Markov 过程复习资料

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1.1 基本概念

1.1.1 随机过程的定义

Definition 1. 设 I 是非空指标集，$(\Omega, \mathcal{F}, \mathbb{P})$ 是概率空间。若 $(X_\alpha : \alpha \in I)$ 是一组定义在 $(\Omega, \mathcal{F}, \mathbb{P})$ 上的随机变量（取值为 $\mathbb{R}^d$），则称 $(X_\alpha : \alpha \in I)$ 为一个**随机过程**。

Definition 2. 假设 $(X_\alpha : \alpha \in I)$ 和 $(Y_\alpha : \alpha \in J)$ 是两个随机过程。若对于任何有限序列 $(s_1, \cdots, s_n) \subset I, (t_1, \cdots, t_m) \subset J$，都有 $(X_{s_1}, \cdots, X_{s_n}) \perp (Y_{t_1}, \cdots, Y_{t_m})$，则称这两个**随机过程独立**。

1.1.2 轨道和修正

Definition 3. 设 $(X_\alpha : \alpha \in I)$ 为随机过程。固定 $\omega \in \Omega$，称 $t \mapsto X_t(\omega)$ 为 X 的一条**轨道**。

Definition 4. 称一个随机过程是（**左连续**//**右连续**//**连续**//**左极右连**//**左连右极**）的，若它的所有轨道都是（左连续//右连续//连续//左极右连//左连右极的）。

Definition 5. 设 $(X_t : t \in I)$ 和 $(Y_t : t \in I)$ 是两个随机过程。若 $\forall t \in I$，有 $\mathbb{P}(X_t = Y_t) = 1$，则称它们互为**修正**。若 $\mathbb{P}(\forall t \in I, X_t = Y_t) = 1$，则称它们是**无区别**的。

Theorem 1. 设 $(X_t : t \geq 0)$ 和 $(Y_t : t \geq 0)$ 是两个右连续的随机过程，而 D 是 $(0, \infty)$ 的可数稠密子集。若 $\forall s \in D, \mathbb{P}(X_s = Y_s) = 1$，则有 $(X_t : t \geq 0)$ 和 $(Y_t : t \geq 0)$ 是无区别的。

1.1.3 有限维分布族

为了简化记号，我们用 $S(I)$ 表示 I 的全体有序有限子集。即：

$$S(I) := \{(t_1, \cdots, t_n) : n \geq 1, t_i \in I, \forall i = 1, \cdots, n\}$$

用 E 表示 $\mathbb{R}^d$，用 $\mathcal{E}$ 表示博雷尔代数。

Definition 6. 设 I 是非空指标集。若对于每个 $J \in S(I)$，都对应一个 $(E^{|}J|, \mathcal{E}^{|}J|)$ 上的概率测度 u_J，则称 $(\mu_J : I \in S(I))$ 为 E 上的**一个有限维分布族**，其中每个 μ_J 称为**一个有限维分布**。设 $X = (X_t : t \in I)$ 是一个随机过程，用 μ_J^X 表示 $(X_{t_1}, \cdots, X_{t_n})$ 的分布。称 $\mathcal{D}_X := \{\mu_J^X : J \in S(I)\}$ 为 X 的有限维分布族，称 μ_J^X 为其中的一个有限维分布。

Definition 7. 给定 $(E, \mathcal{E})$ 上的有限维分布族 $\mathcal{D}$，若存在随机过程 $X = (X_t : t \in I)$ 使得 $\mathcal{D}_X = \mathcal{D}$，则称 X 为 $\mathcal{D}$ 的一个**实现**。若两个随机过程 X, Y 满足 $\mathcal{D}_X = \mathcal{D}_Y$，则称它们为**等价**的。两个等价的过程互称实现。显然，两个互为修正的随机过程一定等价，反过来却未必。

1.1.4 左极右连实现

Definition 8. 状态空间 $E = \mathbb{R}^d$ 上的随机过程有左极右连实现 $\Longleftrightarrow$ 它有左极右连修正。证明见教材 *p5*

PROBLEM 1 Assume $(\mathscr{F}_t : t \geq 0, t \in \mathbb{R})$ is a filtration. For $t \geq 0$ we let $\mathscr{F}_{t+} := \bigcap_{s>t} \mathscr{F}_s$. Prove that $\mathscr{F}_t \subset \mathscr{F}_{t+}$ and $(\mathscr{F}_{t+} : t \geq 0)$ is a filtration.

SOLUTION. To prove $\mathscr{F}_t \subset \mathscr{F}_{t+} = \bigcap_{s>t} \mathscr{F}_s$, we only need to prove $\forall s > t, \mathscr{F}_t \subset \mathscr{F}_s$. By the definition of filtration it's obvious. Now we will prove $(\mathscr{F}_{t+} : t \geq 0)$ is a filtration. Only need to prove $\forall t, s \in \mathbb{R} \wedge t \leq s, \mathscr{F}_{t+} \subset \mathscr{F}_{s+}$. By the definition of $\mathscr{F}_{\dot{c}+}$ we know that $\mathscr{F}_{t+} = \bigcap_{x>t} \mathscr{F}_x = \bigcap_{x>s} \mathscr{F}_x \cap \bigcap_{x:t<x\leq s} \mathscr{F}_x \subset \bigcap_{x>s} \mathscr{F}_x = \mathscr{F}_{s+}$. So $(\mathscr{F}_{t+} : t \geq 0)$ is a filtration. □

PROBLEM 2 Assume $(X_t : t \geq 0, t \in \mathbb{R})$ is a stochastic process on probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Prove that $\forall s, t \geq 0, \varepsilon > 0, \{\rho(X_s, X_t) \geq \varepsilon\} \in \mathscr{F}$.

SOLUTION. Easily $\{\rho(X_s,X_t)\geq\varepsilon\}=\bigcup_{k=1}^{\infty}\{\rho(X_s,X_t)>\varepsilon-\frac{1}{k}\varepsilon\}$. So we only need to prove $\forall k\in\mathbb{N}^+,\{\rho(X_s,X_t)>\varepsilon(1-\frac{1}{k})\}\in\mathscr{F}$. Take $\delta=\varepsilon(1-\frac{1}{k})$, only need to prove $\forall\delta>0,\{\rho(X_s,X_t)>\delta\}\in\mathscr{F}$.

$\forall t\geq 0, X_t:\Omega\to E$ is measurable, where $E\subset\mathbb{R}^d$. So we can find a countable dense set in $\mathbb{R}^d$, write D. We will prove that $\{\rho(X_s,X_t)>\delta\}=\bigcup_{q\in D}\{\rho(X_s,q)-\rho(X_t,q)>\delta\}$. On one hand, easily $\rho(X_s,q)-\rho(X_t,q)>\delta \implies \rho(X_s,X_t)>\delta$ from triangle inequality. So we easily get $\{\rho(X_s,X_t)>\delta\}\supset\bigcup_{q\in D}\{\rho(X_s,q)-\rho(X_t,q)>\delta\}$. On the other hand, assume for certain $\omega\in\Omega$ we have $\rho(X_s(\omega),X_t(\omega))>\delta$, we will prove $\exists q\in D,\rho(X_s(\omega),q)-\rho(X_t(\omega),q)>\delta$. For convenience, we omit (ω) from now on to the end of this paragraph. Since $\rho(X_s,X_t)>\delta$, we know $\gamma:=\frac{\rho(X_s,X_t)-\delta}{2}>0$. Since D is dense, we obtain $\exists q\in D,\rho(X_t,q)<\gamma$. So from triangle inequality we get $\rho(X_s,q)\geq\rho(X_s,X_t)-\rho(X_t,q)>2\gamma+\delta-\gamma=\gamma+\delta$. So we get $\rho(X_s,q)-\rho(X_t,q)>\gamma+\delta-\gamma=\delta$. Finally, we get $\{\rho(X_s,X_t)>\delta\}=\bigcup_{q\in D}\{\rho(X_s,q)-\rho(X_t,q)>\delta\}$.

Noting $\bigcup_{q\in D}\{\rho(X_s,q)-\rho(X_t,q)>\delta\}=\bigcup_{q\in D}\bigcup_{p\in\mathbb{Q}^+}\{\rho(X_s,q)>\delta+p,\rho(X_t,q)<p\}$, and $D,\mathbb{Q}^+$ are countable, so we only need to check $\{\rho(X_s,q)>\delta+p,\rho(X_t,q)<p\}\in\mathscr{F},\forall q\in D,p\in\mathbb{Q}^+$. Noting $\{\rho(X_s,q)>\delta+p,\rho(X_t,q)<p\}=\{\rho(X_s,q)>\delta+p\}\cap\{\rho(X_t,q)<p\}$, and X_s,X_t are measurable from Ω to E, we obtain $\{\rho(X_s,q)>\delta+p\},\{\rho(X_t,q)<p\}\in\mathscr{F}$. So we proved $\{\rho(X_s,X_t)>\delta\}\in\mathscr{F},\forall s,t\geq 0,\forall\delta>0$.

Finally, we obtain $\{\rho(X_s,X_t)\geq\varepsilon\}\in\mathscr{F},\forall s,t\geq 0,\varepsilon>0$. □

PROBLEM 3 Let $\mathscr{D}_X:=\{\mu_J^X:J\in S(I)\}$ be the family of finite-dimentional distributions of a stochastic process $(X_t:t\geq 0,t\in\mathbb{R})$. $\forall(s_1,s_2)\in S(I)$ and $J=(t_1,\cdots,t_n)\in S(I)$, write $K_1:=(s_1,s_2,t_1,\cdots,t_n)\in S(I),K_2:=(s_2,s_1,t_1,\cdots,t_n)\in S(I)$. Take $A_1,A_2\in\mathscr{E},B\in\mathscr{E}^n$, prove that

$$\mu_{K_1}^X(A_1\times A_2\times B)=\mu_{K_2}^X(A_2\times A_1\times B)$$

and

$$\mu_{K_1}^X(E\times E\times B)=\mu_{K_2}^X(E\times E\times B)=\mu_J^X(B)$$

SOLUTION. By the definition of finite-dimentional distributions we get

$$\begin{aligned}\mu_{K_1}^X(A_1\times A_2\times B)&=\mathbb{P}((X_{s_1},X_{s_2},X_{t_1},\cdots,X_{t_n})\in A_1\times A_2\times B)\\&=\mathbb{P}(X_{s_1}\in A_1)\mathbb{P}(X_{s_2}\in A_2)\mathbb{P}((X_{t_1},X_{t_2},\cdots,X_{t_n})\in B)\end{aligned}$$

For the same reason, we obtain

$$\begin{aligned}&\mu_{K_2}^X(A_2\times A_1\times B)\\&\qquad=\mathbb{P}((X_{s_2},X_{s_1},X_{t_1},\cdots,X_{t_n})\in A_2\times A_1\times B)\\&\qquad=\mathbb{P}(X_{s_2}\in A_2)\mathbb{P}(X_{s_1}\in A_1)\mathbb{P}((X_{t_1},X_{t_2},\cdots,X_{t_n})\in B)\end{aligned}$$

So we get

$$\mu_{K_1}^X(A_1\times A_2\times B)=\mu_{K_2}^X(A_2\times A_1\times B)$$

Also, let $A_1 = A_2 = E$, we get

$$\begin{aligned}\mu^X_{K_1}(E \times E \times B) &= \mu^X_{K_2}(E \times E \times B)\\ &= \mathbb{P}(X_{s_1} \in E)\mathbb{P}(X_{s_2} \in E)\mathbb{P}((X_{t_1}, \cdots, X_{t_n}) \in B)\\ &= \mathbb{P}((X_{t_1}, \cdots, X_{t_n}) \in B)\end{aligned}$$

By the definition of finite-dimentional distributions we get

$$\mu^X_J(B) = \mathbb{P}((X_{t_1}, \cdots, X_{t_n}) \in B)$$

So finally we get

$$\mu^X_{K_1}(E \times E \times B) = \mu^X_{K_2}(E \times E \times B) = \mu^X_J(B)$$

□

PROBLEM 4 Assume $(\tau_k : k \in \mathbb{N}^+)$ is an i.i.d sequence of r.v. with exponential distribution with parameter $\alpha > 0$. Let $S_n := \sum_{k=1}^n \tau_k$. For $t \geq 0, t \in \mathbb{R}$, let:

$$N_t := \sum_{n=1}^{\infty} \mathbb{1}_{\{S_n \leq t\}}, X_t := \sum_{n=1}^{\infty} \mathbb{1}_{\{S_n < t\}}$$

Prove that N and X are modifications of each other, but they are not indistinguishable.

SOLUTION. First we prove N and X are modifications of each other. Fix $t \in [0, \infty)$, we need to prove $\mathbb{P}(N_t = X_t) = 1$. Noting $N_t - X_t = \sum_{n=1}^{\infty} \mathbb{1}_{S_n \leq t} - \mathbb{1}_{S_n < 1} = \sum_{n=1}^{\infty} \mathbb{1}_{S_n = t}$, we get $\mathbb{P}(N_t = X_t) = \mathbb{P}(\sum_{n=1}^{\infty} \mathbb{1}_{S_n = t}) = \mathbb{P}(\forall n \in \mathbb{N}^+, S_n \neq t)$. So we only need to prove $\mathbb{P}(S_n = t) = 0, \forall n \in \mathbb{N}^+$. Since $\tau_k, k \in \mathbb{N}^+$ are continuous-distributed, we know $S_n = \sum_{k=1}^n \tau_k$ is continuous-distributed, so $\mathbb{P}(S_n = t) = 0$. So we proved N and X are modifications of each other.

Next we will prove they are not indistinguishable. Only need to prove $\mathbb{P}(\forall t \in [0, \infty), X_t = N_t) = 0 \neq 1$. Since $N_t - X_t = \sum_{n=1}^{\infty} \mathbb{1}_{S_n \leq t} - \mathbb{1}_{S_n < 1} = \sum_{n=1}^{\infty} \mathbb{1}_{S_n = t}$, we know $\forall t, N_t = X_t \iff \forall t, \forall n \in \mathbb{N}^+, S_n \neq t$. But S_n is ranged in $[0, \infty)$, so R.H.S is an impossible event. So we finally get $\mathbb{P}(\forall t \in [0, \infty), X_t = N_t) = 0$ and thus X and N are not indistinguishable. □

PROBLEM 5 Assume T is non-negetive r.v. with distribution function F continuous on $\mathbb{R}$. Let $X_t = \mathbb{1}_{\{T \leq t\}}$. Prove that X is stochastically continuous.

SOLUTION. Only need to check $\forall t \geq 0, X_s \xrightarrow{\mathbb{P}} X_t, s \to t$. Take $\varepsilon > 0$, we need to prove $\lim_{s \to t} \mathbb{P}(\rho(X_s, X_t) > \varepsilon) = 0$. For $u > v \geq 0$, we have $X_u - X_v = \mathbb{1}_{v < T \leq u}$. So $\mathbb{P}(\rho(X_u - X_v) > \varepsilon) \leq \mathbb{P}(X_u \neq X_v) = \mathbb{P}(v < T \leq u) \leq \mathbb{P}(T \in [v, u])$. So we easily get $\lim_{s \to t+} \mathbb{P}(\rho(X_s, X_t) > \varepsilon) \leq \lim_{s \to t+} \mathbb{P}(T \in [t, s]) = 0$ and $\lim_{s \to t-} \mathbb{P}(\rho(X_s, X_t) > \varepsilon) \leq \lim_{s \to t-} \mathbb{P}(T \in [s, t]) = 0$. So $\lim_{s \to t} \mathbb{P}(\rho(X_s, X_t) > \varepsilon) = 0$. □

PROBLEM 6 Assume $I = \mathbb{Z}^+$, then the stochastic process $X = (X_0, X_1, \cdots)$ is a r.v. from Ω to E^∞. Define the distribution of X, μ_X, as follows:

$$\mu_X(A) = \mathbb{P}(X \in A), A \in \mathscr{E}^\infty$$

Then stochastic process X, Y are equivalent $\iff \mu_X = \mu_Y$.

SOLUTION. " $\Longrightarrow$ ":Assume X, Y are equivalent, now we will prove $\mu_X = \mu_Y$. Let $\mathscr{A} := \{A \in \mathscr{P}(E^\infty) : \exists n \in \mathbb{N}^+, A = A_1 \times A_2 \times \cdots \times A_n \times E \times E \times \cdots\}$. Then we can get $\mu_X(A) = \mu^X_{(1,2,\cdots,n)}(A_1 \times \cdots \times A_n)$. So for $A \in \mathscr{A}$ we know $\mu_X(A) = \mu_Y(A)$. By the definition of $\mathscr{E}^\infty = \sigma(\mathscr{A})$, and noting $\mathscr{A}$ is a Semiset algebra, by the Measure extension theorem we get $\mu_X = \mu_Y$.

" $\Longleftarrow$ ": Assume $\mu_X = \mu_Y$, then easily $\mu^X_{(s_1,\cdots,s_n)}(A_{s_1} \times \cdots \times A_{s_n}) = \mu_X(\prod_{k \in \mathbb{N}^+} B_k)$, where $B_k = A_{s_t}$ for $k = s_t$ and $B_k = E$ for $k \neq s_t, \forall t = 1, \cdots, n$. So easily $\mu^X_J = \mu^Y_J, \forall J \subset I \wedge |J| < \infty$. □

1.2 随机游动

Definition 9. 设 $\{\xi_n : n \geq 1\}$ 是独立同分布的 d 维随机变量列，而 X_0 是与之独立的一个 d 维随机变量。令 $X_n := X_0 + \sum_{k=1}^n \xi_k$。称 $(X_n : n \geq 0)$ 为 d 维**随机游动**，并称 $\{\xi_n : n \geq 1\}$ 为其**步长列**。

Definition 10. 若 X_0, ξ_1 均取值与 $\mathbb{Z}^d$，则该随机游动状态空间可以取为 $\mathbb{Z}^d$。特别地，若还有 $\mathbb{P}(|\xi_1| = 1) = 1$，则称其为**简单随机游动**。进一步地，若对于 $\mathbb{Z}^d$ 中的任一单位向量 v，均有 $\mathbb{P}(\xi_1 = v) = \frac{1}{2d}$，则称其为**对称简单随机游动**。

1.2.1 轨道的无界性

方便起见，考虑 $\mathbb{Z}$ 上的简单随机游动 S_n，设其步长列为 $\xi_n : n \geq 1$。设 $\mathbb{P}(\xi_n = 1) = p, \mathbb{P}(\xi_n = -1) = q$，其中 $p, q \in (0,1), p + q = 1$。

Theorem 2. $(S_n : n \geq 1)$ 的轨道是几乎必然无界的。即：

$$\mathbb{P}(\sup_{n \geq 0} |S_n| = \infty) = 1. \tag{1}$$

证明见教材 *p9*

1.2.2 首达时分布

Definition 11. 记 $\mathbb{P}_i(\cdot) = \mathbb{P}(\cdot \mid S_0 = i)$。

Definition 12. 定义 $(S_n : n \geq 0)$ 到达 $x \in \mathbb{Z}$ 的**首达时** $\tau_x := \inf\{n \geq 0 : S_n = x\}$。

Theorem 3. 当 $p = q = \frac{1}{2}$ 时，对于 $a < b, i \in [a,b], a, b, i \in \mathbb{Z}$，有

$$\mathbb{P}_i(\tau_b < \tau_a) = \frac{i-a}{b-a}, \mathbb{P}_i(\tau_a < \tau_b) = \frac{b-i}{b-a} \tag{2}$$

当 $p \neq q$ 时，有

$$\mathbb{P}_i(\tau_b < \tau_a) = \frac{1 - (\frac{q}{p})^{i-a}}{1 - (\frac{q}{p})^{b-a}}, \mathbb{P}_i(\tau_a < \tau_b) = \frac{(\frac{q}{p})^{i-a} - (\frac{q}{p})^{b-a}}{1 - (\frac{q}{p})^{b-a}} \tag{3}$$

证明见教材 *p10*

Theorem 4. 当 $p \geq q$，对 $a \leq i \leq b \in \mathbb{Z}$，有

$$\mathbb{P}_i(\tau_a < \infty) = (\frac{q}{p})^{i-a}, \mathbb{P}_i(\tau_b < \infty) = 1 \tag{4}$$

当 $p \leq q$，有

$$\mathbb{P}_i(\tau_a < \infty) = 1, \mathbb{P}_i(\tau_b < \infty) = (\frac{p}{q})^{b-i} \tag{5}$$

证明见教材 *p11*

PROBLEM 7 Prove that if $(X_n : n \geq 0)$ is a simple random walk, then so is $(-X_n : n \geq 0)$.

SOLUTION . Let $\xi_n := X_n - X_{n-1}$ for $n \in \mathbb{N}^+$. Then Since $(X_n : n \in \mathbb{N})$ is simple random walk we have $X_0, \xi_1, \xi_2, \cdots$ are independent r.v. ranges in $\mathbb{Z}$, and $\xi_i, i = 1, 2 \cdots$ are i.i.d., and $\mathbb{P}(|\xi_i| = 1) = 1$. So we easily get $-X_0, -\xi_1, -\xi_2, \cdots$ are independent r.v. ranges in $\mathbb{Z}$, and $-\xi_i, i = 1, 2, \cdots$ are i.i.d., and $\mathbb{P}(|-\xi_i| = 1) = 1$. Since $-X_n = X_0 + \sum_{k=1}^{n} \xi_k$, by the definition of simple random walk we obtain $(-X_n : n \in \mathbb{N})$ is a simple random walk. $\square$

PROBLEM 8 Let $(X_n : n \geq 0)$ be a d-dimentional random walk with $\mathbb{P}(|\xi_i| \geq 1) > 0$, prove that $\mathbb{P}(\sup_n |X_n| = \infty) = 1$.

SOLUTION . Let $t \in \mathbb{Z}^d, t \neq 0$ and $\mathbb{P}(\xi_i = t) > 0$. Since $\mathbb{P}(\sup_n |X_n| = \infty) = \inf_{k \in \mathbb{N}} \mathbb{P}(\sup_n |X_n| \geq k)$, we only need to prove $\mathbb{P}(\sup_n |X_n| \geq k) = 1$ for every $k \in \mathbb{N}$. Take $K > 3k, K \in \mathbb{N}$. Let $A_s := \{\xi_i = t : i = sK+1, sK+2, \cdots, sK+K-1\}$. Then for $\omega \in A_s$, we have $|X_{sK+K} - X_{sK}| = |\sum_{u=1}^{K-1} t| = K|t| \geq K \geq 3k$. Then $\sup_n |X_n| \geq \max\{|X_{sK+K}|, |X_{sK}|\} \geq \frac{1}{2}|X_{sK+K} - X_{sK}| \geq k$. So we get $\forall s, A_s \subset \{\sup_n |X_n| \geq k\}$. Since ξ_i are independent, easily get $A_s, s = 1, 2, \cdots$ are independent. Noting $\mathbb{P}(A_s) = \mathbb{P}(\xi_i = t)^K > 0$, we get $\sum_{s \in \mathbb{N}} \mathbb{P}(A_s) = \infty$. So from BC-theorem we get $\mathbb{P}(A_s, i.o.) = 1$, thus $\mathbb{P}(\bigcup_{s \in \mathbb{N}} A_s) = 1$. Thus, $\mathbb{P}(\sup_n |X_n| \geq k) = 1$, for every $k \in \mathbb{N}$. Thus, $\mathbb{P}(\sup_n |X_n| = \infty) = \mathbb{P}(\bigcap_{k \in \mathbb{N}} \{\sup_n |X_n| \geq k\}) = 1$. $\square$

PROBLEM 9 Let $(S_n : n \geq 0)$ be a symmetry simple random walk with $S_0 = 0$, for $d = 2$, prove that

$$\mathbb{P}(S_{2n} = 0) = \frac{1}{4^{2n}} \left(\frac{(2n)!}{(n!)^2} \right)^2$$

For $d = 3$, prove that

$$\mathbb{P}(S_{2n} = 0) = \frac{1}{2^{2n}} \frac{(2n)!}{(n!)^2} \sum_{i+j+k=n} \left(\frac{1}{3^n} \frac{n!}{i!j!k!} \right)^2$$

SOLUTION . First we consider $d = 2$. Write $\xi_i = S_i - S_{i-1}$. Then we know S_{2n} occur $\iff$ the number of $(1,0)$ and $(-1,0)$ in $\{\xi_i : i = 1, \cdots, 2n\}$, and the number of $(0,1)$ and $(0,-1)$ in $\{\xi_i : i = 1, \cdots, 2n\}$. We assume there is k pairs of $(1,0), (-1,0)$, then easily there is $n-k$ pairs of $(0,1), (0,-1)$. The probability is $\binom{2n}{k}\binom{2n-k}{k}\binom{2n-2k}{n-k}\frac{1}{4^{2n}}$. So the total probability is $\mathbb{P}(S_{2n} =$

$0) = \sum_{k=0}^{n}\binom{2n}{k}\binom{2n-k}{k}\binom{2n-2k}{n-k}\frac{1}{4^{2n}} = \sum_{k=0}^{n}\frac{(2n)!}{k!k!(n-k)!(n-k)!4^{2n}}$. Noting that $\sum_{k=0}^{n}\frac{(n!)^2}{k!k!(n-k)!(n-k)!} = \sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k} = \binom{2n}{n} = \frac{(2n)!}{n!n!}$, we finally get $\mathbb{P}(S_{2n}=0) = \frac{1}{4^{2n}}\left(\frac{(2n)!}{n!n!}\right)^2$.

Use the same method, consider $d=3$, we have

$$\mathbb{P}(S_{2n}=0) = \sum_{i+j+k=n}\binom{2n}{i}\binom{2n-i}{i}\binom{2n-2i}{j}\binom{2n-2i-j}{j}\binom{2n-2i-2j-k}{k}\frac{1}{6^{2n}}$$

So easily to get $\mathbb{P}(S_{2n}=0) = \frac{1}{2^{2n}}\frac{(2n)!}{(n!)^2}\sum_{i+j+k=n}\left(\frac{1}{3^n}\frac{n!}{i!j!k!}\right)^2$. □

PROBLEM 10 Assume $(S_n : n \geq 0)$ is a symmetry simple random walk with $S_0 = i \in \mathbb{Z}$. Prove that $\forall a \in \mathbb{Z}$, let $\tau_a := \min\{n \in \mathbb{N} : S_n = a\}$, then $\mathbb{P}(\tau_a < \infty) = 1$.

SOLUTION. Without loss of generality assume $a < 0, i = 0$. Take $N \in \mathbb{N}^+$. Consider $\tau := \min\{n \in \mathbb{N} : S_n = a \vee S_n = N\}$. From Problem 21 we can easily know $\mathbb{P}(\tau < \infty) = 1$ because $\{\sup_n |S_n| = \infty\} \subset \{\tau < \infty\}$, a.s. So we get $\{\tau_a = \tau\} \subset \{\tau_a < \infty\}$, a.s. Let $Y_n := S_{n\wedge\tau} := S_{\min\{n,\tau\}}$. Easily $(S_n : n \in \mathbb{N})$ is a martingale, and τ is a stopping time, so we get $(Y_n : n \in \mathbb{N})$ is a martingale, too. And easily $Y_n \in [a, N]$, so Y_n is bounded. So we get $\mathbb{E}(S_\tau) = \lim_{n\to\infty}\mathbb{E}(Y_n) = \mathbb{E}(Y_0) = 0$. Easily to know $\mathbb{E}(S_\tau) = \mathbb{P}(\tau = \tau_a)a + \mathbb{P}(\tau \neq \tau_a)N = 0$. And $\mathbb{P}(\tau = \tau_a) + \mathbb{P}(\tau \neq \tau_a) = 1$, so easily $\mathbb{P}(\tau = \tau_a) = \frac{N}{N-a}$. So $\mathbb{P}(\tau_a < \infty) \geq \frac{N}{N-a}$. Let $N \to \infty$, we get $\mathbb{P}(\tau_a < \infty) = 1$. □

1.3 布朗运动

1.3.1 背景和定义

Definition 13. 假定 $\sigma^2 > 0$，具有连续轨道的实值过程 $(B_t : t \geq 0)$ 满足:

1. $\forall 0 \leq s \leq t$，$B_t - B_s \sim N(0, \sigma^2(t-s))$;
2. $\forall 0 \leq t_0 \leq \cdots \leq t_n$，$B_0, B_1 - B_0, \cdots, B_{t_n} - B_{t_{n-1}}$ 独立，

称 $(B_t : t \geq 0)$ 是以 σ^2 为参数的**布朗运动**。特别的，当 $\sigma^2 = 1$，$(B_t : t \geq 0)$ 为**标准布朗运动**。

Definition 14. 有限维分布为正态分布的随机过程称为**正态过程**。

1.3.2 布朗运动的构造

Theorem 5. 布朗运动是有连续实现的。证明见教材 *p13.*

1.3.3 布朗运动的性质

Theorem 6. 从原点出发的零均值高斯过程 $(B_t : t \geq 0)$ 是标准布朗运动 $\iff$ $\forall s, t \geq 0$，$\mathbb{E}(B_tB_s) = t \wedge s$。证明 *p17.*

Theorem 7. 布朗运动轨道几乎处处不可导。证明 *p17-18.*

Lemma 1. *Assume $(B_t : t \geq 0)$ is a random process ranging in $\mathbb{R}$, $a \in \mathbb{R}^+$, and $\forall s, t : 0 \leq s \leq t, B_t - B_s \sim N(0, a(t-s))$. Assume B_t is continuous about t, a.s. Let $\mathcal{F}_t := \sigma(B_s : 0 \leq s \leq t)$. Then $(B_t : t \geq 0)$ is Brownian motion $\iff \forall 0 \leq s \leq t, B_t - B_s \perp \mathcal{F}_s$.*

证明. “$\implies$”: To prove $B_t - B_s \perp \mathcal{F}_s$, only need to prove for $t_1 < t_2 < \cdots < t_{n-1} = s < t = t_n$, we have $B_t - B_s \perp \sigma(B_{t_k} : k = 1, \cdots, n-1)$. Easily $B_t - B_s \perp \sigma(B_{t_{k+1}} - B_{t_k}, B_{t_1} : k = 1, \cdots, n-2) = \sigma(B_{t_k} : k = 1, \cdots, n-1)$, so $B_t - B_s \perp \mathcal{F}_s$.

“$\impliedby$”: For $t_1 < \cdots < t_n$, we need to prove $B_{t_1}, B_{t_{k+1}} - B_{t_k} : 1 \leq k \leq n-1$ are independent. Use MI to n. When $n = 1$ it's obvious. Assume we have proved it for certain $n \geq 1$, now consider $n+1$. Since $B_{t_{k+1}} - B_{t_k} \in \mathcal{F}_{t_n}, k = 1, \cdots, n-1$, we have $B_{t_{n+1}} - B_{t_n} \perp \sigma(B_{t_1}, B_{t_{k+1}} - B_{t_k} : k = 1, \cdots, n-1)$. So $\mathbb{P}(B_{t_1} \in A_1, B_{t_{k+1}} - B_{t_k} \in A_{k+1}, k = 1, \cdots, n) = \mathbb{P}(B_{t_1} \in A_1, B_{t_{k+1}} - B_{t_k} \in A_{k+1}, k = 1, \cdots, n-1)\mathbb{P}(B_{t_{n+1}} - B_{t_n} \in A_{n+1})$. By Induction assumption we get $\mathbb{P}(B_{t_1} \in A_1, B_{t_{k+1}} - B_{t_k} \in A_{k+1}, k = 1, \cdots, n) = \mathbb{P}(B_{t_1} \in A_1)\prod_{k=1}^{n} \mathbb{P}(B_{t_{k+1}} - B_{t_k} \in A_{k+1})$. So finally we get $B_{t_1}, B_{t_{k+1}} - B_{t_k} : 1 \leq k \leq n$ are independent. □

PROBLEM 11 Assume $(B_t : t \geq 0)$ is Brownian motion, prove that for $r > 0$, we have $(B_{t+r} - B_r : t \geq 0)$ is Brownian motion, too.

SOLUTION. Assume $B_t - B_s \sim N(0, a(t-s)), a > 0$. Let $\mathcal{F}_t := \sigma(B_s : 0 \leq s \leq t)$ and $\mathcal{G}_t := \sigma(B_{r+s} - B_r : 0 \leq s \leq t)$. For $0 \leq s \leq t$, we have $B_{t+r} - B_r - (B_{s+r} - B_r) = B_{t+r} - B_{s+r} \sim N(0, a(t-s))$. And easily to know $B_{r+s} - B_r \in \mathcal{F}_{r+s}$, so $\mathcal{G}_t \subset \mathcal{F}_{t+r}, \forall t \geq 0$. Since $(B_t : t \geq 0)$ is Brownian motion, easily $\mathcal{F}_{s+r} \perp B_{t+r} - B_{s+r}$. Since $\mathcal{G}_t \subset \mathcal{F}_{t+r}$, we obtain $\mathcal{G}_t \perp B_{t+r} - B_{s+r} = B_{t+r} - B_r - (B_{s+r} - B_r)$. Easily since B_t is continuous we get $B_{t+r} - B_r$ is continuous. So $(B_{t+r} : t \geq 0)$ is Brownian motion. □

PROBLEM 12 Assume $(B_t : t \geq 0)$ is standard Brownian motion start at 0. Prove that $\forall c > 0, (cB_{\frac{t}{c^2}} : t \geq 0)$ is standard Brownian motion start at 0, too.

SOLUTION. Since $B_0 = 0$ we get $cB_{\frac{0}{c^2}} = 0$. Let $\mathcal{F}_t := \sigma(B_s : 0 \leq s \leq t)$ and $\mathcal{G}_t := \sigma(cB_{\frac{s}{c^2}} : 0 \leq s \leq t)$. Easily to know $\mathcal{G}_t = \mathcal{F}_{\frac{t}{c^2}}$. For $0 \leq s \leq t$, we have $cB_{\frac{t}{c^2}} - cB_{\frac{s}{c^2}} = c(B_{\frac{t}{c^2}} - B_{\frac{s}{c^2}}) \sim N(0, t-s)$, because $B_{\frac{t}{c^2}} - B_{\frac{s}{c^2}} \sim N(0, \frac{t-s}{c^2})$. And since $(B_t : t \geq 0)$ is Brownian motion, we get $B_{\frac{t}{c^2}} - B_{\frac{s}{c^2}} \perp \mathcal{F}_{\frac{s}{c^2}} = \mathcal{G}_s$. Easily since B_t is continuous we get $cB_{t^2_c}$ is continuous. So $(cB_{\frac{t}{c^2}} : t \geq 0)$ is standard Brownian motion starts at 0, too. □

PROBLEM 13 Assume $(X_t : t \geq 0)$ and $(Y_t : t \geq 0)$ are two independent standard Brownian motion, $a, b \in \mathbb{R}$ and $\sqrt{a^2 + b^2} > 0$. Prove that $(aX_t + bY_t : t \geq 0)$ is a Brownian motion with parameter $c^2 = a^2 + b^2$.

SOLUTION. Let $\mathcal{F}_t := \sigma(X_s : 0 \leq s \leq t)$ and $\mathcal{G}_t := \sigma(Y_s : 0 \leq s \leq t)$. Let $\mathcal{H}_t := \sigma(aX_s + bY_s : 0 \leq s \leq t)$. Since $(X_t : t \geq 0), (Y_t : t \geq 0)$ are two independent Brownian motion, we know $\forall 0 \leq s \leq t, X_t - X_s \perp \mathcal{F}_s, \mathcal{G}_s; Y_t - Y_s \perp \mathcal{F}_s, \mathcal{G}_s$. So we get $aX_t + bY_t - aX_s - bY_s \perp \mathcal{F}_s, \mathcal{G}_s$, thus $aX_t + bY_t - aX_s - bY_s \perp \sigma(\mathcal{F}_s, \mathcal{G}_s)$ Easily $aX_s + bY_s \in \sigma(\mathcal{F}_s, \mathcal{G}_s)$, so $\mathcal{H}_t \subset \sigma(\mathcal{F}_t, \mathcal{G}, t), \forall t \geq 0$. So

$aX_t + bY_t - aX_s - bY_s \perp \mathcal{H}_s$. And easily $a(X_t - X_s) \sim N(0, a^2(t-s)), b(Y_t - Y_s) \sin N(0, b^2(t-s))$, and since $\mathcal{F}_t \perp \mathcal{G}_t$ we get $a(X_t - X_s) \perp b(Y_t - Y_s)$, so $aX_t + bY_t - aX_s - bY_s \sim N(0, (a^2+b^2)(t-s))$. Easily since X_t, Y_t is continuous we get $aX_t + bY_t$ is continuous. So $(aX_t + bY_t : t \geq 0)$ is a Brownian motion with parameter $a^2 + b^2 = c^2$. □

PROBLEM 14 Assume $(B_t : t \geq 0)$ is standard Brownian motion start at 0. Let $X_0 = 0$ and $X_t := tB_{\frac{1}{t}}$. Given

$$\limsup_{t\to\infty} \frac{B_t}{\sqrt{2t \log \log t}} = 1$$

Prove that $(X_t : t \geq 0)$ is standard Brownian motion start at 0.

SOLUTION. First consider the distribution of $X_t - X_s$ for $s < t$. If $s = 0$ then $X_t - X_s = tB_{\frac{1}{t}} \sim N(0, t^2 \times \frac{1}{t}) = N(0, t)$. Else, $s > 0$, then $X_t - X_s = s(B_{\frac{1}{t}} - B_{\frac{1}{s}}) + (t-s)B_{\frac{1}{t}}$. Easily $B_{\frac{1}{s}} - B_{\frac{1}{t}} \sim N(0, \frac{1}{s}) - \frac{1}{t})$, and $B_{\frac{1}{t}} \sim N(0, \frac{1}{t})$, and since $\frac{1}{t} < \frac{1}{s}$ we know $B_{\frac{1}{s}} - B_{\frac{1}{t}} \perp B_{\frac{1}{t}}$. So $X_t - X_s \sim N(0, s^2(\frac{1}{s} - \frac{1}{t}) + (t-s)^2\frac{1}{t}) = N(0, t-s)$.

Second let $\mathcal{G}_t := \sigma(X_s : 0 \leq s \leq t)$, we need to check $X_t - X_s \perp \mathcal{G}_s, \forall 0 \leq s \leq t$. For $s = 0$ we get $\mathcal{G}_s = \{\emptyset, \Omega\}$, so it's obvious. Now assume $s > 0$. Then $\mathcal{G}_s = \sigma(B_{\frac{1}{r}} : 0 \leq r \leq s)$. Only need to prove for any finite set $I \subset [0, s]$, we have $X_t - X_s \perp \sigma(B_{\frac{1}{r}} : r \in I)$. Only need to check $\mathbb{E}(B_{\frac{1}{r}}(X_t - X_s)) = 0$ because they are all normal distributed random variable. Easily $\mathbb{E}(B_{\frac{1}{r}}X_t) = \mathbb{E}((B_{\frac{1}{r}} - B_{\frac{1}{t}})tB_{\frac{1}{t}}) + \mathbb{E}(tB^2_{\frac{1}{t}}) = 1$, and $\mathbb{E}(B_{\frac{1}{r}}X_s) = \mathbb{E}((B_{\frac{1}{r}} - B_{\frac{1}{s}})sB_{\frac{1}{s}}) + \mathbb{E}(sB^2_{\frac{1}{s}}) = 1$ So $\mathbb{E}(B_{\frac{1}{r}}(X_t - X_s)) = 0$.

Finally we need to check X_t is continuous a.s. Easily for $t \neq 0$ we know X_t is continuous at t. Only need to check X_t is continuous at 0 with probability 1. Easily to know $(-B_t : t \geq 0)$ is standard Brownian motion, too. So $\limsup_{t\to\infty} \frac{-B_t}{\sqrt{2t \log \log t}} = 1$. So $\limsup_{t\to\infty} \frac{|B_t|}{2t \log \log t} = 1$. So $\limsup_{t\to 0+} |X_t| = \lim_{t\to\infty} |\frac{1}{t}B_t| \leq \limsup_{t\to\infty} \frac{\sqrt{2t \log \log t}}{t} = 0$. So $\lim_{t\to 0+} |X_t| = 0$. So X_t is continuous with probability 1. □

1.4 普瓦松过程

Definition 15. $(N_t : t \geq 0)$ 是非负整数不降随机过程，$\alpha \geq 0$ 满足:

1. $\forall s, t \geq 0$，$N_{s+t} - N_s \sim P(\alpha t)$，即：$\mathbb{P}(N_{s+t} - N_s = k) = \frac{\alpha^k t^k}{k!} \mathrm{e}^{-\alpha t}$;
2. $\forall 0 \leq t_0 < \cdots < t_n$，$N_0, N_{t_1} - N_{t_0}, \cdots, N_{t_n} - N_{t_{n-1}}$ 相互独立。

称 $(N_t : t \geq 0)$ 是普瓦松过程，参数为 α。

1.4.1 跳跃间隔时间

$(N_t : t \geq 0)$ 以 α 为参数的普瓦松过程，$S_0 = 0, n \geq 1$，$S_n = \inf\{t \geq 0 : N_t - N_0 \geq 0\}$，$\eta_n = S_n - S_{n-1}$。$S_n$ 是 $(N_t : t \geq 0)$ 第 n 次跳跃等待时间，η_n 第 $n-1$ 次跳跃到第 n 次跳跃的间隔时间。

Theorem 8. $\{\eta_n : n \geq 1\}$ 独立同分布，服从 $Exp(\alpha)$。$S_n, n \geq 1$，服从 $\Gamma(1, \alpha)$。证明见 *P19.*

1.4.2 轨道重构

Theorem 9. $\{\eta_n : n \geq 1\}$ 独立同分布，服从 $Exp(\alpha), \gamma > 0$。$S_0 = 0, S_n = \sum_{k=1}^{n} \eta_k$，则：

$$N_t = \sum_{n=1}^{\infty} \mathbb{1}_{S_n \leq t} = \sup\{n \geq 0 : S_n \leq t\}.$$

则随机过程 $(N_t : t \geq 0)$ 是以参数为 α 的普瓦松过程。

1.4.3 长时间极限行为

$(N_t : t \geq 0)$ 以 α 为参数的普瓦松过程。

Theorem 10 (普瓦松过程的强大数定律)**.** $\lim_{t\to\infty} \frac{N_t}{t} \overset{a.s.}{=} \alpha$ 见 *p23.*

Theorem 11 (普瓦松过程的中心极限定理)**.** $\lim_{t\to\infty} \mathbb{P}(\frac{N_t - \alpha t}{\sqrt{\alpha t}} \leq x) = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{x} \mathrm{e}^{-\frac{y^2}{2}} dy$。见 *p23.*

Corollary 1. $s, x > 0$，$\lim_{\lambda\to\infty} \mathrm{e}^{-\lambda s} \sum_{k \leq \lambda x} \frac{(\lambda s)^k}{k!} = \mathbb{1}_{0<s<t} + \frac{1}{2}\mathbb{1}_{\{s=x\}}$。见 *p24.*

Theorem 12 (拉普拉斯变换的反演公式)**.** ξ 是非负随机变量，L 为其拉普拉斯变换，则 $\forall x > 0$，

$$\lim_{\lambda\to\infty} \sum_{k \leq \lambda x} \frac{(-\lambda)^k}{k!} \frac{\mathrm{d}^k}{\mathrm{d}\lambda^k} L(\lambda) = \mathbb{P}(\xi < x) + \frac{1}{2}\mathbb{P}(\xi = x)$$

见 *p24.*

1.4.4 复合普瓦松过程

Definition 16. μ 是 $\mathbb{R}$ 上概率 $\mu(\{0\}) = 0$。$(N_t : t \geq 0)$ 以 $\alpha \geq 0$ 为参数的零初值普瓦松过程，$\{\xi_n : n \geq 1\}$ 与 N_t 独立，具有相同分布 μ，X_0 与 $(N_t), \{\xi_n\}$ 独立。令：$X_t = X_0 + \sum_{n=1}^{N_t} \xi_n, t \geq 0$，则 $(X_t : t \geq 0)$ 是以 α 为跳跃速度，μ 为跳跃分布的**复合普瓦松过程**。

Theorem 13 (复合普瓦松过程的性质)**.** $(X_t : t \geq 0)$ 为如上定义的复合普瓦松过程，则复合普瓦松过程的性质如下：

1. $\forall s, t \geq 0, \theta \in \mathbb{R}$,
$$\mathbb{E}\mathrm{e}^{\mathrm{i}\theta(X_{s+t} - X_s)} = \exp(\alpha t \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}\theta x} - 1)\mu(dx))$$
;

2. $\forall 0 \leq t_0 < \cdots < t_n, X_{t_0}, X_{t_1} - X_{t_0}, \cdots, X_{t_n} - X_{t_{n-1}}$ 相互独立。

Lemma 2. *Assume* $(N_t : t \geq 0)$ *is a random process, and* $\mathbb{P}(\forall t \in [0, \infty), \lim_{s\to t+} N_s = N_t, \forall t \in (0, \infty), \lim_{s\to t-} N_s \in \mathbb{R}) = 1$. *Assume* $\alpha > 0$*, and* $\forall t, s \geq 0$*, we have* $N_{t+s} - N_s \sim Possion(\alpha t)$*. Then* $(N_t : t \geq 0)$ *is a Possion process* $\iff \forall 0 \leq s \leq t, N_t - N_s \perp \mathcal{F}_s$*, where* $\mathcal{F}_s := \sigma(N_x : x \leq s)$*.*

证明. "$\implies$": To prove $N_t - N_s \perp \mathcal{F}_s$, only need to prove for $t_1 < t_2 < \cdots < t_{n-1} = s < t = t_n$, we have $N_t - N_s \perp \sigma(N_{t_k} : k = 1, \cdots, n-1)$. Easily $N_t - N_s \perp \sigma(N_{t_{k+1}} - N_{t_k}, N_{t_1} : k = 1, \cdots, n-2) = \sigma(N_{t_k} : k = 1, \cdots, n-1)$, so $N_t - N_s \perp \mathcal{F}_s$.

"$\impliedby$": For $t_1 < \cdots < t_n$, we need to prove $N_{t_1}, N_{t_{k+1}} - N_{t_k} : 1 \le k \le n-1$ are independent. Use MI to n. When $n = 1$ it's obvious. Assume we have proved it for certain $n \ge 1$, now consider $n+1$. Since $N_{t_{k+1}} - N_{t_k} \in \mathcal{F}_{t_n}, k = 1, \cdots, n-1$, we have $N_{t_{n+1}} - N_{t_n} \perp \sigma(N_{t_1}, N_{t_{k+1}} - N_{t_k} : k = 1, \cdots, n-1)$. So $\mathbb{P}(N_{t_1} \in A_1, N_{t_{k+1}} - N_{t_k} \in A_{k+1}, k = 1, \cdots, n) = \mathbb{P}(N_{t_1} \in A_1, N_{t_{k+1}} - N_{t_k} \in A_{k+1}, k = 1, \cdots, n-1)\mathbb{P}(N_{t_{n+1}} - N_{t_n} \in A_{n+1})$. By Induction assumption we get $\mathbb{P}(N_{t_1} \in A_1, N_{t_{k+1}} - N_{t_k} \in A_{k+1}, k = 1, \cdots, n) = \mathbb{P}(N_{t_1} \in A_1)\prod_{k=1}^{n}\mathbb{P}(N_{t_{k+1}} - N_{t_k} \in A_{k+1})$. So finally we get $N_{t_1}, N_{t_{k+1}} - N_{t_k} : 1 \le k \le n$ are independent. □

PROBLEM 15 Assume $(N_t : t \ge 0)$ is a Possion process with parameter α. Let $P(t) := \mathbb{P}(2 \nmid N_t), Q(t) := \mathbb{P}(2 \mid N_t)$. Prove that $P(t) = \mathrm{e}^{-\alpha t}\sinh(\alpha t), Q(t) = \mathrm{e}^{-\alpha t}\cosh(\alpha t)$.

SOLUTION . Easily to get

$$P(t) = \sum_{k=0}^{\infty}\mathbb{P}(N_t = 2k+1) = \sum_{k=0}^{\infty}\frac{(\alpha t)^{2k=1}}{(2k+1)!}\mathrm{e}^{-\alpha t}$$

Noting that $\sinh(\alpha t) = \frac{1}{2}\left(\sum_{k=0}^{\infty}\frac{(\alpha t)^n}{n!} - \sum_{k=0}^{\infty}\frac{(-\alpha t)^n}{n!}\right) = \sum_{k=0}^{\infty}\frac{(\alpha t)^{2k+1}}{(2k+1)!}$, we easily obtain $P(t) = \mathrm{e}^{-\alpha t}\sinh(\alpha t)$. Noting $P(t) + Q(t) = 1$, we easily get $Q(t) = 1 - P(t) = \mathrm{e}^{-\alpha t}\cosh(\alpha t)$. □

PROBLEM 16 Assume $(N_t : t \ge 0)$ is a Possion process with parameter α. Prove that $\lim_{t\to\infty}\frac{N_t}{t} = \alpha, a.s.$.

Lemma 3. *Assume $(N_t : t \ge 0)$ is a Possion process with parameter α. Then $\mathbb{P}(\forall 0 \le s \le t, N_s \le N_t) = 1$.*

证明. For $s, t \in \mathbb{Q} \wedge 0 \le s \le t$, we have $\mathbb{P}(N_s > N_t) = 0$ since $N_t - N_s \sim Possion(\alpha(t-s))$. So we get $\mathbb{P}(\exists s, t \in \mathbb{Q}, 0 \le s \le t, N_s > N_t) = 0$.

Now we will prove $\exists s, t \in \mathbb{R}, 0 \le s \le t, N_s > N_t \implies \exists a, b \in \mathbb{Q}, 0 \le a \le b, N_a > N_b$. Let $a_n = \frac{\lceil ns \rceil}{n}, b_n = \frac{\lceil nt \rceil}{n}$. Then $\lim a_n = s, \lim b_n = t$. Easily $a_n \ge s, b_n \ge t$. So since N is continuous we get $\lim N_{a_n} = N_s, \lim N_{b_n} = N_t$. Since $N_s > N_t$, we get $\exists n, N_{a_n} > N_{b_n}$. Let $a = a_n, b = b_n$ will work.

So $\mathbb{P}(\forall 0 \le s \le t, N_s \le N_t) = 1 - \mathbb{P}(\exists 0 \le s \le t, N_s > N_t) = 1 - \mathbb{P}(\exists s, t \in \mathbb{Q}, 0 \le s \le t, N_s > N_t) = 1 - 0 = 1$. □

SOLUTION . Consider $N_n : n \in \mathbb{N}$. Let $X_n := N_n - N_{n-1}, n \ge 1$. Then easily $(X_n : n \in \mathbb{N}^+)$ is i.i.d sequence and $X_1 \sim Possion(\alpha)$. So from the strong law of large numbers we get $\lim_{n\to\infty}\frac{N_n}{n} = \alpha$. From Lemma 3 we get $\frac{\lfloor t \rfloor}{t}\frac{N_{\lfloor t \rfloor}}{\lfloor t \rfloor} \le \frac{N_t}{t} \le \frac{N_{\lceil t \rceil}}{\lceil t \rceil}\frac{\lceil t \rceil}{t}, \forall t \in \mathbb{R}$, let $t \to \infty$ we get $\lfloor t \rfloor, \lceil t \rceil \to \infty$, and $\lfloor t \rfloor \sim t \sim \lceil t \rceil$. So finally we get $\lim_{t\to\infty}\frac{N_t}{t} = \alpha$. □

PROBLEM 17 Assume $(N_t : t \geq 0)$ is a Possion process with parameter $\alpha > 0$. Prove that $\frac{N_t - \alpha t}{\sqrt{\alpha t}} \xrightarrow{d} N(0,1)$.

SOLUTION. Consider $N_n : n \in \mathbb{N}$. Let $X_n := N_n - N_{n-1}, n \geq 1$. Then easily $(X_n : n \in \mathbb{N}^+)$ is i.i.d sequence and $X_1 \sim Possion(\alpha)$. Easily $\mathbb{V}(X_n) = \alpha < \infty, \mathbb{E}(X_n) = \alpha$. So from the central limit theorem we get $\frac{N_n - \alpha n}{\sqrt{\alpha n}} \xrightarrow{d} N(0,1)$. Noting $\frac{N_t - \alpha t}{\sqrt{\alpha t}} = \frac{N_{\lfloor t \rfloor} - \alpha \lfloor t \rfloor}{\sqrt{\alpha \lfloor t \rfloor}} \frac{\sqrt{\lfloor t \rfloor}}{\sqrt{t}} + \frac{N_t - N_{\lfloor t \rfloor} - \alpha(t - \lfloor t \rfloor)}{\sqrt{\alpha t}}$. Let $t \to \infty$ we get $\lfloor t \rfloor, \to \infty$, and $\lfloor t \rfloor \sim t$. Noting $N_t - N_{\lfloor t \rfloor} \stackrel{d}{=} N_{t - \lfloor t \rfloor}$, and $t - \lfloor t \rfloor \leq 1$, we easily get $\frac{N_t - N_{\lfloor t \rfloor}}{\sqrt{\alpha t}} \xrightarrow{d} 0$. Easily $\frac{\alpha(t - \lfloor t \rfloor)}{\alpha t} \to 0$, so finally we get that $\frac{N_t - \alpha t}{\sqrt{\alpha t}} \xrightarrow{d} N(0,1)$ □

PROBLEM 18 Assume $(X_t : t \geq 0), (Y_t : t \geq 0)$ are two independent Possion processes with parameter α, β respectively. Prove that $(X_t + Y_t : t \geq 0)$ is Possion process with parameter $\alpha + \beta$.

SOLUTION. Write $Z_t = X_t + Y_t$. First we prove $Z_{t+s} - Z_s \sim Possion((\alpha+\beta)t)$. Since $X_{t+s} - X_s \sim Possion(\alpha t), Y_{t+s} - Y_s \sim Possion(\beta t)$, and $X_{t+s} - X_s \perp Y_{s+t} - Y_s$, easily to get $Z_{t+s} - Z_s = X_{t+s} - X_s + Y_{s+t} - Y_s \sim Possion((\alpha+\beta)t)$.

Second we prove $\forall 0 \leq s \leq t, Z_t - Z_s \perp \mathcal{H}_s$, where $\mathcal{H}_s = \sigma(Z_x : 0 \leq x \leq s)$. Easily $Z_t - Z_s \in \sigma(X_t - X_s, Y_t - Y_s)$. Easily $X_t - X_s \perp \mathcal{F}_s := \sigma(X_x : 0 \leq x \leq s)$ since $(X_x : x \geq 0)$ is Possion process. Since $(X_x : x \geq 0) \perp (Y_x : x \geq 0)$, we get $X_t - X_s \perp \mathcal{G}_s := \sigma(Y_x : 0 \leq x \leq s)$. For the same reason we get $Y_t - Y_s \perp \mathcal{F}_s, \mathcal{G}_s$. So we get $Z_t - Z_s \perp \sigma(F_s, G_s) \supset \mathcal{H}_s$.

Finally, we prove that $\mathbb{P}(\forall t \in [0,\infty), \lim_{s \to t+} Z_s = Z_t, \forall t \in (0,\infty), \lim_{s \to t-} Z_s \in \mathbb{R}) = 1$ Since $Z_t = X_t + Y_t$, and $\mathbb{P}(\forall t \in [0,\infty), \lim_{s \to t+} Y_s = Y_t, \forall t \in (0,\infty), \lim_{s \to t-} Y_s \in \mathbb{R}) = 1$, $\mathbb{P}(\forall t \in [0,\infty), \lim_{s \to t+} X_s = X_t, \forall t \in (0,\infty), \lim_{s \to t-} X_s \in \mathbb{R}) = 1$, obvious we get the requirement.

All in all, $(X_t + Y_t : t \geq 0)$ is a Possion process with parameter $\alpha + \beta$. □

PROBLEM 19 Assume $(\xi_n : n \in \mathbb{N}^+)$ is a sequence of i.i.d. random variable ranging in $\mathbb{Z}^d$. Let $X_n = X_0 + \sum_{k=1}^n \xi_k$, and $X_0 \perp (\xi_n : n \in \mathbb{N}^+)$ ranging in $\mathbb{Z}^d$, too. Assume $(N_t : t \geq 0)$ is a Possion process with parameter $\alpha > 0$. Discuss $\frac{X_{N_t}}{t}$ when $t \to \infty$.

SOLUTION. First we prove that $\lim_{t \to \infty} N_t = \infty, a.s.$. We have $\mathbb{P}(\sup_t N_t \geq n) \geq \mathbb{P}(N_t \geq n), \forall t, \forall n \in \mathbb{N}$. Easily $\lim_{t \to \infty} \mathbb{P}(N_t \geq n) = 1$, so $\mathbb{P}(\sup_t N_t \geq n) = 1, \forall n \in \mathbb{N}$. So $\mathbb{P}(\sup_t N_t = \infty) = 1$. Noting Lemma 3 we easily get $\mathbb{P}(\lim_{t \to \infty} N_t = \infty) = 1$.

Now we can decompose $\frac{X_{N_t}}{t}$ into $\frac{X_{N_t}}{N_t} \frac{N_t}{t}$. We have proved that $\frac{N_t}{t} \to \alpha, a.s.$ in Problem 21, so we only need to find $\frac{X_{N_t}}{N_t}$. Since $N_t \to \infty, a.s.$, we only need to find $\frac{X_n}{n}$ when $n \to \infty$.

If $\mathbb{E}(\xi_1)$ exists, then easily $\frac{X_n}{n} \to \mathbb{E}(\xi_1), a.s.$. Then we easily get $\frac{X_{N_t}}{t} \to \alpha \mathbb{E}(\xi_1), a.s.$. □

1.5 普瓦松随机测度

1.5.1 定义和存在性

$(E, \mathcal{E})$ 为可测空间，μ 为 $(E, \mathcal{E})$ 上的 σ 有限测度。

Definition 17. $\{X(B):B\in\mathcal{E}\}$ 为取非负整数值随机过程，满足:

1. $\forall B\in\mathcal{E}:\mu(B)<\infty$，则 $\mathbb{E}(X(B))=\mu(B)$。

2. $\forall\{B_n:n\geq 1\}\in\mathcal{E}$ 两两不交，则 $X(\bigcup_{k=1}^{\infty}B_k)=\sum_{k=1}^{\infty}X(B_k)$。

称 $\{X(B):B\in\mathcal{E}\}$ 为以 μ 为强度的**整数值随机测度**。

Definition 18. $\{X(B):B\in\mathcal{E}\}$ 为整数值随机测度，满足:

1. $\forall B\in\mathcal{E}:\mu(B)<\infty$，则 $X(B)\sim P(\mu(B))$，即：$\mathbb{P}(X(B)=k)=\frac{\mu(B)^k}{k!}\mathrm{e}^{-\mu(B)},k=0,\cdots,n,\cdots$。

2. $\forall\{B_n:n\geq 1\}\in\mathcal{E}$ 两两不交，则 $\{X(B_k):k\in\mathbb{N}^+\}$ 相互独立。

称 $\{X(B):B\in\mathcal{E}\}$ 为以 α 为强度的**普瓦松随机测度**。

Theorem 14 (普瓦松随机测度的充要条件)**.** *X 为 $(E,\mathcal{E})$ 上以 μ 为强度的整数值随机测度，则 X 为普瓦松随机测度的充要条件是 $\forall n\in\mathbb{N}^+,\xi_k\in\mathbb{R},B_k\in\mathcal{E},k=1,\cdots,n$，$B_i\cap B_j=\varnothing,i\neq j$，当 $\mu(B_k)<\infty,k=1,\cdots,n$，则*

$$\mathbb{E}\exp(\mathrm{i}\sum_{k=1}^{n}\theta_k X(B_k))=\exp(\sum_{k=1}^{n}(\mathrm{e}^{\mathrm{i}\theta_k}-1)\mu(B_k))$$

见 *p28.*

Theorem 15 (普瓦松随机测度的存在性)**.** *μ 为非零有限测度，$\eta\sim P(\mu(E))$，$\{\xi_k:k\in\mathbb{N}^+\}$ i.i.d. 服从 $\mu(E)^{-1}\mu$。$\eta,\xi_1,\cdots,\xi_n,\cdots$ 相互独立。令 $X=\sum_{j=1}^{\eta}\delta_{\xi_j}$，则 X 为以 μ 为强度的普瓦松随机测度。见 p29.*

Theorem 16. *μ 为 σ 有限测度，$\{E_k:k\in\mathbb{N}^+\}\subset\mathcal{E},\mu(E_k)<\infty,k\in\mathbb{N}^+$，$E=\bigcup_{k\in\mathbb{N}^+}E_k$，$E_i\cap E_j=\varnothing,i\neq j$。则存在 X_k 为 E_k 上的普瓦松随机测度强度为 $\mu_k:=\mu|_{E_k},k\in\mathbb{N}^+$。令 $X=\sum_{j=1}^{\infty}X_j$，则 X 为以 μ 为强度的普瓦松随机测度。见 p29.*

1.5.2 积分与补偿的测度

Theorem 17 (普瓦松随机测度的充要条件 2)**.** *X 为 $(E,\mathcal{E})$ 上以 μ 为强度的整数值随机测度，则 X 为普瓦松随机测度的充要条件是 $\forall f\in\mathbb{R}^{\mathbb{R}}:\mu(f)<\infty,$，则*

$$\mathbb{E}\exp(\mathrm{i}X(f))=\exp(\int_E(\mathrm{e}^{\mathrm{i}f(x)}-1)\mu(dx))$$

见 *p28.*

1.5.3 应用

Theorem 18 (复合普瓦松过程构造)**.** ν 为 $\mathbb{R}$ 上非零有限测度。$N(ds,dz)$ 为 $(0,\infty)\times\mathbb{R}$ 上以 $ds\nu(dz)$ 为强度的普瓦松随机测度，ds 为勒贝格测度，X_0 与 $N(ds,dz)$ 独立，

$$X_t = X_0 + \iint_{(0,t]\times\mathbb{R}} zN(ds,dz), t\geq 0$$

则 $(X_t: t\geq 0)$ 具有跳跃速率 α 和跳跃分布 $\mu := \nu(\mathbb{R})^{-1}\nu$ 的复合普瓦松过程。

Theorem 19 (复合普瓦松过程构造 2)**.** μ 为 $\mathbb{R}$ 上非零 σ 有限测度，$\mu(\{0\})=0$，q 为 $\mathbb{R}$ 上非负博雷尔可测函数，$0<\beta:=\mu(q)<\infty$。$N(ds,dz,du)$ 为 $(0,\infty)\times\mathbb{R}\times(0,\infty)$ 上以 $ds\mu(dz)du$ 为强度的普瓦松随机测度，ds 为勒贝格测度，X_0 与 $N(ds,dz)$ 独立，

$$X_t = X_0 + \iiint_{(0,t]\times\mathbb{R}\times[0,q(z)]} zN(ds,dz,du), t\geq 0$$

则 $(X_t: t\geq 0)$ 具有跳跃速率 β 和跳跃分布 $\beta^{-1}q(z)\mu(dz)$ 的复合普瓦松过程。

PROBLEM 20 Assume $(N_t: t\geq 0)$ is Possion process with parameter α, and $\{\xi_n: n\in\mathbb{N}^+\}$ is a sequence of i.i.d random variable. More over, assume $(N_t: t\geq 0)\perp\{\xi_n: n\in\mathbb{N}^+\}$. Let $X_t=\sum_{k=1}^{N_t}\xi_k$. Let $r>0$, prove that:

1. $(N_{t+r}-N_r: t\geq 0)$ is Possion process.
2. $\{\xi_{N_r+n}: n\in\mathbb{N}^+\}$ is also i.i.d sequence with the same distribution of $\{\xi_n: n\in\mathbb{N}^+\}$.
3. $(N_{t+r}-N_r: t\geq 0)\perp(\xi_{N_r+k}: k\in\mathbb{N}^+)$.
4. For $0=t_0<t_1<\cdots<t_n$, we have $(X_{t_1}, X_{t_{k+1}}-X_{t_k}: k=1,2,\cdots,n-1)$ are independent.

SOLUTION.

1. Let $\mathcal{F}_t := \sigma(N_s: 0\leq s\leq t)$ and $\mathcal{G}_t := \sigma(N_{r+s}-N_r: 0\leq s\leq t)$. For $0\leq s\leq t$, we have $N_{t+r}-N_r-(N_{s+r}-N_r)=N_{t+r}-N_{s+r}\sim Possion(\alpha(t-s))$. And easily to know $N_{r+s}-N_r\in\mathcal{F}_{r+s}$, so $\mathcal{G}_t\subset\mathcal{F}_{t+r}, \forall t\geq 0$. Since $(N_t: t\geq 0)$ is Possion process, easily $\mathcal{F}_{s+r}\perp N_{t+r}-N_{s+r}$. Since $\mathcal{G}_t\subset\mathcal{F}_{t+r}$, we obtain $\mathcal{G}_t\perp N_{t+r}-N_{s+r}=N_{t+r}-N_r-(N_{s+r}-N_r)$. Easily since N_t is right-continuous we get $N_{t+r}-N_r$ is right-continuous. For the same reason, we know $\forall s\in[0,\infty), \lim_{t\to s-}N_{t+r}-N_r$ exists. So $(N_{t+r}: t\geq 0)$ is Possion process.
2. Only need to prove that for $m\in\mathbb{N}^+$, the distribution of $(\xi_{N_r+k}: 1\leq k\leq m)$ is same as

that of $(\xi_k : 1 \le k \le m)$. For $A_1, A_2, \cdots, A_m \in \mathcal{B}$, we have:

$$\begin{aligned}
&\mathbb{P}(\xi_{N_r+k} \in A_k, 1 \le k \le m)\\
&=\sum_{t=0}^{\infty}\mathbb{P}(\xi_{N_r+k} \in A_k, 1 \le k \le m, N_r = t)\\
&=\sum_{t=0}^{\infty}\mathbb{P}(\xi_{t+k} \in A_k, 1 \le k \le m, N_r = t)\\
(N_r \perp (\xi_n : n \in \mathbb{N}^+)) &=\sum_{t=0}^{\infty}\mathbb{P}(\xi_{t+k} \in A_k, 1 \le k \le m)\mathbb{P}(N_r = t)\\
&=\sum_{t=0}^{\infty}\prod_{k=1}^{m}\mathbb{P}(\xi_{k+t} \in A_k)\mathbb{P}(N_r = t)\\
&=\sum_{t=0}^{\infty}\prod_{k=1}^{m}\mu(A_k)\mathbb{P}(N_r = t)\\
&=\prod_{k=1}^{m}\mu(A_k)
\end{aligned} \tag{6}$$

where μ is the distribution of ξ_1. So we get for $m \in \mathbb{N}^+$, the distribution of $(\xi_{N_r+k} : 1 \le k \le m)$ is same as that of $(\xi_k : 1 \le k \le m)$.

3. We know that $\forall t \in \mathbb{N}^+, \xi_{N_r+t} \in \sigma(N_r, \xi_k : k \in \mathbb{N}^+)$. So $\sigma(\xi_{N_r+k} : k \in \mathbb{N}^+) \subset \sigma(N_r, \xi_k : k \in \mathbb{N}^+)$. Since $(N_t : t \ge 0)$ is Possion process, we get that $N_{t+r} - N_r \perp N_r, \forall t \ge 0$. So $\sigma(N_{t+r} - N_r : t \ge 0) \perp N_r$. Easily $\sigma(N_{t+r} - N_r : t \ge 0) \perp \sigma(\xi_k : k \in \mathbb{N}^+)$, so finally we get that $\sigma(N_{t+r} - N_r : t \ge 0) \perp \sigma(N_r, \xi_k : k \in \mathbb{N}^+) \supset \sigma(\xi_{N_r+k} : k \in \mathbb{N}^+)$.

4. $\forall 0 = t_0 < t_1 < \cdots < t_n$, then $X_{t_1} = \sum_{i=1}^{N_{t_1}} \xi_i, X_{t_k} - X_{t_{k-1}} = \sum_{i=1}^{N_{t_k} - N_{t_{k-1}}} \xi_{N_{t_{k-1}}+i}\xi_i, k =$

$2, \cdots, n$, then $\forall \{A_k \in \mathscr{E} : k = 1, \cdots, n\}$,

$$
\begin{aligned}
&\mathbb{P}(\bigcap_{k=1}^{n} \sum_{i=1}^{N_{t_k}-N_{t_{k-1}}} \xi_{i+N_{t_{k-1}}} \in A_k) \\
=&\mathbb{P}(\bigcup_{0\le u_1\le\cdots\le u_n} \{ \sum_{i=u_{k-1}+1}^{u_k} \xi_i \in A_k, N_{t_k} = u_k, k = 1, \cdots, n\}) \\
=&\sum_{0\le u_1\le\cdots\le u_n} \mathbb{P}(\sum_{i=u_{k-1}+1}^{u_k} \xi_i \in A_k, k = 1, \cdots, n | N_{t_k} = u_k, k = 1, \cdots, n\})\mathbb{P}(N_{t_k} = u_k, k = 1, \cdots, n) \\
=&\sum_{0\le u_1\le\cdots\le u_n} \mathbb{P}(\sum_{i=u_{k-1}+1}^{u_k} \xi_i \in A_k, k = 1, \cdots, n\})\mathbb{P}(N_{t_k} = u_k, k = 1, \cdots, n) \\
=&\sum_{0\le u_1\le\cdots\le u_n} \prod_{k=1}^{n} \mathbb{P}(\sum_{i=u_{k-1}+1}^{u_k} \xi_i \in A_k) \prod_{j=1}^{n} \mathbb{P}(N_{t_j} = u_j) \\
=&\sum_{0\le u_1\le\cdots\le u_n} \prod_{k=1}^{n} \mathbb{P}(\sum_{i=1}^{u_k-u_{k-1}} \xi_{u_{k-1}+i} \in A_k) \prod_{j=1}^{n} \mathbb{P}(N_{t_j} - N_{t_{j-1}} = u_j - u_{j-1}) \\
=&\sum_{0\le u_1\le\cdots\le u_n} \prod_{k=1}^{n} \mathbb{P}(\sum_{i=1}^{u_k-u_{k-1}} \xi_{u_{k-1}+i} \in A_k)\mathbb{P}(N_{t_k} - N_{t_{k-1}} = u_k - u_{k-1}) \\
=&\sum_{u_1-u_0\in\mathbb{N}} \cdots \sum_{u_n-u_{n-1}\in\mathbb{N}} \prod_{k=1}^{n} \mathbb{P}(\sum_{i=1}^{u_k-u_{k-1}} \xi_{u_{k-1}+i} \in A_k, N_{t_k} - N_{t_{k-1}} = u_k - u_{k-1}) \\
=&\prod_{k=1}^{n} \sum_{u_k-u_{k-1}\in\mathbb{N}} \mathbb{P}(\sum_{i=1}^{u_k-u_{k-1}} \xi_{u_{k-1}+i} \in A_k, N_{t_k} - N_{t_{k-1}} = u_k - u_{k-1}) \\
=&\prod_{k=1}^{n} \mathbb{P}(\sum_{i=1}^{N_{t_k}-N_{t_{k-1}}} \xi_{i+N_{t_{k-1}}} \in A_k)
\end{aligned}
\tag{7}
$$

□

PROBLEM 21 Assume that X is Possion random measure on $(E, \mathcal{E})$ with intensity μ, which is a σ-finite measure. Assume $f : E \to \mathbb{R}$ is measurable and non-negative, prove that:

$$\mathbb{E}(\mathrm{e}^{-X(f)}) = \exp\left\{- \int_E (1 - \mathrm{e}^{-f(x)})\mu(\mathrm{d}x)\right\}$$

SOLUTION. Let $\mathcal{L} := \{g \in \mathcal{M}(E, [0, \infty)) : \mathbb{E}(\mathrm{e}^{-X(f)}) = \exp\left(- \int_E (1 - \mathrm{e}^{-f(x)})\mu(\mathrm{d}x)\right)\}$. First we prove that if g is simple measurable function from E to $[0, \infty)$, then $g \in \mathcal{L}$. Assume $g(x) = \sum_{k=1}^{n} a_k \mathbb{1}_{A_k}(x)$, where $A_k \in \mathcal{E}, a_k > 0, A_i \cap A_j = \emptyset$. Then $\mathbb{E}(\exp(-X(g))) = \mathbb{E}(\prod_{k=1}^{n} \exp(-a_k X(A_k))) = \prod_{k=1}^{n} \mathbb{E}(\exp(-a_k X(A_k)))$, since $X(A_k) : k = 1, \cdots, n$ are independent. Easily to know

$$\mathbb{E}(\exp(-a_k X(A_k))) = \sum_{i=0}^{\infty} \mathbb{P}(X(A_k) = i) \exp(-a_k i) = \sum_{i=0}^{\infty} \frac{\exp(-\mu(A_k))\mu(A_k)^i}{i!} \exp(-a_k i)$$

Noting that

$$\exp(-\int_E(1-\exp(-a_k\mathbb{1}_{A_k}(x)))\mu(\mathrm{d}x)) = \exp(\exp(-a_k)\mu(A_k)-\mu(A_k)) = \exp(-\mu(A_k))\sum_{i=0}^{\infty}\frac{(\exp(-a_k)\mu(A_k))^i}{i!}$$

we get $\mathbb{E}(\exp(-a_kX(A_k))) = \exp(-\int_E(1-\exp(-a_k\mathbb{1}_{A_k}(x)))\mu(\mathrm{d}x))$. Noting $\int_E(1-\exp(-g(x)))\mu(\mathrm{d}x) = \sum_{k=1}^n\int_E(1-\exp(-a_k\mathbb{1}_{A_k}(x)))\mu(\mathrm{d}x)$, we get $\mathbb{E}(\exp(-X(g))) = \exp\left(-\int_E(1-\exp(-g(x)))\mu(\mathrm{d}x)\right)$.

Now for non-negative function f, consider f_n satisfy that $\forall n, f_n$ is simple, and $f_n\nearrow f$ and $f_n\geq 0$. Then easily to know $\mathbb{E}(\exp(-X(f))) = \lim_{n\to\infty}\mathbb{E}(\exp(-X(f_n))) = \lim_{n\to\infty}\exp(-\int_E(1-\exp(-f_n(x)))\mu(\mathrm{d}x)) = \exp(-\int_E(1-\exp(-f(x)))\mu(\mathrm{d}x))$. □

PROBLEM 22 Assume μ is finite measure on $(E,\mathcal{E})$, and X is Possion random measure with intensity μ. Assume $\phi:(E,\mathcal{E})\to(F,\mathcal{F})$ is measurable, prove that $X\circ\phi^{-1}$ is Possion random measure with intensity $\mu\circ\phi^{-1}$.

SOLUTION. Assume $B_k\in\mathcal{F},\forall k\in\mathbb{N}$ and $\forall i\neq j, B_i\cap B_j=\emptyset$. Then $X\circ\phi^{-1}(\bigcup_{k\in\mathbb{N}}B_k) = X(\bigcup_{k\in\mathbb{N}}\phi^{-1}(B_k)) = \sum_{k\in\mathbb{N}}X(\phi^{-1}(B_k))$. Since X is possion random measure with intensity μ, and for $B_1,\cdots,B_n\in\mathcal{F}$ and $B_i\cap B_j=\emptyset$, we have $\phi^{-1}(B_k)$ are disjoint set in $(E,\mathcal{E})$, so $\mathbb{E}(\exp(\mathrm{i}\sum_{k=1}^n\alpha_kX\circ\phi^{-1}(B_k))) = \exp(\sum_{k=1}^n(\exp(\mathrm{i}\alpha_k)-1)\mu\circ\phi^{-1}(B_k))$. So $X\circ\phi^{-1}$ is Possion random measure on $(F,\mathcal{F})$ with intensity $\mu\circ\phi^{-1}$. □

PROBLEM 23 Assume $\alpha\geq 0$, and μ is probability measure on $\mathbb{R}$ with $\mu(\{0\})=0$. Let $N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)$ is Possion random measure on $\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}^+$ with intensity $\mathrm{d}s\mu(\mathrm{d}z)\,\mathrm{d}u$. Let $Y_t = Y_0+\int_0^t\int_{\mathbb{R}}\int_0^\alpha zN(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)$, where $Y_0\perp N$. Prove that $(Y_t: t\geq 0)$ is compound Possion process with rate α and jumping distribution μ.

SOLUTION. We know that $\forall t\geq 0,\forall r: 0\leq r\leq t, Y_r\in\sigma(N(B): B\subset[0,r]\times\mathbb{R}\times[0,\alpha])$. And $\forall w\geq t$, $Y_w-Y_t\in\sigma(N(B): B\subset(t,w]\times\mathbb{R}\times[0,\alpha])$. Easily $(t,w]\cap[0,r]=\emptyset$, so we get $Y_w-Y_t\perp(Y_r: 0\leq r\leq t)$. Now we only need to check the distribution of Y_t+w-Y_t for $t,w\geq 0$.

Easily to know that:

$$\begin{aligned}\mathbb{E}\left(\mathrm{e}^{\mathrm{i}\theta(Y_{t+w}-Y_t)}\right) =&\mathbb{E}\exp\left(\int_s^{s+t}\int_{\mathbb{R}}\int_0^\alpha i\theta zN(\mathrm{d}s,\mathrm{d}u,\mathrm{d}z)\right)\\ =&\exp\left(t\alpha\int_{\mathbb{R}}(\mathrm{e}^{\mathrm{i}\theta z}-1)\mu(\mathrm{d}z)\right)\\ =&\exp(-t\alpha)\sum_{k=0}^{\infty}\frac{1}{k!}(t\alpha\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}\theta z}\mu(\mathrm{d}z))^k\\ =&\mathrm{e}^{-t\alpha}\sum_{k=0}^{\infty}\frac{(\alpha k)^k}{k!}\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}\theta z}\mu^{*k}(\mathrm{d}z)\end{aligned}$$

So we get the result. □

PROBLEM 24 Assume X is Possion random measure on $(E,\mathcal{E})$ with intensity μ, a finite measure. Assume f,g are non-negative measure function on E. Prove that:

1. $\mathbb{E}(X(f)\mathrm{e}^{-X(g)}) = \mu(f\mathrm{e}^{-g})\mathbb{E}(\mathrm{e}^{-X(g)})$.
2. $\mathbb{E}(X(f)^2\mathrm{e}^{-X(g)}) = (\mu(f^2\mathrm{e}^{-g} + \mu(f\mathrm{e}^{-g})'2))\mathbb{E}(\mathrm{e}^{-X(g)})$.

SOLUTION. 1. Let $h(\theta) := \mathbb{E}(\mathrm{e}^{-X(\theta f+g)}) = \exp\left(-\int_E(1-\mathrm{e}^{-\theta f(x)-g(x)}\mu(\mathrm{d}x))\right)$. Then

$$h'(\theta) = \mathbb{E}(X(f)\mathrm{e}^{-X(\theta f+g)}) = \exp\left(-\int_E(1-\mathrm{e}^{-\theta f(x)-g(x)}\mu(\mathrm{d}x))\right)\cdot\int_E f(x)\mathrm{e}^{-\theta f(x)-g(x)}\mu(\mathrm{d}x)$$

Since they are all non-negative, the differential is valid. Let $\theta = 0$, we get $\mathbb{E}(X(f)\mathrm{e}^{-X(g)}) = \mu(f\mathrm{e}^{-g})\mathbb{E}(\mathrm{e}^{-X(g)})$.

2. Take h as above, easily to get $h''(\theta) = \mathbb{E}(X(f)^2)\mathrm{e}^{-X(g)} = \exp\left(-\int_E(1-\mathrm{e}^{-\theta f(x)-g(x)}\mu(\mathrm{d}x))\right)\cdot\left(\int_E f(x)\mathrm{e}^{-\theta f(x)-g(x)}\mu(\mathrm{d}x)\right)^2 + \exp\left(-\int_E(1-\mathrm{e}^{-\theta f(x)-g(x)}\mu(\mathrm{d}x))\right)\cdot\int_E f(x)^2\mathrm{e}^{-\theta f(x)-g(x)}\mu(\mathrm{d}x)$. Let $\theta = 0$, then easily $\mathbb{E}(X(f)^2\mathrm{e}^{-X(g)}) = (\mu(f^2\mathrm{e}^{-g} + \mu(f\mathrm{e}^{-g})'2))\mathbb{E}(\mathrm{e}^{-X(g)})$.

$\square$

2 更新过程及其应用

2.1 更新过程

2.1.1 定义和性质

Definition 19. 设 $\{\xi_n : n \geq 1\}$ 是非负独立同分布随机变量序列。设 F 是它们共同的的分布函数。假设 $F(0) = \mathbb{P}(\xi_n = 0) < 1$。则 $\mu := \mathbb{E}(\xi_n) > 0$。令 $S_n := \sum_{k=1}^{n} \xi_k$。由大数定律可得 $\lim_{n\to\infty} \frac{S_n}{n} = \mu$。令 $N(t) := \sum_{n=1}^{\infty} \mathbb{1}_{\{S_n \leq t\}} = \sup\{n \geq 0 : S_n \leq t\}$。称 $(N(t) : t \geq 0)$ 为**更新过程**。称 $(\xi_n : n \geq 1)$ 为**更新间隔时间**。

Theorem 20. 几乎必然有 $N(\infty) = \infty$。证明见教材 *p74*

2.1.2 更新方程

Definition 20. 称 $m(t) := \mathbb{E}(N(t))$ 为更新过程 $(N(t) : t \geq 0)$ 的**更新函数**。计算可得 $m(t) = \sum_{n=1}^{\infty} \mathbb{P}(N(t) \geq n) = \sum_{n=1}^{\infty} \mathbb{P}(S_n \leq t) = \sum_{n=1}^{\infty} F^{*n}(t)$。

Theorem 21. 对于 $t \geq 0$ 有 $m(t) < \infty$。证明见教材 *p76*

Definition 21. 设 H 为 $[0, \infty)$ 上的右连续的有界变差函数，而 F 为 $[0, \infty)$ 上的概率分布函数。称关于 K 的方程 $K(t) = H(t) + K * F(t), t \geq 0$ 为**更新方程**。更新方程的积分形式为 $K(t) = H(t) + \int_0^t K(t-x)\,\mathrm{d}F(x))$。

Theorem 22. 更新方程存在唯一右连续有界变差函数解 K，且该解具有表达式 $K(t) = H(t) + H * m(t)$。证明见教材 *p76*

Theorem 23. 对于 $0 \leq s \leq t$，有 $\mathbb{P}(S_{N(t)} \leq s) = 1 - F(t) + \int_0^s 1 - F(t-x)\,\mathrm{d}m(x)$。证明见教材 *p77*

Theorem 24 (瓦尔德恒等式)**.** 设 $\{\xi_n : n \geq 1\}$ 为独立同分布的随机变量序列，$\mathcal{F}_n$ 为其自然 σ-代数流。设 $\mathbb{E}(\xi_1)$ 存在。设 τ 为一个停时。则有 $\mathbb{E}(\sum_{k=1}^{\tau} \xi_k) = \mathbb{E}(\tau)\mathbb{E}(\xi_1)$。

Theorem 25. 对于 $t, x > 0$，有 $\mathbb{P}(W(t) > x) = 1 - F(t+x) + \int_0^t 1 - F(t+x-y)\,\mathrm{d}m(y)$，其中 $W(T) := S_{N(t)+1} - t$ 为**待更新时间**。证明见教材 *p78*

PROBLEM 25 Assume $(N(t) : t \geq 0)$ is a renewing process with renewing internal $\{\xi_n : n \geq 1\}$, $S_n = \sum_{k=1}^{n} \xi_k$, $N(t) := \sup\{n : S_n \leq t\}$, calculate $g(t) := \mathbb{E}(N(t)^2)$.

SOLUTION. Let $T_1 = \xi_1$, then $g(t) = \mathbb{E}(\mathbb{E}(N(t)^2 \mid T_1)) = \int_0^t \mathbb{E}(N(t)^2 \mid T_1 = x)dF(x)$. By the independence,

$$\mathbb{E}(N(t)^2 \mid T_1 = x) = \begin{cases} 0 & , x > t \\ \mathbb{E}((1 + N(t-x))^2) & , x \leq t \end{cases}$$

That is $\mathbb{E}(N(t)^2 \mid T_1 = x) = \begin{cases} 0 & ,x > t \\ 1 + 2m(t-x) + g(t-x) & ,x \le t \end{cases}$. Therefore,

$$g(t) = F(t) + 2\int_0^t m(t-x)dF(x) + \int_0^t g(t-x)dF(x)$$

So $g(t) = 2m(t) - F(t) + \int_0^t g(t-x)dF(x)$. Thus, $g(t) = 2m(t) - F(t) + (2m - F) * m(t)$, so $g(t) = m(t) + 2m * m(t)$. □

PROBLEM 26 Assume renewing internal time obey $U(0,1)$. $0 < t < 1$, calculate the distrubution of $S_{N(t)}$ and $\mathbb{E}(S_{N(t)})$.

SOLUTION. By calculating, $m(t) = \mathrm{e}^t - 1, 0 < t < 1$. $\forall 0 \le s \le t < 1$,

$$\mathbb{P}(S_{N(t)} \le s) = 1 - t + \int_0^s (1 - t + x)\mathrm{e}^x dx = 1 - (t-s)\mathrm{e}^s$$

Therefore,

$$\mathbb{E}(S_{N(t)}) = \int_0^t s(1 - t + s)\mathrm{e}^s ds = \mathrm{e}^t - t - 1$$

□

PROBLEM 27 Assume renewing internal time obey random variable X with distrubution function F. Let $\gamma_t = S_{N(t)+1} - t$ be the rest lifetime at time t. Prove:

$$\mathbb{P}(\gamma_t > z) = 1 - F(t+z) + \int_0^t (1 - F(t+z-x))dm(x)$$

SOLUTION. Let $A_z(t) = \mathbb{P}(\gamma_t > z)$, then

$$\mathbb{P}(\gamma_t > z \mid \xi_1 = x) = \begin{cases} 1 & ,x > t+z \\ 0 & ,t < x \le t+z \\ A_z(t-x) & ,0 < x \le t \end{cases}$$

Then,

$$A_z(t) = \int_0^\infty \mathbb{P}(\gamma_t > z \mid \xi_1 = x)dF(x) = 1 - F(t+z) + \int_0^t A_z(t-z)dF(x)$$

Thus,

$$A_z(t) = 1 - F(t+z) + \int_0^\infty (1 - F(t+z-x))dm(x)$$

□

PROBLEM 28 One kind of devices are replaced as they are worn out. Let the lifetime of the devices be sequences $\{\xi_n : n \ge 1\}$, and let $S_n = \sum_{k=1}^n \xi_k$, $N(t) = \sup\{n : S_n \le t\}$. $L(t) = S_{N(t)+1} - S_{N(t)}$. Prove: $\mathbb{P}(L(t) > x) \ge \mathbb{P}(\xi_1 > x)$.

SOLUTION. When $t \le x$, easy to get that $\mathbb{P}(L(t) > x) = \mathbb{P}(\xi_1 > x)$. Now we assume $t > x$.

$$\begin{aligned}
\mathbb{P}(L(t)>x) =& \sum_{k=0}^{\infty} \mathbb{P}\left(\xi_{k+1}>x, N(t)=k\right) \\
=& \sum_{k=0}^{\infty} \mathbb{P}\left(\xi_{k+1}>x, S_k \le t, \xi_{k+1}>t-S_k\right) \\
=& \sum_{k=1}^{\infty} \mathbb{P}\left(\xi_{k+1}>x, t-x<S_k \le t\right) \\
& \quad + \sum_{k=0}^{\infty} \mathbb{P}\left(\xi_{k+1}>t-S_k, S_k \le t-x\right) \\
=& \mathbb{P}\left(\xi_1>x\right) \mathbb{E}[(N(t)-N(t-x))] + \mathbb{P}(N(t)=N(t-x)) \\
=& \mathbb{P}\left(\xi_1>x\right) + \mathbb{P}\left(\xi_1>x\right) \mathbb{E}[(N(t)-N(t-x))-1] \\
& + \mathbb{P}(N(t)=N(t-x)) \\
=& \mathbb{P}\left(\xi_1>x\right) + \mathbb{P}\left(\xi_1>x\right) \mathbb{E}\left[(N(t)-N(t-x)-1) 1_{\{N(t)>N(t-x)\}}\right] \\
& - \mathbb{P}\left(\xi_1>x\right) \mathbb{E}\left(1_{\{N(t)=N(t-x)\}}\right) + \mathbb{P}(N(t)=N(t-x)) \\
\ge& \mathbb{P}\left(\xi_1>x\right).
\end{aligned} \tag{8}$$

□

PROBLEM 29 Toss a coin until we get two successively head, call it a renew. We toss the coin k times, call the number of renews $N(k)$. Find the distribution and expectation of interval time T

SOLUTION. Let $p_n := \mathbb{P}(T = n)$. Then $p_1 = 0, p_2 = \frac{1}{4}$. Easy to find that $p_{n+2} = \frac{1}{2}p_{n+1} + \frac{1}{4}p_n$. The characteristic equation of this sequence is $x^2 = \frac{1}{2}x + \frac{1}{4}$. The roots are $x_1 = \frac{1+\sqrt{5}}{4}, x_2 = \frac{1-\sqrt{5}}{4}$. So $p_n = Ax_1^n + Bx_2^n$. By p_1, p_2, easy to get that $p_n = \frac{1}{2\sqrt{5}}\left(x_1^{n-1} - x_2^{n-1}\right)$. So easily $\mathbb{E}(T) = \sum_{n=1}^{\infty} np_n = 6$. □

2.2 长程极限行为

2.2.1 基本更新定理

Theorem 26. *几乎必然的有* $\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu}$。*证明见教材 p80*

Theorem 27 (基本更新定理)**.** *有* $\lim_{t\to\infty} \frac{m(t)}{t} = \frac{1}{\mu}$。*证明见教材 p81*

2.2.2 中心极限定理

Theorem 28 (中心极限定理)**.** *假设* $\mathbb{D}(\xi_1) < \infty$，*记* $\mu = \mathbb{E}(\xi_1), \sigma^2 = \mathbb{D}(\xi_1)$。*对于* $x \in \mathbb{R}$，*有*

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{N(t) - \frac{t}{\mu}}{\sqrt{\frac{t\sigma^2}{\mu^3}}} \le x\right) = \Phi(x) \tag{9}$$

其中 $\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}\mathrm{e}^{\frac{-y^2}{2}}\,\mathrm{d}y$ 为正态分布的分布函数。证明见教材 *p82*

PROBLEM 30 A radio is powered by one battery, the lifetime of the battery obey the distribution of exponential distribution with parameter $\lambda=\frac{1}{30}$. In long term, in which frequence should we change the battery?

SOLUTION . Easy to get that $\lim_{t\to\infty}\frac{m(t)}{t}=\frac{1}{\mathbb{E}(\xi_1)}=\frac{1}{30}$. So we change battery every 30 hours in average. □

PROBLEM 31 Consider a primitive renewing process with average renewing internal time μ. Assume every renewing time is recorded by probability p, and each record and each renew are independence. Let $N_r(t)$ be the times of renewing by recorded until time t. $\{N_r(t):t\geq 0\}$ is a renewing process or not? And calculate $\lim_{t\to\infty}\frac{N_r(t)}{t}$.

SOLUTION . Assume $X_n:n\in\mathbb{N}$ are i.i.d r.v and $X_0\sim Geo(p)$, and $(X_n:n\in\mathbb{N})\perp(N(t):t\geq 0)$. Let $Y_n:=\sum_{k=1}^{n}X_n$, and $Y_0=0$. Let $\xi_r(n):=\sum_{k=Y_{n-1}+1}^{Y_n}\xi_k$. Then $\xi_r(n):n\in\mathbb{N}^+$ is update time of N_r. Since $(X_n:n\in\mathbb{N})\perp(N(t):t\geq 0)$, we get that $(\xi_r(n):n\in\mathbb{N}^+)$ are i.i.d. And $\mathbb{E}(\xi_r(1))=\mathbb{E}(X_1)\mathbb{E}(\xi_1)=\frac{\mu}{p}$. So $\lim_{t\to\infty}\frac{N_r(t)}{t}=\frac{\mu}{p}$. □

PROBLEM 32 Assume $(U_n:n\in\mathbb{N}^+)$ are i.i.d r.v. and $U_1\sim U(0,1)$. Assume $X_{n,m}:n,m\in\mathbb{N}^+$ are r.v. and $X_{n,m}\mid U_n\sim B(U_n)$. And $(X_{n,m}\mid U_n:m\in\mathbb{N}^+)$ are i.i.d. Let $\xi_n:=\inf\{m\in\mathbb{N}^+: X_{n,m}=1\}$ be the n-th update time of $N(t)$. Find $\lim_{t\to\infty}\frac{N(t)}{t}$.

SOLUTION . Easy to find that $\mathbb{E}(\xi_1)=\int_0^1\mathbb{E}(\xi_1\mid U_1=x)\,\mathrm{d}x=\int_0^1\frac{\mathrm{d}x}{x}=\infty$. So easy to find that $\lim_{t\to\infty}\frac{N(t)}{t}=\infty$. □

PROBLEM 33 Assume $(\xi_n:n\in\mathbb{N}^+)$ is i.i.d r.v. ranging in $\mathbb{N}$ is update time of $N(t)$. Let A_n be the event that at time n there is an update. Assume $a=\lim_{n\to\infty}\mathbb{P}(A_n)$ exists. Prove that $a=\frac{1}{\mathbb{E}(\xi_1)}$.

SOLUTION . Since $N(n)=\sum_{k=1}^{n}\mathbb{1}(A_k)$, we know that $\mathbb{E}(N(n))=\sum_{k=1}^{n}\mathbb{P}(A_k)$. Noting that $\lim_{n\to\infty}\frac{N(n)}{n}=\frac{1}{\mathbb{E}(\xi_1)}$, we obtain that $\lim_{n\to\infty}\mathbb{E}(\frac{N(n)}{n})=\frac{1}{\mathbb{E}(\xi_1)}$. So $\lim_{n\to\infty}\frac{\sum_{k=1}^{n}\mathbb{P}(A_k)}{n}=\frac{1}{\mathbb{E}(\xi_1)}$. By stolz, we can get that $\lim_{n\to\infty}\frac{\sum_{k=1}^{n}\mathbb{P}(A_k)}{n}=a$. So $a=\frac{1}{\mathbb{E}(\xi_1)}$. □

PROBLEM 34 Assume $N_1(t),N_2(t)$ are two independent updating process with update time distribution $E(1),U(0,2)$. Find an estimate of $\mathbb{P}(N_1(100)+N_2(100)\geq 190)$.

SOLUTION . Easy to know the expectation and varience of the update time are $\mu_1=1,\sigma_1^2=1,\mu_2=1,\sigma_2^2=\frac{1}{3}$. So by the central limit theorem of updating process we know that

$$\frac{N_1(100)-100}{\sqrt{100}},\frac{N_2(100)-100}{\sqrt{\frac{100}{3}}}\sim N(0,1)$$

So $\frac{N_1(100)+N_2(100)-200}{\sqrt{\frac{400}{3}}}\sim N(0,1)$. So $\mathbb{P}(N_1(100)+N_2(100)\geq 190)\approx\mathbb{P}(N(0,1)\geq-\frac{\sqrt{3}}{2})$. □

2.3 更新过程的应用

2.3.1 随机游动的爬升时间

Definition 22. 设 $(\xi_n : n \geq 1)$ 是独立同分布的可积随机变量序列且满足 $\mathbb{E}(\xi_1) > 0$。令 $(W_n : n \geq 0)$ 是以 $(\xi_n : n \geq 1)$ 为跳幅的随机游动，其中 $W_0 = 0$。易知 $W_n \to +\infty$。令 $S_0 = 0$，递归地定义停时 S_n，令 $S_n = \inf\{k \geq S_{n-1} : W_k > W_{S_{n-1}}\}$。称每个 S_n 为 (W_n) 的**爬升时间**。

Theorem 29. 对于 $n \geq 1$，令 $\eta_n = S_n - S_{n-1}$。则 (η_n) 是独立同分布的非负随机变量序列。证明见教材 *p85*

Theorem 30. 我们有 $\mathbb{P}(\forall n \geq 1, W_n > 0) = \frac{1}{\mathbb{E}(S_1)}$。证明见教材 *p86*

2.3.2 更新累积过程

Definition 23. 设 $((\xi_n, \eta_n) : n \geq 1)$ 为独立同分布的二维随机变量序列，且 $\xi_n \geq 0$。令 $N(t)$ 为以 ξ_n 为更新间隔时间的更新过程。令 $A(t) = \sum_{n=1}^{N(t)} \eta_n$。称 $(A(t) : t \geq 0)$ 为**更新累积过程**。

Theorem 31. 设 $0 < \mathbb{E}(\xi_1) < \infty, \mathbb{E}(|\eta_1|) < \infty$。则几乎必然有 $\lim_{t\to\infty} \frac{A(t)}{t} = \frac{\mathbb{E}(\eta_1)}{\mathbb{E}(\xi_1)}$。且有 $\lim_{t\to\infty} \frac{\mathbb{E}(A(t))}{t} = \frac{\mathbb{E}(\eta_1)}{\xi_1}$。证明见教材 *p87*

PROBLEM 35 Assume $N(t)$ is updating process. X is the time interval distrabution of $N(t)$. Assume $\mathbb{D}(X) < \infty$. Let $R(t) := S_{N(t)+1} - t$. Find:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T R(t)\, \mathrm{d}t$$

SOLUTION. Easily $N(t) + 1 \geq T \geq N(t)$. So $\int_0^T R(t)\, \mathrm{d}t \leq \sum_{i=1}^{N(T)+1} \int_{S_{i-1}}^{S_i} (S_i - t)\, \mathrm{d}t = \frac{1}{2} \sum_{i=1}^{N(t)+1} (S_i - S_{i-1})^2 = \frac{1}{2} \sum_{i=1}^{N(T)+1} \xi_i^2$. For the same reason, we get that $\int_0^T R(t)\, \mathrm{d}t \geq \frac{1}{2} \sum_{i=1}^{N(T)} \xi_i^2$.

Easy to know that $\lim_{T\to\infty} \frac{1}{T} \sum_{i=1}^{N(T)} \xi_i^2 = \lim_{T\to\infty} \frac{1}{T} \sum_{i=1}^{N(T)+1} \xi_i^2 = \frac{\mathbb{E}(X^2)}{\mathbb{E}(X)^2}$. So finally we get that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T R(t)\, \mathrm{d}t = \frac{\mathbb{E}(X^2)}{2\mathbb{E}(X)^2}$$

. □

PROBLEM 36 Assume the number of people arriving the cinema is distributed as a Possion process with parameter λ. Assume the film begin at a fixed time $t \geq 0$. Let $A(t)$ be the sum of waiting time of all people arriving in $(0, t]$, find $\mathbb{E}(A(t))$.

SOLUTION. Let V_k be the arriving time of k-th people. Let $N(t)$ be the number of people in $(0, t]$. Then $A(t) = \sum_{k=1}^{N(t)} (t - V_k)$. Let $\xi_k := V_k - V_{k-1}$. Then $\sum_{k=1}^{N(t)} V_k = \sum_{k=1}^{N(t)} (N(t) - k)\xi_k = \sum_{k=0}^{N(t)-1} k\xi_{N(t)-k}$. So $\mathbb{E}(A(t)) = t\mathbb{E}(N(t)) - \mathbb{E}(\sum_{k=0}^{N(t)-1} k\xi_{N(t)-k})$. Easy to get that $\mathbb{E}(\sum_{k=0}^{N(t)-1} k\xi_{N(t)-k} \mid N(t) = n) = \frac{nt}{2}$. So $\mathbb{E}(A(t) \mid N(t) = n) = nt - \frac{nt}{2} = \frac{nt}{2}$. So finally we have $\mathbb{E}(A(t)) = \mathbb{E}(\mathbb{E}(A(t) \mid N(t))) = \mathbb{E}(\frac{N(t)t}{2}) = \frac{\lambda t^2}{2}$. □

PROBLEM 37 Assume a machine has life distrabuted p. When machine is broken or has been used T years, we will change a new machine. The price of new machine is C_1, and if the machine is broken, it would cause loss C_2.

1. Give the long-time average fee of this machine.
2. Let $C_1 = 10, C_2 = 0.5$, and $p(x) = \mathbb{1}_{(0,10)}(x)\frac{1}{10}$. Which T can let the fee be minimum.

SOLUTION.

1. Let ξ be the time when the machine will broken. Let $\gamma := \xi \wedge T$. Then the machine will be changed at γ. Obviously $\mathbb{E}(\gamma) = T\mathbb{P}(\xi > T) + \mathbb{E}(\xi\mathbb{1}(\xi \le T)) = T\int_T^\infty p(x)\,\mathrm{d}x + \int_0 Txp(x)\,\mathrm{d}x$. Let η be the fee of this machine, then we have $\eta = C_1\mathbb{1}(\xi > T) + (C_1+C_2)\mathbb{1}(\xi \le T) = C_1 + C_2\mathbb{1}(\xi \le T)$. So $\mathbb{E}(\eta) = C_1 + C_2\int_0^T p(x)\,\mathrm{d}x$. So the long-time average fee is
$$g(T) = \frac{C_1 + C_2\int_0 Tp(x)\,\mathrm{d}x}{T\int_T^\infty p(x)\,\mathrm{d}x + \int_0^T xp(x)\,\mathrm{d}x}$$
.
2. Easy to get that $g(T) = \frac{200+T}{20T-T^2}$ when $T \in (0,10)$. And $g'(T) = \frac{T^2+400T-4000}{(20T-T^2)^2}$. Let $g'(T) = 0$, then $T^2 + 400T - 4000 = 0$, then $T = 20\sqrt{110} - 200 \approx 9.76$. So $T = 9.76$ can make the fee get minimum.

$\square$

3 离散时间马氏链

3.1 马氏性及其等价形式

3.1.1 简单马氏性

Definition 24. $X=(X_n:n\geq 1)$ 关于 σ-代数流 $(\mathcal{G}_n:n\geq 1)$ 是适应的，即 $\forall n, X_n$ 关于 $\mathcal{G}_n$ 可测。令 $\mathcal{F}_n=\sigma(X_0,X_1,\cdots,X_n)$。称 $(\mathcal{F}_n:n\geq 1)$ 为 $(X_n:n\geq 1)$ 的**自然 σ-代数流**。

Definition 25. 称随机过程 X 关于其适应流 $(\mathcal{G}_n)$ 具有**马氏性**或者称它是关于该流的**马氏链**，如果对于任意的 $n\geq 0, i,j\in E, G\in\mathcal{G}_n$，当 $\mathbb{P}(G\cap\{X_n=i\})>0$ 时有 $\mathbb{P}(X_{n+1}=j\mid G\cap\{X_n=i\})=\mathbb{P}(X_{n+1}=j\mid X_n=i)$。该定义等价于 $\mathbb{P}(X_{n+1}=j\mid\mathcal{G}_n)=\mathbb{P}(X_{n+1}=j\mid X_n)$，证明见教材 *p94*

Theorem 32. 过程 X 对其自然流具有马氏性的充要条件是 $\forall n\geq 0, \{i_0,\cdots,i_n=i,i\}\subset E$，当 $\mathbb{P}(X_k=i_k:0\leq k\leq n)>0$ 时有 $\mathbb{P}(X_{n+1}=j\mid X_k=i_k:0\leq k\leq n)=\mathbb{P}(X_{n+1}=j\mid X_n=i)$。证明见教材 *p95*

Theorem 33. 设 X 关于流 $\mathcal{G}_n$ 为马氏链，设 $k>0$，设 $A\in\mathcal{G}_k$。令 $\mathbb{P}_A=\mathbb{P}(\cdot\mid A)$。则 $(X_{k+n}:n\geq 0)$ 关于流 $(\mathcal{G}_{k+n}:n\geq 0)$ 在 $\mathbb{P}_A$ 下有马氏性。证明见教材 *p95*

3.1.2 条件独立性

Theorem 34. 过程 X 关于流 $(\mathcal{G}_n)$ 具有马氏性的充要条件是 $\forall n\geq 0, k\geq 1, \{i,j_1,j_2,\cdots,j_k\}\subset E, G\in\mathcal{G}_n$，有 $\mathbb{P}(X_{n+1}=j_1,\cdots,X_{n+k}=j_k\mid G\cap\{X_n=i\})=\mathbb{P}(X_{n+1}=j_1,\cdots,X_{n+k}=j_k\mid X_n=i)$。证明见教材 *p97*

Theorem 35. 记 $\mathcal{F}^{(n)}=\sigma(X_k:k\geq n)$。则过程 X 关于流 $(\mathcal{G}_n)$ 具有马氏性的充要条件是 $\forall n\geq 0, k\geq 1, i\in E, G\in\mathcal{G}_n, F\in\mathcal{F}^{(n)}$，有 $\mathbb{P}(F\mid G\cap\{X_n=i\})=\mathbb{P}(F\mid X_n=i)$。证明见教材 *p98*。在相同条件下的另一个等价条件为 $\mathbb{P}(G\cap F\mid X_n=i)=\mathbb{P}(G\mid X_n=i)\mathbb{P}(F\mid X_n=i)$，证明见教材 *p98*。从而得出另一个等价条件：在 $\mathbb{P}(\cdot\mid X_n=i)$ 下 $\mathcal{G}_n$ 和 $\mathcal{F}^{(n)}$ 独立。

PROBLEM 38 Assume $(\Omega,\mathcal{F},\mathbb{P})$ is a probability space, and $C\in\mathcal{F}$ satisfy $\mathbb{P}(C)>0$. Let $\mathbb{P}_C:\mathcal{F}\to\mathbb{R}, \mathbb{P}_C(X)=\frac{\mathbb{P}(C\cap X)}{\mathbb{P}(C)}$. Assume $A,B\in\mathcal{F}$, and $\mathbb{P}(B\cap C)>0$, prove that $\mathbb{P}_C(A\mid B)=\mathbb{P}(A\mid B\cap C)$.

SOLUTION. Easily $\mathbb{P}_C(B)=\frac{\mathbb{P}(B\cap C)}{\mathbb{P}(C)}>0$, so $\mathbb{P}_C(A\mid B)$ is well-defined. Easily to get that

$$\mathbb{P}_C(A\mid B)=\frac{\mathbb{P}_C(A\cap B)}{\mathbb{P}_C(B)}=\frac{\frac{\mathbb{P}(A\cap B\cap C)}{\mathbb{P}(C)}}{\frac{\mathbb{P}(B\cap C)}{\mathbb{P}(C)}}=\frac{\mathbb{P}(A\cap B\cap C)}{\mathbb{P}(B\cap C)}=\mathbb{P}(A\mid B\cap C)$$

□

PROBLEM 39 Assume that $(X_n:n\geq 0)$ is 1-dimentional simple symetry random walk, prove that $(|X_n|:n\geq 0)$ is a Markov chain ranges in $\mathbb{N}$.

SOLUTION . Easy to know that $(X_n : n \geq 0)$ is a Markov chain in $\mathbb{Z}$. Let $\mathcal{F}_n := \sigma(X_1, \cdots, X_n), \mathcal{G}_n := \sigma(|X_1|, \cdots, |X_n|)$, then easily $\mathcal{G}_n \subset \mathcal{F}_n$. Then we get that $\mathbb{P}(|X_{n+1}| = i \mid \mathcal{F}_n) = \mathbb{P}(X_{n+1} = i \mid \mathcal{F}_n) + \mathbb{P}(X_{n+1} = -i \mid \mathcal{F}_n) = \mathbb{P}(X_{n+1} = i \mid X_n) + \mathbb{P}(X_{n+1} = -i \mid X_n) = \mathbb{P}(|X_{n+1}| = i \mid X_n) = \frac{1}{2}\mathbb{1}(|X_n - i| = 1)$. Noting $i \geq 0$, we get that $|X_n - i| = 1 \iff ||X_n| - i| = 1$, so $\mathbb{P}(|X_{n+1}| = i \mid \mathcal{F}_n)$ is measureable about $\sigma(|X_n|)$. Since $\sigma(|X_n|) \subset \mathcal{G}_n \subset \mathcal{F}_n$, so we finally get that $\mathbb{P}(|X_{n+1}| = i \mid \mathcal{G}_n) = \mathbb{P}(|X_{n+1}| = i \mid |X_n|)$. So $(|X_n| : n \geq 0)$ is a Markov chain. □

PROBLEM 40 Assume $(X_n : n \geq 0)$ is a Markov chain ranges in E. Assume $\phi : E \to F$ is injection. Prove that $(\phi(X_n) : n \geq 0)$ is a Markov chain ranges in $\phi(E)$.

SOLUTION . Without loss of generality assume $F = \phi(E)$, then ϕ is bijection. Now let $\mathcal{F}_n := \sigma(X_1, \cdots, X_n)$. Since ϕ is bijection we easily get that $\sigma(X_n) = \sigma(\phi(X_n))$, so $\mathcal{F}_n = \sigma(\phi(X_1), \cdots, \phi(X_n))$. Then $\mathbb{P}(\phi(X_{n+1} = i \mid \mathcal{F}_n)) = \mathbb{P}(X_{n+1} = \phi^{-1}(i) \mid \mathcal{F}_n) = \mathbb{P}(X_{n+1} = \phi^{-1}(i) \mid X_{n+1}) = \mathbb{P}(\phi(X_{n+1}) = i \mid \phi(X_n))$. So $(\phi(X_n) : n \geq 0)$ is Markov chain. □

PROBLEM 41 Assume $(X_n : n \geq 0), (Y_n : n \geq 0)$ are two independent Markov chains on E, F respectively. Prove that $((X_n, Y_n) : n \geq 0)$ is Markov chain on $E \times F$.

SOLUTION . Let $\mathcal{F}_n = \sigma(X_0, \cdots, X_n)$ and $\mathcal{G}_n, = \sigma(Y_0, \cdots, Y_n)$, Let $\mathcal{H}_n = \sigma((X_0, Y_0), \cdots, (X_n, Y_n))$. Then easily $\mathcal{H}_n = \sigma(\mathcal{F}_n, \mathcal{G}_n)$. Easy to get that

$$
\begin{aligned}
\mathbb{P}(X_{n+1} = i, Y_{n+1} = j \mid \mathcal{H}_n) =& \mathbb{E}(\mathbb{P}(X_{n+1} = i, Y_{n+1} = j \mid \mathcal{H}_n, X_{n+1}) \mid \mathcal{H}_n) \\
=& \mathbb{E}(\mathbb{1}_i(X_{n+1})\mathbb{P}(Y_{n+1} = j \mid \mathcal{F}_n, \mathcal{G}_n, X_{n+1}) \mid \mathcal{H}_n) \\
(Y_{n+1} \perp \mathcal{F}_n, X_{n+1}) =& \mathbb{E}(\mathbb{1}_i(X_{n+1})\mathbb{P}(Y_{n+1} = j \mid Y_n) \mid \mathcal{H}_n) \\
(Y_n \in \mathcal{H}_n) =& \mathbb{P}(Y_{n+1} = j \mid Y_n)\mathbb{P}(X_{n+1} = i \mid \mathcal{H}_n) \\
=& \mathbb{P}(Y_{n+1} = j \mid Y_n)\mathbb{P}(X_{n+1} = i \mid X_n)
\end{aligned}
$$

So $\mathbb{P}(X_{n+1} = i, Y_{n+1} = j \mid \mathcal{H}_n) \in \sigma(X_n, Y_n) \subset \mathcal{H}_n$. So $\mathbb{P}(X_{n+1} = i, Y_{n+1} = j \mid X_n, Y_n) = \mathbb{P}(X_{n+1} = i, Y_{n+1} = j \mid \mathcal{H}_n)$. So $((X_n, Y_n) : n \geq 0)$ is Markov chain. □

PROBLEM 42 Assume $(X_n : n \geq 0), (Y_n : n \geq 0)$ are two independent Markov chains on E, F respectively. Let $\mathcal{H}_n := \sigma((X_0, Y_0), \cdots, (X_n, Y_n))$. Prove that $(X_n : n \geq 0)$ is Markov chain over $(\mathcal{H}_n : n \geq 0)$.

SOLUTION . Take $\mathcal{F}_n, \mathcal{G}_n$ as above. Obviously $X_n \in \mathcal{F}_n \subset \mathcal{H}_n$. Easily $\mathbb{P}(X_{n+1} = i \mid \mathcal{H}_n) = \mathbb{P}(X_{n+1} = i \mid \mathcal{F}_n, \mathcal{G}_n) = \mathbb{P}(X_{n+1} = i \mid \mathcal{F}_n) = \mathbb{P}(X_{n+1} \mid X_n)$. So $(X_n : n \geq 0)$ is Markov chain over $(\mathcal{H}_n : n \geq 0)$. □

PROBLEM 43 Let μ_0 be a probability distribution on $\mathbb{N}$. For $n \geq 1$, let

$$\mu_n(0) = \mu_{n-1}^{*2}(0) + \mu_{n-1}^{*2}(1), \mu_n(j) = \mu_{n-1}^{*2}(j+1), \forall j \geq 1$$

Where $\mu^{*2} = \mu * \mu$. Let F_n be distribution function of μ_n. Let $F_{n-1}^{-1}(y) := \inf\{x \geq 0 : y \leq F_{n-1}(x)\}$ for $y \in [0,1]$. Assume $X_0 \sim \mu_0$, and $(U_n : n \geq 0)$ are i.i.d r.v. with distribution $U(0,1)$. Let $X_{n+1} := \max\{0, X_n + F_n^{-1}(U_n) - 1\}$. Then $(X_n : n \geq 0)$ is Markov chain.

SOLUTION. Let $\mathcal{F} := \sigma(X_0, \cdots, X_n)$. For $i > 0$, we have

$$\begin{aligned}
\mathbb{P}(X_{n+1} = i \mid \mathcal{F}_n) &= \mathbb{P}(X_n + F_n^{-1}(U_n) - 1 = i \mid \mathcal{F}_n) \\
&= \sum_{k\in\mathbb{Z}} \mathbb{P}(X_n = k, F_n^{-1}(U_n) = i + 1 - k \mid \mathcal{F}_n) \\
&= \sum_{k\in\mathbb{Z}} \mathbb{1}_k(X_n)\mathbb{P}(F_n^{-1}(U_n) = i + 1 - k) \\
&= \sum_{k\in\mathbb{Z}} \mathbb{1}_k(X_n)\mathbb{P}(F_n^{-1}(U_n) = i + 1 - k \mid X_n) \\
&= \sum_{k\in\mathbb{Z}} \mathbb{P}(X_n = k, F_n^{-1}(U_n) = i + 1 - k \mid X_n) \\
&= \mathbb{P}(X_n + F_n^{-1}(U_n) - 1 = i \mid X_n) = \mathbb{P}(X_{n+1} = i \mid X_n)
\end{aligned}$$

For $i = 0$, we have

$$\begin{aligned}
\mathbb{P}(X_{n+1} = 0 \mid \mathcal{F}_n) &= \mathbb{P}(X_n + F_n^{-1}(U_n) - 1 \leq 0 \mid \mathcal{F}_n) \\
&= \sum_{k\in\mathbb{Z}} \mathbb{P}(X_n = k, F_n^{-1}(U_n) \leq 1 - k \mid \mathcal{F}_n) \\
&= \sum_{k\in\mathbb{Z}} \mathbb{1}_k(X_n)\mathbb{P}(F_n^{-1}(U_n) \leq 1 - k) \\
&= \sum_{k\in\mathbb{Z}} \mathbb{1}_k(X_n)\mathbb{P}(F_n^{-1}(U_n) \leq 1 - k \mid X_n) \\
&= \sum_{k\in\mathbb{Z}} \mathbb{P}(X_n = k, F_n^{-1}(U_n) \leq 1 - k \mid X_n) \\
&= \mathbb{P}(X_n + F_n^{-1}(U_n) - 1 \leq 0 \mid X_n) = \mathbb{P}(X_{n+1} = 0 \mid X_n)
\end{aligned}$$

So $(X_n : n \geq 0)$ is Markov chain. □

3.2 转移矩阵

3.2.1 有转移矩阵的马氏链

Definition 26. 称 E 上的矩阵 $P = (p(i,j) : i,j \in E)$ 为一个**转移矩阵**，如果它有下面的性质：

1. $\forall i,j \in E, p(i,j) \geq 0$。
2. $\forall i \in E, \sum_{j\in E} p(i,j) = 1$。

记 $P^n = (p_n(i,j) : i,j \in E)$。以后不区分 $p_n(i,j)$ 和 $p_{i,j}(n)$ 以及 $p_{ij}(n)$。

Definition 27. 称 X 为有转移矩阵 P 的马氏链，如果 $\forall n \geq 0, i,j \in E, G \in \mathcal{G}_n$，有：$\mathbb{P}(X_{n+1} = j \mid G \cap \{X_n = i\}) = p(i,j)$。此性质称为有转移矩阵 P 的马氏性。易知它比马氏性更强。令 μ 为 X_0 的分布，称为 X 的**初始分布**。

3.2.2 有限维分布的性质

Theorem 36. X 为以 P 为转移矩阵的马氏链的充要条件为 $\forall n \geq 0, G \in \mathcal{G}_n, \{i, j_1, \cdots, j_k\} \subset E$，有：$\mathbb{P}(H \mid G \cap \{X_n = i\}) = p(i, j_1)p(j_1, j_2)\cdots p(j_{k-1}, j_k)$，其中 $H = \{X_{n+1} = j_1, \cdots, X_{n+k} = j_k\}$。证明见教材 *p101*

Theorem 37. 过程 X 相对其自然流是以 P 为转移矩阵，μ 为初始分布的马氏链的充要条件是 $\forall n \geq 0, \{i_0, \cdots, i_n\} \subset E, \mathbb{P}(X_k = i_k : 0 \leq k \leq n) = \mu(i_0)p(i_0, i_1)\cdots p(i_{n-1}, i_n)$。证明见教材 *p102*

3.2.3 强马氏性

Definition 28. 设 $(\mathcal{G}_n : n \geq 0)$ 为离散时间流。设 τ 是取值为 $\mathbb{N} \cap \{+\infty\}$ 的随机变量。若 $\forall n \geq 0, \{\tau \leq n\} \in \mathcal{G}_n$，则称 τ 是关于 $(\mathcal{G}_n)$ 的停时。记全体满足 $\forall n, A \cap \{\tau \leq n\} \in \mathcal{G}_n$ 的 A 组成的 σ-代数为 $\mathcal{G}_\tau$。

Theorem 38. 设 X 关于流 $\mathcal{G}$ 为以 P 为转移矩阵的马氏链，而 τ 是停时。则在概率空间 $(\Omega \cap \{\tau < \infty\}, \mathcal{F} \cap \{\tau < \infty\}, \mathbb{P}(\cdot \mid \tau < \infty))$ 下，$(X_{\tau+n} : n \geq 0)$ 为以 P 为转移矩阵的马氏链。证明见教材 *p104*

3.2.4 随机游动的反射原理

Theorem 39 (**反射原理**). 设 $X = (X_n : n \geq 0)$ 是从原点 0 出发的一维简单对称随机游动。则 $\forall x \in \mathbb{Z}_+$，有 $\mathbb{P}(\max_{0 \leq k \leq n} X_k \geq x) = \mathbb{P}(X_n \geq x) + \mathbb{P}(X_n > x)$。证明见教材 *p105*

PROBLEM 44 Let $(X_n : n \geq 0) \perp (Y_n : n \geq 0)$ are Markov chain on E with Transition matrix $(p_{ij} : i, j \in E), (q_{ij} : i, j \in E)$ respectively. Prove: $\{(X_n, Y_n) : n \geq 0\}$ are Markov chain on $E \times E$. And calculate the transition matrix of $(X_n, Y_n) : n \geq 0$.

SOLUTION.

$$
\begin{aligned}
&\mathbb{P}(X_0 = i_0, \cdots, X_{n+1} = i_{n+1}, Y_0 = j_0, \cdots, Y_{n+1} = j_{n+1}) \\
=&\mathbb{P}(X_0 = i_0, \cdots, X_{n+1} = i_{n+1})\mathbb{P}(Y_0 = j_0, \cdots, Y_{n+1} = j_{n+1}) \\
=&\mathbb{P}(X_0 = i_0) \prod_{k=0}^{n} p_{i_k i_{k+1}} \mathbb{P}(Y_0 = j_0) \prod_{k=0}^{n} q_{j_k j_{k+1}} \\
=&\mathbb{P}((X_0, Y_0) = (i_0, j_0)) \prod_{k=0}^{n} p_{i_k i_{k+1}} q_{j_k j_{k+1}} \\
=&\mathbb{P}((X_0, Y_0) = (i_0, j_0)) \prod_{k=0}^{n} \mathbb{P}(X_k = i_k, X_{k+1} = i_{k+1})\mathbb{P}(Y_k = j_k, Y_{k+1} = j_{k+1}) \\
=&\mathbb{P}((X_0, Y_0) = (i_0, j_0)) \prod_{k=0}^{n} \mathbb{P}((X_k, Y_k) = (i_k, j_k), (X_{k+1}, Y_{k+1}) = (i_{k+1}, j_{k+1}))
\end{aligned}
$$

So we get that $((X_n, Y_n) : n \in \mathbb{N})$ is Markov chain with transition matrix $r_{(i,j),(m,n)} = p_{im} q_{jn}$. □

PROBLEM 45 Let S_n be 1-dimensional simple random walk, $a \in \mathbb{Z}$. Let $\tau := \inf\{n \geq 0 : S_n = a\}$. Prove:

1. $(S_{\tau+n} : n \geq 0)$ is a one dimensional simple random walk.
2. $(S_{n\wedge\tau} : n \geq 0)$ is a Markov chain on $\mathbb{Z}$ and give its Transition matrix.
3. $(S_{n\wedge\tau} : n \geq 0) \perp (S_{\tau+n} : n \geq 0)$.

SOLUTION. 1.

$$
\begin{aligned}
&\mathbb{P}(S_\tau = i_0, S_{\tau+1} = i_1, \cdots, S_{\tau+n} = i_n \mid \tau < \infty) \\
=&\sum_{k\in\mathbb{N}} \mathbb{P}(\tau = k, S_\tau = i_0, S_{\tau+1} = i_1, \cdots, S_{\tau+n} = i_n \mid \tau < \infty) \\
=&\sum_{k\in\mathbb{N}} \mathbb{P}(\tau = k, S_k = i_0, S_{k+1} = i_1, \cdots, S_{k+n} = i_n \mid \tau < \infty) \\
=&\mathbb{1}(a = i_0) \sum_{k\in\mathbb{N}} \mathbb{P}(S_0 \neq a, \cdots, S_{k-1} \neq a, S_k = a, S_{k+1} = i_1, \cdots, S_{k+n} = i_n \mid \tau < \infty) \\
=&\frac{\mathbb{1}(a = i_0) \sum_{k\in\mathbb{N}} \mathbb{P}(S_0 \neq a, \cdots, S_{k-1} \neq a, S_k = a, S_{k+1} = i_1, \cdots, S_{k+n} = i_n)}{\mathbb{P}(\tau < \infty)} \\
=&\frac{\mathbb{1}(a = i_0)}{\mathbb{P}(\tau < \infty)} \sum_{k\in\mathbb{N}} \mathbb{P}(S_0 \neq a, \cdots, S_{k-1} \neq a, S_k = a, S_{k+1} = i_1, \cdots, S_{k+n} = i_n) \\
=&\frac{\mathbb{1}(a = i_0)}{\mathbb{P}(\tau < \infty)} \sum_{k\in\mathbb{N}} \mathbb{P}(S_{k+1} = i_1, \cdots, S_{k+n} = i_n \mid S_0 \neq a, \cdots, S_{k-1} \neq a, S_k = a) \\
&\times \mathbb{P}(S_0 \neq a, \cdots, S_{k-1} \neq a, S_k = a) \\
=&\frac{\mathbb{1}(a = i_0)}{\mathbb{P}(\tau < \infty)} \sum_{k\in\mathbb{N}} \mathbb{P}(S_{k+1} = i_1, \cdots, S_{k+n} = i_n \mid S_k = a)\mathbb{P}(\tau = k) \\
=&\frac{\mathbb{1}(a = i_0)}{\mathbb{P}(\tau < \infty)} \sum_{k\in\mathbb{N}} \prod_{l=0}^{n-1} p_{i_l i_{l+1}} \mathbb{P}(\tau = k) = \mathbb{1}(a = i_0) \prod_{l=0}^{n-1} p_{i_l i_{l+1}}
\end{aligned}
$$

Where $p_{ij} : i, j \in \mathbb{Z}$ is the transition matrix of $S_n : n \in \mathbb{N}$. So $(S_{\tau+n} : n \in \mathbb{N})$ is Markov chain with transition matrix same as S_n.

2.
$$
\begin{aligned}
&\mathbb{P}(S_{\tau\wedge 0}=i_0,S_{\tau\wedge 1}=i_1,\cdots,S_{\tau\wedge n}=i_n\mid\tau<\infty)\\
=&\sum_{k\in\mathbb{N}}\mathbb{P}(\tau=k,S_{\tau\wedge 0}=i_0,S_{\tau\wedge 1}=i_1,\cdots,S_{\tau\wedge n}=i_n\mid\tau<\infty)\\
=&\sum_{k\in\mathbb{N}}\mathbb{P}(\tau=k,S_{k\wedge 0}=i_0,S_{k\wedge 1}=i_1,\cdots,S_{k\wedge n}=i_n\mid\tau<\infty)\\
=&\sum_{k\geq n}\mathbb{P}(\tau=k,S_0=i_0,\cdots,S_n=i_n\mid\tau<\infty)\\
+&\sum_{k<n}\mathbb{P}(\tau=k,S_0=i_0,\cdots,S_{k-1}=i_{k-1},S_k=i_k=i_{k+1}=\cdots=i_n\mid\tau<\infty)\\
=&\mathbb{1}(i_0,i_1,\cdots,i_n\neq a)\prod_{k=0}^{n-1}p_{i_ki_{k+1}}+\sum_{k=0}^{n-1}\mathbb{1}(i_0,\cdots,i_{k-1}\neq a,i_k=i_{k+1}=\cdots=i_n=a)\prod_{l=0}^{k-1}p_{i_li_{l+1}}\\
=&\prod_{k=0}^{n-1}(\mathbb{1}(i_k=i_{k+1}=a)+\mathbb{1}(i_k\neq a)p_{i_k,i_{k+1}})
\end{aligned}
$$

So $(S_{n\wedge\tau}:n\in\mathbb{N})$ is Markov chain with transition matrix $q_{i,j}=\mathbb{1}(i=j=a)+\mathbb{1}(i\neq a)p_{i,j}$.

3. By the corollary 3.2.11, we only need to proof τ is stopping time on $(\mathcal{F}_n:n\geq 0)$, Where $\mathcal{F}_n=\sigma(S_k:k\leq n)$. So we only need to prove $\forall n\in\mathbb{N}$, $\{\tau=n\}\in\mathcal{F}_n$. Since $\{\tau=n\}=\{\omega\in\omega:S_0,\cdots,S_{n+1}\neq a,S_n=a\}=\bigcap_{0\leq k\leq n}\{S_k\neq a\}\cap\{S_n=a\}$, And $\{S_k\neq a\}\in\sigma(S_k),\forall 0\leq k\leq n,\{S_n=a\}\in\sigma(S_n)$, Then $\{\tau=n\}\in\mathcal{F}_n$.

□

PROBLEM 46 Let S_n be 1-dimensional symmetry simple random walk starting from zero. Prove: $(|S_n|:n\geq 0)$ is a Markov chain on $\mathbb{Z}^+$ and give its transition matrix.

SOLUTION. Only need to solve problem 57. □

PROBLEM 47 Let S_n be 1-dimensional simple random walk starting from zero. Prove: $(|S_n|:n\geq 0)$ is a Markov chain on $\mathbb{Z}^+$ and give its transition matrix.

SOLUTION. By the definition of $|S_n|$, we can easily get to know $\forall(i_0,\cdots,i_n)\in\mathbb{Z}^+$, $\mathbb{P}(|S_k|=i_k,k=0,\cdots,n)>0\iff i_0=0,|i_k-i_{k-1}|=1,k=1,\cdots,n$. Let $S_n=\sum_{k=1}^n\xi_k$, where $(\xi_n:n\geq)$ are i.i.d. r.v. and $\mathbb{P}(\xi_1=1)=p,\mathbb{P}(\xi_1=-1)=q$. $A:=\{(i_0,\cdots,i_{n+1})\in\mathbb{Z}:i_0=0,|i_k-i_{k-1}|=1,k=1,\cdots,n+1\}$. $\forall(i_0,\cdots,i_{n+1})\in A$, let $r:=\max\{k<n+1:i_k=0\}$. Then $i_r=0,\forall k:n+1>k\geq r+1,i_k\geq 1$.

1. $\forall(i_0,\cdots,i_{n+1})\notin A$, then $\mathbb{P}(|S_k|=i_k,k=0,\cdots,n)=0$, Then we have no need to calculate $\mathbb{P}\Big(|S_{n+1}|=i_{n+1}\Big||S_k|=i_k,k=0,\cdots,n\Big)$.
2. If $(i_0,\cdots,i_{n+1})\in A\wedge r=n$, then $i_n=0,i_{n+1}=1$. Then $|S_n|=0\iff S_n=0\implies S_{n+1}=\pm 1\iff|S_{n+1}|=1$. So we get that $\mathbb{P}\Big(|S_{n+1}|=i_{n+1}\Big||S_k|=i_k,k=1,\cdots,n\Big)=1=\mathbb{P}\Big(|S_{n+1}|=i_{n+1}\Big||S_n|=i_n\Big)$.

3. $\forall(i_0,\cdots,i_{n+1})\in A, i_n\neq 0,$

$$
\begin{aligned}
&\mathbb{P}\Big(|S_k|=i_k, S_n=i_n, k=0,\cdots,n\Big||S_k|=i_k, k=0,\cdots,r\Big)\\
&=\mathbb{P}\Big(|S_k|=i_k, S_n=i_n, k=r+1,\cdots,n\Big||S_k|=i_k, k=0,\cdots,r-1, S_r=0\Big)\\
&=\mathbb{P}\Big(|S_k|=i_k, S_n=i_n, k=r+1,\cdots,n\Big|S_r=0\Big)\\
&=\mathbb{P}\Big(S_k=i_k, S_n=i_n, k=r+1,\cdots,n\Big|S_r=0\Big)\\
&=p^{\frac{n-r+i_n}{2}}q^{\frac{n-r-i_n}{2}}
\end{aligned}
$$

In the same way, we can get

$$
\mathbb{P}\Big(|S_k|=i_k, S_n=-i, k=0,\cdots,n\Big||S_k|=i_k, k=0,\cdots,r\Big)=p^{\frac{n-r-i_n}{2}}q^{\frac{n-r+i}{2}}
$$

So

$$
\begin{aligned}
&\mathbb{P}\Big(S_n=i_n\Big||S_k|=i_k, k=0,\cdots,n\Big)\\
&=\frac{\mathbb{P}(S_n=i_n,|S_k|=i_k,k=0,\cdots,n)}{\mathbb{P}(|S_k|=i_k,k=0,\cdots,n)}\\
&=\frac{\mathbb{P}\Big(S_n=i_n,|S_k|=i_k,k=0,\cdots,n\Big||S_k|=i_k,k=0,\cdots,r\Big)\mathbb{P}(|S_k|=i_k,k=0,\cdots,r)}{\mathbb{P}(|S_k|=i_k,k=0,\cdots,n)}\\
&=\frac{\mathbb{P}\Big(S_n=i_n,|S_k|=i_k,k=0,\cdots,n\Big||S_k|=i_k,k=0,\cdots,r\Big)}{\frac{\mathbb{P}(|S_k|=i_k,k=0,\cdots,n)}{\mathbb{P}(|S_k|=i_k,k=0,\cdots,r)}}\\
&=\frac{\mathbb{P}\Big(S_n=i_n,|S_k|=i_k,k=0,\cdots,n\Big||S_k|=i_k,k=0,\cdots,r\Big)}{\mathbb{P}\Big(|S_k|=i_k,k=0,\cdots,n\Big||S_k|=i_k,k=0,\cdots,r\Big)}\\
&=\frac{\mathbb{P}\Big(S_n=i_n,|S_k|=i_k,k=0,\cdots,n\Big||S_k|=i_k,k=0,\cdots,r\Big)}{\mathbb{P}\Big(|S_k|=i_k,k=r+1,\cdots,n\Big||S_k|=i_k,k=0,\cdots,r\Big)}\\
&=\frac{p^{n-r+\frac{i_n}{2}}q^{n-r-\frac{i_n}{2}}}{p^{n-r+\frac{i_n}{2}}q^{n-r-\frac{i_n}{2}}+p^{n-r-\frac{i_n}{2}}q^{n-r+\frac{i_n}{2}}}\\
&=p^{i_n}(p^{i_n}+q^{i_n})^{-1}
\end{aligned}
$$

In the same way, we can get

$$
\mathbb{P}\Big(S_n=-i_n\Big||S_k|=i_k, k=0,\cdots,n\Big)=q^{i_n}(p^{i_n}+q^{i_n})^{-1}
$$

Then

$$\begin{aligned}
&\mathbb{P}\Big(|S_{n+1}|=i_{n+1}\Big||S_k|=i_k,k=0,\cdots,n\Big)\\
&=\mathbb{P}\Big(|S_{n+1}|=i_{n+1}\mid S_n=i_n,|S_k|=i_k,k=0,\cdots,n)\mathbb{P}(S_n=i_n\Big||S_k|=i_k,k=0,\cdots,n\Big)\\
&+\mathbb{P}\Big(|S_{n+1}|=i_{n+1}\mid S_n=-i_n,|S_k|=i_k,k=0,\cdots,n)\mathbb{P}(S_n=-i_n\Big||S_k|=i_k,k=0,\cdots,n\Big)\\
&=\mathbb{P}\Big(S_{n+1}=i_{n+1}\Big|S_n=i_n\Big)\mathbb{P}\Big(S_n=i_n\Big||S_k|=i_k,k=0,\cdots,n\Big)\\
&+\mathbb{P}\Big(S_{n+1}=-i_{n+1}\Big|S_n=-i_n\Big)\mathbb{P}\Big(S_n=-i_n\Big||S_k|=i_k,k=0,\cdots,n\Big)\\
&=\mathbb{1}(i_{n+1}=i_n+1)(p^{i_n+1}+q^{i_n+1})(p^{i_n}+q^{i_n})^{-1}+\mathbb{1}(i_{n+1}=i_n-1)(p^{i_n}q+pq^{i_n})(p^{i_n}+q^{i_n})^{-1}
\end{aligned}$$

Thus, $(|S_n|:n\geq 0)$ is Markov chain on $\mathbb{Z}^+$, with transition matrix $r_{ij}=\mathbb{1}(0\neq i=j-1)(p^{i+1}+q^{i+1})(p^i+q^i)^{-1}+\mathbb{1}(0\neq i=j+1)(p^iq+pq^i)(p^i+q^i)^{-1}+\mathbb{1}(i=0,j=1)$. When $p=q=\frac{1}{2}$, we get $r_{ij}=\frac{1}{2}\mathbb{1}(i\neq 0,|j-i|=1)+\mathbb{1}(i=0,j=1)$.

□

3.3 状态的分类

方便起见，记 $\mathbb{P}_i=\mathbb{P}(\cdot\mid X_0=i)$，用 $\mathbb{E}_i$ 表示对应的期望。.pr PROBLEM 48 Assume $(X_n:n\geq 0)$ is an irreducible Markov chain on E. Prove that $(X_n:n\geq 0)$ is recurrent (or transient) $\iff$ $\forall i\in E$,

$$\mathbb{P}\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\{X_k=i\}\right)=1(\text{ or }0).$$

SOLUTION . Only need to prove “ $\Longrightarrow$ ”.

First we assume $(X_n:n\in\mathbb{N})$ is recurrent, we should prove $\mathbb{P}\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\{X_k=i\}\right)=1$. Let $\tau_1=\inf\{n>0:X_n=i\}$, and for $n\in\mathbb{N}^+$, we let $\tau_{n+1}=\inf\{n>\tau_n:X_n=i\}$. Since i is recurrent and (X_n) is irreducible, we know that $\tau_1<\infty, a.s.$. Then $(X_{\tau_1+n}:n\in\mathbb{N})$ is a Markov chain with the same transition matrix as (X_n). So we get that $\tau_2-\tau_1<\infty, a.s.$. So $\tau_2<\infty$, a.s.. Use MI, we can easily get that $\forall n\in\mathbb{N}^+,\tau_n<\infty, a.s.$. Easy to get that $\tau_{n+1}>\tau_n$ and $\tau_1>0$, so $\tau_n\geq n$. So $\tau_n<\infty\implies\exists k\geq n, X_k=i$. So $\forall n\in\mathbb{N},\mathbb{P}\left(\bigcup_{k=n}^{\infty}\{X_k=i\}\right)=1$. Thus, $\mathbb{P}\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\{X_k=i\}\right)=1$.

Second we assume $(X_n:n\in\mathbb{N})$ is transient, we should prove that $\mathbb{P}\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\{X_k=i\}\right)=0$. Write $A=\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\{X_k=i\}$. We define τ_n as above. Easy to know $\forall\omega\in A,\forall n\in\mathbb{N}^+,\tau_n<\infty$. And easy to know that $\tau_{n+1}-\tau_n\mid_{\tau_n<\infty}$ has the same distribution for every n. And since (X_n) is transient, we know (X_{τ_k+n}) is transient for every $k\in\mathbb{N}^+$. So we know $\mathbb{P}(\tau_{n+1}-\tau_n<\infty\mid\tau_n<\infty)<1$. Then $\mathbb{P}(A)=\mathbb{P}(\forall n,\tau_n<\infty)\leq\mathbb{P}(\forall n,\tau_{n+1}-\tau_n<\infty)\leq\prod_{n=1}^{\infty}\mathbb{P}(\tau_{n+1}-\tau_n<\infty\mid\tau_n<\infty)=\prod_{n=1}^{\infty}\mathbb{P}(\tau_2-\tau_1<\infty\mid\tau_1<\infty)=0$. □

PROBLEM 49 Assume $(X_n:n\geq 0)$ is Markov chain on E, where E is finite. Prove that $\exists x\in E$, x is recurrent.

SOLUTION . Easily $\sum_{i\in E}\sum_{n=1}^{\infty} p_{ki}(n) = \sum_{n\in\mathbb{N}^+}\sum_{i\in E} p_{ki}(n) = \sum_{n\in\mathbb{N}^+} 1 = +\infty$. Since E is finite, we obtain that there is at least one i such that $\sum_{n\in\mathbb{N}^+} p_{ki}(n) = \infty$, then p_{ki}^*. Then i is recurrent. □

PROBLEM 50 Assume $(X_n : n \geq 0)$ is Markov chain on $\mathbb{Z}$. Prove it is transient $\iff$ $\forall \mu_0$ is primitive distribution, $\lim_{n\to\infty} |X_n| \overset{\text{a.s.}}{=} \infty$.

SOLUTION . Only need to prove that $\forall k \in \mathbb{N}, \liminf_{n\to\infty} |X_n| > k, a.s.$. Consider the event $\liminf_{n\to\infty} |X_n| \leq k$, it means $\forall n \in \mathbb{N}, \exists t \geq n, X_t \in [-k,k]$. So we only need to prove $\mathbb{P}\left(\bigcap_{n=1}^{\infty}\bigcup_{t=n}^{\infty}\{X_t \in [-k,k]\}\right) = 0$. It is sufficient to prove that $\mathbb{P}(\bigcup_{u\in[-k,k]}\bigcap_{n=1}^{\infty}\bigcup_{t=n}^{\infty}\{X_t = u\}) = 0$. Since (X_n) is transient, it has been proved that $\mathbb{P}(\bigcap_{n=1}^{\infty}\bigcup_{t=n}^{\infty}\{X_t = u\}) = 0$. So $\mathbb{P}(\bigcup_{u\in[-k,k]}\bigcap_{n=1}^{\infty}\bigcup_{t=n}^{\infty}\{X_t = u\}) = 0 \leq \sum_{u\in[-k,k]}\mathbb{P}(\bigcap_{n=1}^{\infty}\bigcup_{t=n}^{\infty}\{X_t = u\}) = 0$. □

PROBLEM 51 Assume $\{a_i : i \geq 1\} \subset (0,1)$. Consider $E := \mathbb{N}$, P is a transition matrix on E, where $p_{ij} = a_i \mathbb{1}_{\{j=0\}} + (1-a_i)\mathbb{1}_{\{j=i+1\}}$. Prove:

1. P is irreducible.
2. P is recurrent $\iff$ $\sum_i a_i = \infty$.
3. P is ergodic $\iff$ $\sum_{k=1}^{\infty}\prod_{i=1}^{k-1}(1-a_i) < \infty$.

SOLUTION .

1. Easy to prove that $p_{i0}(1) > 0, \forall i \in \mathbb{N}$. And easily $p_{0i}(i) = \prod_{k=0}^{i-1}(1-a_k) > 0$. So P is irreducible.
2. Since P is irreducible, we only need to consider $X_0 = 0$. Then $\{T_0 > n\} \overset{\text{a.s.}}{=} \{X_k = k, k = 0,\cdots,n\}$. Then $\mathbb{P}_0(T_0 = \infty) = \mathbb{P}_0(\bigcap_n\{T_0 > n\}) = \lim_{n\to\infty}\mathbb{P}_0(X_k = k, k = 0,\cdots,n) = \lim_{n\to\infty}\prod_{k=0}^{n-1}(1-a_k) = \prod_{k=0}^{\infty}(1-a_k)$. Then $\mathbb{P}_0(T_0 = \infty) = 0 \iff \prod_{k=0}^{\infty}(1-a_k) = 0 \iff \sum_k a_k = \infty$.
3. Since $\mathbb{E}_0(T_0) = \sum_{n\in E}\mathbb{P}_0(T_0 > n) = \sum_{n=0}^{\infty}\prod_{k=0}^{n-1}(1-a_k)$, then P is ergodic $\iff$ $\mathbb{E}_0(T_0) < \infty$ $\iff$ $\sum_{n=0}^{\infty}\prod_{k=0}^{n-1}(1-a_k) < \infty$.

□

PROBLEM 52 Assume P is a transition matrix on E and P is irreducible, $j \in E$. Prove: P is recurrent $\iff$ 1 is the minimum non-negative solution of

$$y_i = \sum_{k\neq j} p_{ik}y_k + p_{ij}, i \in E \setminus \{j\} \tag{10}$$

SOLUTION . "$\implies$": If P is recurrent, then the bounded solution of $z_i = \sum_{k\in E} p_{ik}z_k, i \in E\setminus\{j\}$ is constant. Easy to get that 1 is one of solution of Equation (**??**), now we will prove that 1 is the unique solution. Assume there is another solution, then $y_i - 1 = \sum_{k\neq j} p_{ik}(y_k - 1), \forall i \in E \setminus \{j\}$.

Then we let $z_j = 0, z_i = y_i - 1, \forall i \neq j$, we find a non-constant solution, contradiction! So 1 is the unique solution and thus minimum solution.

" $\Longleftarrow$ ": If P is transient, then the bounded solution of $z_i = \sum_{k\in E} p_{ik} z_k, i \in E \setminus \{j\}$ has non-constant solution. Without loss of generality, we can assume $z_j = 0, \forall i \in E, |z_i| \leq 1, \exists i_0 \in E, z_{i_0} < 0$. Let $y_i = 1 + z_i, i \in E$, then $\{y_i : i \in E\}$ is the bounded solution of Equation (**??**). But $y_{i_0} < 1, y_i \geq 0, i \in E$. So 1 is not the minimum solution. □

PROBLEM 53 Let $\{a_k : k \geq 0\}$ satisfies $\sum_{k\geq 0} a_k = 1, a_k \geq 1, a_0 > 0$, $\mu := \sum_{k=1}^{\infty} k a_k > 1$. Define $p_{ij} = \begin{cases} a_j & , i = 0 \\ a_{j-i+1} & , i \geq 1 \wedge j \geq i-1 \\ 0 & , \text{otherwise} \end{cases}$. Prove: P is transient.

SOLUTION. First, we prove that P is irreducible: Since $\sum_{k=1}^{\infty} k a_k > 1$, then $\exists m, a_m > 0$. And $\forall i \geq 1$, $p_{i-1,i} = a_0 > 0$. Then $\forall i, j$, if $i < j$, then $p_{ij}(j-i) = a_0^{j-i} > 0$. If $i \geq j$, let $t \equiv i - j \pmod m, 1 \leq t \leq m$, then $p_{ij}(t+1) = a_0^t a_m > 0$.

Let $\xi_n : n \in \mathbb{N}$ is a sequence of i.i.d r.v with $\mathbb{P}(\xi_0 = i) = a_i$. Since P is irreducible, we only consider the chain begin at 0. Let $X_0 = 0, X_{n+1} = X_n + \xi_n - \mathbb{1}_{X_n > 0}$. Then easily X_n is the Markov chain begin at 0 with transition matrix P. And $X_n = \sum_{k=0}^{n-1} \xi_k - \sum_{k=0}^{n-1} \mathbb{1}_{X_n > 0} \geq \sum_{k=0}^{n-1} (\xi_k - 1)$. So we obtain $\liminf_{n\to\infty} \frac{X_n}{n} \geq \liminf_{n\to\infty} \frac{\sum_{k=0}^{n-1} \xi_k - 1}{n} = \mu - 1 > 0$. So $\liminf_{n\to\infty} X_n = \infty$, so 0 is transient. □

PROBLEM 54 Let $X = \{X(n) : n \geq 0\}$ be Markov chain defined on probability space $(\Omega, \mathscr{F}, \mathbb{P})$, with state space E and transition probability matrix $P = (p(i,j) : i, j \in E)$. Let $a, b \in E$, $\tau_0 = 0$, $\sigma_k = \inf\{n \geq \tau_{k-1} : X(n) = b\}, \tau_k = \inf\{n \geq \sigma_{k-1} : X(n) = a\}$. Prove: $\tau_n, \sigma_n, n \geq 1$ are all stopping time on $(\mathscr{F}_n : n \geq 0)$.

SOLUTION. We use MI to prove it. Easily $\sigma_1 = \inf\{n \geq \tau_0 : X(n) = b\} = \inf\{n \geq 0 : X(n) = b\}$ is stopping time. Assume for certain $n \geq 1$, we have proved that σ_n, τ_{n-1} are stopping times, now we need to prove σ_{n+1}, τ_n are stopping times. Since σ_n is stopping time, we know $\forall k \leq m, \{k \geq \sigma_n\} \in \mathcal{F}_m$. And obviously $\forall k \leq m, \{X(k) = a\} \in \mathcal{F}_m$. So we obtain that $\{\sigma_n \leq m\} = \bigcup_{k=1}^{m} \{k \geq \sigma_n, X(k) = a\} \in \mathcal{F}_m$. So τ_n is stopping time. Since τ_n is stopping time, we know $\forall k \leq m, \{k \geq \tau_n\} \in \mathcal{F}_m$. And obviously $\forall k \leq m, \{X(k) = b\} \in \mathcal{F}_m$. So we obtain that $\{\tau_n \leq m\} = \bigcup_{k=1}^{m} \{k \geq \tau_n, X(k) = b\} \in \mathcal{F}_m$. So σ_{n+1} is stopping time. So we finally obtain that $\forall n \in \mathbb{N}^+, \sigma_n, \tau_n$ are stopping times. □

PROBLEM 55 Let $(X_n : n \geq 0)$ be a one-dimension simple random walk starting at 1. Let $e(n) = \{X_{n\wedge\tau_1} : n \geq 0\}$, where $\tau_1 = \inf\{n \geq 0 : X_n = 0\}$. Find the distribution of $\sup_{n\geq 0} e(n)$.

SOLUTION. Assume $\mathbb{P}(X_{n+1} - X_n = 1) = p, \mathbb{P}(X_{n+1} - X_n = -1) = q$, where $p + q = 1$. Let $E := \sup_{n\geq 0} e(n)$. Let $m \in \mathbb{N}^+$. Let $\sigma := \inf\{n \geq 0 : X_n = 0 \text{ or } X_n = m\}$. Then σ is a stopping time.

First we assume $p \neq q$. Let $Y_n := \left(\frac{q}{p}\right)^{X_n}$, then it's easy to check that Y_n is martingale. So we know that $Y_{n\wedge\sigma}$ is martingale, too. Easy to get that $\sigma < \infty, a.s.$, so $Y_{n\wedge\sigma} \overset{\text{a.s.}}{\to} Y_\sigma$. And $0 \le Y_{n\wedge\sigma} \le m$, so $\mathbb{E}(Y_\sigma) = \mathbb{E}(Y_{n\wedge\sigma}) = \mathbb{E}(Y_0)$. Noting that $\{X_\sigma = 0\} \overset{\text{a.s.}}{=} \{E < m\}$ and $\{X_\sigma = m\} \overset{\text{a.s.}}{=} \{E \ge m\}$, we get two equations:

$$\begin{cases} \mathbb{P}(E < m) + \mathbb{P}(E \ge m) = 1 \\ \mathbb{P}(E < m) + \mathbb{P}(E \ge m)\left(\frac{q}{p}\right)^m = \frac{q}{p} \end{cases}.$$

Solve this equation, we get $\mathbb{P}(E \ge m) = \frac{\frac{q}{p}-1}{(\frac{q}{p})^m - 1}$. Then $\mathbb{P}(E = m) = \frac{(\frac{p}{q})^m(\frac{p}{q}-1)}{((\frac{p}{q})^m-1)((\frac{p}{q})^{m+1}-1)}$. Furthermore, easily $\mathbb{P}(E = \infty) = \lim_{m\to\infty}\mathbb{P}(E \ge m) = \begin{cases} 0 & \frac{q}{p} > 1 \\ 1 - \frac{q}{p} & \frac{q}{p} < 1 \end{cases}$.

Second, we consider $p = q = \frac{1}{2}$. Then easily X_n is martingale. So we know that $X_{n\wedge\sigma}$ is martingale, too. Easy to get that $\sigma < \infty, a.s.$, so $X_{n\wedge\sigma} \overset{\text{a.s.}}{\to} X_\sigma$. And $0 \le X_{n\wedge\sigma} \le m$, so $\mathbb{E}(X_\sigma) = \mathbb{E}(X_{n\wedge\sigma}) = \mathbb{E}(X_0)$. Noting that $\{X_\sigma = 0\} \overset{\text{a.s.}}{=} \{E < m\}$ and $\{X_\sigma = m\} \overset{\text{a.s.}}{=} \{E \ge m\}$, we get two equations:

$$\begin{cases} \mathbb{P}(E < m) + \mathbb{P}(E \ge m) = 1 \\ 0\mathbb{P}(E < m) + m\mathbb{P}(E \ge m) = 1 \end{cases}.$$

Solve this equation, we get $\mathbb{P}(E \ge m) = \frac{1}{m}$. So $\mathbb{P}(E = m) = \frac{1}{m(m+1)}$, and easily $\mathbb{P}(E = \infty) = 0$. □

PROBLEM 56 Prove:

1. When $0 < p \le q$, the reflecting random walk with transition matrix Q_+^a is recurrent.
2. When $0 < q \le p$, the reflecting random walk with transition matrix Q_-^a is recurrent.

SOLUTION. By symmetry, only need to prove 1. By shifting, without loss of generality we can assume $a = 0$. We consider the equation

$$y_0 = y_1, \forall i \ge 1, y_i = qy_{i-1} + py_{i+1}$$

Only need to prove its all bounded solution are all constant. Easy to get $y_{i+2} = \frac{1}{p}y_{i+1} - \frac{q}{p}y_i$. Consider the charasteristic equation of this sequence, $x^2 - \frac{x}{p} + \frac{q}{p} = 0$. We get $x_1 = 1, x_2 = \frac{q}{p} \ge 1$. If $x_2 > 1$, then $y_n = c_1x_1^n + c_2x_2^n$ is bounded $\iff$ $c_2 = 0$, so $y_n = c_1x_1^n = c_1$ is constant. Else, $x_2 = x_1 = 1$, then $y_n = (an+b)x_1^n = an + b$ is bounded $\iff$ $a = 0$, so $y_n = b$ is constant. So the Markov chain is recurrent. □

PROBLEM 57 Prove colloary 4.4.3. i.e., let $\phi_0(n : n \in \mathbb{N}^+)$ be simple random walk begin at $\phi_0(0) \ge a + 1$, let $\zeta_0 := \inf\{m : \phi_0(m) = a + 1\}$, let $Y_n : n \in \mathbb{N}$ be reflecting simple random walk on $\mathbb{Z}_+^a$, starting at $a+1$, indenpendent with ϕ_0. Let $X_n := \begin{cases} \phi_0(n) & n \le \zeta_0 \\ Y_{n-\zeta_0} & n \ge \zeta_0 \end{cases}$. Prove that $X_n : n \in \mathbb{N}$ is reflecting random walk on $\mathbb{Z}_+^a$ begin at $\phi_0(0)$.

SOLUTION. Now we consider $n \in \mathbb{N}^+$ and $i_0, i_1, i_2, \cdots, i_{n+1} \in \mathbb{Z}_+^a$.

1. If $\forall k : 1 \le k \le n, i_k \ne a+1$, then we have

$$\begin{aligned} &\mathbb{P}(X_0 = i_0, \cdots, X_{n+1} = i_{n+1}) \\ &= \mathbb{P}(\phi_0(0) = i_0, \cdots, \phi_0(n+1) = i_{n+1}) \\ &= \mathbb{P}(\phi_0(0) = i_0, \cdots, \phi_0(n) = i_n)\mathbb{P}(\phi_0(n+1) = i_{n+1} \mid \phi_0(n) = i_n) \\ &= \mathbb{P}(X_0 = i_0, \cdots, X_n = i_n) q_+^a(i_n, i_{n+1}) \end{aligned}$$

2. Else, we let $k := \inf\{m : 1 \le m \le n, i_m = a+1\}$. Then we have

$$\begin{aligned} &\mathbb{P}(X_0 = i_0, \cdots, X_{n+1} = i_{n+1}) \\ &= \mathbb{P}(\phi_0(0) = i_0, \cdots, \phi_0(k) = i_k, Y_0 = a+1, Y_1 = i_{k+1}, \cdots, Y_{n-k} = i_n, Y_{n-k+1} = i_{n+1}) \\ &= \mathbb{P}(\phi_0(0) = i_0, \cdots, \phi_0(k) = i_k)\mathbb{P}(Y_0 = a+1, Y_1 = i_{k+1}, \cdots, Y_{n-k} = i_n, Y_{n-k+1} = i_{n+1}) \\ &= \mathbb{P}(\phi_0(0) = i_0, \cdots, \phi_0(k) = i_k)\mathbb{P}(Y_0 = a+1, Y_1 = i_{k+1}, \cdots, Y_{n-k} = i_n) q_+^a(i_n, i_{n+1}) \\ &= \mathbb{P}(X_0 = i_0, \cdots, X_n = i_n) q_+^a(i_n, i_{n+1}) \end{aligned}$$

So we get $(X_n : n \ge 0)$ is reflecting simple random walk on $\mathbb{Z}_+^a$. $\square$

4 分支过程及其应用

4.1 定义和基本构造

4.1.1 分支过程的定义

PROBLEM 58

1. Assume $\{Y_1(n) : n \geq 0\}, \{Y_2(n) : n \geq 0\}$ are two independent migrating branch process with offspring distribution $(p(i) : i \in \mathbb{N})$ and the migrating probobility respectively are $(\gamma_1(i) : i \in \mathbb{Z}_+), (\gamma_2(i) : i \in \mathbb{N})$. Prove: $\{Y_1(n) + Y_2(n) : n \geq 0\}$ is migrating branching process with offspring distribution $p(i) : i \in \mathbb{N}$ and migrating probability $\gamma_1 * \gamma_2$.

2. Let $\{Y(n) : n \in \mathbb{N}\}$ be migrating branch process with offspring distribution $p(j) : j \in \mathbb{N}$ and the migrating distribution $\gamma(i) : i \in \mathbb{N}$. $P_n^\gamma = (p_n^\gamma(i,j); i,j \in \mathbb{N})$ is the n-th transition matrix. Prove: $\forall i, n \geq 1$

$$\sum_{j=0}^{\infty} p_n^\gamma(i,j) z^j = g_n(z)^i \prod_{k=1}^{n} h(g_{k-1}(z)), |z| \leq 1$$

where h is the generating function of $(\gamma(j) : j \in \mathbb{N})$. g is the generating function of $(p(j) : j \in \mathbb{N})$.

3. h, g are defined as above. Assume $m := g'(1-) < \infty, \mu := h'(1-) < \infty$. Prove: $\forall i, n \geq 1$,

$$\mathbb{P}(Y_n \mid Y_0 = i) = im^n + \mu \sum_{k=1}^{n} m^{k-1}$$

SOLUTION . 1. Since Y_1, Y_2 are independent Markov chain, we easily get $\mathbb{P}(Y_1(n+1)+Y_2(n+1) = i \mid \sigma(Y_1(j), Y_2(j) : 0 \leq j \leq n)) = \mathbb{P}(Y_1(n+1) + Y_2(n+1) = i \mid Y_1(n), Y_2(n))$. So to prove $Y_1 + Y_2$ is Markov chain, we only need to prove $\mathbb{P}(Y_1(n+1) + Y_2(n+1) = i \mid Y_1(n), Y_2(n))\mathbb{P}(Y_1(n+1) + Y_2(n+1) = i \mid Y_1(n) + Y_2(n))$.

$$\begin{aligned}
&\mathbb{P}(Y_1(n+1) + Y_2(n+1) = i \mid Y_1(n) = j, Y_2(n) = k) \\
=& \sum_{x+y=i} \mathbb{P}(Y_1(n+1) = x \mid Y_1(n) = j)\mathbb{P}(Y_2(n+1) = y \mid Y_2(n) = k) \\
=& \sum_{x+y=i} p^{*j} * \gamma_1(x) p^{*k} * \gamma_2(y) \\
=& p^{*j} * \gamma_1 * p^{*k} * \gamma_2(i) \\
=& p^{*(j+k)} * \gamma_1 * \gamma_2(i)
\end{aligned}$$

So $\mathbb{P}(Y_1(n+1) + Y_2(n+1) = i \mid Y_1(n), Y_2(n)) = p^{*(Y_1(n)+Y_2(n))} * (\gamma_1 * \gamma_2)(i) \in \sigma(Y_1(n) + Y_2(n)) \subset \sigma(Y_1(n), Y_2(n))$. So $Y_1 + Y_2$ is Markov chain. More over, we have obtained

$\mathbb{P}(Y_1(n+1)+Y_2(n+1)=j \mid Y_1(n)+Y_2(n)=i)=p^{*i}*(\gamma_1*\gamma_2)(j)$. So $\{Y_1(n)+Y_2(n): n\geq 0\}$ is migrating branching process with offspring distribution $p(i): i\in\mathbb{N}$ and migrating probability $\gamma_1*\gamma_2$.

2. Use MI to prove it. Write $G_n(i,z):=\sum_{j=0}^{\infty}p_n^{\gamma}(i,j)z^j$. When $n=0$, we have $p_0^{\gamma}(i,j)=\delta_{ij}$, so $G_0(i,z)=z^i=g_0(z)^i$. When $n=1$, we have $p_1^{\gamma}(i,j)=p^{*i}*\gamma(j)$. So $G_1(i,z)=g(z)^ih(z)$. Assume for certain n we have proved that $G_n(i,z)=g_n(z)^i\prod_{k=1}^n h(g_{k-1}(z))$, Consider $n+1$. Easily $p_{n+1}^{\gamma}(i,j)=\sum_{k\in\mathbb{N}}p_n^{\gamma}(k,j)p(i,\cdot)*\gamma(k)$. So

$$\begin{aligned}G_{n+1}(i,z)=&\sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{N}}p_1^{\gamma}(i,k)p_n^{\gamma}(k,j)z^j\\=&\sum_{k\in\mathbb{N}}p_1^{\gamma}(i,k)G_n(k,z)\\=&\prod_{k=1}^n h(g_{k-1}(z))\sum_{k\in\mathbb{N}}p_1^{\gamma}(i,k)g_n(z)^k\\=&\prod_{k=1}^n h(g_{k-1}(z))G_1(i,g_n(z))\\=&g_{n+1}(z)\prod_{k=1}^{n+1}h(g_{k-1}(z))\end{aligned}$$

3. Easily $\mathbb{P}(Y_n \mid Y_0=i)=D_zG_n(i,z)\mid_{z\to 1-}$. Noting $g(1)=h(1)=1$, easy to get that $\mathbb{P}(Y_n\mid Y_0=i)=im^n+\mu\sum_{k=1}^n m^{k-1}$.

$\square$

PROBLEM 59 Assume $b\in(0,1), p\in(0,1)$. Let $\mu(0)=\frac{1-b-p}{1-p}, \mu(j)=bp^{j-1}, j\geq 1$. Prove:

1. $(\mu(j):j\in\mathbb{N})$ is probability distribution and
$$g(z):=\sum_{j=0}^{\infty}\mu(j)z^j=\frac{1-b-p}{1-p}+\frac{bz}{1-pz}.$$

2. Let $b=(1-p)^2$. Then $g'(1)=1$ and
$$g(z)=p+\frac{(1-p)^2z}{1-pz}=\frac{p-(2p-1)z}{1-pz}.$$
Prove: $\forall n\geq 1$,
$$g_n(z)=\frac{np-((n+1)p-1)z}{1+(n-1)p-npz}.$$

SOLUTION. 1. Easily $\sum_{j=1}^{\infty}\mu(j)=\frac{b}{1-p}$. So $\sum_{j=0}^{\infty}\mu(j)=1$. Easily $\sum_{j=1}^{\infty}\mu(j)z^j=\frac{bz}{1-pz}$. So $g(z)=\mu(0)+\frac{bz}{1-pz}=\frac{1-b-p}{1-p}+\frac{bz}{1-pz}$.

2. $g_{n+1}(z) = g(g_n(z)) = \frac{p-(2p-1)g_n(z)}{1-pg_n(z)}$. So $g_{n+1}(z) - 1 = \frac{(g_n(z)-1)(1-p)}{1-pg_n(z)}$. Thus, we obtain $\frac{1}{g_{n+1}(z)-1} = \frac{1}{g_n(z)-1} - \frac{p}{1-p}$. So $\frac{1}{g_n(z)-1} = \frac{1}{z-1} - \frac{np}{1-p}$, and finally we get $g_n(z) = \frac{np-((n+1)p-1)z}{1+(n-1)p-npz}$. □

PROBLEM 60 Let $\{X(n) : n \in \mathbb{N}\}$ be branch process with offspring distribution $p(j) : j \in \mathbb{N}$. And g is the generating function. Let $m_2 := g'(1) + g''(1) < \infty$. Let $m = g'(1) < \infty$. $\forall k \geq 1$, $X_n^{(k)} = k^{-1}X_n$. Prove: $\forall \varepsilon > 0, i, n \geq 1$, $\mathbb{P}(|X_n^{(k)} - im^n| \geq \varepsilon \mid X_0^{(k)} = i) \to 0, k \to \infty$.

SOLUTION. In fact, we don't need $m_2 < \infty$. We let $(Y(k,n) : n \in \mathbb{N}), k \in \mathbb{N}$ are independent branch process with offspring distribution $p(j) : j \in \mathbb{N}$ and $Y(k,0) = i$. Then $\sum_{j=1}^k Y(j,n)$ is branch process with offspring distribution $p(j) : j \in \mathbb{N}$ and initial value ki. So $\sum_{j=1}^k Y(j,n) \overset{d}{=} X_n \mid X_0^{(k)} = i$. So $\mathbb{P}(|X_n^{(k)} - im^n| \geq \varepsilon \mid X_0^{(k)} = i) = \mathbb{P}(|\frac{\sum_{j=1}^k Y(j,n)}{k} - im^n| \geq \varepsilon)$. By LLN we obtain $\frac{1}{k}\sum_{j=1}^k Y(j,n) \overset{\text{a.s.}}{\to} im^n$. So finally we get $\lim_{k\to\infty} \mathbb{P}(|X_n^{(k)} - im^n| \geq \varepsilon \mid X_0^{(k)} = i) = \lim_{k\to\infty} \mathbb{P}(|\frac{\sum_{j=1}^k Y(j,n)}{k} - im^n| \geq \varepsilon) = 0$. □

PROBLEM 61 Let $\{X(n) : n \in \mathbb{N}\}$ be branch process with offspring distribution $p(j) : j \in \mathbb{N}$. And g is the generating function, where $m := g'(1) \in (1,\infty), m_2 := g'(1) + g''(1) < \infty$. Let $\sigma^2 := m_2 - m^2 = \mathbb{D}(X(1))$. It is well known that $\exists W, \lim_{n\to\infty} \frac{X_n}{m^n} = W$. Prove:

$$\lim_{n\to\infty} \mathbb{E}_1[(m^{-n}X_n - W)^2] = 0, \mathbb{D}_1(W) = \sigma^2 m^{-1}(m-1)^{-1}$$

SOLUTION. For convenience we write $\mathbb{E}, \mathbb{D}$ instead of $\mathbb{E}_1, \mathbb{D}_1$. Easy to get that $\mathbb{E}(m^{-2n}X_n^2) = \frac{\sigma^2(1-m^{-n})}{m^2-m} + 1$. So by Fatou theorem we get that $\mathbb{E}(W^2) \leq \lim_{n\to\infty} \mathbb{E}(m^{-2n}X_n^2) = \frac{\sigma^2}{m^2-m} + 1 < \infty$. And by Doob Stochastic Processes p317 theorem 3.4 we get that $\mathbb{E}(\max_{n\in\mathbb{N}} m^{-2n}X_n^2) < \infty$. Thus, $m^{-2n}X_n^2$ are integrable uniformly, and so do $(m^{-n}X_n - W)^2$. So by LCDT we can get $\lim_{n\to\infty} \mathbb{E}((m^{-n}X_n - W)^2) = 0$. Noting that

$$\mathbb{E}(m^{-2n}X_n^2 - W^2) = \mathbb{E}((m^{-n}X_n + W)(m^{-n}X_n - W)) \leq \sqrt{\mathbb{E}((m^{-n}X_n + W)^2)\mathbb{E}((m^{-n}X_n - W)^2)} \to 0$$

, we get $\mathbb{E}(W^2) = \lim \mathbb{E}(m^{-2n}X_n^2) = \frac{\sigma^2}{m^2-m} + 1$. And $\mathbb{E}(W) = \lim \mathbb{E}(m^{-n}X_n) = 1$. So $\mathbb{D}(W) = \mathbb{E}(W^2) - \mathbb{E}(W)^2 = \frac{\sigma^2}{m(m-1)}$. □

PROBLEM 62 Let $\{Y(n) : n \in \mathbb{N}\}$ be branch process with offspring distribution $p(j) : j \in \mathbb{N}$. And g is the generating function, where $m := g'(1) \leq 1$. Prove $(p^\gamma(j) : j \in \mathbb{N})$ is the steady-state vector of transition matrix P_n^γ, that is $\sum_{i=0}^\infty p^\gamma(i)p_n^\gamma(i,j) = p^\gamma(j), i \geq 0$.

SOLUTION. Since $\lim_{m\to\infty} p_m^\gamma(i,j) = p^\gamma(j)$, and fix $k \in \mathbb{N}$, we have $\sum_{j=0}^\infty p_m^\gamma(k,i)p_n^\gamma(i,j) = p_{n+m}^\gamma(k,j)$, we only need to prove that $\lim_{m\to\infty} \sum_{i=0}^\infty (p_m^\gamma(k,i) - p^\gamma(i))p_n^\gamma(i,j) = 0$. Since $\lim_{m\to\infty} p_m^\gamma(k,i) = p^\gamma(i)$ and $\sum_{i\in\mathbb{N}} p_m^\gamma(k,i) = 1$, we can easily get that $\sum_{i\in\mathbb{N}} p^\gamma(i) = 1$. For $\varepsilon > 0$, we let N large enough such that $\sum_{k=N}^{\infty} p^\gamma(k) < \varepsilon$. Then we let M large enough such that $\forall i : 0 \le i < N, \forall m \ge M, |p_m^\gamma(k,i) - p^\gamma(k)| < \frac{\varepsilon}{N}$. Then

$$\begin{aligned}
&\left|\sum_{i=0}^{\infty}(p_m^\gamma(k,i) - p^\gamma(i))p_n^\gamma(i,j)\right| \\
\le&\sum_{i=0}^{\infty}|p_m^\gamma(k,i) - p^\gamma(i)|p_n^\gamma(i,j) \\
\le&\sum_{i=0}^{N-1}|p_m^\gamma(k,i) - p^\gamma(i)|p_n^\gamma(i,j) + \sum_{i=N}^{\infty}(p_m^\gamma(k,i) + p^\gamma(i))p_n^\gamma(i,j) \\
\le&\sum_{i=0}^{N-1}\frac{\varepsilon}{N} + \sum_{i=N}^{\infty}p_m^\gamma(k,i) + p^\gamma(i) \\
\le&\varepsilon + \sum_{i=N}^{\infty}p^\gamma(i) + 1 - \sum_{i=1}^{N-1}p_m^\gamma(k,i) \\
\le&\varepsilon + \varepsilon + 1 - \sum_{i=1}^{N-1}p^\gamma(i) + \sum_{i=1}^{N-1}|p_m^\gamma(k,i) - p^\gamma(i)| \\
\le&4\varepsilon
\end{aligned}$$

So finally we get $\lim_{m\to\infty}\sum_{i=0}^{\infty}(p_m^\gamma(k,i) - p^\gamma(i))p_n^\gamma(i,j) = 0$. Thus, $\sum_{i=0}^{\infty}p^\gamma(i)p_n^\gamma(i,j) = p^\gamma(j), i \ge 0$. □

PROBLEM 63 Let $\{Y(n) : n \in \mathbb{N}\}$ be branch process with offspring distribution $p(j) : j \in \mathbb{N}$ and migrating distribution γ. And g is the generating function, where $m := g'(1) \le 1$. Discuss $\lim_{n\to\infty}\mathbb{E}(Y_n \mid Y_0 = i)$.

SOLUTION. Easy to get that $\mathbb{E}(Y_n \mid Y_0 = i) = im^n + \mu\sum_{k=1}^{n} m^{k-1}$. When $m = 1$, we know $\mathbb{E}(Y_n \mid Y_0 = i) \to \infty$. When $m < 1$, we know $\mathbb{E}(Y_n \mid Y_0 = i) \to \frac{\mu}{1-m}$. □

PROBLEM 64 Let $S = (S_n : n \ge 0)$ be the one-dimensional symmetry simple random walk with $S_0 = c \ge 0$. Let $k \ge 1$ and τ be the time of the k-th downcrossing 0. X_b is the times of $(S_{n\wedge\tau} : n \ge 0)$ downcrossing b. Prove:

1. $(X_b : b \ge c-1)$ is branch process. And offspring distribution is $Geo(\frac{1}{2})$
2. $(X_{-a} : a \ge 1)$ is branch process. And offspring distribution is $Geo(\frac{1}{2})$
3. $(X_b : 0 \le b \le c-1)$ is migrating branch process. And offspring distribution is $Geo(\frac{1}{2})$

   And the migrating distribution is concentrating on 1.

SOLUTION. For a random walk y, we let $D(n, y)$ be the number of downcrossings of y of n.

1. Fix $b \geq c-1$. Let ϕ_0 be the journey from start point to $b+1$. Let e_n be n-th journey from $b+1$ to b. Let ε_n be n-th journey after ϕ_0 from b to $b+1$. Then we know that e_n, ε_n are independent. Easy to get that $D(e_n, b) = 1$ and $D(\varepsilon_n, b) = 0, D(\varepsilon_n, b+1 = 0)$. Easy to get that $D((S_{n\wedge\tau} : n \in \mathbb{N}), b+1) = \sum_{t=1}^{D((S_{n\wedge\tau}:n\in\mathbb{N}),b)} D(e_t, b+1)$. Noting that $\forall d : c-1 \leq d \leq b, D((S_{n\wedge\tau} : n \in \mathbb{N}), d) \in \sigma(\varepsilon_n : n \in \mathbb{N})$. We easily get that $D(e_t, b+1) \perp \sigma(\varepsilon_n : n \in \mathbb{N})$. So X_b is Markov process. And to prove it's branch process, we only need to prove that $D(e_t, b+1)$ are i.i.d. It has been proved that $D(e_t, b+1)$ are i.i.d and $Geo(\frac{1}{2})$. So the offspring distribution is $Geo(\frac{1}{2})$.

2. Fix $a \geq 1$. Let ϕ_0 be the journey from start point to $-a$. Let e_n be n-th journey from $-a$ to $-a-1$, and ε_n be n-th journey from $-a-1$ to $-a$. Then easy to get that $X_{-a-1} = \sum_{t=1}^{X_{-a}} D(\varepsilon_t, -a-1)$. For the same reason we easily get that $D(\varepsilon_t, -a-1) \perp \sigma(e_n : n \in \mathbb{N})$. And by reflecting easy to get that $D(\varepsilon_t, -a-1) \sim Geo(\frac{1}{2})$, too. So $(X_{-a} : a \geq 1)$ is branch process and offspring distribution is $Geo(\frac{1}{2})$

3. Fix $b < c-1$. Let ϕ_0 be the journey from start point to $b+1$. Let e_n be the n-th journey from $b+1$ to b and ε_n be n-th journey from b to $b+1$. Then easy to prove that $X_{b+1} = D(\phi_0, b+1) + \sum_{t=1}^{X_b} D(e_n, b+1)$. Noting that $D(\phi_0, b+1) = 1$. So for the same reason, we get that $(X_b : 0 \leq b \leq c-1)$ is migrating branch process, with offspring distribution $Geo(\frac{1}{2})$ and migrating distribution δ_1.

□

PROBLEM 65 $c < b \in \mathbb{Z}_+$. Let $W = (W_n : n \geq 0)$ be the one-dimensional reflecting simple random walk with $W_0 = c \geq 0$ on $\mathbb{Z}^{0,b}$, whose transition matrix is $P^{0,b}$, where $a = 0, p, q > 0, p+q = 1$. Let $k \geq 1$ and τ be the time of the k-th downcrossing 0 of (W_n). $0 \leq a \leq b$, X_a is the times of $(S_{n\wedge\tau} : n \geq 0)$ downcrossing a. Prove:

1. $(X_a : c-1 \leq a \leq b-1)$ is branch process. And offspring distribution is $Geo(p)$.

2. $(X_a : 0 \leq a \leq c-1)$ is migrating branch process. And offspring distribution is $Geo(p)$. And the migrating distribution is concentrating on 1.

SOLUTION. For a random walk y, we let $D(n, y)$ be the number of downcrossings of y of n.

1. Fix a such that $c-1 \leq a < b-1$. Let ϕ_0 be the journey from start point to a. Let e_n be the n-th journey from a to $a+1$, and ε_n be the n-th journey from $a+1$ to a. For reflecting simple random walk, we can also prove that e_n, ε_n are independent. Noting that $X_{a+1} = \sum_{t=1}^{X_a} D(\varepsilon_t, a+1)$, we easily get the conclusion.

2. Fix $a : 0 \leq a < c-1$. Let ϕ_0 be the journey from start point to $a+1$. Let e_n be the n-th journey from $a+1$ to a and ε_n be n-th journey from a to $a+1$. Then easy to prove

that $X_{a+1} = D(\phi_0, a+1) + \sum_{t=1}^{X_a} D(e_n, a+1)$. Noting that $D(\phi_0, a+1) = 1$. So for the same reason, we get that $(X_a : 0 \le a \le c-1)$ is migrating branch process, with offspring distribution $Geo(\frac{1}{2})$ and migrating distribution δ_1.

□

PROBLEM 66 Let $W = (W_n : n \ge 0)$ be the one-dimensional simple random walk with $W_0 = 0$, whose transition matrix P given by equation (4.4.3) on textbook, $0 < p < q < 1$. X_a is the times of $(W_{n\wedge\tau} : n \ge 0)$ downcrossing a. $r = \frac{p}{q}$. Prove:

1. $\mathbb{P}(X_0 = i) = r^i(1-r), i \ge 0$;
2. $a \ge 0$, $\mathbb{P}(X_a = 0) = 1 - r^{a+1}$, $\mathbb{P}(X_a = i) = r^{a+1}(1-r), i \ge 1$.

SOLUTION. 1. Since $p < q$, then $W_n \to -\infty, n \to \infty$. Let $\tau_0 = 0, \forall k \ge 1$, $\sigma_k = \inf\{n \ge \tau_{k-1} : W_n = 1\}, \tau_k = \inf\{n \ge \sigma_k : W_n = 0\}$.

(a) If $i = 0$, then $\{X_0 = i\} \overset{\text{a.s.}}{=} \{\sigma_1 = \infty\}$. Then $\mathbb{P}(X_0 = i) = \mathbb{P}(\sigma_1 = \infty) = r$.

(b) If $i \ge 1$, then $\{X_0 = i\} \overset{\text{a.s.}}{=} \{\sigma_i < \infty, \sigma_{i+1} = \infty\}$. Since $\{\tau_i < \infty\} \subset \{\sigma_i \infty\}, \mathbb{P}(\sigma_i < \infty, \tau_i = \infty) = 0$, then by strong markov property,

$$\begin{aligned}
\mathbb{P}(\sigma_{i+1} < \infty \mid \sigma_i < \infty) &= \mathbb{P}(\sigma_{i+1} < \infty \mid \sigma_i < \infty, \tau_i < \infty) \\
&= \mathbb{P}(\sigma_{i+1} < \infty \mid \tau_i < \infty) \\
&= \mathbb{P}(\sigma_{i+1} < \infty \mid \tau_i < \infty, W_{\tau_i} = 0) \\
&= \mathbb{P}(\sigma_1 < \infty) = r
\end{aligned}$$

Therefore,

$$\mathbb{P}(\sigma_{i+1} < \infty) = \mathbb{P}(\sigma_{i+1} < \infty \mid \sigma_i < \infty)\mathbb{P}(\sigma_i < \infty)$$

Then $\mathbb{P}(\sigma_i < \infty) = r^i$. Therefore, $\mathbb{P}(X_0 = i) = \mathbb{P}(\sigma_i < \infty, \sigma_{i+1} = \infty) = \mathbb{P}(\sigma_i < \infty)\mathbb{P}(\sigma_{i+1} = \infty \mid \sigma_i < \infty) = r^i(1-r)$.

2. Let $D_a = \inf(n \ge 0 : W_n = a)$, then $\mathbb{P}(D_a < \infty) = r^a$. By strong markov property, $(W_{D_a+n-a:n\ge 0})$ is a random walk starting from 0 under $\mathbb{P}(\cdot \mid D_a < \infty) = \mathbb{P}(\cdot \mid D_a < \infty, W_{D_a} = a)$. By the conclusion in 1, $\mathbb{P}(X_a = i \mid D_a < \infty) = r^i(1-r), i \ge 0$. Then

$$\begin{aligned}
\mathbb{P}(X_a = 0) &= \mathbb{P}(D_a = \infty) + \mathbb{P}(D_a < \infty, X_a = 0) \\
&= 1 - r^a + \mathbb{P}(D_a < \infty)\mathbb{P}(X_a = 0 \mid D_a < \infty) \\
&= 1 - r^a + r^a(1-r) = 1 - r^{a+1}
\end{aligned}$$

$\forall i \ge 1$,

$$\begin{aligned}
\mathbb{P}(X_a = i) &= \mathbb{P}(D_a < \infty, X_a = i) \\
&= \mathbb{P}(D_a < \infty)\mathbb{P}(X_a = i \mid D_a < \infty) \\
&= r^a r^i(1-r) = r^{a+i}(1-r)
\end{aligned}$$

□

PROBLEM 67 Let $W = (W_n : n \geq 0)$ be the one-dimensional simple random walk with $W_0 = 0$, whose transition matrix P given by equation (4.4.3) on textbook, $0 < p < q < 1$. X_a is the times of $(W_{n\wedge\tau} : n \geq 0)$ downcrossing a. $r = \frac{p}{q}$. Prove: if $a \leq -1$, then $X_a - 1 \sim G(1-r)$, i.e. $\mathbb{P}(X_a = i) = r^{i-1}(1-r), i \geq 1$.

SOLUTION. Let $\tau = \inf\{n \in \mathbb{N} : W_n = a\}$. Then $\tau < \infty, a.s.$, then W_n downcross a at τ. And $(W_{\tau+n} - a : n \in \mathbb{N})$ is simple random walk start at 0. So by 66 we easily get $X_a - 1 \sim Geo(1-r)$. $\square$

5 distribution

name	symbol	range	distribution	mean	variance	generating	characteristic
Bernoulli	$B(1,p), 0<p<1$	$\{0,1\}$	$\mu(0)=1-p, \mu(1)=p$	p	$p(1-p)$	$(1-p+pz)$	$1-p+p\mathrm{e}^{\mathrm{i}t}$
Binomial	$B(n,p), 0<p<1$	$\mathbb{N}\cap[0,n]$	$\mu(k)=\binom{n}{k}p^k(1-p)^{n-k}$	np	$np(1-p)$	$(1-p+pz)^n$	$(1-p+p\mathrm{e}^{\mathrm{i}t})^n$
Geometric	$G(p), 0<p<1$	$\mathbb{N}^+$	$\mu(k)=(1-p)^{k-1}p$	$\frac{1}{p}$	$\frac{1-p}{p^2}$	fuck	$\frac{p\mathrm{e}^{\mathrm{i}t}}{1-(1-p)\mathrm{e}^{\mathrm{i}t}}$
Hypergeometric	$H(n,K,N)$ $N\in\mathbb{N}$ $K\in[0,N]\cap\mathbb{N}$ $n\in[0,N]\cap\mathbb{N}$	$[\max\{0,n+K-N\},$ $\min\{n,K\}]\cap\mathbb{N}$	$\mu(k)=\frac{\binom{K}{k}\binom{N-k}{n-k}}{\binom{N}{n}}$	$n\frac{K}{N}$	$n\frac{K}{N}\frac{N-k}{N}\frac{N-n}{N-1}$	fuck	fuck
Poisson	$P(\lambda), \lambda>0$	$\mathbb{N}$	$\mu(k)=\frac{\mathrm{e}^{-\lambda}\lambda^k}{k!}$	λ	λ	$\mathrm{e}^{\lambda(z-1)}$	$\mathrm{e}^{\lambda(\mathrm{e}^{it}-1)}$
Uniform	$U[a,b], a<b$	$[a,b]$	$f(x)=\frac{1}{b-a}$ $F(x)=\frac{x-a}{b-a}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	-	$\frac{\mathrm{e}^{itb}-\mathrm{e}^{ita}}{\mathrm{i}t(b-a)}$
Exponential	$Exp(\lambda), \lambda>0$	$[0,\infty)$	$f(x)=\lambda\mathrm{e}^{\lambda x}\mathbb{1}_{\{x\geq 0\}}$ $F(x)=(1-\mathrm{e}^{\lambda x})\mathbb{1}_{\{x\geq 0\}}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	-	$(1-\frac{\mathrm{i}t}{\lambda})^{-1}$
Gamma	$\Gamma(\alpha,\beta), \alpha,\beta>0$	$(0,\infty)$	$f(x)=\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\mathrm{e}^{-\beta x}\mathbb{1}_{\{x>0\}}$	$\frac{\alpha}{\beta}$	$\frac{\alpha}{\beta^2}$	-	$(1-\frac{\mathrm{i}t}{\beta})^{-\alpha}$
Normal	$N(\mu,\sigma^2), \mu,\sigma\in\mathbb{R}$	$\mathbb{R}$	$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}$	μ	σ	-	$\mathrm{e}^{\mu\mathrm{i}t-\frac{\sigma^2t^2}{2}}$

6 important lemma

Lemma 4 (Borel-Cantelli). *$(\Omega,\mathcal{F},\mathbb{P})$ is a probability space. $\{A_n : n\in\mathbb{N}\}\subset\mathcal{F}$, then*

1. *if $\sum_{n\in\mathbb{N}}\mathbb{P}(A_n)<\infty$, then $\mathbb{P}(\bigcup_{k=0}^{n}\bigcap_{m=n}^{\infty}A_m)=0$;*
2. *if $\{A_n : n\in\mathbb{N}\}$ are independent, $\sum_{n\in\mathbb{N}}\mathbb{P}(A_n)=\infty$, then $\mathbb{P}(\bigcup_{k=0}^{n}\bigcap_{m=n}^{\infty}A_m)=1$.*

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