

Markov 过程复习资料

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2024 年 6 月 20 日

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2 基本概念和例子

2.1 基本概念

2.1.1 随机过程的定义

Definition 1. 设 I 是非空指标集, $(\Omega, \mathcal{F}, \mathbb{P})$ 是概率空间. 若 $(X_\alpha : \alpha \in I)$ 是一组定义在 $(\Omega, \mathcal{F}, \mathbb{P})$ 上的随机变量 (取值为 \mathbb{R}^d), 则称 $(X_\alpha : \alpha \in I)$ 为一个随机过程。

Definition 2. 假设 $(X_\alpha : \alpha \in I)$ 和 $(Y_\alpha : \alpha \in J)$ 是两个随机过程. 若对于任何有限序列 $(s_1, \dots, s_n) \subset I, (t_1, \dots, t_m) \subset J$, 都有 $(X_{s_1}, \dots, X_{s_n}) \perp (Y_{t_1}, \dots, Y_{t_m})$, 则称这两个随机过程独立。

2.1.2 轨道和修正

Definition 3. 设 $(X_\alpha : \alpha \in I)$ 为随机过程. 固定 $\omega \in \Omega$, 称 $t \mapsto X_t(\omega)$ 为 X 的一条轨道。

Definition 4. 称一个随机过程是 (左连续//右连续//连续//左极右连//左连右极) 的, 若它的所有轨道都是 (左连续//右连续//连续//左极右连//左连右极) 的。

Definition 5. 设 $(X_t : t \in I)$ 和 $(Y_t : t \in I)$ 是两个随机过程. 若 $\forall t \in I$, 有 $\mathbb{P}(X_t = Y_t) = 1$, 则称它们互为修正. 若 $\mathbb{P}(\forall t \in I, X_t = Y_t) = 1$, 则称它们是无区别的。

Theorem 1. 设 $(X_t : t \geq 0)$ 和 $(Y_t : t \geq 0)$ 是两个右连续的随机过程, 而 D 是 $(0, \infty)$ 的可数稠密子集. 若 $\forall s \in D, \mathbb{P}(X_s = Y_s) = 1$, 则有 $(X_t : t \geq 0)$ 和 $(Y_t : t \geq 0)$ 是无区别的。

2.1.3 有限维分布族

为了简化记号, 我们用 $S(I)$ 表示 I 的全体有序有限子集. 即:

$$S(I) := \{(t_1, \dots, t_n) : n \geq 1, t_i \in I, \forall i = 1, \dots, n\}$$

用 E 表示 \mathbb{R}^d , 用 \mathcal{E} 表示博雷尔代数。

Definition 6. 设 I 是非空指标集. 若对于每个 $J \in S(I)$, 都对应一个 $(E^{|J|}, \mathcal{E}^{|J|})$ 上的概率测度 u_J , 则称 $(\mu_J : J \in S(I))$ 为 E 上的一个有限维分布族, 其中每个 μ_J 称为一个有限维分布. 设 $X = (X_t : t \in I)$ 是一个随机过程, 用 μ_J^X 表示 $(X_{t_1}, \dots, X_{t_n})$ 的分布. 称 $\mathcal{D}_X := \{\mu_J^X : J \in S(I)\}$ 为 X 的有限维分布族, 称 μ_J^X 为其中的一个有限维分布。

Definition 7. 给定 (E, \mathcal{E}) 上的有限维分布族 \mathcal{D} , 若存在随机过程 $X = (X_t : t \in I)$ 使得 $\mathcal{D}_X = \mathcal{D}$, 则称 X 为 \mathcal{D} 的一个实现. 若两个随机过程 X, Y 满足 $\mathcal{D}_X = \mathcal{D}_Y$, 则称它们为等价的. 两个等价的过程互称实现. 显然, 两个互为修正的随机过程一定等价, 反过来却未必。

2.1.4 左极右连实现

Definition 8. 状态空间 $E = \mathbb{R}^d$ 上的随机过程有左极右连实现 \iff 它有左极右连修正。证明见教材 p5

PROBLEM I Assume $(\mathcal{F}_t : t \geq 0, t \in \mathbb{R})$ is a filtration. For $t \geq 0$ we let $\mathcal{F}_{t+} := \bigcap_{s>t} \mathcal{F}_s$. Prove that $\mathcal{F}_t \subset \mathcal{F}_{t+}$ and $(\mathcal{F}_{t+} : t \geq 0)$ is a filtration.

SOLUTION. To prove $\mathcal{F}_t \subset \mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_s$, we only need to prove $\forall s > t, \mathcal{F}_t \subset \mathcal{F}_s$. By the definition of filtration it's obvious. Now we will prove $(\mathcal{F}_{t+} : t \geq 0)$ is a filtration. Only need to prove $\forall t, s \in \mathbb{R} \wedge t \leq s, \mathcal{F}_{t+} \subset \mathcal{F}_{s+}$. By the definition of \mathcal{F}_{t+} we know that $\mathcal{F}_{t+} = \bigcap_{x>t} \mathcal{F}_x = \bigcap_{x>s} \mathcal{F}_x \cap \bigcap_{x:t<x\leq s} \mathcal{F}_x \subset \bigcap_{x>s} \mathcal{F}_x = \mathcal{F}_{s+}$. So $(\mathcal{F}_{t+} : t \geq 0)$ is a filtration. \square

PROBLEM II Assume $(X_t : t \geq 0, t \in \mathbb{R})$ is a stochastic process on probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Prove that $\forall s, t \geq 0, \varepsilon > 0, \{\rho(X_s, X_t) \geq \varepsilon\} \in \mathcal{F}$.

SOLUTION. Easily $\{\rho(X_s, X_t) \geq \varepsilon\} = \bigcup_{k=1}^{\infty} \{\rho(X_s, X_t) > \varepsilon - \frac{1}{k}\varepsilon\}$. So we only need to prove $\forall k \in \mathbb{N}^+, \{\rho(X_s, X_t) > \varepsilon(1 - \frac{1}{k})\} \in \mathcal{F}$. Take $\delta = \varepsilon(1 - \frac{1}{k})$, only need to prove $\forall \delta > 0, \{\rho(X_s, X_t) > \delta\} \in \mathcal{F}$.

$\forall t \geq 0, X_t : \Omega \rightarrow E$ is measurable, where $E \subset \mathbb{R}^d$. So we can find a countable dense set in \mathbb{R}^d , write D . We will prove that $\{\rho(X_s, X_t) > \delta\} = \bigcup_{q \in D} \{\rho(X_s, q) - \rho(X_t, q) > \delta\}$. On one hand, easily $\rho(X_s, q) - \rho(X_t, q) > \delta \implies \rho(X_s, X_t) > \delta$ from triangle inequality. So we easily get $\{\rho(X_s, X_t) > \delta\} \supset \bigcup_{q \in D} \{\rho(X_s, q) - \rho(X_t, q) > \delta\}$. On the other hand, assume for certain $\omega \in \Omega$ we have $\rho(X_s(\omega), X_t(\omega)) > \delta$, we will prove $\exists q \in D, \rho(X_s(\omega), q) - \rho(X_t(\omega), q) > \delta$. For convenience, we omit (ω) from now on to the end of this paragraph. Since $\rho(X_s, X_t) > \delta$, we know $\gamma := \frac{\rho(X_s, X_t) - \delta}{2} > 0$. Since D is dense, we obtain $\exists q \in D, \rho(X_t, q) < \gamma$. So from triangle inequality we get $\rho(X_s, q) \geq \rho(X_s, X_t) - \rho(X_t, q) > 2\gamma + \delta - \gamma = \gamma + \delta$. So we get $\rho(X_s, q) - \rho(X_t, q) > \gamma + \delta - \gamma = \delta$. Finally, we get $\{\rho(X_s, X_t) > \delta\} = \bigcup_{q \in D} \{\rho(X_s, q) - \rho(X_t, q) > \delta\}$.

Noting $\bigcup_{q \in D} \{\rho(X_s, q) - \rho(X_t, q) > \delta\} = \bigcup_{q \in D} \bigcup_{p \in \mathbb{Q}^+} \{\rho(X_s, q) > \delta + p, \rho(X_t, q) < p\}$, and D, \mathbb{Q}^+ are countable, so we only need to check $\{\rho(X_s, q) > \delta + p, \rho(X_t, q) < p\} \in \mathcal{F}, \forall q \in D, p \in \mathbb{Q}^+$. Noting $\{\rho(X_s, q) > \delta + p, \rho(X_t, q) < p\} = \{\rho(X_s, q) > \delta + p\} \cap \{\rho(X_t, q) < p\}$, and X_s, X_t are measurable from Ω to E , we obtain $\{\rho(X_s, q) > \delta + p\}, \{\rho(X_t, q) < p\} \in \mathcal{F}$. So we proved $\{\rho(X_s, X_t) > \delta\} \in \mathcal{F}, \forall s, t \geq 0, \forall \delta > 0$.

Finally, we obtain $\{\rho(X_s, X_t) \geq \varepsilon\} \in \mathcal{F}, \forall s, t \geq 0, \varepsilon > 0$. \square

PROBLEM III Let $\mathcal{D}_X := \{\mu_J^X : J \in S(I)\}$ be the family of finite-dimensional distributions of a stochastic process $(X_t : t \geq 0, t \in \mathbb{R})$. $\forall (s_1, s_2) \in S(I)$ and $J = (t_1, \dots, t_n) \in S(I)$, write $K_1 := (s_1, s_2, t_1, \dots, t_n) \in S(I), K_2 := (s_2, s_1, t_1, \dots, t_n) \in S(I)$. Take $A_1, A_2 \in \mathcal{E}, B \in \mathcal{E}^n$, prove that

$$\mu_{K_1}^X(A_1 \times A_2 \times B) = \mu_{K_2}^X(A_2 \times A_1 \times B)$$

and

$$\mu_{K_1}^X(E \times E \times B) = \mu_{K_2}^X(E \times E \times B) = \mu_J^X(B)$$

SOLUTION. By the definition of finite-dimensional distributions we get

$$\begin{aligned} \mu_{K_1}^X(A_1 \times A_2 \times B) &= \mathbb{P}((X_{s_1}, X_{s_2}, X_{t_1}, \dots, X_{t_n}) \in A_1 \times A_2 \times B) \\ &= \mathbb{P}(X_{s_1} \in A_1) \mathbb{P}(X_{s_2} \in A_2) \mathbb{P}((X_{t_1}, X_{t_2}, \dots, X_{t_n}) \in B) \end{aligned}$$

For the same reason, we obtain

$$\begin{aligned} \mu_{K_2}^X(A_2 \times A_1 \times B) &= \mathbb{P}((X_{s_2}, X_{s_1}, X_{t_1}, \dots, X_{t_n}) \in A_2 \times A_1 \times B) \\ &= \mathbb{P}(X_{s_2} \in A_2) \mathbb{P}(X_{s_1} \in A_1) \mathbb{P}((X_{t_1}, X_{t_2}, \dots, X_{t_n}) \in B) \end{aligned}$$

So we get

$$\mu_{K_1}^X(A_1 \times A_2 \times B) = \mu_{K_2}^X(A_2 \times A_1 \times B)$$

Also, let $A_1 = A_2 = E$, we get

$$\begin{aligned}\mu_{K_1}^X(E \times E \times B) &= \mu_{K_2}^X(E \times E \times B) \\ &= \mathbb{P}(X_{s_1} \in E) \mathbb{P}(X_{s_2} \in E) \mathbb{P}((X_{t_1}, \dots, X_{t_n}) \in B) \\ &= \mathbb{P}((X_{t_1}, \dots, X_{t_n}) \in B)\end{aligned}$$

By the definition of finite-dimensional distributions we get

$$\mu_J^X(B) = \mathbb{P}((X_{t_1}, \dots, X_{t_n}) \in B)$$

So finally we get

$$\mu_{K_1}^X(E \times E \times B) = \mu_{K_2}^X(E \times E \times B) = \mu_J^X(B)$$

□

PROBLEM IV Assume $(\tau_k : k \in \mathbb{N}^+)$ is an i.i.d sequence of r.v. with exponential distribution with parameter $\alpha > 0$. Let $S_n := \sum_{k=1}^n \tau_k$. For $t \geq 0, t \in \mathbb{R}$, let:

$$N_t := \sum_{n=1}^{\infty} \mathbb{1}_{\{S_n \leq t\}}, X_t := \sum_{n=1}^{\infty} \mathbb{1}_{\{S_n < t\}}$$

Prove that N and X are modifications of each other, but they are not indistinguishable.

SOLUTION. First we prove N and X are modifications of each other. Fix $t \in [0, \infty)$, we need to prove $\mathbb{P}(N_t = X_t) = 1$. Noting $N_t - X_t = \sum_{n=1}^{\infty} \mathbb{1}_{S_n \leq t} - \mathbb{1}_{S_n < t} = \sum_{n=1}^{\infty} \mathbb{1}_{S_n = t}$, we get $\mathbb{P}(N_t = X_t) = \mathbb{P}(\sum_{n=1}^{\infty} \mathbb{1}_{S_n = t}) = \mathbb{P}(\forall n \in \mathbb{N}^+, S_n \neq t)$. So we only need to prove $\mathbb{P}(S_n = t) = 0, \forall n \in \mathbb{N}^+$. Since $\tau_k, k \in \mathbb{N}^+$ are continuous-distributed, we know $S_n = \sum_{k=1}^n \tau_k$ is continuous-distributed, so $\mathbb{P}(S_n = t) = 0$. So we proved N and X are modifications of each other.

Next we will prove they are not indistinguishable. Only need to prove $\mathbb{P}(\forall t \in [0, \infty), X_t = N_t) = 0 \neq 1$. Since $N_t - X_t = \sum_{n=1}^{\infty} \mathbb{1}_{S_n \leq t} - \mathbb{1}_{S_n < t} = \sum_{n=1}^{\infty} \mathbb{1}_{S_n = t}$, we know $\forall t, N_t = X_t \iff \forall t, \forall n \in \mathbb{N}^+, S_n \neq t$. But S_n is ranged in $[0, \infty)$, so R.H.S is an impossible event. So we finally get $\mathbb{P}(\forall t \in [0, \infty), X_t = N_t) = 0$ and thus X and N are not indistinguishable. □

PROBLEM V Assume T is non-negative r.v. with distribution function F continuous on \mathbb{R} . Let $X_t = \mathbb{1}_{\{T \leq t\}}$. Prove that X is stochastically continuous.

SOLUTION. Only need to check $\forall t \geq 0, X_s \xrightarrow{\mathbb{P}} X_t, s \rightarrow t$. Take $\varepsilon > 0$, we need to prove $\lim_{s \rightarrow t} \mathbb{P}(\rho(X_s, X_t) > \varepsilon) = 0$. For $u > v \geq 0$, we have $X_u - X_v = \mathbb{1}_{v < T \leq u}$. So $\mathbb{P}(\rho(X_u - X_v) > \varepsilon) \leq \mathbb{P}(X_u \neq X_v) = \mathbb{P}(v < T \leq u) \leq \mathbb{P}(T \in [v, u])$. So we easily get $\lim_{s \rightarrow t+} \mathbb{P}(\rho(X_s, X_t) > \varepsilon) \leq \lim_{s \rightarrow t+} \mathbb{P}(T \in [t, s]) = 0$ and $\lim_{s \rightarrow t-} \mathbb{P}(\rho(X_s, X_t) > \varepsilon) \leq \lim_{s \rightarrow t-} \mathbb{P}(T \in [s, t]) = 0$. So $\lim_{s \rightarrow t} \mathbb{P}(\rho(X_s, X_t) > \varepsilon) = 0$. □

PROBLEM VI Assume $I = \mathbb{Z}^+$, then the stochastic process $X = (X_0, X_1, \dots)$ is a r.v. from Ω to E^∞ . Define the distribution of X , μ_X , as follows:

$$\mu_X(A) = \mathbb{P}(X \in A), A \in \mathcal{E}^\infty$$

Then stochastic process X, Y are equivalent $\iff \mu_X = \mu_Y$.

SOLUTION. “ \implies ”: Assume X, Y are equivalent, now we will prove $\mu_X = \mu_Y$. Let $\mathcal{A} := \{A \in \mathcal{P}(E^\infty) : \exists n \in \mathbb{N}^+, A = A_1 \times A_2 \times \dots \times A_n \times E \times E \times \dots\}$. Then we can get $\mu_X(A) = \mu_{(1,2,\dots,n)}^X(A_1 \times \dots \times A_n)$. So for $A \in \mathcal{A}$ we know $\mu_X(A) = \mu_Y(A)$. By the definition of $\mathcal{E}^\infty = \sigma(\mathcal{A})$, and noting \mathcal{A} is a Semiset algebra, by the Measure extension theorem we get $\mu_X = \mu_Y$.

“ \impliedby ”: Assume $\mu_X = \mu_Y$, then easily $\mu_{(s_1,\dots,s_n)}^X(A_{s_1} \times \dots \times A_{s_n}) = \mu_X(\prod_{k \in \mathbb{N}^+} B_k)$, where $B_k = A_{s_t}$ for $k = s_t$ and $B_k = E$ for $k \neq s_t, \forall t = 1, \dots, n$. So easily $\mu_J^X = \mu_J^Y, \forall J \subset I \wedge |J| < \infty$. □

2.2 随机游动

Definition 9. 设 $\{\xi_n : n \geq 1\}$ 是独立同分布的 d 维随机变量列, 而 X_0 是与之独立的一个 d 维随机变量。令 $X_n := X_0 + \sum_{k=1}^n \xi_k$ 。称 $(X_n : n \geq 0)$ 为 d 维随机游动, 并称 $\{\xi_n : n \geq 1\}$ 为其步长列。

Definition 10. 若 X_0, ξ_1 均取值与 \mathbb{Z}^d , 则该随机游动状态空间可以取为 \mathbb{Z}^d 。特别地, 若还有 $\mathbb{P}(|\xi_1| = 1) = 1$, 则称其为简单随机游动。进一步地, 若对于 \mathbb{Z}^d 中的任一单位向量 v , 均有 $\mathbb{P}(\xi_1 = v) = \frac{1}{2d}$, 则称其为对称简单随机游动。

2.2.1 轨道的无界性

方便起见, 考虑 \mathbb{Z} 上的简单随机游动 S_n , 设其步长列为 $\xi_n : n \geq 1$ 。设 $\mathbb{P}(\xi_n = 1) = p, \mathbb{P}(\xi_n = -1) = q$, 其中 $p, q \in (0, 1), p+q = 1$ 。

Theorem 2. $(S_n : n \geq 1)$ 的轨道是几乎必然无界的。即:

$$\mathbb{P}(\sup_{n \geq 0} |S_n| = \infty) = 1. \quad (1)$$

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2.2.2 首达时分布

Definition 11. 记 $\mathbb{P}_i(\cdot) = \mathbb{P}(\cdot | S_0 = i)$ 。

Definition 12. 定义 $(S_n : n \geq 0)$ 到达 $x \in \mathbb{Z}$ 的首达时 $\tau_x := \inf\{n \geq 0 : S_n = x\}$ 。

Theorem 3. 当 $p = q = \frac{1}{2}$ 时, 对于 $a < b, i \in [a, b], a, b, i \in \mathbb{Z}$, 有

$$\mathbb{P}_i(\tau_b < \tau_a) = \frac{i-a}{b-a}, \mathbb{P}_i(\tau_a < \tau_b) = \frac{b-i}{b-a} \quad (2)$$

当 $p \neq q$ 时, 有

$$\mathbb{P}_i(\tau_b < \tau_a) = \frac{1 - (\frac{q}{p})^{i-a}}{1 - (\frac{q}{p})^{b-a}}, \mathbb{P}_i(\tau_a < \tau_b) = \frac{(\frac{q}{p})^{i-a} - (\frac{q}{p})^{b-a}}{1 - (\frac{q}{p})^{b-a}} \quad (3)$$

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Theorem 4. 当 $p \geq q$, 对 $a \leq i \leq b \in \mathbb{Z}$, 有

$$\mathbb{P}_i(\tau_a < \infty) = (\frac{q}{p})^{i-a}, \mathbb{P}_i(\tau_b < \infty) = 1 \quad (4)$$

当 $p \leq q$, 有

$$\mathbb{P}_i(\tau_a < \infty) = 1, \mathbb{P}_i(\tau_b < \infty) = (\frac{p}{q})^{b-i} \quad (5)$$

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PROBLEM VII Prove that if $(X_n : n \geq 0)$ is a simple random walk, then so is $(-X_n : n \geq 0)$.

SOLUTION. Let $\xi_n := X_n - X_{n-1}$ for $n \in \mathbb{N}^+$. Then Since $(X_n : n \in \mathbb{N})$ is simple random walk we have X_0, ξ_1, ξ_2, \dots are independent r.v. ranges in \mathbb{Z} , and $\xi_i, i = 1, 2, \dots$ are i.i.d., and $\mathbb{P}(|\xi_i| = 1) = 1$. So we easily get $-X_0, -\xi_1, -\xi_2, \dots$ are independent r.v. ranges in \mathbb{Z} , and $-\xi_i, i = 1, 2, \dots$ are i.i.d., and $\mathbb{P}(|-\xi_i| = 1) = 1$. Since $-X_n = X_0 + \sum_{k=1}^n \xi_k$, by the definition of simple random walk we obtain $(-X_n : n \in \mathbb{N})$ is a simple random walk. \square

PROBLEM VIII Let $(X_n : n \geq 0)$ be a d -dimensional random walk with $\mathbb{P}(|\xi_i| \geq 1) > 0$, prove that $\mathbb{P}(\sup_n |X_n| = \infty) = 1$.

SOLUTION. Let $t \in \mathbb{Z}^d, t \neq 0$ and $\mathbb{P}(\xi_i = t) > 0$. Since $\mathbb{P}(\sup_n |X_n| = \infty) = \inf_{k \in \mathbb{N}} \mathbb{P}(\sup_n |X_n| \geq k)$, we only need to prove $\mathbb{P}(\sup_n |X_n| \geq k) = 1$ for every $k \in \mathbb{N}$. Take $K > 3k, K \in \mathbb{N}$. Let $A_s := \{\xi_i = t : i = sK + 1, sK + 2, \dots, sK + K - 1\}$. Then for $\omega \in A_s$, we have $|X_{sK+K} - X_{sK}| = |\sum_{u=1}^{K-1} t| = K|t| \geq K \geq 3k$. Then $\sup_n |X_n| \geq \max\{|X_{sK+K}|, |X_{sK}|\} \geq \frac{1}{2}|X_{sK+K} - X_{sK}| \geq k$. So we get $\forall s, A_s \subset \{\sup_n |X_n| \geq k\}$. Since ξ_i are independent, easily get $A_s, s = 1, 2, \dots$ are independent. Noting $\mathbb{P}(A_s) = \mathbb{P}(\xi_i = t)^K > 0$, we get $\sum_{s \in \mathbb{N}} \mathbb{P}(A_s) = \infty$. So from BC-theorem we get $\mathbb{P}(A_s, i.o.) = 1$, thus $\mathbb{P}(\bigcup_{s \in \mathbb{N}} A_s) = 1$. Thus, $\mathbb{P}(\sup_n |X_n| \geq k) = 1$, for every $k \in \mathbb{N}$. Thus, $\mathbb{P}(\sup_n |X_n| = \infty) = \mathbb{P}(\bigcap_{k \in \mathbb{N}} \{\sup_n |X_n| \geq k\}) = 1$. \square

PROBLEM IX Let $(S_n : n \geq 0)$ be a symmetry simple random walk with $S_0 = 0$, for $d = 2$, prove that

$$\mathbb{P}(S_{2n} = 0) = \frac{1}{4^{2n}} \left(\frac{(2n)!}{(n!)^2} \right)^2$$

For $d = 3$, prove that

$$\mathbb{P}(S_{2n} = 0) = \frac{1}{2^{2n}} \frac{(2n)!}{(n!)^2} \sum_{i+j+k=n} \left(\frac{1}{3^n} \frac{n!}{i!j!k!} \right)^2$$

PROOF. First we consider $d = 2$. Write $\xi_i = S_i - S_{i-1}$. Then we know S_{2n} occur \iff the number of $(1, 0)$ and $(-1, 0)$ in $\{\xi_i : i = 1, \dots, 2n\}$, and the number of $(0, 1)$ and $(0, -1)$ in $\{\xi_i : i = 1, \dots, 2n\}$. We assume there is k pairs of $(1, 0), (-1, 0)$, then easily there is $n - k$ pairs of $(0, 1), (0, -1)$. The probability is $\binom{2n}{k} \binom{2n-k}{k} \binom{2n-2k}{n-k} \frac{1}{4^{2n}}$. So the total probability is $\mathbb{P}(S_{2n} = 0) = \sum_{k=0}^n \binom{2n}{k} \binom{2n-k}{k} \binom{2n-2k}{n-k} \frac{1}{4^{2n}} = \sum_{k=0}^n \frac{(2n)!}{k!k!(n-k)!(n-k)!4^{2n}}$. Noting that $\sum_{k=0}^n \frac{(n!)^2}{k!k!(n-k)!(n-k)!} = \sum_{k=0}^n \binom{n}{k} \binom{n}{n-k} = \binom{2n}{n} = \frac{(2n)!}{n!n!}$, we finally get $\mathbb{P}(S_{2n} = 0) = \frac{1}{4^{2n}} \left(\frac{(2n)!}{n!n!} \right)^2$.

Use the same method, consider $d = 3$, we have

$$\mathbb{P}(S_{2n} = 0) = \sum_{i+j+k=n} \binom{2n}{i} \binom{2n-i}{i} \binom{2n-2i}{j} \binom{2n-2i-j}{j} \binom{2n-2i-2j-k}{k} \frac{1}{6^{2n}}$$

So easily to get $\mathbb{P}(S_{2n} = 0) = \frac{1}{2^{2n}} \frac{(2n)!}{(n!)^2} \sum_{i+j+k=n} \left(\frac{1}{3^n} \frac{n!}{i!j!k!} \right)^2$. □

PROBLEM X Assume $(S_n : n \geq 0)$ is a symmetry simple random walk with $S_0 = i \in \mathbb{Z}$. Prove that $\forall a \in \mathbb{Z}$, let $\tau_a := \min\{n \in \mathbb{N} : S_n = a\}$, then $\mathbb{P}(\tau_a < \infty) = 1$.

PROOF. Without loss of generality assume $a < 0, i = 0$. Take $N \in \mathbb{N}^+$. Consider $\tau := \min\{n \in \mathbb{N} : S_n = a \vee S_n = N\}$. From Problem XII we can easily know $\mathbb{P}(\tau < \infty) = 1$ because $\{\sup_n |S_n| = \infty\} \subset \{\tau < \infty\}$, a.s. So we get $\{\tau_a = \tau\} \subset \{\tau_a < \infty\}$, a.s. Let $Y_n := S_{n \wedge \tau} := S_{\min\{n, \tau\}}$. Easily $(S_n : n \in \mathbb{N})$ is a martingale, and τ is a stopping time, so we get $(Y_n : n \in \mathbb{N})$ is a martingale, too. And easily $Y_n \in [a, N]$, so Y_n is bounded. So we get $\mathbb{E}(S_\tau) = \lim_{n \rightarrow \infty} \mathbb{E}(Y_n) = \mathbb{E}(Y_0) = 0$. Easily to know $\mathbb{E}(S_\tau) = \mathbb{P}(\tau = \tau_a)a + \mathbb{P}(\tau \neq \tau_a)N = 0$. And $\mathbb{P}(\tau = \tau_a) + \mathbb{P}(\tau \neq \tau_a) = 1$, so easily $\mathbb{P}(\tau = \tau_a) = \frac{N}{N-a}$. So $\mathbb{P}(\tau_a < \infty) \geq \frac{N}{N-a}$. Let $N \rightarrow \infty$, we get $\mathbb{P}(\tau_a < \infty) = 1$. □

2.3 布朗运动

2.3.1 背景和定义

Definition 13. 假定 $\sigma^2 > 0$, 具有连续轨道的实值过程 $(B_t : t \geq 0)$ 满足:

1. $\forall 0 \leq s \leq t, B_t - B_s \sim N(0, \sigma^2(t-s))$;
2. $\forall 0 \leq t_0 \leq \dots \leq t_n, B_0, B_1 - B_0, \dots, B_{t_n} - B_{t_{n-1}}$ 独立,

称 $(B_t : t \geq 0)$ 是以 σ^2 为参数的布朗运动。特别的, 当 $\sigma^2 = 1$, $(B_t : t \geq 0)$ 为标准布朗运动。

Definition 14. 有限维分布为正态分布的随机过程称为正态过程。

2.3.2 布朗运动的构造

Theorem 5. 布朗运动是有连续实现的。证明见教材 p13.

2.3.3 布朗运动的性质

Theorem 6. 从原点出发的零均值高斯过程 $(B_t : t \geq 0)$ 是标准布朗运动 $\iff \forall s, t \geq 0, \mathbb{E}(B_t B_s) = t \wedge s$. 证明 p17.

Theorem 7. 布朗运动轨道几乎处处不可导。证明 p17-18.

Lemma 1. Assume $(B_t : t \geq 0)$ is a random process ranging in \mathbb{R} , $a \in \mathbb{R}^+$, and $\forall s, t : 0 \leq s \leq t, B_t - B_s \sim N(0, a(t-s))$. Assume B_t is continuous about t , a.s. Let $\mathcal{F}_t := \sigma(B_s : 0 \leq s \leq t)$. Then $(B_t : t \geq 0)$ is Brownian motion $\iff \forall 0 \leq s \leq t, B_t - B_s \perp \mathcal{F}_s$.

证明. “ \implies ”: To prove $B_t - B_s \perp \mathcal{F}_s$, only need to prove for $t_1 < t_2 < \dots < t_{n-1} = s < t = t_n$, we have $B_t - B_s \perp \sigma(B_{t_k} : k = 1, \dots, n-1)$. Easily $B_t - B_s \perp \sigma(B_{t_{k+1}} - B_{t_k}, B_{t_1} : k = 1, \dots, n-2) = \sigma(B_{t_k} : k = 1, \dots, n-1)$, so $B_t - B_s \perp \mathcal{F}_s$.

“ \impliedby ”: For $t_1 < \dots < t_n$, we need to prove $B_{t_1}, B_{t_{k+1}} - B_{t_k} : 1 \leq k \leq n-1$ are independent. Use MI to n . When $n = 1$ it's obvious. Assume we have proved it for certain $n \geq 1$, now consider $n+1$. Since $B_{t_{k+1}} - B_{t_k} \in \mathcal{F}_{t_n}, k = 1, \dots, n-1$, we have $B_{t_{n+1}} - B_{t_n} \perp \sigma(B_{t_1}, B_{t_{k+1}} - B_{t_k} : k = 1, \dots, n-1)$. So $\mathbb{P}(B_{t_1} \in A_1, B_{t_{k+1}} - B_{t_k} \in A_{k+1}, k = 1, \dots, n) = \mathbb{P}(B_{t_1} \in A_1, B_{t_{k+1}} - B_{t_k} \in A_{k+1}, k = 1, \dots, n-1) \mathbb{P}(B_{t_{n+1}} - B_{t_n} \in A_{n+1})$. By Induction assumption we get $\mathbb{P}(B_{t_1} \in A_1, B_{t_{k+1}} - B_{t_k} \in A_{k+1}, k = 1, \dots, n) = \mathbb{P}(B_{t_1} \in A_1) \prod_{k=1}^n \mathbb{P}(B_{t_{k+1}} - B_{t_k} \in A_{k+1})$. So finally we get $B_{t_1}, B_{t_{k+1}} - B_{t_k} : 1 \leq k \leq n$ are independent. □

PROBLEM XI Assume $(B_t : t \geq 0)$ is Brownian motion, prove that for $r > 0$, we have $(B_{t+r} - B_r : t \geq 0)$ is Brownian motion, too.

SOLUTION. Assume $B_t - B_s \sim N(0, a(t-s)), a > 0$. Let $\mathcal{F}_t := \sigma(B_s : 0 \leq s \leq t)$ and $\mathcal{G}_t := \sigma(B_{r+s} - B_r : 0 \leq s \leq t)$. For $0 \leq s \leq t$, we have $B_{t+r} - B_r - (B_{s+r} - B_r) = B_{t+r} - B_{s+r} \sim N(0, a(t-s))$. And easily to know $B_{r+s} - B_r \in \mathcal{F}_{r+s}$, so $\mathcal{G}_t \subset \mathcal{F}_{t+r}, \forall t \geq 0$. Since $(B_t : t \geq 0)$ is Brownian motion, easily $\mathcal{F}_{s+r} \perp B_{t+r} - B_{s+r}$. Since $\mathcal{G}_t \subset \mathcal{F}_{t+r}$, we obtain $\mathcal{G}_t \perp B_{t+r} - B_{s+r} = B_{t+r} - B_r - (B_{s+r} - B_r)$. Easily since B_t is continuous we get $B_{t+r} - B_r$ is continuous. So $(B_{t+r} : t \geq 0)$ is Brownian motion. \square

PROBLEM XII Assume $(B_t : t \geq 0)$ is standard Brownian motion start at 0. Prove that $\forall c > 0, (cB_{\frac{t}{c^2}} : t \geq 0)$ is standard Brownian motion start at 0, too.

SOLUTION. Since $B_0 = 0$ we get $cB_{\frac{0}{c^2}} = 0$. Let $\mathcal{F}_t := \sigma(B_s : 0 \leq s \leq t)$ and $\mathcal{G}_t := \sigma(cB_{\frac{s}{c^2}} : 0 \leq s \leq t)$. Easily to know $\mathcal{G}_t = \mathcal{F}_{\frac{t}{c^2}}$. For $0 \leq s \leq t$, we have $cB_{\frac{t}{c^2}} - cB_{\frac{s}{c^2}} = c(B_{\frac{t}{c^2}} - B_{\frac{s}{c^2}}) \sim N(0, t-s)$, because $B_{\frac{t}{c^2}} - B_{\frac{s}{c^2}} \sim N(0, \frac{t-s}{c^2})$. And since $(B_t : t \geq 0)$ is Brownian motion, we get $B_{\frac{t}{c^2}} - B_{\frac{s}{c^2}} \perp \mathcal{F}_{\frac{s}{c^2}} = \mathcal{G}_s$. Easily since B_t is continuous we get $cB_{\frac{t}{c^2}}$ is continuous. So $(cB_{\frac{t}{c^2}} : t \geq 0)$ is standard Brownian motion starts at 0, too. \square

PROBLEM XIII Assume $(X_t : t \geq 0)$ and $(Y_t : t \geq 0)$ are two independent standard Brownian motion, $a, b \in \mathbb{R}$ and $\sqrt{a^2 + b^2} > 0$. Prove that $(aX_t + bY_t : t \geq 0)$ is a Brownian motion with parameter $c^2 = a^2 + b^2$.

SOLUTION. Let $\mathcal{F}_t := \sigma(X_s : 0 \leq s \leq t)$ and $\mathcal{G}_t := \sigma(Y_s : 0 \leq s \leq t)$. Let $\mathcal{H}_t := \sigma(aX_s + bY_s : 0 \leq s \leq t)$. Since $(X_t : t \geq 0), (Y_t : t \geq 0)$ are two independent Brownian motion, we know $\forall 0 \leq s \leq t, X_t - X_s \perp \mathcal{F}_s, \mathcal{G}_s; Y_t - Y_s \perp \mathcal{F}_s, \mathcal{G}_s$. So we get $aX_t + bY_t - aX_s - bY_s \perp \mathcal{F}_s, \mathcal{G}_s$, thus $aX_t + bY_t - aX_s - bY_s \perp \sigma(\mathcal{F}_s, \mathcal{G}_s)$. Easily $aX_s + bY_s \in \sigma(\mathcal{F}_s, \mathcal{G}_s)$, so $\mathcal{H}_t \subset \sigma(\mathcal{F}_t, \mathcal{G}_t), \forall t \geq 0$. So $aX_t + bY_t - aX_s - bY_s \perp \mathcal{H}_s$. And easily $a(X_t - X_s) \sim N(0, a^2(t-s)), b(Y_t - Y_s) \sim N(0, b^2(t-s))$, and since $\mathcal{F}_t \perp \mathcal{G}_t$ we get $a(X_t - X_s) \perp b(Y_t - Y_s)$, so $aX_t + bY_t - aX_s - bY_s \sim N(0, (a^2 + b^2)(t-s))$. Easily since X_t, Y_t is continuous we get $aX_t + bY_t$ is continuous. So $(aX_t + bY_t : t \geq 0)$ is a Brownian motion with parameter $a^2 + b^2 = c^2$. \square

PROBLEM XIV Assume $(B_t : t \geq 0)$ is standard Brownian motion start at 0. Let $X_0 = 0$ and $X_t := tB_{\frac{1}{t}}$. Given

$$\limsup_{t \rightarrow \infty} \frac{B_t}{\sqrt{2t \log \log t}} = 1$$

Prove that $(X_t : t \geq 0)$ is standard Brownian motion start at 0.

SOLUTION. First consider the distribution of $X_t - X_s$ for $s < t$. If $s = 0$ then $X_t - X_s = tB_{\frac{1}{t}} \sim N(0, t^2 \times \frac{1}{t}) = N(0, t)$. Else, $s > 0$, then $X_t - X_s = s(B_{\frac{1}{t}} - B_{\frac{1}{s}}) + (t-s)B_{\frac{1}{t}}$. Easily $B_{\frac{1}{s}} - B_{\frac{1}{t}} \sim N(0, \frac{1}{s} - \frac{1}{t})$, and $B_{\frac{1}{t}} \sim N(0, \frac{1}{t})$, and since $\frac{1}{t} < \frac{1}{s}$ we know $B_{\frac{1}{s}} - B_{\frac{1}{t}} \perp B_{\frac{1}{t}}$. So $X_t - X_s \sim N(0, s^2(\frac{1}{s} - \frac{1}{t}) + (t-s)^2 \frac{1}{t}) = N(0, t-s)$.

Second let $\mathcal{G}_t := \sigma(X_s : 0 \leq s \leq t)$, we need to check $X_t - X_s \perp \mathcal{G}_s, \forall 0 \leq s \leq t$. For $s = 0$ we get $\mathcal{G}_s = \{\emptyset, \Omega\}$, so it's obvious. Now assume $s > 0$. Then $\mathcal{G}_s = \sigma(B_{\frac{1}{r}} : 0 \leq r \leq s)$. Only need to prove for any finite set $I \subset [0, s]$, we have $X_t - X_s \perp \sigma(B_{\frac{1}{r}} : r \in I)$. Only need to check $\mathbb{E}(B_{\frac{1}{r}}(X_t - X_s)) = 0$ because they are all normal distributed random variable. Easily $\mathbb{E}(B_{\frac{1}{r}}X_t) = \mathbb{E}((B_{\frac{1}{r}} - B_{\frac{1}{t}})tB_{\frac{1}{t}}) + \mathbb{E}(tB_{\frac{1}{t}}^2) = 1$, and $\mathbb{E}(B_{\frac{1}{r}}X_s) = \mathbb{E}((B_{\frac{1}{r}} - B_{\frac{1}{s}})sB_{\frac{1}{s}}) + \mathbb{E}(sB_{\frac{1}{s}}^2) = 1$ So $\mathbb{E}(B_{\frac{1}{r}}(X_t - X_s)) = 0$.

Finally we need to check X_t is continuous a.s. Easily for $t \neq 0$ we know X_t is continuous at t . Only need to check X_t is continuous at 0 with probability 1. Easily to know $(-B_t : t \geq 0)$ is standard Brownian motion, too. So $\limsup_{t \rightarrow \infty} \frac{-B_t}{\sqrt{2t \log \log t}} = 1$. So $\limsup_{t \rightarrow \infty} \frac{|B_t|}{\sqrt{2t \log \log t}} = 1$. So $\limsup_{t \rightarrow 0+} |X_t| = \lim_{t \rightarrow \infty} |\frac{1}{t}B_t| \leq \limsup_{t \rightarrow \infty} \frac{\sqrt{2t \log \log t}}{t} = 0$. So $\lim_{t \rightarrow 0+} |X_t| = 0$. So X_t is continuous with probability 1. \square

2.4 普瓦松过程

Definition 15. $(N_t : t \geq 0)$ 是非负整数不降随机过程, $\alpha \geq 0$ 满足:

1. $\forall s, t \geq 0, N_{s+t} - N_s \sim P(\alpha t)$, 即: $\mathbb{P}(N_{s+t} - N_s = k) = \frac{\alpha^k t^k}{k!} e^{-\alpha t}$;
2. $\forall 0 \leq t_0 < \dots < t_n, N_0, N_{t_1} - N_{t_0}, \dots, N_{t_n} - N_{t_{n-1}}$ 相互独立。

称 $(N_t : t \geq 0)$ 是普瓦松过程, 参数为 α 。

2.4.1 跳跃间隔时间

$(N_t : t \geq 0)$ 以 α 为参数的普瓦松过程, $S_0 = 0, n \geq 1, S_n = \inf\{t \geq 0 : N_t - N_0 \geq n\}$, $\eta_n = S_n - S_{n-1}$. S_n 是 $(N_t : t \geq 0)$ 第 n 次跳跃等待时间, η_n 第 $n-1$ 次跳跃到第 n 次跳跃的间隔时间。

Theorem 8. $\{\eta_n : n \geq 1\}$ 独立同分布, 服从 $Exp(\alpha)$. $S_n, n \geq 1$, 服从 $\Gamma(1, \alpha)$. 证明见 P19.

2.4.2 轨道重构

Theorem 9. $\{\eta_n : n \geq 1\}$ 独立同分布, 服从 $Exp(\alpha), \gamma > 0$. $S_0 = 0, S_n = \sum_{k=1}^n \eta_k$, 则:

$$N_t = \sum_{n=1}^{\infty} \mathbb{1}_{S_n \leq t} = \sup\{n \geq 0 : S_n \leq t\}.$$

则随机过程 $(N_t : t \geq 0)$ 是以参数为 α 的普瓦松过程。

2.4.3 长时间极限行为

$(N_t : t \geq 0)$ 以 α 为参数的普瓦松过程。

Theorem 10 (普瓦松过程的强大数定律). $\lim_{t \rightarrow \infty} \frac{N_t}{t} \stackrel{a.s.}{=} \alpha$ 见 p23.

Theorem 11 (普瓦松过程的中心极限定理). $\lim_{t \rightarrow \infty} \mathbb{P}(\frac{N_t - \alpha t}{\sqrt{\alpha t}} \leq x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{y^2}{2}} dy$. 见 p23.

Corollary 1. $s, x > 0, \lim_{\lambda \rightarrow \infty} e^{-\lambda s} \sum_{k \leq \lambda x} \frac{(\lambda s)^k}{k!} = \mathbb{1}_{0 < s < t} + \frac{1}{2} \mathbb{1}_{\{s=x\}}$. 见 p24.

Theorem 12 (拉普拉斯变换的反演公式). ξ 是非负随机变量, L 为其拉普拉斯变换, 则 $\forall x > 0$,

$$\lim_{\lambda \rightarrow \infty} \sum_{k \leq \lambda x} \frac{(-\lambda)^k}{k!} \frac{d^k}{d\lambda^k} L(\lambda) = \mathbb{P}(\xi < x) + \frac{1}{2} \mathbb{P}(\xi = x)$$

见 p24.

2.4.4 复合普瓦松过程

Definition 16. μ 是 \mathbb{R} 上概率 $\mu(\{0\}) = 0$. $(N_t : t \geq 0)$ 以 $\alpha \geq 0$ 为参数的零初值普瓦松过程, $\{\xi_n : n \geq 1\}$ 与 N_t 独立, 具有相同分布 μ , X_0 与 $(N_t), \{\xi_n\}$ 独立. 令: $X_t = X_0 + \sum_{n=1}^{N_t} \xi_n, t \geq 0$, 则 $(X_t : t \geq 0)$ 是以 α 为跳跃速度, μ 为跳跃分布的复合普瓦松过程。

Theorem 13 (复合普瓦松过程的性质). $(X_t : t \geq 0)$ 为如上定义的复合普瓦松过程, 则复合普瓦松过程的性质如下:

1. $\forall s, t \geq 0, \theta \in \mathbb{R}$,

$$\mathbb{E} e^{i\theta(X_{s+t} - X_s)} = \exp(\alpha t \int_{\mathbb{R}} (e^{i\theta x} - 1) \mu(dx))$$

;

2. $\forall 0 \leq t_0 < \dots < t_n, X_{t_0}, X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_{n-1}}$ 相互独立。

Lemma 2. Assume $(N_t : t \geq 0)$ is a random process, and $\mathbb{P}(\forall t \in [0, \infty), \lim_{s \rightarrow t+} N_s = N_t, \forall t \in (0, \infty), \lim_{s \rightarrow t-} N_s \in \mathbb{R}) = 1$. Assume $\alpha > 0$, and $\forall t, s \geq 0$, we have $N_{t+s} - N_s \sim \text{Poisson}(\alpha t)$. Then $(N_t : t \geq 0)$ is a Poisson process $\iff \forall 0 \leq s \leq t, N_t - N_s \perp \mathcal{F}_s$, where $\mathcal{F}_s := \sigma(N_x : x \leq s)$.

证明. “ \implies ”: To prove $N_t - N_s \perp \mathcal{F}_s$, only need to prove for $t_1 < t_2 < \dots < t_{n-1} = s < t = t_n$, we have $N_t - N_s \perp \sigma(N_{t_k} : k = 1, \dots, n-1)$. Easily $N_t - N_s \perp \sigma(N_{t_{k+1}} - N_{t_k}, N_{t_1} : k = 1, \dots, n-2) = \sigma(N_{t_k} : k = 1, \dots, n-1)$, so $N_t - N_s \perp \mathcal{F}_s$.

“ \impliedby ”: For $t_1 < \dots < t_n$, we need to prove $N_{t_1}, N_{t_{k+1}} - N_{t_k} : 1 \leq k \leq n-1$ are independent. Use MI to n . When $n = 1$ it's obvious. Assume we have proved it for certain $n \geq 1$, now consider $n+1$. Since $N_{t_{k+1}} - N_{t_k} \in \mathcal{F}_{t_n}, k = 1, \dots, n-1$, we have $N_{t_{n+1}} - N_{t_n} \perp \sigma(N_{t_1}, N_{t_{k+1}} - N_{t_k} : k = 1, \dots, n-1)$. So $\mathbb{P}(N_{t_1} \in A_1, N_{t_{k+1}} - N_{t_k} \in A_{k+1}, k = 1, \dots, n) = \mathbb{P}(N_{t_1} \in A_1, N_{t_{k+1}} - N_{t_k} \in A_{k+1}, k = 1, \dots, n-1) \mathbb{P}(N_{t_{n+1}} - N_{t_n} \in A_{n+1})$. By Induction assumption we get $\mathbb{P}(N_{t_1} \in A_1, N_{t_{k+1}} - N_{t_k} \in A_{k+1}, k = 1, \dots, n) = \mathbb{P}(N_{t_1} \in A_1) \prod_{k=1}^n \mathbb{P}(N_{t_{k+1}} - N_{t_k} \in A_{k+1})$. So finally we get $N_{t_1}, N_{t_{k+1}} - N_{t_k} : 1 \leq k \leq n$ are independent. \square

PROBLEM XV Assume $(N_t : t \geq 0)$ is a Poisson process with parameter α . Let $P(t) := \mathbb{P}(2 \nmid N_t)$, $Q(t) := \mathbb{P}(2 \mid N_t)$. Prove that $P(t) = e^{-\alpha t} \sinh(\alpha t)$, $Q(t) = e^{-\alpha t} \cosh(\alpha t)$.

SOLUTION. Easily to get

$$P(t) = \sum_{k=0}^{\infty} \mathbb{P}(N_t = 2k+1) = \sum_{k=0}^{\infty} \frac{(\alpha t)^{2k+1}}{(2k+1)!} e^{-\alpha t}$$

Noting that $\sinh(\alpha t) = \frac{1}{2} \left(\sum_{k=0}^{\infty} \frac{(\alpha t)^k}{k!} - \sum_{k=0}^{\infty} \frac{(-\alpha t)^k}{k!} \right) = \sum_{k=0}^{\infty} \frac{(\alpha t)^{2k+1}}{(2k+1)!}$, we easily obtain $P(t) = e^{-\alpha t} \sinh(\alpha t)$. Noting $P(t) + Q(t) = 1$, we easily get $Q(t) = 1 - P(t) = e^{-\alpha t} \cosh(\alpha t)$. \square

PROBLEM XVI Assume $(N_t : t \geq 0)$ is a Poisson process with parameter α . Prove that $\lim_{t \rightarrow \infty} \frac{N_t}{t} = \alpha, a.s.$

Lemma 3. Assume $(N_t : t \geq 0)$ is a Poisson process with parameter α . Then $\mathbb{P}(\forall 0 \leq s \leq t, N_s \leq N_t) = 1$.

证明. For $s, t \in \mathbb{Q} \wedge 0 \leq s \leq t$, we have $\mathbb{P}(N_s > N_t) = 0$ since $N_t - N_s \sim \text{Poisson}(\alpha(t-s))$. So we get $\mathbb{P}(\exists s, t \in \mathbb{Q}, 0 \leq s \leq t, N_s > N_t) = 0$.

Now we will prove $\exists s, t \in \mathbb{R}, 0 \leq s \leq t, N_s > N_t \implies \exists a, b \in \mathbb{Q}, 0 \leq a \leq b, N_a > N_b$. Let $a_n = \frac{[ns]}{n}, b_n = \frac{[nt]}{n}$. Then $\lim a_n = s, \lim b_n = t$. Easily $a_n \geq s, b_n \geq t$. So since N is continuous we get $\lim N_{a_n} = N_s, \lim N_{b_n} = N_t$. Since $N_s > N_t$, we get $\exists n, N_{a_n} > N_{b_n}$. Let $a = a_n, b = b_n$ will work.

$$\text{So } \mathbb{P}(\forall 0 \leq s \leq t, N_s \leq N_t) = 1 - \mathbb{P}(\exists 0 \leq s \leq t, N_s > N_t) = 1 - \mathbb{P}(\exists s, t \in \mathbb{Q}, 0 \leq s \leq t, N_s > N_t) = 1 - 0 = 1. \quad \square$$

SOLUTION. Consider $N_n : n \in \mathbb{N}$. Let $X_n := N_n - N_{n-1}, n \geq 1$. Then easily $(X_n : n \in \mathbb{N}^+)$ is i.i.d sequence and $X_1 \sim \text{Poisson}(\alpha)$. So from the strong law of large numbers we get $\lim_{n \rightarrow \infty} \frac{N_n}{n} = \alpha$. From Lemma 1 we get $\frac{[t]}{t} \frac{N_{[t]}}{[t]} \leq \frac{N_t}{t} \leq \frac{N_{[t]}}{[t]} \frac{[t]}{t}, \forall t \in \mathbb{R}$, let $t \rightarrow \infty$ we get $[t], [t] \rightarrow \infty$, and $[t] \sim t \sim [t]$. So finally we get $\lim_{t \rightarrow \infty} \frac{N_t}{t} = \alpha$. \square

PROBLEM XVII Assume $(N_t : t \geq 0)$ is a Poisson process with parameter $\alpha > 0$. Prove that $\frac{N_t - \alpha t}{\sqrt{\alpha t}} \xrightarrow{d} N(0, 1)$.

SOLUTION. Consider $N_n : n \in \mathbb{N}$. Let $X_n := N_n - N_{n-1}, n \geq 1$. Then easily $(X_n : n \in \mathbb{N}^+)$ is i.i.d sequence and $X_1 \sim \text{Poisson}(\alpha)$. Easily $\mathbb{V}(X_n) = \alpha < \infty, \mathbb{E}(X_n) = \alpha$. So from the central limit theorem we get $\frac{N_n - \alpha n}{\sqrt{\alpha n}} \xrightarrow{d} N(0, 1)$. Noting $\frac{N_t - \alpha t}{\sqrt{\alpha t}} = \frac{N_{[t]} - \alpha[t]}{\sqrt{\alpha[t]}} \frac{\sqrt{[t]}}{\sqrt{t}} + \frac{N_t - N_{[t]} - \alpha(t - [t])}{\sqrt{\alpha t}}$. Let $t \rightarrow \infty$ we get $[t] \rightarrow \infty$, and $[t] \sim t$. Noting $N_t - N_{[t]} \stackrel{d}{=} N_{t-[t]}$, and $t - [t] \leq 1$, we easily get $\frac{N_t - N_{[t]}}{\sqrt{\alpha t}} \xrightarrow{d} 0$. Easily $\frac{\alpha(t - [t])}{\alpha t} \rightarrow 0$, so finally we get that $\frac{N_t - \alpha t}{\sqrt{\alpha t}} \xrightarrow{d} N(0, 1)$ \square

PROBLEM XVIII Assume $(X_t : t \geq 0), (Y_t : t \geq 0)$ are two independent Poisson processes with parameter α, β respectively. Prove that $(X_t + Y_t : t \geq 0)$ is Poisson process with parameter $\alpha + \beta$.

SOLUTION. Write $Z_t = X_t + Y_t$. First we prove $Z_{t+s} - Z_s \sim \text{Poisson}((\alpha + \beta)t)$. Since $X_{t+s} - X_s \sim \text{Poisson}(\alpha t), Y_{t+s} - Y_s \sim \text{Poisson}(\beta t)$, and $X_{t+s} - X_s \perp Y_{s+t} - Y_s$, easily to get $Z_{t+s} - Z_s = X_{t+s} - X_s + Y_{s+t} - Y_s \sim \text{Poisson}((\alpha + \beta)t)$.

Second we prove $\forall 0 \leq s \leq t, Z_t - Z_s \perp \mathcal{H}_s$, where $\mathcal{H}_s = \sigma(X_x : 0 \leq x \leq s)$. Easily $Z_t - Z_s \in \sigma(X_t - X_s, Y_t - Y_s)$. Easily $X_t - X_s \perp \mathcal{F}_s := \sigma(X_x : 0 \leq x \leq s)$ since $(X_x : x \geq 0)$ is Poisson process. Since $(X_x : x \geq 0) \perp (Y_x : x \geq 0)$, we get $X_t - X_s \perp \mathcal{G}_s := \sigma(Y_x : 0 \leq x \leq s)$. For the same reason we get $Y_t - Y_s \perp \mathcal{F}_s, \mathcal{G}_s$. So we get $Z_t - Z_s \perp \sigma(\mathcal{F}_s, \mathcal{G}_s) \supset \mathcal{H}_s$.

Finally, we prove that $\mathbb{P}(\forall t \in [0, \infty), \lim_{s \rightarrow t+} Z_s = Z_t, \forall t \in (0, \infty), \lim_{s \rightarrow t-} Z_s \in \mathbb{R}) = 1$ Since $Z_t = X_t + Y_t$, and $\mathbb{P}(\forall t \in [0, \infty), \lim_{s \rightarrow t+} Y_s = Y_t, \forall t \in (0, \infty), \lim_{s \rightarrow t-} Y_s \in \mathbb{R}) = 1, \mathbb{P}(\forall t \in [0, \infty), \lim_{s \rightarrow t+} X_s = X_t, \forall t \in (0, \infty), \lim_{s \rightarrow t-} X_s \in \mathbb{R}) = 1$, obvious we get the requirement.

All in all, $(X_t + Y_t : t \geq 0)$ is a Poisson process with parameter $\alpha + \beta$. \square

PROBLEM XIX Assume $(\xi_n : n \in \mathbb{N}^+)$ is a sequence of i.i.d. random variable ranging in \mathbb{Z}^d . Let $X_n = X_0 + \sum_{k=1}^n \xi_k$, and $X_0 \perp (\xi_n : n \in \mathbb{N}^+)$ ranging in \mathbb{Z}^d , too. Assume $(N_t : t \geq 0)$ is a Poisson process with parameter $\alpha > 0$. Discuss $\frac{X_{N_t}}{t}$ when $t \rightarrow \infty$.

SOLUTION. First we prove that $\lim_{t \rightarrow \infty} N_t = \infty, a.s.$ We have $\mathbb{P}(\sup_t N_t \geq n) \geq \mathbb{P}(N_t \geq n), \forall t, \forall n \in \mathbb{N}$. Easily $\lim_{t \rightarrow \infty} \mathbb{P}(N_t \geq n) = 1$, so $\mathbb{P}(\sup_t N_t \geq n) = 1, \forall n \in \mathbb{N}$. So $\mathbb{P}(\sup_t N_t = \infty) = 1$. Noting Lemma 1 we easily get $\mathbb{P}(\lim_{t \rightarrow \infty} N_t = \infty) = 1$.

Now we can decompose $\frac{X_{N_t}}{t}$ into $\frac{X_{N_t}}{N_t} \frac{N_t}{t}$. We have proved that $\frac{N_t}{t} \rightarrow \alpha, a.s.$ in Problem XII, so we only need to find $\frac{X_{N_t}}{N_t}$. Since $N_t \rightarrow \infty, a.s.$, we only need to find $\frac{X_n}{n}$ when $n \rightarrow \infty$.

If $\mathbb{E}(\xi_1)$ exists, then easily $\frac{X_n}{n} \rightarrow \mathbb{E}(\xi_1), a.s.$ Then we easily get $\frac{X_{N_t}}{t} \rightarrow \alpha \mathbb{E}(\xi_1), a.s.$ \square

2.5 普瓦松随机测度

2.5.1 定义和存在性

(E, \mathcal{E}) 为可测空间, μ 为 (E, \mathcal{E}) 上的 σ 有限测度。

Definition 17. $\{X(B) : B \in \mathcal{E}\}$ 为取非负整数值随机过程, 满足:

1. $\forall B \in \mathcal{E} : \mu(B) < \infty$, 则 $\mathbb{E}(X(B)) = \mu(B)$ 。
2. $\forall \{B_n : n \geq 1\} \in \mathcal{E}$ 两两不交, 则 $X(\bigcup_{k=1}^{\infty} B_k) = \sum_{k=1}^{\infty} X(B_k)$ 。

称 $\{X(B) : B \in \mathcal{E}\}$ 为以 μ 为强度的整数值随机测度。

Definition 18. $\{X(B) : B \in \mathcal{E}\}$ 为整数值随机测度, 满足:

1. $\forall B \in \mathcal{E} : \mu(B) < \infty$, 则 $X(B) \sim P(\mu(B))$, 即: $\mathbb{P}(X(B) = k) = \frac{\mu(B)^k}{k!} e^{-\mu(B)}, k = 0, \dots, n, \dots$ 。
2. $\forall \{B_n : n \geq 1\} \in \mathcal{E}$ 两两不交, 则 $\{X(B_k) : k \in \mathbb{N}^+\}$ 相互独立。

称 $\{X(B) : B \in \mathcal{E}\}$ 为以 α 为强度的普瓦松随机测度。

Theorem 14 (普瓦松随机测度的充要条件). X 为 (E, \mathcal{E}) 上以 μ 为强度的整数值随机测度, 则 X 为普瓦松随机测度的充要条件是 $\forall n \in \mathbb{N}^+, \xi_k \in \mathbb{R}, B_k \in \mathcal{E}, k = 1, \dots, n, B_i \cap B_j = \emptyset, i \neq j$, 当 $\mu(B_k) < \infty, k = 1, \dots, n$, 则

$$\mathbb{E} \exp(i \sum_{k=1}^n \theta_k X(B_k)) = \exp(\sum_{k=1}^n (e^{i\theta_k} - 1) \mu(B_k))$$

见 p28.

Theorem 15 (普瓦松随机测度的存在性). μ 为非零有限测度, $\eta \sim P(\mu(E))$, $\{\xi_k : k \in \mathbb{N}^+\}$ *i.i.d.* 服从 $\mu(E)^{-1} \mu$ 。 $\eta, \xi_1, \dots, \xi_n, \dots$ 相互独立。令 $X = \sum_{j=1}^{\eta} \delta_{\xi_j}$, 则 X 为以 μ 为强度的普瓦松随机测度。见 p29.

Theorem 16. μ 为 σ 有限测度, $\{E_k : k \in \mathbb{N}^+\} \subset \mathcal{E}, \mu(E_k) < \infty, k \in \mathbb{N}^+, E = \bigcup_{k \in \mathbb{N}^+} E_k, E_i \cap E_j = \emptyset, i \neq j$ 。则存在 X_k 为 E_k 上的普瓦松随机测度强度为 $\mu_k := \mu|_{E_k}, k \in \mathbb{N}^+$ 。令 $X = \sum_{j=1}^{\infty} X_j$, 则 X 为以 μ 为强度的普瓦松随机测度。见 p29.

2.5.2 积分与补偿的测度

Theorem 17 (普瓦松随机测度的充要条件 2). X 为 (E, \mathcal{E}) 上以 μ 为强度的整数值随机测度, 则 X 为普瓦松随机测度的充要条件是 $\forall f \in \mathbb{R}^{\mathbb{R}} : \mu(f) < \infty$, 则

$$\mathbb{E} \exp(iX(f)) = \exp(\int_E (e^{if(x)} - 1) \mu(dx))$$

见 p28.

2.5.3 应用

Theorem 18 (复合普瓦松过程构造). ν 为 \mathbb{R} 上非零有限测度。 $N(ds, dz)$ 为 $(0, \infty) \times \mathbb{R}$ 上以 $ds\nu(dz)$ 为强度的普瓦松随机测度, ds 为勒贝格测度, X_0 与 $N(ds, dz)$ 独立,

$$X_t = X_0 + \iint_{(0,t] \times \mathbb{R}} z N(ds, dz), t \geq 0$$

则 $(X_t : t \geq 0)$ 具有跳跃速率 α 和跳跃分布 $\mu := \nu(\mathbb{R})^{-1} \nu$ 的复合普瓦松过程。

Theorem 19 (复合普瓦松过程构造 2). μ 为 \mathbb{R} 上非零 σ 有限测度, $\mu(\{0\}) = 0, q$ 为 \mathbb{R} 上非负博雷尔可测函数, $0 < \beta := \mu(q) < \infty$ 。 $N(ds, dz, du)$ 为 $(0, \infty) \times \mathbb{R} \times (0, \infty)$ 上以 $ds\mu(dz)du$ 为强度的普瓦松随机测度, ds 为勒贝格测度, X_0 与 $N(ds, dz)$ 独立,

$$X_t = X_0 + \iiint_{(0,t] \times \mathbb{R} \times [0,q(z)]} z N(ds, dz, du), t \geq 0$$

则 $(X_t : t \geq 0)$ 具有跳跃速率 β 和跳跃分布 $\beta^{-1} q(z) \mu(dz)$ 的复合普瓦松过程。

PROBLEM XX Assume $(N_t : t \geq 0)$ is Poisson process with parameter α , and $\{\xi_n : n \in \mathbb{N}^+\}$ is a sequence of i.i.d random variable. More over, assume $(N_t : t \geq 0) \perp \{\xi_n : n \in \mathbb{N}^+\}$. Let $X_t = \sum_{k=1}^{N_t} \xi_k$. Let $r > 0$, prove that:

1. $(N_{t+r} - N_r : t \geq 0)$ is Poisson process.
2. $\{\xi_{N_r+n} : n \in \mathbb{N}^+\}$ is also i.i.d sequence with the same distribution of $\{\xi_n : n \in \mathbb{N}^+\}$.
3. $(N_{t+r} - N_r : t \geq 0) \perp (\xi_{N_r+k} : k \in \mathbb{N}^+)$.
4. For $0 = t_0 < t_1 < \dots < t_n$, we have $(X_{t_1}, X_{t_{k+1}} - X_{t_k} : k = 1, 2, \dots, n-1)$ are independent.

SOLUTION. 1. Let $\mathcal{F}_t := \sigma(N_s : 0 \leq s \leq t)$ and $\mathcal{G}_t := \sigma(N_{r+s} - N_r : 0 \leq s \leq t)$. For $0 \leq s \leq t$, we have $N_{t+r} - N_r - (N_{s+r} - N_r) = N_{t+r} - N_{s+r} \sim \text{Poisson}(\alpha(t-s))$. And easily to know $N_{r+s} - N_r \in \mathcal{F}_{r+s}$, so $\mathcal{G}_t \subset \mathcal{F}_{t+r}, \forall t \geq 0$. Since $(N_t : t \geq 0)$ is Poisson process, easily $\mathcal{F}_{s+r} \perp N_{t+r} - N_{s+r}$. Since $\mathcal{G}_t \subset \mathcal{F}_{t+r}$, we obtain $\mathcal{G}_t \perp N_{t+r} - N_{s+r} = N_{t+r} - N_r - (N_{s+r} - N_r)$. Easily since N_t is right-continuous we get $N_{t+r} - N_r$ is right-continuous. For the same reason, we know $\forall s \in [0, \infty), \lim_{t \rightarrow s-} N_{t+r} - N_r$ exists. So $(N_{t+r} : t \geq 0)$ is Poisson process.

2. Only need to prove that for $m \in \mathbb{N}^+$, the distribution of $(\xi_{N_r+k} : 1 \leq k \leq m)$ is same as that of $(\xi_k : 1 \leq k \leq m)$. For $A_1, A_2, \dots, A_m \in \mathcal{B}$, we have:

$$\begin{aligned}
& \mathbb{P}(\xi_{N_r+k} \in A_k, 1 \leq k \leq m) \\
&= \sum_{t=0}^{\infty} \mathbb{P}(\xi_{N_r+k} \in A_k, 1 \leq k \leq m, N_r = t) \\
&= \sum_{t=0}^{\infty} \mathbb{P}(\xi_{t+k} \in A_k, 1 \leq k \leq m, N_r = t) \\
& (N_r \perp (\xi_n : n \in \mathbb{N}^+)) = \sum_{t=0}^{\infty} \mathbb{P}(\xi_{t+k} \in A_k, 1 \leq k \leq m) \mathbb{P}(N_r = t) \\
&= \sum_{t=0}^{\infty} \prod_{k=1}^m \mathbb{P}(\xi_{t+k} \in A_k) \mathbb{P}(N_r = t) \\
&= \sum_{t=0}^{\infty} \prod_{k=1}^m \mu(A_k) \mathbb{P}(N_r = t) \\
&= \prod_{k=1}^m \mu(A_k)
\end{aligned} \tag{6}$$

where μ is the distribution of ξ_1 . So we get for $m \in \mathbb{N}^+$, the distribution of $(\xi_{N_r+k} : 1 \leq k \leq m)$ is same as that of $(\xi_k : 1 \leq k \leq m)$.

3. We know that $\forall t \in \mathbb{N}^+, \xi_{N_r+t} \in \sigma(N_r, \xi_k : k \in \mathbb{N}^+)$. So $\sigma(\xi_{N_r+k} : k \in \mathbb{N}^+) \subset \sigma(N_r, \xi_k : k \in \mathbb{N}^+)$. Since $(N_t : t \geq 0)$ is Poisson process, we get that $N_{t+r} - N_r \perp N_r, \forall t \geq 0$. So $\sigma(N_{t+r} - N_r : t \geq 0) \perp N_r$. Easily $\sigma(N_{t+r} - N_r : t \geq 0) \perp \sigma(\xi_k : k \in \mathbb{N}^+)$, so finally we get that $\sigma(N_{t+r} - N_r : t \geq 0) \perp \sigma(N_r, \xi_k : k \in \mathbb{N}^+) \supset \sigma(\xi_{N_r+k} : k \in \mathbb{N}^+)$.
4. $\forall 0 = t_0 < t_1 < \dots < t_n$, then $X_{t_1} = \sum_{i=1}^{N_{t_1}} \xi_i, X_{t_k} - X_{t_{k-1}} = \sum_{i=1}^{N_{t_k} - N_{t_{k-1}}} \xi_{N_{t_{k-1}}+i} \xi_i, k = 2, \dots, n$, then $\forall \{A_k \in \mathcal{E} : k =$

$1, \dots, n\}$,

$$\begin{aligned}
& \mathbb{P}\left(\bigcap_{k=1}^n \sum_{i=1}^{N_{t_k}-N_{t_{k-1}}} \xi_{i+N_{t_{k-1}}} \in A_k\right) \\
&= \mathbb{P}\left(\bigcup_{0 \leq u_1 \leq \dots \leq u_n} \left\{ \sum_{i=u_{k-1}+1}^{u_k} \xi_i \in A_k, N_{t_k} = u_k, k = 1, \dots, n \right\}\right) \\
&= \sum_{0 \leq u_1 \leq \dots \leq u_n} \mathbb{P}\left(\sum_{i=u_{k-1}+1}^{u_k} \xi_i \in A_k, k = 1, \dots, n | N_{t_k} = u_k, k = 1, \dots, n\right) \mathbb{P}(N_{t_k} = u_k, k = 1, \dots, n) \\
&= \sum_{0 \leq u_1 \leq \dots \leq u_n} \mathbb{P}\left(\sum_{i=u_{k-1}+1}^{u_k} \xi_i \in A_k, k = 1, \dots, n\right) \mathbb{P}(N_{t_k} = u_k, k = 1, \dots, n) \\
&= \sum_{0 \leq u_1 \leq \dots \leq u_n} \prod_{k=1}^n \mathbb{P}\left(\sum_{i=u_{k-1}+1}^{u_k} \xi_i \in A_k\right) \prod_{j=1}^n \mathbb{P}(N_{t_j} = u_j) \\
&= \sum_{0 \leq u_1 \leq \dots \leq u_n} \prod_{k=1}^n \mathbb{P}\left(\sum_{i=1}^{u_k-u_{k-1}} \xi_{u_{k-1}+i} \in A_k\right) \prod_{j=1}^n \mathbb{P}(N_{t_j} - N_{t_{j-1}} = u_j - u_{j-1}) \\
&= \sum_{0 \leq u_1 \leq \dots \leq u_n} \prod_{k=1}^n \mathbb{P}\left(\sum_{i=1}^{u_k-u_{k-1}} \xi_{u_{k-1}+i} \in A_k\right) \mathbb{P}(N_{t_k} - N_{t_{k-1}} = u_k - u_{k-1}) \\
&= \sum_{u_1-u_0 \in \mathbb{N}} \dots \sum_{u_n-u_{n-1} \in \mathbb{N}} \prod_{k=1}^n \mathbb{P}\left(\sum_{i=1}^{u_k-u_{k-1}} \xi_{u_{k-1}+i} \in A_k, N_{t_k} - N_{t_{k-1}} = u_k - u_{k-1}\right) \\
&= \prod_{k=1}^n \sum_{u_k-u_{k-1} \in \mathbb{N}} \mathbb{P}\left(\sum_{i=1}^{u_k-u_{k-1}} \xi_{u_{k-1}+i} \in A_k, N_{t_k} - N_{t_{k-1}} = u_k - u_{k-1}\right) \\
&= \prod_{k=1}^n \mathbb{P}\left(\sum_{i=1}^{N_{t_k}-N_{t_{k-1}}} \xi_{i+N_{t_{k-1}}} \in A_k\right)
\end{aligned} \tag{7}$$

□

PROBLEM XXI Assume that X is Poisson random measure on (E, \mathcal{E}) with intensity μ , which is a σ -finite measure. Assume $f : E \rightarrow \mathbb{R}$ is measurable and non-negative, prove that:

$$\mathbb{E}(e^{-X(f)}) = \exp\left\{-\int_E (1 - e^{-f(x)})\mu(dx)\right\}$$

SOLUTION. Let $\mathcal{L} := \{g \in \mathcal{M}(E, [0, \infty)) : \mathbb{E}(e^{-X(g)}) = \exp(-\int_E (1 - e^{-g(x)})\mu(dx))\}$. First we prove that if g is simple measurable function from E to $[0, \infty)$, then $g \in \mathcal{L}$. Assume $g(x) = \sum_{k=1}^n a_k \mathbb{1}_{A_k}(x)$, where $A_k \in \mathcal{E}, a_k > 0, A_i \cap A_j = \emptyset$. Then $\mathbb{E}(\exp(-X(g))) = \mathbb{E}(\prod_{k=1}^n \exp(-a_k X(A_k))) = \prod_{k=1}^n \mathbb{E}(\exp(-a_k X(A_k)))$, since $X(A_k) : k = 1, \dots, n$ are independent. Easily to know

$$\mathbb{E}(\exp(-a_k X(A_k))) = \sum_{i=0}^{\infty} \mathbb{P}(X(A_k) = i) \exp(-a_k i) = \sum_{i=0}^{\infty} \frac{\exp(-\mu(A_k))\mu(A_k)^i}{i!} \exp(-a_k i)$$

Noting that

$$\exp\left(-\int_E (1 - \exp(-a_k \mathbb{1}_{A_k}(x)))\mu(dx)\right) = \exp(\exp(-a_k)\mu(A_k) - \mu(A_k)) = \exp(-\mu(A_k)) \sum_{i=0}^{\infty} \frac{(\exp(-a_k)\mu(A_k))^i}{i!}$$

we get $\mathbb{E}(\exp(-a_k X(A_k))) = \exp(-\int_E (1 - \exp(-a_k \mathbb{1}_{A_k}(x)))\mu(dx))$. Noting $\int_E (1 - \exp(-g(x)))\mu(dx) = \sum_{k=1}^n \int_E (1 - \exp(-a_k \mathbb{1}_{A_k}(x)))\mu(dx)$ we get $\mathbb{E}(\exp(-X(g))) = \exp(-\int_E (1 - \exp(-g(x)))\mu(dx))$.

Now for non-negative function f , consider f_n satisfy that $\forall n, f_n$ is simple, and $f_n \nearrow f$ and $f_n \geq 0$. Then easily to know $\mathbb{E}(\exp(-X(f))) = \lim_{n \rightarrow \infty} \mathbb{E}(\exp(-X(f_n))) = \lim_{n \rightarrow \infty} \exp(-\int_E (1 - \exp(-f_n(x)))\mu(dx)) = \exp(-\int_E (1 - \exp(-f(x)))\mu(dx))$. □

PROBLEM XXII Assume μ is finite measure on (E, \mathcal{E}) , and X is Poisson random measure with intensity μ . Assume $\phi : (E, \mathcal{E}) \rightarrow (F, \mathcal{F})$ is measurable, prove that $X \circ \phi^{-1}$ is Poisson random measure with intensity $\mu \circ \phi^{-1}$.

SOLUTION. Assume $B_k \in \mathcal{F}, \forall k \in \mathbb{N}$ and $\forall i \neq j, B_i \cap B_j = \emptyset$. Then $X \circ \phi^{-1}(\bigcup_{k \in \mathbb{N}} B_k) = X(\bigcup_{k \in \mathbb{N}} \phi^{-1}(B_k)) = \sum_{k \in \mathbb{N}} X(\phi^{-1}(B_k))$. Since X is Poisson random measure with intensity μ , and for $B_1, \dots, B_n \in \mathcal{F}$ and $B_i \cap B_j = \emptyset$, we have $\phi^{-1}(B_k)$ are disjoint set in (E, \mathcal{E}) , so $\mathbb{E}(\exp(i \sum_{k=1}^n \alpha_k X \circ \phi^{-1}(B_k))) = \exp(\sum_{k=1}^n (\exp(i\alpha_k) - 1)\mu \circ \phi^{-1}(B_k))$. So $X \circ \phi^{-1}$ is Poisson random measure on (F, \mathcal{F}) with intensity $\mu \circ \phi^{-1}$. \square

PROBLEM XXIII Assume $\alpha \geq 0$, and μ is probability measure on \mathbb{R} with $\mu(\{0\}) = 0$. Let $N(ds, dz, du)$ is Poisson random measure on $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+$ with intensity $ds\mu(dz)du$. Let $Y_t = Y_0 + \int_0^t \int_{\mathbb{R}} \int_0^\alpha z N(ds, dz, du)$, where $Y_0 \perp N$. Prove that $(Y_t : t \geq 0)$ is compound Poisson process with rate α and jumping distribution μ .

SOLUTION. We know that $\forall t \geq 0, \forall r : 0 \leq r \leq t, Y_r \in \sigma(N(B) : B \subset [0, r] \times \mathbb{R} \times [0, \alpha])$. And $\forall w \geq t, Y_w - Y_t \in \sigma(N(B) : B \subset (t, w] \times \mathbb{R} \times [0, \alpha])$. Easily $(t, w] \cap [0, r] = \emptyset$, so we get $Y_w - Y_t \perp (Y_r : 0 \leq r \leq t)$. Now we only need to check the distribution of $Y_t + w - Y_t$ for $t, w \geq 0$.

Easily to know that:

$$\begin{aligned} \mathbb{E}(e^{i\theta(Y_{t+w}-Y_t)}) &= \mathbb{E} \exp \left(\int_s^{s+t} \int_{\mathbb{R}} \int_0^\alpha i\theta z N(ds, du, dz) \right) \\ &= \exp \left(t\alpha \int_{\mathbb{R}} (e^{i\theta z} - 1)\mu(dz) \right) \\ &= \exp(-t\alpha) \sum_{k=0}^{\infty} \frac{1}{k!} (t\alpha \int_{\mathbb{R}} e^{i\theta z} \mu(dz))^k \\ &= e^{-t\alpha} \sum_{k=0}^{\infty} \frac{(\alpha k)^k}{k!} \int_{\mathbb{R}} e^{i\theta z} \mu^{*k}(dz) \end{aligned}$$

So we get the result. \square

PROBLEM XXIV Assume X is Poisson random measure on (E, \mathcal{E}) with intensity μ , a finite measure. Assume f, g are non-negative measure function on E . Prove that:

1. $\mathbb{E}(X(f)e^{-X(g)}) = \mu(fe^{-g})\mathbb{E}(e^{-X(g)})$.
2. $\mathbb{E}(X(f)^2e^{-X(g)}) = (\mu(f^2e^{-g} + \mu(fe^{-g})'2))\mathbb{E}(e^{-X(g)})$.

SOLUTION. 1. Let $h(\theta) := \mathbb{E}(e^{-X(\theta f + g)}) = \exp \left(- \int_E (1 - e^{-\theta f(x) - g(x)}) \mu(dx) \right)$. Then

$$h'(\theta) = \mathbb{E}(X(f)e^{-X(\theta f + g)}) = \exp \left(- \int_E (1 - e^{-\theta f(x) - g(x)}) \mu(dx) \right) \cdot \int_E f(x) e^{-\theta f(x) - g(x)} \mu(dx)$$

Since they are all non-negative, the differential is valid. Let $\theta = 0$, we get $\mathbb{E}(X(f)e^{-X(g)}) = \mu(fe^{-g})\mathbb{E}(e^{-X(g)})$.

2. Take h as above, easily to get $h''(\theta) = \mathbb{E}(X(f)^2e^{-X(g)}) = \exp \left(- \int_E (1 - e^{-\theta f(x) - g(x)}) \mu(dx) \right) \cdot \left(\int_E f(x) e^{-\theta f(x) - g(x)} \mu(dx) \right)^2 + \exp \left(- \int_E (1 - e^{-\theta f(x) - g(x)}) \mu(dx) \right) \cdot \int_E f(x)^2 e^{-\theta f(x) - g(x)} \mu(dx)$. Let $\theta = 0$, then easily $\mathbb{E}(X(f)^2e^{-X(g)}) = (\mu(f^2e^{-g} + \mu(fe^{-g})'2))\mathbb{E}(e^{-X(g)})$. \square

4 更新过程及其应用

4.1 更新过程

4.1.1 定义和性质

Definition 19. 设 $\{\xi_n : n \geq 1\}$ 是非负独立同分布随机变量序列。设 F 是它们共同的分布函数。假设 $F(0) = \mathbb{P}(\xi_n = 0) < 1$ 。则 $\mu := \mathbb{E}(\xi_n) > 0$ 。令 $S_n := \sum_{k=1}^n \xi_k$ 。由大数定律可得 $\lim_{n \rightarrow \infty} \frac{S_n}{n} = \mu$ 。令 $N(t) := \sum_{n=1}^{\infty} \mathbb{1}_{\{S_n \leq t\}} = \sup\{n \geq 0 : S_n \leq t\}$ 。称 $(N(t) : t \geq 0)$ 为更新过程。称 $(\xi_n : n \geq 1)$ 为更新间隔时间。

Theorem 20. 几乎必然有 $N(\infty) = \infty$ 。证明见教材 p74

4.1.2 更新方程

Definition 20. 称 $m(t) := \mathbb{E}(N(t))$ 为更新过程 $(N(t) : t \geq 0)$ 的更新函数。计算可得 $m(t) = \sum_{n=1}^{\infty} \mathbb{P}(N(t) \geq n) = \sum_{n=1}^{\infty} \mathbb{P}(S_n \leq t) = \sum_{n=1}^{\infty} F^{*n}(t)$ 。

Theorem 21. 对于 $t \geq 0$ 有 $m(t) < \infty$ 。证明见教材 p76

Definition 21. 设 H 为 $[0, \infty)$ 上的右连续的有界变差函数，而 F 为 $[0, \infty)$ 上的概率分布函数。称关于 K 的方程 $K(t) = H(t) + K * F(t), t \geq 0$ 为更新方程。更新方程的积分形式为 $K(t) = H(t) + \int_0^t K(t-x) dF(x)$ 。

Theorem 22. 更新方程存在唯一右连续有界变差函数解 K ，且该解具有表达式 $K(t) = H(t) + H * m(t)$ 。证明见教材 p76

Theorem 23. 对于 $0 \leq s \leq t$ ，有 $\mathbb{P}(S_{N(t)} \leq s) = 1 - F(t) + \int_0^s 1 - F(t-x) dm(x)$ 。证明见教材 p77

Theorem 24 (瓦尔德恒等式). 设 $\{\xi_n : n \geq 1\}$ 为独立同分布的随机变量序列， \mathcal{F}_n 为其自然 σ -代数流。设 $\mathbb{E}(\xi_1)$ 存在。设 τ 为一个停时。则有 $\mathbb{E}(\sum_{k=1}^{\tau} \xi_k) = \mathbb{E}(\tau)\mathbb{E}(\xi_1)$ 。

Theorem 25. 对于 $t, x > 0$ ，有 $\mathbb{P}(W(t) > x) = 1 - F(t+x) + \int_0^t 1 - F(t+x-y) dm(y)$ ，其中 $W(T) := S_{N(t)+1} - t$ 为待更新时间。证明见教材 p78

PROBLEM XXV Assume $(N(t) : t \geq 0)$ is a renewing process with renewing internal $\{\xi_n : n \geq 1\}$, $S_n = \sum_{k=1}^n \xi_k$, $N(t) := \sup\{n : S_n \leq t\}$, calculate $g(t) := \mathbb{E}(N(t)^2)$.

SOLUTION. Let $T_1 = \xi_1$, then $g(t) = \mathbb{E}(\mathbb{E}(N(t)^2 | T_1)) = \int_0^t \mathbb{E}(N(t)^2 | T_1 = x) dF(x)$. By the independence,

$$\mathbb{E}(N(t)^2 | T_1 = x) = \begin{cases} 0 & , x > t \\ \mathbb{E}((1 + N(t-x))^2) & , x \leq t \end{cases}$$

That is $\mathbb{E}(N(t)^2 | T_1 = x) = \begin{cases} 0 & , x > t \\ 1 + 2m(t-x) + g(t-x) & , x \leq t \end{cases}$. Therefore,

$$g(t) = F(t) + 2 \int_0^t m(t-x) dF(x) + \int_0^t g(t-x) dF(x)$$

So $g(t) = 2m(t) - F(t) + \int_0^t g(t-x) dF(x)$. Thus, $g(t) = 2m(t) - F(t) + (2m - F) * m(t)$, so $g(t) = m(t) + 2m * m(t)$. □

PROBLEM XXVI Assume renewing internal time obey $U(0, 1)$. $0 < t < 1$, calculate the distrubution of $S_{N(t)}$ and $\mathbb{E}(S_{N(t)})$.

SOLUTION. By calculating, $m(t) = e^t - 1, 0 < t < 1$. $\forall 0 \leq s \leq t < 1$,

$$\mathbb{P}(S_{N(t)} \leq s) = 1 - t + \int_0^s (1 - t + x) e^x dx = 1 - (t - s) e^s$$

Therefore,

$$\mathbb{E}(S_{N(t)}) = \int_0^t s(1 - t + s) e^s ds = e^t - t - 1$$

□

PROBLEM XXVII Assume renewing internal time obey random variable X with distribution function F . Let $\gamma_t = S_{N(t)+1} - t$ be the rest lifetime at time t . Prove:

$$\mathbb{P}(\gamma_t > z) = 1 - F(t+z) + \int_0^t (1 - F(t+z-x))dm(x)$$

SOLUTION. Let $A_z(t) = \mathbb{P}(\gamma_t > z)$, then

$$\mathbb{P}(\gamma_t > z \mid \xi_1 = x) = \begin{cases} 1 & , x > t+z \\ 0 & , t < x \leq t+z \\ A_z(t-x) & , 0 < x \leq t \end{cases}$$

Then,

$$A_z(t) = \int_0^\infty \mathbb{P}(\gamma_t > z \mid \xi_1 = x)dF(x) = 1 - F(t+z) + \int_0^t A_z(t-x)dF(x)$$

Thus,

$$A_z(t) = 1 - F(t+z) + \int_0^\infty (1 - F(t+z-x))dm(x)$$

□

PROBLEM XXVIII One kind of devices are replaced as they are worn out. Let the lifetime of the devices be sequences $\{\xi_n : n \geq 1\}$, and let $S_n = \sum_{k=1}^n \xi_k$, $N(t) = \sup\{n : S_n \leq t\}$. $L(t) = S_{N(t)+1} - S_{N(t)}$. Prove: $\mathbb{P}(L(t) > x) \geq \mathbb{P}(\xi_1 > x)$.

SOLUTION. When $t \leq x$, easy to get that $\mathbb{P}(L(t) > x) = \mathbb{P}(\xi_1 > x)$. Now we assume $t > x$.

$$\begin{aligned} \mathbb{P}(L(t) > x) &= \sum_{k=0}^{\infty} \mathbb{P}(\xi_{k+1} > x, N(t) = k) \\ &= \sum_{k=0}^{\infty} \mathbb{P}(\xi_{k+1} > x, S_k \leq t, \xi_{k+1} > t - S_k) \\ &= \sum_{k=1}^{\infty} \mathbb{P}(\xi_{k+1} > x, t-x < S_k \leq t) \\ &\quad + \sum_{k=0}^{\infty} \mathbb{P}(\xi_{k+1} > t - S_k, S_k \leq t-x) \\ &= \mathbb{P}(\xi_1 > x) \mathbb{E}[(N(t) - N(t-x))] + \mathbb{P}(N(t) = N(t-x)) \\ &= \mathbb{P}(\xi_1 > x) + \mathbb{P}(\xi_1 > x) \mathbb{E}[(N(t) - N(t-x)) - 1] \\ &\quad + \mathbb{P}(N(t) = N(t-x)) \\ &= \mathbb{P}(\xi_1 > x) + \mathbb{P}(\xi_1 > x) \mathbb{E}[(N(t) - N(t-x) - 1)1_{\{N(t) > N(t-x)\}}] \\ &\quad - \mathbb{P}(\xi_1 > x) \mathbb{E}(1_{\{N(t) = N(t-x)\}}) + \mathbb{P}(N(t) = N(t-x)) \\ &\geq \mathbb{P}(\xi_1 > x). \end{aligned} \tag{8}$$

□

PROBLEM XXIX Toss a coin until we get two successively head, call it a renew. We toss the coin k times, call the number of renews $N(k)$. Find the distribution and expectation of interval time T

SOLUTION. Let $p_n := \mathbb{P}(T = n)$. Then $p_1 = 0, p_2 = \frac{1}{4}$. Easy to find that $p_{n+2} = \frac{1}{2}p_{n+1} + \frac{1}{4}p_n$. The characteristic equation of this sequence is $x^2 = \frac{1}{2}x + \frac{1}{4}$. The roots are $x_1 = \frac{1+\sqrt{5}}{4}, x_2 = \frac{1-\sqrt{5}}{4}$. So $p_n = Ax_1^n + Bx_2^n$. By p_1, p_2 , easy to get that $p_n = \frac{1}{2\sqrt{5}}(x_1^{n-1} - x_2^{n-1})$. So easily $\mathbb{E}(T) = \sum_{n=1}^{\infty} np_n = 6$. □

4.2 长程极限行为

4.2.1 基本更新定理

Theorem 26. 几乎必然的有 $\lim_{t \rightarrow \infty} \frac{N(t)}{t} = \frac{1}{\mu}$ 。证明见教材 p80

Theorem 27 (基本更新定理). 有 $\lim_{t \rightarrow \infty} \frac{m(t)}{t} = \frac{1}{\mu}$ 。证明见教材 p81

4.2.2 中心极限定理

Theorem 28 (中心极限定理). 假设 $\mathbb{D}(\xi_1) < \infty$, 记 $\mu = \mathbb{E}(\xi_1), \sigma^2 = \mathbb{D}(\xi_1)$ 。对于 $x \in \mathbb{R}$, 有

$$\lim_{t \rightarrow \infty} \mathbb{P} \left(\frac{N(t) - \frac{t}{\mu}}{\sqrt{\frac{t\sigma^2}{\mu^3}}} \leq x \right) = \Phi(x) \quad (9)$$

其中 $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{y^2}{2}} dy$ 为正态分布的分布函数。证明见教材 p82

PROBLEM XXX A radio is powered by one battery, the lifetime of the battery obey the distribution of exponential distribution with parameter $\lambda = \frac{1}{30}$. In long term, in which frequency should we change the battery?

SOLUTION. Easy to get that $\lim_{t \rightarrow \infty} \frac{m(t)}{t} = \frac{1}{\mathbb{E}(\xi_1)} = \frac{1}{30}$. So we change battery every 30 hours in average. \square

PROBLEM XXXI Consider a primitive renewing process with average renewing internal time μ . Assume every renewing time is recorded by probability p , and each record and each renew are independence. Let $N_r(t)$ be the times of renewing by recorded until time t . $\{N_r(t) : t \geq 0\}$ is a renewing process or not? And calculate $\lim_{t \rightarrow \infty} \frac{N_r(t)}{t}$.

SOLUTION. Assume $X_n : n \in \mathbb{N}$ are i.i.d r.v and $X_0 \sim \text{Geo}(p)$, and $(X_n : n \in \mathbb{N}) \perp (N(t) : t \geq 0)$. Let $Y_n := \sum_{k=1}^n X_k$, and $Y_0 = 0$. Let $\xi_r(n) := \sum_{k=Y_{n-1}+1}^{Y_n} \xi_k$. Then $\xi_r(n) : n \in \mathbb{N}^+$ is update time of N_r . Since $(X_n : n \in \mathbb{N}) \perp (N(t) : t \geq 0)$, we get that $(\xi_r(n) : n \in \mathbb{N}^+)$ are i.i.d. And $\mathbb{E}(\xi_r(1)) = \mathbb{E}(X_1)\mathbb{E}(\xi_1) = \frac{\mu}{p}$. So $\lim_{t \rightarrow \infty} \frac{N_r(t)}{t} = \frac{\mu}{p}$. \square

PROBLEM XXXII Assume $(U_n : n \in \mathbb{N}^+)$ are i.i.d r.v. and $U_1 \sim U(0, 1)$. Assume $X_{n,m} : n, m \in \mathbb{N}^+$ are r.v. and $X_{n,m} | U_n \sim B(U_n)$. And $(X_{n,m} | U_n : m \in \mathbb{N}^+)$ are i.i.d. Let $\xi_n := \inf\{m \in \mathbb{N}^+ : X_{n,m} = 1\}$ be the n -th update time of $N(t)$. Find $\lim_{t \rightarrow \infty} \frac{N(t)}{t}$.

SOLUTION. Easy to find that $\mathbb{E}(\xi_1) = \int_0^1 \mathbb{E}(\xi_1 | U_1 = x) dx = \int_0^1 \frac{dx}{x} = \infty$. So easy to find that $\lim_{t \rightarrow \infty} \frac{N(t)}{t} = \infty$. \square

PROBLEM XXXIII Assume $(\xi_n : n \in \mathbb{N}^+)$ is i.i.d r.v. ranging in \mathbb{N} is update time of $N(t)$. Let A_n be the event that at time n there is an update. Assume $a = \lim_{n \rightarrow \infty} \mathbb{P}(A_n)$ exists. Prove that $a = \frac{1}{\mathbb{E}(\xi_1)}$.

SOLUTION. Since $N(n) = \sum_{k=1}^n \mathbb{1}(A_k)$, we know that $\mathbb{E}(N(n)) = \sum_{k=1}^n \mathbb{P}(A_k)$. Noting that $\lim_{n \rightarrow \infty} \frac{N(n)}{n} = \frac{1}{\mathbb{E}(\xi_1)}$, we obtain that $\lim_{n \rightarrow \infty} \mathbb{E}(\frac{N(n)}{n}) = \frac{1}{\mathbb{E}(\xi_1)}$. So $\lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n \mathbb{P}(A_k)}{n} = \frac{1}{\mathbb{E}(\xi_1)}$. By stolz, we can get that $\lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n \mathbb{P}(A_k)}{n} = a$. So $a = \frac{1}{\mathbb{E}(\xi_1)}$. \square

PROBLEM XXXIV Assume $N_1(t), N_2(t)$ are two independent updating process with update time distribution $E(1), U(0, 2)$. Find an estimate of $\mathbb{P}(N_1(100) + N_2(100) \geq 190)$.

SOLUTION. Easy to know the expectation and variance of the update time are $\mu_1 = 1, \sigma_1^2 = 1, \mu_2 = 1, \sigma_2^2 = \frac{1}{3}$. So by the central limit theorem of updating process we know that

$$\frac{N_1(100) - 100}{\sqrt{100}}, \frac{N_2(100) - 100}{\sqrt{\frac{100}{3}}} \sim N(0, 1)$$

So $\frac{N_1(100) + N_2(100) - 200}{\sqrt{\frac{400}{3}}} \sim N(0, 1)$. So $\mathbb{P}(N_1(100) + N_2(100) \geq 190) \approx \mathbb{P}(N(0, 1) \geq -\frac{\sqrt{3}}{2})$. \square

4.3 更新过程的应用

4.3.1 随机游动的爬升时间

Definition 22. 设 $(\xi_n : n \geq 1)$ 是独立同分布的可积随机变量序列且满足 $\mathbb{E}(\xi_1) > 0$ 。令 $(W_n : n \geq 0)$ 是以 $(\xi_n : n \geq 1)$ 为跳幅的随机游动，其中 $W_0 = 0$ 。易知 $W_n \rightarrow +\infty$ 。令 $S_0 = 0$ ，递归地定义停时 S_n ，令 $S_n = \inf\{k \geq S_{n-1} : W_k > W_{S_{n-1}}\}$ 。称每个 S_n 为 (W_n) 的爬升时间。

Theorem 29. 对于 $n \geq 1$ ，令 $\eta_n = S_n - S_{n-1}$ 。则 (η_n) 是独立同分布的非负随机变量序列。证明见教材 p85

Theorem 30. 我们有 $\mathbb{P}(\forall n \geq 1, W_n > 0) = \frac{1}{\mathbb{E}(S_1)}$ 。证明见教材 p86

4.3.2 更新累积过程

Definition 23. 设 $((\xi_n, \eta_n) : n \geq 1)$ 为独立同分布的二维随机变量序列，且 $\xi_n \geq 0$ 。令 $N(t)$ 为以 ξ_n 为更新间隔时间的更新过程。令 $A(t) = \sum_{n=1}^{N(t)} \eta_n$ 。称 $(A(t) : t \geq 0)$ 为更新累积过程。

Theorem 31. 设 $0 < \mathbb{E}(\xi_1) < \infty, \mathbb{E}(|\eta_1|) < \infty$ 。则几乎必然有 $\lim_{t \rightarrow \infty} \frac{A(t)}{t} = \frac{\mathbb{E}(\eta_1)}{\mathbb{E}(\xi_1)}$ 。且有 $\lim_{t \rightarrow \infty} \frac{\mathbb{E}(A(t))}{t} = \frac{\mathbb{E}(\eta_1)}{\mathbb{E}(\xi_1)}$ 。证明见教材 p87

PROBLEM XXXV Assume $N(t)$ is updating process. X is the time interval distrabution of $N(t)$. Assume $\mathbb{D}(X) < \infty$. Let $R(t) := S_{N(t)+1} - t$. Find:

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T R(t) dt$$

SOLUTION. Easily $N(t) + 1 \geq T \geq N(t)$. So $\int_0^T R(t) dt \leq \sum_{i=1}^{N(T)+1} \int_{S_{i-1}}^{S_i} (S_i - t) dt = \frac{1}{2} \sum_{i=1}^{N(T)+1} (S_i - S_{i-1})^2 = \frac{1}{2} \sum_{i=1}^{N(T)+1} \xi_i^2$. For the same reason, we get that $\int_0^T R(t) dt \geq \frac{1}{2} \sum_{i=1}^{N(T)} \xi_i^2$.

Easy to know that $\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{i=1}^{N(T)} \xi_i^2 = \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{i=1}^{N(T)+1} \xi_i^2 = \frac{\mathbb{E}(X^2)}{\mathbb{E}(X)^2}$. So finally we get that

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T R(t) dt = \frac{\mathbb{E}(X^2)}{2\mathbb{E}(X)^2}$$

□

PROBLEM XXXVI Assume the number of people arriving the cinema is distributed as a Possion process with parameter λ . Assume the film begin at a fixed time $t \geq 0$. Let $A(t)$ be the sum of waiting time of all people arriving in $(0, t]$, find $\mathbb{E}(A(t))$.

SOLUTION. Let V_k be the arriving time of k -th people. Let $N(t)$ be the number of people in $(0, t]$. Then $A(t) = \sum_{k=1}^{N(t)} (t - V_k)$. Let $\xi_k := V_k - V_{k-1}$. Then $\sum_{k=1}^{N(t)} V_k = \sum_{k=1}^{N(t)} (N(t) - k)\xi_k = \sum_{k=0}^{N(t)-1} k\xi_{N(t)-k}$. So $\mathbb{E}(A(t)) = t\mathbb{E}(N(t)) - \mathbb{E}(\sum_{k=0}^{N(t)-1} k\xi_{N(t)-k})$. Easy to get that $\mathbb{E}(\sum_{k=0}^{N(t)-1} k\xi_{N(t)-k} | N(t) = n) = \frac{nt}{2}$. So $\mathbb{E}(A(t) | N(t) = n) = nt - \frac{nt}{2} = \frac{nt}{2}$. So finally we have $\mathbb{E}(A(t)) = \mathbb{E}(\mathbb{E}(A(t) | N(t))) = \mathbb{E}(\frac{N(t)t}{2}) = \frac{\lambda t^2}{2}$. □

PROBLEM XXXVII Assume a machine has life distrabuted p . When machine is broken or has been used T years, we will change a new machine. The price of new machine is C_1 , and if the machine is broken, it would cause loss C_2 .

1. Give the long-time average fee of this machine.
2. Let $C_1 = 10, C_2 = 0.5$, and $p(x) = \mathbb{1}_{(0,10)}(x)\frac{1}{10}$. Which T can let the fee be minimum.

SOLUTION. 1. Let ξ be the time when the machine will broken. Let $\gamma := \xi \wedge T$. Then the machine will be changed at γ . Obviously $\mathbb{E}(\gamma) = T\mathbb{P}(\xi > T) + \mathbb{E}(\xi \mathbb{1}(\xi \leq T)) = T \int_T^\infty p(x) dx + \int_0^T Txp(x) dx$. Let η be the fee of this machine, then we have $\eta = C_1 \mathbb{1}(\xi > T) + (C_1 + C_2) \mathbb{1}(\xi \leq T) = C_1 + C_2 \mathbb{1}(\xi \leq T)$. So $\mathbb{E}(\eta) = C_1 + C_2 \int_0^T p(x) dx$. So the long-time average fee is

$$g(T) = \frac{C_1 + C_2 \int_0^T Tp(x) dx}{T \int_T^\infty p(x) dx + \int_0^T x p(x) dx}$$

2. Easy to get that $g(T) = \frac{200+T}{20T-T^2}$ when $T \in (0, 10)$. And $g'(T) = \frac{T^2+400T-4000}{(20T-T^2)^2}$. Let $g'(T) = 0$, then $T^2 + 400T - 4000 = 0$, then $T = 20\sqrt{110} - 200 \approx 9.76$. So $T = 9.76$ can make the fee get minimum.

□