QUADRATIC PROGRAMMING WITH QUADRATIC CONSTRAINTS

David P. Baron
Northwestern University

ABSTRACT

A program with a quadratic objective function and quadratic constraints is considered. Two duals to such programs are provided, and an algorithm is presented based upon approximations to the duals. The algorithm consists of a sequence of linear programs and programs involving the optimization of a quadratic function either unconstrained or constrained to the nonnegative orthant. An example involving production planning is presented.

I. INTRODUCTION

Programs with a quadratic objective function and quadratic constraints were first introduced by Kuhn and Tucker in their classic paper [4]. They showed that given certain conditions the solution is equivalent to the solution of a saddle point problem. This paper presents duality results for quadratic programs with quadratic constraints and develops an algorithm for the solution of such programs based upon the approximation of the dual. The algorithm consists of a sequence of linear programs and programs involving the optimization of a quadratic function either unconstrained or constrained to the nonnegative orthant. An example is presented involving production planning.

Duality results for quadratically constrained quadratic programs have been obtained by Peterson and Ecker [7], [8], [9], from the point of view of geometric inequalities. The duality development in this paper is in the spirit of Geoffrion [5]. Sinha [11] has considered programs with a single quadratic constraint, and van de Panne [12] has developed an algorithm for such programs which converges in a finite number of iterations.

II. DUALITY

Consider the following primal program:

$$(P) \qquad z = \text{maximize } c'x + x'Vx$$
$$\text{subject to } a_i x + x'W_i x \leqslant b_i; \qquad i = 1, \ldots, m,$$
$$x \geqslant 0,$$

where c and x are n-component column vectors, V and W_i, $i = 1, \ldots, m$, are symmetric $n \times n$ matrices, b is an m-component vector, a_i, $i = 1, \ldots, m$, are n-component row vectors with

$$A = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix},$$

and $(')$ denotes transpose. Two equivalent programs dual to (P) are

$$(\mathrm{D}_1) \qquad z_1 = \underset{\lambda \geq 0}{\text{minimize}} \left[\underset{x \geq 0}{\text{supremum}}\, L_1(x, \lambda) \right],$$

where $L_1(x, \lambda) = c'x + x'Vx + \sum_{i=1}^{m} \lambda_i (b_i - a_i x - x' W_i x)$, and

$$(\mathrm{D}_2) \qquad z_2 = \underset{\substack{\lambda \geq 0 \\ \gamma \geq 0}}{\text{minimize}} \left[\underset{x \in E^n}{\text{supremum}}\, L_2(x, \lambda, \gamma) \right],$$

where $L_2(x, \lambda, \gamma) = L_1(x, \lambda) + \gamma' x$ and E^n denotes Euclidean n-space.

The algorithm to be presented in the next section utilizes approximations to the computational forms of (D_1) and (D_2) to obtain a solution to (P). As an introduction to the theory on which the algorithm is based, the duality theorems relating (P) with (D_1) and (D_2) will be presented. First a computational form of (D_2) will be given. Assuming that $L_2(x, \lambda, \gamma)$ has a finite maximizer x for any (λ, γ), that maximizer must satisfy

$$(1) \qquad c + 2Vx - A'\lambda - 2 \sum_{i=1}^{m} \lambda_i W_i x + \gamma = 0.$$

(D_2) is thus equivalent to

$$(\mathrm{D}_2^*) \qquad z_2 = \underset{x, \lambda, \gamma}{\text{minimize}} \left[-x'Vx + \sum_{i=1}^{m} \lambda_i x' W_i x + \lambda' b \right]$$

$$\text{subject to } c + 2Vx - A'\lambda - 2\sum_{i=1}^{m} \lambda_i W_i x + \gamma = 0$$

$$\lambda \geq 0, \qquad \gamma \geq 0$$

where the objective function in (D_2^*) is obtained by postmultiplying (1) transposed by x and subtracting that from $L_2(x, \lambda, \gamma)$. The dual program (D_1) restricts x to be in the nonnegative orthant. Theorems 1 and 2 below indicate conditions under which the constraints $x \geq 0$ may be added to the following dual program (analogous to (D_1)) without affecting the solution.

$$(\mathrm{D}_1^*) \qquad z_1 = \underset{x, \lambda}{\text{minimize}} \left[-x'Vx + \sum_{i=1}^{m} \lambda_i x' W_i x + \lambda' b \right]$$

$$\text{subject to } c + 2Vx - A'\lambda - 2\sum \lambda_i W_i x \leq 0$$

$$\lambda \geq 0.$$

The following theorem indicates that (D_1^*) and (D_2^*) are duals of (P) and that solving (D_2^*) solves (D_1^*) and vice versa.

THEOREM 1: Let V be negative semidefinite and W_i, $i = 1, \ldots, m$, be positive semidefinite. If $\hat{x}$ solves (P) and a constraint qualification* is satisfied at $\hat{x}$, there exists $\hat{\lambda} \geq 0$ and $\hat{\gamma} \geq 0$ such that $(\hat{x}, \hat{\lambda}, \hat{\gamma})$ solves (D_2^*) (and equivalently (D_2)), $(\hat{x}, \hat{\lambda})$ solves (D_1^*) (and equivalently (D_1)), and $z = z_1 = z_2$. Conversely, if $(\hat{x}, \hat{\lambda})$ solves (D_1^*), there exists $\hat{\gamma} \geq 0$ such that $(\hat{x}, \hat{\lambda}, \hat{\gamma})$ solves (D_2^*).

---

*For the theorem any of the usual constraint qualifications may be used (see [6]), but since it will be used in the convergence proof, assume that there exists an x^0 such that $a_i x^0 + x^{0\prime} W_i x^0 < b_i$ for all i.

PROOF: The existence of $\hat{\lambda} \geqslant 0$ and $\hat{\gamma} \geqslant 0$ follows from the Kuhn-Tucker Theorem. That $(\hat{x}, \hat{\lambda}, \hat{\gamma})$ solves (D_2^*) and that $z = z_2$ follows from Geoffrion's Theorem 3 [5], since the objective function of (P) is concave and the constraints convex. To show that $(\hat{x}, \hat{\lambda})$ solves (D_1^*), note that $\hat{x} \geqslant 0$ from (P), and $\hat{\gamma} \geqslant 0$ implies that $(\hat{x}, \hat{\lambda})$ is feasible for (D_1^*). Since the objective functions of (D_1^*) and (D_2^*) are the same, and every (x, λ) satisfying the constraints of (D_1^*) is feasible for (D_2^*), $(\hat{x}, \hat{\lambda})$ solves (D_1^*) and $z_2 = z_1$. Conversely, if $(\hat{x}, \hat{\lambda})$ solves (D_1^*), there exists $\hat{\gamma} \geqslant 0$ such that $(\hat{x}, \hat{\lambda}, \hat{\gamma})$ solves (D_2^*), since if (x, λ, γ) is feasible for (D_2^*), then (x, λ) is feasible for (D_1^*). Q.E.D.

The next theorem presents conditions under which a solution to a dual (D_1^*) or (D_2^*) yields a solution to the primal.

THEOREM 2: If either (a) V is negative definite, or (b) some $\bar{\lambda}_i > 0$ with corresponding W_i positive definite, then a solution $(\bar{x}, \bar{\lambda}, \bar{\gamma})$ to (D_2^*) or $(\bar{x}, \bar{\lambda})$ to (D_1^*) is such that $\bar{x}$ solves (P) and $z = z_2 = z_1$.

PROOF: Under the hypotheses of the theorem $L_2(x, \bar{\lambda}, \bar{\gamma})$ is strictly concave in x and thus the maximizer $\bar{x}$ of $L_2(x, \bar{\lambda}, \bar{\gamma})$ is unique, so by Theorem 9 of Geoffrion [5] $\bar{x}$ solves (P) and $z = z_2$. Since $(\bar{x}, \bar{\lambda})$ is feasible for (D_1^*), every (x, λ) feasible for (D_1^*) is feasible for (D_2^*), and the objective functions of (D_1^*) and (D_2^*) are the same, $(\bar{x}, \bar{\lambda})$ solves (D_1^*) and $z_2 = z_1$. A similar argument holds for $L_1(x, \bar{\lambda})$. Q.E.D.

III. AN ALGORITHM

Two variations of a cutting plane algorithm, which approximate the dual programs (D_1^*) and (D_2^*), are offered. The basic algorithm was originally presented by Dantzig [1, ch. 24] as a column generation procedure, and the dual of that procedure has been considered by Zangwill [14, ch. 14] and Eaves and Zangwill [2]. The basic algorithm optimizes a sequence of linear master programs and either an unconstrained quadratic function or a quadratic function over the nonnegative orthant. The algorithm converges if either (a) V is negative definite, (b) at each iteration some $\lambda_i^k > 0$ and the corresponding W_i is positive definite, or (c) the set of all trial points x^k is compact. The optimality test is that the values of the objective functions of the primal and the dual are equal. The convergence proof is given in the appendix and is essentially that offered by Zangwill [14, ch. 14]. Eaves and Zangwill [2] have developed conditions under which certain of the constraints in steps 2_{I} and 2_{II} may be dropped.

ALGORITHM:

1. Choose a finite x^0 feasible* for (P) where the superscript indicates the iteration number.

Variation I:

2_{I}: At the kth iteration solve the linear program,

$$\begin{aligned} &\underset{\lambda,\mu}{\text{minimize}}\ \mu \\ &\text{subject to } L_1(x^{j_I}, \lambda) - \mu \leqslant 0, \qquad j_I = 0, \ldots, k_I - 1 \\ &\lambda \geqslant 0. \end{aligned}$$

Denote the solution by $(\mu^{k_I}, \lambda^{k_I})$.

3_{I}: Maximize $L_1(x, \lambda^{k_I})$

subject to $x \geqslant 0$.

*To prove convergence of the algorithm, the initial trial point x^1 should not be on the boundary of a constraint, since then an infinite λ_i may be optimal in step 2. In practice, any feasible x^1 may be used and a very large value for λ_i used if necessary. A strictly interior point for (P) may be generated by the techniques used with barrier methods (see [3]).

Denote the optimal solution† by x^{kI}. If $\mu^{kI} = L_1(x^{kI}, \lambda^{kI})$, x^{kI} solves (P). Otherwise go to step 2_{I}.

Variation II:

2_{II}: At the kth iteration solve the linear program,

$$\begin{aligned} &\underset{\lambda,\mu,\gamma}{\text{minimize}}\ \mu \\ &\text{subject to } L_2(x^{j_{II}}, \lambda, \gamma) - \mu \leqslant 0, \quad j_{II} = 0, \ldots, k_{II} - 1 \\ &\qquad \lambda \geqslant 0, \qquad \gamma \geqslant 0. \end{aligned}$$

Denote the optimal solution by $(\mu^{kII}, \lambda^{kII}, \gamma^{kII})$.

3_{II}: Maximize $L_2(x, \lambda^{kII}, \gamma^{kII})$

subject to $x \epsilon E^n$.

Denote the solution by x^{kII}. If $\mu^{kII} = L_2(x^{kII}, \lambda^{kII}, \gamma^{kII})$, x^{kII} solves (P). Otherwise go to step 2_{II}.

The advantage of Variation II is that if V is negative definite or if at an iteration $\lambda_i^{kII} > 0$ and W_i is positive definite, x^{kII} is given by the closed form

$$x^{kII} = \tfrac{1}{2}[V - \Sigma \lambda_i^{kII} W_i]^{-1}[A'\lambda^{kII} - c - \gamma^{kII}], \tag{2}$$

which may be easier to obtain than x^{kI} in step 3_{I}. The linear program in 2_{II}, however, contains more variables than that in 2_{I}. Since it may be difficult to predict whether a $\lambda_i^{kII} > 0$ for some W_i positive definite, Variation II may be used whenever $[V - \Sigma\lambda_i^{kII} W_i]$ is nonsingular and then switching to Variation I if it is singular.

The programs in step 2_{I} and 2_{II} approximate the duals (D_1), (D_1^*), (D_2), and (D_2^*), in the sense that

$$\underset{\lambda \geqslant 0}{\text{minimize}}\ [\underset{j_I = 1,\ldots,k_I - 1}{\text{maximize}}\ L_1(x^{j_I}, \lambda)] \tag{3}$$

approximates D_1. Let $\mu = \text{maximum } L_1(x^{j_I}, \lambda)$, so $L_1(x^{j_I}, \lambda) \leqslant \mu$ for all j_I. The program (3) is thus equivalent to that in 2_{I}. Alternatively, observe that (D_2^*) is equivalent to

$$\begin{aligned} &\underset{x,\lambda,\mu,\gamma}{\text{minimize}}\ \mu \\ &\text{subject to } -x'Vx + \Sigma\lambda_i x' W_i x + \lambda' b - \mu \leqslant 0 \\ &\qquad c + 2Vx - A'\lambda - 2\Sigma\lambda_i W_i x + \gamma = 0 \\ &\qquad \lambda \geqslant 0, \qquad \gamma \geqslant 0. \end{aligned}$$

Postmultiply the second constraint (transposed) by x and substitute into the first constraint and the program in step 2_{II} is obtained.

The primal program (P) may be written as

$$\underset{x \geqslant 0}{\max}\ \underset{\lambda \geqslant 0}{\min}\ L_1(x, \lambda) \tag{P$_1$}$$

or as

† Since $L_1(x, \lambda^{k_I})$ is concave in x and the only constraints are $x \geqslant 0$, the procedure of Theil and van de Panne [13] may be used.

$$(\mathrm{P}_2) \qquad \max_{x \in E^n} \min_{\substack{\gamma \geq 0 \\ \lambda \geq 0}} L_2(x, \lambda, \gamma).$$

Steps 3_{I} AND 3_{II} are thus approximations to (P_1) and (P_2), respectively, using λ^k or (λ^k, γ^k). The optimality test is that the objective function for the primal $L_1(x^k, \lambda^k)$ (or $L_2(x^k, \lambda^k, \gamma^k)$) equals the value of the objective function for the dual μ^{kI} (or μ^{kII}).

At any iteration the optimal value of the objective function of (P) is bounded above by $L_1(x^k, \lambda^k)$ (or $L_2(x^k, \lambda^k, \gamma^k)$) by the weak duality Theorem 2 [4]. To determine a lower bound on the optimal value of the objective function, let $t^j \geq 0$ be the dual variable associated with the jth constraint of the linear program in step 2_{I}. Clearly $\sum_{j=1}^{k-1} t^j(c'x^j + (x^j)'Vx^j) = \mu^k$ and $\sum_{j=1}^{k-1} t^j = 1$ by the duality theory for linear programs. Since the objective function is concave, by Jensen's inequality $\mu^k \leq c'(\Sigma t^j x^j) + (\Sigma t^j x^j)'V(\Sigma t^j x^j)$ where the latter is less than or equal to the optimal value of the objective function.

IV. AN EXAMPLE—PRODUCTION PLANNING

Schramm and Damon [10] have proposed a model of production planning involving quadratic constraints which stimulated the following example. Consider a firm that makes three decisions in a period: x_n the number of workers employed in period n, z_n the number of hours worked per employee, and y_n the units sold in period n. Output K_n is given by a quadratic production function

$$K_n = \alpha_n x_n z_n - \beta_n(x_{n-1} - x_n)^2 - \eta_n(z_{n-1} - z_n)^2.$$

The first term is a linear function of the total hours worked and the last two terms represent inefficiencies due to changes in the work force and the hours worked, respectively. The firm carries an inventory I_n between periods, $I_n \equiv I_{n-1} + K_n - y_n$, which is restricted to be nonnegative yielding a constraint quadratic in x_n and z_n. In addition, there may be a restriction on the change in hours worked per man $(z_{n-1} - z_n)^2 \leq B_n$.

The objective is to maximize profit over a finite horizon of N periods. Assume that revenue is given by $[P_n = (a_n - b_n y_n)] \cdot y_n$, where P_n is the price in period n. Costs are the variable cost of labor $\gamma_n x_n z_n$, where γ_n is the wage rate, the cost of hiring and firing $\delta_n(x_{n-1} - x_n)^2$, the cost of changing the hours worked $\xi_n(z_{n-1} - z_n)^2$, and the inventory holding cost $h_n I_n$. The firm's program then is

$$(4) \qquad \max \sum_{n=1}^{N} \left[(a_n - b_n y_n) \cdot y_n - \gamma_n x_n z_n - \delta_n(x_{n-1} - x_n)^2 - \xi_n(z_{n-1} - z_n)^2 - h_n I_n\right],$$

subject to

$$I_n \equiv I_{n-1} + \alpha_n x_n z_n - \beta_n(x_{n-1} - x_n)^2 - \eta_n(z_{n-1} - z_n)^2 - y_n \geq 0$$

$$(z_{n-1} - z_n)^2 \leq B_n$$

$$y_n, x_n, z_n \geq 0$$

$$n = 1, \ldots, N.$$

The initial conditions I_0, z_0, x_0 are assumed fixed.

As an example, (4) was solved using the algorithm for $N=1$ and the following parameters:

$$\beta=1; \quad \eta=1; \quad \alpha=.2; \quad a=20; \quad b=1; \quad \gamma=.2; \quad x_0=1$$

$$z_0=40; \quad \delta=1; \quad B=10; \quad c=45; \quad I_0=5; \quad h=5; \quad \xi=1.$$

The iterations using variation I are:

Iteration k	y^{k-1}	x^{k-1}	z^{k-1}	μ^k	λ_1^k	λ_2^k
1	20.0	1.0	40.0	−4.5	0	0
2	10.5	0.0	40.0	84.5172	2.6897	0
3	8.905	2.495	40.093	91.6621	1.5904	0
4	9.455	1.173	40.005	93.9250	1.2423	0
5	9.629	0.408	39.994	94.7028	1.4010	0
6	9.550	0.792	39.996	94.8446	1.3184	0
7	9.591	0.601	39.994	94.9073	1.2818	0
8	9.609	0.501	39.994	94.9083	1.3022	0
9	9.599	0.561	39.994	94.9092	1.3209	0
10	9.590	0.607	39.994			

The optimal solution is (approximately) $(y, x, z) = (9.590, 0.607, 39.994)$.

APPENDIX

Zangwill [14] has developed a general theory of algorithmic convergence which in many cases, including this one, simplifies proofs of convergence. Only the proof for Variation I will be given here. For cutting-plane algorithms, he has demonstrated that it is sufficient to show the following:

1. All test points (μ^k, λ^k) are contained in a compact set and all points x^k generated by the algorithm at an iteration are contained in a compact set.

2. Define $Y^k=\{\mu, \lambda \mid f(x^j)+\lambda' g(x^j)-\mu \leqslant 0;\ j=1, \ldots, k-1,\ \lambda \geqslant 0\}$, where $f(x^j)=c'x^j+x^{j\prime}Vx^j$ denotes the objective function in (P) and $g(x^j)=b-Ax^j-(x^{j\prime}W_1x^j, \ldots, x^{j\prime}W_mx^j)'$ denotes the constraints in (P) evaluated at the solution at the jth iteration. Then for the optimality test $\mu^k=L(x^k, \lambda^k)$, a point satisfying the test implies $(\mu^k, \lambda^k) \in Y^{k+1}$.

3. The map $\Delta(\lambda^k)$, which is the calculation of x^k in step 3 of the algorithm, is closed.* Also, $f(x)$ and $g(x)$ are continuous.

4. For (μ^k, λ^k) not satisfying the optimality test, and for x^k, it must be shown that

$$(\mu^k, \lambda^k) \notin H(x^k)=\{(u, \lambda) \mid f(x^k)+\lambda' g(x^k)-\mu \leqslant 0,\ \lambda \geqslant 0\}, \qquad \text{and } Y^k \cap H(x^k) \neq \varphi,$$

where φ is the empty set.

If the four conditions are satisfied, the sequence $\{x^k\}$ has a limit point that solves (P).

CONVERGENCE THEOREM: If either (a) V is negative definite, or (b) the set of points x^k is

*A map A from a set into its power set is closed at a limit point x^{∞} of a sequence x^k, $k \in K$ (an infinite subsequence of integers), if (a) $x^k \rightarrow x^{\infty}$, $k \in K$, (b) $y^k \rightarrow y^{\infty}$, $k \in K$, where y^k is the next point generated by the algorithm $A(x^{k-1})$ when x^{k-1} is not optimal, and (c) y^k is a possible value of x^k for a computed x^{k-1}, $k \bullet K$, together imply $y^{\infty} \in A(X^{\infty})$. In this case A is the algorithm generating successive values of x^k in step 3.

compact, or (c) at every iteration some $\lambda_i^k > 0$ and the corresponding W_i is positive definite, and if there exists an x^0 such that $g(x^0) = b - Ax^0 - (x^{0\prime}W_1x^0, \ldots, x^{0\prime}W_mx^0) > 0$, the sequence $\{x^k\}$ converges to a limit point that solves (P).

PROOF: The proof involves the verification of points 1–4.

1. To show that the (μ^k, λ^k) are contained in a compact set, it is sufficient to demonstrate that they are bounded. As indicated in section III, μ^k is bounded above by the optimal value of the objective function of (P). By assumption there exists a point x^0 such that $g(x^0) > 0$. Then for $\lambda = 0, f(x^0) \leq \mu^k$ from the program in step 2_I. Therefore, μ^k is bounded. To show that λ_i^k, $i = 1, \ldots, m$, are bounded, observe that $\lambda_i g_i(x^0) \leq \lambda' g(x^0)$, since $\lambda \geq 0$ and $g(x^0) > 0$. From the program in step 2_I $\lambda' g(x^0) \leq \mu - f(x^0) \leq f(\hat{x}) - f(x^0)$, where $\hat{x}$ is optimal for (P). Then

$$0 \leq \lambda_i \leq \frac{f(\hat{x}) - f(x^0)}{g_i(x^0)},$$

and λ_i is bounded for all i. It will be shown in 3 below that the solutions x^k generated in step 3 will be in a compact set in E^n.

2. At the optimal solution $x^k = \hat{x}$, $\mu^k = f(x^k) + \lambda^{k\prime} g(x^k)$, which is clearly in Y^{k+1}.

3. The map $\Delta(\lambda^k)$ is the calculation of x^k in step 3_I. To show that it is closed, it is sufficient (Zangwill, [14, ch. 7]) to show that the set of feasible solutions to the program in step 3_I is compact or equivalently in this case that the set is bounded. Under assumptions (a) or (c) above, the function $L_I(x, \lambda^k)$ is strictly concave, and a strictly concave, quadratic function must take on its maximum at a finite point. A bound on the set of solutions x^k thus exists and the map $\Delta(\lambda^k)$ is closed. If neither (a) nor (c) holds, (b) is required. Clearly, $f(x)$ and $g(x)$ are continuous.

4. If (μ^k, λ^k) does not satisfy the optimality test, then $\mu^k < f(x^k) + \lambda^{k\prime} g(x^k)$ so $(\mu^k, \lambda^k) \notin H(x^k)$. Finally, it is necessary to show there exists a point $(\mu, \lambda) \in Y^k \cap H(x^k)$. Clearly, $(\lambda = 0, f(\hat{x}) = \mu)$ is such a point. The algorithm converges.

REFERENCES

[1] Dantzig, G. B., *Linear Programming and Extensions* (Princeton University Press, Princeton, New Jersey, 1963).

[2] Eaves, B. Curtis and W. I. Zangwill, "Generalized Cutting Plane Algorithms," Working Paper No. 274, Center for Research in Management Science, University of California, Berkeley (July 1969).

[3] Fiacco, Anthony V. and Garth P. McCormick, "The Sequential Unconstrained Minimization Technique for Nonlinear Programming, A Primal-Dual Method," Management Science *10*, 360–366 (Jan. 1964).

[4] Kuhn, H. W. and A. W. Tucker, "Nonlinear Programming," in *Proc. of the Second Berkeley Symposium on Math. Stat. and Prob.*, J. Neyman, Ed. (U. of California Press, Berkeley, Calif., 1951), pp. 481–492.

[5] Geoffrion, A. M., "Duality in Nonlinear Programming: A Simplified Application-Oriented Development," SIAM Review *13*, 1–37 (Jan. 1971).

[6] Mangasarian, O. L., *Nonlinear Programming* (McGraw-Hill, New York, 1969).

[7] Peterson, E. L., and J. G. Ecker, "Geometric Programming: Duality and l_p-Approximation I," *Proc. Intl. Symp. on Math. Prog.*, H. W. Kuhn and A. W. Tucker, Eds. (Princeton, 1970).

[8] Peterson, E. L., and J. G. Ecker, "Geometric Programming: Duality in Quadratic Programming and l_p-Approximation II (Canonical Programs)," SIAM J. Appl. Math. *17*, 317–340 (Mar. 1969).

[9] Peterson, E. L., and J. G. Ecker, "Geometric Programming: Duality in Quadratic Programming and l_p-Approximation III (Degenerate Programs)," J. Math. Anal. Appl. *29*, 365–383 (Feb. 1970).

[10] Schramm, Richard, and William W. Damon, "A Model of Short-Term Production and Working Capital Management," paper presented at TIMS 10th American Meeting, Atlanta, Georgia (Oct. 1969).

[11] Sinha, J. M., "A Duality Theorem for Nonlinear Programming," Management Science *12*, 385–390 (1966).

[12] van de Panne, C., "Programming with a Quadratic Constraint," Management Science *12*, 709–815 (1966).

[13] Theil, H., and C. van de Panne, "Quadratic Programming as an Extension of Classical Quadratic Maximization," Management Science *7*, 1–20 (Oct. 1960).

[14] Zangwill, W. I., *Nonlinear Programming* (Prentice-Hall, Englewood Cliffs, New Jersey, 1969).