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AN EXTENDED CUTTING PLANE METHOD FOR SOLVING CONVEX MINLP PROBLEMS

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ABSTRACT

An extended version of Kelley's cutting plane method is introduced in the present paper. The extended method can be applied for the solution of convex MINLP (mixed-integer non-linear programming) problems, while Kelley's cutting plane method was originally introduced for the solution of convex NLP (non-linear programming) problems only. The method is suitable for solving large convex MINLP problems with a moderate degree of nonlinearity. The convergence properties of the method are given in the present paper and an example is provided to illustrate the numerical procedure.

KEYWORDS

Optimization, Mixed-Integer Non-Linear Programming, Integer Non-Linear Programming, Extended Cutting Plane.

INTRODUCTION

Cutting plane (CP) methods have been introduced for the solution of different types of optimization problems. The most common cutting plane methods solve convex NLP problems (Kelley (1960)), general ILP (integer linear programming) problems (Gomory (1960)) and some special types of MILP (mixed-integer linear programming) problems (Young (1968), Padberg and Hong (1980)). A mixed-integer CP algorithm and its finite convergence under the assumption that the objective function variable must be an integer was given in Gomory (1960). However, without this assumption, no finite CP algorithm for MILP's is known (Nemhauser and Wolsey (1988)).

In this paper a cutting plane method for solving convex MINLP problems is introduced. The method can be considered as an extension of Kelley's cutting plane method for the solution of convex NLP problems. The extended method does not, however, provide many new perspectives in the solution of MILP problems by cutting planes, since the procedure is based on solving a sequence of MILP problems. The MILP subproblems may, in the present method, be solved for example with the branch and bound method (Land and Doig (1960)) or some other mixed-integer or mixed zero-one algorithms (Roy and Wolsey (1987), Crowder et al. (1983), Schrijver (1986)).

While it does not seem to be simple to obtain a cutting plane method for the solution of general MILP problems, it is interesting to note that Kelley's cutting plane method can be extended from convex NLP problems to convex MINLP in quite a straightforward way. Global convergence of the extended method for convex MINLP problems is shown in the present paper.

An extension of the method, the Extended Cutting Plane (ECP) method, for solving non-convex MINLP problems was already presented in Westerlund, Pettersson and Grossmann (1994). The extended version of the method can be applied to non-convex problems, but global convergence can only be ensured for convex problems. The procedure has been found to be an attractive alternative for the solution of certain examples considered. In order to demonstrate the properties of the extended algorithm, comparisons were made in Westerlund, Pettersson and Grossmann (1994) with the AP/OA/ER (Augmented Penalty Outer Approximation Equality Relaxation) method given in Viswanathan and Grossmann (1990).

In this paper the convergence properties of the ECP method for convex MINLP problems are analyzed in greater detail. The method requires only the solution of a single MILP problem in each iteration, and is found to be well suited for solving large convex MINLP problems with a moderate degree of nonlinearity.

Cutting plane (CP) methods are oftenly criticized because their convergence speed is quite slow. This is especially the case for Kelley's CP method when solving NLP problems with strong nonlinearities. The CP methods have, however, some advantages such as the simplicity and robustness of the solution. The disadvantage with the convergence speed is at least partially avoided in weakly non-linear problems.
Many of the available MINLP algorithms require the solution of both an NLP and an MILP problem in each iteration (Geoffrion (1972), Duran and Grossmann (1986), Kocis and Grossmann (1987), Yuan, Piblouleanu and Domenech (1989), Viswanathan and Grossmann (1990)).
The proposed method bears close resemblance to the Outer Approximation (OA) method proposed by Duran and Grossmann (1986). The main difference between the OA method and the proposed method is that the OA method solves both an MILP problem and an NLP problem in the main iteration loop, while the proposed method does not solve any NLP problems separately. The main iteration loop is, generally, more efficient in the OA method. This is especially the case for strongly non-linear MINLP problems mainly consisting of continuous variables. For pure INLP (Integer Non-Linear Programming) problems no advantage is obtained by the solution of the NLP problem in the OA method. The solution of an NLP problem in each iteration also penalizes the computational work at each iteration. In solving the MILP problem the computational work is, on the other hand, quite moderate in both methods, since the MILP problem to be solved at each iteration is the previous problem with only one additional linear constraint.
In recent years a number of chemical engineering problems have been formulated in terms of mixed-integer non-linear programs, (Kocis and Grossmann (1987), Kravanja and Grossmann (1990), Achenie and Biegler (1990), Westerlund, Pettersson and Grossmann (1994), Brink and Westerlund (1995)). For some problems the computational speed required to obtain the solution is important. For most applied problems, the main difficulty lies in unreliable solutions. Studying the properties of the applied algorithms is thus very important.

FORMULATION OF THE MINLP PROBLEM

The MINLP problem corresponding to the method may be formulated as follows,

$$\min_{\mathbf{x},\mathbf{y}}\{\mathbf{c}_x^T\mathbf{x}+\mathbf{c}_y^T\mathbf{y}\} \tag{1}$$

$$\{\mathbf{x},\mathbf{y}\}\in N$$

where,

$$N=\{\mathbf{x},\mathbf{y}|\mathbf{g}(\mathbf{x},\mathbf{y})\leq 0\} \tag{2}$$

$\mathbf{c}_x$ and $\mathbf{c}_y$ are vectors with constants, $\mathbf{x}$ is a vector of continuous variables, $\mathbf{y}$ is a vector with discrete variables and $\mathbf{g}(\mathbf{x},\mathbf{y})$ is a vector with continuous convex functions, all defined on a set,

$$L=X\cup Y \tag{3}$$

where X is an n-dimensional compact polyhedral convex set,

$$X=\{\mathbf{x}|\mathbf{A}\mathbf{x}\leq\mathbf{a},\mathbf{x}\in R^n\} \tag{4}$$

and Y is a finite discrete set defined by,

$$Y=\{\mathbf{y}|\mathbf{B}\mathbf{y}\leq\mathbf{b},\mathbf{y}\in Z^m\} \tag{5}$$

THE MINLP ALGORITHM

Since the functions, $g_i(\mathbf{x},\mathbf{y})$ in $\mathbf{g}(\mathbf{x},\mathbf{y})$, are convex it follows that for each sequence of points, $\{(\mathbf{x}^k,\mathbf{y}^k)$, $k=0,1,...,K\}$, we have

$$g_i(\mathbf{x}^k,\mathbf{y}^k)+\left(\frac{\partial g_i}{\partial\mathbf{x}}\right)_{\mathbf{x}^k,\mathbf{y}^k}(\mathbf{x}-\mathbf{x}^k)+\left(\frac{\partial g_i}{\partial\mathbf{y}}\right)_{\mathbf{x}^k,\mathbf{y}^k}(\mathbf{y}-\mathbf{y}^k)\leq g_i(\mathbf{x},\mathbf{y}) \tag{6}$$

Also, observe that if,

$$\max_i\{g_i(\mathbf{x}^k,\mathbf{y}^k)\}\leq 0 \tag{7}$$

then all

$$g_i(\mathbf{x}^k,\mathbf{y}^k)\leq 0 \tag{8}$$

Now define at each point, $(\mathbf{x}^k, \mathbf{y}^k)$, a linear function corresponding to the non-linear function satisfying Eq. (6) such that,

$$l_k(\mathbf{x}, \mathbf{y}) = f_k(\mathbf{x}^k, \mathbf{y}^k) + \left(\frac{\partial f_k}{\partial \mathbf{x}}\right)_{\mathbf{x}^k, \mathbf{y}^k} (\mathbf{x} - \mathbf{x}^k) + \left(\frac{\partial f_k}{\partial \mathbf{y}}\right)_{\mathbf{x}^k, \mathbf{y}^k} (\mathbf{y} - \mathbf{y}^k) \tag{9}$$

where $f_k(\mathbf{x}, \mathbf{y})$ is the function, $g_i(\mathbf{x}, \mathbf{y})$, corresponding to $\max_i\{g_i(\mathbf{x}^k, \mathbf{y}^k)\}$.
Then, define for a sequence of points, $\{(\mathbf{x}^k, \mathbf{y}^k), k = 0, 1, ..., K\}$, a corresponding sequence of sets, $\{\Omega_k, k = 0, 1, ..., K\}$, such that,

$$\Omega_K = L \cap \{\mathbf{x}, \mathbf{y} | l_k(\mathbf{x}, \mathbf{y}) \leq 0, k = 0, 1, ..., K-1\} \tag{10}$$

Defining, $\Omega_0 = L$, then each set, Ω_k, can be written as,

$$\Omega_k = \Omega_{k-1} \cap \{\mathbf{x}, \mathbf{y} | l_{k-1}(\mathbf{x}, \mathbf{y}) \leq 0\} \tag{11}$$

A sequence of points, $\{(\mathbf{x}^k, \mathbf{y}^k), k = 0, 1, ..., K\}$, converging to the optimal solution of the problem in Eq. (1) will now be treated. Let the sequence of points be obtained from solving the following sequence of MILP problems,

$$\min_{\mathbf{x}_k, \mathbf{y}_k} \{\mathbf{c}_x^T \mathbf{x}_k + \mathbf{c}_y^T \mathbf{y}_k\} \tag{12}$$

$$\{\mathbf{x}_k, \mathbf{y}_k\} \in \Omega_k$$

$$k = 0, 1, ..., K$$

From the definition of the Ω_k set it follows that,

$$N \subset \Omega_k \tag{13}$$

and

$$\Omega_K \subset \Omega_{K-1} \subset ... \subset \Omega_k \subset ... \subset \Omega_0 \tag{14}$$

From Eq. (14) it also follows that the solutions, $Z_k = \min\{\mathbf{c}_x^T \mathbf{x}_k + \mathbf{c}_y^T \mathbf{y}_k\}$ form a monotonically increasing sequence,

$$Z_K \geq Z_{K-1} \geq ... \geq Z_k \geq ... \geq Z_0 \tag{15}$$

CONVERGENCE OF THE ALGORITHM

We will now consider if the sequence, $\{Z_k, k = 0, 1, ..., K\}$, will converge to the optimal solution in N. Convergence of the MINLP algorithm can be shown, by extending Kelley's (1960) results also to contain integer variables, as follows.
From Eqs. (12-14) it clearly follows that if,

$$(\mathbf{x}_K, \mathbf{y}_K) \in N \tag{16}$$

then $(\mathbf{x}_K, \mathbf{y}_K)$ is the optimal solution to the problem in Eq. (1), the optimal value being Z_K.
If, on the other hand,

$$(\mathbf{x}_K, \mathbf{y}_K) \notin N \tag{17}$$

then it follows from (6-9) that,

$$\max_i\{g_i(\mathbf{x}^K, \mathbf{y}^K)\} = l_K(\mathbf{x}^K, \mathbf{y}^K) > 0 \tag{18}$$

Due to the condition in Eq. (18), the new linearized constraint clearly makes the point $(\mathbf{x}^K, \mathbf{y}^K)$ infeasible when solving the following MILP problem in Eq. (12). It then follows that,

$$(\mathbf{x}^{K+1}, \mathbf{y}^{K+1}) \neq (\mathbf{x}^k, \mathbf{y}^k) \tag{19}$$

$$k = 0, 1, ..., K$$

Thus we find that the procedure generates a new point, $(\mathbf{x}^{K+1}, \mathbf{y}^{K+1})$, in the set, L, such that the new point, $(\mathbf{x}^{K+1}, \mathbf{y}^{K+1})$, will be different from all previous points, $(\mathbf{x}^k, \mathbf{y}^k)$, in the sequence.
From Eqs. (6-9) it further follows that for all linearized constraints from past solutions we have,

$$f_k(\mathbf{x}^k, \mathbf{y}^k) + \left(\frac{\partial f_k}{\partial \mathbf{x}}\right)_{\mathbf{x}^k, \mathbf{y}^k} (\mathbf{x}^{K+1} - \mathbf{x}^k) + \left(\frac{\partial f_k}{\partial \mathbf{y}}\right)_{\mathbf{x}^k, \mathbf{y}^k} (\mathbf{y}^{K+1} - \mathbf{y}^k) \leq 0 \tag{20}$$

$$k = 0, 1, ..., K$$

From Eq. (18) we also find that if an optimal point has not been obtained we have $\{f_k(\mathbf{x}^k, \mathbf{y}^k) > 0, k = 0, 1, ..., K\}$. Then it follows from Eq. (20) that if the procedure converges to a point $(\mathbf{x}^p, \mathbf{y}^p)$ then $f_p(\mathbf{x}^p, \mathbf{y}^p)$ must also converge to zero. Due to the definition of the function f, then all $g_i(\mathbf{x}^p, \mathbf{y}^p) \leq 0$ and the point must be in N.
Now, define a sub-sequence,

$$\Delta \mathbf{x}^j = \mathbf{x}^{k_j} - \mathbf{x}^{K+1} \tag{21}$$

$$\{k_j = k | \mathbf{y}^{k_j} = \mathbf{y}^{K+1}, j = 1, ..., J\}$$

where $\{k_j\}$ corresponds to the subset of indices $\{k_j\} \in \{k = 0, 1, ..., K\}$, where $\mathbf{y}^k = \mathbf{y}^{K+1}$. Since $\mathbf{y} \in Y$, it follows that there only exists a finite number of sub-sequences $\{\Delta \mathbf{x}^j, j = 1, 2, ..., J\}$. Thus, we obtain at each iteration one sub-sequence, $\{\Delta \mathbf{x}^j, j = 1, 2, ..., J\}$, (for some of the finite solutions $\mathbf{y}^{K+1}$), such that J increases.
Now assume that the procedure does not converge. Then it follows from Eq. (18) (all $f_k(\mathbf{x}^k, \mathbf{y}^k) > 0$) and Eq. (20) that there must exist a value, ϵ ($\epsilon > 0$), independent of $K+1$ (for the subsequence where $\mathbf{y}^{k_j} = \mathbf{y}^{K+1}$) such that,

$$\epsilon \leq f_{k_j}(\mathbf{x}^{k_j}, \mathbf{y}^{K+1}) \leq \left(\frac{\partial f_{k_j}}{\partial \mathbf{x}}\right)_{\mathbf{x}^{k_j}, \mathbf{y}^{K+1}} (\mathbf{x}^{k_j} - \mathbf{x}^{K+1}) \leq G \cdot \| \Delta \mathbf{x}^j \| \tag{22}$$

$$j = 1, 2, ..., J$$

where G is a finite constant, such that,

$$G \geq \| \frac{\partial f_{k_j}}{\partial \mathbf{x}} \| \tag{23}$$

for all $\mathbf{x} \in X$ and $\mathbf{y} \in Y$.
From Eq. (22) it follows that if convergence does not occur, then,

$$\| \Delta \mathbf{x}^j \| \geq \frac{\epsilon}{G} \tag{24}$$

for all j. Since $\Delta \mathbf{x}^j = \mathbf{x}^{k_j} - \mathbf{x}^{K+1}$, Eq. (24) implies that $\mathbf{x}^{K+1}$ does not contain a Cauchy sub-sequence. But this is impossible since the set, X, is compact.
Thus the procedure must converge to a point, $(\mathbf{x}^p, \mathbf{y}^p)$, in L. Since we already found that if the procedure converges to a point, $(\mathbf{x}^p, \mathbf{y}^p)$, then $f_p(\mathbf{x}^p, \mathbf{y}^p)$ also converges to zero. Then all $g_i(\mathbf{x}^p, \mathbf{y}^p) \leq 0$ and $(\mathbf{x}^p, \mathbf{y}^p)$ must be in N. Thus Z_p will converge to the optimal solution in N which is the global optimal solution to the convex MINLP problem being considered. For practical computation the condition, $g_i(\mathbf{x}^p, \mathbf{y}^p) \leq 0$, is replaced by $g_i(\mathbf{x}^p, \mathbf{y}^p) \leq \epsilon$ and convergence is then clearly achieved in a finite number of steps. The criterion for termination can, thus, in a numerical algorithm, be written as,

$$f_k(\mathbf{x}^k, \mathbf{y}^k) \leq \epsilon \tag{25}$$

From previous considerations it can also be found that $f_{k_j}(\mathbf{x}^{k_j}, \mathbf{y}^{k_j})$ forms a monotonically decreasing sequence if the function, f, corresponds to the same function, $g_i(\mathbf{x}, \mathbf{y})$. Thus, $f_{k+1}(\mathbf{x}^{k+1}, \mathbf{y}^{k+1}) - f_k(\mathbf{x}^k, \mathbf{y}^k) \geq 0$ only if the function, f, corresponds to a different function, $g_i(\mathbf{x}, \mathbf{y})$, at the iteration steps $k+1$ and k or if $\mathbf{y}^{k+1} \neq \mathbf{y}^k$.
It should, finally, be noted that the convergence proof can simply be extended to the case where the vector of non-linear convex functions, $\mathbf{g}(\mathbf{x}, \mathbf{y})$, is defined on a set, L,

$$L = \cup_k X_k \tag{26}$$

where X_k corresponds to an n–dimensional compact polyhedral convex set, one for each finite discrete point $\mathbf{y} \in Y$.

A NUMERICAL EXAMPLE

In the following, a numerical example from Duran and Grossmann, (1986) is provided to illustrate the numerical procedure. The problem is related to the problem of synthesizing a processing system. The problem is given explicitly in Duran and Grossmann, (1986) as Test Problem 1, and will therefore not be treated in detail in this paper. In terms of the cutting plane method, this problem can be written as,

$$\min_{\mathbf{x}, \mathbf{y}} Z = \{\mathbf{c}_x^T \mathbf{x} + \mathbf{c}_y^T \mathbf{y}\} \tag{27}$$

subject to,

$$\mathbf{g}(\mathbf{x}, \mathbf{y}) \leq 0 \tag{28}$$

where,

$$\mathbf{x} \in \{\mathbf{x}|\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y}^k \leq \mathbf{a}, \mathbf{x} \geq 0\}$$

$$\mathbf{y} \in \{0,1\}^3$$

The vectors, $\mathbf{a}$, $\mathbf{c}_x$, $\mathbf{c}_y$, and the matrices, $\mathbf{A}$ and $\mathbf{B}$, as well as the vector of convex functions, $\mathbf{g}(\mathbf{x},\mathbf{y})$, are defined by,

$$\mathbf{c}_x^T = (0 \quad 0 \quad 0 \quad 1)$$

$$\mathbf{c}_y^T = (5 \quad 6 \quad 8)$$

$$\mathbf{g}(\mathbf{x},\mathbf{y}) = \begin{pmatrix} -0.8\ln(x_2+1) - 0.96\ln(x_1 - x_2 + 1) + 0.8x_3 \\ -\ln(x_2+1) - 1.2\ln(x_1 - x_2 + 1) + x_3 + 2y_3 - 2 \\ 10x_1 - 7x_3 - 18\ln(x_2+1) - 19.2\ln(x_1 - x_2 + 1) + 10 - x_4 \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & -2 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

$$\mathbf{a}^T = (2 \quad 2 \quad 1 \quad 100 \quad 0 \quad 0 \quad 0 \quad 1)$$

The problem is solved such that the algorithm also generates the initial point, $(\mathbf{x}^0, \mathbf{y}^0)$. The termination criterion for convergence in a finite number of steps has been chosen to be $\epsilon = 10^{-5}$. In Table 1, the solution after each iteration step is given. The first column corresponds to the iteration index, columns two to eight to the values of the discrete and the continuous variables and column nine to the value of the objective function obtained from the cutting plane algorithm. Finally, the second last column corresponds to the function value as defined in Eq. (9) and the index, i, corresponds to the non-linear function, $g_i(\mathbf{x},\mathbf{y}) = f_k(\mathbf{x},\mathbf{y})$.
Duran and Grossmann (1986) give the optimal solution, $\mathbf{y}^* = (0,1,0)$, $\mathbf{x}^* = (1.30097, 0, 1, 0.00972)$ and $Z^* = 6.00972$. As can be seen in Table 1, the present procedure converges to the actual solution in 13 steps with the chosen value of ϵ.

k	y_1^k	y_2^k	y_3^k	x_1^k	x_2^k	x_3^k	x_4^k	Z^k	$f_k(\mathbf{x}^k,\mathbf{y}^k)$	i
0	0	0	0	0	0	0	0	0	10.00000	3
1	0	0	0	0	0	1.00000	3.00000	3.00000	0.800000	1
2	1	0	0	0.66667	0.66667	0.66667	0	5.00000	2.805112	3
3	1	0	0	1.25641	1.25641	1.00000	0	5.00000	0.916148	3
4	1	0	0	1.00000	1.00000	1.00000	0.39751	5.39751	0.245482	1
5	0	1	0	0.56818	0	0.68182	0	6.00000	2.270678	3
6	0	1	0	0.78152	0	0.93783	0	6.00000	0.195896	1
7	0	1	0	1.02784	0	0.85888	0	6.00000	0.692392	3
8	0	1	0	1.19336	0	0.97037	0	6.00000	0.060663	3
9	0	1	0	1.21085	0	0.98215	0	6.00000	0.024078	1
10	0	1	0	1.29337	0	0.99684	0	6.00000	0.019391	3
11	0	1	0	1.29919	0	1.00000	0.00674	6.00674	0.000745	1
12	0	1	0	1.30098	0	1.00000	0.00965	6.00965	0.000011	3
13	0	1	0	1.30098	0	1.00000	0.00976	6.00976	0.000004	3

Table 1. Convergence of the proposed method.

DISCUSSION

Convergence properties for an extension of Kelley's cutting plane method, were given in the present paper. The method is suitable for solving large MINLP problems with a moderate degree of nonlinearity. At each iteration, the procedure requires only the improvement of a single MILP solution, obtained at the previous iteration, because of an additional linear constraint. The procedure is, thus, a competitive alternative to algorithms where both an NLP and an MILP problem must be solved in each iteration. The convergence properties of the method for convex MINLP problems were given and a numerical example was finally provided to illustrate the numerical procedure.

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