PROBLEM I Assume $(N_t : t \geq 0)$ is Possion process with parameter α, and $\{\xi_n : n \in \mathbb{N}^+\}$ is a sequence of i.i.d random variable. More over, assume $(N_t : t \geq 0) \perp \{\xi_n : n \in \mathbb{N}^+\}$. Let $X_t = \sum_{k=1}^{N_t} \xi_k$. Let $r > 0$, prove that:

1. $(N_{t+r} - N_r : t \geq 0)$ is Possion process.
2. $\{\xi_{N_r+n} : n \in \mathbb{N}^+\}$ is also i.i.d sequence with the same distribution of $\{\xi_n : n \in \mathbb{N}^+\}$.
3. $(N_{t+r} - N_r : t \geq 0) \perp (\xi_{N_r+k} : k \in \mathbb{N}^+)$.
4. For $0 = t_0 < t_1 < \cdots < t_n$, we have $(X_{t_1}, X_{t_{k+1}} - X_{t_k} : k = 1, 2, \cdots, n-1)$ are independent.

SOLUTION.

1. Let $\mathcal{F}_t := \sigma(N_s : 0 \leq s \leq t)$ and $\mathcal{G}_t := \sigma(N_{r+s} - N_r : 0 \leq s \leq t)$. For $0 \leq s \leq t$, we have $N_{t+r} - N_r - (N_{s+r} - N_r) = N_{t+r} - N_{s+r} \sim Possion(\alpha(t-s))$. And easily to know $N_{r+s} - N_r \in \mathcal{F}_{r+s}$, so $\mathcal{G}_t \subset \mathcal{F}_{t+r}, \forall t \geq 0$. Since $(N_t : t \geq 0)$ is Possion process, easily $\mathcal{F}_{s+r} \perp N_{t+r} - N_{s+r}$. Since $\mathcal{G}_t \subset \mathcal{F}_{t+r}$, we obtain $\mathcal{G}_t \perp N_{t+r} - N_{s+r} = N_{t+r} - N_r - (N_{s+r} - N_r)$. Easily since N_t is right-continuous we get $N_{t+r} - N_r$ is right-continuous. For the same reason, we know $\forall s \in [0, \infty), \lim_{t \to s-} N_{t+r} - N_r$ exists. So $(N_{t+r} : t \geq 0)$ is Possion process.

2. Only need to prove that for $m \in \mathbb{N}^+$, the distribution of $(\xi_{N_r+k} : 1 \leq k \leq m)$ is same as that of $(\xi_k : 1 \leq k \leq m)$. For $A_1, A_2, \cdots, A_m \in \mathcal{B}$, we have:

$$\begin{aligned}
&\mathbb{P}(\xi_{N_r+k} \in A_k, 1 \leq k \leq m) \\
&= \sum_{t=0}^{\infty} \mathbb{P}(\xi_{N_r+k} \in A_k, 1 \leq k \leq m, N_r = t) \\
&= \sum_{t=0}^{\infty} \mathbb{P}(\xi_{t+k} \in A_k, 1 \leq k \leq m, N_r = t) \\
(N_r \perp (\xi_n : n \in \mathbb{N}^+)) &= \sum_{t=0}^{\infty} \mathbb{P}(\xi_{t+k} \in A_k, 1 \leq k \leq m)\mathbb{P}(N_r = t) \\
&= \sum_{t=0}^{\infty} \prod_{k=1}^{m} \mathbb{P}(\xi_{k+t} \in A_k)\mathbb{P}(N_r = t) \\
&= \sum_{t=0}^{\infty} \prod_{k=1}^{m} \mu(A_k)\mathbb{P}(N_r = t) \\
&= \prod_{k=1}^{m} \mu(A_k)
\end{aligned} \tag{1}$$

where μ is the distribution of ξ_1. So we get for $m \in \mathbb{N}^+$, the distribution of $(\xi_{N_r+k} : 1 \leq k \leq m)$ is same as that of $(\xi_k : 1 \leq k \leq m)$.

3. We know that $\forall t \in \mathbb{N}^+, \xi_{N_r+t} \in \sigma(N_r, \xi_k : k \in \mathbb{N}^+)$. So $\sigma(\xi_{N_r+k} : k \in \mathbb{N}^+) \subset \sigma(N_r, \xi_k : k \in \mathbb{N}^+)$. Since $(N_t : t \geq 0)$ is Possion process, we get that $N_{t+r} - N_r \perp N_r, \forall t \geq 0$. So $\sigma(N_{t+r} - N_r : t \geq 0) \perp N_r$. Easily $\sigma(N_{t+r} - N_r : t \geq 0) \perp \sigma(\xi_k : k \in \mathbb{N}^+)$, so finally we get that $\sigma(N_{t+r} - N_r : t \geq 0) \perp \sigma(N_r, \xi_k : k \in \mathbb{N}^+) \supset \sigma(\xi_{N_r+k} : k \in \mathbb{N}^+)$.

4. $\forall 0 = t_0 < t_1 < \cdots < t_n$, then $X_{t_1} = \sum_{i=1}^{N_{t_1}} \xi_i, X_{t_k} - X_{t_{k-1}} = \sum_{i=1}^{N_{t_k}-N_{t_{k-1}}} \xi_{N_{t_{k-1}}+i}\xi_i, k = 2, \cdots, n$, then $\forall \{A_k \in \mathscr{E} : k = 1, \cdots, n\}$,

$$
\begin{aligned}
&\mathbb{P}(\bigcap_{k=1}^{n} \sum_{i=1}^{N_{t_k}-N_{t_{k-1}}} \xi_{i+N_{t_{k-1}}} \in A_k) \\
=&\mathbb{P}(\bigcup_{0\le u_1 \le \cdots \le u_n} \{ \sum_{i=u_{k-1}+1}^{u_k} \xi_i \in A_k, N_{t_k} = u_k, k = 1, \cdots, n\}) \\
=&\sum_{0\le u_1 \le \cdots \le u_n} \mathbb{P}(\sum_{i=u_{k-1}+1}^{u_k} \xi_i \in A_k, k = 1, \cdots, n | N_{t_k} = u_k, k = 1, \cdots, n\}) \mathbb{P}(N_{t_k} = u_k, k = 1, \cdots, n) \\
=&\sum_{0\le u_1 \le \cdots \le u_n} \mathbb{P}(\sum_{i=u_{k-1}+1}^{u_k} \xi_i \in A_k, k = 1, \cdots, n\}) \mathbb{P}(N_{t_k} = u_k, k = 1, \cdots, n) \\
=&\sum_{0\le u_1 \le \cdots \le u_n} \prod_{k=1}^{n} \mathbb{P}(\sum_{i=u_{k-1}+1}^{u_k} \xi_i \in A_k) \prod_{j=1}^{n} \mathbb{P}(N_{t_j} = u_j) \\
=&\sum_{0\le u_1 \le \cdots \le u_n} \prod_{k=1}^{n} \mathbb{P}(\sum_{i=1}^{u_k-u_{k-1}} \xi_{u_{k-1}+i} \in A_k) \prod_{j=1}^{n} \mathbb{P}(N_{t_j} - N_{t_{j-1}} = u_j - u_{j-1}) \\
=&\sum_{0\le u_1 \le \cdots \le u_n} \prod_{k=1}^{n} \mathbb{P}(\sum_{i=1}^{u_k-u_{k-1}} \xi_{u_{k-1}+i} \in A_k) \mathbb{P}(N_{t_k} - N_{t_{k-1}} = u_k - u_{k-1}) \\
=&\sum_{u_1-u_0\in\mathbb{N}} \cdots \sum_{u_n-u_{n-1}\in\mathbb{N}} \prod_{k=1}^{n} \mathbb{P}(\sum_{i=1}^{u_k-u_{k-1}} \xi_{u_{k-1}+i} \in A_k, N_{t_k} - N_{t_{k-1}} = u_k - u_{k-1}) \\
=&\prod_{k=1}^{n} \sum_{u_k-u_{k-1}\in\mathbb{N}} \mathbb{P}(\sum_{i=1}^{u_k-u_{k-1}} \xi_{u_{k-1}+i} \in A_k, N_{t_k} - N_{t_{k-1}} = u_k - u_{k-1}) \\
=&\prod_{k=1}^{n} \mathbb{P}(\sum_{i=1}^{N_{t_k}-N_{t_{k-1}}} \xi_{i+N_{t_{k-1}}} \in A_k)
\end{aligned}
\tag{2}
$$

□

PROBLEM II Assume that X is Possion random measure on $(E, \mathcal{E})$ with intensity μ, which is a σ-finite measure. Assume $f : E \to \mathbb{R}$ is measurable and non-negative, prove that:

$$\mathbb{E}(\mathrm{e}^{-X(f)}) = \exp\left\{ - \int_E (1 - \mathrm{e}^{-f(x)})\mu(\mathrm{d}x) \right\}$$

SOLUTION. Let $\mathcal{L} := \{g \in \mathcal{M}(E, [0, \infty)) : \mathbb{E}(\mathrm{e}^{-X(f)}) = \exp\big(-\int_E (1 - \mathrm{e}^{-f(x)})\mu(\mathrm{d}x)\big)\}$. First we prove that if g is simple measurable function from E to $[0, \infty)$, then $g \in \mathcal{L}$. Assume $g(x) = \sum_{k=1}^{n} a_k \mathbb{1}_{A_k}(x)$, where $A_k \in \mathcal{E}, a_k > 0, A_i \cap A_j = \emptyset$. Then $\mathbb{E}(\exp(-X(g))) = \mathbb{E}(\prod_{k=1}^{n} \exp(-a_k X(A_k))) =$

$\prod_{k=1}^n \mathbb{E}(\exp(-a_k X(A_k)))$, since $X(A_k) : k = 1, \cdots, n$ are independent. Easily to know

$$\mathbb{E}(\exp(-a_k X(A_k))) = \sum_{i=0}^{\infty} \mathbb{P}(X(A_k) = i)\exp(-a_k i) = \sum_{i=0}^{\infty} \frac{\exp(-\mu(A_k))\mu(A_k)^i}{i!}\exp(-a_k i)$$

Noting that

$$\exp(-\int_E (1-\exp(-a_k \mathbb{1}_{A_k}(x)))\mu(\mathrm{d}x)) = \exp(\exp(-a_k)\mu(A_k)-\mu(A_k)) = \exp(-\mu(A_k))\sum_{i=0}^{\infty}\frac{(\exp(-a_k)\mu(A_k))^i}{i!}$$

we get $\mathbb{E}(\exp(-a_k X(A_k))) = \exp(-\int_E(1-\exp(-a_k\mathbb{1}_{A_k}(x)))\mu(\mathrm{d}x))$. Noting $\int_E(1-\exp(-g(x)))\mu(\mathrm{d}x) = \sum_{k=1}^n \int_E(1-\exp(-a_k\mathbb{1}_{A_k}(x)))\mu(\mathrm{d}x)$, we get $\mathbb{E}(\exp(-X(g))) = \exp\left(-\int_E(1-\exp(-g(x)))\mu(\mathrm{d}x)\right)$.

Now for non-negative function f, consider f_n satisfy that $\forall n, f_n$ is simple, and $f_n \nearrow f$ and $f_n \geq 0$. Then easily to know $\mathbb{E}(\exp(-X(f))) = \lim_{n\to\infty}\mathbb{E}(\exp(-X(f_n))) = \lim_{n\to\infty}\exp(-\int_E(1-\exp(-f_n(x)))\mu(\mathrm{d}x)) = \exp(-\int_E(1-\exp(-f(x)))\mu(\mathrm{d}x))$. □

PROBLEM III Assume μ is finite measure on $(E,\mathcal{E})$, and X is Possion random measure with intensity μ. Assume $\phi : (E,\mathcal{E}) \to (F,\mathcal{F})$ is measurable, prove that $X\circ\phi^{-1}$ is Possion random measure with intensity $\mu\circ\phi^{-1}$.

SOLUTION. Assume $B_k \in \mathcal{F}, \forall k \in \mathbb{N}$ and $\forall i \neq j, B_i \cap B_j = \emptyset$. Then $X\circ\phi^{-1}(\bigcup_{k\in\mathbb{N}} B_k) = X(\bigcup_{k\in\mathbb{N}}\phi^{-1}(B_k)) = \sum_{k\in\mathbb{N}} X(\phi^{-1}(B_k))$. Since X is possion random measure with intensity μ, and for $B_1,\cdots,B_n \in \mathcal{F}$ and $B_i\cap B_j = \emptyset$, we have $\phi^{-1}(B_k)$ are disjoint set in $(E,\mathcal{E})$, so $\mathbb{E}(\exp(\mathrm{i}\sum_{k=1}^n \alpha_k X\circ\phi^{-1}(B_k))) = \exp(\sum_{k=1}^n(\exp(\mathrm{i}\alpha_k)-1)\mu\circ\phi^{-1}(B_k))$. So $X\circ\phi^{-1}$ is Possion random measure on $(F,\mathcal{F})$ with intensity $\mu\circ\phi^{-1}$. □

PROBLEM IV Assume $\alpha \geq 0$, and μ is probability measure on $\mathbb{R}$ with $\mu(\{0\}) = 0$. Let $N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)$ is Possion random measure on $\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}^+$ with intensity $\mathrm{d}s\mu(\mathrm{d}z)\,\mathrm{d}u$. Let $Y_t = Y_0 + \int_0^t\int_{\mathbb{R}}\int_0^\alpha zN(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)$, where $Y_0 \perp N$. Prove that $(Y_t : t\geq 0)$ is compound Possion process with rate α and jumping distribution μ.

SOLUTION. We know that $\forall t \geq 0, \forall r : 0 \leq r \leq t, Y_r \in \sigma(N(B) : B \subset [0,r]\times\mathbb{R}\times[0,\alpha])$. And $\forall w \geq t$, $Y_w - Y_t \in \sigma(N(B) : B\subset (t,w]\times\mathbb{R}\times[0,\alpha])$. Easily $(t,w]\cap[0,r] = \emptyset$, so we get $Y_w - Y_t \perp (Y_r : 0 \leq r \leq t)$. Now we only need to check the distribution of $Y_w - Y_t$. Since $\mathrm{d}s\mu(\mathrm{d}z)\,\mathrm{d}u$ has the same distribution with $\mathrm{d}(s-t)\mu(\mathrm{d}z)\,\mathrm{d}u$, we get that $Y_w - Y_t \overset{d}{=} Y_{w-t} - Y_0$. So we only need to check the distribution of $Y_t - Y_0$.

Since the distribution of $Y_t - Y_0$ is determined by the distribution of N on the set $[0,t]\times\mathbb{R}\times[0,\alpha]$, so we can assume N is constructed as 1.5.5, because this will not change the distribution of N on $[0,t]\times\mathbb{R}\times[0,\alpha]$. Let $\eta \sim Possion(\alpha t)$ and $\xi_1,\cdots$ are i.i.d r.v. range in $[0,t]\times\mathbb{R}\times[0,\alpha]$ with distribution $\mathrm{d}s\mu(\mathrm{d}z)\,\mathrm{d}u$, and $\eta\perp(\xi_n : n\in\mathbb{N}'+)$. Assume for $B\subset[0,t]\times\mathbb{R}\times[0,\alpha]$ we have $N(B) = \sum_{j=1}^\eta \delta_{\xi_j}(B)$. Then we get $Y_t - Y_0 = \sum_{j=1}^\eta \xi_j(2)$, where $\xi_j(2)$ is the second ordinate of ξ_j. Easily $\xi_j(2)\sim\mu, \eta\sim Possion(\alpha t)$, so finally we get $(Y_t : t\geq 0)$ is the compound distribution with rate α and jumping distribution μ. □

PROBLEM V Assume X is Possion random measure on $(E,\mathcal{E})$ with intensity μ, a finite measure. Assume f, g are non-negative measure function on E. Prove that:

1. $\mathbb{E}(X(f)\mathrm{e}^{-X(g)}) = \mu(f\mathrm{e}^{-g})\mathbb{E}(\mathrm{e}^{-X(g)})$.
2. $\mathbb{E}(X(f)^2\mathrm{e}^{-X(g)}) = (\mu(f^2\mathrm{e}^{-g} + \mu(f\mathrm{e}^{-g})'2))\mathbb{E}(\mathrm{e}^{-X(g)})$.

SOLUTION. 1. Let $h(\theta) := \mathbb{E}(\mathrm{e}^{-X(\theta f+g)}) = \exp\left(-\int_E (1-\mathrm{e}^{-\theta f(x)-g(x)}\mu(\mathrm{d}x))\right)$. Then

$$h'(\theta) = \mathbb{E}(X(f)\mathrm{e}^{-X(\theta f+g)}) = \exp\left(-\int_E (1-\mathrm{e}^{-\theta f(x)-g(x)}\mu(\mathrm{d}x))\right)\cdot \int_E f(x)\mathrm{e}^{-\theta f(x)-g(x)}\mu(\mathrm{d}x)$$

Since they are all non-negative, the differential is valid. Let $\theta = 0$, we get $\mathbb{E}(X(f)\mathrm{e}^{-X(g)}) = \mu(f\mathrm{e}^{-g})\mathbb{E}(\mathrm{e}^{-X(g)})$.

2. Take h as above, easily to get $h''(\theta) = \mathbb{E}(X(f)^2)\mathrm{e}^{-X(g)} = \exp\left(-\int_E (1-\mathrm{e}^{-\theta f(x)-g(x)}\mu(\mathrm{d}x))\right)\cdot \left(\int_E f(x)\mathrm{e}^{-\theta f(x)-g(x)}\mu(\mathrm{d}x)\right)^2 + \exp\left(-\int_E (1-\mathrm{e}^{-\theta f(x)-g(x)}\mu(\mathrm{d}x))\right)\cdot\int_E f(x)^2\mathrm{e}^{-\theta f(x)-g(x)}\mu(\mathrm{d}x)$. Let $\theta = 0$, then easily $\mathbb{E}(X(f)^2\mathrm{e}^{-X(g)}) = (\mu(f^2\mathrm{e}^{-g} + \mu(f\mathrm{e}^{-g})'2))\mathbb{E}(\mathrm{e}^{-X(g)})$.

$\square$