Temporal Evolution in the Networks of Porous Materials: From Homogenization to Instability

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S1. SIMULATION ALGORITHM

In our simulations, we tested two types of network: a (i) diamond-grid network, and a (ii) topologically random network. In the diamond-grid network, we choose $N_x = 200$ edges in the horizontal direction and $N_y = 100$ edges in the vertical direction. The random network is created using uniformly distributed points with on average $N_x \times N_y$ edges in the horizontal and vertical directions. Note that the randomly distributed points are connected using a Delaunay triangulation. The diameter of each edge is sampled from either a uniform distribution in $r \in [1, 14]$, log-normal distribution with $\mu = 3, \sigma = 0.48$, or truncated normal distribution with $\mathcal{N}(\mu = 7.0, \sigma = 3.6)$. The external flow pressure is applied horizontally from left to right. An external pressure is then considered between the left-most nodes and the rightmost nodes ($p_{\text{left}} = 10, p_{\text{right}} = 0$). Assuming a Poiseuille flow in each edge, the fluid flow q and pressure difference δP_e in each edge is related through $q_e = C_e \delta P_e$, where $C_e = \pi r_e^4 / 8\mu L_e$, L_e is the length of the tube, and μ is the viscosity of the fluid. Let Δ be the transpose of the network's oriented incidence matrix. Note that the orientation of each edge is arbitrary since it only determines the positive direction of flow in that edge. Define $|q_e\rangle$ as the vector of flow through all the edges, then we have $|q_e\rangle = C_e \Delta |P_n\rangle$ where $|P_n\rangle$ is the vector of pressure at all the nodes. With a given boundary condition, we solve the equation through a modified nodal analysis as follows. The conservation of mass at each node can be written as

$$|q_n\rangle = \Delta^\top \mathbf{C} \Delta |P_n\rangle, \tag{S1}$$

where $\mathbf{C} = \text{diag}\left(C_e^{(1)}, C_e^{(2)}, \cdots, C_e^{(N_e)}\right)$ is a diagonal matrix of edge conductances, and $|q_n\rangle$ is the vector of total incoming flow to each node. Note that total incoming flow to an internal node is zero inside the network due to the conservation of mass, and is only non-zero at the boundary nodes. Renumbering the boundary nodes to $1, 2, \cdots, N_B$, we can partition Eq. (S1) to obtain

$$\left[\begin{array}{c|c} {\Delta_b}^\top \mathbf{C} \Delta_b & {\Delta_b}^\top \mathbf{C} \Delta_n \\ \hline {\Delta_n}^\top \mathbf{C} \Delta_b & {\Delta_n}^\top \mathbf{C} \Delta_n \end{array}\right] \left[\begin{array}{c} P_1^{BC} \\ P_2^{BC} \\ \vdots \\ P_{N_B} \\ \hline P_{N_B+1} \\ \vdots \\ P_{N_n} \end{array}\right] = \left[\begin{array}{c} q_1^{BC} \\ q_2^{BC} \\ \vdots \\ q_{N_B}^{BC} \\ \hline 0 \\ \vdots \\ 0 \end{array}\right] \to \begin{bmatrix} \mathbf{A}_{bb} & \mathbf{A}_{bn} \\ \mathbf{A}_{nb} & \mathbf{A}_{nn} \end{bmatrix} \left[\begin{array}{c} P_1^{BC} \\ P_2^{BC} \\ \vdots \\ P_{N_B}^{BC} \\ \hline P_{N_B+1} \\ \vdots \\ P_{N_n} \end{array}\right] = \left[\begin{array}{c} q_1^{BC} \\ q_2^{BC} \\ \vdots \\ q_{N_B}^{BC} \\ \hline 0 \\ \vdots \\ 0 \end{array}\right], \tag{S2}$$

where $\mathbf{A}_{st} = \Delta_s^\top \mathbf{C} \Delta_t^\top$ and $s, t \in \{a, b\}$. In summary the above equations simplifies to

$$\begin{cases} \mathbf{A}_{bb} |P_{BC}\rangle + \mathbf{A}_{bn} |P\rangle = |q_{BC}\rangle, \\ \mathbf{A}_{nb} |P_{BC}\rangle + \mathbf{A}_{nn} |P\rangle = 0. \end{cases} \tag{S3}$$

Solving the above equations results in the nodes' pressure and also the fluid flux at the boundary nodes. With the flux at each edge, q_e, the increase (decrease) of tube radius under erosion (clogging) is obtained as

$$\frac{dr_e}{dt} \propto \pm \frac{q_e}{r_e^n}. \tag{S4}$$

We use simple forward Euler for time integration. For each iteration, we choose the time step dt so that $\max(dr_e) = 0.1 r_0$, where r_0 is the smallest radius among all edges.

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S2. ADDITIONAL NUMERICAL RESULT

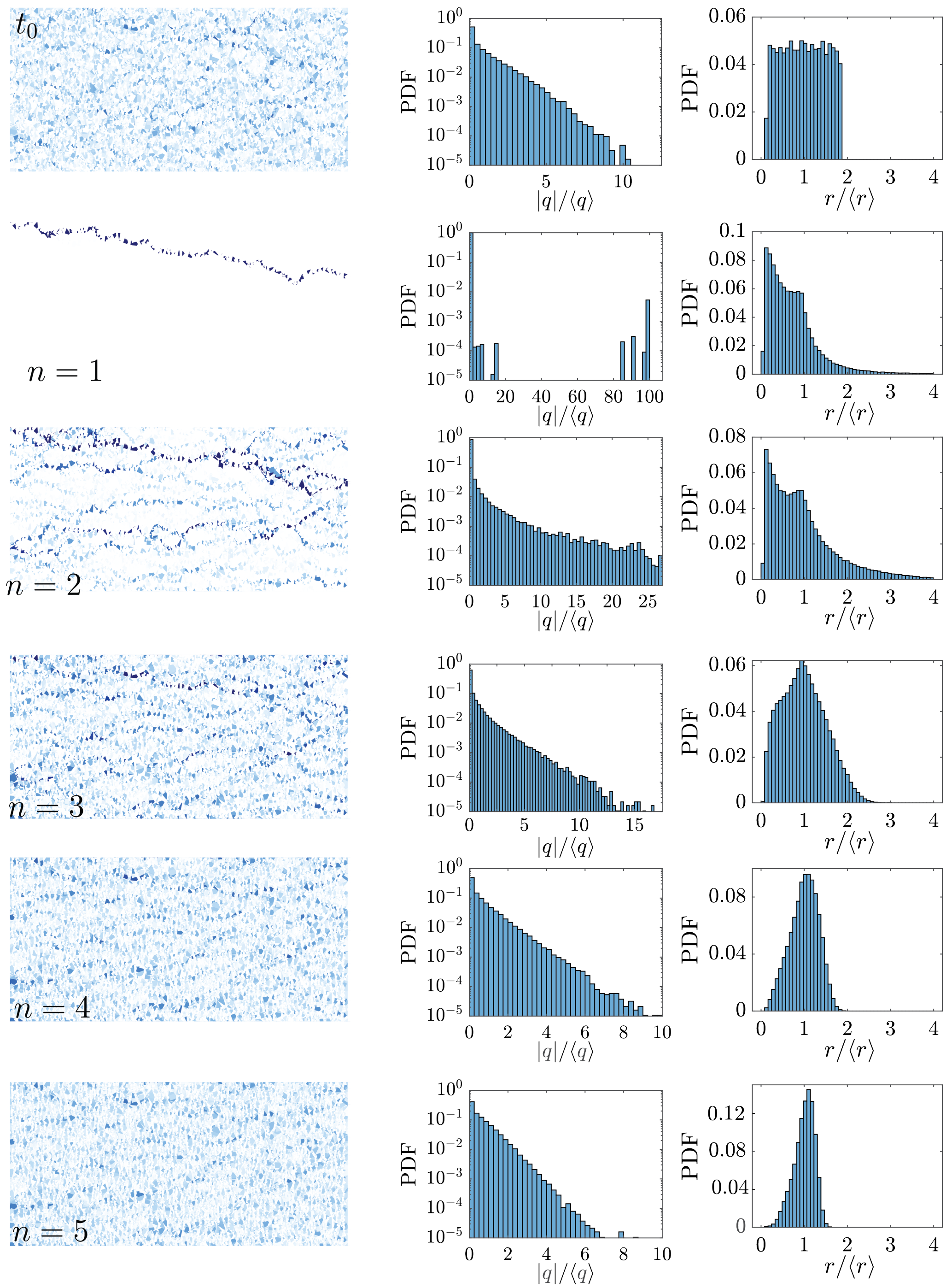

FIG. S1. Ahmad's version: Erosion in a topologically random network of pipes. The first row shows the initial condition at $t = 0$. Each row afterward corresponds to the simulation result after N steps such that $\langle r_{t=N}\rangle = 2r_0$ where $r_0 = \langle r_{t=0}\rangle$. The erosion law is based on Eq. (1) in the main text where different powers of n correspond to different models of erosion. The first column is a snapshot of the pore network, the second column is the PDF of normalized fluid flux $q/\langle q\rangle$, and the last column is the PDF of normalized radius $r/\langle r\rangle$.

S3. ANALYTICAL RESULTS

As described in the main text, the PDF of flow in a topologically disordered network of tubes is the same as a structured diamond grid when the radius of the tubes is highly disordered. In this section, we show that the observed exponential distribution of fluid flux can be described using a mean-field approach on a structured grid. Basically, the random distribution of the diameters along with the conservation of mass in the network are the two main ingredients resulting in an exponential tail distribution. In a diamond grid, the incoming flow to a node is redistributed among the outgoing edges (since fluid mass is conserved). Due to the randomness in the tube's diameter, one can imagine that the redistribution of the incoming flow to a node between the outgoing edges is random and the only condition is that the incoming flux should be equal to the outgoing flux. This model for the flow can be mapped one to one to the problem of force fluctuations in a bead pack [1, 2] as shown in Fig. S2. In a bead pack, the force at each layer is redistributed to the next layer where the total force exerted on the next layer should be in equilibrium with the previous layer. Given the above conditions, the flow at layer $L+1$ at node j can be obtained as

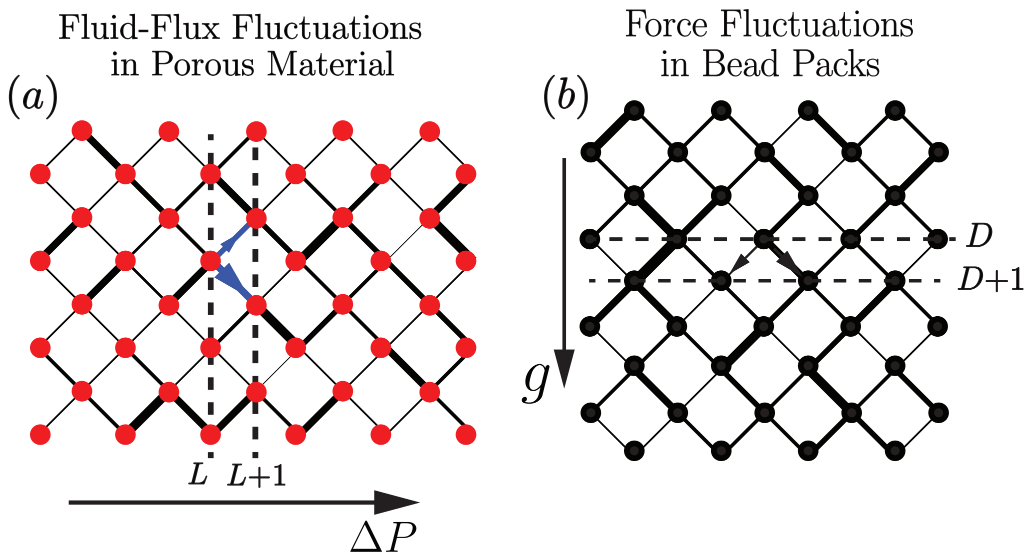

FIG. S2. (a) Schematic of diamond grid network of tubes. The incoming flow to each node is redistributed among the outgoing edges. The thickness of the lines shows the fluid flux transferred in that edge. (b) Schematic diagram showing beads (represented with nodes) and their contacts to the neighboring sites (represented with edges). The thickness of the edges show the weight transferred through that contact.

$$q(L+1,j) = \sum_i w_{ij} q(L,i) = w_{i,i+1} q(L,i+1) + w_{i,i} q(L,i). \tag{S5}$$

where w_{ij} shows the weights by which the flow is redistributed. Note that since the total fluid flux is conserved, then $\sum_j w_{ij} = 1$. Assuming a general distribution of $\eta(w)$, we can use the mean-field approximation to find the distribution of q at the layer L, i.e., $p_L(q)$. The values of $q(L,i)$ are not independent for neighboring sites; however, in our mean-field approximation we ignore such correlations. We find

$$p_L(q) = \prod_{j=1}^{N} \left\{ \int_0^1 dw_j \eta(w_j) \int_0^\infty dq_j p_{L-1}(q_j) \right\} \times \delta\left(\sum_j w_j q_j - q \right), \tag{S6}$$

where N is the number of outgoing edges (e.g., in our structured diamond grid $N=2$) and $\delta(\cdot)$ is the Kronecker delta function. Note that the constraint that q's emanating downward should add up to one is in the definition of $\eta(w)$. Taking the Laplace transform of the above equation and defining $\tilde{p}(s) \equiv \int_0^\infty p(q) e^{-qs} dq$ we obtain

$$\tilde{P}(s) = \left(\int_0^1 dw \eta(w) \tilde{P}(sw) \right)^N \tag{S7}$$

The above equation recursively converges to a distribution. The solution for a structured diamond grid with two neighboring sites becomes $p(q) = 4q\exp(-2q)$ [1–3], which is a distribution with an exponential tail.

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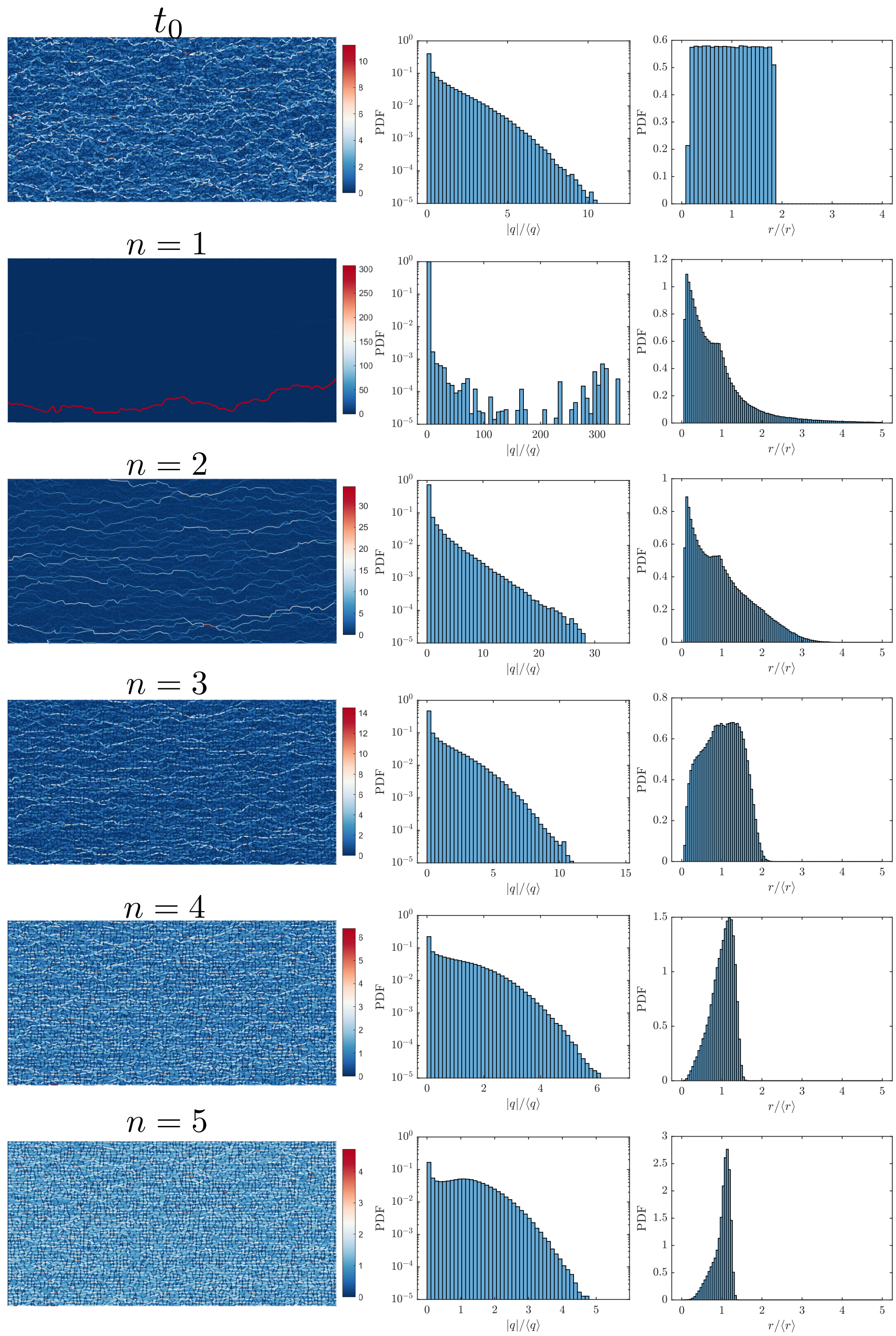

FIG. S3. Deng's version: Erosion in a topologically random network of pipes. The initial condition is shown with the label $t = 0$ in the first row. Each row afterward corresponds to the simulation result after N steps such that $\langle r_{t=N}\rangle = 2r_0$ where $r_0 = \langle r_{t=0}\rangle$ or twice the initial average radius. The erosion law is based on Eq. (1) in the main text where different powers of n correspond to different models of erosion. The first column is a snapshot of the pore network, the second column is the PDF of normalized fluid flux $q/\langle q\rangle$, and the last column is the PDF of normalized radius $r/\langle r\rangle$.