \mathbb{A}^n_k$  is irreducible, and  $p \in V$ . Let  $m_p := \{f \in k[V] : f(p) = 0\}, M_p := \{f \in k[x_1, \dots, x_n] : f(p) = 0\}$ . Prove that  $m_p \cong M_p/\mathbb{I}(V)$ .

SOLTON. Consider  $\theta: M_p \to m_p, \theta(f) := f + \mathbb{I}(V)$ . Since  $p \in V$  we get  $\theta$  is well-defined. And easily we get  $\theta$  is homomorphism. Now consider  $\ker \theta$ . Obviously  $\mathbb{I}(V) \subset \ker \theta$ , now we prove  $\ker \theta \subset \mathbb{I}(V)$ . Assume  $f \in \ker \theta$ , to prove  $f \in \mathbb{I}(V)$ . Since  $\theta(f) = \mathbb{I}(V)$ , we get  $f + \mathbb{I}(V) = \mathbb{I}(V)$ , so  $f \in \mathbb{I}(V)$ . So  $\ker \theta = \mathbb{I}(V)$ . And easily  $\theta$  is surjective, so we get  $m_p = M_p/\ker \theta = M_p/\mathbb{I}(V)$ .  $\square$ 

1.

## ROBEM II Prove that $M_p/M_p^2 + \mathbb{I}(V) \cong m_p/m_p^2$

SOLTON. Consider  $\theta: M_p \to m_p/m_p^2, f \mapsto f|_V + m_p^2$ . Easily  $\theta$  is homomorphism and surjective. Now we prove  $\ker \theta = M_p^2 + \mathbb{I}(V)$ .

On one hand, assume  $f \in \ker \theta$ , i.e.,  $f|_V \in m_p^2$ . Then  $\exists g_1, \dots, g_n, h_1, \dots, h_n \in m_p$  such that  $f|_V = \sum_{k=1}^n g_k h_k$ . Assume  $g_k = G_k|_V$ ,  $h_k = H_k|_V$ . Then  $G_k$ ,  $H_k \in M_p$ . Consider  $f - \sum_{k=1}^n G_k H_k = h \in k[x_1, \dots, x_n]$ , easily to know  $h(x) = 0, \forall x \in V$ . So  $h \in \mathbb{I}(V)$ , thus  $f \in M_p^2 + \mathbb{I}(V)$ .

On the other hand, assume  $f \in M_p^2 + \mathbb{I}(V)$ , to prove  $\theta f = 0$ . Assume  $f = \sum_{k=1}^n G_k H_k + h$ , where  $G_k, H_k \in M_p$  and  $h \in \mathbb{I}(V)$ . Then  $\theta f = \sum_{k=1}^n g_k h_k + m_p^2$ , where  $g_k = G_k|_V, h_k = H_k|_V$ . So we get  $\theta f = m_p^2 = 0$ .

Finally we get  $m_p/m_p^2 = M_p/\ker\theta = M_p/M_p^2 + \mathbb{I}(V)$ .

ROBEM III Assume  $V \subset \mathbb{A}^n_k$  is irreducible, and  $p \in V, f \in k[V], f(p) \neq 0$ . Consider  $V_f := \{x \in V : f(x) \neq 0\}$ . Let  $\theta : V_f \to \mathbb{A}^{n+1}_k, x \mapsto (x, \frac{1}{f(x)})$ . Let  $U = \theta(V_f)$ , prove that  $T_p V \cong T_{\theta(p)} U$ .

SOUTON. Write  $k[A_k^n] = k[x_1, \dots, x_n, y]$ . Assume  $V = \mathbb{V}(I) = \mathbb{V}(f_1, \dots, f_m)$ , where  $f_i \in k[x_1, \dots, x_n]$ . Then  $U = \mathbb{V}(f_1, \dots, f_n, yf - 1)$ . Now consider  $\tau : T_p(V) \to A_k^{n+1}, \tau(x) := (x, \frac{1}{f(p)} - \frac{f_p^{(1)}(x)}{f^2(p)})$ . Now we prove  $\tau(T_pV) = T_{\theta(p)}U$ . Only need to prove  $(yf - 1)_{\theta(p)}^{(1)}(\tau(x)) = 0$ . i.e.,  $\frac{f_p^{(1)}(x)}{f(p)} + f(p)(y - \frac{1}{f(p)}) = 0$ , where  $y = \frac{1}{f(p)} - \frac{f_p^{(1)}}{f^2(p)}$ . Substitute y into the equation, we get is't obvious.

Obviously  $\tau$  is injective, so it's isomorphic.