CMPT361 Assignment 1 Math Portion

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February 17, 2016

1. Since R is orthonormal, $R^{-1} = R^T =$

$$\begin{bmatrix} r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \\ r_{13} & r_{23} & r_{33} \end{bmatrix}$$

Since M is of the form

$$\begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix}$$

, and

$$\begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \times M^{-1} = I$$

M^{-1} should look like this:

$$\begin{bmatrix} R^T & -R^T t \\ 0^T & 1 \end{bmatrix}$$

$M^{-1} =$

$$\begin{bmatrix} r_{11} & r_{21} & r_{31} & -(r_{11}t_1 + r_{12}t_2 + r_{13}t_3) \\ r_{12} & r_{22} & r_{32} & -(r_{21}t_1 + r_{22}t_2 + r_{23}t_3) \\ r_{13} & r_{23} & r_{33} & -(r_{31}t_1 + r_{32}t_2 + r_{33}t_3) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2. Since p_1 and p_2 are vectors in $\mathbb{R}^3$, let $p_1 = [x_1, x_2, x_3]$ and $p_2 = [y_1, y_2, y_3]$ where $x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R}$.

$\Phi(p_1 - p_2) =$

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x_1 - y_1 \\ x_2 - y_2 \\ x_3 - y_3 \\ 1 \end{bmatrix} = \begin{bmatrix} r_{11}(x_1 - y_1) & r_{12}(x_2 - y_2) & r_{13}(x_3 - y_3) & t_1 \\ r_{21}(x_1 - y_1) & r_{22}(x_2 - y_2) & r_{23}(x_3 - y_3) & t_2 \\ r_{31}(x_1 - y_1) & r_{32}(x_2 - y_2) & r_{33}(x_3 - y_3) & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$=$

$$\begin{bmatrix} (r_{11}x_1) - (r_{11}y_1) & (r_{12}x_2) - (r_{12}y_2) & (r_{13}x_3) - (r_{13}y_3) & t_1 \\ (r_{21}x_1) - (r_{21}y_1) & (r_{22}x_2) - (r_{22}y_2) & (r_{23}x_3) - (r_{23}y_3) & t_2 \\ (r_{31}x_1) - (r_{31}y_1) & (r_{32}x_2) - (r_{32}y_2) & (r_{33}x_3) - (r_{33}y_3) & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\Phi(p_1) - \Phi(p_2) =$

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{bmatrix} - \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ 1 \end{bmatrix}$$

$=$

$$\begin{bmatrix} r_{11}x_1 & r_{12}x_2 & r_{13}x_3 & t_1 \\ r_{21}x_1 & r_{22}x_2 & r_{23}x_3 & t_2 \\ r_{31}x_1 & r_{32}x_2 & r_{33}x_3 & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} r_{11}y_1 & r_{12}y_2 & r_{13}y_3 & t_1 \\ r_{21}y_1 & r_{22}y_2 & r_{23}y_3 & t_2 \\ r_{31}y_1 & r_{32}y_2 & r_{33}y_3 & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$=$

$$\begin{bmatrix} (r_{11}x_1) - (r_{11}y_1) & (r_{12}x_2) - (r_{12}y_2) & (r_{13}x_3) - (r_{13}y_3) & 0 \\ (r_{21}x_1) - (r_{21}y_1) & (r_{22}x_2) - (r_{22}y_2) & (r_{23}x_3) - (r_{23}y_3) & 0 \\ (r_{31}x_1) - (r_{31}y_1) & (r_{32}x_2) - (r_{32}y_2) & (r_{33}x_3) - (r_{33}y_3) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since LHS = RHS in $\mathbb{R}^3$, Φ is linear in 3D

3. Let A be an arbitrary vector in $\mathbb{R}^3$, $A = [x, y]$
$\|A\| = \|x - y\| = < [x - y].[x - y] >$ $=[x - y]^T[x - y]$
x' = Rx + t
y' = Ry + t
Therefore, MA = [x'-y'] = R[x-y]
$\|MA\|$=$\|R[x - y]\| = < R[x - y].R[x - y] >$
$= [x - y]^T R^T R[x - y]$
=$[x - y]^T[x - y] = \|A\|$
Hence, M preserves vector length.

Since M preserves vector length. For an arbitrary triangle ABC, AB = MAB, BC = MBC, AC = MAC. Therefore, M preserves area of triangles

Let A,B be 2 vectors in $\mathbb{R}^3$. A = [a,c] B = [b,d] where a,b,c,d are points in $\mathbb{R}^3$.
$< A.B > = < [a - c].[b - d] > = [a - c]^T[a - c]$
a'=Ra + t
b'=Rb + t
c'=Rc + t
d'=Rd + t
$< MA.MB > = < [a' - c'].[b' - d'] > = < R[a - c], R[b - d] > = [a - c]^T R^T R[b - d] = [a - c]^T[a - c] = < A.B >$
Since $< MA.MB > = < A.B >$, $\frac{cos(\angle AB)}{(\|A\|\|b\|)} = \frac{cos(\angle MAMB)}{(\|MA\|\|MB\|)}$. Since we've already established $\|A\|\|b\| = \|MA\|\|MB\|$, $cos(\angle AB) = cos(\angle MAMB)$. Therefore, $\angle AB = \angle MAMB$. M preserves angle.