underGraduate Homework In Mathematics

MarkovProcess 1

白永乐

202011150087

202011150087@mail.bnu.edu.cn

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PROBLEM I Assume $(\mathscr{F}_t : t \geq 0, t \in \mathbb{R})$ is a filtration. For $t \geq 0$ we let $\mathscr{F}_{t+} := \bigcap_{s>t} \mathscr{F}_s$. Prove that $\mathscr{F}_t \subset \mathscr{F}_{t+}$ and $(\mathscr{F}_{t+} : t \geq 0)$ is a filtration.

SOLUTION. To prove $\mathscr{F}_t \subset \mathscr{F}_{t+} = \bigcap_{s>t} \mathscr{F}_s$, we only need to prove $\forall s > t, \mathscr{F}_t \subset \mathscr{F}_s$. By the definition of filtration it's obvious. Now we will prove $(\mathscr{F}_{t+} : t \geq 0)$ is a filtration. Only need to prove $\forall t, s \in \mathbb{R} \wedge t \leq s, \mathscr{F}_{t+} \subset \mathscr{F}_{s+}$. By the definition of $\mathscr{F}_{\dot{c}+}$ we know that $\mathscr{F}_{t+} = \bigcap_{x>t} \mathscr{F}_x = \bigcap_{x>s} \mathscr{F}_x \cap \bigcap_{x:t<x\leq s} \mathscr{F}_x \subset \bigcap_{x>s} \mathscr{F}_x = \mathscr{F}_{s+}$. So $(\mathscr{F}_{t+} : t \geq 0)$ is a filtration. $\square$

PROBLEM II Assume $(X_t : t \geq 0, t \in \mathbb{R})$ is a stochastic process on probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Prove that $\forall s, t \geq 0, \varepsilon > 0, \{\rho(X_s, X_t) \geq \varepsilon\} \in \mathscr{F}$.

SOLUTION. Easily $\{\rho(X_s, X_t) \geq \varepsilon\} = \bigcup_{k=1}^{\infty} \{\rho(X_s, X_t) > \varepsilon - \frac{1}{k}\varepsilon\}$. So we only need to prove $\forall k \in \mathbb{N}^+, \{\rho(X_s, X_t) > \varepsilon(1 - \frac{1}{k})\} \in \mathscr{F}$. Take $\delta = \varepsilon(1 - \frac{1}{k})$, only need to prove $\forall \delta > 0, \{\rho(X_s, X_t) > \delta\} \in \mathscr{F}$.

$\forall t \geq 0, X_t : \Omega \to E$ is measurable, where $E \subset \mathbb{R}^d$. So we can find a countable dense set in $\mathbb{R}^d$, write D. We will prove that $\{\rho(X_s, X_t) > \delta\} = \bigcup_{q \in D} \{\rho(X_s, q) - \rho(X_t, q) > \delta\}$. On one hand, easily $\rho(X_s, q) - \rho(X_t, q) > \delta \implies \rho(X_s, X_t) > \delta$ from triangle inequality. So we easily get $\{\rho(X_s, X_t) > \delta\} \supset \bigcup_{q \in D} \{\rho(X_s, q) - \rho(X_t, q) > \delta\}$. On the other hand, assume for certain $\omega \in \Omega$ we have $\rho(X_s(\omega), X_t(\omega)) > \delta$, we will prove $\exists q \in D, \rho(X_s(\omega), q) - \rho(X_t(\omega), q) > \delta$. For convience we omit (ω) from now on to the end of this paragraph. Since $\rho(X_s, X_t) > \delta$, we know $\gamma := \frac{\rho(X_s, X_t) - \delta}{2} > 0$. Since D is dense, we obtain $\exists q \in D, \rho(X_t, q) < \gamma$. So from triangle inequality we get $\rho(X_s, q) \geq \rho(X_s, X_t) - \rho(X_t, q) > 2\gamma + \delta - \gamma = \gamma + \delta$. So we get $\rho(X_s, q) - \rho(X_t, q) > \gamma + \delta - \gamma = \delta$. So finally we get $\{\rho(X_s, X_t) > \delta\} = \bigcup_{q \in D} \{\rho(X_s, q) - \rho(X_t, q) > \delta\}$.

Noting $\bigcup_{q \in D} \{\rho(X_s, q) - \rho(X_t, q) > \delta\} = \bigcup_{q \in D} \bigcup_{p \in \mathbb{Q}^+} \{\rho(X_s, q) > \delta + p, \rho(X_t, q) < p\}$, and $D, \mathbb{Q}^+$ are countable, so we only need to check $\{\rho(X_s, q) > \delta + p, \rho(X_t, q) < p\} \in \mathscr{F}, \forall q \in D, p \in \mathbb{Q}^+$. Noting $\{\rho(X_s, q) > \delta + p, \rho(X_t, q) < p\} = \{\rho(X_s, q) > \delta + p\} \cap \{\rho(X_t, q) < p\}$, and X_s, X_t are measurable from Ω to E, we obtain $\{\rho(X_s, q) > \delta + p\}, \{\rho(X_t, q) < p\} \in \mathscr{F}$. So we proved $\{\rho(X_s, X_t) > \delta\} \in \mathscr{F}, \forall s, t \geq 0, \forall \delta > 0$.

Finally, we obtain $\{\rho(X_s, X_t) \geq \varepsilon\} \in \mathscr{F}, \forall s, t \geq 0, \varepsilon > 0$. $\square$

PROBLEM III Let $\mathscr{D}_X := \{\mu_J^X : J \in S(I)\}$ be the family of finite-dimentional distributions of a stochastic process $(X_t : t \geq 0, t \in \mathbb{R})$. $\forall (s_1, s_2) \in S(I)$ and $J = (t_1, \cdots, t_n) \in S(I)$, write $K_1 := (s_1, s_2, t_1, \cdots, t_n) \in S(I), K_2 := (s_2, s_1, t_1, \cdots, t_n) \in S(I)$. Take $A_1, A_2 \in \mathscr{E}, B \in \mathscr{E}^n$, prove that

$$\mu_{K_1}^X(A_1 \times A_2 \times B) = \mu_{K_2}^X(A_2 \times A_1 \times B)$$

and

$$\mu_{K_1}^X(E \times E \times B) = \mu_{K_2}^X(E \times E \times B) = \mu_J^X(B)$$