PROBLEM I Let $(X_n : n \geq 0) \perp (Y_n : n \geq 0)$ are Markov chain on E with transition matrix $(p_{ij} : i, j \in E), (q_{ij} : i, j \in E)$ respectively. Prove: $\{(X_n, Y_n) : n \geq 0\}$ are Markov chain on $E \times E$. And calculate the transition matrix of $(X_n, Y_n) : n \geq 0$.

SOLUTION . Easy to get that

$$\begin{aligned}
&\mathbb{P}(X_0 = i_0, \cdots, X_{n+1} = i_{n+1}, Y_0 = j_0, \cdots, Y_{n+1} = j_{n+1}) \\
=&\mathbb{P}(X_0 = i_0, \cdots, X_{n+1} = i_{n+1})\mathbb{P}(Y_0 = j_0, \cdots, Y_{n+1} = j_{n+1}) \\
=&\mathbb{P}(X_0 = i_0) \prod_{k=0}^{n} p_{i_k i_{k+1}} \mathbb{P}(Y_0 = j_0) \prod_{k=0}^{n} q_{j_k j_{k+1}} \\
=&\mathbb{P}((X_0, Y_0) = (i_0, j_0)) \prod_{k=0}^{n} p_{i_k i_{k+1}} q_{j_k j_{k+1}}
\end{aligned}$$

So we get that $((X_n, Y_n) : n \in \mathbb{N})$ is Markov chain with transition matrix $r_{(i,j),(m,n)} = p_{im} q_{jn}$. □

PROBLEM II Let S_n is a one dimensional simple random walk. Let $a \in \mathbb{Z}$. Let $\tau := \inf\{n \geq 0 : S_n = a\}$. Prove:

1. $(S_{\tau+n} : n \geq 0)$ is a one dimensional simple random walk.
2. $(S_{n\wedge\tau} : n \geq 0)$ is a Markov chain on $\mathbb{Z}$ and give its transition matrix.
3. $(S_{n\wedge\tau} : n \geq 0) \perp (S_{\tau+n} : n \geq 0)$.

SOLUTION . 1. Easy to know that

$$\begin{aligned}
&\mathbb{P}(S_\tau = i_0, S_{\tau+1} = i_1, \cdots, S_{\tau+n} = i_n \mid \tau < \infty) \\
=&\sum_{k\in\mathbb{N}} \mathbb{P}(\tau = k, S_\tau = i_0, S_{\tau+1} = i_1, \cdots, S_{\tau+n} = i_n \mid \tau < \infty) \\
=&\sum_{k\in\mathbb{N}} \mathbb{P}(\tau = k, S_k = i_0, S_{k+1} = i_1, \cdots, S_{k+n} = i_n \mid \tau < \infty) \\
=&\mathbb{1}(a = i_0) \sum_{k\in\mathbb{N}} \mathbb{P}(S_0 \neq a, \cdots, S_{k-1} \neq a, S_k = a, S_{k+1} = i_1, \cdots, S_{k+n} = i_n \mid \tau < \infty) \\
=&\frac{\mathbb{1}(a = i_0) \sum_{k\in\mathbb{N}} \mathbb{P}(S_0 \neq a, \cdots, S_{k-1} \neq a, S_k = a, S_{k+1} = i_1, \cdots, S_{k+n} = i_n)}{\mathbb{P}(\tau < \infty)} \\
=&\frac{\mathbb{1}(a = i_0)}{\mathbb{P}(\tau < \infty)} \sum_{k\in\mathbb{N}} \mathbb{P}(S_0 \neq a, \cdots, S_{k-1} \neq a, S_k = a, S_{k+1} = i_1, \cdots, S_{k+n} = i_n) \\
=&\frac{\mathbb{1}(a = i_0)}{\mathbb{P}(\tau < \infty)} \sum_{k\in\mathbb{N}} \mathbb{P}(S_{k+1} = i_1, \cdots, S_{k+n} = i_n \mid S_0 \neq a, \cdots, S_{k-1} \neq a, S_k = a) \\
&\times\mathbb{P}(S_0 \neq a, \cdots, S_{k-1} \neq a, S_k = a) \\
=&\frac{\mathbb{1}(a = i_0)}{\mathbb{P}(\tau < \infty)} \sum_{k\in\mathbb{N}} \mathbb{P}(S_{k+1} = i_1, \cdots, S_{k+n} = i_n \mid S_k = a)\mathbb{P}(\tau = k) \\
=&\frac{\mathbb{1}(a = i_0)}{\mathbb{P}(\tau < \infty)} \sum_{k\in\mathbb{N}} \prod_{l=0}^{n-1} p_{i_l i_{l+1}} \mathbb{P}(\tau = k) = \mathbb{1}(a = i_0) \prod_{l=0}^{n-1} p_{i_l i_{l+1}}
\end{aligned}$$

Where $p_{ij} : i, j \in \mathbb{Z}$ is the transition matrix of $S_n : n \in \mathbb{N}$.

2. Easy to know that

$$
\begin{aligned}
&\mathbb{P}(S_{\tau\wedge 0} = i_0, S_{\tau\wedge 1} = i_1, \cdots, S_{\tau\wedge n} = i_n \mid \tau < \infty) \\
=&\sum_{k\in\mathbb{N}} \mathbb{P}(\tau = k, S_{\tau\wedge 0} = i_0, S_{\tau\wedge 1} = i_1, \cdots, S_{\tau\wedge n} = i_n \mid \tau < \infty) \\
=&\sum_{k\in\mathbb{N}} \mathbb{P}(\tau = k, S_{k\wedge 0} = i_0, S_{k\wedge 1} = i_1, \cdots, S_{k\wedge n} = i_n \mid \tau < \infty) \\
=&\sum_{k\geq n} \mathbb{P}(\tau = k, S_0 = i_0, \cdots, S_n = i_n \mid \tau < \infty) \\
+&\sum_{k<n} \mathbb{P}(\tau = k, S_0 = i_0, \cdots, S_{k-1} = i_{k-1}, S_k = i_k = i_{k+1} = \cdots = i_n \mid \tau < \infty) \\
=&\mathbb{1}(i_0, i_1, \cdots, i_n \neq a) \prod_{k=0}^{n-1} p_{i_k i_{k+1}} + \sum_{k=0}^{n-1} \mathbb{1}(i_0, \cdots, i_{k-1} \neq a, i_k = i_{k+1} = \cdots = i_n = a) \prod_{l=0}^{k-1} p_{i_l i_{l+1}} \\
=&\prod_{k=0}^{n-1} (\mathbb{1}(i_k = i_{k+1} = a) + \mathbb{1}(i_k \neq a) p_{i_k, i_{k+1}})
\end{aligned}
$$

So $(S_{n\wedge\tau} : n \in \mathbb{N})$ is Markov chain with transition matrix $q_{i,j} = \mathbb{1}(i = j = a) + \mathbb{1}(i \neq a) p_{i,j}$.

3.

$\square$

ℝOBℝEM III Let S_n is a one dimensional symmetry simple random walk starting from zero. Prove: $(|S_n| : n \geq 0)$ is a Markov chain on $\mathbb{Z}^+$ and give its transition matrix. ℝOBℝEM IV Let S_n is a one dimensional simple random walk starting from zero. Prove: $(|S_n| : n \geq 0)$ is a Markov chain on $\mathbb{Z}^+$ and give its transition matrix.