Temporal evolution of flow networks?

Erosion Dynamics in Porous Material

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the flow network? (Dated: February 9, 2021)

We study the dynamics of ~~erosion~~ in porous media using a pore-network model. First, We consider a class of erosion dynamics ~~covering different erosion dynamics proportional to~~ flow rate, local velocities, or shear at the walls. assuming a constitutive law for erosion dependent on the We show that depending on the erosion law, the flow may become uniform and homogenized, stay as it is, or become unstable and develop channels. Using a simple model, we identify ~~that the different laws affects flow distribution between parallel ... which ultimately results in different network behavior.~~ a quantitative criteria to distinguish these regimes. ~~Next,~~ By defining an order parameter capturing different behaviors we show that a phase transition happens depending on the erosion law. The extension to clogging dynamics is discussed.

I. INTRODUCTION

a bit too terse

Fluid flow through a porous medium undergoing a dynamical change in its network of micro-structure is a challenging problem and has many environmental and industrial applications, such as oil recovery [1, 2], CO_2 sequestration [3], water filtration/separation [4–6], energy storage [7, 8], or biological applications. Fluid flow throughout a porous medium is often heterogeneously distributed between the pores due to ~~the~~ a highly disordered pore structure. Due to the fluid flow, the pore structure ~~can further~~ change dynamically either through the erosion of pore boundary walls or deposition/sedimentation of material on the boundary of the pores. which is exacerbated due to the strong R^4 dependence of flow on radius Such heterogeneous dynamical changes affect the pore-level fluid flow which in ~~its~~ turn affects the dynamical changes to the pore structure. This feedback mechanism along with the initial heterogeneous fluid flow complicates the dynamical process ~~which~~ and makes it difficult to understand and predict the porous media behavior. It is to be noted that an understanding of the dynamical change is essential to improve any of the porous media applications where the pore network changes over time, applications such as ~~in~~ groundwater remediation and precipitation of minerals in rocks? biofilm growth in water filtration, and protective filters, as well as enhanced oil recovery with polymer flooding, or water-driven erosion.

Despite the importance of this problem, this proble

~~Albeit there is an abundance of applications in understanding the evolution of structures exposed to erosion and deposition~~, it has only been partially understood both theoretically and experimentally. Various models have been used to explore erosion in porous materials proportional to various local flow parameters. Some studies have focused on the erosion of the pore structure proportional to the shear stress at the walls [9], or proportional to the power dissipated [10]. More recently it has been shown that an erosion proportional to the local fluid flux [11] can effectively result in branching/channelization of the porous media. Similarly, in clogging, deposition rate based on local fluid flux, velocity, or even at a constant rate has been used. Despite all of the different models, a

& references?

consensus on the dynamics of erosion/clogging is lacking.

The hydro~~-~~dynamically driven erosion or clogging can be modeled using a phenomenological discrete network-based model [12–15]. (*) The network of pores inside the solid structure is connected ~~together through~~ pore throats that effectively ~~show resistance~~ to the fluid flow between the pores. refer to figure! In the network model, the pores are represented by spherical nodes connected with edges represented by cylindrical tubes. Network-based models have successfully shown to capture key properties of fluid flow in a porous material such as the probability distribution of fluid flux [12], the permeability scaling during clogging [16], or the first fluidized path in a porous structure [17]. In order to study erosion in such network models, the degradation of the solid skeleton or clogging of the pores are taken into account by the change (increase or decrease) in the resistance of the edges between the pores. Using a general local erosion law that can model different dynamics, we show a variety of behaviors are possible. The network can either move toward stability and homogenization, stay as it is, or even becom unstable and develop channels. Using numerical solutions of the resulting erosion laws on the network model we show the emergence of two behaviors and morphologies are possible through selective erosion and subsequent flow enhancement. Using a simplified model we capture the underlying physics and furthermore show a phase transition occurs depending on the parameters of the model.

Finally, we discuss the implications for clogging.

II. METHODS & RESULTS

Repetitive

Flow through pores and throats of a porous medium can be modeled using a pore network model [12–15]. In a network model, the pores/voids of porous medium are represented by a two- or three-dimensional lattice of nodes. The edges between the nodes of the network represent the pore-throat between the pores and are modeled using cylindrical tubes characterized by their radius and length (see Fig. 1(a)). The pore-network model is well suited for the simulation of low-Reynolds fluid flow and has proved to capture the flow dynamics of porous media [12] and also the first fluidized path [17]. We use a structured/unstructured network with randomly distributed nodes and edge length. We consider a low-Reynolds fluid

Repetitive

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(*) shouldn't this be moved to the methods section?

Nice figure.

Can we replace with actual micro-CT (e.g.: ear plugs)

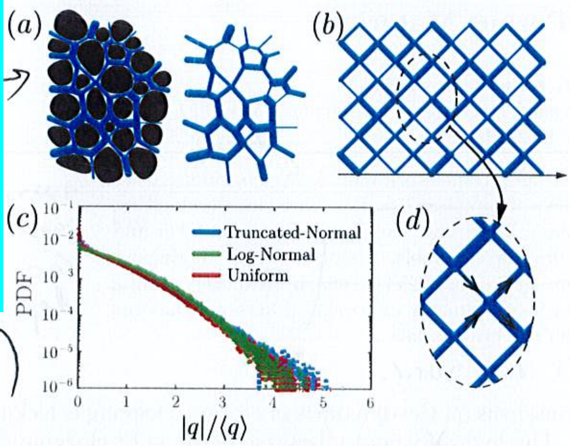

FIG. 1. (a) Schematic of a porous media along with its network ~~model~~ of pores. In the Network model the pores are represented with nodes and the throats between pores are approximated with tubes connecting the pores together. (b) Structured network model used to approximate the network structure. It is to be noted that the in a network model, choosing any random distribution for the diameters results in a universal probability density function of normalized fluid flux as long as the randomness (or disorder) in the tube diameters is large enough. (c) The universal PDF of fluid flux for various distribution of diameters.

size?

flow through the pores, i.e., $\rho ul/\mu \ll 1$ where ρ is the fluid density and μ is the kinematic viscosity, u and l are characteristic fluid velocity and pore length scale. In such conditions, the fluid flow in each tube follows ~~Darcy's law~~ $p_i - p_j = (8\mu l_{ij}/\pi r_{ij}^4) q_{ij}$ where p_i, p_j represents pressures at neighboring nodes i, j, and r_{ij}, l_{ij}, q_{ij} are respectively the radius, length, and fluid flow rate at the tube connecting pores i and j. ~~It is to be noted that~~ the value $\pi r_{ij}^4/8\mu l_{ij}$ can be considered as the conductance C_{ij} of the edge ij in the pore-network between nodes i and j. In addition ~~to Darcy's law, the conservation of mass needs to behold~~ at each node ~~where~~ the total incoming flow should be equal to the total outgoing flow. The conservation of mass at each pore i can be written as $\sum_{j \in \mathcal{N}(i)} q_{ij} = 0$ where $\mathcal{N}(i)$ represents all the neighboring nodes of node i. Considering a random or structured network with a random distribution of tube's radius/length (i.e., effectively their flow resistance) and a pressure difference between nodes at the left and the right boundary, the pressure at the nodes, and the fluid flow rate at the tubes can be obtained. For a large enough randomness in the edge resistances between the pores R_{ij}, the probability distribution of flux in the tubes normalized by the average flux follows a universal curve with an exponential tail i.e. $P(\hat{q}) \propto e^{-\alpha \hat{q}}$ for large $\hat{q}$ where $\hat{q}$ is the normalized fluid flux $\hat{q} = q/\langle q \rangle$ (See Fig. 1c and supplementary material for more details). This PDF of fluid flux q has been shown to correctly captures the exponential tail for the fluid flux in a porous medium [12, 16].

Poiseuille flow

I think that this terminology should be explicitly defined.

by solving the corresponding set of linear equations?

In order to model the erosion in porous media, we consider the abrasion in the throats which correspond to the change in the radius of the edges in the network. We model the dynamics of the change in the radius as

$$\frac{dr_{ij}}{dt} = \alpha \frac{q_{ij}}{r_{ij}^n} \tag{1}$$

where n is an integer number and c is a constant. Different powers of n correspond to different physics in the erosion. Particularly, when (i) $n = 0$ the erosion depends on the amount of flux q_{ij} passing through the edge (AZ: cite studies); (ii) $n = 2$ the erosion depends on the local velocities (AZ: cite studies); (iii) $n = 3$ the erosion depends on the shear force at the boundary of the throat. We first start with a numerical model, where ~~we assume~~ a network of 200×100 tubes in the ~~horizontal~~ and vertical direction on a diamond lattice (similar results for a general random network is presented in the supplemental materials). We initialize the network with a uniform random distribution of diameters within $[r_{\min}, r_{\max}]$ with a large coefficient of variation such that $\sigma_r/r_0 \approx 0.5$ where σ_r is the standard deviation, and $r_0 = \langle r \rangle$ is the the mean ~~of the~~ radius of the tubes. This large coefficient of variation corresponds to the highly disordered porous media. The flow inside the pores, PDF of flux in the tubes, and PDF of tube diameters are shown in Fig. 1. We assume a constant pressure difference between the left and the right boundaries. In each time step, we increase the local radius of the tubes based on the erosion law introduced in Eq. (1). Note that $r_{ij}(t+1) = r_{ij}(t) + (\alpha\delta t) q_{ij}(t)/r_{ij}^n(t)$, where the coefficient $\alpha\delta t$ is chosen such that the maximum change in the radius at each time step is smaller than one-tenth of the smallest tube's radius i.e. $\max(\Delta r_{ij}) \le r_{\min}/10$ of the smallest radius at the initial configuration. This condition guarantees that at each step a small amount of material is eroded and there's no sudden change in the network. We continue the simulations until $\langle r \rangle = 2r_0$.

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You have beautiful motivation for this: I would explain, and discuss the mean-field picture behind Fig 1C.

The results of the simulations for different n are shown in Fig. 2. When $n = 1$ or 2, the erosion develops channels in the porous media, and channeling instability occurs. In such cases, the flow is dominated by a few edges while the rest of the network carries almost no flow. This can also be seen in the PDF of the fluid flux ~~where it separates into two parts~~, a few edges with very high flux values and the rest of the edges with very small flux. The distribution of diameters, similar to the PDF of fluxes, diverges into two parts, one with a few large diameters that carry almost all the flow, while the rest and most of the tubes have very small radii. Contrary to $n = 1, 2$, when $n = 3$ and the erosion depends on shear at the throats' boundary walls, we find that the flow patterns stay very close to the initial shape, however, with larger and exacerbated flux values. Although the maximum fluid flux increases, the PDF of fluid fluxes remains exponential, and the PDF of diameters moves toward larger values. Increasing n to larger values, $n > 3$, we find that the flow pattern in the network moves toward homogenization. Here, the tail of the distribution moves toward smaller values while the PDF of fluid flux ~~keeps its exponential~~ tail. The PDF of

which becomes bimodal

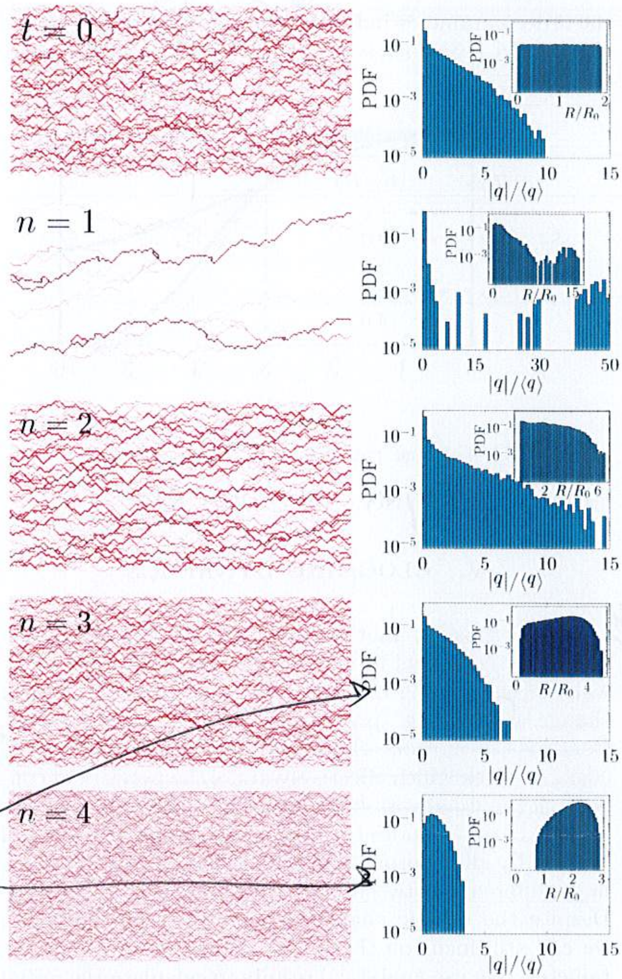
which?

Majority

doesn't look like it in the figure...

[Can we get better statistics?]

I don't think PRL uses such headings? (and they waste space)



How come you are getting less flow overall?

FIG. 2. Erosion in a network of pipes. The initial condition is shown with the label $t = 0$ in the first row. Each row afterward corresponds to the simulation result after N steps such that $\langle R_{t=N} \rangle = 2R_0$ where $R_0 = \langle R_{t=0} \rangle$; in other words the same amount of material is eroded in all cases. The erosion laws used here is $dR/dt \propto Q/R^n$ where n varies for each row. ~~The constant α in all cases are 1 and time step dt is chosen such that the maximum amount of increase in R at each step is smaller than $0.01R_0$.~~

Irrelevant.

the diameters stays uniform, while the standard deviation reduces. Increasing n to even larger values, we find that the fluid flux becomes uniform, where the fluid flux appears to be distributed around the average flux. The PDF of diameters similarly moves toward the same value. If we run the simulations for longer times, we observe that the flow becomes completely uniform where only a single radius appears to carry the flow with the same fluid flux for all the tubes in the network. In summary, we have observed a transition in the network behavior during erosion which happens at $n = 3$: when $n < 3$ the network moves toward developing channels, and when $n > 3$ the erosion results in the homogenization of the flow.

Isn't the CV the more relevant measure

What happens to the average flow & permeability?

where is this shown?

Need more quantitative analysis of the temporal dynamics.

III. SIMPLIFIED MODEL

In order to understand the underlying reason behind the transition in network behavior during erosion for different powers of n, we focus on a simplified model with only two tubes in parallel or series (see Fig. 3a,b). First, assuming two cylindrical tubes with radii r_1, r_2 in series and connected back to back, the flow is the same for the two tubes $q_1 = q_2 = q$ (Fig. 3a). The radius of each tube then changes with $dr_i/dt = \alpha q/r_i^n$ where $i = 1, 2$. Each tube can be considered as a resistor where its conductivity $C_i = \pi r_i^4/8\mu l_i$ changes over time. In similarity with resistor networks, pressure difference ΔP acts analogous to the voltage difference and the fluid flux q becomes analogous to the current where we have $q = C_i \Delta P$. When the cylindrical tubes are in series, we find that the conductivity of each tube changes as $dC_i/dt \propto qC^{(n-3)/4}$. As a result, each tube's conductivity increases with the rate depending on n, i.e. the larger the n the faster it moves toward larger values. If we consider the pressure at the junction between tubes, we find that this middle point pressure moves toward the average value of pressure on both sides. This suggests that when the tubes are in series, the erosion results in achieving a homogenized distribution of pressure. Contrary to tubes in series, when the tubes are in parallel (Fig. 3b), the flow divides between the two tubes based on their conductivity i.e. $q_1/q_2 = C_1/C_2$. Since each tube's radius changes as $dr_i/dt = \alpha q_i/r_i^n$, the evolution of the fluid flow ratio becomes

$$\frac{d}{dt}\left(\frac{C_1}{C_2}\right) \propto \frac{C_1}{C_2^{(n+1)/4}}\left(\left(\frac{C_1}{C_2}\right)^{\frac{n-3}{4}} - 1\right). \quad (2)$$

When $n = 3$ in Eq. (2), the right-hand-side becomes zero and as a result the flow ratio C_1/C_2 remains constant and does not change. However, when $n \neq 3$, we find that $C_1/C_2 = 1$ is an equilibrium point. When $n > 3$ this equilibrium solution is stable and the flow moves toward homogenization; however, when $n < 3$ this equilibrium solution becomes unstable and the solution moves toward $C_1/C_2 \to 0$, or ∞ which means that the whole flow passes through one of the tubes. To summarize, when the tubes are in series any erosion law makes flow become more uniform; however, when the tubes are in parallel depending on the power n the flow in the tubes can move toward becoming more uniform ($n > 3$), stay the same ($n = 3$), or move toward instability and channel development ($n < 3$). This observation is consistent with the numerical simulation results shown in Fig. 2(a)-(e).

add labels to figure or don't refer to them.

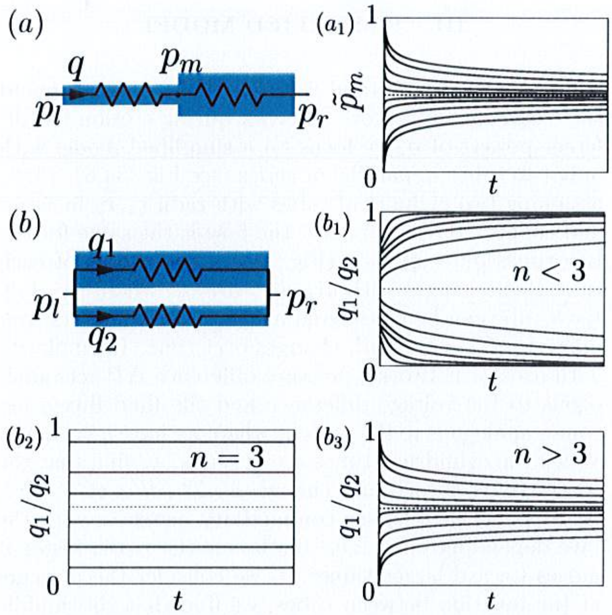
IV. PHASE TRANSITION

In order to quantify the transition of the network between the channeling instability and homogenization, we define an order parameter $\mathcal{O}$ that moves toward 0 or 1 if the network moves toward channeling or homogenization

are you sure you didn't flip the order?

I don't think that's true.

[Continuous n used? Better show points as well.]



They

FIG. 3. Two tubes in (a) series or (b) parallel configuration. When the tubes are in series ~~configuration~~, ~~the tubes have the same fluid flow rate q. The tube radius dynamically change with the erosion law $r_i \propto q/r_i^n$.~~ For any n, the normalized pressure at the junction between the tubes $\tilde{p}_m = (p_m - p_l)/(p_r - p_l)$ approaches 1/2 which results in a homogenized pressure distribution shown in (a_2). When the pipes are in parallel, however, ~~the flow is distributed among the tubes based on their flow conductivity C_1, C_2. The local fluid flow rate then affects the erosion of the tub $r_i \propto q_i/r_i^n$.~~ for ~~When~~ $n < 3$ in the erosion law, the fluid flow eventually passes through a single tube, and the branching occurs. (b_2) when $n = 3$, the flow ratio between the pipes does not change, and when $n > 3$ the flow ratio approaches 1/2 ~~which is the~~ (i.e., homogenization)

(for any n)

respectively. The order parameter is defined as (*)

Why?

$$\mathcal{O} = \frac{1}{N-1}\left(N - \frac{\left(\sum_{ij} q_{ij}^2\right)^2}{\sum_{ij} q_{ij}^4}\right) \tag{3}$$

where N is the number of edges. The order parameter $\mathcal{O} = 0$ when the flux in all the edges become the same $q_{ij} = \bar{q}$. On the other hand, when fluid flux becomes highly localized with only a few edges with non-zero flux, $\mathcal{O} \to 1$. We calculated the order parameters ~~during the simulations~~ for the network of random tubes during erosion ~~and measure the proposed order parameter $\mathcal{O}$~~. The results are shown in Fig. 4 for different amounts of erosion measured by the increase in the average diameter $\langle r\rangle/r_0$. As shown in Fig. 4, at $n = 3$ the order parameter, remains unchanged; however for $n > 3$ the order parameter moves toward 0 where the flow becomes more uniform and for $n < 3$ the order parameter goes toward unity where channels ~~are~~ develop~~ed~~. The transition in the order parameter indicates a second-order phase transition for $n = 3$ which is in agreement with the toy model prediction.

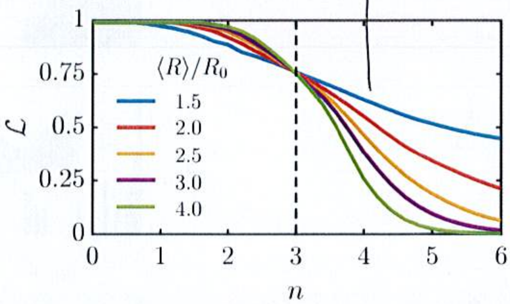


FIG. 4. Phase transition at the power $n = 3$

[Explain details etc.]

V. CLOGGING DYNAMICS

Besides erosion, another change in the network is the deposition/sedimentation of material on the boundary walls of the porous material. We name this dynamical change a "clogging" process as opposed to the erosion. ~~Contrary to erosion, the clogging behavior causes the edges to block which effectively alters the network of connectivity and network behavior.~~ This change in the connection between nodes through edges getting blocked can drastically alter porous structure behavior, e.g., causes a huge difference between effective and true porosity [16]. Despite the drastic change of network with blockages, we can still focus on the initial change in the order parameter. We use analytical results to calculate the initial derivative of the order parameter in the network (see supplementary material for more details) Δ. We plot the initial derivative of the order parameter in clogging in Fig. 4. Similar to the erosion, we find that a transition happens at $n = 3$. When $n < 3$ the network moves toward homogenization during the clogging process and when $n > 3$ the flow moves toward the development of channeling instability. This initial direction, however, might not hold due to the complex changes in the connectivity network during the clogging process.

Rephrase to explain "block" and the ensuing change in network topology

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for longer times?

VI. CONCLUSION

In summary, we analyzed the dynamics of the porous networks during erosion. ~~We showed that depending on different erosion laws various network behaviors are observable~~. We used simple erosion law, inspired by the proposed models so far, and showed that depending on the rate of erosion the network can either move toward homogenization or toward developing channeling instability. Next, we propose an order parameter to capture the phase transition behavior, and using a simplified model of two tubes we described the physical origin of the phase

Acknowledgments?

Thank Kardi + MRSEC

(*) Perhaps mention similarity to PR/IPR and why they are inadequate here.

Δ = Mention that this derivative is same for erosion!

transition behavior. In the case of clogging, since the network connectivity can vary significantly our approach

Need broader discussion.

does not hold ~~in~~ at large times ~~behavior,~~ however, the initial variation can be captured using our model where the results similarly show a behavior change at $n = 3$.

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Update. Does shima plan on uploading to the arxiv?