AlgebraicGeometry 1

白永乐

202011150087

202011150087@mail.bnu.edu.cn

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PROBLEM I P is an ideal of a unitary commutative ring A, then P is prime ideal of $A \iff A/P$ is integral domain.

SOLUTION. $\Rightarrow$:

Since A is a unitary commutative ring, so A/P is unitary commutative ring, too. So we only need to prove $[ab] = [0] \Rightarrow [a] = [0] \vee [b] = [0]$. Obviously $[ab] = 0 \iff ab \in P \iff a \in P \vee b \in P \iff [a] = [0] \vee [b] = [0]$.

$\Leftarrow$:

As the same, $ab \in P \iff [ab] = [0] \Rightarrow [a] = [0] \vee [b] = [0] \iff a \in P \vee b \in P$, so P is prime ideal. $\square$

PROBLEM II M is an ideal of a unitary commutative ring A, then M is maximal ideal of $A \iff A/M$ is a field.

SOLUTION. $\Rightarrow$:

Consider $[a] \in A/M \setminus [0]$, we will prove it has a reverse. Consider $N := \{xm + ya : x, y \in A, m \in M\}$ is the minimum ideal of A contains M and a. Since $[a] \neq [0]$ we know $a \notin M$, so $M \subsetneqq N$. Noting M is maximal, so $N = A$. That means $\exists x, y \in A, m \in M, xm + ya = 1$. So $[xm + ya] = [1]$. Since $[xm] = [0]$ we get $[y][a] = 1$, i.e., $[y] = [a]^{-1}$.

$\Leftarrow$:

Consider $a \in A \setminus M, N := \{xp + ya : x, y \in A, p \in P\}$, we will prove $N = A$, which means M is maximal. Since A/M is field, $\exists y \in A, [y] = [a]^{-1}$. That's means $ay - 1 \in M \subset N$. Noting $ay \in N$, so $1 \in N$, thus $N = A$. $\square$

PROBLEM III A ring A is noetherian, $I \subset A$ is an ideal of A, then A/I is noetherian.

SOLUTION. Consider an ideal $J \subset A/I$, let $M := \{x \in A : [x] \in J\}$. Then $\forall a \in A, x \in M, [ax] = [a][x] \in J$, so $ax \in M$. $\forall a, b \in M, [a - b] = [a] - [b] \in J$, so $a - b \in M$. So M is an ideal of A. Since A is noetherian, we can assume $M = (f_i, i = 1, 2, \cdots n)$. Now we will prove $J = ([f_i], i = 1, 2, \cdots n)$. Consider $[f] \in J$, from definition of M we know $f \in M$, so $f = \sum_{i=1}^n a_i f_i, a_i \in A$, thus $[f] = [\sum_{i=1}^n a_i f_i] = \sum_{i=1}^n [a_i][f_i]$. So $J = ([f_i], i = 1, 2, \cdots n)$. $\square$