under Graduate Homework In Mathematics

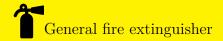
SetTheory 4

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ROBEM I Consider $\mathbb{Q} = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) / \sim$, where $(a,b) \sim (c,d) \iff ad = bc$. Define $+_{\mathbb{Q}}, \cdot_{\mathbb{Q}}$ and verify that your definitions doesn't depend on the choice of representatives.

SOUTION. Let $[(a,b)] +_{\mathbb{Q}} [(c,d)] = [(ad+bc,bd)], [(a,b)] \cdot_{\mathbb{Q}} [(c,d)] = [(ac,bd)],$ and $[(a,b)] <_{\mathbb{Q}} [(c,d)] \iff abd^2 < cdb^2$. Now we prove they are well-defined, i.e., doesn't depend on the choice of representatives.

For $+_{\mathbb{Q}}$, assume $(a,b) \sim (e,f)$, we need to prove $(ad+bc,bd) \sim (ed+fc,df)$. Since af=be, we have $(ad+bc)bf=ad^2f+bdcf=bed^2+bdcf=(ed+fc)bd$. So $+_{\mathbb{Q}}$ is well defined.

For $\cdot_{\mathbb{Q}}$, assume $(a,b) \sim (e,f)$, we need to prove $(ac,bd) \sim (ec,fd)$. Since af = be, we have acfd = bced = bdec. So $\cdot_{\mathbb{Q}}$ is well defined.

For $<_{\mathbb{Q}}$, assume $(a_1, b_1) \sim (a_2, b_2), (c_1, d_1) \sim (c_2, d_2)$ and $(a_1, b_1) < (c_1, d_1)$. Now we need to prove $(a_2, b_2) < (c_2, d_2)$. Since $a_1b_2 = a_2b_1, c_1d_2 = c_2d_1$ we get $a_1b_1d_2^2 < c_2d_2b_1^2$

ROBEM II The set of all continuous functions $f: \mathbb{R} \to \mathbb{R}$ has cardinality \mathfrak{c} (while the set of all functions has cardinality $2^{\mathfrak{c}}$). [A continuous function on \mathbb{R} is determined by its values at rational points.]

SOUTON. Consider $\theta: \mathbb{R} \mathbb{R} \to 2^{\mathbb{Q}}, f \mapsto \{(a,b) \in \mathbb{Q} : f(a) < b\}$. Now we prove f is a injection. Assume $\theta(f) = \theta(g)$, to prove f = g. First we prove for $x \in \mathbb{Q}$ we have f(x) = g(x). We have $f(x) = \sup\{y \in \mathbb{Q} : y < f(x)\} = \sup\{y \in \mathbb{Q} : (x,y) \in \theta(f)\} = \sup\{y \in \mathbb{Q} : (x,y) \in \theta(g)\} = g(x)$. For $x \in \mathbb{R}$, choose a sequence $x_n \in \mathbb{Q}$ such that $x_n \to x$, then $f(x) = \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} g(x_n) = g(x)$. So we get f = g. So $\operatorname{card}^{\mathbb{R}} \mathbb{R} \leq \operatorname{card} 2^{\mathbb{Q}} = 2^{\aleph_0}$. Obviously $\operatorname{card}^{\mathbb{R}} \mathbb{R} \geq 2^{\aleph_0}$, so we get they are equal.

 \mathbb{R}^{OBEM} III There are at least \mathfrak{c} countable order-types of linearly ordered sets.

SOUTION. For every sequence $a = \langle a_n : n \in \mathbb{N} \rangle$ of natural numbers consider the ordertype

$$\tau_a = \{(x, y) \in \mathbb{Z} \times \mathbb{N} : 2 \nmid y \lor 0 < x < a_{\frac{y}{2}}\}$$

And for $(x, y), (z, w) \in \tau_a$ we define $(x, y) < (z, w) \iff y < w \lor y = w, x < z$. Now we will show that if $a \neq b$, then $\tau_a \neq \tau_b$. Assume $\tau_a \cong \tau_b$, we need to prove a = b. assume $\theta : \tau_a \to \tau_b$ is the isomorfism.

We know (x,0) can be defined as $\phi(p) = \exists_{k=1}^{x-1} t_k, \forall_{1 \leq i < j \leq x-1} t_i \neq t_j, \forall k = 1, \dots x-1, t_k < p$. And θ is isomorphism. So $\theta(x,0) = (x,0)$. For (x,1), we let b_0 satisfy $\theta(0,1) = (b_0,m)$. Since the set $\{(x,y):y=1\}$ can be defined by $\psi(p) = \forall r, s(r,s , where <math>\tau(r) := \{s:s < r\}$ and $[r,s] = \{y:r < y < s\}$. we get $\theta[\{(x,y):y=1\}] = \{(x,y):y=1\}$. So we can delete the element whose second coordinary is 0,1, and θ is isomorphism, too. Do this repeatedly, we get $\theta(x,2n+1) = (x,2n+1)$. So $a_n = \operatorname{card}\{(x,2n+1) \in \tau_a\} = \operatorname{card}\{(x,2n+1) \in \tau_b\} = b_n$ and thus a = b.

ROBEM IV The set of all algebraic reals is countable.

SOLTION.

ROBEM V If S is a countable set of reals, then $|\mathbb{R} - S| = \mathfrak{c}$. [Use $\mathbb{R} \times \mathbb{R}$ rather than \mathbb{R} (because $|\mathbb{R} \times \mathbb{R}| = 2^{\aleph_0}$).]