underGraduate Homework In Mathematics

SetTheory 4

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General fire extinguisher

PROBLEM I Consider $\mathbb{Q} = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})/\sim$, where $(a,b) \sim (c,d) \iff ad = bc$. Define $+_{\mathbb{Q}}, \cdot_{\mathbb{Q}}$ and $<_{\mathbb{Q}}$ and verify that your definitions doesn't depend on the choice of representatives.

SOLUTION. Let $[(a,b)] +_{\mathbb{Q}} [(c,d)] = [(ad+bc, bd)], [(a,b)] \cdot_{\mathbb{Q}} [(c,d)] = [(ac,bd)]$, and $[(a,b)] <_{\mathbb{Q}} [(c,d)] \iff abd^2 < cdb^2$. Now we prove they are well-defined, i.e., doesn't depend on the choice of representatives.

For $+_{\mathbb{Q}}$, assume $(a,b) \sim (e,f)$, we need to prove $(ad+bc, bd) \sim (ed+fc, df)$. Since $af = be$, we have $(ad+bc)bf = ad^2f + bdcf = bed^2 + bdcf = (ed+fc)bd$. So $+_{\mathbb{Q}}$ is well defined.

For $\cdot_{\mathbb{Q}}$, assume $(a,b) \sim (e,f)$, we need to prove $(ac,bd) \sim (ec,fd)$. Since $af = be$, we have $acfd = bced = bdec$. So $\cdot_{\mathbb{Q}}$ is well defined.

For $<_{\mathbb{Q}}$, assume $(a_1,b_1) \sim (a_2,b_2), (c_1,d_1) \sim (c_2,d_2)$ and $(a_1,b_1) < (c_1,d_1)$. Now we need to prove $(a_2,b_2) < (c_2,d_2)$. Since $a_1b_2 = a_2b_1, c_1d_2 = c_2d_1$ we get $a_1b_1d_2^2 < c_2d_2b_1^2$ □

PROBLEM II The set of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ has cardinality $\mathfrak{c}$ (while the set of all functions has cardinality $2^{\mathfrak{c}}$). [A continuous function on $\mathbb{R}$ is determined by its values at rational points.]

SOLUTION. Consider $\theta : {}^{\mathbb{R}}\mathbb{R} \to 2^{\mathbb{Q}}, f \mapsto \{(a,b) \in \mathbb{Q} : f(a) < b\}$. Now we prove f is a injection. Assume $\theta(f) = \theta(g)$, to prove $f = g$. First we prove for $x \in \mathbb{Q}$ we have $f(x) = g(x)$. We have $f(x) = \sup\{y \in \mathbb{Q} : y < f(x)\} = \sup\{y \in \mathbb{Q} : (x,y) \in \theta(f)\} = \sup\{y \in \mathbb{Q} : (x,y) \in \theta(g)\} = g(x)$. For $x \in \mathbb{R}$, choose a sequence $x_n \in \mathbb{Q}$ such that $x_n \to x$, then $f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = g(x)$. So we get $f = g$. So $\operatorname{card}{}^{\mathbb{R}}\mathbb{R} \leq \operatorname{card} 2^{\mathbb{Q}} = 2^{\aleph_0}$. Obviously $\operatorname{card}{}^{\mathbb{R}}\mathbb{R} \geq 2^{\aleph_0}$, so we get they are equal. □

PROBLEM III There are at least $\mathfrak{c}$ countable order-types of linearly ordered sets.

SOLUTION. For every sequence $a = \langle a_n : n \in \mathbb{N}\rangle$ of natural numbers consider the ordertype

$$\tau_a = \{(x,y) \in \mathbb{Z} \times \mathbb{N} : 2 \nmid y \wedge 0 < x < a_{\frac{y}{2}}\}$$

And for $(x,y),(z,w) \in \tau_a$ we define $(x,y) < (z,w) \iff y < w \wedge y = w, x < z$. Now we will show that if $a \neq b$, then $\tau_a \neq \tau_b$. Assume $\tau_a \cong \tau_b$, we need to prove $a = b$. assume $\theta : \tau_a \to \tau_b$ is the isomorfism.

We know $(x,0)$ can be defined as $\phi(p) = \exists_{k=1}^{x-1} t_k, \wedge_{1\leq i<j\leq x-1} t_i \neq t_j, \forall k = 1, \cdots x-1, t_k < p$. And θ is isomorphism. So $\theta(x,0) = (x,0)$. For $(x,1)$, we let b_0 satisfy $\theta(0,1) = (b_0, m)$. Since the set $\{(x,y) : y = 1\}$ can be defined by $\psi(p) = \forall r, s(r, s < p \wedge \tau(r) \wedge \tau(s) \to \operatorname{card}[r,s] < \infty)$, where $\tau(r) := \{s : s < r\}$ and $[r,s] = \{y : r < y < s\}$. we get $\theta[\{(x,y) : y = 1\}] = \{(x,y) : y = 1\}$. So we can delete the element whose second coordinary is $0,1$, and θ is isomorphism, too. Do this repeatedly, we get $\theta(x, 2n+1) = (x, 2n+1)$. So $a_n = \operatorname{card}\{(x,2n+1) \in \tau_a\} = \operatorname{card}\{(x,2n+1) \in \tau_b\} = b_n$ and thus $a = b$. □

PROBLEM IV The set of all algebraic reals is countable.

SOLUTION. Assume $\{f_n : n \in \mathbb{N}\}$ is the set of all integral coefficient polynomial. Consider $A_n := \{x \in \mathbb{C} : f(x) = 0\}$ is finite set. Then we get $\cup_{n\in\mathbb{N}} A_n$ is at most countable. Obviouly $\bigcup_{n\in\mathbb{N}} A_n$ is infinite, so it's countable. □

PROBLEM V If S is a countable set of reals, then $|\mathbb{R} - S| = \mathfrak{c}$. [Use $\mathbb{R} \times \mathbb{R}$ rather than $\mathbb{R}$ (because $|\mathbb{R} \times \mathbb{R}| = 2^{\aleph_0}$).]

SOLUTION. Assume $\theta : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ is bijection, and $T = \theta(S)$. Then T is countable. And $\mathrm{card}(\mathbb{R} \setminus S) = \mathrm{card}(\mathbb{R} \times \mathbb{R} \setminus T)$. So we only need to prove $\mathbb{R} \times \mathbb{R} \approx \mathbb{R} \times \mathbb{R} \setminus T$. Obviously $\mathrm{card}\mathbb{R} \times \mathbb{R} \setminus T \leq \mathrm{card}\mathbb{R} \times \mathbb{R}$, so we only need $\mathbb{R} \times \mathbb{R} \setminus T \geq \mathbb{R}$. Since T is countable, we get $\{x : \exists y, (x, y) \in T\}$ is countable. Choose $t \notin \{x : \exists y, (x, y) \in T\}$. Let $f : \mathbb{R} \to \mathbb{R}\times\mathbb{R}\setminus T, x \mapsto (t, x)$. Easily we get f is injection. So $\mathrm{card}\mathbb{R} \times \mathbb{R} \setminus T = \mathfrak{c}$. □

PROBLEM VI Assume T is a tree.

1. If $s, t, u \in T$, then $R_{stu} := \{\delta_{st}, \delta_{tu}, \delta_{us}\}$ has at most 2 elements. And if $p, q \in R_{stu}$, then $p \subset q \vee q \subset p$.
2. $\prec$ is a linear ordering of T which extends $\sqsubset$.
3. For every $t \in T$, Prove $T^t := \{s \in T : t \sqsubset s\}$ is an interval in $(T, \prec)$.

SOLUTION. 1. First we prove for $p, q \in R_{stu}$ we have $p \subset q \vee q \subset p$. Without loss of generality assume $p = \delta_{st}, q = \delta_{tu}$. We have $p, q \subset (\cdot, t)$. Since $(\cdot, t)$ is well ordered, and easily p, q are initial segment, so $p \subset q \vee q \subset p$.

   Now we prove there are at most two elements. From above we know $(R_{stu}, \subset)$ is linear order set, and it's finite. Without loss of generality we assume $\delta_{st} \subset \delta_{tu} \subset \delta_{us}$. Then we get $\delta_{tu} = \delta_{tu} \cap \delta_{us} = (\cdot, t) \cap (\cdot, u) \cap (\cdot, s) \subset \delta_{st}$. That means $\delta_{st} = \delta_{tu}$, so there is at most two elements.

2. Easily to prove $\sqsubset\subset\prec$. Now we prove $\prec$ is linear ordered. Consider a bigger linear ordered set Y is obtained by adding a minimum, $-\infty$, in X. Consider the tree $U := {}^{<\alpha}Y$. We try to make a map from T to B_U. Let $\theta : T \to B_U, \theta(f)(\beta) := \begin{cases} f(\beta), \beta \in \mathrm{dom}\, f \\ -\infty, \beta \notin \mathrm{dom}\, f \end{cases} \quad \forall \beta \in \alpha, f \in T.$

   Then we it's easily to prove θ is injective and $f \prec g \iff \theta(f)(\beta) < \theta(g)(\beta)$, where $\beta = \min\{t \in \alpha : \theta(f)(t) \neq \theta(g)(t)\}$. We define $f, g \in B_U, f < g \iff f(\beta) < g(\beta)$, where $\beta = \min\{t \in \alpha : f(t) \neq g(t)\}$. Now we only need to prove $(B_U, <)$ is linear ordered. Easily $f \not< f, \forall f \in B_U$. And for $f \neq g, f < g \vee g < f$. Assume $f < g < h$, to prove $f < h$.

   If $n_{fg} < n_{gh}$ then we get $f(n_{fg}) < g(n_{fg}) = h(n_{fg})$. So $n_{fh} \leq n_{fg}$. From Item VI.1 we get $n_{fh} = n_{fg} \vee n_{fh} = n_{gh}$. So $n_{fh} = n_{fg}$, and thus $f < h$.

   If $n_{fg} > n_{gh}$, then we get $h(n_{gh}) > g(n_{gh}) = f(n_{gh})$. Same as above we get $n_{fh} = n_{gh}$, so $f < h$.

   If $n_{fg} = n_{gh}$, it's obvious $f < h$.

   So we have proved B_U is linear ordered, and thus $(T, \prec)$ is linear orderd.

3. Only need to prove if $t \sqsubset u, t \sqsubset v, u \prec v$, then $\forall s : u \prec s \prec v, t \sqsubset s$. If $u \sqsubseteq s$ then $t \sqsubset u \sqsubseteq s$. Else we get $u \not\sqsubseteq s$. So we get $s \not\sqsubseteq u \wedge s(n_{su}) > u(n_{su})$. From Item VI.2 we get $t \prec s$. So if

$t \not\sqsubset s$ then $s \not\sqsubseteq t \wedge s(n_{st}) > t(n_{st})$. Since $t \sqsubseteq v$ we get $s(n_{st}) > t(n_{st}) = v(n_{st})$. Since $t \sqsubset v$ we get $n_{st} = n_{sv}$, so $v \prec s$, contradiction! So $t \sqsubset s$.

□

PROBLEM VII

1. Prove that $\prec$ is linear ordered on $T \cup B_T$.
2. For every $t \in T$, prove that $B_t = \{f \in T \cup B_T : t \in f\}$ is interval in $(T \cup B_T, \prec)$.

SOLUTION. 1.

□