PROBLEM I Assume $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, and $C \in \mathcal{F}$ satisfy $\mathbb{P}(C) > 0$. Let $\mathbb{P}_C : \mathcal{F} \to \mathbb{R}, \mathbb{P}_C(X) = \frac{\mathbb{P}(C\cap X)}{\mathbb{P}(C)}$. Assume $A, B \in \mathcal{F}$, and $\mathbb{P}(B \cap C) > 0$, prove that $\mathbb{P}_C(A \mid B) = \mathbb{P}(A \mid B \cap C)$.

SOLUTION. Easily $\mathbb{P}_C(B) = \frac{\mathbb{P}(B\cap C)}{\mathbb{P}(C)} > 0$, so $\mathbb{P}_C(A \mid B)$ is well-defined. Easily to get that

$$\mathbb{P}_C(A \mid B) = \frac{\mathbb{P}_C(A \cap B)}{\mathbb{P}_C(B)} = \frac{\frac{\mathbb{P}(A\cap B\cap C)}{\mathbb{P}(C)}}{\frac{\mathbb{P}(B\cap C)}{\mathbb{P}(C)}} = \frac{\mathbb{P}(A \cap B \cap C)}{\mathbb{P}(B \cap C)} = \mathbb{P}(A \mid B \cap C)$$

$\square$

PROBLEM II Assume that $(X_n : n \geq 0)$ is 1-dimentional simple symetry random walk, prove that $(|X_n| : n \geq 0)$ is a Markov chain ranges in $\mathbb{N}$.

SOLUTION. Easy to know that $(X_n : n \geq 0)$ is a Markov chain in $\mathbb{Z}$. Let $\mathcal{F}_n := \sigma(X_1, \cdots, X_n), \mathcal{G}_n := \sigma(|X_1|, \cdots, |X_n|)$, then easily $\mathcal{G}_n \subset \mathcal{F}_n$. Then we get that $\mathbb{P}(|X_{n+1}| = i \mid \mathcal{F}_n) = \mathbb{P}(X_{n+1} = i \mid \mathcal{F}_n) + \mathbb{P}(X_{n+1} = -i \mid \mathcal{F}_n) = \mathbb{P}(X_{n+1} = i \mid X_n) + \mathbb{P}(X_{n+1} = -i \mid X_n) = \mathbb{P}(|X_{n+1}| = i \mid X_n) = \frac{1}{2}\mathbb{1}(|X_n - i| = 1)$. Noting $i \geq 0$, we get that $|X_n - i| = 1 \iff ||X_n| - i| = 1$, so $\mathbb{P}(|X_{n+1}| = i \mid \mathcal{F}_n)$ is measureable about $\sigma(|X_n|)$. Since $\sigma(|X_n|) \subset \mathcal{G}_n \subset \mathcal{F}_n$, so we finally get that $\mathbb{P}(|X_{n+1}| = i \mid \mathcal{G}_n) = \mathbb{P}(|X_{n+1}| = i \mid |X_n|)$. So $(|X_n| : n \geq 0)$ is a Markov chain. $\square$

PROBLEM III Assume $(X_n : n \geq 0)$ is a Markov chain ranges in E. Assume $\phi : E \to F$ is injection. Prove that $(\phi(X_n) : n \geq 0)$ is a Markov chain ranges in $\phi(E)$.

SOLUTION. Without loss of generality assume $F = \phi(E)$, then ϕ is bijection. Now let $\mathcal{F}_n := \sigma(X_1, \cdots, X_n)$. Since ϕ is bijection we easily get that $\sigma(X_n) = \sigma(\phi(X_n))$, so $\mathcal{F}_n = \sigma(\phi(X_1), \cdots, \phi(X_n))$. Then $\mathbb{P}(\phi(X_{n+1} = i \mid \mathcal{F}_n)) = \mathbb{P}(X_{n+1} = \phi^{-1}(i) \mid \mathcal{F}_n) = \mathbb{P}(X_{n+1} = \phi^{-1}(i) \mid X_{n+1}) = \mathbb{P}(\phi(X_{n+1}) = i \mid \phi(X_n))$. So $(\phi(X_n) : n \geq 0)$ is Markov chain. $\square$

PROBLEM IV Assume $(X_n : n \geq 0), (Y_n : n \geq 0)$ are two independent Markov chains on E, F respectively. Prove that $((X_n, Y_n) : n \geq 0)$ is Markov chain on $E \times F$.

SOLUTION. Let $\mathcal{F}_n = \sigma(X_0, \cdots, X_n)$ and $\mathcal{G}_n, = \sigma(Y_0, \cdots, Y_n)$, Let $\mathcal{H}_n = \sigma((X_0, Y_0), \cdots, (X_n, Y_n))$. Then easily $\mathcal{H}_n = \sigma(\mathcal{F}_n, \mathcal{G}_n)$. Easy to get that

$$\begin{aligned}
\mathbb{P}(X_{n+1} = i, Y_{n+1} = j \mid \mathcal{H}_n) =& \mathbb{E}(\mathbb{P}(X_{n+1} = i, Y_{n+1} = j \mid \mathcal{H}_n, X_{n+1}) \mid \mathcal{H}_n) \\
=& \mathbb{E}(\mathbb{1}_i(X_{n+1})\mathbb{P}(Y_{n+1} = j \mid \mathcal{F}_n, \mathcal{G}_n, X_{n+1}) \mid \mathcal{H}_n) \\
(Y_{n+1} \perp \mathcal{F}_n, X_{n+1}) =& \mathbb{E}(\mathbb{1}_i(X_{n+1})\mathbb{P}(Y_{n+1} = j \mid Y_n) \mid \mathcal{H}_n) \\
(Y_n \in \mathcal{H}_n) =& \mathbb{P}(Y_{n+1} = j \mid Y_n)\mathbb{P}(X_{n+1} = i \mid \mathcal{H}_n) \\
=& \mathbb{P}(Y_{n+1} = j \mid Y_n)\mathbb{P}(X_{n+1} = i \mid X_n)
\end{aligned}$$

So $\mathbb{P}(X_{n+1} = i, Y_{n+1} = j \mid \mathcal{H}_n) \in \sigma(X_n, Y_n) \subset \mathcal{H}_n$. So $\mathbb{P}(X_{n+1} = i, Y_{n+1} = j \mid X_n, Y_n) = \mathbb{P}(X_{n+1} = i, Y_{n+1} = j \mid \mathcal{H}_n)$. So $((X_n, Y_n) : n \geq 0)$ is Markov chain. $\square$

PROBLEM V Assume $(X_n : n \geq 0), (Y_n : n \geq 0)$ are two independent Markov chains on E, F respectively. Let $\mathcal{H}_n := \sigma((X_0, Y_0), \cdots, (X_n, Y_n))$. Prove that $(X_n : n \geq 0)$ is Markov chain over $(\mathcal{H}_n : n \geq 0)$.

SOLUTION. Take $\mathcal{F}_n, \mathcal{G}_n$ as above. Obviously $X_n \in \mathcal{F}_n \subset \mathcal{H}_n$. Easily $\mathbb{P}(X_{n+1} = i \mid \mathcal{H}_n) = \mathbb{P}(X_{n+1} = i \mid \mathcal{F}_n, \mathcal{G}_n) = \mathbb{P}(X_{n+1} = i \mid \mathcal{F}_n) = \mathbb{P}(X_{n+1} \mid X_n)$. So $(X_n : n \geq 0)$ is Markov chain over $(\mathcal{H}_n : n \geq 0)$. $\square$

PROBLEM VI Let μ_0 be a probability distribution on $\mathbb{N}$. For $n \geq 1$, let

$$\mu_n(0) = \mu_{n-1}^{*2}(0) + \mu_{n-1}^{*2}(1), \mu_n(j) = \mu_{n-1}^{*2}(j+1), \forall j \geq 1$$

Where $\mu^{*2} = \mu * \mu$. Let F_n be distribution function of μ_n. Let $F_{n-1}^{-1}(y) := \inf\{x \geq 0 : y \leq F_{n-1}(x)\}$ for $y \in [0,1]$. Assume $X_0 \sim \mu_0$, and $(U_n : n \geq 0)$ are i.i.d r.v. with distribution $U(0,1)$. Let $X_{n+1} := \max\{0, X_n + F_n^{-1}(U_n) - 1\}$. Then $(X_n : n \geq 0)$ is Markov chain.

SOLUTION. Let $\mathcal{F} := \sigma(X_0, \cdots, X_n)$. For $i > 0$, we have

$$\begin{aligned}
\mathbb{P}(X_{n+1} = i \mid \mathcal{F}_n) &= \mathbb{P}(X_n + F_n^{-1}(U_n) - 1 = i \mid \mathcal{F}_n) \\
&= \sum_{k \in \mathbb{Z}} \mathbb{P}(X_n = k, F_n^{-1}(U_n) = i + 1 - k \mid \mathcal{F}_n) \\
&= \sum_{k \in \mathbb{Z}} \mathbb{1}_k(X_n) \mathbb{P}(F_n^{-1}(U_n) = i + 1 - k) \\
&= \sum_{k \in \mathbb{Z}} \mathbb{1}_k(X_n) \mathbb{P}(F_n^{-1}(U_n) = i + 1 - k \mid X_n) \\
&= \sum_{k \in \mathbb{Z}} \mathbb{P}(X_n = k, F_n^{-1}(U_n) = i + 1 - k \mid X_n) \\
&= \mathbb{P}(X_n + F_n^{-1}(U_n) - 1 = i \mid X_n) = \mathbb{P}(X_{n+1} = i \mid X_n)
\end{aligned}$$

For $i = 0$, we have

$$\begin{aligned}
\mathbb{P}(X_{n+1} = 0 \mid \mathcal{F}_n) &= \mathbb{P}(X_n + F_n^{-1}(U_n) - 1 \leq 0 \mid \mathcal{F}_n) \\
&= \sum_{k \in \mathbb{Z}} \mathbb{P}(X_n = k, F_n^{-1}(U_n) \leq 1 - k \mid \mathcal{F}_n) \\
&= \sum_{k \in \mathbb{Z}} \mathbb{1}_k(X_n) \mathbb{P}(F_n^{-1}(U_n) \leq 1 - k) \\
&= \sum_{k \in \mathbb{Z}} \mathbb{1}_k(X_n) \mathbb{P}(F_n^{-1}(U_n) \leq 1 - k \mid X_n) \\
&= \sum_{k \in \mathbb{Z}} \mathbb{P}(X_n = k, F_n^{-1}(U_n) \leq 1 - k \mid X_n) \\
&= \mathbb{P}(X_n + F_n^{-1}(U_n) - 1 \leq 0 \mid X_n) = \mathbb{P}(X_{n+1} = 0 \mid X_n)
\end{aligned}$$

So $(X_n : n \geq 0)$ is Markov chain. $\square$