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EQUILIBRIUM POINTS OF BIMATRIX GAMES*

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Abstract. An algebraic proof is given of the existence of equilibrium points for bimatrix (or two-person, non-zero-sum) games. The proof is constructive, leading to an efficient scheme for computing an equilibrium point. In a nondegenerate case, the number of equilibrium points is finite and odd. The proof is valid for any ordered field.

A. Introduction. The two-person matrix game is defined as follows: The two players are designated M and N. Player M has m pure strategies at his disposal, and N has n. On any one play of the game, if M plays his ith pure strategy, and N plays his jth pure strategy, the payoff to M is $a_{i,j}$, and the payoff to N is $b_{i,j}$. Denote by A and B the m by n matrices whose (i, j)-elements are $a_{i,j}$ and $b_{i,j}$ respectively. The game is completely specified when the payoff matrices A and B are given.

A mixed strategy for M is a column x of nonnegative elements x_i, which represent the relative frequency with which M will play his ith pure strategy. Thus $x_1 + x_2 + \cdots + x_m = 1$. Likewise, a mixed strategy for N is a column y whose nonnegative components y_j sum to 1.

If on each play of the game, M and N select a pure strategy randomly, according to the probability distributions given by x and y, the expected payoffs to M and N respectively are

$$(1) \qquad \sum_{i=1}^{m} \sum_{j=1}^{n} x_i a_{i,j} y_j \quad \text{and} \quad \sum_{i=1}^{m} \sum_{j=1}^{n} x_i b_{i,j} y_j .$$

Let e denote the column of 1's (whose order will be understood from the context), and T denote matric transposition. If C is a matrix with components $c_{i,j}$, $C = 0$ means that $c_{i,j} = 0$, and $C \geqq 0$ means that $c_{i,j} \geqq 0$, for all values of i and j. In matric terms, a pair (x, y) of mixed strategies is defined by

$$(2) \qquad e^T x = e^T y = 1, \quad \text{and} \quad x \geqq 0, y \geqq 0,$$

and the corresponding payoffs may be expressed as

$$(3) \qquad x^T A y \quad \text{and} \quad x^T B y.$$

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An equilibrium point for the game is a pair (x_0, y_0) satisfying (2) such that, for all pairs (x, y) satisfying (2),

$$(4) \qquad x_0^T A y_0 \geqq x^T A y_0, \quad \text{and} \quad x_0^T B y_0 \geqq x_0^T B y.$$

Nash has shown [6], by a fixed point argument, valid for the real field, that an equilibrium point exists for each pair (A, B). Recently [2] A. Robinson has shown, by a metamathematical argument, that the proof of Nash is valid in an arbitrary ordered field. In a recent paper [3], Kuhn (assuming the existence) extended results of Vorobjev and characterized the set of all equilibrium points in a way which suggests a computational scheme for computing this set. Mills [5] poses the problem as a mixed integer programming problem. [4] contains a similar formulation. It may be noted that it has long been known that an equilibrium point for the zero-sum matrix game, namely the case $B = -A$, may be recognized as a pair of optimal solutions to an associated dual pair of linear programming problems (see for example Tucker's recent treatment [7]). From the presentation given below it would appear that a linear formulation in the general case is not possible.

In this paper it is shown, by an algebraic argument, that an equilibrium point lies on a path joining a sequence of adjacent extreme points of a certain convex polyhedron. Such a path, and hence an equilibrium point, is readily computed within the usual format of linear programming computations. It is worth remarking on the adaptation used here of the two concepts so useful in linear programming; namely the device of perturbing a convex polyhedron, and the generation of adjacent-extreme-point paths. These are fully described in [1], for example.

In the sequel, the sequence of steps will be (1) to prove, for an equivalent problem, that an equilibrium point exists for the problem in so-called nondegenerate form, and that (2) in this case the number of equilibrium points is odd, and all are extreme points of a certain convex polyhedron. An explicit extreme-point path which contains an equilibrium point is defined in the nondegenerate case, which in turn gives an equilibrium point for the general (not necessarily nondegenerate) case.

B. An equivalent formulation. A well-known equivalent formulation is derived here, which will be more suitable to the discussion. Let e_i be a column vector having 1 as its ith component and zeroes elsewhere. Observe first that, since the e_i satisfy (2), an equilibrium point (x_0, y_0) for the game must satisfy

$$(5) \qquad B^T x_0 \leqq (x_0^T B y_0) e \quad \text{and} \quad A y_0 \leqq (x_0^T A y_0) e.$$

Conversely, if (5) holds for the pair (x_0, y_0) satisfying (2), then for (x, y)

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satisfying (2), taking the matric product of x^T with both sides of the second expression, and the matric product of y^T with both sides of the first expression preserves the two inequalities. Hence, (2) and (5) are equivalent to (2) and (4).

Now let $E = ee^T$ be the matrix with all 1's, so that for (x, y) satisfying (2), $Ey = e$, and $x^TEy = 1$. Let k be fixed and large enough so that $kE - B^T > 0$, and $kE - A > 0$, and consider solutions (x, y) to

$$(6) \qquad \begin{aligned} &(kE - B^T)x \geqq e, x \geqq 0, \\ &(kE - A)y \geqq e, y \geqq 0, \end{aligned} \quad \text{and} \quad \begin{aligned} &y^T[(kE - B^T)x - e] = 0, \\ &x^T[(kE - A)y - e] = 0. \end{aligned}$$

There is then a one-to-one correspondence between the points (x_0, y_0) satisfying (2) and (5), and points (x, y) satisfying (6). Indeed, it is immediately verified that the correspondence is given by

$$x_0 = \left(\frac{1}{x^Te}\right) x \quad \text{and} \quad y_0 = \left(\frac{1}{y^Te}\right) y,$$

with

$$k - \left(\frac{1}{x^Te}\right) = x_0{}^TBy_0, \qquad k - \left(\frac{1}{y^Te}\right) = x_0{}^TAy_0.$$

C. The nondegenerate problem. In this section the following problem, namely that given in (6), will be considered: given matrices A and B of order m by n whose elements are all positive, find pairs (x, y) satisfying

$$(7) \qquad \begin{aligned} &B^Tx - e \geqq 0, x \geqq 0, \\ &Ay - e \geqq 0, y \geqq 0, \end{aligned} \quad \text{and} \quad \begin{aligned} &y^T(B^Tx - e) = 0, \\ &x^T(Ay - e) = 0. \end{aligned}$$

A pair (x, y) satisfying (7) will also be called an *equilibrium point.*

Geometric considerations. Let

$$(8) \qquad X = \{x: x \geqq 0, B^Tx - e \geqq 0\}.$$

Denote the identity matrix by I, and write

$$(9) \qquad B = (b_1, b_2, \cdots, b_n), \quad I = (e_1, e_2, \cdots, e_m),$$

so that b_j is the jth column of B, and e_i is the ith column of I. X is thus the set of points x satisfying the $m + n$ inequalities

$$(10) \qquad \begin{aligned} e_i{}^Tx &\geqq 0, & i &= 1, 2, \cdots, m, \\ b_j{}^Tx - 1 &\geqq 0, & j &= 1, 2, \cdots, n, \end{aligned}$$

with each of which one may therefore associate the corresponding column of the m by $(m + n)$ matrix (B, I).

Points on the boundary of X are points of X satisfying at least one of the

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$m + n$ linear relations

$$(11)\qquad \begin{aligned} e_i^T x \phantom{{}-1} &= 0, &\qquad i &= 1, 2, \cdots, m, \\ b_j^T x - 1 &= 0, &\qquad j &= 1, 2, \cdots, n. \end{aligned}$$

With any x is associated a unique matrix $M(x)$ consisting of columns of (B, I) such that b_j (resp., e_i) is a column of $M(x)$ if and only if $b_j^T x - 1 = 0$ (resp., $e_i^T x = 0$). The ordering of the columns of $M(x)$ will not be important.

If $\bar{B}$ is a matrix of m rows, whose columns are columns of (B, I), there is at most one x for which $\bar{B} = M(x)$ if rank $\bar{B} = m$. If further such an x is in X, it will be called an *extreme point* of X. It is of great convenience, if not of computational necessity, to ensure in what follows that for an extreme point x, $M(x)$ has precisely m columns. To this end, for the remainder of this section it will be assumed that X satisfies the

NONDEGENERACY ASSUMPTION. *Let $\bar{B}$ be a matrix of order m by r whose columns are columns of (B, I). If there is an x such that $\bar{B} = M(x)$, then $\bar{B}$ has rank r.*

We may remark that ways of perturbing the data defining a given convex polyhedron, such as X, in such a way that the perturbed data define a nondegenerate polyhedron, are well-known. In the next section, an explicit example of such a perturbation is given, and it is shown that such perturbation does not disturb the existence proof, and indeed facilitates the computation of an equilibrium point. Some relevant effects of the assumption are noted here.

Let x_0 be a point of X, and $M(x_0)$ have order m by r. Then rank $M(x_0) = r \leqq m$, so that the columns of $M(x_0)$ form a linearly independent set. Write $M(x_0) = (d_1, d_2, \cdots, d_r)$ (we include the possibility that $M(x_0)$ has no columns!) One may adjoin columns to $M(x_0)$ extending it to a nonsingular matrix which we shall label C: $C = (d_1, d_2, \cdots, d_m)$. Denote the inverse of C^T by C^{-T} and write: $C^{-T} = (d^1, d^2, \cdots, d^m)$. The d^i thus satisfy uniquely

$$d_i^T d^j = \delta_i^j,$$

where $\delta_i^j = 1$ if $i = j$, and $\delta_i^j = 0$ if $i \neq j$.

Consider points of the form

$$(12)\qquad x = x_0 + \sum_{i=1}^{m} t_i d^i,$$

where the t_i are scalars.

LEMMA. *There is a $k > 0$ such that for* (i) $\sum_{i=1}^m t_i^2 \leqq k$, *and* (ii) $t_i \geqq 0$ *for* $1 \leqq i \leqq r$, *points x of the form* (12) *are in X.*

Proof. Let d denote any column of (B, I), and consider

$$d^T x = d^T x_0 + \sum_{i=1}^{m} t_i (d^T d^i). \tag{13}$$

If $d = d_s$, where $1 \leqq s \leqq r$, the right side reduces to $\theta + t_s$, where θ is 1 if d is some b_j and is 0 if d is some e_i. Hence $\theta + t_s \geqq \theta$ if and only if $t_s \geqq 0$, for $1 \leqq s \leqq r$. If d is any other column of (B, I), then one has $d^T x_0 > \theta$, and hence $d^T x > \theta$, for t_i sufficiently small.

The Lemma may be considered the main effect of the assumption of nondegeneracy. From it may readily be deduced the following, which are used in the sequel.

(i) If x_0 is an extreme point of X, so that $M(x)$ is nonsingular, points of the form $x = x_0 + t_i d^i$ are in X, for t_i nonnegative and small enough; for those points with t_i positive, $M(x) = M_i$, where M_i is obtained from $M(x_0)$ by deleting the ith column d_i. The set of such points is an open edge of X with endpoint x_0.

(ii) If x_0 is a point of X for which rank $M(x_0) = m - 1$, then, referring to (12), for points of the form $x = x_0 + t_m d^m$ and $|\, t_m \,| \leqq k$, for some positive k, one has $M(x) = M(x_0)$. An *open edge* of X is defined as the (nonempty) set of all such points. Since for any x, points near x can only satisfy those linear relations (11) satisfied by x, we conclude that, with reference to (i) above, there are precisely m open edges of X having a given extreme point as an endpoint.

(iii) There are precisely m unbounded edges of X, and each has one endpoint; namely points of the form $x = ke_i$, for k positive and large enough. This follows from the fact that B has only positive elements. Any other open edge of X has two endpoints, which, by definition, form a pair of *adjacent extreme points*. Thus, two extreme points of X are adjacent if and only if their associated matrices differ in but one column.

Next, let

$$Y = \{y: y \geqq 0,\, Ay - e \geqq 0\}. \tag{14}$$

Points on the boundary of Y are points of Y satisfying at least one of the $m + n$ linear relations

$$\begin{aligned} a_i^T y - 1 &= 0, \qquad & i &= 1, 2, \cdots, m, \\ e_j^T y &= 0, \qquad & j &= 1, 2, \cdots, n, \end{aligned} \tag{15}$$

where $A^T = (a_1, a_2, \cdots, a_m)$.

As for X, it is assumed that Y satisfies the nondegeneracy assumption. Remarks made for X apply throughout for Y, with m and n interchanged,

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and (B, I) replaced by (I, A^T). Finally, $N(y)$ designates the matrix for y, as $M(x)$ for x.

Let $Z = (X, Y)$, the cartesian product of X and Y. A point $z = (x, y)$ will be called an extreme point of Z if and only if x is an extreme point of X, and y is an extreme point of Y; and z will be said to lie on an open edge of Z if and only if just one of x and y is an extreme point, and the other lies on an open edge.

Proof and construction. Now with reference to (7), the equilibrium conditions for an equilibrium point $z = (x, y)$ are that z is in Z, and

$$(16) \qquad \begin{aligned} (e_i^T x)(a_i^T y - 1) &= 0, \qquad & i &= 1, 2, \cdots, m, \\ (e_j^T y)(b_j^T x - 1) &= 0, \qquad & j &= 1, 2, \cdots, n. \end{aligned}$$

LEMMA. *An equilibrium point of the nondegenerate problem is an extreme point of Z.*

Proof. If $z = (x, y)$ is a point of Z satisfying the $m + n$ equilibrium conditions (16), then at least $m + n$ of the linear conditions (15) and (11) must hold. But, by nondegeneracy, x can satisfy at most m, and y can satisfy at most n. Hence x satisfies exactly m, and is thus an extreme point, and y satisfies exactly n, and is thus an extreme point.

Thus, an equilibrium point z is recognized as follows: for each r, $1 \leqq r \leqq m + n$, either the rth column of (B, I) is a column of $M(x)$, or the rth column of (I, A^T) is a column of $N(y)$, and (hence) not both.

Now, for fixed r, let S_r be the set of points of Z such that all $m + n$ of the equilibrium conditions (16), except possibly $(e_r^T y)(b_r^T x - 1) = 0$, are satisfied.

LEMMA. *Each point of S_r is either an extreme point of Z, or a point on an open edge of Z.*

Proof. If z is a point of S_r, then z satisfies at least $m + n - 1$ equilibrium conditions; hence at least $m + n - 1$ of the linear conditions (15) and (11). Any point of Z satisfies at most $m + n$ of the linear conditions. If z satisfies $m + n$, it is an extreme point of Z; if it satisfies $m + n - 1$, it is on an open edge of Z.

LEMMA. *There is precisely one unbounded open edge of Z composed of points of S_r.*

Proof. Consider $y = ke_r$. For k large enough, y is in Y. If k_0 denotes the least such value of k, then (a) $y_0 = k_0 e_r$ is an extreme point of Y, and (b) for just one value of j, say $j = s$, does one have $a_j^T y_0 = 1$.

Then consider points x of the form $x = ke_s$. It is then clear that, for k large enough, points (x, y_0), forming an unbounded edge of Z, are points of S_r. If k_1 denotes the least value for which $k_1 e_s$ is in X, and we set $x_0 = k_1 e_s$, then (x_0, y_0) is the endpoint of that edge. For future reference, we refer to this edge as E_0.

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Now one readily checks that a point on an edge of Z may not belong to both S_r and $S_{r'}$, for $r \neq r'$. In particular, one may therefore associate one and only one unbounded edge of Z with a given value of r.

LEMMA. *Let z be an extreme point of Z and a point of S_r. There are then one or two open edges of Z, consisting wholly of points of S_r, which have z as endpoint. z is an equilibrium point if and only if there is one such edge.*

Proof. The proof is based on merely counting linear conditions satisfied and equilibrium conditions satisfied.

Let z be an extreme point of Z and a point of S_r. There are then two cases, depending on whether $(e_r^T y)(b_r^T x - 1)$ equals 0 or is positive, where $z = (x, y)$.

Case I. $(e_r^T y)(b_r^T x - 1) = 0$. Then z is an equilibrium point. Conversely, any equilibrium point is a point of S_r. Hence for each of the equilibrium conditions, being $m + n$ in number, just one of the factors vanish. In particular, either $e_r^T y = 0$ or $b_r^T x - 1 = 0$, and not both. Suppose $e_r^T y = 0$ (the case $b_r^T x = 1$ is similarly treated). There are $m + n$ edges of Z which have z as common endpoint. Along any one of these precisely one of the $m + n$ linear conditions satisfied by z is violated. Only that edge along which $e_r^T y = 0$ is violated will consist of points of S_r.

Case II. $(e_r^T y)(b_r^T x - 1) > 0$. In this case, since z is an extreme point, and must thus satisfy $m + n$ of the linear conditions, there is precisely one equilibrium condition for which both factors vanish. If this condition is $e_t^T x = 0$, and $a_t^T y - 1 = 0$, for some t, then the two edges, one of which violates only the condition $e_t^T x = 0$, and the other of which violates only the condition $a_t^T y - 1 = 0$, consist wholly of points of S_r. Along any other edge with endpoint z some one equilibrium condition defining S_r must be violated.

Call two distinct open edges of Z *adjacent* if they have a common endpoint. We will consider sequences of adjacent open edges of S_r which, together with their endpoints, we shall call *r-paths*.

By the preceding lemmas, there exists an extreme point of Z which is a point of S_r, and starting from such an extreme point one may move either along one or along two edges of Z consisting of points of S_r. In any case, starting from such a point, call it z, one may move along an edge of Z consisting of points of S_r. Then either that edge ends in an extreme point, say z_1, of Z, or that edge is the single unbounded edge E_0 of points of S_r. In the former case, either z_1 is an equilibrium point, in which case one cannot proceed, or else z_1 is not an equilibrium point, in which case there is another edge, with endpoint z_1, along which one may continue. Continuing this process, therefore, starting from z we shall terminate the process if one either (i) enters the unbounded edge E_0 of S_r, (ii) reaches an equilibrium point (other than z itself), or (iii) returns to some point previously traversed.

But note that (i) no edge of S_r may be traversed twice in this process (since this would ultimately imply an extreme point of the path which is the endpoint of three edges of the path), and (ii) the number of extreme points of Z is finite. In particular, with reference to (iii) of the preceding paragraph, starting from z one may only return to the path at z, and then only if z is not an equilibrium point. In summary, starting from z either one returns to z or not. If one does the resulting path is an r-path which we call a *closed r-path*. If one does not return to z, the process ends either in an equilibrium point or in the edge E_0. There are then two cases: either z is an equilibrium point, in which case the process defines a complete r-path, or z is not an equilibrium point, in which case one may repeat the process along the other edge of points of S_r which has z as endpoint. We thus have:

LEMMA. *S_r is nonempty. S_r is the union of a finite number of disjoint r-paths. Each r-path is either a closed r-path (and contains no equilibrium points), or contains one or two equilibrium points.*

Now let P_0 be that r-path which contains the unbounded edge E_0.

THEOREM 1. *P_0 contains precisely one equilibrium point. This point may be computed by traversing P_0 starting with the unbounded edge E_0. The number of equilibrium points is finite and odd.*

Proof. Starting with E_0, the unique path through the r-path P_0 must terminate, and in an extreme point, which is then an equilibrium point.

Any path, other than P_0, which is not a closed r-path will have two endpoints, each of which is an equilibrium point. These endpoints must be distinct.

This completes the discussion of the nondegenerate case. In the next section the resolution of degeneracy is considered.

D. The general case. Resolving degeneracy. In this section it is shown that (i) the original data may be perturbed, using a procedure familiar from linear programming theory, to yield a nondegenerate problem for which the results of the preceding section are valid; (ii) the perturbation scheme ensures that any extreme point of the perturbed polyhedron defines a definite extreme point of the original polyhedron. In particular, an equilibrium point for the perturbed problem defines an equilibrium point for the original problem. The particular perturbation scheme is not unique. A computational scheme for computing an extreme point path using the perturbed data is well-known, and is not discussed here.

Consider $\epsilon > 0$ in the underlying ordered field. Let $e(\epsilon)$ denote a column, appropriate order assumed, whose ith component is the ith power of ϵ. Consider sets

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$$\begin{aligned} X(\epsilon) &= \{x : x \geqq 0, \quad B^T x - e + e(\epsilon) \geqq 0\}, \\ Y(\epsilon) &= \{y : y \geqq 0, \quad Ay - e + e(\epsilon) \geqq 0\}, \\ Z(\epsilon) &= (X(\epsilon), Y(\epsilon)). \end{aligned} \tag{17}$$

Clearly, $Z(\epsilon)$ contains $Z = Z(0)$.

LEMMA 1. *There exists ϵ_0 such that for $0 < \epsilon < \epsilon_0$, the set $X(\epsilon)$ satisfies the nondegeneracy assumption.*

Proof. Observe that if d denotes any column of (B, I), the linear condition corresponding to (11) may be written

$$d^T x - 1 + \epsilon^k = 0, \tag{18}$$

with the understanding that $k = 0$ if $d = e_i$ for any i, and that $k = j$ if $d = b_j$.

Let $\bar{B} = (d_1, d_2, \cdots, d_r)$ be an m by r matrix of rank r whose columns are columns of (B, I). Let d be any column of (B, I) which is not a column of $\bar{B}$, and suppose that d is a (unique) linear combination of $\bar{B}$:

$$d = \sum_{i=1}^{r} t_i d_i. \tag{19}$$

Suppose further that there is an x satisfying

$$d^T x = 1 - \epsilon^k; \quad \text{and} \quad d_i^T x = 1 - \epsilon^{k_i}, \quad i = 1, 2, \cdots, r, \tag{20}$$

for the appropriate values of k, k_i. Note that any two positive integers from k, k_i are distinct. From (19) and (20),

$$0 = d^T x - \sum_{i=1}^{r} t_i d_i^T x = (1 - \epsilon^k) - \sum_{i=1}^{r} t_i (1 - \epsilon^{k_i}), \tag{21}$$

a polynomial equation of degree no greater than m. We assert that the right-hand-side is not identically 0, for if it were then necessarily (i) $k = 0$, so that $d = e_j$, some j, and (ii) $k_i = 0$ for those i for which t_i is not 0. But then, referring to (19), e_j would be expressed as a linear combination of other e_i, which is not possible. It follows that the right-hand-side is equal to 0 for at most m values of ϵ.

Since (i) for the given $\bar{B}$, a d from (B, I) satisfying (19) and (20) is possible for at most a finite number of values of ϵ, and (ii) the number of such $\bar{B}$'s is finite, it follows that $X(\epsilon)$ satisfies the nondegeneracy assumption for all but a finite number of values of ϵ. There is thus some open range $(0, \epsilon_0)$, such that for any ϵ in the range, $X(\epsilon)$ is nondegenerate.

LEMMA 2. *There is a range $(0, \epsilon_1)$ for which the following holds. Let M be any nonsingular matrix (with columns from (B, I)), and let $x_0(\epsilon)$ denote*

the unique point for which $M = M(x_0(\epsilon))$. *If for some* $\bar{\epsilon}$ *in the range,* $x_0(\bar{\epsilon})$ *is an extreme point of* $X(\bar{\epsilon})$ *then*
(i) $x_0(\epsilon)$ *is an extreme point of* $X(\epsilon)$ *for all* ϵ *in the range, and*
(ii) $x_0 = x_0(0)$ *is an extreme point of* X.

Proof. As a preliminary, let $f(\epsilon) = \epsilon^n + a_{n-1}\epsilon^{n-1} + \cdots + a_1\epsilon + a_0$ be any polynomial. The triangle inequality yields

$$|f(\epsilon)| \leqq \sum_{i=0}^{n} |a_i| \quad \text{for} \quad |\epsilon| < 1. \tag{22}$$

Next, for a given polynomial, let r be the least value for which $a_r \neq 0$. We may then write

$$f(\epsilon) = \epsilon^r(a_r + \epsilon g(\epsilon)). \tag{23}$$

Applying (22) to g, it is clear that there is a range $(0, \bar{\epsilon})$ within which f is either 0 or takes the sign of a_r.

Next, for a given $\bar{\epsilon}$ let $x_0(\bar{\epsilon})$ be an extreme point of $X(\bar{\epsilon})$, and let $M = M(x_0(\bar{\epsilon}))$. Then M, as a nonsingular matrix, defines, for positive ϵ, and with reference to $X(\epsilon)$, a point $x_0(\epsilon)$. Then $x_0(\epsilon)$ is an extreme point of $X(\epsilon)$ if and only if it is a point of $X(\epsilon)$. Next, let d be a column of (B, I) which is not a column of M, and consider

$$d^T x_0(\epsilon) - 1 + \epsilon^k = f(\epsilon). \tag{24}$$

With reference to Lemma 1, for ϵ in the range $(0, \epsilon_0)$, $f(\epsilon)$ is a polynomial in ϵ which does not vanish identically. With reference to (23), the "dominant coefficient" a_r of f is independent of ϵ. Hence, given M and d, f is uniquely defined and there is an $\bar{\epsilon} < \epsilon_0$ such that for ϵ in the range $(0, \bar{\epsilon})$, f is not 0 and the sign of f is the sign of a_r.

The number of such pairs M, d is finite. There is therefore some $\epsilon_1 < \epsilon_0$ such that for ϵ in the range $(0, \epsilon_1)$, for any pair M, d, the corresponding f retains its sign.

In particular for this range, if x is a point of $X(\epsilon)$, we have $f(\epsilon) > 0$, for all such f's, and part (i) of the lemma follows.

For part (ii), (22) gives $f(0) = 0$ ($r \neq 0$, a definite assertion of degeneracy), or $f(0) = a_r$ ($r = 0$). In any case, $f(0)$ may not take the sign opposite to a_r, and hence x_0 is in X, and hence is an extreme point.

THEOREM 2. *There exists an equilibrium point for the problem defined by* (7).

Proof. The proof follows from the Lemmas. With ϵ_1 as in Lemma 2, by the results of §B (assuming ϵ_1 holds equally well for $Y(\epsilon)$), $Z(\epsilon)$ has a (computable) equilibrium point. Any equilibrium point $z_0(\epsilon)$ defines a pair M and N and hence x_0 and y_0 such that $z_0 = (x_0, y_0)$ is an extreme point and an equilibrium point of Z.

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Let Q denote the set of equilibrium points for Z. We close this section with a few statements relating to Q. Q is bounded and may be characterized by $\bar{Q}$, the set of extreme point equilibrium points: Q is the union of the convex hulls of certain subsets of $\bar{Q}$ (as discussed in [3]).

E. Remarks. The results presented above give a prescription for computing efficiently one equilibrium point. It is not clear at this time how or in what manner these results may be extended to an efficient scheme for computing $\bar{Q}$. For example, $\bar{Q}(\epsilon) = Q(\epsilon)$, and one may locate all equilibrium points of $Q(\epsilon)$ by, for example, traversing all paths of S_r. But a scheme for finding all r-paths (excluding closed r-paths) is not in evidence. Further, it is seen by simple examples that $\bar{Q}(\epsilon)$ may *not* yield all of $\bar{Q}$; specifically in the event that Z is degenerate.

Care has been taken to keep all arguments algebraic, so that the results hold in an arbitrary ordered field.

It may be observed that a "dual" development is possible, initiated by the introduction of "slack variables", and reposing (7) in the, perhaps more popular, form:

$$\begin{pmatrix} 0 & B^T \\ A & 0 \end{pmatrix}\begin{pmatrix} y \\ x \end{pmatrix} - \begin{pmatrix} w \\ u \end{pmatrix} = \begin{pmatrix} e \\ e \end{pmatrix}; \quad \begin{pmatrix} y \\ x \end{pmatrix} \geqq 0,$$

$$\begin{pmatrix} w \\ u \end{pmatrix} \geqq 0, \quad \text{and} \quad \begin{pmatrix} y \\ x \end{pmatrix}^T \begin{pmatrix} w \\ u \end{pmatrix} = 0.$$

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