underGraduate Homework In Mathematics

AlgebraicGeometry 13

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General fire extinguisher

PROBLEM I Assume $\Omega \subset \mathbb{C}$ is a domain. Prove that f is meromorphic map over Ω is equiv for Ω as opensubset of $\mathbb{C}$ and as Remian surface.

SOLUTION. First we assume $f : \Omega \to \mathbb{C}_\infty$ is holomorphic. Let $T := \{x \in \Omega : f(x) = \infty\}$. Now we prove f is meromorphic over Ω. Since $\forall x \in T, f(x) = \infty$ and f is continous, we get $\forall x \in T, \lim_{y \to x} f(y) = \infty$. Now we only need to prove every $x \in T$ is isolated point. If not, we will prove $f \equiv \infty$. Let $V := \{x \in \Omega : f(x) = \infty \wedge \exists x_n \in \Omega, x_n \neq x, x_n \to x, f(x_n) = \infty\} \neq \varnothing$. Easily V is closed in Ω, now we prove it's open, too. Assume $x \in V$, and $x_n \in \Omega, x_n \neq x, x_n \to x, f(x_n) = \infty$. Since f is holomorphic, we get $\exists V : \infty \in V \subset \mathbb{C}_\infty$ is open, $\exists U : x \in U \subset \Omega$ is open, such that $g := \phi \circ f|_U \circ \mathrm{id} : U \to \mathbb{C}$ is holomorphic, where $\phi(x) = \frac{1}{x}$. Then $g(x) = g(x_n) = 0$. So $g|_U \equiv 0$. So we get $f|_U \equiv 0$. So $U \subset V$. So V is open in Ω. Since Ω is connected, we get $V = \Omega$. So $f \equiv \infty$.

Second we assume $f : W \to \mathbb{C}$ is holomorphic, where $W \subset \Omega$ is open, and $\forall x \in T := \Omega \setminus W, \lim_{y \to x} f(y) = \infty$, and x is isolated point. Now let $h : \Omega \to \mathbb{C}_\infty, h|_W = f, h|_T \equiv \infty$. We only need to prove h is holomorphic from Ω(as Remian surface) to $\mathbb{C}_\infty$. Only need to prove h is holomorphic on $x \in T$. Let $\phi : \mathbb{C} \setminus \{0\} \to \mathbb{C}, x \mapsto \frac{1}{x}$. Let $U := h^{-1}(\mathbb{C} \setminus \{0\}) \subset \Omega$ is a neibor of x. Now we prove $g := \phi \circ h \circ \mathrm{id} : U \to \mathbb{C}$ is holomorphic. Easily g is well-defined since $0 \notin h(U)$. And f is meromorphic on Ω, so g is holomorphic. $\square$