# Markov 过程复习资料

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2024 年 6 月 20 日

## 目录

# 2 基本概念和例子

## 2.1 基本概念

### 2.1.1 随机过程的定义

**Definition 1.** 设 $I$ 是非空指标集，$(\Omega, \mathcal{F}, \mathbb{P})$ 是概率空间。若 $(X_\alpha : \alpha \in I)$ 是一组定义在 $(\Omega, \mathcal{F}, \mathbb{P})$ 上的随机变量（取值为 $\mathbb{R}^d$），则称 $(X_\alpha : \alpha \in I)$ 为一个**随机过程**。

**Definition 2.** 假设 $(X_\alpha : \alpha \in I)$ 和 $(Y_\alpha : \alpha \in J)$ 是两个随机过程。若对于任何有限序列 $(s_1, \cdots, s_n) \subset I, (t_1, \cdots, t_m) \subset J$，都有 $(X_{s_1}, \cdots, X_{s_n}) \perp (Y_{t_1}, \cdots, Y_{t_m})$，则称这两个**随机过程独立**。

### 2.1.2 轨道和修正

**Definition 3.** 设 $(X_\alpha : \alpha \in I)$ 为随机过程。固定 $\omega \in \Omega$，称 $t \mapsto X_t(\omega)$ 为 $X$ 的一条**轨道**。

**Definition 4.** 称一个随机过程是（**左连续**//**右连续**//**连续**//**左极右连**//**左连右极**）的，若它的所有轨道都是（左连续//右连续//连续//左极右连//左连右极的）。

**Definition 5.** 设 $(X_t : t \in I)$ 和 $(Y_t : t \in I)$ 是两个随机过程。若 $\forall t \in I$，有 $\mathbb{P}(X_t = Y_t) = 1$，则称它们互为**修正**。若 $\mathbb{P}(\forall t \in I, X_t = Y_t) = 1$，则称它们是**无区别**的。

**Theorem 1.** 设 $(X_t : t \geq 0)$ 和 $(Y_t : t \geq 0)$ 是两个右连续的随机过程，而 $D$ 是 $(0, \infty)$ 的可数稠密子集。若 $\forall s \in D, \mathbb{P}(X_s = Y_s) = 1$，则有 $(X_t : t \geq 0)$ 和 $(Y_t : t \geq 0)$ 是无区别的。

### 2.1.3　有限维分布族

为了简化记号，我们用 $S(I)$ 表示 $I$ 的全体有序有限子集。即:

$$S(I) := \{(t_1, \cdots, t_n) : n \geq 1, t_i \in I, \forall i = 1, \cdots, n\}$$

用 $E$ 表示 $\mathbb{R}^d$，用 $\mathcal{E}$ 表示博雷尔代数。

**Definition 6.** 设 $I$ 是非空指标集。若对于每个 $J \in S(I)$，都对应一个 $(E^{|J|}, \mathcal{E}^{|J|})$ 上的概率测度 $u_J$，则称 $(\mu_J : I \in S(I))$ 为 $E$ 上的**一个有限维分布族**，其中每个 $\mu_J$ 称为**一个有限维分布**。设 $X = (X_t : t \in I)$ 是一个随机过程，用 $\mu_J^X$ 表示 $(X_{t_1}, \cdots, X_{t_n})$ 的分布。称 $\mathcal{D}_X := \{\mu_J^X : J \in S(I)\}$ 为 $X$ 的有限维分布族，称 $\mu_J^X$ 为其中的一个有限维分布。

**Definition 7.** 给定 $(E, \mathcal{E})$ 上的有限维分布族 $\mathcal{D}$，若存在随机过程 $X = (X_t : t \in I)$ 使得 $\mathcal{D}_X = \mathcal{D}$，则称 $X$ 为 $\mathcal{D}$ 的**一个实现**。若两个随机过程 $X, Y$ 满足 $\mathcal{D}_X = \mathcal{D}_Y$，则称它们为**等价的**。两个等价的过程互称实现。显然，两个互为修正的随机过程一定等价，反过来却未必。

### 2.1.4　左极右连实现

**Definition 8.** 状态空间 $E = \mathbb{R}^d$ 上的随机过程有左极右连实现 $\iff$ 它有左极右连修正。证明见教材 *p5*

PROBLEM 1 Assume $(\mathscr{F}_t : t \geq 0, t \in \mathbb{R})$ is a filtration. For $t \geq 0$ we let $\mathscr{F}_{t+} := \bigcap_{s>t} \mathscr{F}_s$. Prove that $\mathscr{F}_t \subset \mathscr{F}_{t+}$ and $(\mathscr{F}_{t+} : t \geq 0)$ is a filtration.

SOLUTION. To prove $\mathscr{F}_t \subset \mathscr{F}_{t+} = \bigcap_{s>t} \mathscr{F}_s$, we only need to prove $\forall s > t, \mathscr{F}_t \subset \mathscr{F}_s$. By the definition of filtration it's obvious. Now we will prove $(\mathscr{F}_{t+} : t \geq 0)$ is a filtration. Only need to prove $\forall t, s \in \mathbb{R} \wedge t \leq s, \mathscr{F}_{t+} \subset \mathscr{F}_{s+}$. By the definition of $\mathscr{F}_{\dot{c}+}$ we know that $\mathscr{F}_{t+} = \bigcap_{x>t} \mathscr{F}_x = \bigcap_{x>s} \mathscr{F}_x \cap \bigcap_{x:t<x\leq s} \mathscr{F}_x \subset \bigcap_{x>s} \mathscr{F}_x = \mathscr{F}_{s+}$. So $(\mathscr{F}_{t+} : t \geq 0)$ is a filtration. □

PROBLEM 2 Assume $(X_t : t \geq 0, t \in \mathbb{R})$ is a stochastic process on probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Prove that $\forall s, t \geq 0, \varepsilon > 0, \{\rho(X_s, X_t) \geq \varepsilon\} \in \mathscr{F}$.

SOLUTION. Easily $\{\rho(X_s, X_t) \geq \varepsilon\} = \bigcup_{k=1}^{\infty} \{\rho(X_s, X_t) > \varepsilon - \frac{1}{k}\varepsilon\}$. So we only need to prove $\forall k \in \mathbb{N}^+, \{\rho(X_s, X_t) > \varepsilon(1 - \frac{1}{k})\} \in \mathscr{F}$. Take $\delta = \varepsilon(1 - \frac{1}{k})$, only need to prove $\forall \delta > 0, \{\rho(X_s, X_t) > \delta\} \in \mathscr{F}$.

$\forall t \geq 0, X_t : \Omega \to E$ is measurable, where $E \subset \mathbb{R}^d$. So we can find a countable dense set in $\mathbb{R}^d$, write $D$. We will prove that $\{\rho(X_s, X_t) > \delta\} = \bigcup_{q \in D} \{\rho(X_s, q) - \rho(X_t, q) > \delta\}$. On one hand, easily $\rho(X_s, q) - \rho(X_t, q) > \delta \implies \rho(X_s, X_t) > \delta$ from triangle inequality. So we easily get $\{\rho(X_s, X_t) > \delta\} \supset \bigcup_{q \in D} \{\rho(X_s, q) - \rho(X_t, q) > \delta\}$. On the other hand, assume for certain $\omega \in \Omega$ we have $\rho(X_s(\omega), X_t(\omega)) > \delta$, we will prove $\exists q \in D, \rho(X_s(\omega), q) - \rho(X_t(\omega), q) > \delta$. For convenience, we omit $(\omega)$ from now on to the end of this paragraph. Since $\rho(X_s, X_t) > \delta$, we know $\gamma := \frac{\rho(X_s, X_t) - \delta}{2} > 0$. Since $D$ is dense, we obtain $\exists q \in D, \rho(X_t, q) < \gamma$. So from triangle inequality we get $\rho(X_s, q) \geq \rho(X_s, X_t) - \rho(X_t, q) > 2\gamma + \delta - \gamma = \gamma + \delta$. So we get $\rho(X_s, q) - \rho(X_t, q) > \gamma + \delta - \gamma = \delta$. Finally, we get $\{\rho(X_s, X_t) > \delta\} = \bigcup_{q \in D}\{\rho(X_s, q) - \rho(X_t, q) > \delta\}$.

Noting $\bigcup_{q\in D}\{\rho(X_s, q) - \rho(X_t, q) > \delta\} = \bigcup_{q\in D}\bigcup_{p\in\mathbb{Q}^+}\{\rho(X_s, q) > \delta + p, \rho(X_t, q) < p\}$, and $D, \mathbb{Q}^+$ are countable, so we only need to check $\{\rho(X_s, q) > \delta + p, \rho(X_t, q) < p\} \in \mathscr{F}, \forall q \in D, p \in \mathbb{Q}^+$. Noting $\{\rho(X_s, q) > \delta + p, \rho(X_t, q) < p\} = \{\rho(X_s, q) > \delta + p\} \cap \{\rho(X_t, q) < p\}$, and $X_s, X_t$ are measurable from $\Omega$ to $E$, we obtain $\{\rho(X_s, q) > \delta + p\}, \{\rho(X_t, q) < p\} \in \mathscr{F}$. So we proved $\{\rho(X_s, X_t) > \delta\} \in \mathscr{F}, \forall s, t \geq 0, \forall \delta > 0$.

Finally, we obtain $\{\rho(X_s, X_t) \geq \varepsilon\} \in \mathscr{F}, \forall s, t \geq 0, \varepsilon > 0$. $\square$

PROBLEM 3 Let $\mathscr{D}_X := \{\mu_J^X : J \in S(I)\}$ be the family of finite-dimentional distributions of a stochastic process $(X_t : t \geq 0, t \in \mathbb{R})$. $\forall (s_1, s_2) \in S(I)$ and $J = (t_1, \cdots, t_n) \in S(I)$, write $K_1 := (s_1, s_2, t_1, \cdots, t_n) \in S(I), K_2 := (s_2, s_1, t_1, \cdots, t_n) \in S(I)$. Take $A_1, A_2 \in \mathscr{E}, B \in \mathscr{E}^n$, prove that

$$\mu_{K_1}^X(A_1 \times A_2 \times B) = \mu_{K_2}^X(A_2 \times A_1 \times B)$$

and

$$\mu_{K_1}^X(E \times E \times B) = \mu_{K_2}^X(E \times E \times B) = \mu_J^X(B)$$

SOLUTION . By the definition of finite-dimentional distributions we get

$$\begin{aligned}
\mu_{K_1}^X(A_1 \times A_2 \times B) &= \mathbb{P}((X_{s_1}, X_{s_2}, X_{t_1}, \cdots, X_{t_n}) \in A_1 \times A_2 \times B) \\
&= \mathbb{P}(X_{s_1} \in A_1)\mathbb{P}(X_{s_2} \in A_2)\mathbb{P}((X_{t_1}, X_{t_2}, \cdots, X_{t_n}) \in B)
\end{aligned}$$

For the same reason, we obtain

$$\begin{aligned}
&\mu_{K_2}^X(A_2 \times A_1 \times B) \\
&\qquad = \mathbb{P}((X_{s_2}, X_{s_1}, X_{t_1}, \cdots, X_{t_n}) \in A_2 \times A_1 \times B) \\
&\qquad = \mathbb{P}(X_{s_2} \in A_2)\mathbb{P}(X_{s_1} \in A_1)\mathbb{P}((X_{t_1}, X_{t_2}, \cdots, X_{t_n}) \in B)
\end{aligned}$$

So we get

$$\mu_{K_1}^X(A_1 \times A_2 \times B) = \mu_{K_2}^X(A_2 \times A_1 \times B)$$

Also, let $A_1 = A_2 = E$, we get

$$\begin{aligned}
\mu_{K_1}^X(E \times E \times B) &= \mu_{K_2}^X(E \times E \times B) \\
&= \mathbb{P}(X_{s_1} \in E)\mathbb{P}(X_{s_2} \in E)\mathbb{P}((X_{t_1}, \cdots, X_{t_n}) \in B) \\
&= \mathbb{P}((X_{t_1}, \cdots, X_{t_n}) \in B)
\end{aligned}$$

By the definition of finite-dimentional distributions we get

$$\mu_J^X(B) = \mathbb{P}((X_{t_1}, \cdots, X_{t_n}) \in B)$$

So finally we get

$$\mu^X_{K_1}(E \times E \times B) = \mu^X_{K_2}(E \times E \times B) = \mu^X_J(B)$$

□

PROBLEM 4 Assume $(\tau_k : k \in \mathbb{N}^+)$ is an i.i.d sequence of r.v. with exponential distribution with parameter $\alpha > 0$. Let $S_n := \sum_{k=1}^n \tau_k$. For $t \geq 0, t \in \mathbb{R}$, let:

$$N_t := \sum_{n=1}^{\infty} \mathbb{1}_{\{S_n \leq t\}}, X_t := \sum_{n=1}^{\infty} \mathbb{1}_{\{S_n < t\}}$$

Prove that $N$ and $X$ are modifications of each other, but they are not indistinguishable.

SOLUTION . First we prove $N$ and $X$ are modifications of each other. Fix $t \in [0, \infty)$, we need to prove $\mathbb{P}(N_t = X_t) = 1$. Noting $N_t - X_t = \sum_{n=1}^{\infty} \mathbb{1}_{S_n \leq t} - \mathbb{1}_{S_n < 1} = \sum_{n=1}^{\infty} \mathbb{1}_{S_n = t}$, we get $\mathbb{P}(N_t = X_t) = \mathbb{P}(\sum_{n=1}^{\infty} \mathbb{1}_{S_n = t}) = \mathbb{P}(\forall n \in \mathbb{N}^+, S_n \neq t)$. So we only need to prove $\mathbb{P}(S_n = t) = 0, \forall n \in \mathbb{N}^+$. Since $\tau_k, k \in \mathbb{N}^+$ are continuous-distributed, we know $S_n = \sum_{k=1}^n \tau_k$ is continuous-distributed, so $\mathbb{P}(S_n = t) = 0$. So we proved $N$ and $X$ are modifications of each other.

Next we will prove they are not indistinguishable. Only need to prove $\mathbb{P}(\forall t \in [0, \infty), X_t = N_t) = 0 \neq 1$. Since $N_t - X_t = \sum_{n=1}^{\infty} \mathbb{1}_{S_n \leq t} - \mathbb{1}_{S_n < 1} = \sum_{n=1}^{\infty} \mathbb{1}_{S_n = t}$, we know $\forall t, N_t = X_t \iff \forall t, \forall n \in \mathbb{N}^+, S_n \neq t$. But $S_n$ is ranged in $[0, \infty)$, so R.H.S is an impossible event. So we finally get $\mathbb{P}(\forall t \in [0, \infty), X_t = N_t) = 0$ and thus $X$ and $N$ are not indistinguishable. □

PROBLEM 5 Assume $T$ is non-negetive r.v. with distribution function $F$ continuous on $\mathbb{R}$. Let $X_t = \mathbb{1}_{\{T \leq t\}}$. Prove that $X$ is stochastically continuous.

SOLUTION . Only need to check $\forall t \geq 0, X_s \xrightarrow{\mathbb{P}} X_t, s \to t$. Take $\varepsilon > 0$, we need to prove $\lim_{s \to t} \mathbb{P}(\rho(X_s, X_t) > \varepsilon) = 0$. For $u > v \geq 0$, we have $X_u - X_v = \mathbb{1}_{v < T \leq u}$. So $\mathbb{P}(\rho(X_u - X_v) > \varepsilon) \leq \mathbb{P}(X_u \neq X_v) = \mathbb{P}(v < T \leq u) \leq \mathbb{P}(T \in [v, u])$. So we easily get $\lim_{s \to t+} \mathbb{P}(\rho(X_s, X_t) > \varepsilon) \leq \lim_{s \to t+} \mathbb{P}(T \in [t, s]) = 0$ and $\lim_{s \to t-} \mathbb{P}(\rho(X_s, X_t) > \varepsilon) \leq \lim_{s \to t-} \mathbb{P}(T \in [s, t]) = 0$. So $\lim_{s \to t} \mathbb{P}(\rho(X_s, X_t) > \varepsilon) = 0$. □

PROBLEM 6 Assume $I = \mathbb{Z}^+$, then the stochastic process $X = (X_0, X_1, \cdots)$ is a r.v. from $\Omega$ to $E^\infty$. Define the distribution of $X$, $\mu_X$, as follows:

$$\mu_X(A) = \mathbb{P}(X \in A), A \in \mathscr{E}^\infty$$

Then stochastic process $X, Y$ are equivalent $\iff \mu_X = \mu_Y$.

SOLUTION . " $\Longrightarrow$ ":Assume $X, Y$ are equivalent, now we will prove $\mu_X = \mu_Y$. Let $\mathscr{A} := \{A \in \mathscr{P}(E^\infty) : \exists n \in \mathbb{N}^+, A = A_1 \times A_2 \times \cdots \times A_n \times E \times E \times \cdots\}$. Then we can get $\mu_X(A) = \mu^X_{(1,2,\cdots,n)}(A_1 \times \cdots \times A_n)$. So for $A \in \mathscr{A}$ we know $\mu_X(A) = \mu_Y(A)$. By the definition of $\mathscr{E}^\infty = \sigma(\mathscr{A})$, and noting $\mathscr{A}$ is a Semiset algebra, by the Measure extension theorem we get $\mu_X = \mu_Y$.

" $\Longleftarrow$ ": Assume $\mu_X = \mu_Y$, then easily $\mu^X_{(s_1,\cdots,s_n)}(A_{s_1} \times \cdots \times A_{s_n}) = \mu_X(\prod_{k \in \mathbb{N}^+} B_k)$, where $B_k = A_{s_t}$ for $k = s_t$ and $B_k = E$ for $k \neq s_t, \forall t = 1, \cdots, n$. So easily $\mu^X_J = \mu^Y_J, \forall J \subset I \wedge |J| < \infty$. □

## 2.2 随机游动

**Definition 9.** 设 $\{\xi_n : n \geq 1\}$ 是独立同分布的 $d$ 维随机变量列，而 $X_0$ 是与之独立的一个 $d$ 维随机变量。令 $X_n := X_0 + \sum_{k=1}^{n} \xi_k$。称 $(X_n : n \geq 0)$ 为 $d$ **维随机游动**，并称 $\{\xi_n : n \geq 1\}$ 为其**步长列**。

**Definition 10.** 若 $X_0, \xi_1$ 均取值与 $\mathbb{Z}^d$，则该随机游动状态空间可以取为 $\mathbb{Z}^d$。特别地，若还有 $\mathbb{P}(|\xi_1| = 1) = 1$，则称其为**简单随机游动**。进一步地，若对于 $\mathbb{Z}^d$ 中的任一单位向量 $v$，均有 $\mathbb{P}(\xi_1 = v) = \frac{1}{2d}$，则称其为**对称简单随机游动**。

### 2.2.1 轨道的无界性

方便起见，考虑 $\mathbb{Z}$ 上的简单随机游动 $S_n$，设其步长列为 $\xi_n : n \geq 1$。设 $\mathbb{P}(\xi_n = 1) = p, \mathbb{P}(\xi_n = -1) = q$，其中 $p, q \in (0,1), p + q = 1$。

**Theorem 2.** $(S_n : n \geq 1)$ 的轨道是几乎必然无界的。即：

$$\mathbb{P}(\sup_{n \geq 0} |S_n| = \infty) = 1. \tag{1}$$

证明见教材 *p9*

### 2.2.2 首达时分布

**Definition 11.** 记 $\mathbb{P}_i(\cdot) = \mathbb{P}(\cdot \mid S_0 = i)$。

**Definition 12.** 定义 $(S_n : n \geq 0)$ 到达 $x \in \mathbb{Z}$ 的**首达时** $\tau_x := \inf\{n \geq 0 : S_n = x\}$。

**Theorem 3.** 当 $p = q = \frac{1}{2}$ 时，对于 $a < b, i \in [a, b], a, b, i \in \mathbb{Z}$，有

$$\mathbb{P}_i(\tau_b < \tau_a) = \frac{i - a}{b - a}, \mathbb{P}_i(\tau_a < \tau_b) = \frac{b - i}{b - a} \tag{2}$$

当 $p \neq q$ 时，有

$$\mathbb{P}_i(\tau_b < \tau_a) = \frac{1 - (\frac{q}{p})^{i-a}}{1 - (\frac{q}{p})^{b-a}}, \mathbb{P}_i(\tau_a < \tau_b) = \frac{(\frac{q}{p})^{i-a} - (\frac{q}{p})^{b-a}}{1 - (\frac{q}{p})^{b-a}} \tag{3}$$

证明见教材 *p10*

**Theorem 4.** 当 $p \geq q$，对 $a \leq i \leq b \in \mathbb{Z}$，有

$$\mathbb{P}_i(\tau_a < \infty) = (\frac{q}{p})^{i-a}, \mathbb{P}_i(\tau_b < \infty) = 1 \tag{4}$$

当 $p \leq q$，有

$$\mathbb{P}_i(\tau_a < \infty) = 1, \mathbb{P}_i(\tau_b < \infty) = (\frac{p}{q})^{b-i} \tag{5}$$

证明见教材 *p11*

ℝOBLEM 7 Prove that if $(X_n : n \geq 0)$ is a simple random walk, then so is $(-X_n : n \geq 0)$.

SOLUTION . Let $\xi_n := X_n - X_{n-1}$ for $n \in \mathbb{N}^+$. Then Since $(X_n : n \in \mathbb{N})$ is simple random walk we have $X_0, \xi_1, \xi_2, \cdots$ are independent r.v. ranges in $\mathbb{Z}$, and $\xi_i, i = 1, 2 \cdots$ are i.i.d., and $\mathbb{P}(|\xi_i| = 1) = 1$. So we easily get $-X_0, -\xi_1, -\xi_2, \cdots$ are independent r.v. ranges in $\mathbb{Z}$, and $-\xi_i, i = 1, 2, \cdots$ are i.i.d., and $\mathbb{P}(|-\xi_i| = 1) = 1$. Since $-X_n = X_0 + \sum_{k=1}^{n} \xi_k$, by the definition of simple random walk we obtain $(-X_n : n \in \mathbb{N})$ is a simple random walk. $\square$

PROBLEM 8 Let $(X_n : n \geq 0)$ be a $d$-dimentional random walk with $\mathbb{P}(|\xi_i| \geq 1) > 0$, prove that $\mathbb{P}(\sup_n |X_n| = \infty) = 1$.

SOLUTION . Let $t \in \mathbb{Z}^d, t \neq 0$ and $\mathbb{P}(\xi_i = t) > 0$. Since $\mathbb{P}(\sup_n |X_n| = \infty) = \inf_{k \in \mathbb{N}} \mathbb{P}(\sup_n |X_n| \geq k)$, we only need to prove $\mathbb{P}(\sup_n |X_n| \geq k) = 1$ for every $k \in \mathbb{N}$. Take $K > 3k, K \in \mathbb{N}$. Let $A_s := \{\xi_i = t : i = sK+1, sK+2, \cdots, sK+K-1\}$. Then for $\omega \in A_s$, we have $|X_{sK+K} - X_{sK}| = |\sum_{u=1}^{K-1} t| = K|t| \geq K \geq 3k$. Then $\sup_n |X_n| \geq \max\{|X_{sK+K}|, |X_{sK}|\} \geq \frac{1}{2}|X_{sK+K} - X_{sK}| \geq k$. So we get $\forall s, A_s \subset \{\sup_n |X_n| \geq k\}$. Since $\xi_i$ are independent, easily get $A_s, s = 1, 2, \cdots$ are independent. Noting $\mathbb{P}(A_s) = \mathbb{P}(\xi_i = t)^K > 0$, we get $\sum_{s \in \mathbb{N}} \mathbb{P}(A_s) = \infty$. So from BC-theorem we get $\mathbb{P}(A_s, i.o.) = 1$, thus $\mathbb{P}(\bigcup_{s \in \mathbb{N}} A_s) = 1$. Thus, $\mathbb{P}(\sup_n |X_n| \geq k) = 1$, for every $k \in \mathbb{N}$. Thus, $\mathbb{P}(\sup_n |X_n| = \infty) = \mathbb{P}(\bigcap_{k \in \mathbb{N}} \{\sup_n |X_n| \geq k\}) = 1$. $\square$

PROBLEM 9 Let $(S_n : n \geq 0)$ be a symmetry simple random walk with $S_0 = 0$, for $d = 2$, prove that

$$\mathbb{P}(S_{2n} = 0) = \frac{1}{4^{2n}} \left( \frac{(2n)!}{(n!)^2} \right)^2$$

For $d = 3$, prove that

$$\mathbb{P}(S_{2n} = 0) = \frac{1}{2^{2n}} \frac{(2n)!}{(n!)^2} \sum_{i+j+k=n} \left( \frac{1}{3^n} \frac{n!}{i!j!k!} \right)^2$$

SOLUTION . First we consider $d = 2$. Write $\xi_i = S_i - S_{i-1}$. Then we know $S_{2n}$ occur $\iff$ the number of $(1,0)$ and $(-1,0)$ in $\{\xi_i : i = 1, \cdots, 2n\}$, and the number of $(0,1)$ and $(0,-1)$ in $\{\xi_i : i = 1, \cdots, 2n\}$. We assume there is $k$ pairs of $(1,0), (-1,0)$, then easily there is $n-k$ pairs of $(0,1), (0,-1)$. The probability is $\binom{2n}{k}\binom{2n-k}{k}\binom{2n-2k}{n-k}\frac{1}{4^{2n}}$. So the total probability is $\mathbb{P}(S_{2n} = 0) = \sum_{k=0}^{n} \binom{2n}{k}\binom{2n-k}{k}\binom{2n-2k}{n-k}\frac{1}{4^{2n}} = \sum_{k=0}^{n} \frac{(2n)!}{k!k!(n-k)!(n-k)!4^{2n}}$. Noting that $\sum_{k=0}^{n} \frac{(n!)^2}{k!k!(n-k)!(n-k)!} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \binom{2n}{n} = \frac{(2n)!}{n!n!}$, we finally get $\mathbb{P}(S_{2n} = 0) = \frac{1}{4^{2n}} \left( \frac{(2n)!}{n!n!} \right)^2$.

Use the same method, consider $d = 3$, we have

$$\mathbb{P}(S_{2n} = 0) = \sum_{i+j+k=n} \binom{2n}{i}\binom{2n-i}{i}\binom{2n-2i}{j}\binom{2n-2i-j}{j}\binom{2n-2i-2j-k}{k}\frac{1}{6^{2n}}$$

So easily to get $\mathbb{P}(S_{2n} = 0) = \frac{1}{2^{2n}} \frac{(2n)!}{(n!)^2} \sum_{i+j+k=n} \left( \frac{1}{3^n} \frac{n!}{i!j!k!} \right)^2$. $\square$

PROBLEM 10 Assume $(S_n : n \geq 0)$ is a symmetry simple random walk with $S_0 = i \in \mathbb{Z}$. Prove that $\forall a \in \mathbb{Z}$, let $\tau_a := \min\{n \in \mathbb{N} : S_n = a\}$, then $\mathbb{P}(\tau_a < \infty) = 1$.

SOLUTION. Without loss of generality assume $a < 0, i = 0$. Take $N \in \mathbb{N}^+$. Consider $\tau := \min\{n \in \mathbb{N} : S_n = a \vee S_n = N\}$. From Problem 21 we can easily know $\mathbb{P}(\tau < \infty) = 1$ because $\{\sup_n |S_n| = \infty\} \subset \{\tau < \infty\}$, a.s. So we get $\{\tau_a = \tau\} \subset \{\tau_a < \infty\}$, a.s. Let $Y_n := S_{n\wedge\tau} := S_{\min\{n,\tau\}}$. Easily $(S_n : n \in \mathbb{N})$ is a martingale, and $\tau$ is a stopping time, so we get $(Y_n : n \in \mathbb{N})$ is a martingale, too. And easily $Y_n \in [a, N]$, so $Y_n$ is bounded. So we get $\mathbb{E}(S_\tau) = \lim_{n\to\infty} \mathbb{E}(Y_n) = \mathbb{E}(Y_0) = 0$. Easily to know $\mathbb{E}(S_\tau) = \mathbb{P}(\tau = \tau_a)a + \mathbb{P}(\tau \neq \tau_a)N = 0$. And $\mathbb{P}(\tau = \tau_a) + \mathbb{P}(\tau \neq \tau_a) = 1$, so easily $\mathbb{P}(\tau = \tau_a) = \frac{N}{N-a}$. So $\mathbb{P}(\tau_a < \infty) \geq \frac{N}{N-a}$. Let $N \to \infty$, we get $\mathbb{P}(\tau_a < \infty) = 1$. □

## 2.3 布朗运动

### 2.3.1 背景和定义

**Definition 13.** 假定 $\sigma^2 > 0$，具有连续轨道的实值过程 $(B_t : t \geq 0)$ 满足:

1. $\forall 0 \leq s \leq t$，$B_t - B_s \sim N(0, \sigma^2(t-s))$;
2. $\forall 0 \leq t_0 \leq \cdots \leq t_n$，$B_0, B_1 - B_0, \cdots, B_{t_n} - B_{t_{n-1}}$ 独立，

称 $(B_t : t \geq 0)$ 是以 $\sigma^2$ 为参数的**布朗运动**。特别的，当 $\sigma^2 = 1$，$(B_t : t \geq 0)$ 为**标准布朗运动**。

**Definition 14.** 有限维分布为正态分布的随机过程称为**正态过程**。

### 2.3.2 布朗运动的构造

**Theorem 5.** 布朗运动是有连续实现的。证明见教材 *p13.*

### 2.3.3 布朗运动的性质

**Theorem 6.** 从原点出发的零均值高斯过程 $(B_t : t \geq 0)$ 是标准布朗运动 $\iff$ $\forall s, t \geq 0$，$\mathbb{E}(B_t B_s) = t \wedge s$。证明 *p17.*

**Theorem 7.** 布朗运动轨道几乎处处不可导。证明 *p17-18.*

**Lemma 1.** *Assume $(B_t : t \geq 0)$ is a random process ranging in $\mathbb{R}$, $a \in \mathbb{R}^+$, and $\forall s, t : 0 \leq s \leq t, B_t - B_s \sim N(0, a(t-s))$. Assume $B_t$ is continuous about $t$, a.s. Let $\mathcal{F}_t := \sigma(B_s : 0 \leq s \leq t)$. Then $(B_t : t \geq 0)$ is Brownian motion $\iff \forall 0 \leq s \leq t, B_t - B_s \perp \mathcal{F}_s$.*

证明. "$\Longrightarrow$": To prove $B_t - B_s \perp \mathcal{F}_s$, only need to prove for $t_1 < t_2 < \cdots < t_{n-1} = s < t = t_n$, we have $B_t - B_s \perp \sigma(B_{t_k} : k = 1, \cdots, n-1)$. Easily $B_t - B_s \perp \sigma(B_{t_{k+1}} - B_{t_k}, B_{t_1} : k = 1, \cdots, n-2) = \sigma(B_{t_k} : k = 1, \cdots, n-1)$, so $B_t - B_s \perp \mathcal{F}_s$.

"$\Longleftarrow$": For $t_1 < \cdots < t_n$, we need to prove $B_{t_1}, B_{t_{k+1}} - B_{t_k} : 1 \leq k \leq n-1$ are independent. Use MI to $n$. When $n = 1$ it's obvious. Assume we have proved it for certain $n \geq 1$, now consider $n+1$. Since $B_{t_{k+1}} - B_{t_k} \in \mathcal{F}_{t_n}, k = 1, \cdots, n-1$, we have $B_{t_{n+1}} - B_{t_n} \perp \sigma(B_{t_1}, B_{t_{k+1}} - B_{t_k} : k = 1, \cdots, n-1)$. So $\mathbb{P}(B_{t_1} \in A_1, B_{t_{k+1}} - B_{t_k} \in A_{k+1}, k = 1, \cdots, n) = \mathbb{P}(B_{t_1} \in$

$A_1, B_{t_{k+1}} - B_{t_k} \in A_{k+1}, k = 1, \cdots, n-1)\mathbb{P}(B_{t_{n+1}} - B_{t_n} \in A_{n+1})$. By Induction assumption we get $\mathbb{P}(B_{t_1} \in A_1, B_{t_{k+1}} - B_{t_k} \in A_{k+1}, k = 1, \cdots, n) = \mathbb{P}(B_{t_1} \in A_1) \prod_{k=1}^{n} \mathbb{P}(B_{t_{k+1}} - B_{t_k} \in A_{k+1})$. So finally we get $B_{t_1}, B_{t_{k+1}} - B_{t_k} : 1 \leq k \leq n$ are independent. $\square$

PROBLEM 11 Assume $(B_t : t \geq 0)$ is Brownian motion, prove that for $r > 0$, we have $(B_{t+r} - B_r : t \geq 0)$ is Brownian motion, too.

SOLUTION. Assume $B_t - B_s \sim N(0, a(t-s)), a > 0$. Let $\mathcal{F}_t := \sigma(B_s : 0 \leq s \leq t)$ and $\mathcal{G}_t := \sigma(B_{r+s} - B_r : 0 \leq s \leq t)$. For $0 \leq s \leq t$, we have $B_{t+r} - B_r - (B_{s+r} - B_r) = B_{t+r} - B_{s+r} \sim N(0, a(t-s))$. And easily to know $B_{r+s} - B_r \in \mathcal{F}_{r+s}$, so $\mathcal{G}_t \subset \mathcal{F}_{t+r}, \forall t \geq 0$. Since $(B_t : t \geq 0)$ is Brownian motion, easily $\mathcal{F}_{s+r} \perp B_{t+r} - B_{s+r}$. Since $\mathcal{G}_t \subset \mathcal{F}_{t+r}$, we obtain $\mathcal{G}_t \perp B_{t+r} - B_{s+r} = B_{t+r} - B_r - (B_{s+r} - B_r)$. Easily since $B_t$ is continuous we get $B_{t+r} - B_r$ is continuous. So $(B_{t+r} : t \geq 0)$ is Brownian motion. $\square$

PROBLEM 12 Assume $(B_t : t \geq 0)$ is standard Brownian motion start at 0. Prove that $\forall c > 0, (cB_{\frac{t}{c^2}} : t \geq 0)$ is standard Brownian motion start at 0, too.

SOLUTION. Since $B_0 = 0$ we get $cB_{\frac{0}{c^2}} = 0$. Let $\mathcal{F}_t := \sigma(B_s : 0 \leq s \leq t)$ and $\mathcal{G}_t := \sigma(cB_{\frac{s}{c^2}} : 0 \leq s \leq t)$. Easily to know $\mathcal{G}_t = \mathcal{F}_{\frac{t}{c^2}}$. For $0 \leq s \leq t$, we have $cB_{\frac{t}{c^2}} - cB_{\frac{s}{c^2}} = c(B_{\frac{t}{c^2}} - B_{\frac{s}{c^2}}) \sim N(0, t-s)$, because $B_{\frac{t}{c^2}} - B_{\frac{s}{c^2}} \sim N(0, \frac{t-s}{c^2})$. And since $(B_t : t \geq 0)$ is Brownian motion, we get $B_{\frac{t}{c^2}} - B_{\frac{s}{c^2}} \perp \mathcal{F}_{\frac{s}{c^2}} = \mathcal{G}_s$. Easily since $B_t$ is continuous we get $cB_{t^2_c}$ is continuous. So $(cB_{\frac{t}{c^2}} : t \geq 0)$ is standard Brownian motion starts at 0, too. $\square$

PROBLEM 13 Assume $(X_t : t \geq 0)$ and $(Y_t : t \geq 0)$ are two independent standard Brownian motion, $a, b \in \mathbb{R}$ and $\sqrt{a^2 + b^2} > 0$. Prove that $(aX_t + bY_t : t \geq 0)$ is a Brownian motion with parameter $c^2 = a^2 + b^2$.

SOLUTION. Let $\mathcal{F}_t := \sigma(X_s : 0 \leq s \leq t)$ and $\mathcal{G}_t := \sigma(Y_s : 0 \leq s \leq t)$. Let $\mathcal{H}_t := \sigma(aX_s + bY_s : 0 \leq s \leq t)$. Since $(X_t : t \geq 0), (Y_t : t \geq 0)$ are two independent Brownian motion, we know $\forall 0 \leq s \leq t, X_t - X_s \perp \mathcal{F}_s, \mathcal{G}_s; Y_t - Y_s \perp \mathcal{F}_s, \mathcal{G}_s$. So we get $aX_t + bY_t - aX_s - bY_s \perp \mathcal{F}_s, \mathcal{G}_s$, thus $aX_t + bY_t - aX_s - bY_s \perp \sigma(\mathcal{F}_s, \mathcal{G}_s)$ Easily $aX_s + bY_s \in \sigma(\mathcal{F}_s, \mathcal{G}_s)$, so $\mathcal{H}_t \subset \sigma(\mathcal{F}_t, \mathcal{G}, t), \forall t \geq 0$. So $aX_t + bY_t - aX_s - bY_s \perp \mathcal{H}_s$. And easily $a(X_t - X_s) \sim N(0, a^2(t-s)), b(Y_t - Y_s) \sin N(0, b^2(t-s))$, and since $\mathcal{F}_t \perp \mathcal{G}_t$ we get $a(X_t - X_s) \perp b(Y_t - Y_s)$, so $aX_t + bY_t - aX_s - bY_s \sim N(0, (a^2+b^2)(t-s))$. Easily since $X_t, Y_t$ is continuous we get $aX_t + bY_t$ is continuous. So $(aX_t + bY_t : t \geq 0)$ is a Brownian motion with parameter $a^2 + b^2 = c^2$. $\square$

PROBLEM 14 Assume $(B_t : t \geq 0)$ is standard Brownian motion start at 0. Let $X_0 = 0$ and $X_t := tB_{\frac{1}{t}}$. Given

$$\limsup_{t \to \infty} \frac{B_t}{\sqrt{2t \log \log t}} = 1$$

Prove that $(X_t : t \geq 0)$ is standard Brownian motion start at 0.

SOLUTION. First consider the distribution of $X_t - X_s$ for $s < t$. If $s = 0$ then $X_t - X_s = tB_{\frac{1}{t}} \sim N(0, t^2 \times \frac{1}{t}) = N(0, t)$. Else, $s > 0$, then $X_t - X_s = s(B_{\frac{1}{t}} - B_{\frac{1}{s}}) + (t-s)B_{\frac{1}{t}}$. Easily $B_{\frac{1}{s}} - B_{\frac{1}{t}} \sim N(0, \frac{1}{s}) - \frac{1}{t})$, and $B_{\frac{1}{t}} \sim N(0, \frac{1}{t})$, and since $\frac{1}{t} < \frac{1}{s}$ we know $B_{\frac{1}{s}} - B_{\frac{1}{t}} \perp B_{\frac{1}{t}}$. So $X_t - X_s \sim N(0, s^2(\frac{1}{s} - \frac{1}{t}) + (t-s)^2\frac{1}{t}) = N(0, t-s)$.

Second let $\mathcal{G}_t := \sigma(X_s : 0 \leq s \leq t)$, we need to check $X_t - X_s \perp \mathcal{G}_s, \forall 0 \leq s \leq t$. For $s = 0$ we get $\mathcal{G}_s = \{\emptyset, \Omega\}$, so it's obvious. Now assume $s > 0$. Then $\mathcal{G}_s = \sigma(B_{\frac{1}{r}} : 0 \leq r \leq s)$. Only need to prove for any finite set $I \subset [0, s]$, we have $X_t - X_s \perp \sigma(B_{\frac{1}{r}} : r \in I)$. Only need to check $\mathbb{E}(B_{\frac{1}{r}}(X_t - X_s)) = 0$ because they are all normal distributed random variable. Easily $\mathbb{E}(B_{\frac{1}{r}}X_t) = \mathbb{E}((B_{\frac{1}{r}} - B_{\frac{1}{t}})tB_{\frac{1}{t}}) + \mathbb{E}(tB_{\frac{1}{t}}^2) = 1$, and $\mathbb{E}(B_{\frac{1}{r}}X_s) = \mathbb{E}((B_{\frac{1}{r}} - B_{\frac{1}{s}})sB_{\frac{1}{s}}) + \mathbb{E}(sB_{\frac{1}{s}}^2) = 1$ So $\mathbb{E}(B_{\frac{1}{r}}(X_t - X_s)) = 0$.

Finally we need to check $X_t$ is continuous a.s. Easily for $t \neq 0$ we know $X_t$ is continuous at $t$. Only need to check $X_t$ is continuous at 0 with probability 1. Easily to know $(-B_t : t \geq 0)$ is standard Brownian motion, too. So $\limsup_{t\to\infty} \frac{-B_t}{\sqrt{2t\log\log t}} = 1$. So $\limsup_{t\to\infty} \frac{|B_t|}{2t\log\log t} = 1$. So $\limsup_{t\to 0+} |X_t| = \lim_{t\to\infty} |\frac{1}{t}B_t| \leq \limsup_{t\to\infty} \frac{\sqrt{2t\log\log t}}{t} = 0$. So $\lim_{t\to 0+} |X_t| = 0$. So $X_t$ is continuous with probability 1. □

## 2.4 普瓦松过程

**Definition 15.** $(N_t : t \geq 0)$ 是非负整数不降随机过程，$\alpha \geq 0$ 满足:

1. $\forall s, t \geq 0$，$N_{s+t} - N_s \sim P(\alpha t)$，即：$\mathbb{P}(N_{s+t} - N_s = k) = \frac{\alpha^k t^k}{k!}\mathrm{e}^{-\alpha t}$;
2. $\forall 0 \leq t_0 < \cdots < t_n$，$N_0, N_{t_1} - N_{t_0}, \cdots, N_{t_n} - N_{t_{n-1}}$ 相互独立。

称 $(N_t : t \geq 0)$ 是**普瓦松过程**，参数为 $\alpha$。

### 2.4.1 跳跃间隔时间

$(N_t : t \geq 0)$ 以 $\alpha$ 为参数的普瓦松过程，$S_0 = 0, n \geq 1$，$S_n = \inf\{t \geq 0 : N_t - N_0 \geq 0\}$，$\eta_n = S_n - S_{n-1}$。$S_n$ 是 $(N_t : t \geq 0)$ 第 $n$ 次跳跃等待时间，$\eta_n$ 第 $n-1$ 次跳跃到第 $n$ 次跳跃的间隔时间。

**Theorem 8.** $\{\eta_n : n \geq 1\}$ 独立同分布，服从 $Exp(\alpha)$。$S_n, n \geq 1$，服从 $\Gamma(1, \alpha)$。证明见 *P19.*

### 2.4.2 轨道重构

**Theorem 9.** $\{\eta_n : n \geq 1\}$ 独立同分布，服从 $Exp(\alpha), \gamma > 0$。$S_0 = 0, S_n = \sum_{k=1}^n \eta_k$，则:

$$N_t = \sum_{n=1}^{\infty} \mathbb{1}_{S_n \leq t} = \sup\{n \geq 0 : S_n \leq t\}.$$

则随机过程 $(N_t : t \geq 0)$ 是以参数为 $\alpha$ 的普瓦松过程。

### 2.4.3 长时间极限行为

$(N_t : t \geq 0)$ 以 $\alpha$ 为参数的普瓦松过程。

**Theorem 10** (普瓦松过程的强大数定律)**.** $\lim_{t\to\infty} \frac{N_t}{t} \overset{a.s.}{=} \alpha$ 见 *p23.*

**Theorem 11** (普瓦松过程的中心极限定理)**.** $\lim_{t\to\infty} \mathbb{P}(\frac{N_t-\alpha t}{\sqrt{\alpha t}} \leq x) = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{x} \mathrm{e}^{-\frac{y^2}{2}}dy$。见 *p23.*

**Corollary 1.** $s, x > 0$，$\lim_{\lambda\to\infty} \mathrm{e}^{-\lambda s}\sum_{k\leq\lambda x} \frac{(\lambda s)^k}{k!} = \mathbb{1}_{0<s<t} + \frac{1}{2}\mathbb{1}_{\{s=x\}}$。见 *p24.*

**Theorem 12** (拉普拉斯变换的反演公式)**.** $\xi$ 是非负随机变量，$L$ 为其拉普拉斯变换，则 $\forall x > 0$，

$$\lim_{\lambda\to\infty}\sum_{k\leq\lambda x}\frac{(-\lambda)^k}{k!}\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}L(\lambda) = \mathbb{P}(\xi < x) + \frac{1}{2}\mathbb{P}(\xi = x)$$

见 *p24.*

### 2.4.4 复合普瓦松过程

**Definition 16.** $\mu$ 是 $\mathbb{R}$ 上概率 $\mu(\{0\}) = 0$。$(N_t : t \geq 0)$ 以 $\alpha \geq 0$ 为参数的零初值普瓦松过程，$\{\xi_n : n \geq 1\}$ 与 $N_t$ 独立，具有相同分布 $\mu$，$X_0$ 与 $(N_t), \{\xi_n\}$ 独立。令：$X_t = X_0 + \sum_{n=1}^{N_t}\xi_n, t \geq 0$，则 $(X_t : t \geq 0)$ 是以 $\alpha$ 为跳跃速度，$\mu$ 为跳跃分布的**复合普瓦松过程**。

**Theorem 13** (复合普瓦松过程的性质)**.** $(X_t : t \geq 0)$ 为如上定义的复合普瓦松过程，则复合普瓦松过程的性质如下：

1. $\forall s, t \geq 0, \theta \in \mathbb{R}$,
$$\mathbb{E}\mathrm{e}^{\mathrm{i}\theta(X_{s+t}-X_s)} = \exp(\alpha t\int_{\mathbb{R}}(\mathrm{e}^{\mathrm{i}\theta x} - 1)\mu(dx))$$
;

2. $\forall 0 \leq t_0 < \cdots < t_n, X_{t_0}, X_{t_1} - X_{t_0}, \cdots, X_{t_n} - X_{t_{n-1}}$ 相互独立。

**Lemma 2.** *Assume* $(N_t : t \geq 0)$ *is a random process, and* $\mathbb{P}(\forall t \in [0,\infty), \lim_{s\to t+} N_s = N_t, \forall t \in (0,\infty), \lim_{s\to t-} N_s \in \mathbb{R}) = 1$*. Assume* $\alpha > 0$*, and* $\forall t, s \geq 0$*, we have* $N_{t+s} - N_s \sim Possion(\alpha t)$*. Then* $(N_t : t \geq 0)$ *is a Possion process* $\iff \forall 0 \leq s \leq t, N_t - N_s \perp \mathcal{F}_s$*, where* $\mathcal{F}_s := \sigma(N_x : x \leq s)$*.*

证明. “$\implies$”: To prove $N_t - N_s \perp \mathcal{F}_s$, only need to prove for $t_1 < t_2 < \cdots < t_{n-1} = s < t = t_n$, we have $N_t - N_s \perp \sigma(N_{t_k} : k = 1, \cdots, n-1)$. Easily $N_t - N_s \perp \sigma(N_{t_{k+1}} - N_{t_k}, N_{t_1} : k = 1, \cdots, n-2) = \sigma(N_{t_k} : k = 1, \cdots, n-1)$, so $N_t - N_s \perp \mathcal{F}_s$.

“$\impliedby$”: For $t_1 < \cdots < t_n$, we need to prove $N_{t_1}, N_{t_{k+1}} - N_{t_k} : 1 \leq k \leq n-1$ are independent. Use MI to $n$. When $n = 1$ it's obvious. Assume we have proved it for certain $n \geq 1$, now consider $n+1$. Since $N_{t_{k+1}} - N_{t_k} \in \mathcal{F}_{t_n}, k = 1, \cdots, n-1$, we have $N_{t_{n+1}} - N_{t_n} \perp \sigma(N_{t_1}, N_{t_{k+1}} - N_{t_k} : k = 1, \cdots, n-1)$. So $\mathbb{P}(N_{t_1} \in A_1, N_{t_{k+1}} - N_{t_k} \in A_{k+1}, k = 1, \cdots, n) = \mathbb{P}(N_{t_1} \in A_1, N_{t_{k+1}} - N_{t_k} \in A_{k+1}, k = 1, \cdots, n-1)\mathbb{P}(N_{t_{n+1}} - N_{t_n} \in A_{n+1})$. By Induction assumption we get $\mathbb{P}(N_{t_1} \in A_1, N_{t_{k+1}} - N_{t_k} \in A_{k+1}, k = 1, \cdots, n) = \mathbb{P}(N_{t_1} \in A_1)\prod_{k=1}^{n}\mathbb{P}(N_{t_{k+1}} - N_{t_k} \in A_{k+1})$. So finally we get $N_{t_1}, N_{t_{k+1}} - N_{t_k} : 1 \leq k \leq n$ are independent. □

PROBLEM 15 Assume $(N_t : t \geq 0)$ is a Possion process with parameter $\alpha$. Let $P(t) := \mathbb{P}(2 \nmid N_t), Q(t) := \mathbb{P}(2 \mid N_t)$. Prove that $P(t) = \mathrm{e}^{-\alpha t}\sinh(\alpha t), Q(t) = \mathrm{e}^{-\alpha t}\cosh(\alpha t)$.

SOLUTION. Easily to get

$$P(t) = \sum_{k=0}^{\infty} \mathbb{P}(N_t = 2k+1) = \sum_{k=0}^{\infty} \frac{(\alpha t)^{2k=1}}{(2k+1)!}\mathrm{e}^{-\alpha t}$$

Noting that $\sinh(\alpha t) = \frac{1}{2}\left(\sum_{k=0}^{\infty} \frac{(\alpha t)^n}{n!} - \sum_{k=0}^{\infty} \frac{(-\alpha t)^n}{n!}\right) = \sum_{k=0}^{\infty} \frac{(\alpha t)^{2k+1}}{(2k+1)!}$, we easily obtain $P(t) = \mathrm{e}^{-\alpha t}\sinh(\alpha t)$. Noting $P(t) + Q(t) = 1$, we easily get $Q(t) = 1 - P(t) = \mathrm{e}^{-\alpha t}\cosh(\alpha t)$. □

PROBLEM 16 Assume $(N_t : t \geq 0)$ is a Possion process with parameter $\alpha$. Prove that $\lim_{t\to\infty} \frac{N_t}{t} = \alpha, a.s.$.

**Lemma 3.** *Assume $(N_t : t \geq 0)$ is a Possion process with parameter $\alpha$. Then $\mathbb{P}(\forall 0 \leq s \leq t, N_s \leq N_t) = 1$.*

证明. For $s, t \in \mathbb{Q} \wedge 0 \leq s \leq t$, we have $\mathbb{P}(N_s > N_t) = 0$ since $N_t - N_s \sim Possion(\alpha(t-s))$. So we get $\mathbb{P}(\exists s, t \in \mathbb{Q}, 0 \leq s \leq t, N_s > N_t) = 0$.

Now we will prove $\exists s, t \in \mathbb{R}, 0 \leq s \leq t, N_s > N_t \implies \exists a, b \in \mathbb{Q}, 0 \leq a \leq b, N_a > N_b$. Let $a_n = \frac{\lceil ns \rceil}{n}, b_n = \frac{\lceil nt \rceil}{n}$. Then $\lim a_n = s, \lim b_n = t$. Easily $a_n \geq s, b_n \geq t$. So since $N$ is continuous we get $\lim N_{a_n} = N_s, \lim N_{b_n} = N_t$. Since $N_s > N_t$, we get $\exists n, N_{a_n} > N_{b_n}$. Let $a = a_n, b = b_n$ will work.

So $\mathbb{P}(\forall 0 \leq s \leq t, N_s \leq N_t) = 1 - \mathbb{P}(\exists 0 \leq s \leq t, N_s > N_t) = 1 - \mathbb{P}(\exists s, t \in \mathbb{Q}, 0 \leq s \leq t, N_s > N_t) = 1 - 0 = 1$. □

SOLUTION. Consider $N_n : n \in \mathbb{N}$. Let $X_n := N_n - N_{n-1}, n \geq 1$. Then easily $(X_n : n \in \mathbb{N}^+)$ is i.i.d sequence and $X_1 \sim Possion(\alpha)$. So from the strong law of large numbers we get $\lim_{n\to\infty} \frac{N_n}{n} = \alpha$. From Lemma 3 we get $\frac{\lfloor t \rfloor}{t}\frac{N_{\lfloor t \rfloor}}{\lfloor t \rfloor} \leq \frac{N_t}{t} \leq \frac{N_{\lceil t \rceil}}{\lceil t \rceil}\frac{\lceil t \rceil}{t}, \forall t \in \mathbb{R}$, let $t \to \infty$ we get $\lfloor t \rfloor, \lceil t \rceil \to \infty$, and $\lfloor t \rfloor \sim t \sim \lceil t \rceil$. So finally we get $\lim_{t\to\infty} \frac{N_t}{t} = \alpha$. □

PROBLEM 17 Assume $(N_t : t \geq 0)$ is a Possion process with parameter $\alpha > 0$. Prove that $\frac{N_t - \alpha t}{\sqrt{\alpha t}} \xrightarrow{d} N(0, 1)$.

SOLUTION. Consider $N_n : n \in \mathbb{N}$. Let $X_n := N_n - N_{n-1}, n \geq 1$. Then easily $(X_n : n \in \mathbb{N}^+)$ is i.i.d sequence and $X_1 \sim Possion(\alpha)$. Easily $\mathbb{V}(X_n) = \alpha < \infty, \mathbb{E}(X_n) = \alpha$. So from the central limit theorem we get $\frac{N_n - \alpha n}{\sqrt{\alpha n}} \xrightarrow{d} N(0, 1)$. Noting $\frac{N_t - \alpha t}{\sqrt{\alpha t}} = \frac{N_{\lfloor t \rfloor} - \alpha \lfloor t \rfloor}{\sqrt{\alpha \lfloor t \rfloor}}\frac{\sqrt{\lfloor t \rfloor}}{\sqrt{t}} + \frac{N_t - N_{\lfloor t \rfloor} - \alpha(t - \lfloor t \rfloor)}{\sqrt{\alpha t}}$. Let $t \to \infty$ we get $\lfloor t \rfloor, \to \infty$, and $\lfloor t \rfloor \sim t$. Noting $N_t - N_{\lfloor t \rfloor} \stackrel{d}{=} N_{t - \lfloor t \rfloor}$, and $t - \lfloor t \rfloor \leq 1$, we easily get $\frac{N_t - N_{\lfloor t \rfloor}}{\sqrt{\alpha t}} \xrightarrow{d} 0$. Easily $\frac{\alpha(t - \lfloor t \rfloor)}{\alpha t} \to 0$, so finally we get that $\frac{N_t - \alpha t}{\sqrt{\alpha t}} \xrightarrow{d} N(0, 1)$ □

PROBLEM 18 Assume $(X_t : t \geq 0), (Y_t : t \geq 0)$ are two independent Possion processes with parameter $\alpha, \beta$ respectively. Prove that $(X_t + Y_t : t \geq 0)$ is Possion process with parameter $\alpha + \beta$.

SOLUTION . Write $Z_t = X_t + Y_t$. First we prove $Z_{t+s} - Z_s \sim Possion((\alpha+\beta)t)$. Since $X_{t+s} - X_s \sim Possion(\alpha t), Y_{t+s} - Y_s \sim Possion(\beta t)$, and $X_{t+s} - X_s \perp Y_{s+t} - Y_s$, easily to get $Z_{t+s} - Z_s = X_{t+s} - X_s + Y_{s+t} - Y_s \sim Possion((\alpha+\beta)t)$.

Second we prove $\forall 0 \le s \le t, Z_t - Z_s \perp \mathcal{H}_s$, where $\mathcal{H}_s = \sigma(Z_x : 0 \le x \le s)$. Easily $Z_t - Z_s \in \sigma(X_t - X_s, Y_t - Y_s)$. Easily $X_t - X_s \perp \mathcal{F}_s := \sigma(X_x : 0 \le x \le s)$ since $(X_x : x \ge 0)$ is Possion process. Since $(X_x : x \ge 0) \perp (Y_x : x \ge 0)$, we get $X_t - X_s \perp \mathcal{G}_s := \sigma(Y_x : 0 \le x \le s)$. For the same reason we get $Y_t - Y_s \perp \mathcal{F}_s, \mathcal{G}_s$. So we get $Z_t - Z_s \perp \sigma(F_s, G_s) \supset \mathcal{H}_s$.

Finally, we prove that $\mathbb{P}(\forall t \in [0,\infty), \lim_{s\to t+} Z_s = Z_t, \forall t \in (0,\infty), \lim_{s\to t-} Z_s \in \mathbb{R}) = 1$ Since $Z_t = X_t + Y_t$, and $\mathbb{P}(\forall t \in [0,\infty), \lim_{s\to t+} Y_s = Y_t, \forall t \in (0,\infty), \lim_{s\to t-} Y_s \in \mathbb{R}) = 1$, $\mathbb{P}(\forall t \in [0,\infty), \lim_{s\to t+} X_s = X_t, \forall t \in (0,\infty), \lim_{s\to t-} X_s \in \mathbb{R}) = 1$, obvious we get the requirement.

All in all, $(X_t + Y_t : t \ge 0)$ is a Possion process with parameter $\alpha + \beta$. □

PROBLEM 19 Assume $(\xi_n : n \in \mathbb{N}^+)$ is a sequence of i.i.d. random variable ranging in $\mathbb{Z}^d$. Let $X_n = X_0 + \sum_{k=1}^n \xi_k$, and $X_0 \perp (\xi_n : n \in \mathbb{N}^+)$ ranging in $\mathbb{Z}^d$, too. Assume $(N_t : t \ge 0)$ is a Possion process with parameter $\alpha > 0$. Discuss $\frac{X_{N_t}}{t}$ when $t \to \infty$.

SOLUTION . First we prove that $\lim_{t\to\infty} N_t = \infty, a.s.$. We have $\mathbb{P}(\sup_t N_t \ge n) \ge \mathbb{P}(N_t \ge n), \forall t, \forall n \in \mathbb{N}$. Easily $\lim_{t\to\infty} \mathbb{P}(N_t \ge n) = 1$, so $\mathbb{P}(\sup_t N_t \ge n) = 1, \forall n \in \mathbb{N}$. So $\mathbb{P}(\sup_t N_t = \infty) = 1$. Noting Lemma 3 we easily get $\mathbb{P}(\lim_{t\to\infty} N_t = \infty) = 1$.

Now we can decompose $\frac{X_{N_t}}{t}$ into $\frac{X_{N_t}}{N_t}\frac{N_t}{t}$. We have proved that $\frac{N_t}{t} \to \alpha, a.s.$ in Problem 21, so we only need to find $\frac{X_{N_t}}{N_t}$. Since $N_t \to \infty, a.s.$, we only need to find $\frac{X_n}{n}$ when $n \to \infty$.

If $\mathbb{E}(\xi_1)$ exists, then easily $\frac{X_n}{n} \to \mathbb{E}(\xi_1), a.s.$. Then we easily get $\frac{X_{N_t}}{t} \to \alpha\mathbb{E}(\xi_1), a.s.$. □

## 2.5 普瓦松随机测度

### 2.5.1 定义和存在性

$(E, \mathcal{E})$ 为可测空间，$\mu$ 为 $(E, \mathcal{E})$ 上的 $\sigma$ 有限测度。

**Definition 17.** $\{X(B) : B \in \mathcal{E}\}$ 为取非负整数值随机过程，满足:

1. $\forall B \in \mathcal{E} : \mu(B) < \infty$，则 $\mathbb{E}(X(B)) = \mu(B)$。
2. $\forall \{B_n : n \ge 1\} \in \mathcal{E}$ 两两不交，则 $X(\bigcup_{k=1}^\infty B_k) = \sum_{k=1}^\infty X(B_k)$。

称 $\{X(B) : B \in \mathcal{E}\}$ 为以 $\mu$ 为强度的**整数值随机测度**。

**Definition 18.** $\{X(B) : B \in \mathcal{E}\}$ 为整数值随机测度，满足:

1. $\forall B \in \mathcal{E} : \mu(B) < \infty$，则 $X(B) \sim P(\mu(B))$，即: $\mathbb{P}(X(B) = k) = \frac{\mu(B)^k}{k!}\mathrm{e}^{-\mu(B)}, k = 0, \cdots, n, \cdots$。
2. $\forall \{B_n : n \ge 1\} \in \mathcal{E}$ 两两不交，则 $\{X(B_k) : k \in \mathbb{N}^+\}$ 相互独立。

称 $\{X(B) : B \in \mathcal{E}\}$ 为以 $\alpha$ 为强度的**普瓦松随机测度**。

**Theorem 14** (普瓦松随机测度的充要条件)**.** $X$ 为 $(E,\mathcal{E})$ 上以 $\mu$ 为强度的整数值随机测度，则 $X$ 为普瓦松随机测度的充要条件是 $\forall n\in\mathbb{N}^+,\xi_k\in\mathbb{R},B_k\in\mathcal{E},k=1,\cdots,n$，$B_i\cap B_j=\varnothing,i\neq j$，当 $\mu(B_k)<\infty,k=1,\cdots,n$，则

$$\mathbb{E}\exp(\mathrm{i}\sum_{k=1}^{n}\theta_k X(B_k))=\exp(\sum_{k=1}^{n}(\mathrm{e}^{\mathrm{i}\theta_k}-1)\mu(B_k))$$

*见 p28.*

**Theorem 15** (普瓦松随机测度的存在性)**.** $\mu$ 为非零有限测度，$\eta\sim P(\mu(E))$，$\{\xi_k:k\in\mathbb{N}^+\}$ *i.i.d.* 服从 $\mu(E)^{-1}\mu$。$\eta,\xi_1,\cdots,\xi_n,\cdots$ 相互独立。令 $X=\sum_{j=1}^{\eta}\delta_{\xi_j}$，则 $X$ 为以 $\mu$ 为强度的普瓦松随机测度。见 *p29.*

**Theorem 16.** $\mu$ 为 $\sigma$ 有限测度，$\{E_k:k\in\mathbb{N}^+\}\subset\mathcal{E},\mu(E_k)<\infty,k\in\mathbb{N}^+$，$E=\bigcup_{k\in\mathbb{N}^+}E_k$，$E_i\cap E_j=\varnothing,i\neq j$。则存在 $X_k$ 为 $E_k$ 上的普瓦松随机测度强度为 $\mu_k:=\mu|_{E_k},k\in\mathbb{N}^+$。令 $X=\sum_{j=1}^{\infty}X_j$，则 $X$ 为以 $\mu$ 为强度的普瓦松随机测度。见 *p29.*

### 2.5.2 积分与补偿的测度

**Theorem 17** (普瓦松随机测度的充要条件 2)**.** $X$ 为 $(E,\mathcal{E})$ 上以 $\mu$ 为强度的整数值随机测度，则 $X$ 为普瓦松随机测度的充要条件是 $\forall f\in\mathbb{R}^{\mathbb{R}}:\mu(f)<\infty$,，则

$$\mathbb{E}\exp(\mathrm{i}X(f))=\exp(\int_E(\mathrm{e}^{\mathrm{i}f(x)}-1)\mu(dx))$$

*见 p28.*

### 2.5.3 应用

**Theorem 18** (复合普瓦松过程构造)**.** $\nu$ 为 $\mathbb{R}$ 上非零有限测度。$N(ds,dz)$ 为 $(0,\infty)\times\mathbb{R}$ 上以 $ds\nu(dz)$ 为强度的普瓦松随机测度，$ds$ 为勒贝格测度，$X_0$ 与 $N(ds,dz)$ 独立，

$$X_t=X_0+\iint_{(0,t]\times\mathbb{R}}zN(ds,dz),t\geq 0$$

则 $(X_t:t\geq 0)$ 具有跳跃速率 $\alpha$ 和跳跃分布 $\mu:=\nu(\mathbb{R})^{-1}\nu$ 的复合普瓦松过程。

**Theorem 19** (复合普瓦松过程构造 2)**.** $\mu$ 为 $\mathbb{R}$ 上非零 $\sigma$ 有限测度，$\mu(\{0\})=0$，$q$ 为 $\mathbb{R}$ 上非负博雷尔可测函数，$0<\beta:=\mu(q)<\infty$。$N(ds,dz,du)$ 为 $(0,\infty)\times\mathbb{R}\times(0,\infty)$ 上以 $ds\mu(dz)du$ 为强度的普瓦松随机测度，$ds$ 为勒贝格测度，$X_0$ 与 $N(ds,dz)$ 独立，

$$X_t=X_0+\iiint_{(0,t]\times\mathbb{R}\times[0,q(z)]}zN(ds,dz,du),t\geq 0$$

则 $(X_t:t\geq 0)$ 具有跳跃速率 $\beta$ 和跳跃分布 $\beta^{-1}q(z)\mu(dz)$ 的复合普瓦松过程。

PROBLEM 20 Assume $(N_t:t\geq 0)$ is Possion process with parameter $\alpha$, and $\{\xi_n:n\in\mathbb{N}^+\}$ is a sequence of i.i.d random variable. More over, assume $(N_t:t\geq 0)\perp\{\xi_n:n\in\mathbb{N}^+\}$. Let $X_t=\sum_{k=1}^{N_t}\xi_k$. Let $r>0$, prove that:

1. $(N_{t+r} - N_r : t \geq 0)$ is Possion process.

2. $\{\xi_{N_r+n} : n \in \mathbb{N}^+\}$ is also i.i.d sequence with the same distribution of $\{\xi_n : n \in \mathbb{N}^+\}$.

3. $(N_{t+r} - N_r : t \geq 0) \perp (\xi_{N_r+k} : k \in \mathbb{N}^+)$.

4. For $0 = t_0 < t_1 < \cdots < t_n$, we have $(X_{t_1}, X_{t_{k+1}} - X_{t_k} : k = 1, 2, \cdots, n-1)$ are independent.

SOLUTION. 1. Let $\mathcal{F}_t := \sigma(N_s : 0 \leq s \leq t)$ and $\mathcal{G}_t := \sigma(N_{r+s} - N_r : 0 \leq s \leq t)$. For $0 \leq s \leq t$, we have $N_{t+r} - N_r - (N_{s+r} - N_r) = N_{t+r} - N_{s+r} \sim Possion(\alpha(t-s))$. And easily to know $N_{r+s} - N_r \in \mathcal{F}_{r+s}$, so $\mathcal{G}_t \subset \mathcal{F}_{t+r}, \forall t \geq 0$. Since $(N_t : t \geq 0)$ is Possion process, easily $\mathcal{F}_{s+r} \perp N_{t+r} - N_{s+r}$. Since $\mathcal{G}_t \subset \mathcal{F}_{t+r}$, we obtain $\mathcal{G}_t \perp N_{t+r} - N_{s+r} = N_{t+r} - N_r - (N_{s+r} - N_r)$. Easily since $N_t$ is right-continuous we get $N_{t+r} - N_r$ is right-continuous. For the same reason, we know $\forall s \in [0, \infty), \lim_{t \to s-} N_{t+r} - N_r$ exists. So $(N_{t+r} : t \geq 0)$ is Possion process.

2. Only need to prove that for $m \in \mathbb{N}^+$, the distribution of $(\xi_{N_r+k} : 1 \leq k \leq m)$ is same as that of $(\xi_k : 1 \leq k \leq m)$. For $A_1, A_2, \cdots, A_m \in \mathcal{B}$, we have:

$$
\begin{aligned}
&\mathbb{P}(\xi_{N_r+k} \in A_k, 1 \leq k \leq m) \\
&= \sum_{t=0}^{\infty} \mathbb{P}(\xi_{N_r+k} \in A_k, 1 \leq k \leq m, N_r = t) \\
&= \sum_{t=0}^{\infty} \mathbb{P}(\xi_{t+k} \in A_k, 1 \leq k \leq m, N_r = t) \\
(N_r \perp (\xi_n : n \in \mathbb{N}^+)) &= \sum_{t=0}^{\infty} \mathbb{P}(\xi_{t+k} \in A_k, 1 \leq k \leq m)\mathbb{P}(N_r = t) \\
&= \sum_{t=0}^{\infty} \prod_{k=1}^{m} \mathbb{P}(\xi_{k+t} \in A_k)\mathbb{P}(N_r = t) \\
&= \sum_{t=0}^{\infty} \prod_{k=1}^{m} \mu(A_k)\mathbb{P}(N_r = t) \\
&= \prod_{k=1}^{m} \mu(A_k)
\end{aligned} \tag{6}
$$

where $\mu$ is the distribution of $\xi_1$. So we get for $m \in \mathbb{N}^+$, the distribution of $(\xi_{N_r+k} : 1 \leq k \leq m)$ is same as that of $(\xi_k : 1 \leq k \leq m)$.

3. We know that $\forall t \in \mathbb{N}^+, \xi_{N_r+t} \in \sigma(N_r, \xi_k : k \in \mathbb{N}^+)$. So $\sigma(\xi_{N_r+k} : k \in \mathbb{N}^+) \subset \sigma(N_r, \xi_k : k \in \mathbb{N}^+)$. Since $(N_t : t \geq 0)$ is Possion process, we get that $N_{t+r} - N_r \perp N_r, \forall t \geq 0$. So $\sigma(N_{t+r} - N_r : t \geq 0) \perp N_r$. Easily $\sigma(N_{t+r} - N_r : t \geq 0) \perp \sigma(\xi_k : k \in \mathbb{N}^+)$, so finally we get that $\sigma(N_{t+r} - N_r : t \geq 0) \perp \sigma(N_r, \xi_k : k \in \mathbb{N}^+) \supset \sigma(\xi_{N_r+k} : k \in \mathbb{N}^+)$.

4. $\forall 0=t_0<t_1<\cdots<t_n$, then $X_{t_1}=\sum_{i=1}^{N_{t_1}}\xi_i, X_{t_k}-X_{t_{k-1}}=\sum_{i=1}^{N_{t_k}-N_{t_{k-1}}}\xi_{N_{t_{k-1}}+i}\xi_i, k=2,\cdots,n$, then $\forall\{A_k\in\mathscr{E}:k=1,\cdots,n\}$,

$$
\begin{aligned}
&\mathbb{P}(\bigcap_{k=1}^{n}\sum_{i=1}^{N_{t_k}-N_{t_{k-1}}}\xi_{i+N_{t_{k-1}}}\in A_k)\\
=&\mathbb{P}(\bigcup_{0\le u_1\le\cdots\le u_n}\{\sum_{i=u_{k-1}+1}^{u_k}\xi_i\in A_k, N_{t_k}=u_k, k=1,\cdots,n\})\\
=&\sum_{0\le u_1\le\cdots\le u_n}\mathbb{P}(\sum_{i=u_{k-1}+1}^{u_k}\xi_i\in A_k, k=1,\cdots,n|N_{t_k}=u_k,k=1,\cdots,n\})\mathbb{P}(N_{t_k}=u_k,k=1,\cdots,n)\\
=&\sum_{0\le u_1\le\cdots\le u_n}\mathbb{P}(\sum_{i=u_{k-1}+1}^{u_k}\xi_i\in A_k, k=1,\cdots,n\})\mathbb{P}(N_{t_k}=u_k,k=1,\cdots,n)\\
=&\sum_{0\le u_1\le\cdots\le u_n}\prod_{k=1}^{n}\mathbb{P}(\sum_{i=u_{k-1}+1}^{u_k}\xi_i\in A_k)\prod_{j=1}^{n}\mathbb{P}(N_{t_j}=u_j)\\
=&\sum_{0\le u_1\le\cdots\le u_n}\prod_{k=1}^{n}\mathbb{P}(\sum_{i=1}^{u_k-u_{k-1}}\xi_{u_{k-1}+i}\in A_k)\prod_{j=1}^{n}\mathbb{P}(N_{t_j}-N_{t_{j-1}}=u_j-u_{j-1})\\
=&\sum_{0\le u_1\le\cdots\le u_n}\prod_{k=1}^{n}\mathbb{P}(\sum_{i=1}^{u_k-u_{k-1}}\xi_{u_{k-1}+i}\in A_k)\mathbb{P}(N_{t_k}-N_{t_{k-1}}=u_k-u_{k-1})\\
=&\sum_{u_1-u_0\in\mathbb{N}}\cdots\sum_{u_n-u_{n-1}\in\mathbb{N}}\prod_{k=1}^{n}\mathbb{P}(\sum_{i=1}^{u_k-u_{k-1}}\xi_{u_{k-1}+i}\in A_k, N_{t_k}-N_{t_{k-1}}=u_k-u_{k-1})\\
=&\prod_{k=1}^{n}\sum_{u_k-u_{k-1}\in\mathbb{N}}\mathbb{P}(\sum_{i=1}^{u_k-u_{k-1}}\xi_{u_{k-1}+i}\in A_k, N_{t_k}-N_{t_{k-1}}=u_k-u_{k-1})\\
=&\prod_{k=1}^{n}\mathbb{P}(\sum_{i=1}^{N_{t_k}-N_{t_{k-1}}}\xi_{i+N_{t_{k-1}}}\in A_k)
\end{aligned}
\tag{7}
$$

□

PROBLEM 21 Assume that $X$ is Possion random measure on $(E,\mathcal{E})$ with intensity $\mu$, which is a $\sigma$-finite measure. Assume $f:E\to\mathbb{R}$ is measurable and non-negative, prove that:

$$\mathbb{E}(\mathrm{e}^{-X(f)})=\exp\left\{-\int_E(1-\mathrm{e}^{-f(x)})\mu(\mathrm{d}x)\right\}$$

SOLUTION. Let $\mathcal{L}:=\{g\in\mathcal{M}(E,[0,\infty)):\mathbb{E}(\mathrm{e}^{-X(f)})=\exp\left(-\int_E(1-\mathrm{e}^{-f(x)})\mu(\mathrm{d}x)\right)\}$. First we prove that if $g$ is simple measurable function from $E$ to $[0,\infty)$, then $g\in\mathcal{L}$. Assume $g(x)=\sum_{k=1}^{n}a_k\mathbb{1}_{A_k}(x)$, where $A_k\in\mathcal{E}, a_k>0, A_i\cap A_j=\emptyset$. Then $\mathbb{E}(\exp(-X(g)))=\mathbb{E}(\prod_{k=1}^{n}\exp(-a_kX(A_k)))=$

$\prod_{k=1}^n \mathbb{E}(\exp(-a_k X(A_k)))$, since $X(A_k): k=1,\cdots,n$ are independent. Easily to know

$$\mathbb{E}(\exp(-a_k X(A_k))) = \sum_{i=0}^{\infty} \mathbb{P}(X(A_k)=i)\exp(-a_k i) = \sum_{i=0}^{\infty} \frac{\exp(-\mu(A_k))\mu(A_k)^i}{i!}\exp(-a_k i)$$

Noting that

$$\exp(-\int_E (1-\exp(-a_k \mathbb{1}_{A_k}(x)))\mu(\mathrm{d}x)) = \exp(\exp(-a_k)\mu(A_k)-\mu(A_k)) = \exp(-\mu(A_k))\sum_{i=0}^{\infty}\frac{(\exp(-a_k)\mu(A_k))^i}{i!}$$

we get $\mathbb{E}(\exp(-a_k X(A_k))) = \exp(-\int_E(1-\exp(-a_k\mathbb{1}_{A_k}(x)))\mu(\mathrm{d}x))$. Noting $\int_E(1-\exp(-g(x)))\mu(\mathrm{d}x) = \sum_{k=1}^n \int_E(1-\exp(-a_k\mathbb{1}_{A_k}(x)))\mu(\mathrm{d}x)$, we get $\mathbb{E}(\exp(-X(g))) = \exp\left(-\int_E(1-\exp(-g(x)))\mu(\mathrm{d}x)\right)$.

Now for non-negative function $f$, consider $f_n$ satisfy that $\forall n, f_n$ is simple, and $f_n \nearrow f$ and $f_n \geq 0$. Then easily to know $\mathbb{E}(\exp(-X(f))) = \lim_{n\to\infty}\mathbb{E}(\exp(-X(f_n))) = \lim_{n\to\infty}\exp(-\int_E(1-\exp(-f_n(x)))\mu(\mathrm{d}x)) = \exp(-\int_E(1-\exp(-f(x)))\mu(\mathrm{d}x))$. □

PROBLEM 22 Assume $\mu$ is finite measure on $(E,\mathcal{E})$, and $X$ is Possion random measure with intensity $\mu$. Assume $\phi:(E,\mathcal{E})\to(F,\mathcal{F})$ is measurable, prove that $X\circ\phi^{-1}$ is Possion random measure with intensity $\mu\circ\phi^{-1}$.

SOLUTION. Assume $B_k\in\mathcal{F},\forall k\in\mathbb{N}$ and $\forall i\neq j, B_i\cap B_j=\emptyset$. Then $X\circ\phi^{-1}(\bigcup_{k\in\mathbb{N}}B_k) = X(\bigcup_{k\in\mathbb{N}}\phi^{-1}(B_k)) = \sum_{k\in\mathbb{N}}X(\phi^{-1}(B_k))$. Since $X$ is possion random measure with intensity $\mu$, and for $B_1,\cdots,B_n\in\mathcal{F}$ and $B_i\cap B_j=\emptyset$, we have $\phi^{-1}(B_k)$ are disjoint set in $(E,\mathcal{E})$, so $\mathbb{E}(\exp(\mathrm{i}\sum_{k=1}^n\alpha_k X\circ\phi^{-1}(B_k))) = \exp(\sum_{k=1}^n(\exp(\mathrm{i}\alpha_k)-1)\mu\circ\phi^{-1}(B_k))$. So $X\circ\phi^{-1}$ is Possion random measure on $(F,\mathcal{F})$ with intensity $\mu\circ\phi^{-1}$. □

PROBLEM 23 Assume $\alpha\geq 0$, and $\mu$ is probability measure on $\mathbb{R}$ with $\mu(\{0\})=0$. Let $N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)$ is Possion random measure on $\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}^+$ with intensity $\mathrm{d}s\mu(\mathrm{d}z)\,\mathrm{d}u$. Let $Y_t = Y_0+\int_0^t\int_{\mathbb{R}}\int_0^{\alpha} zN(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)$, where $Y_0\perp N$. Prove that $(Y_t:t\geq 0)$ is compound Possion process with rate $\alpha$ and jumping distribution $\mu$.

SOLUTION. We know that $\forall t\geq 0,\forall r: 0\leq r\leq t, Y_r\in\sigma(N(B):B\subset[0,r]\times\mathbb{R}\times[0,\alpha])$. And $\forall w\geq t$, $Y_w-Y_t\in\sigma(N(B):B\subset(t,w]\times\mathbb{R}\times[0,\alpha])$. Easily $(t,w]\cap[0,r]=\emptyset$, so we get $Y_w-Y_t\perp(Y_r:0\leq r\leq t)$. Now we only need to check the distribution of $Y_t+w-Y_t$ for $t,w\geq 0$.

Easily to know that:

$$\begin{aligned}
\mathbb{E}\left(\mathrm{e}^{\mathrm{i}\theta(Y_{t+w}-Y_t)}\right) =& \mathbb{E}\exp\left(\int_s^{s+t}\int_{\mathbb{R}}\int_0^{\alpha} i\theta zN(\mathrm{d}s,\mathrm{d}u,\mathrm{d}z)\right)\\
=&\exp\left(t\alpha\int_{\mathbb{R}}(\mathrm{e}^{\mathrm{i}\theta z}-1)\mu(\mathrm{d}z)\right)\\
=&\exp(-t\alpha)\sum_{k=0}^{\infty}\frac{1}{k!}(t\alpha\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}\theta z}\mu(\mathrm{d}z))^k\\
=&\mathrm{e}^{-t\alpha}\sum_{k=0}^{\infty}\frac{(\alpha k)^k}{k!}\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}\theta z}\mu^{*k}(\mathrm{d}z)
\end{aligned}$$

So we get the result. □

PROBLEM 24 Assume $X$ is Possion random measure on $(E,\mathcal{E})$ with intensity $\mu$, a finite measure. Assume $f,g$ are non-negative measure function on $E$. Prove that:

1. $\mathbb{E}(X(f)\mathrm{e}^{-X(g)}) = \mu(f\mathrm{e}^{-g})\mathbb{E}(\mathrm{e}^{-X(g)})$.
2. $\mathbb{E}(X(f)^2\mathrm{e}^{-X(g)}) = (\mu(f^2\mathrm{e}^{-g} + \mu(f\mathrm{e}^{-g})'2))\mathbb{E}(\mathrm{e}^{-X(g)})$.

SOLUTION. 1. Let $h(\theta) := \mathbb{E}(\mathrm{e}^{-X(\theta f+g)}) = \exp\left(-\int_E(1-\mathrm{e}^{-\theta f(x)-g(x)}\mu(\mathrm{d}x))\right)$. Then

$$h'(\theta) = \mathbb{E}(X(f)\mathrm{e}^{-X(\theta f+g)}) = \exp\left(-\int_E(1-\mathrm{e}^{-\theta f(x)-g(x)}\mu(\mathrm{d}x))\right)\cdot\int_E f(x)\mathrm{e}^{-\theta f(x)-g(x)}\mu(\mathrm{d}x)$$

Since they are all non-negative, the differential is valid. Let $\theta=0$, we get $\mathbb{E}(X(f)\mathrm{e}^{-X(g)}) = \mu(f\mathrm{e}^{-g})\mathbb{E}(\mathrm{e}^{-X(g)})$.

2. Take $h$ as above, easily to get $h''(\theta) = \mathbb{E}(X(f)^2)\mathrm{e}^{-X(g)} = \exp\left(-\int_E(1-\mathrm{e}^{-\theta f(x)-g(x)}\mu(\mathrm{d}x))\right)\cdot\left(\int_E f(x)\mathrm{e}^{-\theta f(x)-g(x)}\mu(\mathrm{d}x)\right)^2 + \exp\left(-\int_E(1-\mathrm{e}^{-\theta f(x)-g(x)}\mu(\mathrm{d}x))\right)\cdot\int_E f(x)^2\mathrm{e}^{-\theta f(x)-g(x)}\mu(\mathrm{d}x)$. Let $\theta=0$, then easily $\mathbb{E}(X(f)^2\mathrm{e}^{-X(g)}) = (\mu(f^2\mathrm{e}^{-g} + \mu(f\mathrm{e}^{-g})'2))\mathbb{E}(\mathrm{e}^{-X(g)})$.

□

# 4 更新过程及其应用

## 4.1 更新过程

### 4.1.1 定义和性质

**Definition 19.** 设 $\{\xi_n : n \geq 1\}$ 是非负独立同分布随机变量序列。设 $F$ 是它们共同的的分布函数。假设 $F(0) = \mathbb{P}(\xi_n = 0) < 1$。则 $\mu := \mathbb{E}(\xi_n) > 0$。令 $S_n := \sum_{k=1}^{n} \xi_k$。由大数定律可得 $\lim_{n\to\infty} \frac{S_n}{n} = \mu$。令 $N(t) := \sum_{n=1}^{\infty} \mathbb{1}_{\{S_n \leq t\}} = \sup\{n \geq 0 : S_n \leq t\}$。称 $(N(t) : t \geq 0)$ 为**更新过程**。称 $(\xi_n : n \geq 1)$ 为**更新间隔时间**。

**Theorem 20.** *几乎必然有 $N(\infty) = \infty$。证明见教材 p74*

### 4.1.2 更新方程

**Definition 20.** 称 $m(t) := \mathbb{E}(N(t))$ 为更新过程 $(N(t) : t \geq 0)$ 的**更新函数**。计算可得 $m(t) = \sum_{n=1}^{\infty} \mathbb{P}(N(t) \geq n) = \sum_{n=1}^{\infty} \mathbb{P}(S_n \leq t) = \sum_{n=1}^{\infty} F^{*n}(t)$。

**Theorem 21.** *对于 $t \geq 0$ 有 $m(t) < \infty$。证明见教材 p76*

**Definition 21.** 设 $H$ 为 $[0, \infty)$ 上的右连续的有界变差函数，而 $F$ 为 $[0, \infty)$ 上的概率分布函数。称关于 $K$ 的方程 $K(t) = H(t) + K * F(t), t \geq 0$ 为**更新方程**。更新方程的积分形式为 $K(t) = H(t) + \int_0^t K(t-x)\,\mathrm{d}F(x))$。

**Theorem 22.** *更新方程存在唯一右连续有界变差函数解 $K$，且该解具有表达式 $K(t) = H(t) + H * m(t)$。证明见教材 p76*

**Theorem 23.** *对于 $0 \leq s \leq t$，有 $\mathbb{P}(S_{N(t)} \leq s) = 1 - F(t) + \int_0^s 1 - F(t-x)\,\mathrm{d}m(x)$。证明见教材 p77*

**Theorem 24** (瓦尔德恒等式)**.** *设 $\{\xi_n : n \geq 1\}$ 为独立同分布的随机变量序列，$\mathcal{F}_n$ 为其自然 $\sigma$-代数流。设 $\mathbb{E}(\xi_1)$ 存在。设 $\tau$ 为一个停时。则有 $\mathbb{E}(\sum_{k=1}^{\tau} \xi_k) = \mathbb{E}(\tau)\mathbb{E}(\xi_1)$。*

**Theorem 25.** *对于 $t, x > 0$，有 $\mathbb{P}(W(t) > x) = 1 - F(t+x) + \int_0^t 1 - F(t+x-y)\,\mathrm{d}m(y)$，其中 $W(T) := S_{N(t)+1} - t$ 为**待更新时间**。证明见教材 p78*

PROBLEM 25 Assume $(N(t) : t \geq 0)$ is a renewing process with renewing internal $\{\xi_n : n \geq 1\}$, $S_n = \sum_{k=1}^{n} \xi_k$, $N(t) := \sup\{n : S_n \leq t\}$, calculate $g(t) := \mathbb{E}(N(t)^2)$.

SOLUTION. Let $T_1 = \xi_1$, then $g(t) = \mathbb{E}(\mathbb{E}(N(t)^2 \mid T_1)) = \int_0^t \mathbb{E}(N(t)^2 \mid T_1 = x)dF(x)$. By the independence,

$$\mathbb{E}(N(t)^2 \mid T_1 = x) = \begin{cases} 0 & , x > t \\ \mathbb{E}((1 + N(t-x))^2) & , x \leq t \end{cases}$$

That is $\mathbb{E}(N(t)^2 \mid T_1 = x) = \begin{cases} 0 & ,x > t \\ 1 + 2m(t-x) + g(t-x) & ,x \le t \end{cases}$. Therefore,

$$g(t) = F(t) + 2\int_0^t m(t-x)dF(x) + \int_0^t g(t-x)dF(x)$$

So $g(t) = 2m(t) - F(t) + \int_0^t g(t-x)dF(x)$. Thus, $g(t) = 2m(t) - F(t) + (2m - F) * m(t)$, so $g(t) = m(t) + 2m * m(t)$. □

PROBLEM 26 Assume renewing internal time obey $U(0,1)$. $0 < t < 1$, calculate the distrubution of $S_{N(t)}$ and $\mathbb{E}(S_{N(t)})$.

SOLUTION. By calculating, $m(t) = \mathrm{e}^t - 1, 0 < t < 1$. $\forall 0 \le s \le t < 1$,

$$\mathbb{P}(S_{N(t)} \le s) = 1 - t + \int_0^s (1 - t + x)\mathrm{e}^x dx = 1 - (t-s)\mathrm{e}^s$$

Therefore,

$$\mathbb{E}(S_{N(t)}) = \int_0^t s(1-t+s)\mathrm{e}^s ds = \mathrm{e}^t - t - 1$$

□

PROBLEM 27 Assume renewing internal time obey random variable $X$ with distrubution function $F$. Let $\gamma_t = S_{N(t)+1} - t$ be the rest lifetime at time $t$. Prove:

$$\mathbb{P}(\gamma_t > z) = 1 - F(t+z) + \int_0^t (1 - F(t+z-x))dm(x)$$

SOLUTION. Let $A_z(t) = \mathbb{P}(\gamma_t > z)$, then

$$\mathbb{P}(\gamma_t > z \mid \xi_1 = x) = \begin{cases} 1 & ,x > t+z \\ 0 & ,t < x \le t+z \\ A_z(t-x) & ,0 < x \le t \end{cases}$$

Then,

$$A_z(t) = \int_0^\infty \mathbb{P}(\gamma_t > z \mid \xi_1 = x)dF(x) = 1 - F(t+z) + \int_0^t A_z(t-z)dF(x)$$

Thus,

$$A_z(t) = 1 - F(t+z) + \int_0^\infty (1 - F(t+z-x))dm(x)$$

□

PROBLEM 28 One kind of devices are replaced as they are worn out. Let the lifetime of the devices be sequences $\{\xi_n : n \ge 1\}$, and let $S_n = \sum_{k=1}^n \xi_k$, $N(t) = \sup\{n : S_n \le t\}$. $L(t) = S_{N(t)+1} - S_{N(t)}$. Prove: $\mathbb{P}(L(t) > x) \ge \mathbb{P}(\xi_1 > x)$.

SOLUTION. When $t \le x$, easy to get that $\mathbb{P}(L(t) > x) = \mathbb{P}(\xi_1 > x)$. Now we assume $t > x$.

$$\begin{aligned}
\mathbb{P}(L(t) > x) =& \sum_{k=0}^{\infty} \mathbb{P}\left(\xi_{k+1} > x, N(t) = k\right) \\
=& \sum_{k=0}^{\infty} \mathbb{P}\left(\xi_{k+1} > x, S_k \le t, \xi_{k+1} > t - S_k\right) \\
=& \sum_{k=1}^{\infty} \mathbb{P}\left(\xi_{k+1} > x, t - x < S_k \le t\right) \\
& \quad + \sum_{k=0}^{\infty} \mathbb{P}\left(\xi_{k+1} > t - S_k, S_k \le t - x\right) \\
=& \mathbb{P}\left(\xi_1 > x\right) \mathbb{E}[(N(t) - N(t-x))] + \mathbb{P}(N(t) = N(t-x)) \\
=& \mathbb{P}\left(\xi_1 > x\right) + \mathbb{P}\left(\xi_1 > x\right) \mathbb{E}[(N(t) - N(t-x)) - 1] \\
& + \mathbb{P}(N(t) = N(t-x)) \\
=& \mathbb{P}\left(\xi_1 > x\right) + \mathbb{P}\left(\xi_1 > x\right) \mathbb{E}\left[(N(t) - N(t-x) - 1) 1_{\{N(t) > N(t-x)\}}\right] \\
& - \mathbb{P}\left(\xi_1 > x\right) \mathbb{E}\left(1_{\{N(t) = N(t-x)\}}\right) + \mathbb{P}(N(t) = N(t-x)) \\
\ge& \mathbb{P}\left(\xi_1 > x\right).
\end{aligned} \tag{8}$$

□

PROBLEM 29 Toss a coin until we get two successively head, call it a renew. We toss the coin $k$ times, call the number of renews $N(k)$. Find the distribution and expectation of interval time $T$

SOLUTION. Let $p_n := \mathbb{P}(T = n)$. Then $p_1 = 0, p_2 = \frac{1}{4}$. Easy to find that $p_{n+2} = \frac{1}{2}p_{n+1} + \frac{1}{4}p_n$. The characteristic equation of this sequence is $x^2 = \frac{1}{2}x + \frac{1}{4}$. The roots are $x_1 = \frac{1+\sqrt{5}}{4}, x_2 = \frac{1-\sqrt{5}}{4}$. So $p_n = Ax_1^n + Bx_2^n$. By $p_1, p_2$, easy to get that $p_n = \frac{1}{2\sqrt{5}}\left(x_1^{n-1} - x_2^{n-1}\right)$. So easily $\mathbb{E}(T) = \sum_{n=1}^{\infty} np_n = 6$. □

## 4.2 长程极限行为

### 4.2.1 基本更新定理

**Theorem 26.** *几乎必然的有* $\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu}$。*证明见教材 p80*

**Theorem 27** (基本更新定理)**.** *有* $\lim_{t\to\infty} \frac{m(t)}{t} = \frac{1}{\mu}$。*证明见教材 p81*

### 4.2.2 中心极限定理

**Theorem 28** (中心极限定理)**.** *假设* $\mathbb{D}(\xi_1) < \infty$，*记* $\mu = \mathbb{E}(\xi_1), \sigma^2 = \mathbb{D}(\xi_1)$。*对于* $x \in \mathbb{R}$，*有*

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{N(t) - \frac{t}{\mu}}{\sqrt{\frac{t\sigma^2}{\mu^3}}} \le x\right) = \Phi(x) \tag{9}$$

其中 $\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}\mathrm{e}^{\frac{-y^2}{2}}\,\mathrm{d}y$ 为正态分布的分布函数。证明见教材 *p82*

PROBLEM 30 A radio is powered by one battery, the lifetime of the battery obey the distribution of exponential distribution with parameter $\lambda=\frac{1}{30}$. In long term, in which frequence should we change the battery?

SOLUTION . Easy to get that $\lim_{t\to\infty}\frac{m(t)}{t}=\frac{1}{\mathbb{E}(\xi_1)}=\frac{1}{30}$. So we change battery every 30 hours in average. □

PROBLEM 31 Consider a primitive renewing process with average renewing internal time $\mu$. Assume every renewing time is recorded by probability $p$, and each record and each renew are independence. Let $N_r(t)$ be the times of renewing by recorded until time $t$. $\{N_r(t):t\geq 0\}$ is a renewing process or not? And calculate $\lim_{t\to\infty}\frac{N_r(t)}{t}$.

SOLUTION . Assume $X_n:n\in\mathbb{N}$ are i.i.d r.v and $X_0\sim Geo(p)$, and $(X_n:n\in\mathbb{N})\perp(N(t):t\geq 0)$. Let $Y_n:=\sum_{k=1}^{n}X_n$, and $Y_0=0$. Let $\xi_r(n):=\sum_{k=Y_{n-1}+1}^{Y_n}\xi_k$. Then $\xi_r(n):n\in\mathbb{N}^+$ is update time of $N_r$. Since $(X_n:n\in\mathbb{N})\perp(N(t):t\geq 0)$, we get that $(\xi_r(n):n\in\mathbb{N}^+)$ are i.i.d. And $\mathbb{E}(\xi_r(1))=\mathbb{E}(X_1)\mathbb{E}(\xi_1)=\frac{\mu}{p}$. So $\lim_{t\to\infty}\frac{N_r(t)}{t}=\frac{\mu}{p}$. □

PROBLEM 32 Assume $(U_n:n\in\mathbb{N}^+)$ are i.i.d r.v. and $U_1\sim U(0,1)$. Assume $X_{n,m}:n,m\in\mathbb{N}^+$ are r.v. and $X_{n,m}\mid U_n\sim B(U_n)$. And $(X_{n,m}\mid U_n:m\in\mathbb{N}^+)$ are i.i.d. Let $\xi_n:=\inf\{m\in\mathbb{N}^+:X_{n,m}=1\}$ be the $n$-th update time of $N(t)$. Find $\lim_{t\to\infty}\frac{N(t)}{t}$.

SOLUTION . Easy to find that $\mathbb{E}(\xi_1)=\int_0^1\mathbb{E}(\xi_1\mid U_1=x)\,\mathrm{d}x=\int_0^1\frac{\mathrm{d}x}{x}=\infty$. So easy to find that $\lim_{t\to\infty}\frac{N(t)}{t}=\infty$. □

PROBLEM 33 Assume $(\xi_n:n\in\mathbb{N}^+)$ is i.i.d r.v. ranging in $\mathbb{N}$ is update time of $N(t)$. Let $A_n$ be the event that at time $n$ there is an update. Assume $a=\lim_{n\to\infty}\mathbb{P}(A_n)$ exists. Prove that $a=\frac{1}{\mathbb{E}(\xi_1)}$.

SOLUTION . Since $N(n)=\sum_{k=1}^{n}\mathbb{1}(A_k)$, we know that $\mathbb{E}(N(n))=\sum_{k=1}^{n}\mathbb{P}(A_k)$. Noting that $\lim_{n\to\infty}\frac{N(n)}{n}=\frac{1}{\mathbb{E}(\xi_1)}$, we obtain that $\lim_{n\to\infty}\mathbb{E}(\frac{N(n)}{n})=\frac{1}{\mathbb{E}(\xi_1)}$. So $\lim_{n\to\infty}\frac{\sum_{k=1}^{n}\mathbb{P}(A_k)}{n}=\frac{1}{\mathbb{E}(\xi_1)}$. By stolz, we can get that $\lim_{n\to\infty}\frac{\sum_{k=1}^{n}\mathbb{P}(A_k)}{n}=a$. So $a=\frac{1}{\mathbb{E}(\xi_1)}$. □

PROBLEM 34 Assume $N_1(t),N_2(t)$ are two independent updating process with update time distribution $E(1),U(0,2)$. Find an estimate of $\mathbb{P}(N_1(100)+N_2(100)\geq 190)$.

SOLUTION . Easy to know the expectation and varience of the update time are $\mu_1=1,\sigma_1^2=1,\mu_2=1,\sigma_2^2=\frac{1}{3}$. So by the central limit theorem of updating process we know that

$$\frac{N_1(100)-100}{\sqrt{100}},\frac{N_2(100)-100}{\sqrt{\frac{100}{3}}}\sim N(0,1)$$

So $\frac{N_1(100)+N_2(100)-200}{\sqrt{\frac{400}{3}}}\sim N(0,1)$. So $\mathbb{P}(N_1(100)+N_2(100)\geq 190)\approx\mathbb{P}(N(0,1)\geq-\frac{\sqrt{3}}{2})$. □

## 4.3 更新过程的应用

### 4.3.1 随机游动的爬升时间

**Definition 22.** 设 $(\xi_n : n \geq 1)$ 是独立同分布的可积随机变量序列且满足 $\mathbb{E}(\xi_1) > 0$。令 $(W_n : n \geq 0)$ 是以 $(\xi_n : n \geq 1)$ 为跳幅的随机游动，其中 $W_0 = 0$。易知 $W_n \to +\infty$。令 $S_0 = 0$，递归地定义停时 $S_n$，令 $S_n = \inf\{k \geq S_{n-1} : W_k > W_{S_{n-1}}\}$。称每个 $S_n$ 为 $(W_n)$ 的**爬升时间**。

**Theorem 29.** 对于 $n \geq 1$，令 $\eta_n = S_n - S_{n-1}$。则 $(\eta_n)$ 是独立同分布的非负随机变量序列。证明见教材 *p85*

**Theorem 30.** 我们有 $\mathbb{P}(\forall n \geq 1, W_n > 0) = \frac{1}{\mathbb{E}(S_1)}$。证明见教材 *p86*

### 4.3.2 更新累积过程

**Definition 23.** 设 $((\xi_n, \eta_n) : n \geq 1)$ 为独立同分布的二维随机变量序列，且 $\xi_n \geq 0$。令 $N(t)$ 为以 $\xi_n$ 为更新间隔时间的更新过程。令 $A(t) = \sum_{n=1}^{N(t)} \eta_n$。称 $(A(t) : t \geq 0)$ 为**更新累积过程**。

**Theorem 31.** 设 $0 < \mathbb{E}(\xi_1) < \infty, \mathbb{E}(|\eta_1|) < \infty$。则几乎必然有 $\lim_{t\to\infty} \frac{A(t)}{t} = \frac{\mathbb{E}(\eta_1)}{\mathbb{E}(\xi_1)}$。且有 $\lim_{t\to\infty} \frac{\mathbb{E}(A(t))}{t} = \frac{\mathbb{E}(\eta_1)}{\xi_1}$。证明见教材 *p87*

PROBLEM 35 Assume $N(t)$ is updating process. $X$ is the time interval distrabution of $N(t)$. Assume $\mathbb{D}(X) < \infty$. Let $R(t) := S_{N(t)+1} - t$. Find:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T R(t)\,\mathrm{d}t$$

SOLUTION. Easily $N(t)+1 \geq T \geq N(t)$. So $\int_0^T R(t)\,\mathrm{d}t \leq \sum_{i=1}^{N(T)+1} \int_{S_{i-1}}^{S_i} (S_i - t)\,\mathrm{d}t = \frac{1}{2}\sum_{i=1}^{N(t)+1}(S_i - S_{i-1})^2 = \frac{1}{2}\sum_{i=1}^{N(T)+1} \xi_i^2$. For the same reason, we get that $\int_0^T R(t)\,\mathrm{d}t \geq \frac{1}{2}\sum_{i=1}^{N(T)} \xi_i^2$.

Easy to know that $\lim_{T\to\infty} \frac{1}{T}\sum_{i=1}^{N(T)} \xi_i^2 = \lim_{T\to\infty} \frac{1}{T}\sum_{i=1}^{N(T)+1} \xi_i^2 = \frac{\mathbb{E}(X^2)}{\mathbb{E}(X)^2}$. So finally we get that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T R(t)\,\mathrm{d}t = \frac{\mathbb{E}(X^2)}{2\mathbb{E}(X)^2}$$

. □

PROBLEM 36 Assume the number of people arriving the cinema is distributed as a Possion process with parameter $\lambda$. Assume the film begin at a fixed time $t \geq 0$. Let $A(t)$ be the sum of waiting time of all people arriving in $(0, t]$, find $\mathbb{E}(A(t))$.

SOLUTION. Let $V_k$ be the arriving time of $k$-th people. Let $N(t)$ be the number of people in $(0, t]$. Then $A(t) = \sum_{k=1}^{N(t)}(t - V_k)$. Let $\xi_k := V_k - V_{k-1}$. Then $\sum_{k=1}^{N(t)} V_k = \sum_{k=1}^{N(t)}(N(t) - k)\xi_k = \sum_{k=0}^{N(t)-1} k\xi_{N(t)-k}$. So $\mathbb{E}(A(t)) = t\mathbb{E}(N(t)) - \mathbb{E}(\sum_{k=0}^{N(t)-1} k\xi_{N(t)-k})$. Easy to get that $\mathbb{E}(\sum_{k=0}^{N(t)-1} k\xi_{N(t)-k} \mid N(t) = n) = \frac{nt}{2}$. So $\mathbb{E}(A(t) \mid N(t) = n) = nt - \frac{nt}{2} = \frac{nt}{2}$. So finally we have $\mathbb{E}(A(t)) = \mathbb{E}(\mathbb{E}(A(t) \mid N(t))) = \mathbb{E}(\frac{N(t)t}{2}) = \frac{\lambda t^2}{2}$. □

PROBLEM 37 Assume a machine has life distrabuted $p$. When machine is broken or has been used $T$ years, we will change a new machine. The price of new machine is $C_1$, and if the machine is broken, it would cause loss $C_2$.

1. Give the long-time average fee of this machine.

2. Let $C_1 = 10, C_2 = 0.5$, and $p(x) = \mathbb{1}_{(0,10)}(x)\frac{1}{10}$. Which $T$ can let the fee be minimum.

SOLUTION. 1. Let $\xi$ be the time when the machine will broken. Let $\gamma := \xi \wedge T$. Then the machine will be changed at $\gamma$. Obviously $\mathbb{E}(\gamma) = T\mathbb{P}(\xi > T) + \mathbb{E}(\xi \mathbb{1}(\xi \leq T)) = T\int_T^\infty p(x)\,\mathrm{d}x + \int_0 Txp(x)\,\mathrm{d}x$. Let $\eta$ be the fee of this machine, then we have $\eta = C_1\mathbb{1}(\xi > T) + (C_1 + C_2)\mathbb{1}(\xi \leq T) = C_1 + C_2\mathbb{1}(\xi \leq T)$. So $\mathbb{E}(\eta) = C_1 + C_2\int_0^T p(x)\,\mathrm{d}x$. So the long-time average fee is

$$g(T) = \frac{C_1 + C_2\int_0 Tp(x)\,\mathrm{d}x}{T\int_T^\infty p(x)\,\mathrm{d}x + \int_0^T xp(x)\,\mathrm{d}x}$$

.

2. Easy to get that $g(T) = \frac{200+T}{20T-T^2}$ when $T \in (0, 10)$. And $g'(T) = \frac{T^2+400T-4000}{(20T-T^2)^2}$. Let $g'(T) = 0$, then $T^2 + 400T - 4000 = 0$, then $T = 20\sqrt{110} - 200 \approx 9.76$. So $T = 9.76$ can make the fee get minimum.

□

# 7 分支过程及其应用

## 7.1 定义和基本构造

### 7.1.1 分支过程的定义

PROBLEM 38

1. Assume $\{Y_1(n) : n \geq 0\}, \{Y_2(n) : n \geq 0\}$ are two independent migrating branch process with offspring distribution $(p(i) : i \in \mathbb{N})$ and the migrating probobility respectively are $(\gamma_1(i) : i \in \mathbb{Z}_+), (\gamma_2(i) : i \in \mathbb{N})$. Prove: $\{Y_1(n) + Y_2(n) : n \geq 0\}$ is migrating branching process with offspring distribution $p(i) : i \in \mathbb{N}$ and migrating probability $\gamma_1 * \gamma_2$.

2. Let $\{Y(n) : n \in \mathbb{N}\}$ be migrating branch process with offspring distribution $p(j) : j \in \mathbb{N}$ and the migrating distribution $\gamma(i) : i \in \mathbb{N}$. $P_n^\gamma = (p_n^\gamma(i,j); i, j \in \mathbb{N})$ is the $n$-th transition matrix. Prove: $\forall i, n \geq 1$

$$\sum_{j=0}^{\infty} p_n^\gamma(i,j) z^j = g_n(z)^i \prod_{k=1}^{n} h(g_{k-1}(z)), |z| \leq 1$$

where $h$ is the generating function of $(\gamma(j) : j \in \mathbb{N})$. $g$ is the generating function of $(p(j) : j \in \mathbb{N})$.

3. $h, g$ are defined as above. Assume $m := g'(1-) < \infty, \mu := h'(1-) < \infty$. Prove: $\forall i, n \geq 1$,

$$\mathbb{P}(Y_n \mid Y_0 = i) = im^n + \mu \sum_{k=1}^{n} m^{k-1}$$

SOLUTION. 1. Since $Y_1, Y_2$ are independent Markov chain, we easily get $\mathbb{P}(Y_1(n+1)+Y_2(n+1) = i \mid \sigma(Y_1(j), Y_2(j) : 0 \leq j \leq n)) = \mathbb{P}(Y_1(n+1) + Y_2(n+1) = i \mid Y_1(n), Y_2(n))$. So to prove $Y_1 + Y_2$ is Markov chain, we only need to prove $\mathbb{P}(Y_1(n+1) + Y_2(n+1) = i \mid Y_1(n), Y_2(n))\mathbb{P}(Y_1(n+1) + Y_2(n+1) = i \mid Y_1(n) + Y_2(n))$.

$$\begin{aligned}
&\mathbb{P}(Y_1(n+1) + Y_2(n+1) = i \mid Y_1(n) = j, Y_2(n) = k) \\
=& \sum_{x+y=i} \mathbb{P}(Y_1(n+1) = x \mid Y_1(n) = j)\mathbb{P}(Y_2(n+1) = y \mid Y_2(n) = k) \\
=& \sum_{x+y=i} p^{*j} * \gamma_1(x) p^{*k} * \gamma_2(y) \\
=& p^{*j} * \gamma_1 * p^{*k} * \gamma_2(i) \\
=& p^{*(j+k)} * \gamma_1 * \gamma_2(i)
\end{aligned}$$

So $\mathbb{P}(Y_1(n+1) + Y_2(n+1) = i \mid Y_1(n), Y_2(n)) = p^{*(Y_1(n)+Y_2(n))} * (\gamma_1 * \gamma_2)(i) \in \sigma(Y_1(n) + Y_2(n)) \subset \sigma(Y_1(n), Y_2(n))$. So $Y_1 + Y_2$ is Markov chain. More over, we have obtained

$\mathbb{P}(Y_1(n+1)+Y_2(n+1)=j \mid Y_1(n)+Y_2(n)=i) = p^{*i} * (\gamma_1 * \gamma_2)(j)$. So $\{Y_1(n)+Y_2(n) : n \geq 0\}$ is migrating branching process with offspring distribution $p(i) : i \in \mathbb{N}$ and migrating probability $\gamma_1 * \gamma_2$.

2. Use MI to prove it. Write $G_n(i,z) := \sum_{j=0}^{\infty} p_n^{\gamma}(i,j) z^j$. When $n=0$, we have $p_0^{\gamma}(i,j) = \delta_{ij}$, so $G_0(i,z) = z^i = g_0(z)^i$. When $n=1$, we have $p_1^{\gamma}(i,j) = p^{*i} * \gamma(j)$. So $G_1(i,z) = g(z)^i h(z)$. Assume for certain $n$ we have proved that $G_n(i,z) = g_n(z)^i \prod_{k=1}^{n} h(g_{k-1}(z))$, Consider $n+1$. Easily $p_{n+1}^{\gamma}(i,j) = \sum_{k\in\mathbb{N}} p_n^{\gamma}(k,j) p(i,\cdot) * \gamma(k)$. So

$$\begin{aligned}
G_{n+1}(i,z) &= \sum_{j\in\mathbb{N}} \sum_{k\in\mathbb{N}} p_1^{\gamma}(i,k) p_n^{\gamma}(k,j) z^j \\
&= \sum_{k\in\mathbb{N}} p_1^{\gamma}(i,k) G_n(k,z) \\
&= \prod_{k=1}^{n} h(g_{k-1}(z)) \sum_{k\in\mathbb{N}} p_1^{\gamma}(i,k) g_n(z)^k \\
&= \prod_{k=1}^{n} h(g_{k-1}(z)) G_1(i, g_n(z)) \\
&= g_{n+1}(z) \prod_{k=1}^{n+1} h(g_{k-1}(z))
\end{aligned}$$

3. Easily $\mathbb{P}(Y_n \mid Y_0 = i) = D_z G_n(i,z) \mid_{z\to 1-}$. Noting $g(1) = h(1) = 1$, easy to get that $\mathbb{P}(Y_n \mid Y_0 = i) = im^n + \mu \sum_{k=1}^{n} m^{k-1}$.

□

PROBLEM 39 Assume $b \in (0,1), p \in (0,1)$. Let $\mu(0) = \frac{1-b-p}{1-p}, \mu(j) = bp^{j-1}, j \geq 1$. Prove:

1. $(\mu(j) : j \in \mathbb{N})$ is probability distribution and

$$g(z) := \sum_{j=0}^{\infty} \mu(j) z^j = \frac{1-b-p}{1-p} + \frac{bz}{1-pz}.$$

2. Let $b = (1-p)^2$. Then $g'(1) = 1$ and

$$g(z) = p + \frac{(1-p)^2 z}{1-pz} = \frac{p-(2p-1)z}{1-pz}.$$

Prove: $\forall n \geq 1$,

$$g_n(z) = \frac{np - ((n+1)p-1)z}{1+(n-1)p-npz}.$$

SOLUTION. 1. Easily $\sum_{j=1}^{\infty} \mu(j) = \frac{b}{1-p}$. So $\sum_{j=0}^{\infty} \mu(j) = 1$. Easily $\sum_{j=1}^{\infty} \mu(j) z^j = \frac{bz}{1-pz}$. So $g(z) = \mu(0) + \frac{bz}{1-pz} = \frac{1-b-p}{1-p} + \frac{bz}{1-pz}$.

2. $g_{n+1}(z) = g(g_n(z)) = \frac{p-(2p-1)g_n(z)}{1-pg_n(z)}$. So $g_{n+1}(z) - 1 = \frac{(g_n(z)-1)(1-p)}{1-pg_n(z)}$. Thus, we obtain $\frac{1}{g_{n+1}(z)-1} = \frac{1}{g_n(z)-1} - \frac{p}{1-p}$. So $\frac{1}{g_n(z)-1} = \frac{1}{z-1} - \frac{np}{1-p}$, and finally we get $g_n(z) = \frac{np-((n+1)p-1)z}{1+(n-1)p-npz}$. □

PROBLEM 40 Let $\{X(n) : n \in \mathbb{N}\}$ be branch process with offspring distribution $p(j) : j \in \mathbb{N}$. And $g$ is the generating function. Let $m_2 := g'(1) + g''(1) < \infty$. Let $m = g'(1) < \infty$. $\forall k \geq 1$, $X_n^{(k)} = k^{-1}X_n$. Prove: $\forall \varepsilon > 0, i, n \geq 1$, $\mathbb{P}(|X_n^{(k)} - im^n| \geq \varepsilon \mid X_0^{(k)} = i) \to 0, k \to \infty$.

SOLUTION. In fact, we don't need $m_2 < \infty$. We let $(Y(k,n) : n \in \mathbb{N}), k \in \mathbb{N}$ are independent branch process with offspring distribution $p(j) : j \in \mathbb{N}$ and $Y(k,0) = i$. Then $\sum_{j=1}^k Y(j,n)$ is branch process with offspring distribution $p(j) : j \in \mathbb{N}$ and initial value $ki$. So $\sum_{j=1}^k Y(j,n) \overset{d}{=} X_n \mid X_0^{(k)} = i$. So $\mathbb{P}(|X_n^{(k)} - im^n| \geq \varepsilon \mid X_0^{(k)} = i) = \mathbb{P}(|\frac{\sum_{j=1}^k Y(j,n)}{k} - im^n| \geq \varepsilon)$. By LLN we obtain $\frac{1}{k}\sum_{j=1}^k Y(j,n) \overset{\text{a.s.}}{\to} im^n$. So finally we get $\lim_{k\to\infty} \mathbb{P}(|X_n^{(k)} - im^n| \geq \varepsilon \mid X_0^{(k)} = i) = \lim_{k\to\infty} \mathbb{P}(|\frac{\sum_{j=1}^k Y(j,n)}{k} - im^n| \geq \varepsilon) = 0$. □

PROBLEM 41 Let $\{X(n) : n \in \mathbb{N}\}$ be branch process with offspring distribution $p(j) : j \in \mathbb{N}$. And $g$ is the generating function, where $m := g'(1) \in (1, \infty), m_2 := g'(1) + g''(1) < \infty$. Let $\sigma^2 := m_2 - m^2 = \mathbb{D}(X(1))$. It is well known that $\exists W, \lim_{n\to\infty} \frac{X_n}{m^n} = W$. Prove:

$$\lim_{n\to\infty} \mathbb{E}_1[(m^{-n}X_n - W)^2] = 0, \mathbb{D}_1(W) = \sigma^2 m^{-1}(m-1)^{-1}$$

SOLUTION. For convenience we write $\mathbb{E}, \mathbb{D}$ instead of $\mathbb{E}_1, \mathbb{D}_1$. Easy to get that $\mathbb{E}(m^{-2n}X_n^2) = \frac{\sigma^2(1-m^{-n})}{m^2-m} + 1$. So by Fatou theorem we get that $\mathbb{E}(W^2) \leq \lim_{n\to\infty} \mathbb{E}(m^{-2n}X_n^2) = \frac{\sigma^2}{m^2-m} + 1 < \infty$. And by Doob Stochastic Processes p317 theorem 3.4 we get that $\mathbb{E}(\max_{n\in\mathbb{N}} m^{-2n}X_n^2) < \infty$. Thus, $m^{-2n}X_n^2$ are integrable uniformly, and so do $(m^{-n}X_n - W)^2$. So by LCDT we can get $\lim_{n\to\infty} \mathbb{E}((m^{-n}X_n - W)^2) = 0$. Noting that

$$\mathbb{E}(m^{-2n}X_n^2 - W^2) = \mathbb{E}((m^{-n}X_n + W)(m^{-n}X_n - W)) \leq \sqrt{\mathbb{E}((m^{-n}X_n + W)^2)\mathbb{E}((m^{-n}X_n - W)^2)} \to 0$$

, we get $\mathbb{E}(W^2) = \lim \mathbb{E}(m^{-2n}X_n^2) = \frac{\sigma^2}{m^2-m} + 1$. And $\mathbb{E}(W) = \lim \mathbb{E}(m^{-n}X_n) = 1$. So $\mathbb{D}(W) = \mathbb{E}(W^2) - \mathbb{E}(W)^2 = \frac{\sigma^2}{m(m-1)}$. □

PROBLEM 42 Let $\{Y(n) : n \in \mathbb{N}\}$ be branch process with offspring distribution $p(j) : j \in \mathbb{N}$. And $g$ is the generating function, where $m := g'(1) \leq 1$. Prove $(p^\gamma(j) : j \in \mathbb{N})$ is the steady-state vector of transition matrix $P_n^\gamma$, that is $\sum_{i=0}^\infty p^\gamma(i)p_n^\gamma(i,j) = p^\gamma(j), i \geq 0$.

SOLUTION. Since $\lim_{m\to\infty} p_m^\gamma(i,j) = p^\gamma(j)$, and fix $k \in \mathbb{N}$, we have $\sum_{j=0}^\infty p_m^\gamma(k,i)p_n^\gamma(i,j) = p_{n+m}^\gamma(k,j)$, we only need to prove that $\lim_{m\to\infty} \sum_{i=0}^\infty (p_m^\gamma(k,i) - p^\gamma(i))p_n^\gamma(i,j) = 0$. Since $\lim_{m\to\infty} p_m^\gamma(k,i) = p^\gamma(i)$ and $\sum_{i\in\mathbb{N}} p_m^\gamma(k,i) = 1$, we can easily get that $\sum_{i\in\mathbb{N}} p^\gamma(i) = 1$. For $\varepsilon > 0$, we let $N$ large enough such that $\sum_{k=N}^{\infty} p^\gamma(k) < \varepsilon$. Then we let $M$ large enough such that $\forall i : 0 \le i < N, \forall m \ge M, |p_m^\gamma(k,i) - p^\gamma(k)| < \frac{\varepsilon}{N}$. Then

$$
\begin{aligned}
&\left|\sum_{i=0}^{\infty}(p_m^\gamma(k,i) - p^\gamma(i))p_n^\gamma(i,j)\right| \\
\le&\sum_{i=0}^{\infty}|p_m^\gamma(k,i) - p^\gamma(i)|p_n^\gamma(i,j) \\
\le&\sum_{i=0}^{N-1}|p_m^\gamma(k,i) - p^\gamma(i)|p_n^\gamma(i,j) + \sum_{i=N}^{\infty}(p_m^\gamma(k,i) + p^\gamma(i))p_n^\gamma(i,j) \\
\le&\sum_{i=0}^{N-1}\frac{\varepsilon}{N} + \sum_{i=N}^{\infty}p_m^\gamma(k,i) + p^\gamma(i) \\
\le&\varepsilon + \sum_{i=N}^{\infty}p^\gamma(i) + 1 - \sum_{i=1}^{N-1}p_m^\gamma(k,i) \\
\le&\varepsilon + \varepsilon + 1 - \sum_{i=1}^{N-1}p^\gamma(i) + \sum_{i=1}^{N-1}|p_m^\gamma(k,i) - p^\gamma(i)| \\
\le&4\varepsilon
\end{aligned}
$$

So finally we get $\lim_{m\to\infty}\sum_{i=0}^{\infty}(p_m^\gamma(k,i) - p^\gamma(i))p_n^\gamma(i,j) = 0$. Thus, $\sum_{i=0}^{\infty}p^\gamma(i)p_n^\gamma(i,j) = p^\gamma(j), i \ge 0$. □

PROBLEM 43 Let $\{Y(n) : n \in \mathbb{N}\}$ be branch process with offspring distribution $p(j) : j \in \mathbb{N}$ and migrating distribution $\gamma$. And $g$ is the generating function, where $m := g'(1) \le 1$. Discuss $\lim_{n\to\infty}\mathbb{E}(Y_n \mid Y_0 = i)$.

SOLUTION. Easy to get that $\mathbb{E}(Y_n \mid Y_0 = i) = im^n + \mu\sum_{k=1}^{n}m^{k-1}$. When $m = 1$, we know $\mathbb{E}(Y_n \mid Y_0 = i) \to \infty$. When $m < 1$, we know $\mathbb{E}(Y_n \mid Y_0 = i) \to \frac{\mu}{1-m}$. □

PROBLEM 44 Let $S = (S_n : n \ge 0)$ be the one-dimensional symmetry simple random walk with $S_0 = c \ge 0$. Let $k \ge 1$ and $\tau$ be the time of the $k$-th downcrossing 0. $X_b$ is the times of $(S_{n\wedge\tau} : n \ge 0)$ downcrossing $b$. Prove:

1. $(X_b : b \ge c-1)$ is branch process. And offspring distribution is $Geo(\frac{1}{2})$
2. $(X_{-a} : a \ge 1)$ is branch process. And offspring distribution is $Geo(\frac{1}{2})$
3. $(X_b : 0 \le b \le c-1)$ is migrating branch process. And offspring distribution is $Geo(\frac{1}{2})$

   And the migrating distribution is concentrating on 1.

SOLUTION. For a random walk $y$, we let $D(n, y)$ be the number of downcrossings of $y$ of $n$.

1. Fix $b \geq c-1$. Let $\phi_0$ be the journey from start point to $b+1$. Let $e_n$ be $n$-th journey from $b+1$ to $b$. Let $\varepsilon_n$ be $n$-th journey after $\phi_0$ from $b$ to $b+1$. Then we know that $e_n, \varepsilon_n$ are independent. Easy to get that $D(e_n, b) = 1$ and $D(\varepsilon_n, b) = 0, D(\varepsilon_n, b+1 = 0)$. Easy to get that $D((S_{n\wedge\tau} : n \in \mathbb{N}), b+1) = \sum_{t=1}^{D((S_{n\wedge\tau}:n\in\mathbb{N}),b)} D(e_t, b+1)$. Noting that $\forall d : c-1 \leq d \leq b, D((S_{n\wedge\tau} : n \in \mathbb{N}), d) \in \sigma(\varepsilon_n : n \in \mathbb{N})$. We easily get that $D(e_t, b+1) \perp \sigma(\varepsilon_n : n \in \mathbb{N})$. So $X_b$ is Markov process. And to prove it's branch process, we only need to prove that $D(e_t, b+1)$ are i.i.d. It has been proved that $D(e_t, b+1)$ are i.i.d and $Geo(\frac{1}{2})$. So the offspring distribution is $Geo(\frac{1}{2})$.

2. Fix $a \geq 1$. Let $\phi_0$ be the journey from start point to $-a$. Let $e_n$ be $n$-th journey from $-a$ to $-a-1$, and $\varepsilon_n$ be $n$-th journey from $-a-1$ to $-a$. Then easy to get that $X_{-a-1} = \sum_{t=1}^{X_{-a}} D(\varepsilon_t, -a-1)$. For the same reason we easily get that $D(\varepsilon_t, -a-1) \perp \sigma(e_n : n \in \mathbb{N})$. And by reflecting easy to get that $D(\varepsilon_t, -a-1) \sim Geo(\frac{1}{2})$, too. So $(X_{-a} : a \geq 1)$ is branch process and offspring distribution is $Geo(\frac{1}{2})$

3. Fix $b < c-1$. Let $\phi_0$ be the journey from start point to $b+1$. Let $e_n$ be the $n$-th journey from $b+1$ to $b$ and $\varepsilon_n$ be $n$-th journey from $b$ to $b+1$. Then easy to prove that $X_{b+1} = D(\phi_0, b+1) + \sum_{t=1}^{X_b} D(e_n, b+1)$. Noting that $D(\phi_0, b+1) = 1$. So for the same reason, we get that $(X_b : 0 \leq b \leq c-1)$ is migrating branch process, with offspring distribution $Geo(\frac{1}{2})$ and migrating distribution $\delta_1$.

□

PROBLEM 45 $c < b \in \mathbb{Z}_+$. Let $W = (W_n : n \geq 0)$ be the one-dimensional reflecting simple random walk with $W_0 = c \geq 0$ on $\mathbb{Z}^{0,b}$, whose transition matrix is $P^{0,b}$, where $a = 0, p, q > 0, p+q = 1$. Let $k \geq 1$ and $\tau$ be the time of the $k$-th downcrossing 0 of $(W_n)$. $0 \leq a \leq b$, $X_a$ is the times of $(S_{n\wedge\tau} : n \geq 0)$ downcrossing $a$. Prove:

1. $(X_a : c-1 \leq a \leq b-1)$ is branch process. And offspring distribution is $Geo(p)$.

2. $(X_a : 0 \leq a \leq c-1)$ is migrating branch process. And offspring distribution is $Geo(p)$. And the migrating distribution is concentrating on 1.

SOLUTION. For a random walk $y$, we let $D(n, y)$ be the number of downcrossings of $y$ of $n$.

1. Fix $a$ such that $c-1 \leq a < b-1$. Let $\phi_0$ be the journey from start point to $a$. Let $e_n$ be the $n$-th journey from $a$ to $a+1$, and $\varepsilon_n$ be the $n$-th journey from $a+1$ to $a$. For reflecting simple random walk, we can also prove that $e_n, \varepsilon_n$ are independent. Noting that $X_{a+1} = \sum_{t=1}^{X_a} D(\varepsilon_t, a+1)$, we easily get the conclusion.

2. Fix $a : 0 \leq a < c-1$. Let $\phi_0$ be the journey from start point to $a+1$. Let $e_n$ be the $n$-th journey from $a+1$ to $a$ and $\varepsilon_n$ be $n$-th journey from $a$ to $a+1$. Then easy to prove

that $X_{a+1} = D(\phi_0, a+1) + \sum_{t=1}^{X_a} D(e_n, a+1)$. Noting that $D(\phi_0, a+1) = 1$. So for the same reason, we get that $(X_a : 0 \le a \le c-1)$ is migrating branch process, with offspring distribution $Geo(\frac{1}{2})$ and migrating distribution $\delta_1$.

$\square$

PROBLEM 46 Let $W = (W_n : n \ge 0)$ be the one-dimensional simple random walk with $W_0 = 0$, whose transition matrix $P$ given by equation (4.4.3) on textbook, $0 < p < q < 1$. $X_a$ is the times of $(W_{n\wedge\tau} : n \ge 0)$ downcrossing $a$. $r = \frac{p}{q}$. Prove:

1. $\mathbb{P}(X_0 = i) = r^i(1-r), i \ge 0$;
2. $a \ge 0$, $\mathbb{P}(X_a = 0) = 1 - r^{a+1}$, $\mathbb{P}(X_a = i) = r^{a+1}(1-r), i \ge 1$.

SOLUTION. 1. Since $p < q$, then $W_n \to -\infty, n \to \infty$. Let $\tau_0 = 0, \forall k \ge 1$, $\sigma_k = \inf\{n \ge \tau_{k-1} : W_n = 1\}, \tau_k = \inf\{n \ge \sigma_k : W_n = 0\}$.

(a) If $i = 0$, then $\{X_0 = i\} \overset{\text{a.s.}}{=} \{\sigma_1 = \infty\}$. Then $\mathbb{P}(X_0 = i) = \mathbb{P}(\sigma_1 = \infty) = r$.

(b) If $i \ge 1$, then $\{X_0 = i\} \overset{\text{a.s.}}{=} \{\sigma_i < \infty, \sigma_{i+1} = \infty\}$. Since $\{\tau_i < \infty\} \subset \{\sigma_i \infty\}, \mathbb{P}(\sigma_i < \infty, \tau_i = \infty) = 0$, then by strong markov property,

$$\begin{aligned}\mathbb{P}(\sigma_{i+1} < \infty \mid \sigma_i < \infty) &= \mathbb{P}(\sigma_{i+1} < \infty \mid \sigma_i < \infty, \tau_i < \infty)\\ &= \mathbb{P}(\sigma_{i+1} < \infty \mid \tau_i < \infty)\\ &= \mathbb{P}(\sigma_{i+1} < \infty \mid \tau_i < \infty, W_{\tau_i} = 0)\\ &= \mathbb{P}(\sigma_1 < \infty) = r\end{aligned}$$

Therefore,

$$\mathbb{P}(\sigma_{i+1} < \infty) = \mathbb{P}(\sigma_{i+1} < \infty \mid \sigma_i < \infty)\mathbb{P}(\sigma_i < \infty)$$

Then $\mathbb{P}(\sigma_i < \infty) = r^i$. Therefore, $\mathbb{P}(X_0 = i) = \mathbb{P}(\sigma_i < \infty, \sigma_{i+1} = \infty) = \mathbb{P}(\sigma_i < \infty)\mathbb{P}(\sigma_{i+1} = \infty \mid \sigma_i < \infty) = r^i(1-r)$.

2. Let $D_a = \inf(n \ge 0 : W_n = a)$, then $\mathbb{P}(D_a < \infty) = r^a$. By strong markov property, $(W_{D_a+n-a:n\ge 0})$ is a random walk starting from 0 under $\mathbb{P}(\cdot \mid D_a < \infty) = \mathbb{P}(\cdot \mid D_a < \infty, W_{D_a} = a)$. By the conclusion in 1, $\mathbb{P}(X_a = i \mid D_a < \infty) = r^i(1-r), i \ge 0$. Then

$$\begin{aligned}\mathbb{P}(X_a = 0) &= \mathbb{P}(D_a = \infty) + \mathbb{P}(D_a < \infty, X_a = 0)\\ &= 1 - r^a + \mathbb{P}(D_a < \infty)\mathbb{P}(X_a = 0 \mid D_a < \infty)\\ &= 1 - r^a + r^a(1-r) = 1 - r^{a+1}\end{aligned}$$

$\forall i \ge 1$,

$$\begin{aligned}\mathbb{P}(X_a = i) &= \mathbb{P}(D_a < \infty, X_a = i)\\ &= \mathbb{P}(D_a < \infty)\mathbb{P}(X_a = i \mid D_a < \infty)\\ &= r^a r^i(1-r) = r^{a+i}(1-r)\end{aligned}$$

$\square$

PROBLEM 47 Let $W = (W_n : n \geq 0)$ be the one-dimensional simple random walk with $W_0 = 0$, whose transition matrix $P$ given by equation (4.4.3) on textbook, $0 < p < q < 1$. $X_a$ is the times of $(W_{n\wedge\tau} : n \geq 0)$ downcrossing $a$. $r = \frac{p}{q}$. Prove: if $a \leq -1$, then $X_a - 1 \sim G(1-r)$, i.e. $\mathbb{P}(X_a = i) = r^{i-1}(1-r), i \geq 1$.

SOLUTION. Let $\tau = \inf\{n \in \mathbb{N} : W_n = a\}$. Then $\tau < \infty, a.s.$, then $W_n$ downcross $a$ at $\tau$. And $(W_{\tau+n} - a : n \in \mathbb{N})$ is simple random walk start at 0. So by 46 we easily get $X_a - 1 \sim Geo(1-r)$. $\square$

# 101 distribution

| name | symbol | range | distribution | mean | variance | generating | characteristic |
|---|---|---|---|---|---|---|---|
| Bernoulli | $B(1,p), 0<p<1$ | $\{0,1\}$ | $\mu(0)=1-p, \mu(1)=p$ | $p$ | $p(1-p)$ | $(1-p+pz)$ | $1-p+p\mathrm{e}^{\mathrm{i}t}$ |
| Binomial | $B(n,p), 0<p<1$ | $\mathbb{N}\cap[0,n]$ | $\mu(k)=\binom{n}{k}p^k(1-p)^{n-k}$ | $np$ | $np(1-p)$ | $(1-p+pz)^n$ | $(1-p+p\mathrm{e}^{\mathrm{i}t})^n$ |
| Geometric | $G(p), 0<p<1$ | $\mathbb{N}^+$ | $\mu(k)=(1-p)^{k-1}p$ | $\frac{1}{p}$ | $\frac{1-p}{p^2}$ | fuck | $\frac{p\mathrm{e}^{\mathrm{i}t}}{1-(1-p)\mathrm{e}^{\mathrm{i}t}}$ |
| Hypergeometric | $H(n,K,N)$ $N\in\mathbb{N}$ $K\in[0,N]\cap\mathbb{N}$ $n\in[0,N]\cap\mathbb{N}$ | $[\max\{0,n+K-N\},$ $\min\{n,K\}]\cap\mathbb{N}$ | $\mu(k)=\frac{\binom{K}{k}\binom{N-k}{n-k}}{\binom{N}{n}}$ | $n\frac{K}{N}$ | $n\frac{K}{N}\frac{N-k}{N}\frac{N-n}{N-1}$ | fuck | fuck |
| Poisson | $P(\lambda), \lambda>0$ | $\mathbb{N}$ | $\mu(k)=\frac{\mathrm{e}^{-\lambda}\lambda^k}{k!}$ | $\lambda$ | $\lambda$ | $\mathrm{e}^{\lambda(z-1)}$ | $\mathrm{e}^{\lambda(\mathrm{e}^{it}-1)}$ |
| Uniform | $U[a,b], a<b$ | $[a,b]$ | $f(x)=\frac{1}{b-a}$ $F(x)=\frac{x-a}{b-a}$ | $\frac{a+b}{2}$ | $\frac{(b-a)^2}{12}$ | - | $\frac{\mathrm{e}^{itb}-\mathrm{e}^{ita}}{\mathrm{i}t(b-a)}$ |
| Exponential | $Exp(\lambda), \lambda>0$ | $[0,\infty)$ | $f(x)=\lambda\mathrm{e}^{\lambda x}\mathbb{1}_{\{x\geq 0\}}$ $F(x)=(1-\mathrm{e}^{\lambda x})\mathbb{1}_{\{x\geq 0\}}$ | $\frac{1}{\lambda}$ | $\frac{1}{\lambda^2}$ | - | $(1-\frac{\mathrm{i}t}{\lambda})^{-1}$ |
| Gamma | $\Gamma(\alpha,\beta), \alpha,\beta>0$ | $(0,\infty)$ | $f(x)=\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\mathrm{e}^{-\beta x}\mathbb{1}_{\{x>0\}}$ | $\frac{\alpha}{\beta}$ | $\frac{\alpha}{\beta^2}$ | - | $(1-\frac{\mathrm{i}t}{\beta})^{-\alpha}$ |
| Normal | $N(\mu,\sigma^2), \mu,\sigma\in\mathbb{R}$ | $\mathbb{R}$ | $f(x)=\frac{1}{\sigma\sqrt{2\pi}}\mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}$ | $\mu$ | $\sigma$ | - | $\mathrm{e}^{\mu\mathrm{i}t-\frac{\sigma^2t^2}{2}}$ |

# 102 important lemma

**Lemma 4** (**Borel-Cantelli**). *$(\Omega,\mathcal{F},\mathbb{P})$ is a probability space. $\{A_n : n\in\mathbb{N}\}\subset\mathcal{F}$, then*

1. *if $\sum_{n\in\mathbb{N}}\mathbb{P}(A_n)<\infty$, then $\mathbb{P}(\bigcup_{k=0}^{n}\bigcap_{m=n}^{\infty}A_m)=0$;*
2. *if $\{A_n : n\in\mathbb{N}\}$ are independent, $\sum_{n\in\mathbb{N}}\mathbb{P}(A_n)=\infty$, then $\mathbb{P}(\bigcup_{k=0}^{n}\bigcap_{m=n}^{\infty}A_m)=1$.*

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