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How to cite this article:
Howard Brenner and John Happel (1958). Slow viscous flow past a sphere in a cylindrical tube.
Journal of Fluid Mechanics,4, pp 195-213 doi:10.1017/S0022112058000392

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Slow viscous flow past a sphere in a cylindrical tube

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SUMMARY

A theoretical treatment is presented for the slow flow of a viscous fluid through a cylindrical container within which a small spherical particle is confined. The sphere is situated in an arbitrary position within the cylinder and moves at constant velocity parallel to the walls. Approximate expressions are derived which give the frictional drag, rotational couple, and permanent pressure drop caused by the presence of this obstacle in the original Poiseuillian field of flow. The primary parameters involved are the ratio of sphere to cylinder radius and fractional distance of the particle from the longitudinal axis of the cylinder. With appropriate modifications, the results are also applicable to a sphere settling in a quiescent fluid. This yields the necessary boundary corrections to Stokes law arising in connection with devices such as the falling ball viscometer when the sphere is eccentrically located.

1. INTRODUCTION

The problem of viscous flow through a cylindrical duct containing particles of approximately spherical shape is of interest in connection with a variety of processes, such as fluidization, elutriation, and flow through fixed and moving beds of solids. A logical start towards elucidation of the hydrodynamic behaviour of these systems is undertaken here by considering the slow translation of a single spherical particle moving parallel to the longitudinal axis of an infinitely long circular cylinder through which a viscous fluid is flowing. The sphere may occupy any preassigned position. This is in contrast to the work of previous investigators who have concerned themselves exclusively with the axisymmetrical case in which the sphere is restricted to the cylinder axis. Haberman (1956, 1957) has recently reviewed the literature on this subject.

In a companion article by the authors (Happel & Brenner 1957), the results of the present theoretical investigation have been employed to discuss the behaviour of dilute fluidized beds at low Reynolds numbers.

2. NOMENCLATURE AND BOUNDARY CONDITIONS

In the treatment to follow, it is necessary to resort to a variety of different coordinate systems. These are: Cartesian coordinates (x, y, z), spherical coordinates (r, θ, ϕ), and cylindrical coordinates (ρ, ϕ, z), each having a

common origin at the sphere centre. It is also necessary to utilize Cartesian coordinates (X, Y, Z), and cylindrical coordinates (R, Φ, Z), both originating along the cylinder axis and chosen so as to make $z = Z$. Relations between these various systems of coordinates are depicted in figure 1.
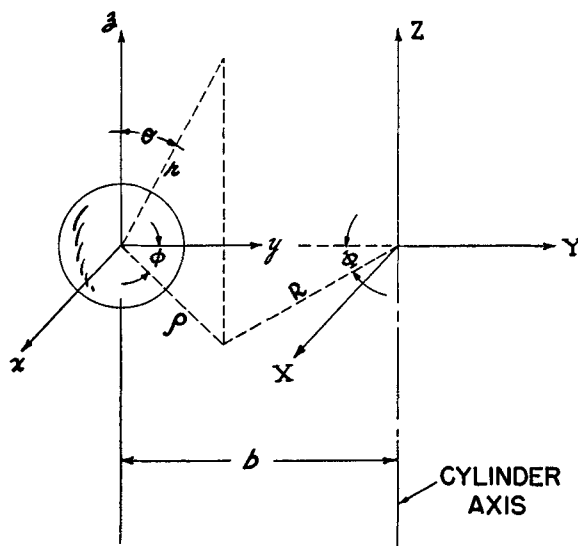
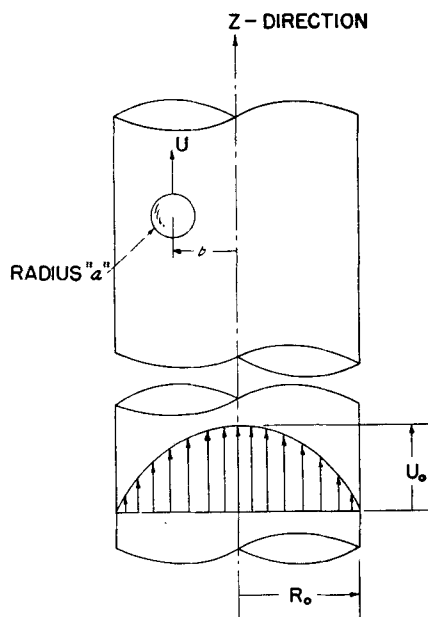Figure 1. Relations between different coordinate systems employed.

Figure 2. Definition sketch.

The sphere moves with an arbitrary constant velocity U relative to the cylinder wall in the direction of z-positive, parallel to the cylinder axis. At the same time, the fluid flows in laminar flow with a mean velocity of $\frac{1}{2}U_0$, in the same direction. The sphere radius is a, the cylinder radius is R_0, and the centre of the sphere is situated at a distance b from the cylinder axis, as shown in figure 2.

In terms of a coordinate system which moves with the sphere, the usual hypothesis of no relative motion at fluid–solid interfaces results in the following boundary conditions which define the fluid velocity field $\mathbf{v}$:

$$\mathbf{v} = 0 \quad \text{at } r = a, \tag{2.1}$$

and

$$\mathbf{v} = -\mathbf{i}_z U \quad \text{at } R = R_0. \tag{2.2}$$

At large distances from the sphere, $z = \pm \infty$, the disturbance propagated by the sphere vanishes and the fluid velocity distribution becomes Poiseuillian. This gives rise to the additional boundary condition,

$$\mathbf{v} = \mathbf{i}_Z\{U_0(1 - R^2/R_0^2) - U\} \quad \text{at } z = \pm \infty. \tag{2.3}$$

The equations of motion to be satisfied are

$$\nabla^2 \mathbf{v} = (1/\mu)\nabla p, \tag{2.4}$$

together with the continuity equation for incompressible fluids,

$$\nabla . \mathbf{v} = 0. \tag{2.5}$$

Here μ is the viscosity and p the viscous pressure. Use of the linearized creeping motion equations restricts the validity of the final results to situations in which the relative particle Reynolds number,

$$2a|U_0(1 - b^2/R_0^2) - U|/\eta,$$

is small, where η is the kinematic viscosity.

As in a previous study (Happel & Byrne 1954), the above boundary-value problem is solved by the method of 'reflections'. Thus, the solution consists of the sum of a series of velocity fields, all of which satisfy (2.4) and (2.5), and each partially satisfying the boundary conditions as follows:

$$\mathbf{v}^{(0)} = \mathbf{i}_Z\{U_0(1 - R^2/R_0^2) - U\}, \tag{2.6}$$

$$\mathbf{v}^{(1)} = \begin{cases} -\mathbf{v}^{(0)} & \text{at } r = a, \\ 0 & \text{at } z = \pm \infty \quad \text{(i.e. } r = \infty\text{),} \end{cases} \tag{2.7}$$

$$\mathbf{v}^{(2)} = \begin{cases} -\mathbf{v}^{(1)} & \text{at } R = R_0, \\ 0 & \text{at } Z = \pm \infty, \end{cases} \tag{2.8}$$

$$\mathbf{v}^{(3)} = \begin{cases} -\mathbf{v}^{(2)} & \text{at } r = a, \\ 0 & \text{at } z = \pm \infty \quad \text{(i.e. } r = \infty\text{),} \end{cases} \tag{2.9}$$

etc., with as many fields taken as needed for an appropriate degree of approximation. The field $\mathbf{v}$ satisfying the boundary conditions (2.1) to (2.3) is then obtained in the form

$$\mathbf{v} = \mathbf{v}^{(0)} + \mathbf{v}^{(1)} + \mathbf{v}^{(2)} + \mathbf{v}^{(3)} + ..., \tag{2.10}$$

and the corresponding pressure field is given by

$$p = p^{(0)} + p^{(1)} + p^{(2)} + p^{(3)} + \dots . \tag{2.11}$$

The success of this method of solution depends upon the linearity of the equations of motion. Its advantage resides in the fact that it is only necessary to consider boundary conditions associated with one surface at a time.

If we let $\mathbf{W}$ and $\mathbf{L}$, respectively, represent the frictional force and rotational moment experienced by the sphere, there are analogous relations of the form

$$\mathbf{W} = \mathbf{W}^{(0)} + \mathbf{W}^{(1)} + \mathbf{W}^{(2)} + \mathbf{W}^{(3)} + \dots, \tag{2.12a}$$

and

$$\mathbf{L} = \mathbf{L}^{(0)} + \mathbf{L}^{(1)} + \mathbf{L}^{(2)} + \mathbf{L}^{(3)} + \dots . \tag{2.12b}$$

Furthermore, if ΔP_S represents the additional pressure drop (above that due to the original Poiseuillian field, ΔP_0) experienced by the fluid as a result of the presence of the sphere, then

$$\Delta P_S = \Delta P_1 + \Delta P_2 + \Delta P_3 + \Delta P_4 + \dots, \tag{2.13}$$

where the pressure drop, ΔP_i, associated with field i is simply the difference between the viscous pressure $p^{(i)}$ at $Z = -\infty$ and $Z = +\infty$. There is no ambiguity in this definition of pressure drop, since as $Z \to \pm\infty$ the pressure becomes constant across the tube and, hence, independent of Φ and R. It will prove convenient, therefore, in calculating pressure drop to evaluate the pressures $p^{(i)}$ at $R = R_0$. Except for an arbitrary additive constant, which can be taken to be zero without loss of generality, the conditions of symmetry about the plane $Z = 0$ demand that the pressure be an odd function of Z. These observations lead to the formula

$$\Delta P_i = -2 \lim_{Z \to +\infty} (p^{(i)})_{R = R_0}, \tag{2.14}$$

where the subscript denotes evaluation of the function at the cylinder wall.

3. The first reflected field

The unperturbed field, $\mathbf{v}^{(0)}$, is defined in (2.6). It is easy to demonstrate that

$$\mathbf{W}^{(0)} = 0, \tag{3.1}$$

and

$$\mathbf{L}^{(0)} = 0. \tag{3.2}$$

The first reflected field $\mathbf{v}^{(1)}$, defined in (2.7), has already been obtained by Simha (1936) in connection with suspension viscosity. Using Lamb's (1932, p. 594) general solution in spherical coordinates, Simha's result can be expressed in the form

$$\mathbf{v}^{(1)} = \nabla \times (\mathbf{r}\chi_{-2}^{(1)}) + \nabla(\Phi_{-2}^{(1)} + \Phi_{-3}^{(1)} + \Phi_{-4}^{(1)} + (1/2\mu)r^2 p_{-2}^{(1)} - (1/30\mu)r^2 p_{-4}^{(1)}) +$$
$$+ (1/\mu)\mathbf{r}(p_{-2}^{(1)} + \tfrac{1}{2}p_{-3}^{(1)} + \tfrac{1}{3}p_{-4}^{(1)}), \tag{3.3}$$

and

$$p^{(1)} = p_{-2}^{(1)} + p_{-3}^{(1)} + p_{-4}^{(1)}, \tag{3.4}$$

where $\mathbf{r}$ is the radius vector drawn from the sphere origin, and $p_{-(n+1)}$,

$\Phi_{-(n+1)}$ and $\chi_{-(n+1)}$ are solid spherical harmonics of degree $-(n+1)$ defined as follows:

$$p^{(1)}_{-2}/\mu = Ar^{-2}P_1(\cos\theta), \qquad p^{(1)}_{-3}/\mu = BR_0 r^{-3}\cos\phi P^1_2(\cos\theta),$$

$$p^{(1)}_{-4}/\mu = CR_0^2 r^{-4}P_3(\cos\theta), \qquad \Phi^{(1)}_{-2} = DR_0^2 r^{-2}P_1(\cos\theta),$$

$$\Phi^{(1)}_{-3} = ER_0^3 r^{-3}\cos\phi P^1_2(\cos\theta), \qquad \Phi^{(1)}_{-4} = FR_0^4 r^{-4}P_3(\cos\theta), \tag{3.5}$$

$$\chi^{(1)}_{-2} = GR_0 r^{-2}\sin\phi P^1_1(\cos\theta),$$

with the constants given by

$$A = \tfrac{3}{2}a[U - U_0\{1 - (b^2/R_0^2) - \tfrac{2}{3}(a^2/R_0^2)\}],$$

$$B = -\tfrac{10}{3}aU_0(b/R_0)(a/R_0)^2, \qquad C = -\tfrac{7}{2}aU_0(a/R_0)^4,$$

$$D = \tfrac{1}{4}a(a/R_0)^2[U - U_0\{1 - (b^2/R_0^2) - \tfrac{6}{5}(a^2/R_0^2)\}], \tag{3.6}$$

$$E = -\tfrac{1}{3}aU_0(b/R_0)(a/R_0)^4, \qquad F = -\tfrac{1}{4}aU_0(a/R_0)^6,$$

$$G = aU_0(b/R_0)(a/R_0)^2.$$

The quantities $P_n(\cos\theta)$ and $P^m_n(\cos\theta)$ are Legendre polynomials of order n, and associated Legendre polynomials (of the first kind) of order n and rank m, respectively. For reference, the following values appear in equation (3.5):

$$P_1(\cos\theta) = \cos\theta, \qquad P^1_1(\cos\theta) = \sin\theta,$$

$$P^1_2(\cos\theta) = 3\sin\theta\cos\theta, \qquad P_3(\cos\theta) = \tfrac{1}{2}(5\cos^3\theta - 3\cos\theta). \tag{3.7}$$

The drag and rotational moments experienced by the sphere can be obtained from the relations

$$\mathbf{W}^{(1)} = -4\pi\nabla(r^3 p^{(1)}_{-2}), \tag{3.8}$$

and

$$\mathbf{L}^{(1)} = -8\pi\nabla(r^3\chi^{(1)}_{-2}). \tag{3.9}$$

These relations apply quite generally to Lamb's solution. Noting from (3.5) that $r^3 p^{(1)}_{-2} = \mu Ar\cos\theta = \mu Az$ and introducing the value of A from (3.6) results in

$$\mathbf{W}^{(1)} = -\mathbf{i}_z 6\pi\mu a[U - U_0\{1 - (b^2/R_0^2) - \tfrac{2}{3}(a^2/R_0^2)\}]. \tag{3.10}$$

In a like manner, application of the relation $r\sin\phi\sin\theta = x$ yields

$$\mathbf{L}^{(1)} = -\mathbf{i}_x 8\pi\mu a^2 U_0(b/R_0)(a/R_0). \tag{3.11}$$

If figure 2 represents a meridian plane passing through both the sphere origin and cylinder axis, the tendency of the couple is to rotate the sphere in a clockwise direction.

We shall limit ourselves in the subsequent development to an approximate solution in which the ratio of sphere-to-cylinder radius a/R_0 is small. This covers most situations of practical interest. The terms in (3.5) are such that a final expression for the drag, correct to zeroth and first powers of a/R_0, can ultimately be obtained by retaining only the $p^{(1)}_{-2}$ harmonic in the present solution. Thus, in place of the previous results,

we now resort to the approximate solution

$$\mathbf{v}^{(1)} \doteq (1/2\mu)\nabla(r^2 p^{(1)}_{-2}) + (1/\mu)\mathbf{r}p^{(1)}_{-2}, \qquad (3.12)$$

and

$$p^{(1)} \doteq p^{(1)}_{-2}, \qquad (3.13)$$

where

$$p^{(1)}_{-2}/\mu = Hr^{-2}\cos\theta = H(z/r^3), \qquad (3.14)$$

with the constant H defined by

$$H = \tfrac{3}{2}a[U - U_0\{1 - (b^2/R_0^2)\}]. \qquad (3.15)$$

To the present degree of approximation, the drag is now given by

$$\mathbf{W}^{(1)} \doteq -\mathbf{i}_Z\, 6\pi\mu a[U - U_0\{1 - (b^2/R_0^2)\}], \qquad (3.16)$$

while the rotational moment remains unaltered.

4. Transformation to cylindrical coordinates

In order to compute the field $\mathbf{v}^{(2)}$, defined in (2.8), it is necessary to establish the form taken by $\mathbf{v}^{(1)}$ at the cylinder wall, $R = R_0$. This is best done by transforming the latter field to cylindrical coordinates (R, Φ, Z), originating along the cylinder axis. With the expression for $p^{(1)}_{-2}/\mu$, given in (3.14), and the aid of the relation

$$\nabla(1/r) = -\mathbf{r}/r^3, \qquad (4.1)$$

it is easy to show that an alternative form for the first reflected field is

$$\mathbf{v}^{(1)} = \tfrac{1}{2}H[2(1/r)\mathbf{i}_z - \nabla(z/r)]. \qquad (4.2)$$

In terms of the desired coordinates we have

$$\mathbf{i}_z = \mathbf{i}_Z \qquad (4.3)$$

and

$$\nabla = \mathbf{i}_R\frac{\partial}{\partial R} + \mathbf{i}_\Phi\frac{1}{R}\frac{\partial}{\partial\Phi} + \mathbf{i}_Z\frac{\partial}{\partial Z}, \qquad (4.4)$$

so that our objective may be attained by transforming the scalar functions $1/r$ and z/r to these coordinates.

Watson (1922) gives the relation

$$1/r = (\rho^2 + z^2)^{-1/2} = (2/\pi)\int_0^\infty K_0(\lambda\rho)\cos\lambda z\, d\lambda, \qquad (4.5)$$

where K_0 is the modified Bessel function of the second kind of order zero. Although this transformation is only valid for $\rho > 0$, the restriction is inconsequential since it is sufficient for our immediate purposes to evaluate $\mathbf{v}^{(1)}$ only in the vicinity of the cylinder wall. Noting that

$$\rho = (R^2 + b^2 - 2bR\cos\Phi)^{1/2}, \qquad (4.6)$$

we can avail ourselves of the further transformation,

$$K_0(\lambda\rho) = \sum_{k=-\infty}^{\infty} K_k(\lambda R)I_k(\lambda b)\cos k\Phi, \qquad (4.7)$$

given by Watson. Here, I_k and K_k are modified Bessel functions of the first and second kinds, respectively, of order k. The relation is valid only

for $R > b$. For the reason outlined above, this restriction is of no consequence. These expressions combine to give

$$\frac{1}{r} = \frac{2}{\pi} \sum_{k=-\infty}^{\infty} \cos k\Phi \int_0^{\infty} K_k(\lambda R) I_k(\lambda b) \cos \lambda Z \, d\lambda. \qquad (4.8)$$

To obtain the second scalar transformation we observe, upon performing the indicated differentiations in the following equation, that the following identity is valid:

$$zK_k(\lambda R)I_k(\lambda b)\cos\lambda z = \frac{\partial}{\partial\lambda}[K_k(\lambda R)I_k(\lambda b)\sin\lambda Z] - \sin\lambda Z\frac{\partial}{\partial\lambda}[K_k(\lambda R)I_k(\lambda b)].$$

Thus, if (4.8) is multiplied by z, we obtain with the aid of the above

$$\frac{z}{r} = \frac{2}{\pi} \sum_{k=-\infty}^{\infty} \cos k\Phi [K_k(\lambda R)I_k(\lambda b)\sin\lambda Z]_{\lambda=0}^{\lambda=\infty} -$$

$$- \frac{2}{\pi} \sum_{k=-\infty}^{\infty} \cos k\Phi \int_0^{\infty} \frac{\partial}{\partial\lambda}[K_k(\lambda R)I_k(\lambda b)]\sin\lambda Z \, d\lambda.$$

The first term in brackets vanishes at the upper and lower limits of evaluation, $\lambda = 0$ and ∞. Upon performing the indicated differentiation in the second expression, we arrive at the requisite transformation,

$$\frac{z}{r} = -\frac{2}{\pi} \sum_{k=-\infty}^{\infty} \cos k\Phi \int_0^{\infty} [RK_k'(\lambda R)I_k(\lambda b) + bK_k(\lambda R)I_k'(\lambda b)]\sin\lambda Z \, d\lambda. \quad (4.9)$$

The differentiations denoted by primes are with respect to the entire argument.

It is now a relatively simple matter to obtain the desired transformation,

$$[\mathbf{v}^{(1)}]_{R_0} = \mathbf{i}_R(1/\pi) \sum_{k=-\infty}^{\infty} \cos k\Phi \int_0^{\infty} \alpha_k(\lambda)\sin\lambda Z \, d\lambda +$$

$$+ \mathbf{i}_\Phi(1/\pi) \sum_{k=-\infty}^{\infty} \sin k\Phi \int_0^{\infty} \beta_k(\lambda)\sin\lambda Z \, d\lambda +$$

$$+ \mathbf{i}_Z(1/\pi) \sum_{k=-\infty}^{\infty} \cos k\Phi \int_0^{\infty} \gamma_k(\lambda)\cos\lambda Z \, d\lambda, \quad (4.10)$$

where we have written

$$a_k(\lambda) = H[\lambda R_0 K_k(\lambda R_0)I_k(\lambda b) +$$

$$+ \lambda b K_k'(\lambda R_0)I_k'(\lambda b) + \{k^2/(\lambda R_0)\}K_k(\lambda R_0)I_k(\lambda b)], \quad (4.11)$$

$$\beta_k(\lambda) = -Hk[(b/R_0)K_k(\lambda R_0)I_k'(\lambda b) + K_k'(\lambda R_0)I_k(\lambda b)], \qquad (4.12)$$

$$\gamma_k(\lambda) = H[\lambda R_0 K_k'(\lambda R_0)I_k(\lambda b) +$$

$$+ \lambda b K_k(\lambda R_0)I_k'(\lambda b) + 2K_k(\lambda R_0)I_k(\lambda b)]. \quad (4.13)$$

These relations have been simplified to some extent by the use of Bessel's modified equation,

$$K_k''(\lambda R) = -K_k'(\lambda R)/(\lambda R) + \{1 + k^2/(\lambda^2 R^2)\}K_k(\lambda R). \qquad (4.14)$$

An analogous expression for the initial pressure field at the cylinder wall can be obtained from (3.13), (3.14) and (4.8) with the assistance of the relation $z/r^3 = -\partial(1/r)/\partial z$, yielding

$$[p^{(1)}]_{R_0} = (2\mu H/\pi) \sum_{k=-\infty}^{\infty} \cos k\Phi \int_0^{\infty} \lambda K_k(\lambda R_0)I_k(\lambda b)\sin\lambda Z \, d\lambda. \quad (4.15)$$

5. The second reflection, $\mathbf{v}^{(2)}$

A general solution of (2.4) and (2.5) in cylindrical coordinates, suitable for the field $\mathbf{v}^{(2)}$, is given by

$$\mathbf{v}^{(2)} = \sum_{k=-\infty}^{\infty} \left[\nabla \times (\mathbf{i}_Z \Omega_k^{(2)}) + \nabla \Psi_k^{(2)} + R \frac{\partial}{\partial R} \nabla \Pi_k^{(2)} + \mathbf{i}_Z \frac{\partial \Pi_k^{(2)}}{\partial Z} \right], \qquad (5.1)$$

and

$$p^{(2)} = -2\mu \sum_{k=-\infty}^{\infty} \frac{\partial^2 \Pi_k^{(2)}}{\partial Z^2}, \qquad (5.2)$$

where Ω_k, Ψ_k and Π_k are arbitrary cylindrical harmonic functions of order k in (R, Φ, Z) coordinates. That this is, indeed, a simultaneous solution of the creeping motion and continuity equations is best verified by making the substitution $R\,\partial/\partial R = X(\partial/\partial X) + Y(\partial/\partial Y)$ in (5.1) and carrying out the operations indicated by (2.4) and (2.5) in Cartesian coordinates.

These equations are perfectly general and would be applicable for the fields $\mathbf{v}^{(4)}$, $\mathbf{v}^{(6)}$, etc. In the present instance, the harmonic functions are assumed to have the form

$$\Pi_k^{(2)} = -(1/\pi)\cos k\Phi \int_0^\infty \lambda^{-1} \pi_k(\lambda) I_k(\lambda R) \sin \lambda Z \, d\lambda,$$

$$\Psi_k^{(2)} = -(1/\pi)\cos k\Phi \int_0^\infty \lambda^{-1} \psi_k(\lambda) I_k(\lambda R) \sin \lambda Z \, d\lambda, \qquad (5.3)$$

$$\Omega_k^{(2)} = -(1/\pi)\sin k\Phi \int_0^\infty \lambda^{-1} \omega_k(\lambda) I_k(\lambda R) \sin \lambda Z \, d\lambda.$$

The functions $\pi_k(\lambda)$, $\psi_k(\lambda)$ and $\omega_k(\lambda)$ are to be determined from the boundary conditions expressed by (2.8). Upon substituting values from (5.3) into (5.1), we eventually find

$$\mathbf{v}^{(2)} = -\mathbf{i}_R (1/\pi) \sum_{k=-\infty}^{\infty} \cos k\Phi \int_0^\infty \left[\omega_k(\lambda) \frac{k I_k(\lambda R)}{\lambda R} + \psi_k(\lambda) I_k'(\lambda R) + \right.$$

$$\left. + \pi_k(\lambda) \lambda R I_k''(\lambda R) \right] \sin \lambda Z \, d\lambda -$$

$$- \mathbf{i}_\Phi (1/\pi) \sum_{k=-\infty}^{\infty} \sin k\Phi \int_0^\infty \left[\pi_k(\lambda) \frac{k I_k(\lambda R)}{\lambda R} - \pi_k(\lambda) k I_k'(\lambda R) - \right.$$

$$\left. - \omega_k(\lambda) I_k'(\lambda R) - \psi_k(\lambda) \frac{k I_k(\lambda R)}{\lambda R} \right] \sin \lambda Z \, d\lambda -$$

$$- \mathbf{i}_Z (1/\pi) \sum_{k=-\infty}^{\infty} \cos k\Phi \int_0^\infty [\psi_k(\lambda) I_k(\lambda R) + \pi_k(\lambda) \lambda R I_k'(\lambda R) + $$

$$+ \pi_k(\lambda) I_k(\lambda R)] \cos \lambda Z \, d\lambda. \qquad (5.4)$$

Evaluation of the above at $R = R_0$, and comparison to (4.10) by means of the boundary condition,

$$[\mathbf{v}^{(2)}]_{R_0} = -[\mathbf{v}^{(1)}]_{R_0}, \qquad (5.5)$$

leads to three simultaneous equations involving the functions $\pi_k(\lambda)$, $\psi_k(\lambda)$ and $\omega_k(\lambda)$ in terms of $\alpha_k(\lambda)$, $\beta_k(\lambda)$ and $\gamma_k(\lambda)$. Solving these equations and introducing values for the latter functions from (4.11) to (4.13) gives:

$$\pi_k(\lambda) = H K_k(\lambda R_0) I_k(\lambda b) \{ I_k(\lambda R_0) \}^{-1} + \xi_k(\lambda), \qquad (5.6)$$

$$\psi_k(\lambda) = -[1 + (\lambda R_0)I_k'(\lambda R_0)\{I_k(\lambda R_0)\}^{-1}]\pi_k(\lambda) +$$
$$+ H\{I_k(\lambda R_0)\}^{-1}[\lambda R_0\, K_k'(\lambda R_0)I_k(\lambda b) +$$
$$+ \lambda b K_k(\lambda R_0)I_k'(\lambda b) + 2K_k(\lambda R_0)I_k(\lambda b)], \quad (5.7)$$

and

$$\omega_k(\lambda) = 2k\{\lambda R_0\, I_k'(\lambda R_0)\}^{-1}[I_k(\lambda R_0)\pi_k(\lambda) - HK_k(\lambda R_0)I_k(\lambda b)], \quad (5.8)$$

where $\xi_k(\lambda)$ in (5.6) is given by

$$\xi_k(\lambda) = H[I_k(\lambda b)I_k'(\lambda R_0)\{I_k(\lambda R_0)\}^{-1} - (b/R_0)I_k'(\lambda b)] \times$$
$$\times [2k^2 I_k^3(\lambda R_0)\{(\lambda R_0)^2 I_k'(\lambda R_0)\}^{-1} + \{\lambda R_0 + (k^2/(\lambda R_0))\}I_k^2(\lambda R_0) -$$
$$- 2I_k(\lambda R_0)I_k'(\lambda R_0) - \lambda R_0[I_k'(\lambda R_0)]^2]^{-1}. \quad (5.9)$$

These equations have been simplified to some extent by means of the relations

$$I_k''(\lambda R_0) = -I_k'(\lambda R_0)/(\lambda R_0) + \{1 + k^2/(\lambda^2 R_0^2)\}I_k(\lambda R_0),$$

and

$$K_k'(\lambda R_0)I_k(\lambda R_0) - K_k(\lambda R_0)I_k'(\lambda R_0) = -1/(\lambda R_0). \quad (5.10)$$

The additional boundary condition, $\mathbf{v}^{(2)} \to 0$ as $Z \to \pm \infty$, is met because $\mathbf{v}^{(1)} \to 0$ as $Z \to \pm \infty$ and $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$ are related by (5.5).

The pressure drop due to the first two reflected fields can now be computed. Substitution of the value of $\Pi_k^{(2)}$, given in (5.3), into the expression for $p^{(2)}$, given by (5.2), and evaluation at $R = R_0$, using the value of $\pi_k(\lambda)$ from (5.6), yields

$$[p^{(2)}]_{R_0} = -\frac{2\mu H}{\pi} \sum_{k=-\infty}^{\infty} \cos k\Phi \int_0^\infty \lambda K_k(\lambda R_0)I_k(\lambda b)\sin\lambda Z\, d\lambda -$$
$$- \frac{2\mu}{\pi} \sum_{k=-\infty}^{\infty} \cos k\Phi \int_0^\infty \lambda\xi_k(\lambda)I_k(\lambda R_0)\sin\lambda Z\, d\lambda. \quad (5.11)$$

As evidenced by (4.15), the lead term in the above expression is simply $-[p^{(1)}]_{R_0}$, so that

$$[p^{(1)} + p^{(2)}]_{R_0} = -(2\mu/\pi) \sum_{k=-\infty}^{\infty} \cos k\Phi \int_0^\infty \lambda\xi_k(\lambda)I_k(\lambda R_0)\sin\lambda Z\, d\lambda. \quad (5.12)$$

From (2.14), the pressure drop due to these first two reflections is, therefore,

$$(\Delta P_1 + \Delta P_2) = -\frac{4\mu}{\pi R_0^2} \lim_{Z\to\infty} \sum_{k=-\infty}^{\infty} \cos k\Phi \int_0^\infty \eta_k(\lambda)\frac{\sin\lambda Z}{\lambda}\, d\lambda, \quad (5.13)$$

where we have written

$$\eta_k(\lambda) = (\lambda R_0)^2\xi_k(\lambda)I_k(\lambda R_0). \quad (5.14)$$

By Dirichlet's theorem,

$$\lim_{Z\to\infty} \int_0^\infty \eta_k(\lambda)\frac{\sin\lambda Z}{\lambda}\, d\lambda = \frac{\pi}{2}\,\eta_k(0+). \quad (5.15)$$

This limit, as $\lambda \to 0+$, can easily be obtained by expanding the Bessel functions in series for small values of their arguments; whence,

$$\eta_k(0+) = \eta_k(0) = \begin{cases} -4H(1 - b^2/R_0^2) & \text{for } k = 0, \\ 0 & \text{for } k \neq 0, \end{cases} \quad (5.16)$$

from which, with the definition of H in (3.15), we finally obtain

$$(\Delta P_1 + \Delta P_2) = (12\mu a/R_0^2)(1 - b^2/R_0^2)[U_0(1 - b^2/R_0^2) - U]. \qquad (5.17)$$

A more physically meaningful representation is

$$[(\Delta P_1 + \Delta P_2)\pi R_0^2]\tfrac{1}{2}U_0 = [6\pi\mu a\{U_0(1 - b^2/R_0^2) - U\}][U_0(1 - b^2/R_0^2)]. \qquad (5.18)$$

For the small values of (a/R_0) for which the present approximation is valid, the first term in brackets represents the additional force (above Poiseuille's law) required to force fluid through the tube. The second term is the mean velocity with which fluid traverses the cylinder. Their product gives the additional energy dissipation incurred by the presence of the sphere in the original field of flow. On the right-hand side of this expression, the first term in brackets is the frictional force experienced by the sphere (equation (3.16)), whereas the second bracketed term corresponds to the *local* velocity of the unperturbed parabolic field in the vicinity of the sphere. It appears then that, for a sufficiently small sphere, the product of the drag and *local* velocity also gives the additional energy dissipation caused by the presence of an obstacle in the field of flow.

Within the volume of space presently occupied by the particle, the field $\mathbf{v}^{(2)}$ has no singularities. The sphere can therefore experience no frictional forces or couples in virtue of this field, and we have

$$\mathbf{W}^{(2)} = 0, \qquad (5.19)$$

and

$$\mathbf{L}^{(2)} = 0. \qquad (5.20)$$

6. The third reflection

The results obtained thus far for drag and pressure drop may be regarded as the *zeroth* approximation in powers of a/R_0; that for rotational moment is correct to the first power. To evaluate each of these quantities correctly to the next highest powers in a/R_0, it is necessary to consider the third and fourth reflections. Fortunately, exact solutions for $\mathbf{v}^{(3)}$ and $\mathbf{v}^{(4)}$ are not required to arrive at exact values for these initial corrections.

The frictional resistance and couple associated with $\mathbf{v}^{(3)}$ can be calculated exactly by means of Faxen's (1927) laws (see, also, Peres (1929)). In the present application, these laws take the form

$$\mathbf{W}^{(3)} = 6\pi\mu a[\mathbf{v}^{(2)}]_0 + \pi a^3[\nabla p^{(2)}]_0, \qquad (6.1)$$

and

$$\mathbf{L}^{(3)} = 4\pi\mu a^3[\nabla \times \mathbf{v}^{(2)}]_0, \qquad (6.2)$$

where the subscript zero indicates that the function in brackets is to be evaluated at the sphere centre. The relations are valid for arbitrary fields, $\mathbf{v}^{(2)}$ and $\mathbf{v}^{(3)}$, provided that each is in accord with the equations of slow motion and they are related by (2.9). To preserve the consistency of the present results regarding powers of a/R_0, we omit the last term of (6.1).

In terms of the (R, Φ, Z) system of coordinates, the centre of the sphere is situated at the point $(R = b, \Phi = 0, Z = 0)$. Thus, from (5.4) and (6.1)

$$\mathbf{W}^{(3)} = -\mathbf{i}_Z 6\mu(a/R_0) \sum_{k=-\infty}^{\infty} \int_0^\infty [\psi_k(\lambda)I_k(\lambda b) + \pi_k(\lambda)\lambda b I_k'(\lambda b) + \\ + \pi_k(\lambda)I_k(\lambda b)]d(\lambda R_0). \qquad (6.3)$$

Employing the values of $\psi_k(\lambda)$ and $\pi_k(\lambda)$, given by (5.6) and (5.7), and putting $\alpha = \lambda R_0$, this becomes

$$\mathbf{W}^{(3)} = -\mathbf{i}_z\, 6\pi\mu a [U - U_0(1 - b^2/R_0^2)] f(b/R_0)(a/R_0), \qquad (6.4)$$

where

$$f(b/R_0) = f(\beta) = -\frac{3}{2\pi} \sum_{k=-\infty}^{\infty} \int_0^{\infty} \left\{ \delta_k(\alpha) \left[\frac{I_k'(\alpha) I_k(\alpha\beta)}{I_k(\alpha)} - \beta I_k'(\alpha\beta) \right]^2 + \right.$$
$$\left. + \left[\frac{I_k(\alpha\beta)}{I_k(\alpha)} \right]^2 - \frac{2K_k(\alpha) I_k(\alpha\beta)}{I_k(\alpha)} [I_k(\alpha\beta) + \alpha\beta I_k'(\alpha\beta)] \right\} d\alpha, \qquad (6.5)$$

in which

$$\delta_k(\alpha) = \left[\frac{2k^2 I_k^3(\alpha)}{\alpha^3 I_k'(\alpha)} + \left(1 + \frac{k^2}{\alpha^2}\right) I_k^2(\alpha) - \frac{2I_k(\alpha) I_k'(\alpha)}{\alpha} - [I_k'(\alpha)]^2 \right]^{-1}, \qquad (6.6)$$

and for brevity we have put $\beta = (b/R_0)$. Equation (5.10) has been employed in simplifying the above.

Using the method indicated in the Appendix, $f(b/R_0)$ has been evaluated as a power series in even powers of b/R_0 and the first two terms, obtained by numerical integration, are

$$f(b/R_0) = 2 \cdot 1047 - 0 \cdot 6977(b/R_0)^2 + \dots. \qquad (6.7)$$

This development is valid for values of $b/R_0 \to 0$, near the cylinder axis. In the vicinity of the cylinder wall, where $b/R_0 \to 1$, it is possible to make an exact calculation, thereby obtaining (see the Appendix)

$$\lim_{b/R_0 \to 1} (1 - b/R_0) f(b/R_0) = \tfrac{3}{4}. \qquad (6.8)$$

This limit is approached asymptotically. A tentative plot of the function $(1 - b/R_0) f(b/R_0)$ *vs* b/R_0, derived from these limiting expressions, is presented in figure 3. This should be of some assistance in the extrapolation of values of $f(b/R_0)$ beyond those for which (6.7) is strictly applicable. In view of the inadequacy of the formulae for intermediate values of the eccentricity b/R_0, such a plot can only be regarded as preliminary.

Equation (5.4) results in the expression

$$[\nabla \times \mathbf{v}^{(2)}]_0 = \mathbf{i}_x(1/\pi) \sum_{k=-\infty}^{\infty} \int_0^{\infty} [2\lambda\pi_k(\lambda) I_k'(\lambda b) - kb^{-1}\omega_k(\lambda) I_k(\lambda b)]\, d\lambda, \qquad (6.9)$$

where we have noted that

$$[\mathbf{i}_\Phi]_0 = \mathbf{i}_x. \qquad (6.10)$$

Substituting in (6.2) and employing both (5.6) and (5.8) eventually gives

$$\mathbf{L}^{(3)} = -\mathbf{i}_x\, 8\pi\mu a^2 [U - U_0(1 - b^2/R_0^2)] g(b/R_0)(a/R_0)^2, \qquad (6.11)$$

where

$$g(b/R_0) = g(\beta) = -(3/2\pi) \sum_{k=-\infty}^{\infty} \int_0^{\infty} \left\{ \alpha K_k(\alpha) I_k(\alpha\beta) I_k'(\alpha\beta) \{I_k(\alpha)\}^{-1} + \right.$$
$$+ \delta_k(\alpha) \left[I_k'(\alpha\beta) - \frac{k^2 I_k(\alpha) I_k(\alpha\beta)}{\alpha^2 \beta I_k'(\alpha)} \right] \left[\frac{I_k(\alpha\beta) I_k'(\alpha)}{I_k(\alpha)} - \right.$$
$$\left. \left. - \beta I_k'(\alpha\beta) \right] \right\} d\alpha. \qquad (6.12)$$

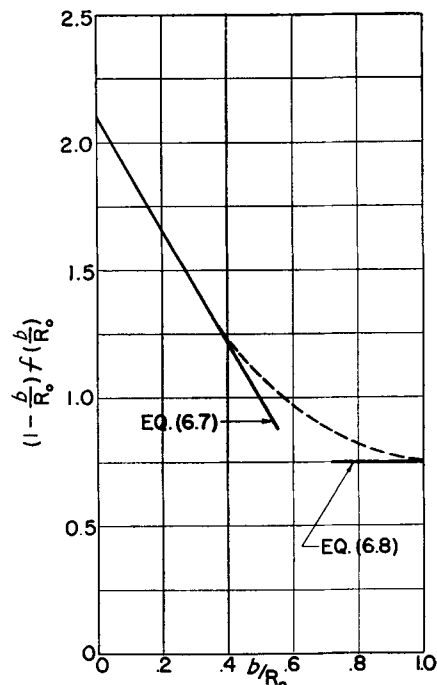The function $\delta_k(\alpha)$ is defined in (6.6).

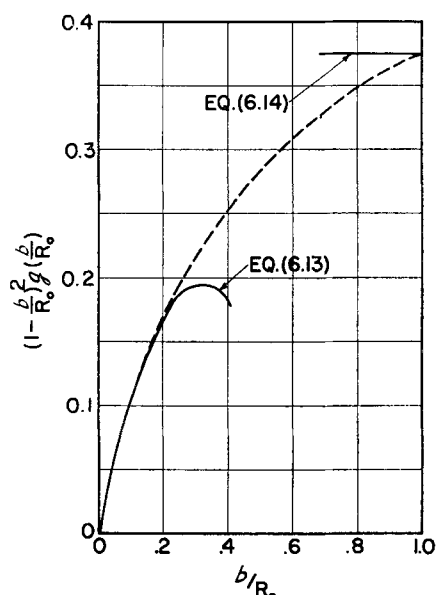Figure 3. Tentative plot of equation (6.5) for calculating drag and pressure drop.

Figure 4. Tentative plot of equation (6.12) for calculating rotational moment.

For small values of the eccentricity b/R_0, numerical integration of the foregoing yields

$$g(b/R_0) = 1 \cdot 296(b/R_0) + \dots , \qquad (6.13)$$

and the expansion proceeds in odd powers of b/R_0. In the limiting case, as the wall is approached, an exact calculation gives

$$\lim_{b/R_0 \to 1} (1 - b/R_0)^2 g(b/R_0) = \tfrac{3}{8}. \qquad (6.14)$$

Again, this limit is achieved asymptotically. A preliminary graph of $(1 - b/R_0)^2 g(b/R_0)$ *vs* b/R_0, prepared with the aid of these limiting forms, is shown in figure 4.

7. The fourth reflection

In order to obtain the first correction to the *zeroth* approximation for pressure drop it is necessary to consider the field $\mathbf{v}^{(4)}$. Here, again, an exact calculation to the first power in (a/R_0) can be made without a corresponding knowledge of $\mathbf{v}^{(4)}$. This can be seen from the following argument. If we resort to Lamb's general solution for the field $\mathbf{v}^{(3)}$, reflected from the sphere, then by analogy to (3.8), we have

$$4\pi\nabla(r^3 p^{(3)}_{-2}) = -\mathbf{W}^{(3)}. \qquad (7.1)$$

This can be solved for the solid spherical harmonic function $p^{(3)}_{-2}$ by multiplying scalarly by $\mathbf{r}$; whence,

$$4\pi r \frac{\partial}{\partial r} (r^3 p^{(3)}_{-2}) = -\mathbf{r} \cdot \mathbf{W}^{(3)}.$$

However, $p^{(3)}_{-2}$ is, among other things, a homogeneous polynomial of order -2. Thus, by Euler's theorem on homogeneous polynomials,

$$r \frac{\partial}{\partial r} (p^{(3)}_{-2}) = -2p^{(3)}_{-2}.$$

This makes
$$p^{(3)}_{-2} = -(4\pi r^3)^{-1}\mathbf{r} \cdot \mathbf{W}^{(3)}.$$

Using the expression for $\mathbf{W}^{(3)}$ in (6.4), and noting that $\mathbf{r} \cdot \mathbf{i}_z = z$, yields the harmonic function,

$$(1/\mu)p^{(3)}_{-2} = Jr^{-2}\cos\theta, \qquad (7.2)$$

where we have written

$$J = \tfrac{3}{2}a[U - U_0(1 - b^2/R_0^2)]f(b/R_0)(a/R_0). \qquad (7.3)$$

As in equations (3.12) and (3.13), that part of the entire velocity and pressure fields, $\mathbf{v}^{(3)}$ and $p^{(3)}$, associated with this particular harmonic function are

$$\mathbf{v}^{(3)} \doteqdot (1/(2\mu))\nabla(r^2 p^{(3)}_{-2}) + (1/\mu)\mathbf{r}p^{(3)}_{-2}, \qquad (7.4)$$

and
$$p^{(3)} \doteqdot p^{(3)}_{-2}. \qquad (7.5)$$

Of all the terms in Lamb's general solution, the only harmonic function which introduces the sphere radius a to the first power in the velocity and pressure fields is the p_{-2} harmonic. This is the dominant term as $r \to \infty$. Thus, as regards powers of a and, ultimately, in the next reflection, powers of (a/R_0), this is the only term which need be retained to obtain results which are correct to the first power in (a/R_0). To this degree of approximation, (7.4) and (7.5) may be regarded as exact representations of the

velocity and pressure fields. Comparing these expressions to those given in (3.12) to (3.15), we find

$$\mathbf{v}^{(3)} = (J/H)\mathbf{v}^{(1)}, \qquad p^{(3)} = (J/H)p^{(1)}. \tag{7.6}$$

To the same degree of approximation, it is easy to see that

$$\mathbf{v}^{(4)} = (J/H)\mathbf{v}^{(2)}, \qquad p^{(4)} = (J/H)p^{(2)}. \tag{7.7}$$

This immediately leads to the result

$$(\Delta P_3 + \Delta P_4) = (J/H)(\Delta P_1 + \Delta P_2). \tag{7.8}$$

With the values for J and H, and the expression for the pressure drop due to the first two reflected fields, quoted in (5.17), this becomes

$$(\Delta P_3 + \Delta P_4) = (12\mu a/R_0^2)(1 - b^2/R_0^2)[U_0(1 - b^2/R_0^2) - U]f(b/R_0)(a/R_0). \tag{7.9}$$

The fourth reflected field can have no singularities within the volume occupied by the sphere, so that

$$\mathbf{W}^{(4)} = 0, \quad \text{and} \quad \mathbf{L}^{(4)} = 0. \tag{7.10}$$

8. Final results

Upon summing the individual results for drag, rotational moment, and pressure drop we find

$$\mathbf{W} = \mathbf{i}_z 6\pi\mu a[U_0(1 - b^2/R_0^2) - U][1 + f(b/R_0)(a/R_0) + ...], \tag{8.1}$$

$$\mathbf{L} = -\mathbf{i}_x 8\pi\mu a^2\{U_0(b/R_0)(a/R_0) - [U_0(1 - b^2/R_0^2) - U]g(b/R_0)(a/R_0)^2 + ...\}, \tag{8.2}$$

and

$$\Delta P_S = (12\mu a/R_0^2)(1 - b^2/R_0^2)[U_0(1 - b^2/R_0^2) - U][1 + f(b/R_0)(a/R_0) + ...]. \tag{8.3}$$

These expressions are correct to the highest powers of (a/R_0) quoted. The case of a sedimenting sphere in a quiescent fluid is obtained by putting $U_0 = 0$ in the above.

A form of the above equations suitable for examining situations in which the sphere is near the container walls can be obtained by expressing the previous results in terms of the ratio of sphere radius to minimum distance (of the sphere centre) from the wall, $a/(R_0 - b)$. This results in

$$\mathbf{W} = \mathbf{i}_z 6\pi\mu a[U_0(1 - b^2/R_0^2) - U]\left[1 + (1 - b/R_0)f(b/R_0)\left(\frac{a}{R_0 - b}\right) + ...\right], \tag{8.4}$$

$$\mathbf{L} = -\mathbf{i}_x 8\pi\mu a^2\left\{U_0(b/R_0)(1 - b/R_0)\left(\frac{a}{R_0 - b}\right) - \right.$$
$$\left. - [U_0(1 - b^2/R_0^2) - U](1 - b/R_0)^2 g(b/R_0)\left(\frac{a}{R_0 - b}\right)^2 + ...\right\}, \tag{8.5}$$

$$\Delta P_S = (12\mu a/R_0^2)(1 - b^2/R_0^2)[U_0(1 - b^2/R_0^2) - U] \times$$
$$\times \left[1 + (1 - b/R_0)f(b/R_0)\left(\frac{a}{R_0 - b}\right) + ...\right]. \tag{8.6}$$

As previously observed, the quantities $(1 - b/R_0)f(b/R_0)$ and $(1 - b/R_0)^2 g(b/R_0)$ attain limiting values of $\frac{3}{4}$ and $\frac{3}{8}$, respectively, as $b/R_0 \to 1$. It should be noted that

$$\frac{a}{R_0 - b} \leqslant 1, \tag{8.7}$$

where the equality sign applies when the sphere touches the wall of the cylinder.

9. Discussion of results

When the sphere is at the cylinder axis, $b/R_0 = 0$, equations (8.1) to (8.3) reduce to results previously given by Happel & Byrne (1954) and, independently, by Wakiya (1953) and Haberman (1957).

It appears from the results of this investigation that both the drag and pressure drop are even functions of the eccentricity b/R_0. This is to be expected, since symmetry requires that if the sphere is moved from its present location, $R = b$, to the opposite side of the cylinder axis, $R = -b$, neither the direction nor the magnitude of the drag should be altered. Likewise, the pressure drop should be unaffected by this transition, in accord with present calculations. On the other hand, equation (8.2) shows that the rotational moment is an odd function of b/R_0. This conclusion, again, coincides with intuition, since the sphere will tend to rotate in a direction opposite to its original direction, without alteration in the magnitude of the rotational couple, when it is placed in mirror-image position on the opposite side of the cylinder axis. Of course, at the cylinder axis, we have

$$\mathbf{L}(0) = 0. \tag{9.1}$$

Further support for the validity of the present results is provided by observing that, in the absence of external forces acting on the sphere, the viscous pressure drop, due to the presence of a particle in the original field of flow, is an essentially positive quantity. This stems from a one-to-one correspondence between pressure drop in rectilinear flow and energy dissipation. Thus, equation (8.3) shows that, for net flow in the $+z$-direction, a positive pressure drop implies the inequality

$$U_0(1 - b^2/R_0^2) \geqslant U.$$

Inasmuch as $U_0(1 - b^2/R_0^2)$ is the *local* fluid velocity, this shows that the sphere necessarily lags the fluid, an obviously correct inference. Moreover, equation (8.1) correctly indicates that, in these circumstances, the frictional force on the sphere is always in the direction of net flow, and the sphere is thus dragged along by the fluid.

There are some remarkable conclusions to be drawn from the present calculations. The most interesting of these is, perhaps, the fact that the drag experienced by a small sphere sedimenting in a quiescent fluid does not increase monotonically as we proceed outward from the cylinder axis towards the wall. Rather, it attains a minimum value at some

intermediate point. This can be seen quite clearly from (8.1) which, in present circumstances, has the form

$$\mathbf{W} = -\mathbf{i}_z \, 6\pi\mu a U[1 + f(b/R_0)(a/R_0) + ...]. \qquad (9.2)$$

As we proceed away from the axis, the function, $f(b/R_0)$, and hence the drag, decreases initially as required by (6.7); however, as the wall is approached, equation (6.8) shows that $f(b/R_0)$ ultimately increases like $1/(1 - b/R_0)$. Evidently, there is some intermediate value of eccentricity for which the drag is least. More extensive calculations are needed to locate this point with certainty. Experimental verification of the existence of a minimum may be a matter of considerable difficulty if it occurs near the axis, because of the relative magnitude of the 'eccentricity' term in (6.7) (for small values of (b/R_0)) contrasted with the lead term.

One might suspect that the results for the sedimentation of a small particle through a quiescent fluid, near the cylinder wall, would be exactly the same as for a sphere settling in the vicinity of a plane wall. Surprisingly, this is not the case. Lorentz's (1907) results for the latter problem are

$$\mathbf{W} = -\mathbf{i}_z \, 6\pi\mu a U \left[1 + \frac{9}{16} \left(\frac{a}{h} \right) + ... \right], \qquad (9.3)$$

while Faxen (1923) gives

$$\mathbf{L} = -\mathbf{i}_x \, 8\pi\mu a^2 U \left[\frac{3}{32} \left(\frac{a}{h} \right)^4 - \frac{45}{512} \left(\frac{a}{h} \right)^5 + ... \right], \qquad (9.4)$$

where h is the perpendicular distance from the sphere centre to the wall. These should be compared with equations (8.4) and (8.5), respectively, which, when we set $U_0 = 0$ and utilize (6.8) and (6.14), take the following form near the cylinder walls:

$$\mathbf{W} = -\mathbf{i}_z \, 6\pi\mu a U \left[1 + \frac{3}{4} \left(\frac{a}{R_0 - b} \right) + ... \right], \qquad (9.5)$$

and

$$\mathbf{L} = -\mathbf{i}_x \, 8\pi\mu a^2 U \left[\frac{3}{8} \left(\frac{a}{R_0 - b} \right)^2 + ... \right]. \qquad (9.6)$$

The distances from the wall, h and $R_0 - b$, are comparable. Thus, in the case of frictional resistance, the difference between the two results is simply a numerical coefficient as regards the first power of the ratio of sphere radius to distance from wall. The discrepancy in rotational moment is, however, of a more fundamental nature, since it involves differences in the exponents of the ratios, a/h and $a/(R_0 - b)$.

Experimental studies aimed at the verification of the foregoing equations are currently being conducted at New York University.

The treatment of the present problem by means of the linearized Navier–Stokes equations fails to reveal the presence of sidewise forces tending to move the sphere towards the tube centre. In any real situation, 'Bernoulli' forces tending to produce this result must exist. This lack of sidewise forces is a characteristic failing of the creeping motion equations, and stems from a neglect of fluid inertia in the original equations of motion. Since our present results show the force to be absent in the limiting case

as the Reynolds number tends to zero, it appears that, at low Reynolds numbers, this force should be proportional to the relative particle Reynolds number discussed in §2. Other considerations show this force to be an odd function of b/R_0.

It would be interesting to repeat the present study using the Oseen (1928) equations, which partially take account of fluid inertia. This, in conjunction with a corresponding experimental investigation of sidewise forces, would provide a stringent test of the Oseen equations. Here, the test would not be obscured by the fact that this approximate inertial effect is merely a second-order effect superimposed upon the primary viscous effect, as in Oseen's (1928) correction of Stokes's law. In this connection, it is of interest to note that the Oseen equations have already been employed by Faxen (1923) (see, also, Oseen (1928)), for the case of a sphere moving along the axis of a cylinder through a quiescent fluid. This, of course, does not provide a test in the sense of the above.

Appendix

Calculation of $f(\beta)$ and $g(\beta)$ as $\beta \to 0$ and $\beta \to 1$

The expression for $f(\beta)$ given in (6.5) can be put in more tractable form. This is done by multiplying each side of the identity,

$$\frac{d}{d\alpha}\left[K_k(\alpha)/I_k(\alpha)\right] = -1/[\alpha I_k^2(\alpha)] \tag{A 1}$$

by $\alpha I_k^2(\alpha\beta)$ and partially integrating the result, thereby obtaining

$$\int_0^\infty \frac{K_k(\alpha)}{I_k(\alpha)}\left[I_k^2(\alpha\beta) + 2\alpha\beta I_k(\alpha\beta)I_k'(\alpha\beta)\right]d\alpha = \left[\alpha I_k^2(\alpha\beta)\frac{K_k(\alpha)}{I_k(\alpha)}\right]_0^\infty +$$
$$+ \int_0^\infty \left[\frac{I_k(\alpha\beta)}{I_k(\alpha)}\right]^2 d\alpha.$$

The function in brackets vanishes at the upper and lower limits, and thus

$$f(\beta) = \frac{3}{2\pi}\sum_{k=-\infty}^\infty \int_0^\infty \left\{\frac{K_k(\alpha)I_k^2(\alpha\beta)}{I_k(\alpha)} - \right.$$
$$\left. - \frac{\delta_k(\alpha)}{I_k^2(\alpha)}[I_{k+1}(\alpha)I_k(\alpha\beta) - \beta I_k(\alpha)I_{k+1}(\alpha\beta)]^2\right\} d\alpha. \tag{A 2}$$

A series development of $f(\beta)$ can be carried out by expanding the Bessel functions of argument $\alpha\beta$ in power series and evaluating the remaining integrals involving α by numerical techniques. An expression correct to powers in β^2 requires that we only retain the terms corresponding to $k = 0, 1$ and -1 in the infinite summation. To this degree of approximation, we obtain

$$f(\beta) = f(0) + \tfrac{3}{2}(\beta^2/\pi)(\tfrac{1}{6}Q_1 - \tfrac{1}{2}Q_2 + Q_3 - \tfrac{1}{2}Q_4).$$

The lead term, $f(0)$, has already been evaluated by Happel & Byrne (1954) as well as by Haberman (1956), in connection with the motion of a sphere along the cylinder axis. Their result (which has been checked) is

$$f(0) = \frac{3}{2\pi} \int_0^\infty \left[\frac{1 - \delta_0(\alpha)I_1^2(\alpha)}{I_0^2(\alpha)} \right] d\alpha = 2 \cdot 1047.$$

Numerical integration of the remaining integrals gives

$$Q_1 = \int_0^\infty \alpha^2 [\{I_0^2(\alpha)\}^{-1} + \{I_1^2(\alpha)\}^{-1}] \, d\alpha = 9 \cdot 4488,$$

$$Q_2 = \int_0^\infty \frac{\delta_0(\alpha)\alpha^2 I_1^2(\alpha)}{I_0^2(\alpha)} \, d\alpha = -5 \cdot 4094,$$

$$Q_3 = \int_0^\infty \frac{\delta_0(\alpha)\alpha I_1(\alpha)}{I_0(\alpha)} \, d\alpha = -7 \cdot 2013,$$

$$Q_4 = \int_0^\infty \frac{\delta_1(\alpha)\alpha^2 I_2^2(\alpha)}{I_1^2(\alpha)} \, d\alpha = -2 \cdot 9211.$$

These combine to give the expression for $f(\beta)$ cited in (6.7). The modified Bessel functions of the second kind, which otherwise cause difficulty in numerical integration because of their behaviour at $\alpha = 0$, have been eliminated by resorting to (A1) and variations thereof.

To obtain the limiting form as $\beta \to 1$, we note from (A2) that

$$\lim_{\beta \to 1}(1 - \beta)f(\beta) = \frac{3}{2\pi} \lim_{\beta \to 1}(1 - \beta) \sum_{k=-\infty}^{\infty} \int_0^\infty \frac{K_k(\alpha)I_k^2(\alpha\beta)}{I_k(\alpha)} \, d\alpha.$$

But, because of the monotonicity of the function $I_k(x)$ in conjunction with its essentially positive nature, one can demonstrate that

$$K_0[\alpha(1 - \beta)] \geq \sum_{k=-\infty}^{\infty} \frac{K_k(\alpha)I_k^2(\alpha\beta)}{I_k(\alpha)}, \tag{A3}$$

by putting $\Phi = 0$ in (4.7) and (4.6). However,

$$\int_0^\infty K_0[\alpha(1 - \beta)] \, d\alpha = \frac{\pi}{2(1 - \beta)}, \tag{A4}$$

from which the limiting form given in (6.8) follows without difficulty.

A series expansion of $g(\beta)$, defined in (6.12), correct to the first power of β gives

$$g(\beta) = -\tfrac{3}{2}(\beta/\pi)(\tfrac{1}{6}Q_1 + \tfrac{1}{2}Q_3 + \tfrac{1}{2}Q_5),$$

where

$$Q_5 = \int_0^\infty \frac{\delta_1(\alpha)\alpha^2 I_2^2(\alpha)}{I_1(\alpha)[I_1(\alpha) + \alpha I_2(\alpha)]} \, d\alpha = -1 \cdot 3788.$$

This results in (6.13).

Finally, from (6.12),

$$\lim_{\beta \to 1}(1 - \beta)^2 g(\beta) = -\frac{3}{4\pi} \lim_{\beta \to 1}(1 - \beta)^2 \frac{\partial}{\partial \beta} \sum_{k=-\infty}^{\infty} \int_0^\infty \frac{K_k(\alpha)I_k^2(\alpha\beta)}{I_k(\alpha)} \, d\alpha.$$

Resorting to (A3) and (A4) it is now an easy matter to verify the correctness of (6.16).

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