

under Graduate Homework In Mathematics

Set Theory 4

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General fire extinguisher

PROBLEM I Consider $\mathbb{Q} = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) / \sim$, where $(a, b) \sim (c, d) \iff ad = bc$. Define $+_{\mathbb{Q}}$, $\cdot_{\mathbb{Q}}$ and $<_{\mathbb{Q}}$ and verify that your definitions doesn't depend on the choice of representatives.

SOLUTION. Let $[(a, b)] +_{\mathbb{Q}} [(c, d)] = [(ad + bc, bd)]$, $[(a, b)] \cdot_{\mathbb{Q}} [(c, d)] = [(ac, bd)]$, and $[(a, b)] <_{\mathbb{Q}} [(c, d)] \iff abd^2 < cdb^2$. Now we prove they are well-defined, i.e., doesn't depend on the choice of representatives.

For $+_{\mathbb{Q}}$, assume $(a, b) \sim (e, f)$, we need to prove $(ad + bc, bd) \sim (ed + fc, df)$. Since $af = be$, we have $(ad + bc)bf = ad^2f + bdcf = bed^2 + bdcf = (ed + fc)bd$. So $+_{\mathbb{Q}}$ is well defined.

For $\cdot_{\mathbb{Q}}$, assume $(a, b) \sim (e, f)$, we need to prove $(ac, bd) \sim (ec, fd)$. Since $af = be$, we have $acfd = bced = bdec$. So $\cdot_{\mathbb{Q}}$ is well defined.

For $<_{\mathbb{Q}}$, assume $(a_1, b_1) \sim (a_2, b_2)$, $(c_1, d_1) \sim (c_2, d_2)$ and $(a_1, b_1) < (c_1, d_1)$. Now we need to prove $(a_2, b_2) < (c_2, d_2)$. Since $a_1b_2 = a_2b_1$, $c_1d_2 = c_2d_1$ we get $a_1b_1d_2^2 < c_2d_2b_1^2$ \square

PROBLEM II The set of all continuous functions $f : \mathbb{R} \rightarrow \mathbb{R}$ has cardinality \mathfrak{c} (while the set of all functions has cardinality $2^{\mathfrak{c}}$). [A continuous function on \mathbb{R} is determined by its values at rational points.]

SOLUTION. Consider $\theta : {}^{\mathbb{R}}\mathbb{R} \rightarrow 2^{\mathbb{Q}}, f \mapsto \{(a, b) \in \mathbb{Q} : f(a) < b\}$. Now we prove f is a injection. Assume $\theta(f) = \theta(g)$, to prove $f = g$. First we prove for $x \in \mathbb{Q}$ we have $f(x) = g(x)$. We have $f(x) = \sup\{y \in \mathbb{Q} : y < f(x)\} = \sup\{y \in \mathbb{Q} : (x, y) \in \theta(f)\} = \sup\{y \in \mathbb{Q} : (x, y) \in \theta(g)\} = g(x)$. For $x \in \mathbb{R}$, choose a sequence $x_n \in \mathbb{Q}$ such that $x_n \rightarrow x$, then $f(x) = \lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} g(x_n) = g(x)$. So we get $f = g$. So $\text{card}^{\mathbb{R}}\mathbb{R} \leq \text{card} 2^{\mathbb{Q}} = 2^{\aleph_0}$. Obviously $\text{card}^{\mathbb{R}}\mathbb{R} \geq 2^{\aleph_0}$, so we get they are equal. \square

PROBLEM III There are at least \mathfrak{c} countable order-types of linearly ordered sets.

SOLUTION. For every sequence $a = \langle a_n : n \in \mathbb{N} \rangle$ of natural numbers consider the ordertype

$$\tau_a = \{(x, y) \in \mathbb{Z} \times \mathbb{N} : 2 \nmid y \wedge 0 < x < a_{\frac{y}{2}}\}$$

And for $(x, y), (z, w) \in \tau_a$ we define $(x, y) < (z, w) \iff y < w \wedge y = w, x < z$. Now we will show that if $a \neq b$, then $\tau_a \neq \tau_b$. Assume $\tau_a \cong \tau_b$, we need to prove $a = b$. assume $\theta : \tau_a \rightarrow \tau_b$ is the isomorphism.

We know $(x, 0)$ can be defined as $\phi(p) = \exists_{k=1}^{x-1} t_k, \wedge_{1 \leq i < j \leq x-1} t_i \neq t_j, \forall k = 1, \dots, x-1, t_k < p$. And θ is isomorphism. So $\theta(x, 0) = (x, 0)$. For $(x, 1)$, we let b_0 satisfy $\theta(0, 1) = (b_0, m)$. Since the set $\{(x, y) : y = 1\}$ can be defined by $\psi(p) = \forall r, s(r, s < p \wedge \tau(r) \wedge \tau(s) \rightarrow \text{card}[r, s] < \infty)$, where $\tau(r) := \{s : s < r\}$ and $[r, s] = \{y : r < y < s\}$. we get $\theta[\{(x, y) : y = 1\}] = \{(x, y) : y = 1\}$. So we can delete the element whose second coordinary is 0, 1, and θ is isomorphism, too. Do this repeatedly, we get $\theta(x, 2n+1) = (x, 2n+1)$. So $a_n = \text{card}\{(x, 2n+1) \in \tau_a\} = \text{card}\{(x, 2n+1) \in \tau_b\} = b_n$ and thus $a = b$. \square

PROBLEM IV The set of all algebraic reals is countable.

SOLUTION. Assume $\{f_n : n \in \mathbb{N}\}$ is the set of all integral coefficient polynomial. Consider $A_n := \{x \in \mathbb{C} : f_n(x) = 0\}$ is finite set. Then we get $\bigcup_{n \in \mathbb{N}} A_n$ is at most countable. Obviously $\bigcup_{n \in \mathbb{N}} A_n$ is infinite, so it's countable. \square

PROBLEM V If S is a countable set of reals, then $|\mathbb{R} - S| = \mathfrak{c}$. [Use $\mathbb{R} \times \mathbb{R}$ rather than \mathbb{R} (because $|\mathbb{R} \times \mathbb{R}| = 2^{\aleph_0}$).]

SOLUTION. Assume $\theta : \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}$ is bijection, and $T = \theta(S)$. Then T is countable. And $\text{card}(\mathbb{R} \setminus S) = \text{card}(\mathbb{R} \times \mathbb{R} \setminus T)$. So we only need to prove $\mathbb{R} \times \mathbb{R} \approx \mathbb{R} \times \mathbb{R} \setminus T$. Obviously $\text{card} \mathbb{R} \times \mathbb{R} \setminus T \leq \text{card} \mathbb{R} \times \mathbb{R}$, so we only need $\mathbb{R} \times \mathbb{R} \setminus T \geq \mathbb{R}$. Since T is countable, we get $\{x : \exists y, (x, y) \in T\}$ is countable. Choose $t \notin \{x : \exists y, (x, y) \in T\}$. Let $f : \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R} \setminus T, x \mapsto (t, x)$. Easily we get f is injection. So $\text{card} \mathbb{R} \times \mathbb{R} \setminus T = \mathfrak{c}$. \square

PROBLEM VI Assume T is a tree.

1. If $s, t, u \in T$, then $R_{stu} := \{\delta_{st}, \delta_{tu}, \delta_{us}\}$ has at most 2 elements. And if $p, q \in R_{stu}$, then $p \subset q \vee q \subset p$.
2. \prec is a linear ordering of T which extends \sqsubset .
3. For every $t \in T$, Prove $T^t := \{s \in T : t \sqsubset s\}$ is an interval in (T, \prec) .

SOLUTION. 1. First we prove for $p, q \in R_{stu}$ we have $p \subset q \vee q \subset p$. Without loss of generality assume $p = \delta_{st}, q = \delta_{tu}$. We have $p, q \subset (\cdot, t)$. Since (\cdot, t) is well ordered, and easily p, q are initial segment, so $p \subset q \vee q \subset p$.

Now we prove there are at most two elements. From above we know (R_{stu}, \subset) is linear order set, and it's finite. Without loss of generality we assume $\delta_{st} \subset \delta_{tu} \subset \delta_{us}$. Then we get $\delta_{tu} = \delta_{tu} \cap \delta_{us} = (\cdot, t) \cap (\cdot, u) \cap (\cdot, s) \subset \delta_{st}$. That means $\delta_{st} = \delta_{tu}$, so there is at most two elements.

2. Easily to prove $\sqsubset \subset \prec$. Now we prove \prec is linear ordered. Consider a bigger linear ordered set Y is obtained by adding a minimum, $-\infty$, in X . Consider the tree $U := {}^{<\alpha}Y$. We try to

make a map from T to B_U . Let $\theta : T \rightarrow B_U, \theta(f)(\beta) := \begin{cases} f(\beta), \beta \in \text{dom } f \\ -\infty, \beta \notin \text{dom } f \end{cases} \quad \forall \beta \in \alpha, f \in T$.

Then we it's easily to prove θ is injective and $f \prec g \iff \theta(f)(\beta) < \theta(g)(\beta)$, where $\beta = \min\{t \in \alpha : \theta(f)(t) \neq \theta(g)(t)\}$. We define $f, g \in B_U, f < g \iff f(\beta) < g(\beta)$, where $\beta = \min\{t \in \alpha : f(t) \neq g(t)\}$. Now we only need to prove $(B_U, <)$ is linear ordered. Easily $f \not\prec f, \forall f \in B_U$. And for $f \neq g, f < g \vee g < f$. Assume $f < g < h$, to prove $f < h$.

If $n_{fg} < n_{gh}$ then we get $f(n_{fg}) < g(n_{fg}) = h(n_{fg})$. So $n_{fh} \leq n_{fg}$. From VI.1 we get $n_{fh} = n_{fg} \vee n_{fh} = n_{gh}$. So $n_{fh} = n_{fg}$, and thus $f < h$.

If $n_{fg} > n_{gh}$, then we get $h(n_{gh}) > g(n_{gh}) = f(n_{gh})$. Same as above we get $n_{fh} = n_{gh}$, so $f < h$.

If $n_{fg} = n_{gh}$, it's obvious $f < h$.

So we have proved B_U is linear ordered, and thus (T, \prec) is linear orderd.

3. Only need to prove if $t \sqsubset u, t \sqsubset v, u \prec v$, then $\forall s : u \prec s \prec v, t \sqsubset s$. If $u \sqsubseteq s$ then $t \sqsubset u \sqsubseteq s$. Else we get $u \not\sqsubseteq s$. So we get $s \not\sqsubseteq u \wedge s(n_{su}) > u(n_{su})$. From VI.2 we get $t \prec s$. So if $t \not\sqsubseteq s$

then $s \not\sqsubseteq t \wedge s(n_{st}) > t(n_{st})$. Since $t \sqsubseteq v$ we get $s(n_{st}) > t(n_{st}) = v(n_{st})$. Since $t \sqsubset v$ we get $n_{st} = n_{sv}$, so $v \prec s$, contradiction! So $t \sqsubset s$.

□

PROBLEM VII

1. Prove that \prec is linear ordered on $T \cup B_T$.
2. For every $t \in T$, prove that $B_t = \{f \in B_T : t \in f\} \cup \{f \in T : t \sqsubset f\}$ is interval in $(T \cup B_T, \prec)$.

SOLUTION. 1. consider a bigger tree U . For $f \in T$ let $\theta(f) = f$, for $f \in B_T$ we let $\theta(f)$ is a map from $ordertype(\text{dom}(f))$ to X , and $\theta(f)(\beta) := g(\beta)$, where $g \in f$ and $\beta \in \text{dom}(g)$. Let $U = \theta(T \cup B_T)$. Then easily $T \subset U$. Now we prove θ is isomorphic from $(T \cup B_T, \prec)$ to (U, \prec) . Easily for $f, g \in T$ we have $f \sqsubset g \iff \theta(f) \sqsubset \theta(g)$.

And for $f \in T, g \in B_T$ we have $f \in g \iff \theta(f) \sqsubset \theta(g)$. So from the definition of \prec we get θ is isomorphic. Since we have proved (U, \prec) is linear order(VI.2), we get $(T \cup B_T, \prec)$ is linear order, too.

2. Since $\theta(B_t) = U^{\theta(t)}$ is an interval(VI.3), we get B_t is interval, too.

□

PROBLEM VIII Prove that if X is Suslinnnnn line, then X^2 is not c.c.c.

SOLUTION. Let $\mathcal{A} := \{Y \subset X : \text{card}(X \setminus Y) \leq \omega\}$. For every $Y \in \mathcal{A}$, let $\theta(Y) := \{(a, c) \subset X : (a, c) \cap Y = \emptyset \wedge a < c\}$. Since X is not separable, we know Y is not dense in X , so $\theta(Y) \neq \emptyset$. So we can use AC to get a map $\tau : \mathcal{A} \rightarrow X \times X \times X, Y \mapsto (a, b, c)$, where $(a, c) \cap Y = \emptyset$ and $a < b < c$. Now we give a map from ω_1 to $X \times X \times X$ recursively. Assume for some cardinary α we have defined $f(\beta)$ for every $\beta < \alpha$, now we give $f(\alpha)$. Let $f(\alpha) := \tau(\{b : \exists a, c \in X, (a, b, c) \in f[\alpha]\})$. Since $\alpha < \omega_1$ is countable, we get the set $\{b : \exists a, c \in X, (a, b, c) \in f[\alpha]\}$ is countalbe, so it's in $\text{dom}(\tau)$. So the definition of f is well.

Now consider $\{(a, b) \times (b, c) : (a, b, c) \in \text{ran}(f)\}$. From the definition of f we easily get they are disjoint open interval. And easily we get f is injective, so $\text{ran}(f)$ is uncountable. So we get a class of uncountable disjoint open interval in $X \times X$. So we obtain $X \times X$ is not c.c.c. □

PROBLEM IX Assume P is a perfect set and $P \cap (a, b) \neq \emptyset$. Prove that $\text{card}(P \cap (a, b)) = \mathfrak{c}$.

Lemma 1. If P is a perfect set and $P \cap (a, b) \neq \emptyset$, then $\exists c \in (a, b) \cap P$ such that $P \cap (a, c), P \cap (c, b) \neq \emptyset$.

证明. Since $P \cap (a, b) \neq \emptyset$, take $x \in P \cap (a, b)$. Since P is perfect set, and (a, b) is open, we get $\exists y \neq x, y \in P \cap (a, b)$. without loss of generality we assume $x < y$. Then $y \in (x, c) \cap P$, so $\exists z \neq y, z \in (x, c) \cap P$ for the same reason. Without loss of generality we assume $y < z$. Now let $c = y$, we get $c \in P \cap (a, b)$ and $x \in (a, c) \cap P, z \in (c, b) \cap P$. So the lemma have been proved. □

SOLUTION. To prove $\text{card}(P \cap (a, b)) = \mathfrak{c}$, we only need to find a injective from $(0, 1)$ to $(P \cap (a, b))$. First we consider the tree $T := {}^{<\omega}2$. We define $\tau : T \rightarrow \{(m, n) \in X \times X : m < n, \exists x \in P, m < x < n\}$ recursively. Let $\tau(\emptyset) = (a, b)$. Assume for $x \in T$ we have defined τ on $(\cdot, x)_T$. Now we need to give $\tau(x)$. Assume $\text{ordertype}(x) = n + 1$ for some $n \in \mathbb{N}$. Then $\text{dom}(x) = n + 1$. Consider $y = x|_n <_T x$, assume $\tau(y) = (m, n)$. From ?? we know $\exists l \in P \cap (m, n)$ such that $P \cap (m, l), P \cap (l, n) \neq \emptyset$. If $x(n) = 0$ we let $\tau(x) = (m, l)$; else we get $x(n) = 1$, we let $\tau(x) = (l, n)$.

From the definition of τ we can easily get $f \sqsubset g \rightarrow \tau(f)_1 \leq \tau(g)_1 \leq \tau(g)_2 \leq \tau(f)_2$. Now we construct $\theta : (0, 1) \rightarrow P \cap (a, b)$. For $x = \sum_{k=1}^{\infty} \frac{a_k}{2^k} \in (0, 1)$, let $f : \omega \rightarrow 2, f(k) = a_k$. Consider $f|_n \in T$, assume $\theta(f|_n) = (m_k, n_k)$. Then we get $\bigcap_{k \in \mathbb{N}} [m_k, n_k] \cap P \neq \emptyset$ since they are descending closed interval. Let $\theta(x) := \min \bigcap_{k \in \mathbb{N}} [m_k, n_k] \cap P$.

First we prove $\theta(x) \in (a, b) \cap P$. Obviously $\theta(x) \in P \cap [a, b]$, since $[m_k, n_k] \subset [a, b]$. Since $x \in (0, 1), \exists k, a_k = 0$. Then $[m_k, n_k] = [m_{k-1}, l_{k-1}]$, where $l_{k-1} < m_{k-1} \leq b$. So we get $\theta(x) < b$. For the same reason we get $\theta(x) > a$.

Now we only need to prove θ is injective. Assume $x \neq y$. Assume $x = \sum_{k=1}^{\infty} \frac{a_k}{2^k}, y = \sum_{k=1}^{\infty} \frac{b_k}{2^k}$. Without loss of generality we assume $x < y$. And we can assume $\lim a_k, \lim b_k \neq 1$. Assume $t = \min\{k : a_k < b_k\}$. Let $f(k) = a_k, g(k) = b_k$, and $\tau(f|_{t+1}) = (m, n), \tau(g|_{t+1}) = (n, l)$. Then $\theta(x) \leq n \leq \theta(y)$. Now we only need to prove $\theta(x) < n$. Since $\lim a_k \neq 1$, we obtain $\exists i > t, a_i = 0$. Then $\tau(f|_{i+1})_2 < n$, and thus $\theta(x) < n \leq \theta(y)$.

So we have found a injective from $(0, 1)$ to $P \cap (a, b)$. And obviously $(a, b) \cap P \subset \mathbb{R}$, we get $\text{card}((a, b) \cap P) = \mathfrak{c}$. \square