Beam Vibrations With Time-Dependent Boundary Conditions

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A procedure is described for extending the method of separation of variables to the solution of beam-vibration problems with time-dependent boundary conditions. The procedure is applicable to a wide variety of time-dependent boundary-value problems in systems governed by linear partial differential equations.

Introduction

It happens frequently that vibration problems have to deal with continuous systems of which one or more boundaries are constrained to undergo displacements or tractions which vary with time. Simple examples include a cantilevered beam whose clamped end experiences a lateral-displacement pulse, a rod subjected at its ends to varying longitudinal pressure, or a plate whose edge is twisted by an oscillating edge moment. In other cases the time-dependence of the boundary condition is itself the object of investigation as when one structural element interacts with another, or when we wish to infer the state of strain at one end of a structure from continuous measurements made at another end. All of these problems are characterized by the fact that the boundary conditions are not "stationary" and on this account, solutions are not, in general, obtainable by the classical method of separation of variables, although solutions do appear in the literature for a number of special problems of this type.[3]

The present paper includes these solutions as special cases. In it is developed a general method by means of which the time-dependence is removed from the boundary conditions. The remaining problem then can be solved by any one of the classical or numerical methods available for handling the free- or forced-vibration problem. The method is here developed for and applied to the problem of the flexural vibrations of beams. It is equally applicable to time-dependent boundary-value problems in the flexural vibrations of plates, the torsional vibrations of shafts and, in fact, in a wide variety of systems governed by linear partial differential equations.

Problems of this type have been solved by the method of Laplace transform, but the present method employs more elementary mathematical techniques and is applicable in certain cases where the transforms do not exist or their inverses are not known.

Statement of Problem

Consider a structure whose displacement is governed by the equation

$$a^2 \frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 w}{\partial t^2} = \frac{q(x)p(t)}{\rho A} \qquad [1]$$

This is the case if the structure is a prismatic beam executing flexural vibrations according to the simple Bernoulli-Euler or classical theory of flexure. Then the symbols have the meanings

w = deflection of beam
x = position along beam; $x = 0$ is one end of beam, $x = l$ is the other end
ρ = density
A = cross-sectional area of beam
a^2 = $EI/\rho A$, where E and I are Young's modulus and the second moment of area of the cross section of the beam, respectively
$q(x)p(t)$ = external load per unit length of beam. If the load does not vary with time, $p(t) = 1$

The method to be described applies equally well if the rotatory inertia, shear, damping, and elastic foundation terms are included in the differential equation, but they are omitted for simplicity.

In the following development, the symbol D_i is used to represent a linear differential operator of order 0, 1, 2, or 3, as the boundary conditions of the problem dictate, that is, $D_i[w]$ stands for w, or $\partial w/\partial x$, or $\partial^2 w/\partial x^2$, or $\partial^3 w/\partial x^3$, or a linear combination of these operations. With this notation the boundary conditions can be written

$$\left.\begin{aligned} D_i[w(0, t)] &= f_i(t), \quad i = 1,2 \\ D_i[w(l, t)] &= f_i(t), \quad i = 3,4 \end{aligned}\right\} \qquad [2]$$

For example, if the beam in question is a cantilever clamped at $x = 0$

$$D_1 = 1,\ D_2 = \partial/\partial x,\ D_3 = \partial^2/\partial x^2,\ D_4 = \partial^3/\partial x^3$$

while the displacement and slope at $x = 0$, and the moment and shear at $x = l$ may be required to vary with time according to $f_i(t)$. Linear combinations of the operators occur, for example, when the end of the beam is spring-restrained against displacement or rotation.

The initial conditions of the motion are specified by two arbitrary functions

$$\left.\begin{aligned} w(x, 0) &= w_0(x) \\ \left.\frac{\partial w}{\partial t}\right]_{t=0} &= \dot{w}_0(x) \end{aligned}\right\} \qquad [3]$$

A difficulty in solving problems of this type by the method of separation of variables arises when not all of the functions $f_i(t)$ vanish. The method of separation of variables breaks down when applied directly because it is not possible to satisfy Equations [2] by adjustment of the x-dependent function.

Method of Solution

The foregoing difficulty may be resolved by separating the solution into two parts, one of which is later adjusted so as to simplify the boundary conditions on the other. We take

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$$w = \zeta(x, t) + \sum_{i=1}^{4} f_i(t)g_i(x) \qquad [4]$$

Substituting Equation [4] into Equation [1] we find that ζ must satisfy the differential equation

$$a^2 \frac{\partial^4 \zeta}{\partial x^4} + \frac{\partial^2 \zeta}{\partial t^2} = \frac{q(x)p(t)}{\rho A} - \sum_{i=1}^{4} (a^2 f_i g_i^{\text{IV}} + \ddot{f}_i g_i) \qquad [5]$$

where Roman superscripts indicate differentiations with respect to x, and dots indicate differentiations with respect to time.

In addition, the assumed expression for w, Equation [4], must satisfy the boundary conditions, Equations [2]. Hence

$$\left.\begin{aligned} D_i[\zeta(0, t)] &= f_i(t) - \sum_{j=1}^{4} D_i[g_j(0)]f_j(t), \ i = 1,2 \\ D_i[\zeta(l, t)] &= f_i(t) - \sum_{j=1}^{4} D_i[g_j(l)]f_j(t), \ i = 3,4 \end{aligned}\right\} \qquad [6]$$

Finally, the initial conditions, Equations [3], become

$$\left.\begin{aligned} \zeta(x, 0) &= w_0 - \sum_{i=1}^{4} f_i(0)g_i(x) \\ \left.\frac{\partial \zeta}{\partial t}\right]_{t=0} &= \dot{w}_0 - \sum_{i=1}^{4} \dot{f}_i(0)g_i(x) \end{aligned}\right\} \qquad [7]$$

The functions $g_i(x)$ are now chosen so as to reduce to zero the right-hand sides of Equations [6]. To assure this, it is sufficient to satisfy the 16 conditions

$$\left.\begin{aligned} D_1[g_1(0)] &= 1 & D_1[g_2(0)] &= 0 \\ D_2[g_1(0)] &= 0 & D_2[g_2(0)] &= 1 \\ D_3[g_1(l)] &= 0 & D_3[g_2(l)] &= 0 \\ D_4[g_1(l)] &= 0 & D_4[g_2(l)] &= 0 \\ D_1[g_3(0)] &= 0 & D_1[g_4(0)] &= 0 \\ D_2[g_3(0)] &= 0 & D_2[g_4(0)] &= 0 \\ D_3[g_3(l)] &= 1 & D_3[g_4(l)] &= 0 \\ D_4[g_3(l)] &= 0 & D_4[g_4(l)] &= 1 \end{aligned}\right\} \qquad [8]$$

or, in more abbreviated notation

$$\left.\begin{aligned} D_j[g_i(0)] &= \delta_{ij}, \quad j = 1,2, \quad i = 1,2,3,4 \\ D_j[g_i(l)] &= \delta_{ij}, \quad j = 3,4, \quad i = 1,2,3,4 \end{aligned}\right\} \qquad [9]$$

where $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ij} = 1$ for $i = j$. Each column of Equations [8] provides four conditions on one of the four functions g_i. In order to be certain of being able to satisfy these conditions in all cases the g_i are taken to be polynomials of the fifth degree in x

$$g_i = a_i + b_i x + c_i x^2 + d_i x^3 + e_i x^4 + f_i x^5, \ i = 1,2,3,4 \qquad [10]$$

choosing the coefficients according to the following rule:

"Substitute each of the g_i in the appropriate (ith) column of Equations [8]. In each case, there will result a set of four linear algebraic equations governing the coefficients of the particular function g_i. If more than four of the constants $a_i \ldots f_i$ appear in these expressions, reduce to four the number which do appear by setting equal to zero the coefficient of the term of highest degree in x and also, if necessary, the coefficient of the term of second highest degree. If any of the constants $a_i \ldots f_i$ does not appear, set it equal to zero." In this way the four conditions on the coefficients of each of the g_i will always be expressed by means of four linear, independent, algebraic equations in four unknowns. These equations determine the constants of Equations [10] so as to satisfy Equations [8]. With this choice of the functions g_i, the boundary conditions expressed by Equations [6] become those of a "stationary" problem

$$\left.\begin{aligned} D_i[\zeta(0, t)] &= 0 \qquad i = 1,2 \\ D_i[\zeta(l, t)] &= 0 \qquad i = 3,4 \end{aligned}\right\} \qquad [11]$$

Usually only a third-degree polynomial is required for the functions g_i. The additional terms are included to accommodate exceptional cases as, for example, when time-dependent moment and shear are prescribed at both ends of the beam. It should be noticed, also, that it is necessary only to compute those of the g_i for which the corresponding $f_i(t)$ do not vanish.

It remains to find the function $\zeta(x, t)$ satisfying the differential equation, Equation [5], the boundary conditions, Equations [11], and the initial conditions, Equations [7]. This can be done in the classical manner. We seek a solution in the form

$$\zeta = \sum_{n=1}^{\infty} X_n T_n \qquad [12]$$

where

$$X_n = X_n(x), \quad T_n = T_n(t)$$

and assume that the functions X_n will be orthogonal[4] with respect to the interval $0,l$ so that $q(x)$, $g_i(x)$ and $g_i^{\text{IV}}(x)$ can be expanded in series of functions X_n by means of the expansion formulas

$$\left.\begin{aligned} q(x) &= \sum_{n=1}^{\infty} Q_n X_n \\ g_i(x) &= \sum_{n=1}^{\infty} G_{in} X_n \\ g_i^{\text{IV}}(x) &= \sum_{n=1}^{\infty} G_{in}^* X_n \end{aligned}\right\} \qquad [13]$$

where the constants Q_n, G_{in}, and G_{in}^* are given by the expressions

$$\left.\begin{aligned} Q_n &= \frac{\int_0^l q(x)X_n dx}{\int_0^l X_n^2 dx} \\ G_{in} &= \frac{\int_0^l g_i(x)X_n dx}{\int_0^l X_n^2 dx}, \quad i = 1, 2, 3, 4 \\ G_{in}^* &= \frac{\int_0^l g_i^{\text{IV}}(x)X_n dx}{\int_0^l X_n^2 dx}, \quad i = 1, 2, 3, 4 \end{aligned}\right\} \qquad [14]$$

After substituting Equations [12] and [13] in Equation [5] and separating variables, the equations governing X_n and T_n are readily solved

$$X_n = C_n \cos \frac{m_n x}{l} + D_n \sin \frac{m_n x}{l} + E_n \cosh \frac{m_n x}{l} + F_n \sinh \frac{m_n x}{l} \qquad [15]$$

$$T_n = A_n \cos \omega_n t + B_n \sin \omega_n t + \frac{1}{\omega_n} \int_0^t P_n(\tau) \sin \omega_n(t - \tau)d\tau \qquad [16]$$

[4] The conditions under which the functions X_n are orthogonal with respect to the interval $0,l$ are well known (5). It is sufficient to note that this is the case if the ends of the beam are fixed or free or simply supported or restrained against translation or rotation by linear springs. These include all the types of end conditions which can arise from Equation [11]. In a private communication, Dr. R. W. Hamming has shown how to extend the method to nonorthogonal functions by introducing the adjoint equation.

where

$$P_n(\tau) = Q_n \frac{p(\tau)}{\rho A} - \sum_{i=1}^{4} [G_{in} \ddot{f}_i(\tau) + a^2 G_{in}^* f_i(\tau)], \quad m_n^2 = \frac{\omega_n l^2}{a}$$

Equations [15], [16], and [12] determine the form of ζ. The constants of integration C_n, D_n, E_n, and F_n are adjusted to satisfy the simple boundary conditions, Equations [11], which also determine the transcendental equation governing m_n and hence the natural frequencies ω_n. This part of the solution is identical with the free-vibration case. The initial conditions given in Equations [7] serve to determine the constants of integration A_n and B_n. Since

$$\zeta(x, 0) = \sum_{n=1}^{\infty} A_n X_n$$

$$\left.\frac{\partial \zeta}{\partial t}\right]_{t=0} = \sum_{n=1}^{\infty} B_n X_n \omega_n$$

we must have

$$\left.\begin{aligned} A_n &= \frac{\int_0^l \left[w_0 - \sum_{i=1}^{4} f_i(0) g_i(x) \right] X_n dx}{\int_0^l X_n^2 dx} \\ B_n &= \frac{\int_0^l \left[\dot{w}_0 - \sum_{i=1}^{4} \dot{f}_i(0) g_i(x) \right] X_n dx}{\omega_n \int_0^l X_n^2 dx} \end{aligned}\right\} \ldots\ldots [17]$$

This completes the formal solution of the problem.

Applications of the Method

Example 1. As a first illustration, consider the vibrations which arise when the "free" end of a cantilever beam is actuated by a cam. Taking the fixed end of the beam at $x = l$, the boundary conditions are

$$w(0, t) = f_1(t), \quad \left.\frac{\partial^2 w}{\partial x^2}\right]_{x=0} = \left.\frac{\partial w}{\partial x}\right]_{x=l} = w(l, t) = 0$$

where $f_1(t)$ is the transverse displacement enforced by the cam. For convenience take $w_0 = \dot{w}_0 = 0$. Applying the rule of the previous section we have

$$g_1 = 1 - \frac{3x}{2l} + \frac{x^3}{2l^3} \ldots\ldots\ldots\ldots\ldots [18]$$

Application of the boundary conditions to Equation [15] yields

$$X_n = \sinh m_n \sin \frac{m_n x}{l} - \sin m_n \sinh \frac{m_n x}{l} \ldots\ldots [19]$$

where

$$\tan m_n = \tanh m_n$$

From Equation [14]

$$G_{in} = 2/[m_n(\sinh m_n - \sin m_n)]$$

and from Equations [17]

$$A_n = -f_1(0) G_{1n}$$

$$B_n = -\dot{f}_1(0) G_{1n}/\omega_n$$

so that a general solution of this problem is

$$w = f_1(t)\left(1 - \frac{3x}{2l} + \frac{x^3}{2l^3}\right) - \sum_{n=1}^{\infty} \frac{2X_n \left[f_1(t) - \omega_n \int_0^t f_1(\tau) \sin \omega_n(t - \tau) d\tau \right]}{m_n(\sinh m_n - \sin m_n)} \ldots\ldots [20]$$

Special cases of this general solution may be obtained by substituting the corresponding values of f_1 in Equation [20] (6).

Example 2. The clamped end of a cantilever beam undergoes a damped sine-wave displacement transient, starting from rest. As before, we take the clamped end of the beam at $x = l$; then we are given

$$D_1 = \frac{\partial^2}{\partial x^2}, \quad D_2 = \frac{\partial^3}{\partial x^3}, \quad D_3 = 1, \quad D_4 = \frac{\partial}{\partial x}$$

$$f_1 = f_2 = f_4 = 0, \quad f_3 = Ae^{-\beta t} \sin \omega_f t$$

From Equations [15] and [16]

$$X_n = \left(\sin \frac{m_n x}{l} + \sinh \frac{m_n x}{l}\right)(\cos m_n + \cosh m_n) - \left(\cos \frac{m_n x}{l} + \cosh \frac{m_n x}{l}\right)(\sin m_n + \sinh m_n)$$

$$\cos m_n \cosh m_n = -1$$

$$g_3 = 1$$

$$G_{3n} = [(\coth m_n - \cot m_n) m_n]^{-1}$$

$$A_n = 0$$

$$B_n = -AG_{3n} \frac{\omega_f}{\omega_n}$$

$$T_n = -A \frac{\omega_f}{\omega_n} G_{3n} \sin \omega_n t - \frac{1}{\omega_n} G_{3n} \int_0^t \ddot{f}_3(\tau) \sin \omega_n(t - \tau) d\tau$$

and a complete solution is

$$w = Ae^{-\beta t} \sin \omega_f t + \sum_{n=1}^{\infty} \frac{AX_n}{m_n(\coth m_n - \cot m_n)} [\omega_f L \sin(\omega_n t - \varphi_1) + \omega_n L e^{-\beta t} \sin(\omega_f t + \varphi_2) - e^{-\beta t} \sin \omega_f t] \ldots\ldots [21]$$

$$L = \omega_n \left\{ [\beta^2 + (\omega_f + \omega_n)^2][\beta^2 + (\omega_f - \omega_n)^2] \right\}^{-1/2}$$

$$\varphi_1 = \tan^{-1} 2\beta\omega_n/(\beta^2 + \omega_f^2 - \omega_n^2)$$

$$\varphi_2 = \tan^{-1} 2\beta\omega_f/(\beta^2 + \omega_n^2 - \omega_f^2)$$

A Special Case

When the time-dependent functions in the boundary conditions are simple circular or hyperbolic or exponential functions, it is sometimes advantageous to choose the $g_i(x)$ in other than polynomial forms. Although these cases can be handled by the general method described in the preceding sections, the solution may be put in a form more suitable for computation by means of the following device:

If, for example, the $f_i(t)$ are simple trigonometric functions, then

$$\ddot{f}_i(t) = -k^2 f_i(t)$$

If we choose

$$g_i(x) = a_i \sin \sqrt{\frac{k}{a}}\, x + b_i \cos \sqrt{\frac{k}{a}}\, x + c_i \cosh \sqrt{\frac{k}{a}}\, x + d_i \sinh \sqrt{\frac{k}{a}}\, x \quad \ldots\ldots [22]$$

then the right-hand side of Equation [5] will reduce to zero (except for the distributed-load term, if any), and at the same time the constants $a_i \ldots\ldots d_i$ of Equations [22] may be adjusted so as to satisfy the 16 conditions given in Equations [8]. Having removed the time-dependence in this manner, the solution proceeds exactly as before except that the summations with respect to i do not appear in Equation [16].

This variant of the general method takes advantage of a special form of the time-dependence of the boundary conditions, and therefore cannot be used in general. Where it can be employed, it serves to remove from the summation sign the forced part of the solution. This part of the solution is expressed in closed form.

Example 3. As a simple example of the special method, consider a simply supported beam, initially at rest, one of whose ends $x = l$, is subjected to an oscillating bending moment of amplitude M and frequency ω_f. In this case the given quantities are

$$D_1 = 1\,,\ D_2 = \frac{\partial^2}{\partial x^2}\,,\ D_3 = 1\,,\ D_4 = EI\,\frac{\partial^2}{\partial x^2}$$

$$f_1 = f_2 = f_3 = 0\,,\ f_4 = -M \sin \omega_f t$$

$$w_0 = \dot{w}_0 = 0$$

The constants of Equation [22] are to be adjusted so as to satisfy the conditions expressed by Equations [8]. When this is done

$$Mg_4(x) = \frac{M}{EI}\frac{a}{2\omega_f}\left[\sinh \sqrt{\frac{\omega_f}{a}}\, x \operatorname{cosech} \sqrt{\frac{\omega_f}{a}}\, l - \sin \sqrt{\frac{\omega_f}{a}}\, x \operatorname{cosec} \sqrt{\frac{\omega_f}{a}}\, l\right] \quad \ldots\ldots [23]$$

Equation [23] gives directly, in closed form, the amplitude of the steady state or forced vibration. It remains to complete the solution by writing out the transient. From Equation [15], together with the boundary conditions of Equations [11]

$$X_n = \sin \frac{n\pi x}{l}\,,\quad \omega_n = \frac{n^2\pi^2 a}{l^2}$$

From Equations [16] and [17]

$$A_n = 0$$

$$B_n = \frac{M}{EI}\frac{2\omega_f a}{n\pi}\frac{(-1)^n}{\omega_n^2 - \omega_f^2}$$

$$T_n = B_n \sin \omega_n t$$

The complete solution of the problem is therefore

$$w = -\frac{M}{EI}\frac{a}{2\omega_f}\left[\sinh \sqrt{\frac{\omega_f}{a}}\, x \operatorname{cosech} \sqrt{\frac{\omega_f}{a}}\, l - \sin \sqrt{\frac{\omega_f}{a}}\, x \operatorname{cosec} \sqrt{\frac{\omega_f}{a}}\, l\right] \sin \omega_f t + \frac{M}{EI}\frac{a}{2\omega_f}\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n\left(\frac{\omega_n^2}{\omega_f^2} - 1\right)} \sin \frac{n\pi x}{l} \sin \frac{n^2\pi^2 a t}{l^2} \quad \ldots [24]$$

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