SET THEORY

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1 Question

PROBLEM I Let $(U, \leq), (V, \prec)$ be two well-orderings. Consider $f := \{(x, y) : x \in U \wedge y \in V \wedge (U_x, \leq) \cong (V_y, \prec)\}$, prove f is isomorphism from some initial segment of U to some initial segment of V.

SOLUTION. First we need to prove $\mathrm{dom}(f)$ is initial segment of U. Only need to prove $\forall a \in \mathrm{dom}(f), U_a \subset \mathrm{dom}(f)$. Assume $h : U_a \to V_y$ is isomorphism, consider $b < a$. Since h is isomorphism, so $h[U_b]$ is initial segment of V_y and thus is initial segment of V(Because the property "isInitialSegment" is definable). So $b \in \mathrm{dom}(f)$, too. So $\mathrm{dom}(f)$ is initial segment of U. For the same reason we know $\mathrm{ran}(f)$ is initial of V.

Second we will prove f is a map. Assume $U_x \cong V_{y_1} \cong V_{y_2}$, since well-order set can't be isomorphic to it's proper initial segment, so $y_1 = y_2$. So f is a map. For the same reason f_{-1} is a map, too. So f is bijection from some initial segment of U to some initial segment of V. $\square$

PROBLEM II The relation "$(P, \leq) \cong (Q, \leq)$" is an equivalence relation (on the class of all partially ordered sets).

SOLUTION. First we prove $\cong$ has reflexivity. Obviously $\mathrm{id} : P \to P$ is isomorphism.

Second we prove $\cong$ has symmetry. If $f : P \to Q$ is isomorphism, then $f_{-1} : Q \to P$ is isomorphism, too.

Finally we prove $\cong$ has transitivity. If $f : P \to Q, g : Q \to R$ are isomorphisms, then $g \circ f : P \to R$ is isomorphism from P to R. $\square$

PROBLEM III Let $\mathcal{A}$ denote the class of all well orderings. For any $a, b \in \mathcal{A}$, define $a \prec b \iff a$ is isomorphic to an initial segment of b. Show that $\prec$ is a well ordering on $\mathcal{A}/\cong$, where $\cong$ is the equivalence relation given in Problem II.

SOLUTION. Obviously $\prec$ is partial order, so we only need to prove every nonempty subclass of $\mathcal{A}/\cong$ has minimum. Assume $\emptyset \neq \mathcal{B} \subset \mathcal{A}/\cong$, assume $[a] \in \mathcal{B}$, where $[a] = \{b : b \cong a\}$. Let $B = \mathrm{ini}(a) \cap \bigcup \mathcal{B}$, where $\text{ini}(a)$ means all of initial segment of a. Then $B \subset \text{ini}(a)$ is a subset of $\text{ini}(a)$, and $\text{ini}(a)$ is a well ordered set, so it has minimum. assume $b = \min B \in B$. Then we will prove $[b] = \min \mathcal{B}$.

Consider $[c] \in \mathcal{B}$, if $[a] \prec [c]$, then since $[b] \prec [a]$ we get $[b] \prec [c]$. Else, we get $[c] \prec [a]$. So there is a isomorphism from c to some d in $\text{ini}(a)$. Then $d \in [c]$ and $d \in B$. So $b \prec d$ and thus $[b] \prec [d]$. So $[b]$ is the minimum of $\mathcal{B}$. □

PROBLEM IV

1. If $(W, <)$ is a well ordering and $U \subset W$, then $(U, < \cap (U \times U))$ is a well ordering.
2. If $(W_1, <_1)$ and $(W_2, <_2)$ are two well orderings and $W_1 \cap W_2 = \varnothing$, then $W_1 \oplus W_2 = (W_1 \cup W_2, \prec)$ is a well ordering, where
$$\prec = <_1 \cup <_2 \cup \{(a, b) \mid a \in W_1 \wedge b \in W_2\}$$
3. If $(W_1, <_1)$ and $(W_2, <_2)$ are two well orderings, then $W_1 \otimes W_2 = (W_1 \times W_2, \prec)$ is a well ordering, where
$$(a_1, b_1) \prec (a_2, b_2) \leftrightarrow b_1 <_2 b_2 \vee (b_1 = b_2 \wedge a_1 <_1 a_2)$$

SOLUTION.

1. Obviously V is partial ordered. Consider nonempty set $V \subset U$, we know $V \subset W$, so $\min V$ exists.
2. First we need to prove $\prec$ is partial order.
   - Reflexivity: For $a \in W_1 \cup W_2$, if $a \in W_1$ then $(a, a) \notin \leq_1$. Obviously $(a, a) \notin \leq_2, W_1 \times W_2$, so $(a, a) \notin \prec$. If $a \in W_2$ for the same reason we get $(a, a) \notin \prec$. So $a \not\prec a$.
   - Transitivity: Consider $a \prec b, b \prec c$. Only need to prove $a \prec c$. If $a \in W_1, c \in W_2$ then obvious $a \prec c$. So we can assume $a, c \in W_i$, where $i = 1$ or $i = 2$. Since $a \prec b \prec c$ we can get $b \in W_i$, too. So we get $a <_i b <_i c$ and thus $a <_i c$. So $a \prec c$.

   Second we prove $\prec$ is well order. For nonempty set $U \subset W_1 \cap W_2$, if $U \cap W_1 \neq \emptyset$, then $\min U = \min W_1 \cap U$ exists (W_1 is well-order). Else, $U \subset W_2$, so $\min U$ exists.
3. As same as above we can easily get $\prec$ is partial order, so we only need prove $\prec$ is well order. For nonempty $U \subset W_1 \times W_2$, consider $\operatorname{ran} U \subset W_2$, we get $b = \min \operatorname{ran}(U)$ exists. Then consider $U_{-1}[b] \subset W_1$, we get $a = \min U_{-1}[b]$ exists. Now we will prove $(a, b) = \min U$. Obviously $(a, b) \in U$. If $(x, y) \in U$, then $y \in \operatorname{ran}(U)$, so $y \geq_2 b$. If $b <_2 y$ then $(a, b) \prec (x, y)$, else $b = y$, so $x \in U_{-1}[b]$ and thus $x \geq_1 a$, so $(x, y) \not\prec (a, b)$. So $(a, b) = \min U$.

□

PROBLEM V Show that the following are equivalent:

1. T is transitive;

2. $\bigcup T \subseteq T$;

3. $T \subseteq \mathscr{P}(T)$.

SOLUTION. 1. $V.1 \Rightarrow V.2$:

   $\forall x \in \bigcup T, \exists y \in T, x \in y$. Since T is transitive, we get $y \in T \to y \subset T$, so $x \in y \subset T, x \in T$. So $\bigcup T \subset T$.

2. $V.2 \Rightarrow V.3$:

   $\forall x \in T, \forall y \in x, y \in \bigcup T \subset T$. So $x \subset T$, that's means $x \in \mathscr{P}(T)$. So $T \subset \mathscr{P}(T)$.

3. $V.3 \Rightarrow V.1$:

   $\forall x \in T$, since $T \subset \mathscr{P}(T)$, we have $x \subset T$. So T is transitive.

$\square$

PROBLEM VI Let $\alpha, \beta, \gamma \in \text{Ord}$ and let $\alpha < \beta$. Then

a $\alpha + \gamma \leq \beta + \gamma$.

b $\alpha \cdot \gamma \leq \beta \cdot \gamma$.

c $\alpha^\gamma \leq \beta^\gamma$.

Given examples to show that $\leq$ cannot be replaced by $<$ in either inequality.

SOLUTION. a If not, let $c := \min\{\gamma \in \text{Ord} : \exists \alpha, \beta \in \text{Ord}, \alpha < \beta, \alpha + \gamma > \beta + \gamma\}$. Obviously $c \neq 0$. If c is successor, then assume $c = d + 1$. Then $\alpha + d \leq \beta + d$. Obviously $\alpha + 1 = \alpha \cup \{\alpha\} \subset \beta \cup \{\beta\}$, so $c > 1$. So $(\alpha + d) + 1 \leq (\beta + d) + 1$, i.e., $\alpha + c \leq \beta + c$, contradiction! Else, c is limit. So $\alpha + c = \sup\{\alpha + d : d < c\} \leq \sup\{\beta + d : d < c\} = \beta + c$, contradiction, too.

b If not, let $c := \min\{\gamma \in \text{Ord} : \exists \alpha, \beta \in \text{Ord}, \alpha < \beta, \alpha \cdot \gamma > \beta \cdot \gamma\}$. Obviously $c \neq 0$. If c is successor, then assume $c = d + 1$. Then $\alpha \cdot d \leq \beta \cdot d$. From VI.a we get $(\alpha d) + \alpha \leq (\beta d) + \alpha \leq \beta d + \beta$. i.e., $\alpha \cdot c \leq \beta \cdot c$, contradiction! Else, c is limit. So $\alpha c = \sup\{\alpha d : d < c\} \leq \sup\{\beta d : d < c\} = \beta c$, contradiction, too.

c If not, let $c := \min\{\gamma \in \text{Ord} : \exists \alpha, \beta \in \text{Ord}, \alpha < \beta, \alpha^\gamma > \beta^\gamma\}$. Obviously $c \neq 0$. If c is successor, then assume $c = d + 1$. Then $\alpha^d \leq \beta^d$. From VI.b we get $\alpha^d \alpha \leq \beta^d \alpha \leq \beta^d \beta$. i.e., $\alpha^c \leq \beta^c$, contradiction! Else, c is limit. So $\alpha^c = \sup\{\alpha^d : d < c\} \leq \sup\{\beta^d : d < c\} = \beta^c$, contradiction, too.

$\square$

EXAMPLE VI. a Let $\alpha = 0, \beta = 1, \gamma = \omega$, then $\alpha < \beta$ but $\alpha + \gamma = \omega = 1 + \omega = \beta + \gamma$.

b Let $\alpha = 1, \beta = 2, \gamma = \omega$, then $\alpha \cdot \gamma = \omega = 2 \cdot \omega = \omega$.

c Let $\alpha = 2, \beta = 3, \gamma = \omega$, then $\alpha^\gamma = \beta^\gamma$.

PROBLEM VII Show that the following rules do not hold for all $\alpha, \beta, \gamma \in \mathrm{Ord}$:

a If $\alpha + \gamma = \beta + \gamma$ then $\alpha = \beta$.

b If $\gamma > 0$ and $\alpha \cdot \gamma = \beta \cdot \gamma$ then $\alpha = \beta$.

c $(\beta + \gamma) \cdot \alpha = \beta \cdot \alpha + \gamma \cdot \alpha$.

SOLUTION. a ExampleVI.a

b ExampleVI.b

c $(1+1)\omega = \omega \neq \omega \cdot 2 = \omega + \omega = 1 \cdot \omega + 1 \cdot \omega$.

□

PROBLEM VIII Find a set $A \subset \mathbb{Q}$ such that $(A, <_{\mathbb{Q}}) \cong (\alpha, \in)$, where

a $\alpha = \omega + 1$,

b $\alpha = \omega \cdot 2$,

c $\alpha = \omega \cdot \omega$,

d $\alpha = \omega^\omega$,

e $\alpha = \varepsilon_0$.

f α is any ordinal $< \omega_1$.

SOLUTION. a Let $A = \{1 - \frac{1}{2^n} : n \in \mathbb{N}\} \cup \{1\}$. Then $1 - \frac{1}{2^n} \mapsto n, 1 \mapsto \omega$ is the isomorphism.

b Let $A = \{1 - \frac{1}{2^n} : n \in \mathbb{N}\} \cup \{2 - \frac{1}{2^n} : n \in \mathbb{N}\}$. Then $1 - \frac{1}{2^n} \mapsto n, 2 - \frac{1}{2^n} \mapsto \omega + n$ is the isomorphism.

c Let $A = \{m - \frac{1}{2^n} : m \in \mathbb{N}^+, n \in \mathbb{N}\}$. Then $m - \frac{1}{2^n} \mapsto \omega \cdot (m-1) + n$ is the isomorphism.

d Obviously $\omega^\omega = \sup\{\omega^n : n \in \mathbb{N}\} = \sum_{n=k}^{\infty} \omega^n, \forall k \in \mathbb{N}$. Consider $A_n := \{n - \frac{1}{2^{k_1}} - \frac{1}{2^{k_1+k_2}} - \cdots - 2^{k_1+k_2\cdots+k_n} : k_t \in \mathbb{N}^+, t = 1, 2, \cdots n\}$. We can easily get $A_n \cong \omega^n$. Then let $A := \bigcup_{k=1}^{\infty} A_n$, we get $A \cong \sum_{k=1}^{\infty} \omega^k = \omega^\omega$.

e Consider finite $\mathbb{N}$ sequence $\mathcal{A} := \bigcup_{k=1}^{\infty} \mathbb{N}^k$. For $a, b \in \mathcal{A}$ define $a + b = (a_1, a_2, \cdots a_n, b_1, b_2, \cdots b_m)$, where $(a_1, a_2, \cdots a_n) = a, (b_1, b_2, \cdots b_m) = b$. Define $a \leq b \iff \mathrm{len}(a) \leq (\mathrm{len}(b) \wedge \forall k \leq \mathrm{len}(a), a_k = b_k) \vee (\exists k \leq \mathrm{len}(a), a_k < b_k \wedge \forall j \leq k, a_j \leq b_j)$. Consider $\phi : \mathcal{A} \to \mathbb{Q}, (a_1, a_2, \cdots a_n) \mapsto 1 - \sum_{k=1}^{n} 2^{-k} \frac{1}{\sum_{t=1}^{k} a_t} - 2^{-n} \frac{1}{\sum_{t=1}^{n} a_t}$. Easy to prove ϕ is isomorphism from $(\mathcal{A}, \leq)$ to $(\mathbb{Q}, \leq)$. So we only need to find $B \subset \mathcal{A}$ such that $B \cong \varepsilon_0$. Let $b_0 = \omega, b_{n+1} = \omega^{b_n}$, then $\varepsilon_0 = \sum_{k=0}^{\infty} b_n$. If we have found $B_n \subset \mathcal{A}$ such that $B_n \cong b_n$ then $B := \bigcup_{k=0}^{\infty} \{n + a : a \in B_n\} \cong \varepsilon_0$. So we only need to find B_n.

□

PROBLEM IX An ordinal α is a limit ordinal iff $\alpha = \omega \cdot \beta$ for some $\beta \in \mathrm{Ord}$.

SOLUTION . First we prove $\omega \cdot \beta$ is limit ordinal. If not, consider the least β such that $\omega \cdot \beta$ is successor. If β is successor, then $\omega \cdot \beta = \omega \cdot \alpha + \omega$. Thus $\omega \cdot \beta$ is limit ordinal. If β is limit ordinal, we get $\omega \cdot \beta = \bigcup_{\alpha<\beta} \omega \cdot \alpha$. Obviously $\beta > \alpha \to \omega \cdot \beta > \omega \cdot \alpha$, so $\omega \cdot \beta$ is limit ordinal.

Second we prove every limit ordinal has the form $\omega \cdot \beta$. Assume γ is a limit ordinal. Let $B := \{x \in \gamma : x \text{ is limit ordinal}\}$. Let $f : \gamma \to B, f := \{(x, y) : \exists n \in \mathbb{N}, x = y + n\}$. Obviously $\inf\{y \in \gamma : \exists n \in \mathbb{N}, x = y + n\}$ is a limit ordinal, and for different limit ordinal y_1, y_2, we have $y_1 + n \neq y_2 + m$. So f is a map. Let $\beta := \mathrm{ordertype}(B)$. Then to peove $\omega \cdot \beta = \gamma$, we only need to prove $\omega \otimes B \cong \gamma$. Let $g : \gamma \to \omega \times B, x \mapsto (n, f(x))$, where $f(x) + n = x$. Easy to prove g is isomorphism, so $\omega \times \beta = \gamma$. □

PROBLEM X Find the first three $\alpha > 0$ s.t. $\xi + \alpha = \alpha$ for all $\xi < \alpha$.

SOLUTION . The first is 0 because there is no ordinal less than 0. The second is 1 because $0+1 = 1$. The third is ω, because on one hand if $\alpha < \omega$ then $1 + \alpha \neq \alpha$ on the other hand $\xi + \omega = \omega, \forall \xi < \omega$. □

PROBLEM XI Find the least ξ such that

a $\omega + \xi = \xi$.

b $\omega \cdot \xi = \xi, \xi \neq 0$.

c $\omega^{\xi} = \xi$.

(Hint for (1): Consider a sequence $\langle \xi_n \rangle$ s.t. $\xi_{n+1} = \omega + \xi_n$.)

Lemma 1. If $f : \mathrm{Ord} \to \mathrm{Ord}$ and $a \le b \to f(a) \le f(b)$ and $f(\sup B) = \sup f(B)$ for any B is subset of Ord, let $a_0 = 0, a_{n+1} = f(a_n)$, then $\xi = \sup\{a_n : n \in \mathbb{N}\}$ is the least ξ such that $f(\xi) = \xi$.

证明. First we prove $a_{n+1} \ge a_n$. Use MI it's obvious.

Second we prove $f(\xi) = \xi$. Obviously $f(\xi) = f(\sup\{a_n\}) = \sup\{f(a_n)\} = \sup\{a_{n+1}\} = \lim a_{n+1} = \lim a_n = \xi$.

Finally we prove ξ is the least. Assume $f(\alpha) = \alpha$, then use MI we can easily prove $\alpha \ge a_n \forall n < \omega$. So $\alpha \ge \sup\{a_n\} = \xi$. □

SOLUTION .

1. Let $f(x) = \omega + x$. From Lemma 1, we can let $a_n = \omega \cdot n$, then $a_0 = 0$ and $a_{n+1} = f(a_n)$. So $\xi = \sup\{a_n\} = \omega \cdot \omega = \omega^2$.

2. Let $f(x) = \omega \cdot x$. From Lemma 1, we can let $a_0 = 0, a_n = \omega^{n-1}, \forall n \ge 1$. Then $a_{n+1} = f(a_n)$. So $\xi = \sup\{a_n\} = \omega^{\omega}$.

3. Let $f(x) = \omega^x$. From Lemma 1, we can let $a_0 = 0, a_{n+1} = f(a_n) = \omega^{a_n}$, then $\xi = \sup\{a_n\} = \varepsilon_0$.

□