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# A New Wide Neighborhood Primal–Dual Infeasible-Interior-Point Method for Symmetric Cone Programming

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**Abstract** We present a new infeasible-interior-point method, based on a wide neighborhood, for symmetric cone programming. The convergence is shown for a commutative class of search directions, which includes the Nesterov–Todd direction and the $xs$ and $sx$ directions. Moreover, we derive the complexity bound of the wide neighborhood infeasible interior-point methods that coincides with the currently best known theoretical complexity bounds for the short step path-following algorithm.

**Keywords** Jordan algebra · Symmetric cone programming · Wide neighborhood · Infeasible-interior-point method · Polynomial complexity

## 1 Introduction

Since Karmarkar's ground-breaking paper [1], interior-point methods (IPMs) became one of the most active research areas. For a survey we refer to the books [2–4]. It is broadly accepted today that a primal–dual algorithm is the most efficient IPM. The first extension of primal–dual IPMs to a more general setting than linear programming was achieved by Nesterov and Todd [5, 6]. These authors developed the concept of self-scaled barriers and self-scaled cones (those cones that are endowed with

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a self-scaled barrier) and showed the iteration complexity of a particular primal–dual algorithm (see [7]). Güler [8] observed that the self-scaled cones are precisely the symmetric cones, thus the Nesterov–Todd (NT) algorithm was the first primal–dual IPM for optimization over symmetric cones. It is well known that symmetric cone programming (SCP) includes linear programming (LP), semidefinite programming (SDP) and second order cone programming (SOCP) as special cases. Thus, more and more attention has been focused on the programming problems over symmetric cones. Faybusovich [9] first extended primal–dual IPMs to SCP through Euclidean Jordan algebraic tools. Schmieta and Alizadeh [10] proved polynomial iteration complexities for variants of the short-, semi-long-, and long-step path-following algorithms over symmetric cones. Vieira [11, 12] proposed primal–dual IPMs for SCP based on the kernel functions. Recently, Wang and Bai [13] generalized Darvay's full-Newton step primal–dual path-following IPM for LP in [14] and presented a new full NT step primal–dual path-following IPM for SCP. Liu et al. [15] and Zhang and Zhang [16] proposed IPMs with the second-order corrector step for SCP and showed the polynomial convergence.

The aforementioned algorithms concern on the so-called feasible IPMs, which require the starting point be strictly feasible. This requirement may be difficult to obtain in practical implementation. However, infeasible-interior-point methods (IIPMs), unlike feasible-IPMs, do not require that the iterates be feasible to the relevant linear systems, but only be in the interior of the cone constraints. As such, infeasible points are easy to obtain, IIPMs are known as practically efficient algorithms among numerous variations of the primal–dual IPMs. At the same time, the analysis of IIPMs presents significant difficulties due to the non-orthogonality of search directions. The IIPMs were first proposed by Lustig [17] and Tanabe [18]. Mizuno [19] introduced a primal–dual IIPM for LP and proved global convergence of the algorithm. Potra and Sheng [20, 21] independently investigated the superlinear convergence of primal–dual IIPMs for SDP. Later, Zhang [22] analyzed the convergence of an IIPM for SDP using the $xs$ and $sx$ search directions. Rangarajan and Todd [23] established the polynomial complexity of a long step IIPM for SCP. Rangarajan [24] proposed an IIPM for SCP using a wide neighborhood of the central path and proved the convergence for a commutative family of search directions. Motivated by their work, we present a new IIPM for SCP.

In fact, many early primal–dual IPMs are based on the small neighborhood or the so-called negative infinity wide neighborhood. Later, Ai and Zhang [25] give a new wide neighborhood for linear complementarity problem (LCP). Based on the new wide neighborhood, the derived iteration bound of their algorithm yields the first wide neighborhood path-following algorithm having the same theoretical complexity as a small neighborhood algorithm. Later, Li and Terlaky [26] generalized the Ai–Zhang's idea to SDP and showed that the iteration complexity of their algorithm is the same as that of Ai and Zhang [25]. Analogously, we extend the Ai–Zhang's wide neighborhood to symmetric cone and obtain our wide neighborhood. Using the proposed wide neighborhood, we design a new IIPM over symmetric cones and show the convergence for the commutative class of search direction. The complexity of the wide neighborhood IIPM has the same theoretical complexity bound as the best short step path-following algorithm IPMs.

The outline of this paper is as follows. Some key theories of the symmetric cones are given in Sect. 2. We provide some preliminaries about the new IIPM in Sects. 3, 4 and 5. We propose the new IIPM and establish polynomial-time convergence for infeasible and feasible starting points in Sect. 6. Some conclusions are given in Sect. 7.

## 2 Euclidean Jordan Algebra and Symmetric Cones

In this section, we describe briefly some concepts, properties, and results from Euclidean Jordan algebras (for more details, see [27]).

Let $(\mathcal{J}, \langle\cdot,\cdot\rangle)$ be an $n$-dimensional inner product space over the real field $\mathbb{R}$ with a bilinear mapping $\circ : \mathcal{J} \times \mathcal{J} \mapsto \mathcal{J}$. Then the triple $(\mathcal{J}, \circ, \langle\cdot,\cdot\rangle)$ is a Euclidean Jordan algebra if the following conditions hold:

(i) $x \circ y = y \circ x$ for all $x, y \in \mathcal{J}$;
(ii) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ for all $x, y \in \mathcal{J}$, where $x^2 := x \circ x$;
(iii) $\langle x \circ y, z\rangle = \langle y, x \circ z\rangle$ for all $x, y, z \in \mathcal{J}$.

A Jordan algebra $\mathcal{J}$ has an identity element, if there exists an element $e \in \mathcal{J}$ such that $x \circ e = e \circ x = x$ for all $x \in \mathcal{J}$. The set $\mathcal{K} := \{x^2 : x \in \mathcal{J}\}$ is called the cone of squares of Euclidean Jordan algebra $\mathcal{J}$. A cone is symmetric if and only if it is the cone of squares of some Euclidean Jordan algebra. $\operatorname{int}\mathcal{K}$ denotes the interior of the symmetric cone $\mathcal{K}$.

For $x \in \mathcal{J}$, let $r$ be the smallest integer such that the set $\{e, x, x^2, \ldots, x^r\}$ is linearly dependent. Then $r$ is called the degree of $x$ and denoted by $\deg(x)$. The rank of $\mathcal{J}$ is the maximum of $\deg(x)$ over all members $x \in \mathcal{J}$.

An idempotent $c$ is a nonzero element of $\mathcal{J}$ such that $c^2 = c$. A complete system of orthogonal idempotents is a set $\{c_1, \ldots, c_k\}$ of idempotents, where $c_i \circ c_j = 0$ for all $i \neq j$, and $c_1 + \cdots + c_k = e$. An idempotent is primitive if it is not the sum of two other idempotents. A complete system of orthogonal primitive idempotents is called a Jordan frame.

**Theorem 2.1** (Spectral decomposition [27, Theorem III.1.2]) *Let $\mathcal{J}$ be a Euclidean Jordan algebra with rank $r$. Then for every $x \in \mathcal{J}$, there exist a Jordan frame $\{c_1, \ldots, c_r\}$ and real numbers $\lambda_1, \ldots, \lambda_r$ such that*

$$x = \lambda_1 c_1 + \cdots + \lambda_r c_r.$$

*The numbers $\lambda_i$ are called the* (*spectral*) *eigenvalues of $x$.*

Given the spectral decomposition $x = \lambda_1 c_1 + \cdots + \lambda_r c_r$, some functions in the eigenvalues can be generated as follows:

(i) The inverse $x^{-1} := \lambda_1^{-1} c_1 + \cdots + \lambda_r^{-1} c_r$, whenever all $\lambda_i \neq 0$;
(ii) The square root $\sqrt{x} := \sqrt{\lambda_1} c_1 + \cdots + \sqrt{\lambda_r} c_r$, whenever all $\lambda_i \geq 0$;
(iii) The trace $\operatorname{tr}(x) := \lambda_1 + \cdots + \lambda_r$;
(iv) The determinant $\det(x) := \lambda_1 \cdots \lambda_r$;
(v) The Frobenius norm $\|x\|_F := \sqrt{\langle x, x\rangle} = \sqrt{\sum \lambda_i^2}$;

(vi) The metric projection $x^+ := \lambda_1^+ c_1 + \cdots + \lambda_r^+ c_r$, where $\lambda_i^+ = \max\{\lambda_i, 0\}$ for $i = 1, 2, \ldots, r$. Moreover, $x^- := x - x^+$.

Since "$\circ$" is bilinear for every $x \in \mathcal{J}$, there exists a linear operator $L_x$ such that for every $y \in \mathcal{J}$, $x \circ y := L_x y$. In particular, $L_x e = x$ and $L_x x = x^2$. For each $x, y \in \mathcal{J}$ define $Q_{x,y} := L_x L_y + L_y L_x - L_{x\circ y}$, $Q_x := Q_{x,x} = 2L_x^2 - L_{x^2}$. $Q_x$ is called the quadratic representation of $x$. In particular, one has $Q_x e = x^2$. The following are two useful propositions about the quadratic representation.

**Proposition 2.1** ([27, Proposition III.2.2]) *If* $x, y \in \operatorname{int}\mathcal{K}$, *then* $Q_x y \in \operatorname{int}\mathcal{K}$.

**Proposition 2.2** ([10, Proposition 21]) *Let* $x, y, p \in \operatorname{int}\mathcal{K}$. *Moreover, define* $\tilde{x} := Q_p x$ *and* $\tilde{y} := Q_{p^{-1}} y$. *Then*

(i) $Q_{x^{1/2}} y$ *and* $Q_{y^{1/2}} x$ *have the same spectrum.*
(ii) $Q_{x^{1/2}} y$ *and* $Q_{\tilde{x}^{1/2}} \tilde{y}$ *have the same spectrum.*

**Proposition 2.3** ([28, Proposition 2.9]) *Let* $x, y \in \operatorname{int}\mathcal{K}$. *If* $x$ *and* $y$ *operator commute, then* $Q_{x^{1/2}} y = Q_{y^{1/2}} x = x \circ y$.

Moreover, we give two important lemmas.

**Lemma 2.1** ([29, Lemma 2.15]) *If* $x \circ s \in \operatorname{int}\mathcal{K}$, *then* $\det(x) \neq 0$.

**Lemma 2.2** ([24, Lemma 2.9]) *For* $x, y \in \mathcal{J}$, *then* $\|x \circ y\|_F \leq \|x\|_F \|y\|_F$.

## 3 Problem Background

Let $\mathcal{J}$ be a Euclidean Jordan algebra of dimension $n$ and rank $r$, and $\mathcal{K}$ be its cone of squares. Consider the following primal and dual problem:

$$\text{(P)} \quad \min\langle c, x\rangle, \quad \text{s.t. } Ax = b,\ x \in \mathcal{K},$$

and

$$\text{(D)} \quad \max\langle b, y\rangle, \quad \text{s.t. } A^* y + s = c, s \in \mathcal{K},\ y \in \mathbb{R}^m,$$

where $c \in \mathcal{J}$, $b \in \mathbb{R}^m$ and $A$ is a linear operator that maps $\mathcal{J}$ into $\mathbb{R}^m$ and $A^*$ is its adjoint operator such that $\langle x, A^* y\rangle = \langle Ax, y\rangle$ for all $x \in \mathcal{J}$, $y \in \mathbb{R}^m$.

For convenience of reference, we define the following two sets:

$$\mathcal{F} := \left\{(x, y, s) \in \mathcal{K} \times \mathbb{R}^m \times \mathcal{K} : Ax = b,\ A^* y + s = c\right\},$$

$$\mathcal{F}^0 := \left\{(x, y, s) \in \operatorname{int}\mathcal{K} \times \mathbb{R}^m \times \operatorname{int}\mathcal{K} : Ax = b,\ A^* y + s = c\right\}.$$

We call $\mathcal{F}$ and $\mathcal{F}^0$, respectively, the (primal–dual) feasibility set and strict feasibility set of (P) and (D). $(x, y, s)$ is said to be feasible if $(x, y, s) \in \mathcal{F}$ and strictly feasible if it $(x, y, s) \in \mathcal{F}^0$. We denote the sets of optimal solutions of (P) and (D) by $\mathcal{P}^*$ and $\mathcal{D}^*$, respectively. In this paper, we assume that $A$ is surjective and $\mathcal{F}^0 \neq \emptyset$.

Faybusovich [9] showed that $x^*$ and $(y^*, s^*)$ are optimal solutions if and only if they satisfy the following system

$$\begin{aligned} Ax &= b, \quad x \in \mathcal{K}, \\ A^*y + s &= c, \quad s \in \mathcal{K}, \ y \in \mathbb{R}^m, \\ x \circ s &= 0. \end{aligned} \tag{1}$$

The perturbed system of (1) is obtained by replacing $x \circ s = 0$ with $x \circ s = \tau\mu e$. Applying Newton's method for the perturbed system leads to the following Newton's equations:

$$\begin{aligned} A\Delta x &= b - Ax, \\ A^*\Delta y + \Delta s &= c - s - A^*y, \\ s \circ \Delta x + x \circ \Delta s &= \tau\mu e - x \circ s, \end{aligned} \tag{2}$$

where $\tau \in ]0, 1[$ is called a centering parameter. Moreover, the primal and dual residuals are denoted by $r_p$ and $r_d$ and are defined as follows:

$$r_p = b - Ax \quad \text{and} \quad r_d = c - s - A^*y.$$

Due to the fact that $x$ and $s$ do not operator commute in general, (2) doesn't always have a unique solution. It is well known that this difficulty can be solved by applying a scaling scheme as follows [10, Lemma 28].

The scaled Newton's system follows directly from (2) as

$$\begin{aligned} \tilde{A}\Delta\tilde{x} &= b - \tilde{A}\tilde{x}, \\ \tilde{A}^*\Delta y + \Delta\tilde{s} &= \tilde{c} - \tilde{A}^*y - \tilde{s}, \\ \tilde{s} \circ \Delta\tilde{x} + \tilde{x} \circ \Delta\tilde{s} &= \tau\mu e - \tilde{x} \circ \tilde{s}, \end{aligned} \tag{3}$$

where $\tilde{A} = AQ_{p^{-1}}$, $\tilde{c} = Q_{p^{-1}}c$ and $\tilde{x} = Q_p x$, $\tilde{s} = Q_{p^{-1}}s$ and $\Delta\tilde{x} = Q_p\Delta x$, $\Delta\tilde{s} = Q_{p^{-1}}\Delta s$ and scaling $p$ is selected from the commutative class

$$\mathcal{C}(x, s) := \{p \in \operatorname{int}\mathcal{K} : Q_p x \text{ and } Q_{p^{-1}}s \text{ operator commute}\}.$$

In particular, choosing $p = s^{1/2}$ and $p = x^{-1/2}$, we get the $xs$ and $sx$ search directions, respectively. Moreover, for the choice of

$$p = \left[Q_{x^{1/2}}(Q_{x^{1/2}}s)^{-1/2}\right]^{-1/2} = \left[Q_{s^{-1/2}}(Q_{s^{1/2}}x)^{1/2}\right]^{-1/2},$$

we obtain the NT search direction.

In this paper, we restrict the scaling $p \in \mathcal{C}(x, s)$ and replace the Newton's system (3) with the following system:

$$\begin{aligned} \tilde{A}\Delta\tilde{x} &= b - \tilde{A}\tilde{x}, \\ \tilde{A}^*\Delta y + \Delta\tilde{s} &= \tilde{c} - \tilde{A}^*y - \tilde{s}, \\ \tilde{s} \circ \Delta\tilde{x} + \tilde{x} \circ \Delta\tilde{s} &= (\tau\mu e - \tilde{x} \circ \tilde{s})^- + \sqrt{r}(\tau\mu e - \tilde{x} \circ \tilde{s})^+. \end{aligned} \tag{4}$$

Let $\alpha \in [0, 1]$ be the step sizes taken along $(\Delta\tilde{x}, \Delta y, \Delta\tilde{s})$. Then, the new iterate is given by

$$\big(\tilde{x}(\alpha), y(\alpha), \tilde{s}(\alpha)\big) := (\tilde{x}, y, \tilde{s}) + \alpha(\Delta\tilde{x}, \Delta y, \Delta\tilde{s}). \tag{5}$$

Moreover, we have

$$r_p(\alpha) = b - AQ_{p^{-1}}\tilde{x}(\alpha) = b - A(x + \alpha\Delta x) = (1-\alpha)r_p. \tag{6}$$

$$\begin{aligned} r_d(\alpha) &= c - Q_p\tilde{s}(\alpha) - A^* y(\alpha) = c - (s + \alpha\Delta s) - A^*(y + \alpha\Delta y) \\ &= (1-\alpha)r_d. \end{aligned} \tag{7}$$

$$\begin{aligned} \tilde{\mu}(\alpha) &= \big\langle \tilde{x}(\alpha), \tilde{s}(\alpha)\big\rangle / r = \big\langle Q_p(x + \alpha\Delta x), Q_{p^{-1}}(s + \alpha\Delta s)\big\rangle / r \\ &= \langle x + \alpha\Delta x, s + \alpha\Delta s\rangle / r = \mu(\alpha). \end{aligned}$$

## 4 New Wide Neighborhood

The so-called negative infinity neighborhood that is a wide neighborhood is defined as $\mathcal{N}_\infty^-(1-\gamma) := \{(x, y, s) \in \operatorname{int}\mathcal{K} \times \mathbb{R}^m \times \operatorname{int}\mathcal{K} : \lambda_{\min}(Q_{x^{1/2}}s) \geq \gamma\mu\}$, where $\gamma \in ]0, 1[$ and $\mu = \langle x, s\rangle / r$ is the normalized duality gap.

Ai and Zhang in [25] introduced a new neighborhood of the central path for LCP. Later, Li and Terlaky [26] extended it to SDP problems. Analogously, we define our neighborhood for SCP as follows:

$$\mathcal{N}(\tau, \tau_1, \eta) = \mathcal{N}_\infty^-(1-\tau_1) \cap \big\{(x, y, s) \in \mathcal{F}^0 : \big\|(\tau\mu e - Q_{x^{1/2}}s)^+\big\|_F \leq \eta(\tau - \tau_1)\mu\big\},$$

where $\eta \geq 1$ and $0 < \tau_1 < \tau < 1$. Obviously, $\mathcal{N}(\tau, \tau_1, \eta)$ is even wider than a given $\mathcal{N}_\infty^-(1-\tau)$ neighborhood.

In particular, if we choose $\eta = 1$, then $\|(\tau\mu e - Q_{x^{1/2}}s)^+\|_F \leq (\tau - \tau_1)\mu$ implies $\lambda_{\min}(Q_{x^{1/2}}s) \geq \tau_1\mu$, which is $\mathcal{N}_\infty^-(1-\tau_1) \subset \mathcal{N}(\tau, \tau_1, 1)$. Hence we have $\mathcal{N}(\tau, \tau_1, 1) = \{(x, y, s) \in \mathcal{F}^0 : \|(\tau\mu e - Q_{x^{1/2}}s)^+\|_F \leq (\tau - \tau_1)\mu\}$.

To facilitate our analysis, we represent $\mathcal{N}(\tau, \tau_1, 1)$ as $\mathcal{N}(\tau, \beta)$, that is,

$$\mathcal{N}(\tau, \beta) := \big\{(x, y, s) \in \operatorname{int}\mathcal{K} \times \mathbb{R}^m \times \operatorname{int}\mathcal{K} : \big\|(\tau\mu e - w)^+\big\|_F \leq \beta\tau\mu\big\}, \tag{8}$$

where $w = Q_{x^{1/2}}s$, $\beta = (\tau - \tau_1)/\tau$ and $\mu = \langle x, s\rangle / r$.

*Remark 4.1* Note that by part (i) of Proposition 2.2, $Q_{x^{1/2}}s$ and $Q_{s^{1/2}}x$ have the same spectrum, thus $\mathcal{N}(\tau, \tau_1, \eta)$ is symmetric with respect to $x$ and $s$.

The neighborhood $\mathcal{N}(\tau, \beta)$ can be defined in terms of eigenvalues of $w$. Let $\tilde{w} = Q_{\tilde{x}^{1/2}}\tilde{s}$. By part (ii) of Proposition 2.2, $w$ and $\tilde{w}$ have the same eigenvalues. Therefore, using the proof techniques of Proposition 3.2 in [24], one has the following lemma.

**Lemma 4.1** *The neighborhood $\mathcal{N}(\tau, \beta)$ is scaling invariant, that is, $(x, y, s)$ is in the neighborhood if and only if $(\tilde{x}, y, \tilde{s})$ is.*

## 5 Step Size Selection

In this section, we discuss how to choose a step size $\alpha$. Our choice of the step size $\alpha$ is based on several considerations as follows:

A1. Sufficient reduction of the duality gap $\mu(\alpha)$;
A2. Ensuring $(\tilde{x}(\alpha), y(\alpha), \tilde{s}(\alpha)) \in \mathcal{N}(\tau, \beta)$;
A3. Sufficient reduction of the residual norms $\|r_p(\alpha)\|_F$ and $\|r_d(\alpha)\|_F$.

In order to achieve this purpose, we need some technical results.

### 5.1 Technical Lemmas

First, using (4) and (5), we easily obtain the following expression:

$$\begin{aligned}\tilde{x}(\alpha) \circ \tilde{s}(\alpha) &= \tilde{x} \circ \tilde{s} + \alpha\big[(\tau\mu e - \tilde{x} \circ \tilde{s})^- + \sqrt{r}(\tau\mu e - \tilde{x} \circ \tilde{s})^+\big] + \alpha^2 \Delta\tilde{x} \circ \Delta\tilde{s} \\ &= T(\alpha) + \alpha^2 \Delta\tilde{x} \circ \Delta\tilde{s}, \end{aligned} \tag{9}$$

where

$$T(\alpha) := \tilde{x} \circ \tilde{s} + \alpha\big[(\tau\mu e - \tilde{x} \circ \tilde{s})^- + \sqrt{r}(\tau\mu e - \tilde{x} \circ \tilde{s})^+\big]. \tag{10}$$

$$\begin{aligned}\big\langle \tilde{x}(\alpha), \tilde{s}(\alpha)\big\rangle = \mu r + \alpha\big[(\tau - 1)\mu r + (\sqrt{r} - 1)\operatorname{tr}(\tau\mu e - \tilde{x} \circ \tilde{s})^+\big] \\ + \alpha^2 \operatorname{tr}(\Delta\tilde{x} \circ \Delta\tilde{s}). \end{aligned} \tag{11}$$

$$\begin{aligned}\tilde{\mu}(\alpha) = \mu(\alpha) = \mu + \alpha\left[(\tau - 1)\mu + \frac{\sqrt{r} - 1}{r}\operatorname{tr}(\tau\mu e - \tilde{x} \circ \tilde{s})^+\right] \\ + \frac{\alpha^2}{r}\operatorname{tr}(\Delta\tilde{x} \circ \Delta\tilde{s}). \end{aligned} \tag{12}$$

We begin with several lemmas that are frequently used in the section.

**Lemma 5.1** *Let* $(\tilde{x}, y, \tilde{s}) \in \mathcal{N}(\tau, \beta)$. *Then* $\|(\tau\mu e - \tilde{x} \circ \tilde{s})^+\|_F \le \beta\tau\mu$.

*Proof* Since $\tilde{x}$ and $\tilde{s}$ operator commute and by Proposition 2.3, we get

$$\tilde{\omega} = Q_{\tilde{x}^{1/2}}\tilde{s} = \tilde{x} \circ \tilde{s},$$

which implies $\|(\tau\mu e - \tilde{x} \circ \tilde{s})^+\|_F = \|(\tau\mu e - \tilde{\omega})^+\|_F \le \beta\tau\mu$, where the inequality follows from (8). □

**Lemma 5.2** *Let* $(\tilde{x}, y, \tilde{s}) \in \mathcal{N}(\tau, \beta)$. *Then* $\operatorname{tr}(\tau\mu e - \tilde{x} \circ \tilde{s})^+ \le \sqrt{r}\beta\tau\mu$.

*Proof* Using the Cauchy–Schwarz inequality and Lemma 5.1, we have

$$\operatorname{tr}(\tau\mu e - \tilde{x} \circ \tilde{s})^+ = \operatorname{tr}\big(e \circ (\tau\mu e - \tilde{x} \circ \tilde{s})^+\big) \le \sqrt{r}\big\|(\tau\mu e - \tilde{x} \circ \tilde{s})^+\big\|_F \le \sqrt{r}\beta\tau\mu,$$

which completes the proof. □

Let $\tilde{x} \circ \tilde{s}$ have the spectral decomposition $\tilde{x} \circ \tilde{s} = \lambda_1 c_1 + \cdots + \lambda_r c_r$, where $\{c_1, \ldots, c_r\}$ is a Jordan frame and the spectral eigenvalues satisfy

$$\tau\mu - \lambda_1 \le \tau\mu - \lambda_2 \le \cdots \le \tau\mu - \lambda_k \le 0 < \tau\mu - \lambda_{k+1} \le \cdots \le \tau\mu - \lambda_r.$$

Then, one has

$$\begin{aligned} (\tau\mu e - \tilde{x} \circ \tilde{s})^+ &= \sum_{i=k+1}^{r} (\tau\mu - \lambda_i) c_i, \\ (\tau\mu e - \tilde{x} \circ \tilde{s})^- &= \sum_{i=1}^{k} (\tau\mu - \lambda_i) c_i. \end{aligned} \tag{13}$$

**Lemma 5.3** *Let $G = L_{\tilde{s}}^{-1} L_{\tilde{x}}$, $\beta \le 1/2$ and $\tau_1 = (1-\beta)\tau$, then*

$$\left\| (L_{\tilde{x}} L_{\tilde{s}})^{-1/2} \left[ (\tau\mu e - \tilde{x} \circ \tilde{s})^- + \sqrt{r} (\tau\mu e - \tilde{x} \circ \tilde{s})^+ \right] \right\|_F^2 \le (1 + \beta\tau)\mu r.$$

*Proof* Since $\tilde{x}$ and $\tilde{s}$ operator commute, $\tilde{x}$, $\tilde{s}$ and $\tilde{x} \circ \tilde{s}$ share a common Jordan frame and for $i = 1, \ldots, r$, $L_{\tilde{x}} L_{\tilde{s}} c_i = \tilde{x} \circ (\tilde{s} \circ c_i) = \lambda_i c_i$, which implies $(L_{\tilde{x}} L_{\tilde{s}})^{-1} c_i = \lambda_i^{-1} c_i$.

By (13), we have

$$\begin{aligned} &\left\| (L_{\tilde{x}} L_{\tilde{s}})^{-\frac{1}{2}} \left( (\tau\mu e - \tilde{x} \circ \tilde{s})^- + \sqrt{r} (\tau\mu e - \tilde{x} \circ \tilde{s})^+ \right) \right\|_F^2 \\ &\quad = \left\| (L_{\tilde{x}} L_{\tilde{s}})^{-\frac{1}{2}} \sum_{i=1}^{k} (\tau\mu - \lambda_i) c_i + \sqrt{r} (L_{\tilde{x}} L_{\tilde{s}})^{-\frac{1}{2}} \sum_{i=k+1}^{r} (\tau\mu - \lambda_i) c_i \right\|_F^2 \\ &\quad = \left\| \sum_{i=1}^{k} (\tau\mu - \lambda_i) (L_{\tilde{x}} L_{\tilde{s}})^{-\frac{1}{2}} c_i + \sqrt{r} \sum_{i=k+1}^{r} (\tau\mu - \lambda_i) (L_{\tilde{x}} L_{\tilde{s}})^{-\frac{1}{2}} c_i \right\|_F^2 \\ &\quad = \left\| \sum_{i=1}^{k} (\tau\mu - \lambda_i) \lambda_i^{-\frac{1}{2}} c_i + \sqrt{r} \sum_{i=k+1}^{r} (\tau\mu - \lambda_i) \lambda_i^{-\frac{1}{2}} c_i \right\|_F^2 \\ &\quad = \sum_{i=1}^{k} \frac{(\tau\mu - \lambda_i)^2}{\lambda_i} + r \sum_{i=k+1}^{r} \frac{(\tau\mu - \lambda_i)^2}{\lambda_i} \\ &\quad = \sum_{i=1}^{k} (\lambda_i - \tau\mu) \frac{\lambda_i - \tau\mu}{\lambda_i} + r \sum_{i=k+1}^{r} \frac{(\tau\mu - \lambda_i)^2}{\lambda_i} \\ &\quad \le \sum_{i=1}^{k} (\lambda_i - \tau\mu) + \frac{r}{\tau_1 \mu} \sum_{i=k+1}^{r} (\tau\mu - \lambda_i)^2 \\ &\quad \le \sum_{i=1}^{r} \lambda_i + \frac{r}{\tau_1 \mu} \left\| (\tau\mu e - \tilde{x} \circ \tilde{s})^+ \right\|_F^2 \le \mu r + \frac{\beta^2 \tau^2 \mu^2 r}{\tau_1 \mu} \le (1 + \beta\tau)\mu r, \end{aligned}$$

where the first inequality follows from $0 \le (\lambda_i - \tau\mu)/\lambda_i \le 1$, $\forall i = 1, \dots, k$ and $\lambda_{\min} \ge \tau_1 \mu$, the third inequality follows from Lemma 5.1, the last inequality is due to $\beta \le 1/2$ and $\tau_1 = (1-\beta)\tau$. □

**Lemma 5.4** *Let* $G = L_{\tilde{s}}^{-1} L_{\tilde{x}}$. *Then* $\operatorname{tr}(\Delta\tilde{x} \circ \Delta\tilde{s}) \le (1+\beta\tau)\mu r/4$.

*Proof* Multiplying the last equation of (4) by $(L_{\tilde{x}} L_{\tilde{s}})^{-\frac{1}{2}}$ and computing the squares of norms of both sides, we have

$$\begin{aligned}
&\big\|(L_{\tilde{x}} L_{\tilde{s}})^{-1/2}\big[(\tau\mu e - \tilde{x}\circ\tilde{s})^- + \sqrt{r}(\tau\mu e - \tilde{x}\circ\tilde{s})^+\big]\big\|_F^2 \\
&\quad = \big\|G^{-1/2}\Delta\tilde{x} + G^{1/2}\Delta\tilde{s}\big\|_F^2 \\
&\quad = \big\|G^{-1/2}\Delta\tilde{x}\big\|_F^2 + \big\|G^{-1/2}\Delta\tilde{s}\big\|_F^2 + 2\operatorname{tr}(\Delta\tilde{x}\circ\Delta\tilde{s}) \\
&\quad = \big\|G^{-1/2}\Delta\tilde{x} - G^{-1/2}\Delta\tilde{s}\big\|_F^2 + 4\operatorname{tr}(\Delta\tilde{x}\circ\Delta\tilde{s}).
\end{aligned}$$

Combined with Lemma 5.3, we have

$$\begin{aligned}
\operatorname{tr}(\Delta\tilde{x}\circ\Delta\tilde{s}) &\le \frac{1}{4}\big\|(L_{\tilde{x}} L_{\tilde{s}})^{-1/2}\big[(\tau\mu e - \tilde{x}\circ\tilde{s})^- + \sqrt{r}(\tau\mu e - \tilde{x}\circ\tilde{s})^+\big]\big\|_F^2 \\
&\le \frac{1}{4}(1+\beta\tau)\mu r,
\end{aligned}$$

which completes the proof. □

## 5.2 Choosing the Step Size to Satisfy A1

Let $\alpha_g := \arg\min_{\alpha\in[0,1]}\{\mu(\alpha)\}$. Then the following lemma shows that $\alpha_g = 1$.

**Lemma 5.5** *Let* $\beta \le 1/2$, $\tau \le 1/4$. *Then* $\mu(\alpha)$ *is strictly monotonically decreasing in* $\alpha \in [0,1]$.

*Proof* From (12), we have

$$\mu(\alpha)' = \tau\mu - \mu + \frac{\sqrt{r}-1}{r}\operatorname{tr}(\tau\mu e - \tilde{x}\circ\tilde{s})^+ + \frac{2\alpha}{r}\operatorname{tr}(\Delta\tilde{x}\circ\Delta\tilde{s}). \tag{14}$$

By Lemma 5.2, one has

$$\begin{aligned}
\tau\mu - \mu + \frac{\sqrt{r}-1}{r}\operatorname{tr}(\tau\mu e - \tilde{x}\circ\tilde{s})^+ &\le \tau\mu - \mu + \frac{\sqrt{r}-1}{r}\sqrt{r}\beta\tau\mu \\
&\le (\tau + \beta\tau - 1)\mu < 0.
\end{aligned} \tag{15}$$

*Case 1.* $\operatorname{tr}(\Delta\tilde{x}\circ\Delta\tilde{s}) \le 0$. By (14) and (15), we have $\mu(\alpha)' < 0$ in $[0,1]$.
*Case 2.* $\operatorname{tr}(\Delta\tilde{x}\circ\Delta\tilde{s}) > 0$. By (15) and Lemma 5.4, we have

$$\frac{[\mu - \tau\mu - \frac{\sqrt{r}-1}{r}\operatorname{tr}(\tau\mu e - \tilde{x}\circ\tilde{s})^+]r}{2\operatorname{tr}(\Delta\tilde{x}\circ\Delta\tilde{s})} \geq \frac{(1-\tau-\beta\tau)\mu r}{2\operatorname{tr}(\Delta\tilde{x}\circ\Delta\tilde{s})} \geq \frac{2(1-\tau-\beta\tau)\mu r}{(1+\beta\tau)\mu r} \geq \frac{10}{9} > 1,$$

which implies

$$\frac{2\alpha}{r}\operatorname{tr}(\Delta\tilde{x}\circ\Delta\tilde{s}) \leq \frac{2}{r}\operatorname{tr}(\Delta\tilde{x}\circ\Delta\tilde{s}) \leq \mu - \tau\mu - \frac{\sqrt{r}-1}{r}\operatorname{tr}(\tau\mu e - \tilde{x}\circ\tilde{s})^+,$$

which is equivalent to $\mu(\alpha)' < 0$ in [0,1].

Thus, taking two cases into account, we obtain the desired result. □

### 5.3 Choosing the Step Size to Satisfy A2

To find an $\alpha$ satisfying A2, we give the following illustration. First, define

$$g(\alpha) := \begin{cases} \frac{\alpha}{\sqrt{r}}\|(\Delta\tilde{x}\circ\Delta\tilde{s})^-\|_F - \beta\tau\mu(\alpha), & \text{if } \alpha < 1/\sqrt{r}, \\ \alpha^2\|(\Delta\tilde{x}\circ\Delta\tilde{s})^-\|_F - \beta\tau\mu(\alpha), & \text{if } \alpha \geq 1/\sqrt{r}. \end{cases} \tag{16}$$

From Lemma 5.5, it is easy to see that $g(\alpha)$ is strictly monotonically increasing in [0, 1]. Let

$$\alpha_c := \max\bigl\{\alpha : g(\alpha) \leq 0, \text{ for all } \alpha \in [0,1]\bigr\}, \tag{17}$$

which implies for $\alpha \in [0, \alpha_c]$

$$\frac{\alpha}{\sqrt{r}}\bigl\|(\Delta\tilde{x}\circ\Delta\tilde{s})^-\bigr\|_F \leq \beta\tau\mu(\alpha), \quad \text{for } \alpha < 1/\sqrt{r}, \tag{18}$$

$$\alpha^2\bigl\|(\Delta\tilde{x}\circ\Delta\tilde{s})^-\bigr\|_F \leq \beta\tau\mu(\alpha), \quad \text{for } \alpha \geq 1/\sqrt{r}, \tag{19}$$

and $\mu(\alpha) \geq 0$.

To show that for each $\alpha \in [0, \alpha_c]$ one has $(\tilde{x}(\alpha), y(\alpha), \tilde{s}(\alpha)) \in \mathcal{N}(\tau, \beta)$, we need the following three lemmas.

**Lemma 5.6** ([30, Lemma 5.10]) *Let $\mathcal{J}$ be a Euclidean Jordan algebra. If $x, y \in \mathcal{J}$, $x, y \in \operatorname{int}\mathcal{K}$, $w = Q_{x^{1/2}}y$, then $\|(\tau\mu e - w)^+\|_F \leq \|(\tau\mu e - x\circ y)^+\|_F$, where $\tau \in ]0,1[$ and $\mu > 0$.*

**Lemma 5.7** ([30, Lemma 5.12]) *If $x, y \in \mathcal{J}$, then*

$$\bigl\|(x+y)^+\bigr\|_F \leq \bigl\|x^+ + y^+\bigr\|_F \leq \bigl\|x^+\bigr\|_F + \bigl\|y^+\bigr\|_F.$$

Using the spectral decomposition of $T(\alpha)$, (10) and (13), we have

$$\begin{aligned}
T(\alpha) &= \tilde{x} \circ \tilde{s} + \alpha(\tau\mu e - \tilde{x} \circ \tilde{s})^- + \alpha\sqrt{r}(\tau\mu e - \tilde{x} \circ \tilde{s})^+ \\
&= \sum_{i=1}^{r} \lambda_i c_i + \alpha \sum_{i=1}^{k} (\tau\mu - \lambda_i) c_i + \alpha\sqrt{r} \sum_{i=k+1}^{r} (\tau\mu - \lambda_i) c_i \\
&= \sum_{i=1}^{k} \big[(1-\alpha)\lambda_i + \alpha\tau\mu\big] c_i + \sum_{i=k+1}^{r} \big[\lambda_i + \alpha\sqrt{r}(\tau\mu - \lambda_i)\big] c_i,
\end{aligned}$$

which implies $T(\alpha) \succeq 0$.

**Lemma 5.8** *Let* $\mu(\alpha) > 0$. *Then for all* $\alpha \in [0, 1]$ *we have*

$$\big\|\big[\tau\mu(\alpha)e - T(\alpha)\big]^+\big\|_F \le (1 - \alpha\sqrt{r})\beta\tau\mu(\alpha), \quad \alpha < 1/\sqrt{r}, \tag{20}$$

$$\big\|\big[\tau\mu(\alpha)e - T(\alpha)\big]^+\big\|_F = 0, \quad \alpha \ge 1/\sqrt{r}. \tag{21}$$

*Proof* From Lemma 5.5 for all $\alpha \in [0, 1]$, one has $\mu(\alpha) \le \mu$.

Due to $0 < \mu(\alpha) \le \mu$ and $T(\alpha) \succeq 0$, we have

$$\begin{aligned}
&\big\|\big[\tau\mu(\alpha)e - T(\alpha)\big]^+\big\|_F^2 \\
&\quad \le \left\|\left[\tau\mu(\alpha)e - \frac{\mu(\alpha)}{\mu} T(\alpha)\right]^+\right\|_F^2 \\
&\quad = \left[\frac{\mu(\alpha)}{\mu}\right]^2 \big\|\big[\tau\mu e - T(\alpha)\big]^+\big\|_F^2 \\
&\quad = \left[\frac{\mu(\alpha)}{\mu}\right]^2 \left[\sum_{i=1}^{k} \big[\big[(1-\alpha)(\tau\mu - \lambda_i)\big]^+\big]^2 + \sum_{i=k+1}^{r} \big[\big[(1-\alpha\sqrt{r})(\tau\mu - \lambda_i)\big]^+\big]^2\right] \\
&\quad = \left[\frac{\mu(\alpha)}{\mu}\right]^2 \sum_{i=k+1}^{r} \big[\big[(1-\alpha\sqrt{r})(\tau\mu - \lambda_i)\big]^+\big]^2.
\end{aligned} \tag{22}$$

For $\alpha < 1/\sqrt{r}$, using (22) and $1 - \alpha\sqrt{r} > 0$, one has

$$\begin{aligned}
\big\|\big[\tau\mu(\alpha)e - T(\alpha)\big]^+\big\|_F^2 &\le (1-\alpha\sqrt{r})^2 \left[\frac{\mu(\alpha)}{\mu}\right]^2 \big\|(\tau\mu e - \tilde{x} \circ \tilde{s})^+\big\|_F^2 \\
&\le (1-\alpha\sqrt{r})^2 \big[\beta\tau\mu(\alpha)\big]^2,
\end{aligned}$$

where the last inequality follows from Lemma 5.1.

For $\alpha \ge 1/\sqrt{r}$, using (22) and $1 - \alpha\sqrt{r} \le 0$, one has

$$\big\|\big[\tau\mu(\alpha)e - T(\alpha)\big]^+\big\|_F^2 = 0.$$

The proof is completed. □

The following lemma shows our main result in this section.

**Lemma 5.9** *Let $\alpha_c$ be defined by (17). For all $\alpha \in [0, \alpha_c]$, we have*

$$\big(\tilde{x}(\alpha), y(\alpha), \tilde{s}(\alpha)\big) \in \mathcal{N}(\tau, \beta).$$

*Proof*
*Case 1.* $\alpha_c \geq 1/\sqrt{r}$.
For $\alpha \in [0, 1/\sqrt{r}\,[$, using Lemmas 5.6, 5.7, (20) and (18), we have

$$\begin{aligned}
&\big\|\big[\tau\mu(\alpha)e - Q_{\tilde{x}(\alpha)^{1/2}}\tilde{s}(\alpha)\big]^+\big\|_F \\
&\quad \leq \big\|\big[\tau\mu(\alpha)e - \tilde{x}(\alpha)\circ\tilde{s}(\alpha)\big]^+\big\|_F \\
&\quad \leq \big\|\big[\tau\mu(\alpha)e - T(\alpha)\big]^+\big\|_F + \alpha^2\big\|(\Delta\tilde{x}\circ\Delta\tilde{s})^-\big\|_F \\
&\quad \leq (1-\alpha\sqrt{r})\beta\tau\mu(\alpha) + \alpha^2\big\|(\Delta\tilde{x}\circ\Delta\tilde{s})^-\big\|_F \\
&\quad \leq (1-\alpha\sqrt{r})\beta\tau\mu(\alpha) + \alpha\sqrt{r}\beta\tau\mu(\alpha) \\
&\quad = \beta\tau\mu(\alpha).
\end{aligned} \tag{23}$$

For $\alpha \in [1/\sqrt{r}, \alpha_c]$, using Lemmas 5.6, 5.7, (21) and (19), one has

$$\begin{aligned}
&\big\|\big[\tau\mu(\alpha)e - Q_{\tilde{x}(\alpha)^{1/2}}\tilde{s}(\alpha)\big]^+\big\|_F \\
&\quad \leq \big\|\big[\tau\mu(\alpha)e - \tilde{x}(\alpha)\circ\tilde{s}(\alpha)\big]^+\big\|_F \\
&\quad \leq \big\|\big[\tau\mu(\alpha)e - T(\alpha)\big]^+\big\|_F + \alpha^2\big\|(\Delta\tilde{x}\circ\Delta\tilde{s})^-\big\|_F \\
&\quad = \alpha^2\big\|(\Delta\tilde{x}\circ\Delta\tilde{s})^-\big\|_F \\
&\quad \leq \beta\tau\mu(\alpha).
\end{aligned}$$

Therefore, for all $\alpha \in [0, \alpha_c]$, we have $\|[\tau\mu(\alpha)e - \tilde{x}(\alpha)\circ\tilde{s}(\alpha)]^+\|_F \leq \beta\tau\mu(\alpha)$.
*Case 2.* $\alpha_c < 1/\sqrt{r}$.
For $\alpha \in [0, \alpha_c)$, by the proof of (23), we also have

$$\big\|\big[\tau\mu(\alpha)e - Q_{\tilde{x}(\alpha)^{1/2}}\tilde{s}(\alpha)\big]^+\big\|_F \leq \beta\tau\mu(\alpha).$$

Together with *Case 1* and *Case 2*, for all $\alpha \in [0, \alpha_c]$, one has

$$\big\|\big[\tau\mu(\alpha)e - \tilde{x}(\alpha)\circ\tilde{s}(\alpha)\big]^+\big\|_F \leq \beta\tau\mu(\alpha),$$

which implies $\tilde{x}(\alpha)\circ\tilde{s}(\alpha) \in \operatorname{int}\mathcal{K}$.
From Lemma 2.1, we have $\det(\tilde{x}(\alpha)) \neq 0$ and $\det(\tilde{s}(\alpha)) \neq 0$. Furthermore, since $\tilde{x} \in \operatorname{int}\mathcal{K}$, $\tilde{s} \in \operatorname{int}\mathcal{K}$, by continuity it follows that both $\tilde{x}(\alpha) \in \operatorname{int}\mathcal{K}$ and $\tilde{s}(\alpha) \in \operatorname{int}\mathcal{K}$ in $[0, \alpha_c]$. The proof is completed. □

### 5.4 Choosing the Step Size to Satisfy A3

To reduce $\|r_p\|_F$ and $\|r_d\|_F$, we require the following condition:

$$\frac{\mu(\alpha)}{\mu} \geq \max\left\{\frac{\|r_p(\alpha)\|_F}{\|r_p\|_F}, \frac{\|r_d(\alpha)\|_F}{\|r_d\|_F}\right\} = 1-\alpha. \tag{24}$$

The requirement may prevent $\mu(\alpha)$ from getting too close to the boundary of the symmetric cones too early.

Therefore, the step size that reduces the residual norm is defined by

$$\alpha_f := \max\{\alpha : \langle \tilde{x}(\alpha), \tilde{s}(\alpha)\rangle \geq (1-\alpha)\langle \tilde{x}, \tilde{s}\rangle, \text{ for all } \alpha \in [0,1]\}. \tag{25}$$

Finally, the step size is given by

$$\hat{\alpha} := \max\{\alpha : (\tilde{x}(\alpha), y(\alpha), \tilde{s}(\alpha)) \in \mathcal{N}(\tau, \beta), \text{ for all } \alpha \in [0, \alpha_f]\}. \tag{26}$$

## 6 Algorithm and Its Convergence

Based on the analysis, we give a general algorithmic framework.

**Algorithm 1** Let $\varepsilon > 0$, $\tau \leq 1/4$, $\beta \leq 1/2$, and $(x^0, y^0, s^0) \in \mathcal{N}(\tau, \beta)$. Set $\mu_0 = \langle x^0, s^0\rangle / r$, $k := 0$.

Step 1 If $\mu_k \leq \varepsilon \mu_0$, then stop.

Step 2 Choose a scaling element $p \in \mathcal{C}(x^k, s^k)$ and compute $(\tilde{x}^k, \tilde{s}^k)$.

Step 3 Compute the directions $(\Delta\tilde{x}^k, \Delta y^k, \Delta\tilde{s}^k)$ by solving system (4).

Step 4 Compute $\alpha_c$ and $\alpha_f$ by (17) and (25), respectively.

Step 5 If $\alpha_c \geq \alpha_f$, set $\hat{\alpha}^k = \alpha_f$, $(\tilde{x}^{k+1}, y^{k+1}, \tilde{s}^{k+1}) = (\tilde{x}(\hat{\alpha}^k), y(\hat{\alpha}^k), \tilde{s}(\hat{\alpha}^k))$. Otherwise, compute the largest $\hat{\alpha}^k \in [\alpha_c, \alpha_f]$ such that $(\tilde{x}^{k+1}, y^{k+1}, \tilde{s}^{k+1}) = (\tilde{x}(\hat{\alpha}^k), y(\hat{\alpha}^k), \tilde{s}(\hat{\alpha}^k)) \in \mathcal{N}(\tau, \beta)$.

Step 6 Let $(x^{k+1}, y^{k+1}, s^{k+1}) = (Q_{p^{-1}}\tilde{x}^{k+1}, y^{k+1}, Q_p\tilde{s}^{k+1})$ and compute $\mu_{k+1}$ by $\mu_{k+1} = \langle x^{k+1}, s^{k+1}\rangle / r$. Set $k := k+1$ and go to Step 1.

For Algorithm 1, the following proposition is readily verified.

**Proposition 6.1** *Let $\{(x^k, y^k, s^k)\}$ be generated by Algorithm 1. Then for $k \geq 0$, $Ax^{k+1} - b = \nu^{k+1}(Ax^0 - b)$, and $A^*y^{k+1} + s^{k+1} - c = \nu^{k+1}(A^*y^0 + s^0 - c)$, where $\nu^0 = 1$ and $\nu^{k+1} = (1-\hat{\alpha}^k)\nu^k = \prod_{i=0}^k (1-\hat{\alpha}^i) \in [0,1]$.*

From Proposition 6.1, we have $\nu^k = \frac{\|Ax^k - b\|_F}{\|Ax^0 - b\|_F} = \frac{\|A^*y^k + s^k - c\|_F}{\|A^*y^0 + s^0 - c\|_F}$, which implies that $\nu^k$ represents the relative infeasibility at $(x^k, y^k, s^k)$. By (25) and Algorithm 1, one has

$$\langle x^{k+1}, s^{k+1}\rangle \geq (1-\hat{\alpha}^k)\langle x^k, s^k\rangle \geq \nu^{k+1}\langle x^0, s^0\rangle, \tag{27}$$

which ensures that the infeasibility approaches zero as the complementarity $\langle x, s\rangle$ approaches zero.

Using the idea of Rangarajan in [24], we now specify a particular starting point for Algorithm 1. This choice was first proposed by Zhang [31, 32] for LCP, and then was extended by Rangarajan [24] to symmetric cones.

Let $u^0$ and $(r^0, v^0)$ be the minimum-norm solutions to the linear systems $Ax = b$ and $A^*y + s = c$, respectively. That is,

$$u^0 = \arg\min\{\|u\|_F : Au = b\}, \qquad (r^0, v^0) = \arg\min\{\|v\|_F : A^*r + v = c\}.$$

We choose $(x^0, y^0, s^0)$ such that $x^0 = s^0 = \rho^0 e$, $\rho^0 \geq \max\{\|u^0\|_2, \|v^0\|_2\}$. This implies that $x^0, s^0 \in \operatorname{int}\mathcal{K}$, $x^0 - u^0 \in \mathcal{K}$, and $s^0 - v^0 \in \mathcal{K}$.

Let $\rho^* = \min\{\max(\|x^*\|_2, \|s^*\|_2) : x^* \in \mathcal{P}^*, (y^*, s^*) \in \mathcal{D}^*\}$. In addition, we assume that for some constant $\Psi > 0$, it has $\rho^0 \geq \rho^*/\Psi$ (note that we can always increase $\rho^0$).

We construct an auxiliary sequence $\{(u^k, r^k, v^k)\}$ as follows:

$$\begin{aligned}\left(u^{k+1}, r^{k+1}, v^{k+1}\right) = {}& \left(x^{k+1}, y^{k+1}, s^{k+1}\right) \\ & - \left(1 - \hat{\alpha}^k\right)\left(x^k - u^k, y^k - r^k, s^k - v^k\right). \end{aligned} \tag{28}$$

The auxiliary sequence will be used in our analysis of complexity and need not be actually computed in Algorithm 1. The following lemma gives useful properties of the auxiliary sequence $\{(u^k, r^k, v^k)\}$. This proof is obtained directly by substitution.

**Lemma 6.1** *Let $\{(x^k, y^k, s^k)\}$ be generated by Algorithm 1, $\{(u^k, r^k, v^k)\}$ be given by (28), and $\{\nu^k\}$ be given by Proposition 6.1. Then for $k \geq 0$*

(i) *$Au^k = b$ and $A^*r^k + v^k = c$;*
(ii) *$x^k - u^k = \nu^k(x^0 - u^0) \in \mathcal{K}$ and $s^k - v^k = \nu^k(s^0 - v^0) \in \mathcal{K}$.*

In the rest of this section, we establish our main complexity result for Algorithm 1. Before proceeding to the complexity result, we first give two key lemmas.

**Lemma 6.2** *Let $\beta \leq 1/2$, $\tau \leq 1/4$. Then for all $\alpha \in [0, 1]$, we have*

$$\mu(\alpha) \leq (1 - \xi\alpha)\mu \leq \mu, \quad \textit{where } \xi = \frac{3}{4} - \tau - \frac{5}{4}\beta\tau.$$

*Proof* By (11) and Lemma 5.2, we have

$$\begin{aligned}\mu(\alpha) &= \mu + \alpha\left[\tau\mu - \mu + \frac{\sqrt{r} - 1}{r}\operatorname{tr}(\tau\mu e - \tilde{x} \circ \tilde{s})^+\right] + \frac{\alpha^2}{r}\operatorname{tr}(\Delta\tilde{x} \circ \Delta\tilde{s}) \\ &\leq \mu + \alpha\left[\tau\mu - \mu + (1 - 1/\sqrt{r})\beta\tau\mu + \frac{\alpha}{r}\operatorname{tr}(\Delta\tilde{x} \circ \Delta\tilde{s})\right] \\ &\leq \mu + \alpha\left[\tau\mu - \mu + \beta\tau\mu + \frac{\alpha}{r}\operatorname{tr}(\Delta\tilde{x} \circ \Delta\tilde{s})\right] \\ &= \left[1 - \left[1 - \tau - \beta\tau - \frac{\alpha}{\mu r}\operatorname{tr}(\Delta\tilde{x} \circ \Delta\tilde{s})\right]\alpha\right]\mu. \end{aligned} \tag{29}$$

If $\operatorname{tr}(\Delta\tilde{x} \circ \Delta\tilde{s}) \leq 0$, by (29) we have

$$\mu(\alpha) \leq \left[1 - (1 - \tau - \beta\tau)\alpha\right]\mu \leq (1 - \xi\alpha)\mu.$$

If $\operatorname{tr}(\Delta\tilde{x} \circ \Delta\tilde{s}) > 0$, by (29) and Lemma 5.4, we also have

$$\mu(\alpha) \leq \left[1 - \left[1 - \tau - \beta\tau - \frac{1}{4}\alpha(1+\beta\tau)\right]\alpha\right]\mu$$
$$\leq \left[1 - \left(\frac{3}{4} - \tau - \frac{5}{4}\beta\tau\right)\alpha\right]\mu \leq (1-\xi\alpha)\mu.$$

This proves the lemma. □

**Lemma 6.3** ([10, Lemma 33]) *Let $p, q \in \mathcal{J}$ and $G$ be a positive definite matrix which is symmetric with respect to the scalar product $\langle\cdot,\cdot\rangle$. Then*

$$\|p\|_F\|q\|_F \leq \sqrt{\operatorname{cond}(G)}\left\|G^{-1/2}p\right\|_F\left\|G^{1/2}q\right\|_F$$
$$\leq \frac{1}{2}\sqrt{\operatorname{cond}(G)}\left(\left\|G^{-1/2}p\right\|_F^2 + \left\|G^{1/2}q\right\|_F^2\right),$$

*where* $\operatorname{cond}(G) = \lambda_{\max}(G)/\lambda_{\min}(G)$ *is the condition number of* $G$.

*Remark 6.1* It is clear that $\|\cdot\|_G$ defined by

$$\left\|(u,v)\right\|_G = \left(\left\|G^{-1/2}u\right\|_F^2 + \left\|G^{1/2}v\right\|_F^2\right)^{1/2}, \quad \text{for } u, v \in \mathcal{J},$$

is a norm on $\mathcal{J} \times \mathcal{J}$.

In what follows, we estimate the upper bound on $\|\Delta\tilde{x}\|_F\|\Delta\tilde{s}\|_F$. Using Lemma 5.3 and the proof techniques of Lemma 5.16 in [30], one has

$$\left\|(\Delta\tilde{x}, \Delta\tilde{s})\right\|_G \leq \sqrt{(1+\beta\tau)r\mu} + (1+\sqrt{2})\zeta. \tag{30}$$

Using the particular starting point choice and the proof techniques of Lemma A.1 in [30], we have

$$\zeta \leq (5+4\Psi)r\sqrt{\mu}/\sqrt{\tau_1}. \tag{31}$$

By (30) and (31), we obtain the following lemma.

**Lemma 6.4** *Let $G = L_{\tilde{s}}^{-1}L_{\tilde{x}}$. Then we have $\|(\Delta\tilde{x}, \Delta\tilde{s})\|_G^2 \leq \omega^2 r^2\mu$, where $\omega = (\sqrt{(1+\beta\tau)\tau_1/r} + (1+\sqrt{2})(5+4\Psi))/\sqrt{\tau_1} \geq 20$.*

By Lemmas 6.3 and 6.4, it is easy to obtain the following corollary.

**Corollary 6.1** *Let $G = L_{\tilde{s}}^{-1}L_{\tilde{x}}$. Then $\|\Delta\tilde{x}\|_F\|\Delta\tilde{s}\|_F \leq \frac{1}{2}\sqrt{\operatorname{cond}(G)}\omega^2 r^2\mu$.*

The following two lemmas provide lower bounds on $\alpha_c$ and $\alpha_f$.

**Lemma 6.5** *Let $\alpha_c$ be defined by (17). Then $\alpha_c \geq \frac{4\beta\tau}{3\sqrt{\operatorname{cond}(G)}\omega^2 r^{3/2}}$.*

*Proof* If $1/\sqrt{r} \leq \alpha_c$, we immediately obtain the lower bound on $\alpha_c$. Thus, we only mainly consider $\alpha_c < 1/\sqrt{r}$.

By Lemma 2.2 and Corollary 6.1, one has

$$\left\|(\Delta\tilde{x} \circ \Delta\tilde{s})^-\right\|_F \leq \|\Delta\tilde{x} \circ \Delta\tilde{s}\|_F \leq \|\Delta\tilde{x}\|_F\|\Delta\tilde{s}\|_F \leq \sqrt{\operatorname{cond}(G)}\omega^2 r^2\mu/2. \tag{32}$$

By (11), Lemma 5.2 and Corollary 6.1, one has

$$\begin{aligned}\mu(\alpha) &\geq \mu + \alpha(\tau\mu - \mu) + \frac{\alpha^2}{r}\operatorname{tr}(\Delta\tilde{x} \circ \Delta\tilde{s})\\ &\geq \left[1 + (\tau - 1)\alpha - \alpha^2\sqrt{\operatorname{cond}(G)}\omega^2 r/2\right]\mu. \end{aligned}\tag{33}$$

Combined with (32) and (33), we have

$$\begin{aligned} g(\alpha) &= \frac{\alpha}{\sqrt{r}}\left\|(\Delta\tilde{x} \circ \Delta\tilde{s})^-\right\|_F - \beta\tau\mu(\alpha)\\ &\leq \frac{\alpha}{\sqrt{r}}\left\|(\Delta\tilde{x} \circ \Delta\tilde{s})^-\right\|_F - \beta\tau\left[1 + (\tau - 1)\alpha - \alpha^2\sqrt{\operatorname{cond}(G)}\omega^2 r/2\right]\mu\\ &\leq \alpha\sqrt{\operatorname{cond}(G)}\omega^2 r^2\mu/(2\sqrt{r}) - \beta\tau\left[1 + (\tau - 1)\alpha - \alpha^2\sqrt{\operatorname{cond}(G)}\omega^2 r/2\right]\mu\\ &= \frac{1}{2}\beta\tau\mu H(\alpha), \end{aligned}$$

where

$$\begin{aligned} H(\alpha) &= \alpha^2\sqrt{\operatorname{cond}(G)}\omega^2 r + \alpha\left[\sqrt{\operatorname{cond}(G)}\omega^2 r^{3/2}/\beta\tau + 2(1 - \tau)\right] - 2\\ &:= a_1\alpha^2 + b_1\alpha + c_1. \end{aligned}$$

To obtain the largest $\alpha$ such that $g(\alpha) \leq 0$, we consider the positive root of the quadratic function $H(\alpha)$. We have

$$\begin{aligned} \alpha_+ &= \frac{2c_1}{-b_1 - \sqrt{b_1^2 - 4a_1c_1}}\\ &\geq \frac{4\beta\tau}{\sqrt{\operatorname{cond}(G)}\omega^2 r^{3/2}\left(1 + \frac{1}{4\omega^2} + \sqrt{(1 + \frac{1}{4\omega^2})^2 + \frac{1}{8\omega^2}}\right)}\\ &\geq \frac{4\beta\tau}{3\sqrt{\operatorname{cond}(G)}\omega^2 r^{3/2}}, \end{aligned}$$

where the first inequality follows from

$$\begin{aligned} b_1 &= \frac{1}{\beta\tau}\sqrt{\operatorname{cond}(G)}\omega^2 r^{3/2} + 2(1 - \tau)\\ &= \frac{1}{\beta\tau}\sqrt{\operatorname{cond}(G)}\omega^2 r^{3/2}\left(1 + \frac{2(1 - \tau)\beta\tau}{\sqrt{\operatorname{cond}(G)}\omega^2 r^{3/2}}\right) \end{aligned}$$

$$\leq \frac{1}{\beta\tau}\sqrt{\mathrm{cond}\,(G)}\omega^2 r^{3/2}\left(1+\frac{1}{4\omega^2}\right),$$

and

$$\begin{aligned}\sqrt{b_1^2-4a_1c_1} &= \sqrt{\left[\sqrt{\mathrm{cond}\,(G)}\omega^2 r^{3/2}/\beta\tau+2(1-\tau)\right]^2+8\sqrt{\mathrm{cond}(G)}\omega^2 r} \\ &\leq \frac{1}{\beta\tau}\sqrt{\mathrm{cond}\,(G)}\omega^2 r^{3/2}\sqrt{\left(1+\frac{1}{4\omega^2}\right)^2+\frac{8\beta^2\tau^2}{\sqrt{\mathrm{cond}(G)}\omega^2 r^2}} \\ &\leq \frac{1}{\beta\tau}\sqrt{\mathrm{cond}\,(G)}\omega^2 r^{3/2}\sqrt{\left(1+\frac{1}{4\omega^2}\right)^2+\frac{1}{8\omega^2}}\,.\end{aligned}$$

Hence, $\alpha_c \geq 4\beta\tau/(3\sqrt{\mathrm{cond}(G)}\omega^2 r^{3/2})$. □

**Lemma 6.6** *Let $\alpha_f$ be defined by (25). Then $\alpha_f \geq \frac{2\tau}{\sqrt{\mathrm{cond}(G)}\omega^2 r}$.*

*Proof* Using (11) and (33), one has

$$\big\langle x(\alpha), s(\alpha)\big\rangle = \mu(\alpha)r \geq (1-\alpha)\mu r + \alpha\tau\mu r - \alpha^2\sqrt{\mathrm{cond}(G)}\omega^2 r^2\mu/2.$$

It is easily shown that if $\alpha \leq 2\tau/\sqrt{\mathrm{cond}(G)}\omega^2 r$, one has

$$\alpha\big(\tau\mu r - \alpha\sqrt{\mathrm{cond}(G)}\omega^2 r^2\mu/2\big) \geq 0,$$

which implies $\langle x(\alpha), s(\alpha)\rangle \geq (1-\alpha)\mu r$.

From the definition of $\alpha_f$ in (25), we have $\alpha_f \geq 2\tau/(\sqrt{\mathrm{cond}(G)}\omega^2 r)$. □

## 6.1 Complexity for Infeasible Starting Points

Similarly to [10, Lemma 36], we give the following lemma, which gives a bound on $\mathrm{cond}(G)$ for some specific search directions.

**Lemma 6.7** *For the NT direction*, $\mathrm{cond}(G) = 1$. *For the xs and sx directions*, $\mathrm{cond}(G) \leq r/\tau_1$.

The following theorem gives an upper bound for the number of iterations in which Algorithm 1 stops with an $\varepsilon$-approximate solution.

**Theorem 6.1** *Suppose that $\sqrt{\mathrm{cond}(G)}$ is bounded from above by $\kappa < \infty$ for all iterations. Then Algorithm 1 will terminate in $\mathcal{O}(\kappa r^{1.5}\log\varepsilon^{-1})$ iterations such that $\|Ax^k - b\| \leq \varepsilon\|Ax^0 - b\|$, $\|A^*y^k + s^k - c\| \leq \varepsilon\|A^*y^0 + s^0 - c\|$ and $\langle x^k, s^k\rangle \leq \varepsilon\langle x^0, s^0\rangle$.*

*Proof* Let $\hat{\alpha}^0 = 4\beta\tau/(3\sqrt{\mathrm{cond}(G)}\omega^2 r^{3/2})$.

Since $\beta \le 1/2, \tau \le 1/4, \mathrm{cond}(G) \ge 1, \omega \ge 20$, we may conclude that

$$\hat{\alpha}^0 = \frac{4\beta\tau}{3\sqrt{\mathrm{cond}(G)}\omega^2 r} \le \frac{2\tau}{\sqrt{\mathrm{cond}(G)}\omega^2 r} \le \alpha_f,$$

which implies $\hat{\alpha}^0 \le \min\{\alpha_c, \alpha_f\}$.

From the definition of $\hat{\alpha}$ in (26), we have $\hat{\alpha}^0 \le \hat{\alpha}$. By Lemma 5.5, we have

$$\begin{aligned}\mu(\hat{\alpha}) &\le \mu(\hat{\alpha}_0) \le (1 - \xi\hat{\alpha}_0)\mu \\ &= \left[1 - \frac{4\xi\beta\tau}{3\sqrt{\mathrm{cond}(G)}\omega^2 r^{3/2}}\right]\mu \\ &\le \left[1 - \frac{4\xi\beta\tau}{3\kappa\omega^2 r^{3/2}}\right]\mu,\end{aligned}$$

where the third inequality follows from Lemma 6.2, and the last inequality follows from $\sqrt{\mathrm{cond}(G)} \le \kappa$.

It is clear that if $k \ge (3\kappa\omega^2 r^{3/2} \log \varepsilon^{-1})/(4\xi\beta\tau)$, one has

$$\left[1 - \frac{4\xi\beta\tau}{3\kappa\omega^2 r^{3/2}}\right]^k \mu_0 \le \varepsilon\mu_0,$$

which implies $\mu(\hat{\alpha}) \le \varepsilon\mu_0$.

Therefore, Algorithm 1 terminates after at most $\lceil \frac{3\kappa\omega^2 r^{3/2}}{4\xi\beta\tau} \log \varepsilon^{-1} \rceil$ steps. Using (27), one has $\nu^k \le \varepsilon$, which implies $\nu^k = \frac{\|Ax^k - b\|_F}{\|Ax^0 - b\|_F} = \frac{\|A^* y^k + s^k - c\|_F}{\|A^* y^0 + s^0 - c\|_F} \le \varepsilon$.

This completes the proof. □

By using the bound on $\sqrt{\mathrm{cond}(G)}$ in Lemma 6.7 for the NT, $xs$ and $sx$ search directions, we have the following iteration complexities.

**Corollary 6.2** *If the NT search direction is used, the iteration complexity of Algorithm 1 is $\mathcal{O}(r^{1.5} \log \varepsilon^{-1})$. If the $xs$ and $sx$ search directions are used, the iteration complexities of Algorithm 1 are $\mathcal{O}(r^2 \log \varepsilon^{-1})$.*

## 6.2 Complexity for Feasible Starting Points

If strictly feasible starting points are used, then the step size $\alpha_f$ will be omitted. In this section, we will show the complexity for using strictly feasible starting points. Since the proof techniques are exactly the same as in the infeasible starting point case, we will only give a brief outline of the proof, omitting the details.

If strictly feasible starting points are used, we rewrite (11) as

$$\langle \tilde{x}(\alpha), \tilde{s}(\alpha) \rangle = \mu r + \alpha\left[(\tau - 1)\mu r + (\sqrt{r} - 1)\,\mathrm{tr}(\tau\mu e - \tilde{x} \circ \tilde{s})^+\right].$$

As a result, corresponding to Corollary 6.1, Lemmas 6.5 and 6.2, we have the following lemmas.

**Lemma 6.8** *Let* $G = L_{\tilde{s}}^{-1} L_{\tilde{x}}$. *Then* $\|\Delta\tilde{x}\|_F \|\Delta\tilde{s}\|_F \le \frac{5}{8}\sqrt{\mathrm{cond}(G)} r\mu$.

**Lemma 6.9** *Let* $\alpha_c$ *be defined by* (17). *Then* $\alpha_c \ge \frac{\beta\tau}{\sqrt{\mathrm{cond}(G)}\sqrt{r}}$.

**Lemma 6.10** *Let* $\beta \le 1/2$, $\tau \le 1/4$. *Then for all* $\alpha \in [0, 1]$, *we have*

$$\mu(\alpha) \le \left[1 - (1 - \tau - \beta\tau)\alpha\right]\mu \le \mu.$$

Similarly to Theorem 6.1 and using Lemmas 6.10 and 6.9, we have the following iteration complexity bound.

**Theorem 6.2** *Let a feasible point* $(x^0, y^0, s^0) \in \mathcal{N}(\tau, \beta)$, *and suppose the condition number of* $G$ *such that* $\sqrt{\mathrm{cond}(G)} \le \kappa < \infty$ *for all iterations. Then the feasible algorithm terminates in at most* $\mathcal{O}(\kappa\sqrt{r}\log\varepsilon^{-1})$ *iterations.*

**Corollary 6.3** *If the NT search direction is used, the iteration complexity of Algorithm* 1 *is* $\mathcal{O}(\sqrt{r}\log\varepsilon^{-1})$. *If the* $xs$ *and* $sx$ *search directions are used, the iteration complexities of Algorithm* 1 *are* $\mathcal{O}(r\log\varepsilon^{-1})$.

## 7 Conclusion

In this paper, we have established polynomial complexity bounds of infeasible and feasible IPMs for the commutative class of search directions. The Euclidean Jordan algebra is a basic tool in our analysis. Compared with the results in Rangarajan [24], the complexity bound is reduced by a factor of $r^{0.5}$. As a special case, when starting with a feasible point, the corresponding complexity bounds can be reduced by a factor of $r^{1.5}$. In particular, the complexity bound for the NT direction coincides with the bound obtained for LP by Zhang and Zhang [32]. Moreover, the proposed IIPM is based on a wide neighborhood, which is extension of the neighborhood given by Ai and Zhang [25].

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