Network Evolution

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In this report we study the evolution of a network of porous media through erosion/clogging process.

1 Toy Model 1 - Erosion

In each pipe with radius R which acts as a resistor, we have Poissoulle flow equation that relates the flow Q to pressure difference ΔP as

$$Q = \frac{-\pi r^4}{8\mu L}\Delta P \tag{1}$$

$$Q = -C(r)\Delta P, \quad C(r) = \frac{\pi r^4}{8\mu L} = \beta r^4 \tag{2}$$

We now change the radius of the pipes according to

$$\frac{dr}{dt} = \alpha \frac{Q}{r^n} \tag{3}$$

1.1 Two in Series

Now assume two pipes back to back as shown in Fig. 1. If the two pipes are in series, then

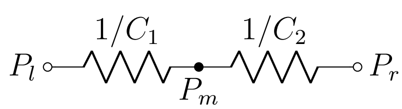

Figure 1: Two resistors at series

Q is always a constant. Therefore

$$\frac{dr}{dt} = \alpha\frac{Q}{r^n} \tag{4}$$

$$r^n dr = \alpha Q dt \tag{5}$$

$$\frac{1}{n+1}r^{n+1} - \frac{1}{n+1}r_0^{n+1} = \alpha Q t \tag{6}$$

$$r = \left((n+1)\alpha Q t + r_0^{n+1}\right)^{\frac{1}{n+1}} \tag{7}$$

$$C_i = \beta\left((n+1)\alpha Q t + r_{i,0}^{n+1}\right)^{\frac{4}{n+1}} \tag{8}$$

So the pressure at the middle point becomes as

$$Q = \frac{1}{1/C_1 + 1/C_2}\Delta P = \text{const.}, \tag{9}$$

$$P_m - P_l = \frac{Q}{C_1} \tag{10}$$

$$P_r - P_l = \frac{1}{1/C_1 + 1/C_2}Q \tag{11}$$

$$\frac{P_m - P_l}{P_r - P_l} = \frac{1/C_1}{1/C_1 + 1/C_2} = \frac{C_2}{C_1 + C_2} \tag{12}$$

Combining the results we find that

$$\boxed{\frac{P_m - P_l}{P_r - P_l} = \frac{\beta\left((n+1)\alpha Q t + r_{2,0}^{n+1}\right)^{\frac{4}{n+1}}}{\beta\left((n+1)\alpha Q t + r_{1,0}^{n+1}\right)^{\frac{4}{n+1}} + \beta\left((n+1)\alpha Q t + r_{2,0}^{n+1}\right)^{\frac{4}{n+1}}}} \tag{13}$$

$$\lim_{t\to\infty}\frac{P_m - P_l}{P_r - P_l} = \frac{1}{2} \tag{14}$$

Note that if $n+1 < 4$ (i.e. $n < 3$) then the growth is much faster, and the limit is reached much faster; however, if $n \geq 3$ the growth becomes slower. The result of simulation is shown in the Fig.

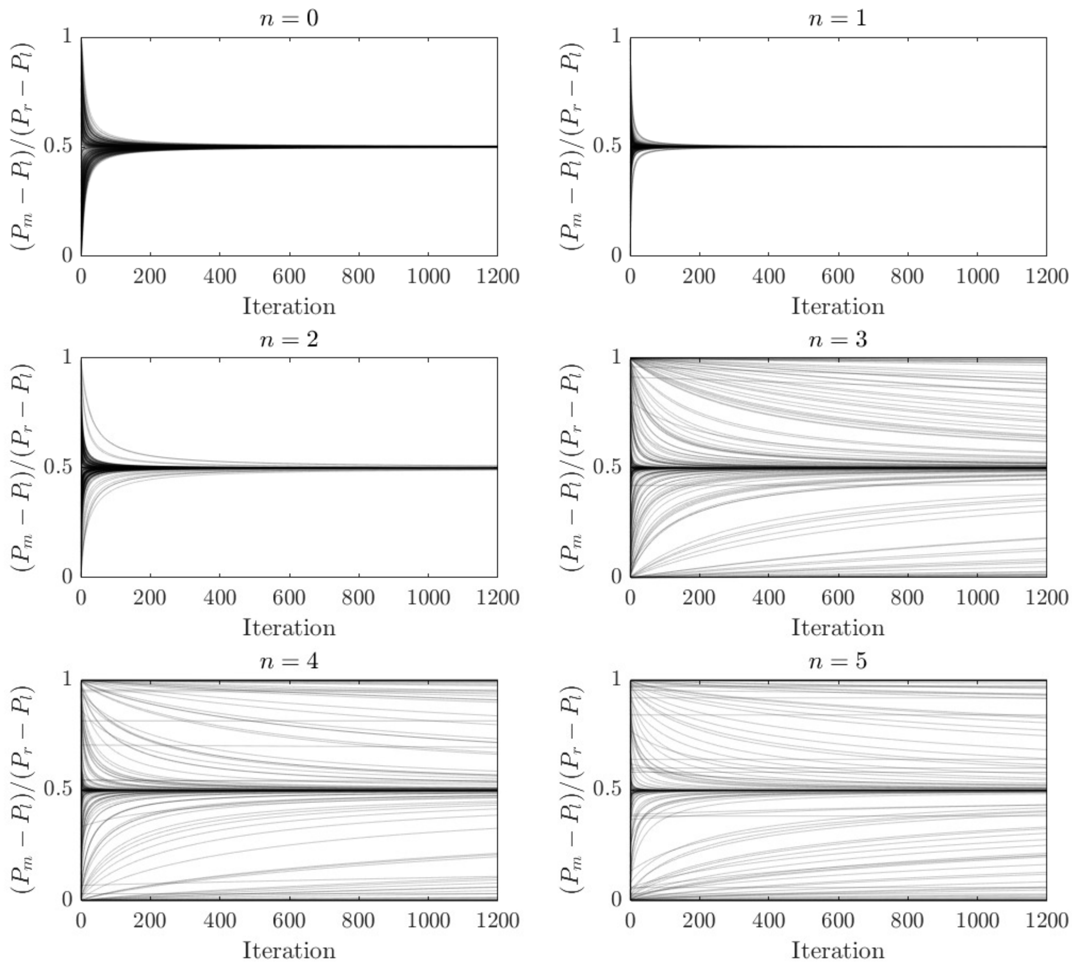

Figure 2: Toy model for two pipes in seris. In each iteration, we change the radius of the pipe as $r_{\text{new}} = r_{\text{old}} + \alpha Q / r_{\text{old}}^n$

1.2 Two Parallel Pipes Case

Now assume two pipes in parallel as shown in Fig. 3. In this case, first we have

$$Q_1 = C_1 \Delta P, \quad \text{and} \quad Q_2 = C_2 \Delta P \tag{15}$$

$$Q = (C_1 + C_2)\Delta P \tag{16}$$

We change the radius of each pipe according to its flow, as a result, we have

$$\frac{dr_1}{dt} = \alpha \frac{Q_1}{r_1^n}, \qquad \frac{dr_2}{dt} = \alpha \frac{Q_2}{r_2^n} \tag{17}$$

$$r_1^n \frac{dr_1}{dt} + r_2^n \frac{dr_2}{dt} = \alpha (Q_1 + Q_2) = \alpha Q \tag{18}$$

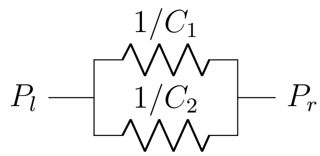

Figure 3: Two resistors parallel together

On the other hand, we know that

$$\Delta P = \frac{Q_1}{C_1} = \frac{Q_2}{C_2} \tag{19}$$

$$\Delta P = \frac{Q_1}{\beta r_1^4} = \frac{Q_2}{\beta r_2^4} \tag{20}$$

$$\frac{dr_i}{dt} = \alpha \frac{Q_i}{r_i^n} = \frac{\alpha\beta\Delta P}{r_i^{n-4}} \tag{21}$$

We have

$$r_1^{n-4}\frac{dr_1}{dt} = r_2^{n-4}\frac{dr_2}{dt} \tag{22}$$

$$\text{if } n = 3, \quad \rightarrow \log(\frac{r_1}{r_{1,0}}) = \log(\frac{r_2}{r_{2,0}}), \rightarrow r_1 = \gamma r_2 \tag{23}$$

$$\text{if } n \neq 3, \quad r_1^{n-3} - r_2^{n-3} = \gamma \tag{24}$$

where γ is a constant. In the above equation, without loss of generality, we can assume that $r_1 > r_2$ and γ is positive. Now, looking at the other part of the equation, we have

$$\frac{dr_2}{dt} = \alpha \frac{Q_2}{r_2^4} = \frac{\alpha\beta\Delta P}{r_2^{n-4}} \tag{25}$$

$$= \frac{\alpha\beta}{r_2^{n-4}} \frac{Q}{\beta(r_2^4 + r_1^4)} = \frac{\alpha Q}{r_2^n + r_2^{n-4} r_1^4} \tag{26}$$

$$= \frac{\alpha Q}{r_2^n + r_2^{n-4}\left(r_2^{n-3} + \gamma\right)^{\frac{4}{n-3}}} \tag{27}$$

$$\left[r_2^n + r_2^{n-4}\left(r_2^{n-3} + \gamma\right)^{\frac{4}{n-3}}\right] dr_2 = \alpha Q dt \tag{28}$$

$$\frac{1}{n+1}\left[r_2^{n+1} + \left(r_2^{n-3} + \gamma\right)^{\frac{n+1}{n-3}}\right] = \alpha Q t + C \tag{29}$$

$$r_2^{n+1} + \left(r_2^{n-3} + \gamma\right)^{\frac{n+1}{n-3}} = (n+1)\left(\alpha Q t + C\right) \tag{30}$$

$$r_2^{n+1} + r_1^{n+1} = At + B \tag{31}$$

So the two set of equations summarizes to

$$r_1^{n-3} - r_2^{n-3} = \gamma \tag{32}$$

$$r_1^{n+1} + r_2^{n+1} = At + B \tag{33}$$

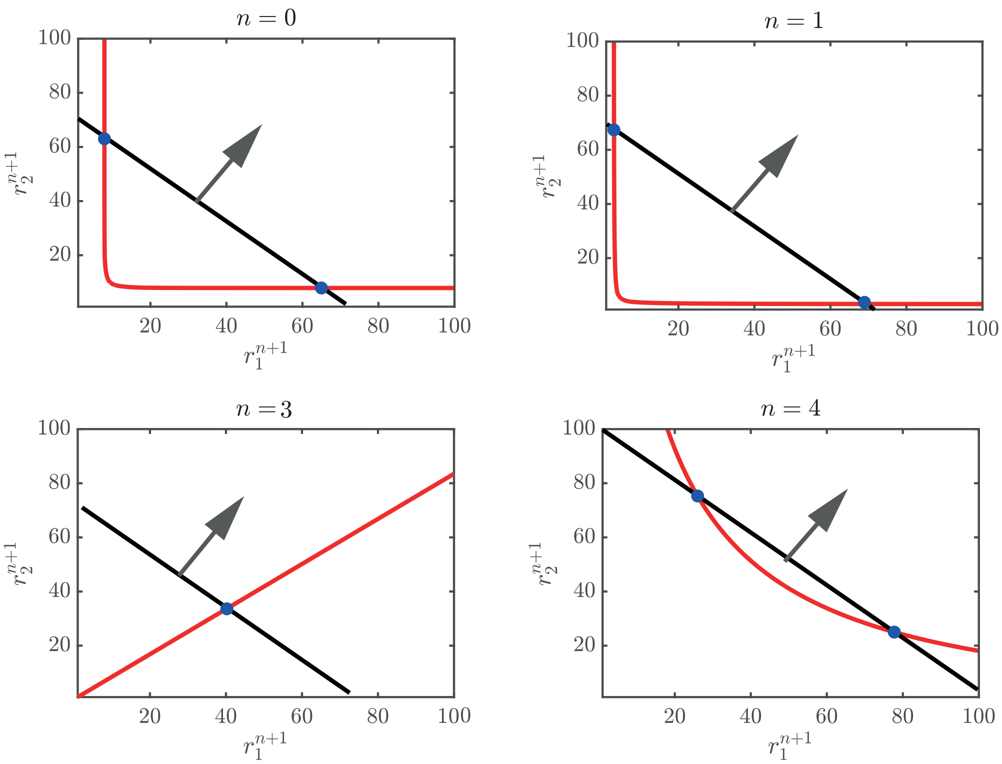

Figure 4: Analytical solution for r_1 and r_2 for parallel resistors. For different n the solution changes. Specifically, if $n < 3$, then one of the diameters stops varying while the other one increases in size. If $n = 3$, the both grow together, and if $n > 3$ then they both increase in size.

The plot for the solution to the above equations for different values of $n \neq 3$ and also the separate solution for $n = 3$ is shown in the following figure.

The result for numerical simulation is shown in Fig. 5.

Another way of seeing this and also generalize the results is to study the change of relative radius $\tilde{r}_1 = \frac{r_1}{r_1+r_2}$ under the assumptions

$$C_i = r_i^m, \; \frac{dr_i}{dt} = \alpha \frac{Q_i}{r_i^n}. \tag{34}$$

It can be easily derived that

$$\frac{d\tilde{r_1}}{dt} \propto \tilde{r}_1(1-\tilde{r}_1)\frac{r_1^{m-n-1} - r_2^{m-n-1}}{r_1^m + r_2^m}, \tag{35}$$

which has a fixed point at either 1. $\tilde{r}_1 = 0$ or 1 or 2.$r_1 = r_2$. The first case is the channeling and the second case is the uniform. It can also be shown that if $m - n - 1 > 0$ the system ends at channelling, otherwise uniform except when $m - n - 1 = 0$ it will end up with no change of the relative radius.

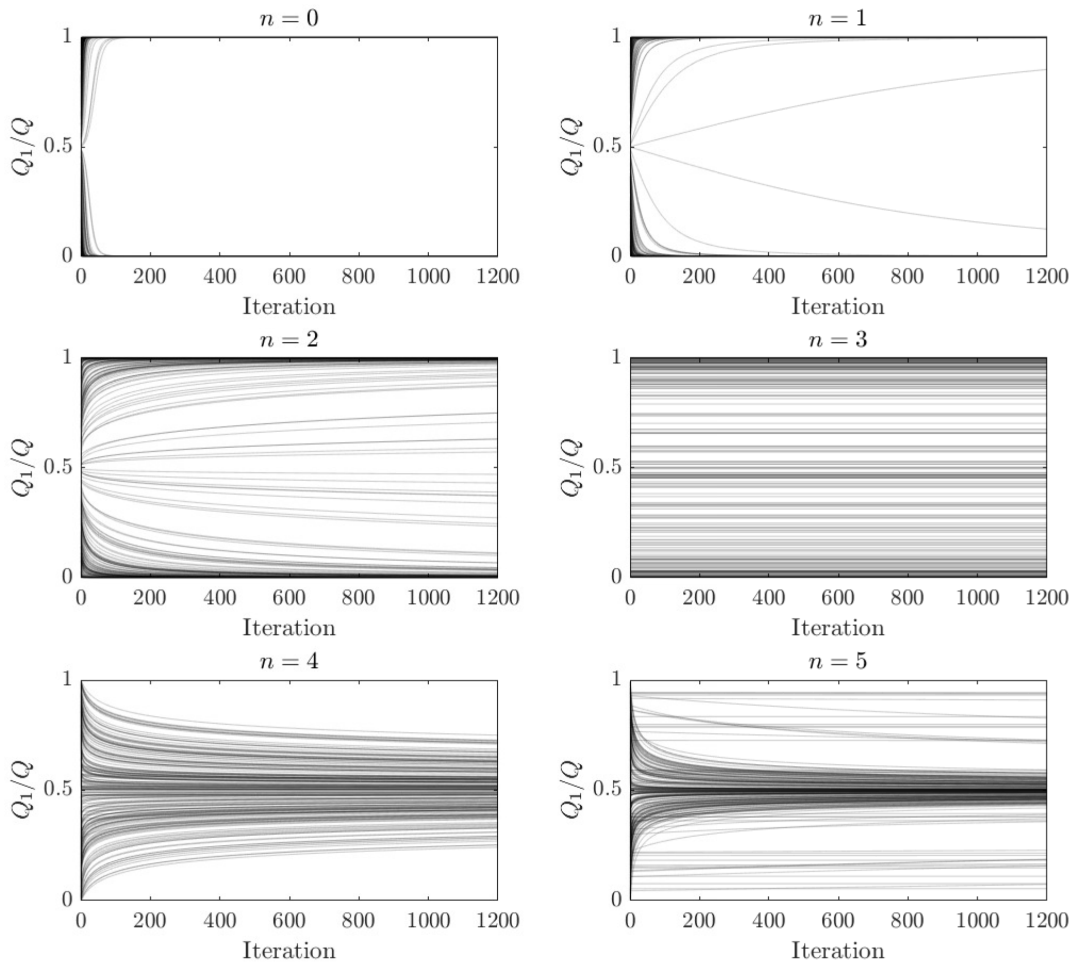

Figure 5: Toy model for two pipes in parallel. In each iteration, we change the radius of the pipe as $r_{\text{new}} = r_{\text{old}} + \alpha Q_i / r_{\text{old}}^n$

1.3 Parallel and Series

Now, we assume a combination of both parallel and series case as shown in Fig. 6. The top and bottom series part, become more and more homogeneous for any flow that passes through them. However, the flow between top and bottom separates based on the power of n.

1.3.1 Corrections (Deng)

Numerical simulations show different result here for more complex cases other than naive parallel or series cases. For example for the following case Fig **??**. The simulations are similar to previous ones that keep the flow constant while increasing the radius proportional to Q/r^n. It now shows that the flow does not need to be homogeneous through top and bottom tubes.

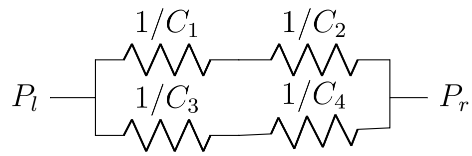

Figure 6: Parallel/Series Case

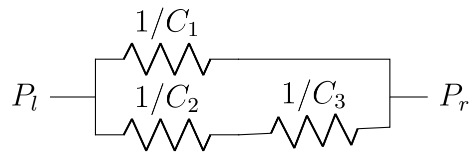

Figure 7: Parallel/Series Case (Deng)

A special case can be easily analytically calculated. Set $C_{10} = 2, C_{20} = C_{30} = 1$, and let $n = 7$, in this case

$$\frac{dC_i}{dt} = Q/C_i \tag{36}$$

and $C_2 = C_3$ will always hold, and it is not difficult to solve C_1 and C_2

$$\frac{dC_1}{dt} = \frac{2Q}{C_2 + C_1} \tag{37}$$

$$\frac{dC_2}{dt} = \frac{Q}{C_2 + C_1}, \tag{38}$$

where Q is the total flow. In fact we do not need to solve these simple equations and it is clear that at the end we get $C_2 = C_3 = 1/2C_1$ since $\frac{dC_1}{dC_2} = 2$. However, this means at the end $Q_1/Q = 4/5$ instead of $1/2$.

2 Toy Model - Clogging

$$\frac{dr}{dt} = -\alpha \frac{Q}{r^n} \tag{39}$$

2.1 Series - Numerical result

We assume the following pattern for the resistors 7 The numerical simulation results are

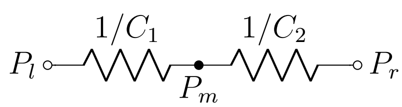

Figure 8: Two resistors at series

shown in Fig. 8.

2.2 Parallel - Numerical result

We assume the following pattern for the resistors 9

The numerical simulation results are shown in Fig. 10.

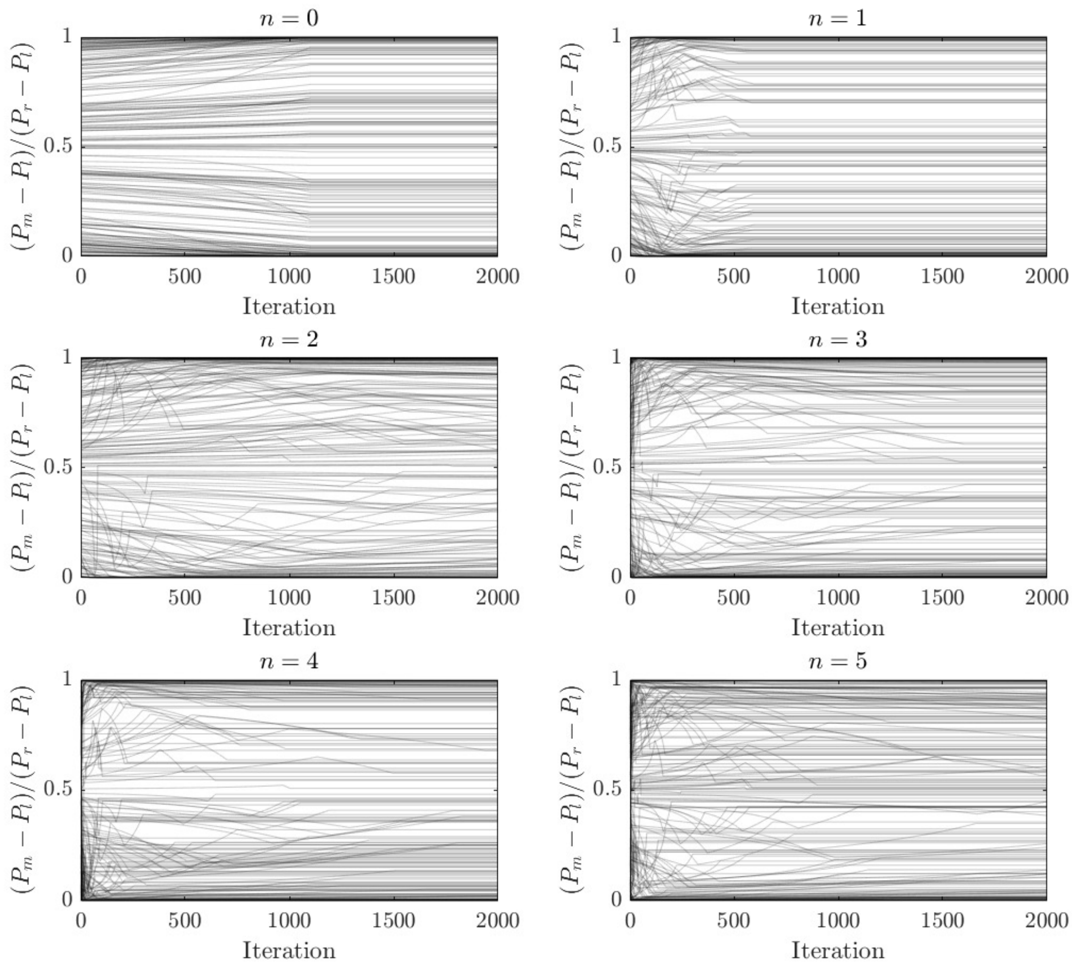

Figure 9: Toy model for two pipes in series. In each iteration, we change the radius of the pipe as $r_{\text{new}} = r_{\text{old}} - \alpha Q_i / r_{\text{old}}^n$. Note that we set a maximum change of $\Delta r = \min 1, r$.

3 Localization Degree

In channelling, sparsity of flow occurs. In other words, most of the flow passes through few pipes. The measures already used to calculate sparsity are the following

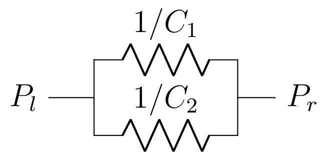

Figure 10: Two resistors parallel together

Measure	Definition	Description
ℓ^0_ϵ	$\#\{j, q_j \leq \epsilon\}$	
$-\ell^p$	$-\left(\sum_j q_j\right)^{1/p}$	
ℓ^2/ℓ^1	$\sqrt{\sum_j q_j^2}/\sum_j q_j$	
κ_4	$\sum_j q_j^4/\left(\sum_j q_j^2\right)^2$	
$-\log$	$-\sum_j \log\left(1+q_j^2\right)$	
$-\tanh_{a,b}$	$-\sum_j \tanh\left(aq_j^b\right)$	

The following measures I propose based on our problem

$$\mathcal{L}_1 = \frac{1}{N-1}\left(N - \frac{\left(\sum_j q_j\right)^2}{\sum_j q_j^2}\right) \tag{40}$$

$$\mathcal{L}_2 = \frac{1}{N-1}\left(N - \frac{\left(\sum_j q_j^2\right)^2}{\sum_j q_j^4}\right) \tag{41}$$

In the above measures if q is uniform and the same everywhere, then the measure is $\mathcal{L}_1 = \mathcal{L}_\in = 0$. The extreme heterogeneity case the measure goes toward 1. The resulting videos are Movie S1 and S2.

4 Generalization of conductance erosion law (Deng)

Here is a naive guess of how the exponent of erosion law can change the network topology. We can assume the erosion law

$$C = ar^m \tag{42}$$

$$\frac{dr}{dt} = \frac{bQ}{r^n}, \tag{43}$$

where a, b, m, n are some constants. It can be derived that it is equivalent to

$$\frac{dC}{dt} \propto QC^{1-\frac{n+1}{m}}. \tag{44}$$

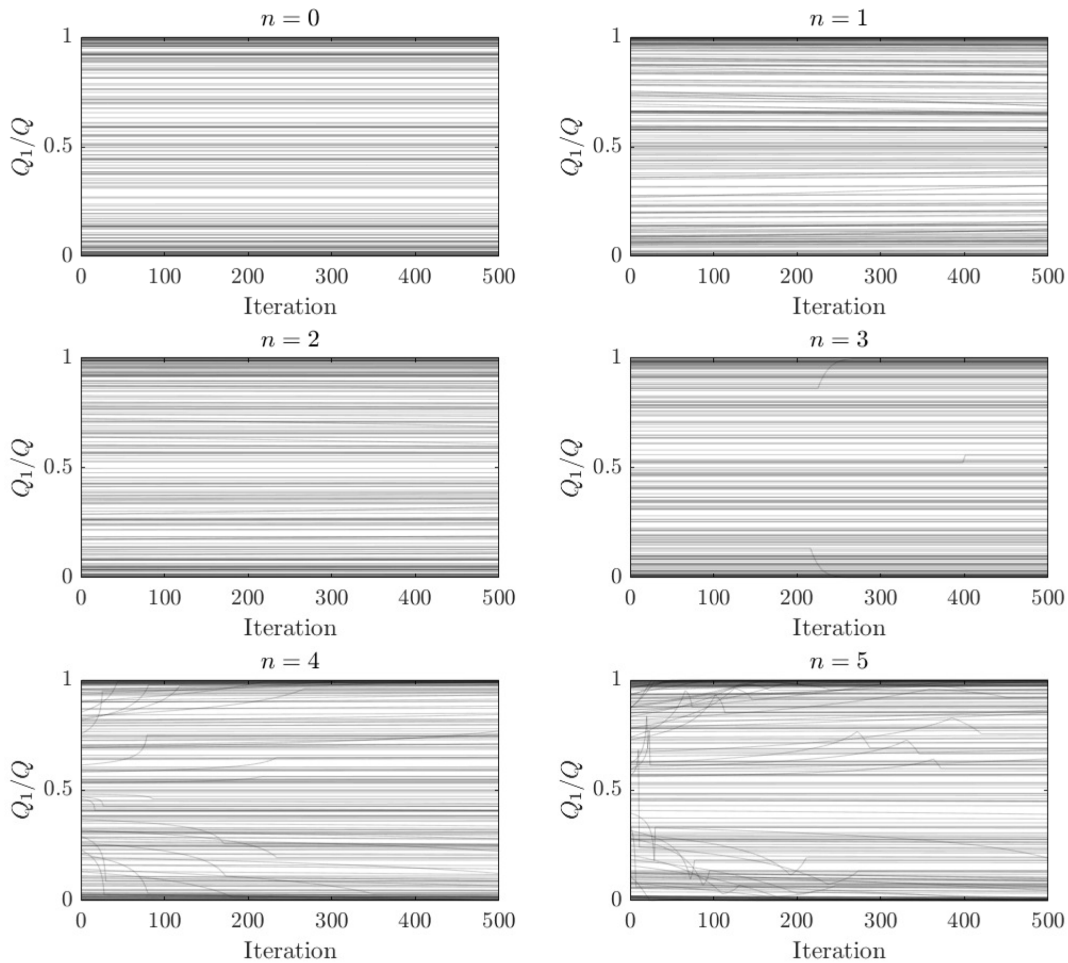

Figure 11: Toy model for two pipes in series. In each iteration, we change the radius of the pipe as $r_{\text{new}} = r_{\text{old}} - \alpha Q_i / r_{\text{old}}^n$. Note that we set a maximum change of $\Delta r = \min 1, r$.

Particularly in previous series/parallel cases, $m = 4$ and n varies. We observed that $n < 3$ leads to channeling while $n > 3$ leads to homogeneity. Therefore, a reasonable guess is with the erosion law

$$\frac{dC}{dt} \propto Q^{\alpha} C^{\beta}, \tag{45}$$

when $\alpha = 1$, the sign of β will determine the channeling behavior. And the simulations as shown in Fig. **??** confirm the guess.

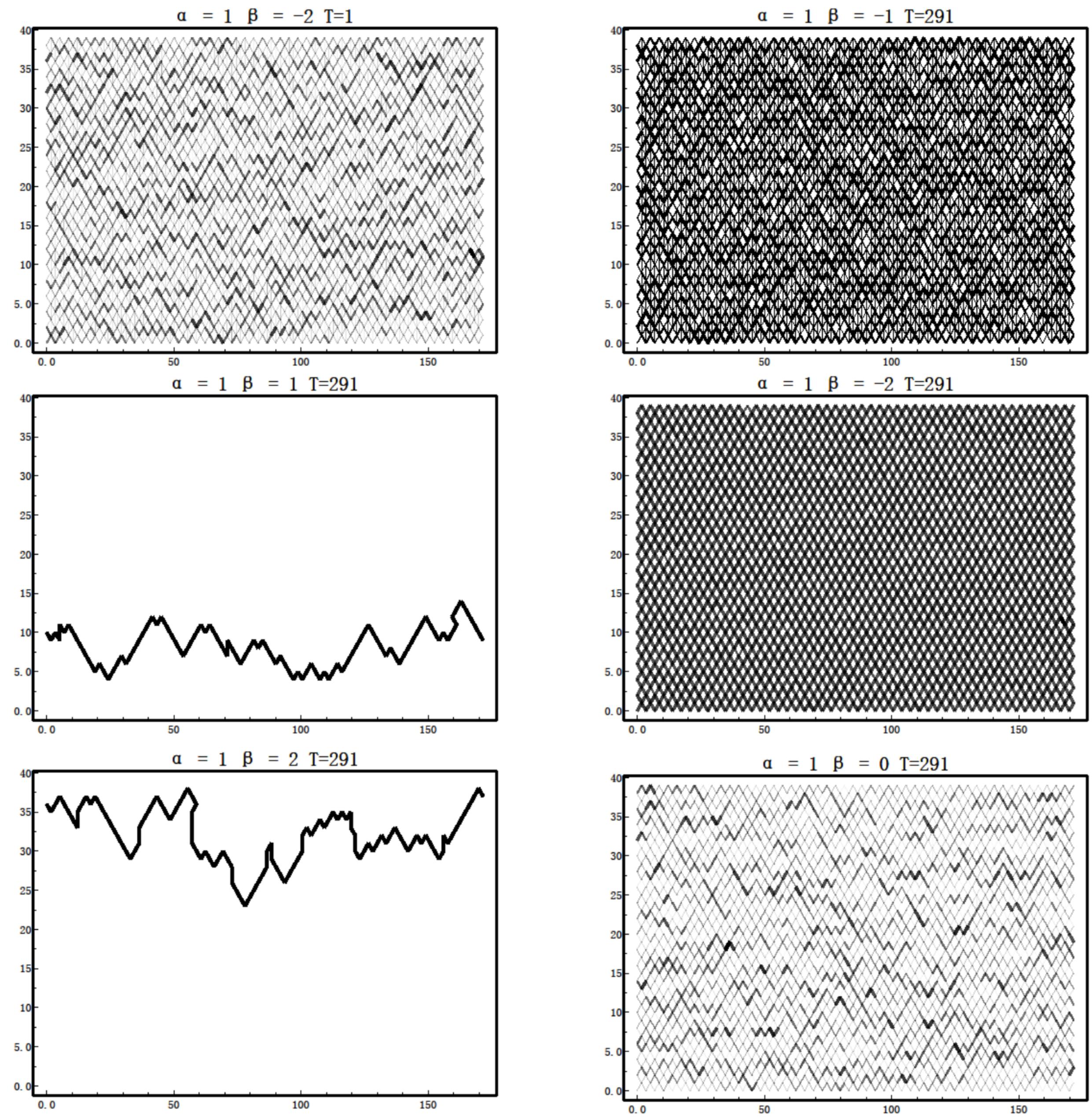

Figure 12: A grid network erosion simulations with different β exponents. Top left is the initial flow distribution for every simulation (T=1) and the rest are final distributions of the flow.