Everything You Need To Know About K-Theory

Nathanael Chwojko-Srawley

February 1, 2025

Contents

All ring swill be associative with $1 \neq 0$. An R-module will be a right R-module unless specified. We do this so that a homomorphism $R^n \to R^m$ may be thought of as an $m \times n$ matrix with entries in R acting on column in R^n.

1

Projective modules and Vector bundles

If R is a field or a divisor ring, every R-module is a vector-space which is algebraically characterized up to dimension. More generally, a free R-module is a module that is R-module isomorphic to $R^{\oplus I}$ for same indexing set I. Any R-module homomorphism between two free modules is based on where chosen basis of the domain is represented by the chosen basis of the codomain; namely it can be represented by a matrix (or the generalization thereof for infinite dimension).

1.1 IBP, Stably Free, and Projective Modules

If R is a field or a divisor ring, then $R^n \cong R^m$ implies $n = m$. If R is commutative this is still the case (consider $R/\mathfrak{m}$ by a maximal ideal), group rings (for finite groups), and any (left)-Noetherian ring satisfies this property as well (using a version of Nakayama's lemma can prove this, see [2]). This property is *not* true for general rings:

Example 1.1: Failure of Invariant Basis Number

let R be a commutative ring and let M be the $\mathbb{N} \times \mathbb{N}$ matrices where each column has finitely many nonzero entries, this is sometimes denoted $CFM_{\mathbb{N}}(R)$. Then $CFM_{\mathbb{N}}(R) \cong CFM_{\mathbb{N}}(R)^2$ via the map:

$$\varphi : CFM_{\mathbb{N}}(R) \to CFM_{\mathbb{N}}(R)^2 \qquad M \mapsto (\text{odd columns of } M,\ \text{even columns of } M)$$

Another example the reader could work through is $\mathrm{End}_k(k^{\oplus \mathbb{N}})$.

Definition 1.1.1: Invariant Basis Property

Let R be a ring. Then we say that R satisfies the (right) *invariant basis property* (IBP) if

$$R^n \cong R^m \qquad \Rightarrow \qquad n = m$$

If R satisfies IBP, then the rank of a free R-module M is an invariant, independent of the choice of basis.

Proposition 1.1.1: Necessary Condition For Ibp

Let R be a ring. Then R satisfies IBP if there exists a ring map $f : R \to F$ from R to a field (or division ring) F

Proof **:**
Given such a map, notice that any basis for a free R-module M is the basis for he vector-space $V = M \otimes_R F$, and V is independent of choice of basis, and hence any two bases must have the same cardinality.

Kernel's of a surjective R-module homomorphisms $\varphi : R^n \to R^m$ need not be free, however they must be stably free:

Definition 1.1.2: Stably Free

Let P be an R-module. Then P is called *stably free* of rank $n - m$ if there exists n, m such that

$$P \oplus R^m \cong R^n$$

If R satisfies IBP, then the rank of a stably free module is independent of the choice of n and m.

Certainly, the kernel stably free as R^m is free as we must have a split sequence (see [2]). Not all R-modules are stably free (or more interestingly, not all stably free modules are free). The notion of stably free seems very similar to *projective*, however we have not specified the condition for the 2nd summand in the definition of projective modules and hence it is a slightly weaker condition. We shall get to projective modules shortly.

Let us take a moment to consider the case where an R-module P is stably free of rank 1 , that is $P \oplus R \cong R^n$. In this case, we may ask when can we "cancel" the R on each side, a fruitful discussion considering the K-theory we are building up to. In this case, let σ be a row vector of the right hand side, and call it the *unimodular row*. We get the following interesting set of equivalences:

1. σ is unimodular
2. $R^n \cong P \oplus R$ where $P = \ker(\pi)$ where $\pi : R^n \to R$ (given by σ)
3. $R = r_1 R + \cdots + r_n R$
4. $1 = r_1 s_1 + \cdots + r_n s_n$ for appropriate $s_i \in R$

Note that if P where free in $R^n \cong P \oplus R$, then a basis of P would give a new basis for R^n, which corresponds to an invertible matrix (say g) whose first row is the unimodular row $\sigma : R^n \to R$ corresponding to P. From this we get the following citation:

Proposition 1.1.2: Free and Unimodular

Let P be an R-module that stably free of rank 1. Then P is free if and only if the corresponding unimodulr row ma ybe completed to an invertible matrix.

Proof **:**
combine the above discussion.

If R is commutative, then every unimodular row of length 2 may be completed: if $r_1 s_1 + r_2 s_2 = 1$, then the invertible matrix is:

$$\begin{pmatrix} r_1 & r_2 \\ -s_2 & s_1 \end{pmatrix}$$

And so, if$R^2 \cong R \oplus P$, then $P \cong R$. We shall get stronger versions of this result later (note here though that commutativity was important, this fails for unimodular row of length to wit ha non-commutative ring).

In dimension 3, we can already find counter example:

Example 1.2: Stably Free But Not Free

Take

$$R = \frac{\mathbb{R}[x, y, z]}{(x^2 + y^2 + z^2 - 1)}$$

Note that ever element $(f, g, h) \in R^3$ gives a vector-field in $\mathbb{R}^3$. Let σ be the unimodular row $\sigma = (x, y, z)$. Then $1s$ is the vector-field pointing radially outward. If P were free, a basis of P would yield two tangent vector fields on S^2 which are linearly independent at every point of S^2 (since together with σ, they span the tangent space of 3-space at each point). But this contradicts the infamous Hairy-sphere theorem[a].

[a] Or more humorously *Hairy ball theorem*

Let us now return to the discussion of projective modules.

Definition 1.1.3: Projective Module

Let P be an R-module. Then P is *projective* if there exists an R-module Q such that $P \oplus Q$ is a free R-module.

See [2] for details on the equivalence and examples, on important one is that there is always a lift in the following commutative diagram:

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists!"'] \arrow[d, "g"] & \\ M \arrow[r, "f"] & N \arrow[r, "0"] & 0 \end{tikzcd}$$

where f is surjective. Naturally, free and stably free module are projective. A more nuanced example is that all $M_n(k)$-modules (with $n > 1$) are projective but not free (which can be deduced using

Artin-Wederburn). A more canonical set of examples comes from vector bundles. Take $R = C^0(X)$ to be the set of continuous real valued function on a compact topological space X, and let $\Psi : E \to X$ be a vector-bundle. Then the set $\Gamma(E)$ of continuous sections naturally forms a projective R-module. Let us show this (see [1] for the algebraic geometry perspective of the equivalence between vector bundle and locally free sheaves). The trivial bundle is $\mathbb{R}^n \times X \to X$ and we get $\Gamma(\mathbb{R}^n \times X) = R^n$. Now assume $\Psi : E \to X$ is a non-trivial vector-bundles (up to vector-bundle isomorphism). Then if $\Gamma(E)$ were free, there would be sections $\{s_1, ..., s_n\}$ that would for ma basis $f : T^n \to E$ which would give an isomorphism $\Gamma(T^n) = \Gamma(E)$. As the kernel and cokernel of bundles of f have no nonzero sections, the must banish ,and so f would be an isomorphism, and so $\Psi : E \to X$ would in fact be trivial.

If X is compact, then the famous *Swan's Theorem* gives an equivalence between the category of projective modules over $C^0(X)$ and the category of vector bundles over X, showing that vector bundles are the geometric interpretation of projective modules, which motivates similar discussions over sheaves of module for a scheme.

If P is finitely generated, then $P \oplus Q \cong R^n$, meaning there is an idempotent element in P which can be associated to an element of $M_n(R)$. To find this element, we can take the natural projection-inclusion

$$R^n \to P \to R^n$$

which gives an idempotent element $e \in M_n(R)$ (when representing this composition as a matrix under a choice of basis). Labeling this composition e, we see that $e(R^n) = P$.

If P is a projective R-module over a PID, then it is in fact free (as given by the structure theorem for modules over PID[1]). Another famous class of modules where all finitely generated projectives are free comes from Quillen and Suslin stating that finitely generated modules over a polynomial ring (or a Laurent polynomial ring) over a PID is free[2]. This then gives the result mentioned earlier about group rings over finite abelian groups, for $\mathbb{Z}[G]$ is the Laurent polynomial ring $\mathbb{Z}[x, x^{-1}, ..., z, z^{-1}]$. On the other hand, if G is *not* abelian (but has the conditions of being torsion-free and nilpotent), then any projective $\mathbb{Z}[G]$-module P is stably free but not free, in particular that:

$$P \oplus \mathbb{Z}[G] \cong (\mathbb{Z}[G])^2$$

As of writing this book, it is unknown of the condition "nilpotent" can be dropped and produce a similar (or the same) result.

Another important class of finitely generated projective R-module that are always free are those over *local rings*. For informative purposes, the result can in fact be extended to infinitely generated projective R-modules over local rings (Kaplansky gave such a proof). The following is given as a reminder:

Proposition 1.1.3: Projective Over Local Ring, then Free

Let R be a local ring. Then every finitely generated projective Rmodule is free, where

$$P \cong R^n \qquad p = \dim_{R/\mathfrak{m}}(P/\mathfrak{m}P)$$

***Proof* :**
see [2, chapter 20.3]

[1] Note this even works for infinitely generated projective modules, something briefly outlined in some exercises in [2]
[2] This result appears in [1] where it is used to show that all locally free $\mathcal{O}_X$-modules on affine space must be free

Proposition 1.1.4: Free At Stalks

Let R be a commutative ring, $\mathfrak{p}$ a prime ideal, and P a finitely generated projective R-module. Then the localization $P_{\mathfrak{p}}$ is isomorphic to $(R_{\mathfrak{p}})^n$ for some $n \geq 0$. Furthermore, there exists some $s \in R \setminus \mathfrak{p}$ where the localization away from s is free:

$$P\left[\frac{1}{s}\right] \cong R\left[\frac{1}{s}\right]^n$$

And we further get that $P_{\mathfrak{p}'} \cong (R_{\mathfrak{p}'})^n$ for every prime ideal $\mathfrak{p}'$ of R not containing S

Proof **:**
As P is projective, let $P \oplus Q = R^m$. Then $P_{\mathfrak{p}} \oplus Q_{\mathfrak{p}} = (R_{\mathfrak{p}})^m$, hence $P_{\mathfrak{p}}$ is a finitely generated projective $R_{\mathfrak{p}}$-module. As $R_{\mathfrak{p}}$ is local, $P_{\mathfrak{p}}$ is in fact free. Now, every element of $P_{\mathfrak{p}}$ is of the form p/s, for $p \in P$ and $s \in R \setminus \mathfrak{p}$. By choosing a basis for $P_{\mathfrak{p}}$, we may clear the denominators to get an R-module homomorphism $\varphi : R^n \to P$ that become an isomorphism when localizing at $\mathfrak{p}$.

Now, as $\operatorname{coker}(\varphi)$ is a finitely generated R-module which vanishes upon localization, it must be annihilated by some $s \in R \setminus \mathfrak{p}$. Given this s, we get a surjective map:

$$\varphi\left[\frac{1}{s}\right] : \left(R\left[\frac{1}{s}\right]\right)^n \to P\left[\frac{1}{s}\right]$$

As $P\left[\frac{1}{s}\right]$ is projective (as localization preserves projectiveness), $\left(R\left[\frac{1}{s}\right]\right)^n$ is isomorphic to the direct sum of $P\left[\frac{1}{s}\right]$ and a finitely generated $R\left[\frac{1}{s}\right]$-module M (where $M_{\mathfrak{p}} = 0$). As M is annihilated by some $t \in R \setminus \mathfrak{p}$, we get:

$$\varphi\left[\frac{1}{st}\right] : \left(R\left[\frac{1}{st}\right]\right)^n \xrightarrow{\cong} P\left[\frac{1}{st}\right]$$

completing the proof.

We should next define the "vector bundle" structure that we can put on modules. To that end, we will have to assume some algebraic geometry, nothing more than the first chapter on spectra. Let $\operatorname{Spec} R$ be the collection of prime ideals of R, and define the Zariski topology with basic open sets given by:

$$D(s) = \{\mathfrak{p} \in \operatorname{Spec} R \mid s \notin \mathfrak{p}\} \cong \operatorname{Spec} R\left[\frac{1}{s}\right]$$

Definition 1.1.4: Rank At Prime Ideal

Let R be a commutative ring and M be a finitely generated R-module. Then the *rank* at $\mathfrak{p}$ is:

$$\operatorname{rank}_{\mathfrak{p}}(M) = \dim_{\kappa(\mathfrak{p})} M \otimes_R \kappa(\mathfrak{p})$$

Note that $M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}} \cong \kappa(\mathfrak{p})^{\operatorname{rank}_{\mathfrak{p}}(M)}$, hence $\operatorname{rank}_{\mathfrak{p}}(M)$ is the minimal number of generators for $M_{\mathfrak{p}}$. If P is a finitely generated projective R-module, then $\operatorname{rank}(P) : \mathfrak{p} \mapsto \operatorname{rank}_{\mathfrak{p}}(P)$ is a continuous function between the $\operatorname{Spec} R$ with the Zariski topology and $\mathbb{N}$ with the discrete topology.

If $\operatorname{Spec} R$ is connected, then $\operatorname{rank}(P) : \operatorname{Spec} R \to \mathbb{N}$ must always be a constant function, hence every finitely generated projective R-module has constant rank. In the simplest case, R is an integral domain, then the rank is constant with:

$$\operatorname{rank}(P) = \dim_{\operatorname{Frac}(R)}(P \otimes_R \operatorname{Frac}(R))$$

Note that if P is of constant rank, then it is finitely generated[3]. If M is note projective, then $\operatorname{rank}(M)$ need not be continuous: consider $R = \mathbb{Z}$ with $M = \mathbb{Z}/p\mathbb{Z}$.

There are a few more interesting algebraic results that I will ommit for now that could be useful, for example we only need to concern ourselves to projective modules of rank $\leq \operatorname{kdim} R$. It should also be noted that a finitely generated R-module is locally free if and only if it is projective

1.2 Picard Group of a Commutative Ring

Let us now tackle the following. To fully understand the interplay between the algebraic and the geometric, the following theorem must be quoted:

Theorem 1.2.1: Serre-Swan's Theorem

1. **(Serre's Theorem)** Let R be a commutative Noetherian ring. Then the category of finitely generated projective R-modules is equivalent to the category of algebraic vector bundles[a] on $\operatorname{Spec} R$
2. **(Swan's Theorem)** Let X be a compact Hausdorff space. Then the category of finitely generated projective R-modules over the continuous functions $C(X)$ is equivalent to the category of finite-rank vector bundles on X, the equivalence being established by sending vector bundles its module of continuous section.

[a] locally free sheaves of structure sheaf-modules of constant finite rank

Proof **:**
See Serre's paper and Swan's paper.

Due to these connections, we have the following definition:

Definition 1.2.1: Algebraic Line Bundle

Let R be a commutative ring. Then an *algebraic line bundle* L over R is a finitely generated projective R-module of constant rank 1.

In this section, we shall just say *line bundle*. As line bundles are locally 1 dimensional, it is easy to verify that the tensor product of line bundles is again a line bundle (hint: $\dim(V \otimes W) = \dim(V) \cdot \dim(W)$). Furthermore, it should be immediate to the current reader that $L \otimes_R R \cong L$ for any line bundle (even projective module). Then $\otimes_R$ forms a commutative monoid. If we show there is an inverse, we have a group, and indeed:

[3] Prove it for the case of rank 1, then for the rank r case show that $\Lambda^r(P)$ is finitely generated

Lemma 1.2.1: Inverse Line Bundle

Let L be a line bundle and $L^\vee = \mathrm{Hom}_R(L, R)$. Then

$$L \otimes_R L^\vee \cong R$$

Proof **:**
First, $L^\vee$ is line bundle as $\mathrm{rank}\,(L) = \mathrm{rank}\,(L^\vee)$. Next, take:

$$f \otimes_R \ell \mapsto f(\ell)$$

If $L = R$, this is certainly an isomorphism. Hence in the general case, localizing at $\mathfrak{p}$ we get:

$$(L^\vee \otimes_R L)_\mathfrak{p} \cong L^\vee_\mathfrak{p} \otimes_{R_\mathfrak{p}} L_\mathfrak{p} \to R_\mathfrak{p}$$

which is then certainly an isomorphism. As being an isomorphism is a local property, we get the global isomorphism, completing the proof.

Definition 1.2.2: Picard Group

Let R be a commutative ring. Then the *picard group* of R is the set of isomorphism classes of line bundles on R with the operation being the tensor product $\otimes_R$ and inverses $L^{-1} = L^\vee$.

Proposition 1.2.1: Picard Group Homomorphism

Let $R \to S$ be a ring homomorphism between commutative rings. Then $\mathrm{Pic}\,(R) \to \mathrm{Pic}\,(S)$ is a group homomorphism mapping $L \mapsto L \otimes_R S$

Proof **:**
Certainly $L \otimes_R S$ is a line bundle over S. Then it is an exercise in algebra to see that:

$$(L \otimes_R M) \otimes_R S \cong (L \otimes_R S) \otimes_R (M \otimes_R S)$$

showing that $\otimes_R S$ is a group-homomorphism, as we sought to show.

We prove the following representation of line bundles for later:

Lemma 1.2.2: Representation of Line Bundle

Let L be a line bundle over R. Then

$$\mathrm{End}_R(L) \cong R$$

Proof **:**
Multiplication by elements of r give a map $R \to \mathrm{End}_R(L)$. This map is locally an isomorphism, and hence an isomorphism, completing the proof.

Example 1.3: Picard Groups of Commutative Rings

1. Let $R = k$. Then as k is a field, every projective k-module is free. But then any projective k module of rank 1 is a dimension 1 vector-space, namely it is isomorphic to k. Hence:
$$\operatorname{Pic}(k) = 0$$
2. Similarly, as every module over a PID is free, if $R = \mathbb{Z}$, then:
$$\operatorname{Pic}(\mathbb{Z}) = 0$$
3. Let R be a Dedekind domain (usually a number ring). We shall heavily be relying on the algebra presented in [2, chapter 22.6]. First, recall that each ideal $I \subseteq R$ is a projective R-module, and it is certainly locally of rank 1 (every ideal is generated by at most 2 elements, and localizing at the appropriate prime makes the ideal principal). Next, recall that any torsion-free R-module of rank n is isomorphic to:
$$M \cong R^{n-1} \oplus I$$
for an ideal I that can be replaced by $J = (a)I$ for $a \in R$. As all projective modules are torsion free, and we are looking at rank 1 modules, we see that the line bundles on R are in correspondence to the ideals of R. As principal ideals are isomorphic to R, we have that:
$$\operatorname{Pic}(R) \cong \operatorname{Cl}(R)$$
4. (If you know some divisor theory) Building off the work from the last point, If X is a nonsingular affine variety over an algebraically closed field $\overline{k}$ then $\operatorname{Pic}(k[X])$ (the Picard group of its coordinate ring) is isomorphic to the divisor class group of X.
5. it was proven by Quillen and Suslin that every projective module over polynomial ring is free. Thus, the picard group of $k[x_1, ..., x_n]$ is always trivial:
$$\operatorname{Pic}(k[x_1, ..., x_n]) = 0$$
6. (foreshadowing for elliptic curve theory). An elliptic curve is a smooth algebraic curve (over an algebraically closed field $\overline{k}$) whose genus is 1[a]. We shall see that there is a natural subgroup of the Picard group, Pic^0, and that if E is an elliptic curve associated to $R = \overline{k}[x, y, z]/I$, then:
$$\operatorname{Pic}^0(R) \cong R$$
Which is a rather interesting idempotence! See cite:AlgGeo for more details.

[a] Every nonsingular curve maps onto a compact Riemann surface of genus g

We shall next define a natural notion known as the *determinant line bundle.* Its key feature of interest to us is that all finitely generated projective R-modules are determined by their rank and determinant. First, recall the following properties of the exterior product:

Proposition 1.2.2: Properties of The Exterior Product

Let R be a commutative ring and M an R-module. Then:

1. $\Lambda^k(R^n)$ is a free R-module of rank $\binom{n}{k}$ generated by $ei_1 \wedge \cdots \wedge e_{i_n}$ where $i_1 \leq \cdots \leq i_n$. Note that $\Lambda^n(R^n) \cong R$
2. If $R \to S$ is a ring map, there is an isomorphism:
$$\big(\Lambda^k M\big) \otimes_R S \cong \Lambda^k(M \otimes_R S)$$
3. There is a natural isomorphism:
$$\bigwedge^k(P \oplus Q) \cong \bigoplus_{i=0}^{k}(\bigwedge^i P) \otimes (\bigwedge^{k-i} Q)$$
4. If P is projective, $\Lambda^k(P)$ is projective (as it is locally free).

Proof **:**
exercise, or see [2, chapter 15.4]

Recall that $\Lambda^n(M)$ for an n-rank R-module is generated by a single element, and by moving all the coefficients to the front we get the determinant. This motivates the following terminology:

Definition 1.2.3: Determinant Bundle

Let P be a finitely generated projective R-module of constant rank n. Then $\Lambda^n(P)$ is called the *determinant line bundle*, and is denoted $\det(L)$.

If P is not constant rank, we define it component-wise. If $g : P \to P$ is an endomorphism, $\det(-)$ is a functor $\deg(g) : \det(P) \to \det(P)$, and by lemma 1.2.2 we can think of $\deg(g) \in R$.

We shall state, but not prove, the following useful classification result:

Proposition 1.2.3: Classification of Projective Module Over 1-dimensional Ring

Let R be a commutative Noetherian 1-dimensional ring. Then all finitely generated projective R-modules are completely classified by their rank and determinant. In particular, every finitely generated projective R-module P of rank ≥ 1 is isomorphic to $L \oplus R^f$, where $L = \det(P)$ and $f = \text{rank}\,(P) - 1$.

Proof **:**
hint: Show that R is the only stably free-line bundle.

Let us next focus on line bundles over integral domains. We shall see that all line bundles on integral domains are given by *invertible ideal*:

Proposition 1.2.4: Classification of Projective Modules over Integral Domains

Let R be a commutative integral domain. L is a line bundle if and only if it is (isomorphic to) an invertible ideal. If I and J are fractional ideals and I is invertible, then $I \otimes_R J \cong IJ$. Furthermore, there is an exact sequence of abelian groups:

$$1 \to R^\times \to F^\times \xrightarrow{\text{Div}} J_K(R) \to \operatorname{Pic}(R) \to 0.$$

where $J_K(R)$ is the ideal group.

***Proof* :**
If I and J are invertible ideals such that $IJ \subseteq R$, then we can interpret elements of J as homomorphisms $I \to R$. If $IJ = R$, then we can find $x_i \in I$ and $y_i \in J$ so that $x_1y_1+\cdots+x_ny_n = 1$. The $\{x_i\}$ assemble to give a map $R^n \to I$ and the $\{y_i\}$ assemble to give a map $I \to R^n$. The composite $I \to R^n \to I$ is the identity, because it sends $r \in I$ to $\sum x_i y_i r = r$. Thus I is a summand of R^n, i.e., I is a finitely generated projective module.

As R is an integral domain and $I \subseteq F$, $\operatorname{rank}(I)$ is the constant $\dim_F(I \otimes_R F) = \dim_F(F) = 1$. Hence I is a line bundle.

This construction gives a set map $J_K(R) \to \operatorname{Pic}(R)$; to show that it is a group homomorphism, it suffices to show that $I \otimes_R J \cong IJ$ for invertible ideals. Suppose that I is a submodule of F which is also a line bundle over R. As I is projective, $I \otimes_R -$ is an exact functor. Thus if J is an R-submodule of F, then $I \otimes_R J$ is a submodule of $I \otimes_R F$. The map $I \otimes_R F \to F$ given by multiplication in F is an isomorphism because I is locally free and F is a field. Therefore the composite

$$I \otimes_R J \subseteq I \otimes_R F \xrightarrow{\text{multiply}} F$$

sends $\sum x_i \otimes y_i$ to $\sum x_i y_i$. Hence $I \otimes_R J$ is isomorphic to its image $IJ \subseteq F$. This proves the third assertion.

The kernel of $J_K(R) \to \operatorname{Pic}(R)$ is the set of invertible ideals I having an isomorphism $I \cong R$. If $f \in I$ corresponds to $1 \in R$ under such an isomorphism, then $I = fR = \div(f)$. This proves exactness of the sequence at $J_K(R)$.

Clearly the units $R^\times$ of R inject into $F^\times$. If $f \in F^\times$, then $fR = R$ if and only if $f \in R$ and f is in no proper ideal, i.e., if and only if $f \in R^\times$. This proves exactness at $R^\times$ and $F^\times$.

Finally, we have to show that every line bundle L is isomorphic to an invertible ideal of R. Since $\operatorname{rank}(L) = 1$, there is an isomorphism $L \otimes_R F \cong F$. This gives an injection $L \cong L \otimes_R R \subseteq L \otimes_R F \cong F$, i.e., an isomorphism of L with an R-submodule I of F. Since L is finitely generated, I is a fractional ideal. Choosing an isomorphism $\tilde{L} \cong J$, Lemma 3.1 yields

$$R \cong L \otimes_R \tilde{L} \cong I \otimes_R J \cong IJ.$$

Hence $IJ = fR$ for some $f \in F^\times$, and $I(f^{-1}J) = R$, so I is invertible.

There is further discussion different interesting divisor groups, class groups, and their relation to the Picard group, but we leave that material to cite:algGeo.

2

K-theory

We saw the Picard group allowed us to treat the collection of (isomorphism classes of) line bundles of constant rank one as a group under $\otimes$. We shall now be looking at a similar situation with the direct sum. We shall see that the inverting process for $\oplus$ shall be done in a less constructive ways for reasons that I will get back to when I better understand.

2.1 Algebraic Build-up

Let M be an abelian monoid. Then we universally augment it into an abelian group, $M^{-1}M$ through the usual free-quotient construction. In particular, take $\mathcal{F}(M \times M)/\left([m+n]-[m]-[n]\right)$. This is called the *Grothendieck Completion*. A *Cancellation monoid* is a monoid where $m+a=n+a$ if and only if $m=n$.

Proposition 2.1.1: Properties of Monoid Completion

Let M be a monoid, $M^{-1}M$ be its completion. Then:

1. Every element of $M^{-1}M$ can be written as $[m]-[n]$ for $m, n \in M$
2. For each $[m], [n] \in M^{-1}M$, $[m]=[n]$ if and only if there exists a p such that $m+p=n+p$.
3. The map $M \times M \to M^{-1}M$ where $(m, n) \mapsto [m]-[n]$ is surjective
4. M injects into $M^{-1}M$ if and only if M is a cancellation monoid.
5. The Grothendieck Completion is a functor that is left adjoint to the forgetful functor from the category of abelian group to the category of abelian monoids.

Proof **:**
these are all simple exercises, but you can take a look at [2, chatper 15].

If we augment a monoid with a 2nd operation that satisfies the ring axioms, then we get a *semiring*. The semiring $\mathbb{N}$ is the initial semiring in this category. Then the Grothendeick functor can also be seen as a functor form semirings to rings.

When we shall be studying $K_0(R)$ groups, we shall find that the free R-modules play an important role. These elements are important enough so that we consider the following special type of subgroup of a monoid

Definition 2.1.1: Cofinal Monoid

Let M be a monoid. Then $L \subseteq M$ is called *cofinal* if for every $m \in M$, there exists a $m' \in M$ such that $m + m' \in L$.

The terminology *cofinal* comes from order theory, were a subset $A \subseteq B$ of an preorder $(B, \leq)$ is cofinal if for every $b \in B$, there is an $a \in A$ such that $b \leq a$.

Lemma 2.1.1: Cofinal Monoids

Let $L \subseteq m$ be a cofinal monoid. Then:

1. $L^{-1}L \subseteq M^{-1}M$ is a subgroup
2. Every element $[m] \in M^{-1}M$ can be written as $[m'] - [\ell]$ for $m' \in M$ and $\ell \in L$.
3. If $[m] = [m'] \in M^{-1}M$, then $m + \ell = m' + \ell$ for some $\ell \in L$.

Proof **:**
this is a matter of checking definitions

Example 2.1: Grothendeick Completion

1. some good examples in the book!

2.2 K_0 of a ring

Definition 2.2.1: First K-ring: $K_0(R)$

Let R be a [not necessarily commutative] ring, $P(R)$ the collection of (isomorphism classes of) finitely generated projective R-modules. Then combining it with $\oplus$ and appending an identity element 0, we get an abelian monoid. It's Grothendeick completion is the group $K_0(R)$.

When R is commutative, then along with $\otimes_R$, this forms a ring.

We restrict to finitely generated modules simply because the usual behaviour of infinitely generated modules trivializes the structure, for example if P is any finitely generated projective R-module then,

then if we had to consider the infinitely generated module R^∞ we get:

$$P \oplus R^\infty \cong R^\infty$$

Therefore $[P] = 0$ and $K_0(R) = 0$, and so we naturally stick to the finitely generated case.

If R is commutative, then $K_0(R)$ is commutative with identity $1 = [R]$. This is indeed the identity, for $\oplus$ and $\otimes$ distribute, and $P \otimes_R R \cong P$. The canonical trivial example is when $R = k$ (or any division ring). Then $K_0(k) \cong \mathbb{Z}$. Similarity, if R is a local ring or a PID, then $K_0(R) \cong \mathbb{Z}$ (as these are sufficient conditions for a projective module to be free). We shall give non-trivial example after we have developed some more theory to be able to identify which properties lead to structural results for $K_0(R)$.

It should be noted that $K_0(-)$ is a functor from **Ring** to **Ab** or from **CRing** to **CRing**, namely if $\varphi : R \to S$ is a ring homomorphism, then $\otimes_R S : P(R) \to P(S)$ is a monoid map which induces the map $K_0(R) \to K_0(S)$. If R, S are commutative, then $\otimes_R S : P(R) \to P(S)$ is a semiring map as:

$$(P \otimes_R Q) \otimes_R S \cong (P \otimes_R S) \otimes_R (Q \otimes_R S)$$

and hence we get a ring homomorphism. This shall become handy when trying to translate properties from known $K_0(R)$ groups to one's we are trying to study.

In $K_0(R)$, the free-modules play a particularity important role. First of all, they are *cofinal* in $P(R)$ (see definition 2.1.1). Then every element of $K_0(R)$ can be written as:

$$[P] - [R^n]$$

Furthermore, by lemma 2.1.1, we have $[P] = [Q]$ if and only if:

$$P \oplus R^n \cong Q \oplus R^m$$

that is, P, Q are stably isomorphic, an $[P] = [R^n]$ if P is stably free. If R satisfies the IBP, then $L \subseteq P(R)$ is isomorphic to $\mathbb{N}$. If R respect IBP, we get the following useful characterization of trivial K_0 groups:

Lemma 2.2.1: Ibp and Trivial K_0

Let $\mathbb{N} \to P(R)$ be the monoid map sending $n \to R^n$ inducing the group homomorphism $\mathbb{Z} \to K_0(R)$. Then:

1. this map is injective if and only if R satisfies IBP
2. If R satisfies IBP, then

   $K_0(R) \cong \mathbb{Z}$ if and only if every finitely generated projective R-module is stably free

Proof **:**

1. This is immediate by IBP
2. $P \oplus \mathbb{R}^n \cong \mathbb{R}^m$, and so each $[P] = [R^m]$ for some n.

Example 2.2: K_0 of some rings

1. As mentioned before, fields/division rings, PIDs, and local rings all have $K_0(R) = \mathbb{Z}$

2. If $R = R_1 \times R_2$, then by [2, chatper 23] we get $P(R) = P(R_1) \times P(R_2)$, and its completion gives $K_0(R) = K_0(R_1) \times K_0(R_2)$. We may thus compute $K_0(R)$ component-wise.

3. Let $R = M_n(F)$ Then R is a simple ring, and so every $M_n(F)$-module is projective. Recall that an $M_n(F)$-module is some direct sums of F, and the *length* of an $M_n(F)$-module is the number of copies. Then the length gives a natural map:

$$K_0(M_n(F)) \cong \mathbb{Z}$$

Note that $M_n(F)$ has length n, and so the subgroup generated by the free modules have index n, so the natural inclusion map $\mathbb{Z} \to K_0(M_n(F))$ does not split!

4. Let R be a semi-simple ring so that by Artin-Wedderburn:

$$R \cong M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$$

Thus, we get:

$$K_0(R) = \prod_i K_0(M_{n_i}(D_i)) \cong \mathbb{Z}^k$$

5. If R is a commutative ring, we have a natural map from R to a field, $R \to R\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$; call this field F. Then this induces a natural map $K_0(R) \to K(F)$, that is $K_0(R) \to \mathbb{Z}$ which maps $[R]$ to 1. Note that there doesn't always exist, such a map, we shall see an example where $K_0(R) \cong \mathbb{Q}$.

6. (If you know some algebraic geometry) Recall that a sheaf is *flasque* if all restriction maps are surjective. The key interesting property of Flasque sheaves is that they have trivial cohomology, $H^i(X, \mathcal{F}) = 0$ for $i > 0$ and $\mathcal{F}$ a flasque sheaf. We shall link the $K_0(R)$ group to (co)homological considerations, and will find the following definition to be the appropriate purely algebraic counter-part. We say a ring R is *flasque* if there is an R-bimodule M that is finitely generated and projective as a right R-module, such that there is a R-bimodule isomorphism:

$$\varphi : R \oplus M \xrightarrow{\sim} M$$

From this, we get that $K_0(R) = 0$. To see this, note that:

$$P \oplus (P \otimes_R M) \cong P \otimes_R (R \oplus M) \cong (P \otimes_R M)$$

Hence $[P] + [P \otimes_R M] = [P \otimes_R M]$, implying $[P] = [0]$. If R is flasque and the right R-module structure of M is R, then R is said to be an *infinite sum ring*. The isomorphism φ becomes $R^2 \cong R$. An example of such a ring is $\mathrm{End}_R(R^\infty)$.

7. (There is an example on p.76 of the K-theory textbook I'm following on Von Neumann regular rings, which are rings where for every $x \in R$, there exists a $y \in R$ such that $xyx = x$. These are rings in which inverse need not exist, but they almost do, for example each field is Von Neumann regular rings because we can let $y = x^{-1}$. No integral domain is Von Neumann regular, but $M_n(F)$ is Von Neumann regular. There is a natural lattice structure that comes out of its idempotence that $K_0(R)$ is isomorphic to if R is a Von Neumann regular rings)

8. let $R = \varinjlim_i R_i$ where $\{R_i\}$ is a filtered system. Then every finitely generated projective R-module is of the form:
$$R \otimes_{R_i} P_i$$
where P_i is some finitely generated projective R_i-module. Any isomorphism:
$$R \otimes_{R_i} P_i \cong R \otimes_{R_i} P_i'$$
must set a finite set of generators to a finite set of generators, and so for j large enough this reduces to:
$$R_j \otimes_{R_i} P_i \cong R_j \otimes_{R_i} P_i'$$
But that means that $P(R)$ is then a filtered colimit of $P(R_i)$! We thus have the following wonderful result about colimits passing through K-groups:
$$K_0(R) = \varinjlim K_0(R_i)$$
If R is commutative, we can show that every R is the colimit of finitely generated subrings! As every finitely generated commutative ring is Neotherian with finite normalization (the integral closure is finitely generated over the ring), we may reduce studying $K_0(R)$ to the study of $K_0(R_i)$ where the R_i have nicer properties!

The following is very useful and allows for interesting geometric interpretations. For the following, we shall say that I is *complete* if every Cauchy sequence $\sum_i^\infty a_i$ where $a_i \in R/I^n$ converges to a unique element in I.

Theorem 2.2.1: K-group and Nilpotence

Let R be a ring and $I \subseteq R$ be nilpotent, or more generally complete (see definition above). Then:
$$K_0(R) \cong K_0(R/I)$$
And if R is commutative, then:
$$K_0(R) \cong K_0(R_{\text{red}})$$

***Proof* :**
We'll show $P(R) \cong P(R/I)$, which shall complete the result. We first need to build-up some important result:

lemma 1: Let P_1, P_2 be finitely generated projective R-modules and $I \subseteq R$ a radical idea. Then $P_1 \cong_R P_2$ if and only if $P_1/IP_1 \cong_R P_2/IP_2$

Proof. Certainly $P_1 \cong_R P_2$ implies $P_1/IP_1 \cong_R P_2/IP_2$, so we do the converse. There is a natural map $P_1 \to P_1/IP_1$, and then there is an isomorphism $\varphi : P_1/IP_1 \to P_2/IP_2$. Finally, there is a natural projection $\pi : P_2 \to P_2/IP_2$:

$$\begin{tikzcd} P_1 \arrow[d, "\pi"'] \arrow[dr, "\varphi\circ\pi"] & P_2 \arrow[d, "\pi"] \\ P_1/IP_1 \arrow[r, "\varphi"'] & P_2/IP_2 \end{tikzcd}$$

As P_1 is projective, there is a natural lift making the following diagram commute:

$$\begin{tikzcd} P_1 \arrow[r, "\tilde{\varphi}"] \arrow[d, "\pi"'] & P_2 \arrow[d, "\pi"] \\ P_1/IP_1 \arrow[r, "\varphi"'] & P_2/IP_2 \end{tikzcd}$$

If we show $\widetilde{\varphi}$ is an isomorphism, we're done. Note that as φ is an isomorphism, in particular it is surjective, then as I is radical it is a consequence of Nakayama's lemma that $\varphi : P_1 \to P_2$ is surjective (see exercises in [2, Chapter 20.3] or checkout [3]). Let's thus show injectivity. To that end, let us localize at each $\mathfrak{p} \subseteq R$ and consider:

$$\begin{tikzcd} P_{1,\mathfrak{p}} \arrow[r, "\tilde{\varphi}_{\mathfrak{p}}"] \arrow[d, "\pi"'] & P_{2,\mathfrak{p}} \arrow[d, "\pi"] \\ P_{1,\mathfrak{p}}/I_{\mathfrak{p}}P_{1,\mathfrak{p}} \arrow[r, "\varphi_{\mathfrak{p}}"'] & P_{2,\mathfrak{p}}/I_{\mathfrak{p}}P_{2,\mathfrak{p}} \end{tikzcd}$$

As P is a finitely generated projective module, its localization at a prime ideal is free, hence each $P_{i,\mathfrak{p}}$ has a basis. The key is that the projection onto $P_{1,\mathfrak{p}}/I_{\mathfrak{p}}P_{1,\mathfrak{p}}$ is still a basis. If we show then as $\varphi_{\mathfrak{p}}$ is an isomorphism, by commutativity the image of the basis in $\widetilde{\varphi}_{\mathfrak{p}}$ must be linearly independent which shows the map is injective. To that end, if $\{e_1, ..., e_n\}$ is a basis for $P_{1,\mathfrak{p}}$, its projection spans so let's show it is linearly independent. Say

$$\sum_i r_i \overline{e}_i = 0$$

Then $\sum_i r_i e_i \in I_{\mathfrak{p}}P_{1,\mathfrak{p}}$. By properties of ideals, $r_i \in I_{\mathfrak{p}}$, but then we must have that in the quotient $r_i = 0$ for all i, which by our previous reasoning allows us to conclude injectivity.

Hence, $\widetilde{\varphi}_{\mathfrak{p}}$ is injective for each prime $\mathfrak{p} \subseteq R$, and so it is a classical algebra result that $\widetilde{\varphi}$ is injective (see [2, chapter 20.3]. □

Now, let us show that if I is a nilpotent ideal, then there is a bijection between the finitely generated projective R-modules and the finitely generated projective R/I-modules, which shall then complete the proof. Without loss of generality we can assume that $I^2 = 0$ (we can then use induction to for the result for $I^n = 0$ for $n \geq 2)^a$. Let us first show that an idempotent $\overline{e} \in R/I$ lifts to an idempotent $e \in R$. If r is a lift of e, then we can verify that:

$$e = r + rxr + (1-r)y(1-r)$$

where $x, y \in I$. Verify that $e^2 = e$ and it indeed maps to $\overline{e}$. This lift is unique up to conjugation by an element $u \in 1 + I$. With this lift, we shall show that every finitely generated R/I-module $\overline{P}$ is isomorphic to P/IP for an appropriate finitely generated projective module P. The key is that idempotent matrices correspond to summands of free modules. In particular if $\overline{P}$ is a finitely generated projective R/I-module, then there exists an idempotent $\overline{e} \in M_n(R/I)$ such that $\mathrm{im}(\overline{e}) = \overline{P}$. Then this e can be lifted to $e \in M_n(R)$ where $e^2 = e$ and $e \equiv \overline{e} \mod I$, so by taking $\mathrm{im}(e) = P$ we get:

$$\overline{P} \cong P/IP$$

Finally, we are at the stage where we reduced our problem of finitely generated projective R-modules to modules of the form P/IP. Then by our lemma, $P_1/IP_1 \cong P_2/IP_2$ if and only if $P_1 \cong P_2$, showing that the isomorphism classes are in bijective corresponds, completing the proof.

[a] This is where the completeness generalization also comes in, as we can consider $\varprojlim R/I^n$

Example 2.3: 0-dimensional Commutative Ring

Let R be a commutative ring and R_{red} be its reduction. Then it is an easy exercise to show that R_{red} is Artinian if and only if $\operatorname{Spec}(R)$ is finite and discrete. More generally (in the "non-finite case"), if a ring is von Neumann regular (for each $x \in R$, there exists a $y \in R$ such that $yxy = x$; this is a "weak inverse"), then R has Krull dimension 0 and is reduced (in fact, this is an equivalent characterization of von-Neumann regular rings, giving a good geometric characterization of such rings[a])

[a]Note that there are many non-commutative examples, such matrix rings, left principal ideals generated by idempotents, and $\operatorname{End}_k(V)$ for a vector space, and as they are closed under direct product that makes every semi-simple von-Neumann regular ring semi-simple

2.3 Connection to H_0

We start connecting the K_0 ring to (co)homological results, starting with the most simple one of the $H_0(R)$ group. which we shall take as $\operatorname{Hom}(\operatorname{Spec}(R), \mathbb{Z})$. To see this connection, consider the following warm-up result:

Theorem 2.3.1: Pierce's Theorem

Let R be a 0-dimensional commutative ring. Then:

$$K_0(R) = \operatorname{Hom}(\operatorname{Spec}(R), \mathbb{Z})$$

Note that no assumption was made on R be finitely generated, for example it may be an infinite product of fields.

Proof **:**

Recall that as R is 0-dimensional all its prime ideals are maximal, and $\operatorname{Spec}(R)$ has the discrete topology. Then by generalizing the classification of Artinian rings to 0-dimensional rings (in particular, every finitely generated projective R-module will be a finite product of modules of the form $R/\mathfrak{m}$), we can consider the rank of each $R/\mathfrak{m}$ individually which gives us the exact information of the $K_0(R)$ group, completing the proof.

We shall thus push this one step further and see the relation for more general commutative rings:

Definition 2.3.1: H_0 Group

Let R be a commutative ring. Then:

$$H_0(R) = [\operatorname{Spec}(R), \mathbb{Z}] = \operatorname{Hom}(\operatorname{Spec}(R), \mathbb{Z})$$

As $\operatorname{Spec}(R)$ is (quasi)compact, $H_0(R)$ will always be a free abelian group. If furthermore R is Noetherian, then:

$$H_0(R) \cong \mathbb{Z}^n$$

If on top of that R is an (integral) domain (more generally $\operatorname{Spec}(R)$ is connected), then:

$$H_0(R) \cong \mathbb{Z}$$

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