白永乐

202011150087

202011150087@mail.bnu.edu.cn

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General fire extinguisher

$f(v) \in f[x_{n+1}]$. Since f is injective, $f[x_0] \cap f[x_{n+1}] = \varnothing$. So $f(u) \neq f(v)$. Else, we know $g(v) = v \notin f[x_0] \ni f(u)$.

- $u \in f[x_m], v \in f[x_n]$: If $m = n$ then $g(u) = f(m+1, u) \neq f(n+1, v) = g(v)$. Else $m \neq n$, then $g(u) \in f[x_{m+1}], g(v) \in f[x_{n+1}], f[x_{m+1}] \cap f[x_{n+1}] = \varnothing$. So $g(u) \neq g(v)$.
- $u \in x_n, v \notin x, \forall m, v \notin f[x_m]$: Easily $g(u) \in f[x_{n+1}]$ and $g(v) = v \notin f[x_{n+1}]$.
- $u, v \notin x, \forall n, u, v \notin f[x_n]$: Easily $g(u) = u \neq v = g(v)$.

Second we prove g is surjective. i.e., $\forall u \in y, \exists t \in x \cup y, g(t) = u$.

- $u \in f[x_n]$ for some n: If $n = 0$ then $y = f(0, t)$ for some $t \in x$. Then $g(t) = u$. Else $n \geq 1$, write $n = m + 1$. Then $y = f(m+1, t)$ for some $t \in x$. So $g(t) = u$.
- $u \notin f[x_n], \forall n$: Easily we get $g(u) = u$.

So all in all g is bijective.

$\square$

PROBLEM II

1. A subset of a finite set is finite.

2. The union of a finite set of finite sets is finite.

3. The power set of a finite set is finite.

4. The image of a finite set (under a mapping) is finite.

$isFinite(\bigcup x))$ for $n \in \omega$. When $n = 0, 1, 2$ it's obvious. Assume for certain $n \geq 2$ we have $\varphi(n)$, then we prove $\varphi(n+1)$. Assume $f : x \to n+1$ is bijective, let $y = f^{-1}[n] \subset x$. Then $y \approx n$, by induction we know $\bigcup y$ is finite. Since $x = y \cup \{f^{-1}(n)\}$ we get $\bigcup x = (\bigcup y) \cup f^{-1}(n)$. So $\bigcup x$ is finite, too.

3. Use MI of the card. For $x \approx 0$ we know $\mathscr{P}(x) = \{\varnothing\} \approx 1$. Assume for certain n we have $\forall x \approx n, isFinite(\mathscr{P}(x))$, then for $x \approx n+1$: Assume $f : x \to n+1$ is bijection. Let $y = f^{-1}[n]$ and $t = f^{-1}(n)$. Then $y \approx n$. Let $\theta : \mathscr{P}(x) \setminus \mathscr{P}(y) \to \mathscr{P}(y), \theta(a) := a \setminus \{t\}$. Easily θ is bijective, so $\mathscr{P}(x) \setminus \mathscr{P}(y) \approx \mathscr{P}(y)$ is finite. From II.2 we know $\mathscr{P}(x) = \mathscr{P}(y) \cup (\mathscr{P}(x) \setminus \mathscr{P}(y))$ is finite.

4. Use MI by card. For $A \approx 0$ it's obvious. Assume for $A \approx n$ it's right, now we prove for $A \approx n+1$ it's right, too. Let $f : A \to n+1$ is a bijection, and $g : A \to \mathbb{S}\mathrm{et}$ is a map on A. Let $B := f^{-1}[n] \subset A, t = f^{-1}(n) \in A$. Then $B \approx n$, so by induction we know $g[B]$ is finite. Since $A = B \cup \{t\}$ we get $g[A] = g[B] \cup g[\{t\}] = g[B] \cup \{g(t)\}$. Noting $\{g(t)\} \approx 1$ is finite, from II.2 we get $g[A]$ is finite, too.

$\square$

PROBLEM III

1. A subset of a countable set is at most countable.
2. The union of a finite set of countable sets is countable.
3. The image of a countable set (under a mapping) is at most countable.

SOLUTION. 1. Assume A is countable and $\theta : A \to \omega$ is bijection. For $B \subset A$, we have $B \approx \theta[B]$. So we only need to prove every subset of ω is at most countable. Let $x \subset \omega$. If x is finite, contridiction! So h is surjective.

Now we assume for certain $n \geq 2$ we have union of n countable sets is countable, we need to prove so do $n+1$ sets. Assume $A \approx n+1$ and $\forall x \in A, x \approx \omega$. Assume $f : A \to n+1$ is bijection, and let $B := f^{-1}[n], t = f^{-1}(n)$, then $\bigcup A = (\bigcup B) \cup t$. By induction we know $\bigcup B$ is countable. And we have proved union of two countable sets is countable. So finally we get $\bigcup A$ is countable.

3. Only need to prove image of ω is at most countable. For $f : \omega \to \mathbb{S}\mathrm{et}$ is a map, we need to prove $\operatorname{ran}(f)$ is at most countable. Let $h : \operatorname{ran}(f) \to \omega, t \mapsto \min f^{-1}[\{t\}]$. Obviously h is a injective, so $\operatorname{ran}(f)$ is at most countable.

$\square$

PROBLEM IV $\mathbb{N} \times \mathbb{N}$ is countable.

SOLUTION. We will prove $f : \mathbb{N}^2 \to \mathbb{N}, (m,n) \mapsto 2^m(2n+1) - 1$ is bijection. First we prove it's injection. Assume $f(a,b) = f(c,d)$, then $2^a(2b+1) = 2^c(2d+1)$. If $a \neq c$, assume $a < c$, then $2b+1 = x^{c-a}(2d+1)$. But $2 \mid x^{c-a}(2d+1), 2 \nmid 2b+1$, contridiction! So $a = c$. Then we get $2b+1 = 2d+1$, so $b = d$. So f is injective.

Second we prove f is surjective. For $t \in \mathbb{N}$, let $m := \sup\{k : 2^k \mid t+1\}$. Since $0 < t+1 < \omega$ and $2^k \mid t+1 \to 2^k \leq t+1$ we get $m < \omega$. Assume $t+1 = 2^m \cdot l$, then easily $2 \nmid l$. So we can assume $l = 2n+1$. Then $t = f(m,n)$. All in all, we get f is bijective. $\square$

PROBLEM V Prove that $\kappa^\kappa \leq 2^{\kappa \times \kappa}$.

only need to prove h is injective. Assume $U, V \subseteq A$ and $h(U) = h(V)$. We get $f^{-1}[U] = f^{-1}[V]$. If $U \neq V$, assume $U \setminus V \neq \varnothing$ and $x \in U \setminus V$, then since f is surjective we get $\exists t \in A, f(t) = x$. So $t \in f^{-1}[U]$ but $t \notin f^{-1}[V]$, contridiction! So h is injective. Then $\mathscr{P}(A) \preccurlyeq \mathscr{P}(B)$. $\square$

PROBLEM VIII Let X be a set. If there is an injective function $f : X \to X$ such that $\operatorname{ran}(f) \subsetneq X$, then X is infinite.

SOLUTION. Use MI to prove $\forall n \in \omega, X \not\approx n$. For $n = 0$, if $X \approx n$ then $X = 0$. So $X \subset \operatorname{ran}(f)$, contridiction! Assume for certain $n \geq 1$ we get $\forall m < n, X \not\approx m$, then we need to prove $X \not\approx n$. If not, assume $h : X \to n$ is bijection. Consider $h[\operatorname{ran}(f)] \subsetneq n$, we get $\exists m < n, h[\operatorname{ran}(f)] \approx m$. Since f is injective, and h is bijection, we get $X \approx m$. Contridiction to the induction! So we finally proved $\forall n \in \omega, X \not\approx n$. $\square$