"What is your intuition on complex analysis?" It is a question I asked many times, often receiving answers depending on the domain of mathematics the person seem to enjoy the most: "It's about solving complicated integrals", "It is a special case of harmonic analysis", "it is useful for analytic number theory", "it is the study of Riemann surfaces". None of these answer felt satisfactory to me: many answers felt like it deferred the responsibility for its purpose to another field of mathematics. Though each of these has given me some motivation for its study, these explanations did not feel like they gave a good reason for why complex analysis produces such rigid, one may even sometimes say magical, results. This requirement that the intuitions should explain the "beauty" of complex analysis has kept me wobbly on my footing for complex analysis, never feeling I can justify calling such a field *analysis* when compared to the flexible results offered in real analysis.

I believe I came across an explanation that finally grounds me and unifies all previous explanations given to me. This article will expand the following sentence

Complex analysis is the completion of polynomials with the tools of real analysis applied to it.

Level of Math Knowledge This article shall try its best to be as accessible as possible. A lot of the intuition shall stem from an understanding of what is real analysis and the basic properties of polynomials. In fact, the reader who has already taken a course in complex analysis may find this article most useful to unify their framework on complex analysis.

1 Polynomials

To understand complex analysis, we start with the initial objects of study: polynomials. They naturally arise in many contexts in mathematics:

1. (Algebra) Polynomials are the objects that hold *relation information*: given some variables $x_1, ..., x_n$, by using $+$ and $\cdot$ we combine them to form polynomials which represent *relations* in algebra. For those more familiar with some abstract algebra, any ring is the quotient of a free ring where the kernel consists of the polynomials representing all relationships on the generators. Thus, understanding the properties of polynomials is of keen interest to algebraist.
2. (Analysis) Polynomials can approximate continuous functions on any (compact) neighbourhood to an arbitrary degree of precision (by the Stone–Weierstrass theorem). In particular, on compact intervals we can take a *sequence* of polynomials that uniformally converges to the continuous function. Thus, understanding the properties of polynomials is of keen interest to analysists
3. (Geometry) Polynomials can inherently create shapes from their zero sets and their graphs[1]. As polynomials are determined by their coefficients, they are determined by only finitely many values. From this perspective, polynomials form some of the "simplest" shapes that can be studied. Thus, understanding the properties of polynomials is of keen interest to geometers.

[1]which themselves are just zero sets of a polynomial

Let us focus on the special case where there is only one variable. Any polynomial $p(z)$ is naturally studied when over the complex numbers $\mathbb{C}$: every polynomial over $\mathbb{C}$ breaks down into linear terms:

$$p(z) = c(z - a_1)(z - a_2) \cdots (z - a_n)$$

Thus, the information about $p(z)$ is encoded in the multi-set $\{c, a_1, ..., a_n\}$[2] where c is some constant multiple. Thus, the complex numbers are the natural domain to study polynomials.

It is thus productive to use all the tools of mathematics in understanding the properties of polynomials. We may continue the study of polynomials over $\mathbb{C}$[3] in a purely algebraic direction and go towards results such as *galois theory* and *number theory*. We may also treat them geometrically, leading to *algebraic geometry* and *scheme theory*. From an analysis perspective, we may treat polynomials as a family of functions and ask multiple analytic questions. This approach will require a bit more care due to the "infinite" nature of analysis as compared to the "finite" nature of polynomials.

2 From Polynomials to Complex Analysis

We wish to be able to study polynomials through the traditional analytic tools (limits, convergence, family of functions, bounding, interpolation, etc). We may want to study the family of polynomials as well as define a natural topology for the important open/closed sets given by polynomials; this is called the *Zariski Topology*[4]. However, this approach has a couple of blockers, namely:

- The Zariski topology is too coarse for analysis. The euclidean topology is "made" for the ideas of analysis: limits have single points, it is a metric space[5], properties such as compactness can be described using simple analytic properties such as limits or closed and bounded, and continuous functions have many flexible properties (consider results such as Urysohn's lemma, that any closed set is the zero-set of a continuous function, or a functions is continuous if and only if it preserves limits). These properties are *not* present in the natural topology induced by polynomials, the Zariski topology: most open sets are *dense*, limits usually converge to infinitely many points, compactness becomes a much weaker property (the stronger notion of *properness* needs to be used), and so forth.
- Polynomials are not *complete*: The limit of polynomials can be an arbitrary continuous function on a compact interval (in the euclidean topology). Even more simply, if we keep adding a term $a_n x^n$ to a polynomial $a_0 + a_1 x + a_2 x^2 + \cdots$, the result is *not* a polynomial.
- Similarly to the previous point, there are "not enough" polynomials for some fundamental results expected in analysis: the subset of polynomials functions on the unit ball is not closed (there are not enough polynomials to converge to) and there are not enough polynomials to have limit when uniformally converging[6].

[2] We may want to recover this set by taking $p(z) = 0$, but there is a complication here where if $a_i = a_j$ for some i, j. Then the zero set "forgets" this information; for this the notion of *schemes* more naturally keeps track of multiplicity information

[3] Or more algebraically, over an algebraically closed field $\overline{k}$ of characteristic 0

[4] It is given by taking the zero sets of polynomials as the collection of closed sets

[5] Most spaces of functions shall be metric spaces, though often they have the further structure of a Banach or Fréchet spaces

[6] In particular, uniformally converging on compact sets, a technicality due to unstable boundary behaviour which would be covered in a course in complex analysis

The solution to this is by taking the "right completion" of polynomials in such a way that polynomials can be embedded in this completion and retain most of their properties. The completion ought to be as minimal as possible to be "as close" to polynomials as possible, while being large enough so that we can work with our usual real-analysis tools:

1. We first allow *infinite polynomials*, namely power series. For those who are comfortable with limits from category theory, we take the set of polynomials and add all the limits of the diagrams:
$$\varinjlim_{n\to\infty} p_n(x)$$
where $p_n(x)$ is a polynomial of degree n whose coefficients match the coefficients of the polynomials of degree $k < n$.

   It must be noted that since we are not working with infinite sums, we must specify that we are going to work with *convergent* power series. Convergent power series shall depend on the domain of convergence, which need not be all of $\mathbb{C}$ (ex. $\sum_n^\infty \frac{z^{n!}}{n^2}$).

2. The next condition comes from how much of our results depend on an open (or closed) subset of our space (the above examples gives an example of why this is important). For this reason, instead of looking at functions that are power series, it would be better to focus on functions that are *locally convergent power series*, that is there exists a neighborhood U of a point such that $f|_U$ is equal to a power series.

   Functions that are locally a convergent power series are called *analytic*.

These two conditions give enough functions for the collection to be closed under uniform convergence, namely if f_n is a sequence of analytic functions and $f_n \to f$ uniformally (on compact sets), then f shall still be analytic by Weiestrass Convergence Theorem. This family of functions on an open subset $U \subseteq \mathbb{C}$ is called the *Holomorphic functions*, and we shall denote it as $\mathcal{H}(U)$. The family $\mathcal{H}(U)$ forms a complete metric space[7], giving us a natural space in which we may ask analytic questions!

The first thing we should do is to see which properties of polynomials extend to properties of holomorphic functions. As it turns out, we have done a good job finding a proper embedding for our polynomials and they share many similar properties:

- Polynomials over $\mathbb{C}$ are certainly unbounded (they shall go to infinity). By Louisville's Theorem, the same applies to holomorphic functions
- If a holomorphic function f is bounded by a polynomial in growth (i.e. $|f(z)| \leq |p(z)|$), then f is a polynomial. This mirrors how a polynomial cannot be bounded by another polynomial unless it is of lower order or equal to it.
- A polynomial is determined by its roots up to a constant multiple. By *Weierstrass factorization theorem*, every holomorphic decomposes into an infinite product of the form:
$$f(z) = z^m e^{g(z)} \prod_k^\infty \left[\left(1 - \frac{z}{a_k}\right) e^{p(z)}\right]$$
where $g(z)$ is a polynomial.

[7]Even stronger, it forms a Fréchet space, which ought to be thought as spaces that are not nice enough to define a single norm, but nice enough to replace the norm by a sequence of compatible semi-norms

- Though a holomorphic functions can decomposed as above, it is not determined by its roots (for example, e^z has no roots). The next best result is *Cauchy's integral formula* which allows us to "extract" information from holomorphic functions about the value at points.
- Polynomials form an integral domains and rational polynomials form a field. Similarly, the family of holomorphic functions forms an integral domain, while the family of meromorphic functions forms a field. From an analysis perspective this is *very* surprising: if f, g are real differentiable function and $fg = 0$, then it is very much not the case that we can conclude that $f = 0$ or $g = 0$. This reflects how holomorphic functions do indeed "naturally" extend the polynomials and the algebraic structures they form!
- The polynomial automorphisms of $\mathbb{C}$ are linear, and the holomorphic automorphisms of $\mathbb{C}$ are also linear. Hence, extending to holomorphic functions does not add additional automorphisms of $\mathbb{C}$.
- The open subsets of the Zariski topology are usually *dense* where its compliment is of codimension ≤ 1. In the case of 1 dimension, only a finite set of points is outside any (nontrivial) open set. Similarly, many holomorphic functions have an *analytic continuation* that extends the definition of a holomorphic function defined on a (euclidean) open $U \subseteq \mathbb{C}$ to all of $\mathbb{C}$ minus possibly a countable set of (non-clustered) points[8]. Though it must be noted there is an analytic blocker to extend this result to all power series (consider again $\sum_n^\infty z^{n!}/n^2$ which grows too quickly to be defined on $\mathbb{C}$).

These results should give a sense of how we have done a good job in finding the right analytic space to embed polynomials!

We can ask if there is any results from complex analysis that comes back to the study of polynomials. This is indeed the case in multiple ways, but let us highlight one (infamous) example: the *Riemann-zeta function* is the power series function:

$$\zeta(s) = \sum_k^\infty \frac{1}{s^k}$$

Through it, we can find many important results about the polynomials that govern the properties of primes. Unfortunately, going into details would require going over the connection between polynomials and primes, a topic better left for another blog, (or to get the full details see [ChwNAa, chapter 3] for my exposition on this topic). More generally number theory, through the use of analytic number theory and elliptic functions, has produced some incredible results in the theory of polynomials from the theory of complex analysis.

3 From Complex Analysis to Other Fields

In a complex analysis class, holomorphic functions are not introduced as the completion of polynomials: we say that $f : \mathbb{C} \to \mathbb{C}$ is *complex differentiable* or *holomorphic* at $z \in \mathbb{C}$ if the derivative-limit exists, that is holomorphic function are locally complex-linear. It is then proven that a function is holomorphic on open $U \subseteq \mathbb{C}$ if and only if it is locally a power series. This approach is certainly pedagogically better as a lot of the intuition given by the above extensively leans on knowledge of

[8] a point is said to be a *cluster point* if any open neighborhood around the point contains infinitely many points. It is non-clustered, or sparse, otherwise

real analysis, while latter relies only on calculus. An advantage of the pedagogical approach is the direct connection to the geometric properties of holomorphic functions (as they allow us to define differential forms), and the direct connection to the "dynamics" of holomorphic functions (they contribute to the theory of harmonic functions). As holomorphic functions is the analytic setting of polynomials, we can ask how complex analysis allows us to be a bridge between polynomials and geometry and harmonic analysis. This is a very deep connection and cannot be comprehensively summarized, but for the readers enjoyment I've compiled some interesting snippets that should demonstrate how powerful a tool complex analysis is.

3.1 To Geometry

The approach via differentiation naturally shows that such functions have a natural *geometric* structure. Namely, we may define *holomorphic forms* $f(z)dz$. Then any first course in complex analysis shows that f is holomorphic on U if and only if $f(z)dz$ is closed (that is, a 1-form ω on $\mathbb{R}^2$ is closed if and only if $\omega = f(z)dz$ for a holomorphic function f). Thus, (1-dimensional) holomorphic functions can be defined purely geometrically!

We may hope that the study of 1-dimensional complex manifolds, historically called Riemann surfaces, gives information about polynomials. In fact, an absolutely incredible connection happens. First, when working with *compact sets* we usually expect a level of finiteness with what we are working with, be it boundedness, growth, extreme values, and so forth. In the case of compact Riemann surfaces, the functions and characterization of them reduces to the study of polynomials! In particular:

Every compact Riemann surface is the zero set of an algebraic curve, and
meromorphic functions reduce to rational polynomials

These results have major consequences in both algebra in geometry! For example, though not immediately obvious, the study of equations of the form

$$y^2 = x^3 + ax + b$$

is in fact the study of Compact Riemann surfaces of genus 1, i.e. the study of tori! Such connections are heavily exploited in the modern theory of algebraic geometry and would be further explored in a class on algebraic curves or complex geometry.

3.2 To Harmonic Analysis

One of the first results in the theory of holomorphic functions is that a holomorphic function f satisfies the *Cauchy-Riemann equations*

$$\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$$

In particular, if $f = u + iv$ is the decomposition of f into to real functions, u and v are *harmonic function*:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \Delta u = 0$$

where Δ is the Laplace operator. A classical exercise is to show that every harmonic function is the real part of a holomorphic function. Thus, holomorphic functions have the *dynamics* of harmonic

PDEs, that is, given a point $z \in U \subseteq \mathbb{C}$, the properties of the paths as this point travels in the field given by f will behave like those functions satisfying the Laplace operator. Hence, complex functions have *harmonic dynamics*! A couple of striking consequences are shown in every first course in complex analysis:

- Every harmonic function is smooth, hence every holomorphic functions is smooth. Namely, it suffices to show it is once complex-differentiable to show it is infinitely complex-differentiable
- It satisfies the mean value property: The value at any point $f(z)$ is equal to the average around the point:
$$f(a) = \int_{|z-a|=r} f(\gamma) d\gamma$$
- From this, we can also deduce that if $K \subseteq \mathbb{C}$ is closed and U is holomorphic on $\int K$, it is harmonic on K and it achieves it maximum/minimum on the boundary ∂K!

Using these results, we can "see" why $z^2 + 1$ must have a roots in $\mathbb{C}$. Taking a circle of radius 1, we see that the functions "rises" when going in the x-direction up to 2. By the Cauchy-Riemann equations and the mean value property, it must similarly "dip" in the y-direction[9]; we can see how it would have roots at $\pm i$; see this video for an excellent visualization of this phenomenon!

4 Conclusion

Complex analysis touches so many fields. Depending on the readers interests, different aspects of complex analysis will resonate to them. What is ultimately satisfying to me is that the "weirdness" of holomorphic functions and their rather extreme rigidity as compared to real smooth functions can be put at the feet of the rigidity of polynomials and the closeness between polynomials and holomorphic functions. This article scratches the surface of this connection, for example no time has been spent elucidating the connection between elliptic curves and harmonic functions[10], and almost no time has been spent elucidating the bridges algebra and geometry its connection to complex analysis[11]. The reader interested in seeing the details of these ideas expanded upon can checkout the multitude of textbook on complex analysis, or you can checkout [ChwNAb] to see my exposition on the topics.

References

Chwojko-Srawley, Nathanael. *Everything You Need To Know About Class Field Theory*. NA. URL: `https://nathanaelsrawley.com/assets/pdfs/notes/EYNTKA_class_field_theory.pdf`.

– *Everything You Need To Know About Complex Analysis*. 1st ed. NA. URL: `https://nathanaelsrawley.com/assets/pdfs/notes/EYNTKA_complex_analysis.pdf`.

[9] This intuition is a crude intuition and ultimately is relies on the rigidity of $z^2 + 1$, but it should still be an enlightening example

[10] Fourier analysis makes some very interesting appearances!

[11] The connection to Riemann surfaces shows a small part of this connection, showing how a crude invariant of curves is their genus number, however the Riemann-Roch theorem had no time to make an appearance!