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A METHOD OF ANALYTIC CENTERS FOR QUADRATICALLY CONSTRAINED CONVEX QUADRATIC PROGRAMS*

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Abstract. An interior point method is developed for maximizing a concave quadratic function under convex quadratic constraints. The algorithm constructs a sequence of nested convex sets and finds their approximate centers using a partial Newton step. Given the first convex set and its approximate center, the total arithmetic operations required to converge to an approximate solution are of order $O(\sqrt{m}(m+n)n^2 \ln \varepsilon)$, where m is the number of constraints, n is the number of variables, and ε is determined by the desired tolerance of the optimal value and the size of the first convex set. A method to initialize the algorithm is also proposed so that the algorithm can start from an arbitrary (perhaps infeasible) point.

Key words. analytic center, quadratic programming, interior point methods, Karmarkar's algorithm, method of centers

AMS(MOS) subject classification. 90C20

1. Introduction.

We consider the quadratically constrained convex quadratic program

$$(1.1.1)\quad (\text{QCQP}): \begin{cases} \text{maximize} & q(x) \equiv \frac{1}{2}x^TQx + c^Tx \\ \text{subject to} & x \in P \equiv \{x \mid q_i(x) = \frac{1}{2}x^TQ_ix + c_i^Tx \geqq b_i,\ i = 1, 2, \cdots, m\}, \end{cases}$$

where $x \in \mathfrak{R}^n$, c, $c_i \in \mathfrak{R}^n$, and Q, $Q_i \in \mathfrak{R}^{n\times n}$ are symmetric negative semidefinite matrices. We assume that the set of feasible solutions P in (QCQP) is bounded and it has a nonempty interior.

The problem (QCQP) has been extensively studied in the literature and several algorithms have been proposed to solve it. In a series of papers Peterson and Ecker [23]-[25] studied the duality theory for (QCQP) using a geometric programming approach. The dual formulation of (QCQP) given by Peterson and Ecker was later used by Ecker and Niemi [3] and Fang and Rajasekera [4] to develop dual algorithms to solve this problem. Baron [1] proposed a cutting plane approach to solve the Lagrangian dual of (QCQP). A triangularization method was given in Hao [26]. Baron [1] and Hao [26] also provided several applications of (QCQP).

The proposed algorithm for (QCQP) in this paper is motivated from some recent advances in mathematical programming that came from studies [2], [19], [27], [31] on Karmarkar's algorithm [15] for linear programming. The two approaches that have been studied extensively in recent years are the logarithmic barrier function method, attributed to Frisch [6] (see also [5]), and the method of centers due to Huard [11]. Gonzaga [10], Monteiro and Adler [21], Renegar [27], Kojima, Mizuno, and Yoshise [16], and Vaidya [31] developed polynomial time algorithms for solving linear programs by effectively controlling the parameters in logarithmic barrier method or the method of centers. The initial results for linear programming have been extended by Goldfarb and Liu [9], Kojima, Mizuno, and Yoshise [17], Mehrotra and Sun [20], Monteiro and Adler [22], and Ye [32] for solving convex quadratic programs with linear constraints.

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In this paper we extend the method developed by us in [20] to (QCQP). We study a path of centers defined by a logarithmic function associated with (QCQP) and show that this path could be followed by using partial Newton steps. The convergence rate of the method is the same as the one obtained by Renegar [27] and Vaidya [31] for linear programs. We have noticed that in a recent paper Jarre [13] has independently analyzed an algorithm similar to the one given here. However, his approaches of proof are different from ours.

This paper is organized as follows. In the next section we develop our algorithm for (QCQP) that we intend to analyze. Section 3 contains the analysis. Finally, in § 4 we outline an approach that could be used to satisfy the initialization assumption of the algorithm.

2. The basic algorithm. Let z^* stand for the optimal value of (QCQP) and consider the convex set

$$P_z \equiv \{x \in \Re^n \mid q(x) \geqq z,\ q_i(x) \geqq b_i,\ i=1,2,\cdots,m\}, \qquad z < z^*. \tag{2.1.1}$$

The assumptions on set P imply (cf. [28]) that P_z is bounded and the set $\operatorname{int} P_z = \{x \in \Re^n \mid q(x) > z,\ q_i(x) > b_i,\ i=1,2,\cdots,m\}$ is nonempty.

A point $\omega \in \Re^n$ is called an *analytic center* of P_z if it maximizes

$$F(x,z) \equiv m \ln(q(x)-z) + \sum_{i=1}^{m} \ln(q_i(x)-b_i) \tag{2.1.2}$$

subject to $x \in \operatorname{int} P_z$. The function $F(x,z)$ defined in (2.1.2) is called the *potential function.*

Let $F(x,z)$ be as in (2.1.2) and ω be a corresponding analytic center. Let $f(x,z) \equiv F(\omega,z) - F(x,z)$. The function $f(x,z)$ is called the *normalized potential function.* Note that ω is an analytic center if and only if it minimizes $f(x,z)$ over $\operatorname{int} P_z$. Let $\nabla f(x,z)$ and $\nabla^2 f(x,z)$ represent the gradient and Hessian of $f(x,z)$ with respect to x. It is easy to see that

$$\nabla f(x,z) = -\frac{m\nabla q(x)}{q(x)-z} - \sum_{i=1}^{m} \frac{\nabla q_i(x)}{q_i(x)-b_i}$$

and

$$\nabla^2 f(x,z) = \frac{m}{(q(x)-z)^2}\nabla q(x)(\nabla q(x))^T + \sum_{i=1}^{m} \frac{1}{(q_i(x)-b_i)^2}\nabla q_i(x)(\nabla q_i(x))^T$$
$$-\frac{m}{q(x)-z}Q - \sum_{i=1}^{m}\frac{1}{q_i(x)-b_i}Q_i.$$

In this paper we are concerned with the following algorithm for solving (QCQP).

ALGORITHM 2.1.
Initialization: Let $x^0 \in \operatorname{int} P_{z^0}$, be such that $f(x^0, z^0) \leqq .003$.
For $k = 0, 1, \cdots$ **until** $q(x^k) - z^k \leqq 2^{-\theta}$ is achieved for a given positive number θ **do**:
Determine a step direction p by solving

$$\nabla^2 f(x^k, z^k)p = -\nabla f(x^k, z^k).$$

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Let

$$x^{k+1} \leftarrow x^k + \frac{\beta}{\sqrt{p^T \nabla^2 f(x^k, z^k) p}} p,$$

where $\beta = .03$.

$$z^{k+1} \leftarrow z^k + \frac{\alpha}{\sqrt{m}} (q(x^{k+1}) - z^k)$$

with $\alpha = .0024$.

End

The values $\beta = .03$ and $\alpha = .0024$ are chosen for our analysis. In practice β may be obtained by performing one-dimensional line search to maximize the potential function $F(x, z^k)$. An approach to satisfy the initialization assumption $f(x^0, z^0) \leqq .003$ is given in § 4. The parameter θ used in the stopping criterion is a large positive constant which ensures desired accuracy in the objective function. It is taken so that it also satisfies $z^* - z^0 \leqq 2^\theta$.

The motivation of this algorithm is as follows. At the beginning of iteration k we have an approximation x^k of the analytic center ω^k of the convex set P_{z^k}. The normalized potential function provides the metric that is used to measure the closeness. We take a partial Newton step at x^k to reduce this function and move to x^{k+1}. The point x^{k+1} is closer to ω^k than x^k. Finally z^k is increased to z^{k+1} such that x^{k+1} also serves as an approximation to ω^{k+1}. The sequence of nested convex sets $P_{z^0}, P_{z^1}, \cdots, P_{z^k}, \cdots$ will shrink towards the set of optimal solution(s).

The main computational work in the implementation of the Algorithm 2.1 involves solving a system of linear equations:

$$\nabla^2 f(x, z) p = -\nabla f(x, z). \tag{2.1.3}$$

The matrix $\nabla^2 f(x, z)$ defining the system of linear equations (2.1.3) is symmetric and positive definite. Direct (e.g., symmetric Gaussian elimination) and iterative methods (e.g., preconditioned conjugate gradient method) may be used to solve (2.1.3).

The remainder of this paper is devoted to the analysis of Algorithm 2.1. We show that, starting from x^0, the algorithm takes $O(\sqrt{m}\,\theta)$ iterations to ensure that $z^* - z^k \leqq 2^{-\theta} \equiv \varepsilon$ for any given (large) constant $\theta > 0$. Hence the total arithmetic operations to achieve such an approximate solution are of order $O(\sqrt{m}(m+n)n^2 \ln \varepsilon)$.

Note that the stopping criterion stated in Algorithm 2.1 is in terms of quantities that are computable at each iteration. As a simple consequence of Lemma 3.6 of next section we know that $z^* - z^k \leqq 4(q(x^k) - z^k)$, hence the algorithm produces an interior solution that has a near optimal value.

In order to ensure that $z^* - z^k \leqq 2^{-\theta}$ in $O(\sqrt{m}\,\theta)$ iterations, it is sufficient to show that

$$\frac{z^* - z^{k+1}}{z^* - z^k} \leqq 1 - \frac{.001}{\sqrt{m}}$$

for all iterations of Algorithm 2.1. This is established in the next section.

3. The convergence theorem and its proof. In this section we prove the following convergence theorem for Algorithm 2.1.

THEOREM 3.1. *Let z^* denote the optimal objective value of* (QCQP). *Then Algorithm* 2.1 *is well defined and at iteration k we have*

$$\frac{z^* - z^{k+1}}{z^* - z^k} \leqq 1 - \frac{.001}{\sqrt{m}}. \tag{3.1.1}$$

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We now give several properties of the normalized potential function and the analytic centers that are used in the sequel. We first show that the normalized potential function $f(x, z)$ is a strictly convex function and, therefore, the analytic center ω is unique.

LEMMA 3.2. *Let P_z, defined as in* (2.1.1), *be a bounded convex set and let* int $P_z \equiv \{x \in \Re^n \mid q(x) > z,\ q_i(x) > b_i,\ i = 1, 2, \cdots, m\}$ *be nonempty. Then the normalized potential function $f(x, z)$ is a strictly convex function on* int P_z.

Proof. It is sufficient to show that the Hessian of $f(x, z)$ is a positive-definite matrix at all the points in int P_z. We assume the contrary, that is, there is an $x \in \text{int } P_z$ at which the Hessian is only positive semidefinite. Let v be a nonzero vector (direction) such that

$$v^T\left[\sum_{i=1}^m \frac{-1}{q_i(x)-z} Q_i + \frac{1}{(q_i(x)-z)^2} \nabla q_i(x) \nabla q_i(x)^T\right] v$$
$$+ v^T\left[\frac{-m}{q(x)-z} Q + \frac{m}{(q(x)-z)^2} \nabla q(x) \nabla q(x)^T\right] v = 0.$$

The positive semidefiniteness of $-Q_i$ and $\nabla q_i(x) \nabla q_i(x)^T$ implies that $v^T Q_i v = 0$ and $\nabla q_i(x)^T v = 0$ for $i = 1, 2, \cdots, m$. Since

$$q_i(x+v) = q_i(x) + \tfrac{1}{2} v^T Q_i v + \nabla q_i(x)^T v,$$

all the points on the line $x + \theta v$ $(-\infty < \theta < \infty)$ are feasible. This contradicts the boundedness of P_z. $\square$

The following lemma gives the Taylor expansion of $f(x, z)$ at $y \in \text{int } P_z$.

LEMMA 3.3. *Let $y, y' \in$* int P_z *and $h = y' - y$. The Taylor expansion of $f(y', z)$ at y is given by*

$$f(y', z) - f(y, z) = \sum_{j=1}^\infty \frac{1}{j!} \sum_{k=0}^{\lfloor j/2 \rfloor} \alpha_{jk} \left[\sum_{i=1}^m (A_i(y))^{j-2k} (B_i(y))^k + m(A(y))^{j-2k} (B(y))^k\right],$$

where $A(y)$, $A_i(y)$, $B(y)$, and $B_i(y)$ are obtained by evaluating

$$A(x) = -\frac{\nabla q(x)^T h}{q(x)-z}, \qquad A_i(x) = -\frac{\nabla q_i(x)^T h}{q_i(x)-b_i},$$
$$B(x) = -\frac{h^T Q h}{q(x)-z}, \qquad B_i(x) = -\frac{h^T Q_i h}{q_i(x)-b_i},$$

at $x = y$, respectively. Moreover, the coefficients α_{jk} are defined by the following recursive relationships:

$$\begin{aligned}
&\alpha_{1,0} = 1, && \\
&\alpha_{j+1,0} = j\alpha_{j,0}, && j = 1, 2, \cdots, \\
&\alpha_{j+1,k} = (j-k)\alpha_{jk} + (j-2k+2)\alpha_{j,k-1}, && j = 1, 2, \cdots,\ k = 1, 2, \cdots, \lfloor j/2 \rfloor, \\
&\alpha_{jk} = 0, && j = 1, 2, \cdots,\ k > \lfloor j/2 \rfloor.
\end{aligned}$$

Proof. Since

$$\text{(3.3.1)} \qquad f(y', z) - f(y, z) = \sum_{j=1}^\infty \frac{1}{j!} \left(\frac{\partial}{\partial x_1} h_1 + \cdots + \frac{\partial}{\partial x_n} h_n\right)^j f(x, z)\bigg|_{x=y},$$

we only need to show that for $i = 0, \cdots, m$,

$$\text{(3.3.2)} \qquad \left(\frac{\partial}{\partial x_1} h_1 + \cdots + \frac{\partial}{\partial x_n} h_n\right)^j \ln(q_i(x) - b_i) = -\sum_{k=0}^{\lfloor j/2 \rfloor} \alpha_{jk} (A_i(x))^{j-2k} (B_i(x))^k.$$

It can be easily verified that (3.3.2) is valid for $j=1$ and $j=2$. Now suppose that (3.3.2) is valid for j. We have

$$\left(\frac{\partial}{\partial x_1}h_1+\cdots+\frac{\partial}{\partial x_n}h_n\right)^{j+1}\ln(q_i(x)-b_i)$$
$$=\left(\frac{\partial}{\partial x_1}h_1+\cdots+\frac{\partial}{\partial x_n}h_n\right)\left(-\sum_{k=0}^{\lfloor j/2\rfloor}\alpha_{jk}(A_i(x))^{j-2k}(B_i(x))^k\right).$$

The identities

$$\left(\frac{\partial}{\partial x_1}h_1+\cdots+\frac{\partial}{\partial x_n}h_n\right)A_i(x)=(A_i(x))^2+B_i(x)$$

and

$$\left(\frac{\partial}{\partial x_1}h_1+\cdots+\frac{\partial}{\partial x_n}h_n\right)B_i(x)=A_i(x)B_i(x)$$

are used in the sequel.

If $j=2l$, we get

$$\left(\frac{\partial}{\partial x_1}h_1+\cdots+\frac{\partial}{\partial x_n}h_n\right)\left[-\sum_{k=0}^{\lfloor j/2\rfloor}\alpha_{jk}(A_i(x))^{j-2k}(B_i(x))^k\right]$$
$$=-\alpha_{2l,0}(2l)(A_i(x))^{2l-1}(A_i^2(x)+B_i(x))$$
$$-\sum_{k=1}^{l-1}\alpha_{2l,k}[(2l-2k)(A_i(x))^{2l-2k-1}(B_i(x))^k(A_i^2(x)+B_i(x))$$
$$+k(A_i(x))^{2l-2k}(B_i(x))^{k-1}A_i(x)B_i(x)]-l(B_i(x))^{l-1}A_i(x)B_i(x)$$
$$=-2l\alpha_{2l,0}(A_i(x))^{2l+1}$$
$$-\sum_{k=1}^{l}[(2l-k)\alpha_{2l,k}+(2l-2k+2)\alpha_{2l,k-1}](A_i(x))^{2l-2k+1}(B_i(x))^k$$
$$=-\sum_{k=0}^{\lfloor (j+1)/2\rfloor}\alpha_{j+1,k}(A_i(x))^{j+1-2k}(B_i(x))^k,$$

and if $j=2l+1$, we get

$$\left(\frac{\partial}{\partial x_1}h_1+\cdots+\frac{\partial}{\partial x_n}h_n\right)\left(-\sum_{k=0}^{\lfloor j/2\rfloor}\alpha_{jk}(A_i(x))^{k-2l}(B_i(x))^k\right)$$
$$=-\alpha_{2l+1,0}(2l+1)(A_i(x))^{2l}(A_i^2(x)+B_i(x))$$
$$-\sum_{k=1}^{l}\alpha_{2l+1,k}[(2l+1-2k)(A_i(x))^{2l-2k}(B_i(x))^k(A_i^2(x)+B_i(x))$$
$$+k(A_i(x))^{2l+1-2k}(B_i(x))^{k-1}A_i(x)B_i(x)]$$
$$=-(2l+1)\alpha_{2l+1,0}(A_i(x))^{2l+2}$$
$$-\sum_{k=1}^{l}[(2l+1-k)\alpha_{2l+1,k}+(2l+1-2k+2)\alpha_{2l+1,k-1}](A_i(x))^{2l-2k+2}(B_i(x))^k$$
$$-\alpha_{2l+1,l}(B_i(x))^{l+1}$$
$$=-\sum_{k=0}^{l}\alpha_{2l+2,k}(A_i(x))^{2l+2-2k}(B_i(x))^k-\alpha_{2l+1,l}(B_i(x))^{l+1}$$
$$=-\sum_{k=0}^{\lfloor (j+1)/2\rfloor}\alpha_{j+1,k}(A_i(x))^{j+1-2k}(B_i(x))^k.$$

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This completes the induction. The proof of Lemma 3.3 follows by combining (3.3.1) and (3.3.2). □

LEMMA 3.4. *Let* α_{jk}*'s be defined as in Lemma* 3.3; *then*

$$\sum_{k=0}^{\lfloor j/2 \rfloor} \alpha_{jk} \leqq 2^{j-1}(j-1)!.$$

Proof. If $j=2l$, we have

$$\sum_{k=0}^{l} \alpha_{2l,k} = \sum_{k=1}^{l-1} [(2l-1-k)\alpha_{2l-1,k} + (2l-2k+1)\alpha_{2l-1,k-1}] + (2l-1)\alpha_{2l-1,0} + \alpha_{2l-1,l-1}$$

$$= \sum_{k=0}^{l-1} (4l-2-3k)\alpha_{2l-1,k} \leqq (4l-2) \sum_{k=0}^{l-1} \alpha_{2l-1,k} \qquad \text{(because all } \alpha_{jk} \geqq 0).$$

Otherwise $j=2l+1$, and we have

$$\sum_{k=0}^{l} \alpha_{2l+1,k} = \sum_{k=1}^{l} [(2l-k)\alpha_{2l,k} + (2l+2-2k)\alpha_{2l,k-1} + \alpha_{2l,k-1}] + 2l\alpha_{2l,0}$$

$$= \sum_{k=0}^{l} (4l-3k)\alpha_{2l,k} \leqq 4l \sum_{k=0}^{l} \alpha_{2l,k}.$$

Hence, by letting $\beta_j = \sum_{k=0}^{\lfloor j/2 \rfloor} \alpha_{jk}$, we have

$$\beta_j \leqq (2j-2)\beta_{j-1}.$$

Therefore,

$$\beta_j \leqq 2^{j-1}(j-1)!\beta_1 = 2^{j-1}(j-1)!. \qquad \square$$

Let $E(y, r) \equiv \{x \mid (x-y)^T \nabla^2 f(y, z)(x-y) \leqq r^2\}$ be the ellipsoid of radius r around y. In the following lemma we show that the ellipsoid $E(y, 0.5)$ at any point $y \in \operatorname{int} P_z$ is contained in P_z. This lemma also provides simple estimations about the values of the objective and constraint functions at $x \in E(y, r)$.

LEMMA 3.5. *If* $x \in E(y, r)$, *then*

$$\frac{|q_i(y) - q_i(x)|}{q_i(y) - b_i} \leqq r + \frac{r^2}{2}, \quad i = 1, 2, \cdots, m \tag{3.5.1}$$

and

$$\frac{|q(y) - q(x)|}{q(y) - z} \leqq \frac{r}{\sqrt{m}} + \frac{r^2}{2m}. \tag{3.5.2}$$

Furthermore, $E(y, 0.5) \subset P_z$.

Proof. Since $x \in E(y, r)$, for $i = 1, 2, \cdots, m$ we have

$$\frac{|\nabla q_i(y)^T (x-y)|}{q_i(y) - b_i} \leqq r,$$

and

$$\frac{|(x-y)^T Q_i (x-y)|}{q_i(y) - b_i} \leqq r^2.$$

Inequalities (3.5.1) follow by using

$$\frac{|q_i(y) - q_i(x)|}{q_i(y) - b_i} \leqq \frac{|\nabla q_i(y)^T (x-y)|}{q_i(y) - b_i} + \frac{1}{2} \frac{|(x-y)^T Q_i (x-y)|}{q_i(y) - b_i}.$$

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Inequality (3.5.2) can be obtained in a similar way. Now to see that $E(y, 0.5) \subset P_z$, note that

$$\frac{q_i(x)-b_i}{q_i(y)-b_i} = 1 + \frac{q_i(x)-q_i(y)}{q_i(y)-b_i} \geqq 1 - r - \frac{r^2}{2} > 0,$$

for $r \leqq 0.5$ and $i = 0, 1, \cdots, m$. □

We now provide a lemma that will be used in the proof of Theorem 3.1, as well as in the proofs of Lemma 3.9 and Lemma 3.12.

LEMMA 3.6. *Let ω be the analytic center of the convex set P_z and $0 < r \leqq 0.5$. Then*

$$0 < 1 - \frac{r}{\sqrt{m}} - \frac{r^2}{2m} \leqq \frac{q(x)-z}{q(\omega)-z} \leqq 1 + \frac{r}{\sqrt{m}} + \frac{r^2}{2m} \quad \forall x \in E(\omega, r) \tag{3.6.1}$$

and

$$\frac{q(x)-z}{q(\omega)-z} \leqq 2 \quad \forall x \in P_z. \tag{3.6.2}$$

Proof. The proof of (3.6.1) is similar to the proof of Lemma 3.5. To show (3.6.2) note that

$$\begin{aligned}
&\frac{1}{m}\sum_{i=1}^{m} \frac{q_i(x)-b_i}{q_i(\omega)-b_i} + \frac{q(x)-z}{q(\omega)-z} \\
&= \frac{1}{m}\sum_{i=1}^{m}\left[1 + \frac{\nabla q_i(\omega)^T(x-\omega)}{q_i(\omega)-b_i} + \frac{1}{2}\frac{(x-\omega)^T Q_i (x-\omega)}{q_i(\omega)-b_i}\right] \\
&\qquad + 1 + \frac{\nabla q(\omega)^T(x-\omega)}{q(\omega)-z} + \frac{1}{2}\frac{(x-\omega)^T Q (x-\omega)}{q(\omega)-z} \\
&= 1 - \frac{\nabla f(\omega)^T(x-\omega)}{m} + 1 + \frac{1}{2m}\sum_{i=1}^{m}\frac{(x-\omega)^T Q_i(x-\omega)}{q_i(\omega)-b_i} + \frac{1}{2}\frac{(x-\omega)^T Q(x-\omega)}{q(\omega)-z} \\
&= 2 + \frac{1}{2m}\sum_{i=1}^{m}\frac{(x-\omega)^T Q_i(x-\omega)}{q_i(\omega)-b_i} + \frac{1}{2}\frac{(x-\omega)^T Q(x-\omega)}{q(\omega)-z} \qquad (\text{since } \nabla f(\omega, z) = 0).
\end{aligned}$$

The inequality (3.6.2) follows from the facts that $[q_i(x)-b_i]/[q_i(\omega)-b_i] \geqq 0$ for all i and any $x \in P_z$, and that all Q_i including Q are negative semidefinite. □

The next lemma estimates the residual of the normalized potential function in the ellipse $E(y, 0.5)$.

LEMMA 3.7. *If $x \in E(y, r)$, $0 \leqq r < 0.5$, then*

$$\left| f(x, z) - f(y, z) - \nabla f(y, z)^T(x-y) - \frac{1}{2}(x-y)^T \nabla^2 f(y, z)(x-y) \right| \leqq \frac{4r^3}{3(1-2r)}.$$

Proof. It suffices to show that

$$\left| \sum_{j=3}^{\infty} \left[\frac{1}{j!}\sum_{i=1}^{m}\sum_{k=0}^{\lfloor j/2 \rfloor} \alpha_{jk}(A_i(y))^{j-2k}(B_i(y))^k + \frac{m}{j!}\sum_{k=0}^{\lfloor j/2 \rfloor} \alpha_{jk}(A(y))^{j-2k}(B(y))^k \right] \right| \leqq \frac{4r^3}{3(1-2r)},$$

where

$$A_i(y) = -\frac{\nabla q_i(y)^T(x-y)}{q_i(y)-b_i}, \quad B_i(y) = -\frac{(x-y)^T Q_i(x-y)}{q_i(y)-b_i}, \quad i = 0, \cdots, m,$$

with $A_0(y)=A(y)$, $B_0(y)=B(y)$, $b_0=z$, and $Q_0=Q$. Now, since $x\in E(y,r)$, we have

$$\sum_{i=0}^{m}[(A_i(y))^2+B_i(y)]+m(A(y))^2+mB(y)\leqq r^2,$$

and therefore,

$$m(A(y))^2+mB(y)\leqq r^2,\quad (A_i(y))^2+B_i(y)\leqq r^2,\quad i=1,\cdots,m.$$

Let $\beta_j=\sum_{k=0}^{\lfloor j/2\rfloor}\alpha_{jk}$, where α_{jk} are defined as in Lemma 3.3. Since $\beta_j\geqq 0$ and $(A_i(y))^{-2}B_i(y)\geqq 0$, we have

$$\begin{aligned}
&\left|\sum_{j=3}^{\infty}\frac{1}{j!}\left[\sum_{i=1}^{m}\sum_{k=0}^{\lfloor j/2\rfloor}\alpha_{jk}(A_i(y))^{j-2k}(B_i(y))^k+m\sum_{k=0}^{\lfloor j/2\rfloor}\alpha_{jk}(A(y))^{j-2k}(B(y))^k\right]\right|\\
&\leqq\sum_{j=3}^{\infty}\frac{\beta_j}{j!}\left[\sum_{i=1}^{m}\sum_{k=0}^{\lfloor j/2\rfloor}|A_i(y)|^j((A_i(y))^{-2}B_i(y))^k+m\sum_{k=0}^{\lfloor j/2\rfloor}|A(y)|^j((A(y))^{-2}B(y))^k\right]\\
&\leqq\sum_{j=3}^{\infty}\frac{\beta_j}{j!}\left[\sum_{i=1}^{m}|A_i(y)|^j(1+(A_i(y))^{-2}B_i(y))^{\lfloor j/2\rfloor}+m|A(y)|^j(1+(A(y))^{-2}B(y))^{\lfloor j/2\rfloor}\right]\\
&\leqq\sum_{j=3}^{\infty}\frac{\beta_j}{j!}\left[\sum_{i=1}^{m}((A_i(y))^2+B_i(y))^{j/2}+m((A(y))^2+B(y))^{j/2}\right]\\
&\qquad\qquad\text{(because } (1+(A_i(y))^{-2}B_i(y))^{\lfloor j/2\rfloor}\leqq(1+(A_i(y))^{-2}B_i(y))^{j/2})\\
&\leqq\sum_{j=3}^{\infty}\frac{\beta_j}{j!}(r^2)^{(i/2)-1}\left[\sum_{i=1}^{m}((A_i(y))^2+B_i(y))+m((A(y))^2+B(y))\right]\\
&\leqq\sum_{j=3}^{\infty}\frac{2^{j-1}}{j}r^j\leqq\frac{r}{3}\sum_{j=3}^{\infty}(2r)^{j-1}=\frac{4r^3}{3(1-2r)}.
\end{aligned}$$

□

The next lemma shows that the closeness of x and ω can be measured by the value of potential function.

LEMMA 3.8. *Let* $0\leqq\delta<0.5$, *and*

$$f(x,z)\leqq\frac{\delta^2}{2}-\frac{4\delta^3}{3(1-2\delta)};$$

then $x\in E(\omega,\delta)$.

Proof. Since $f(x,z)$ is a strictly convex function and $f(\omega,z)=0$, its minimum value over the region $\{x\in R^n\,|\,x\in P_z,\ x\notin\operatorname{int}E(\omega,\delta)\}$ occurs on the boundary of $E(\omega,\delta)$. It is therefore sufficient to show that $f(x,z)\geqq\delta^2/2-4\delta^3/3(1-2\delta)$ for all the points on the boundary of $E(\omega,\delta)$. The Taylor expansion of $f(x,z)$ at ω and Lemma 3.7 gives

$$\begin{aligned}
f(x,z)&\geqq f(\omega,z)+\nabla f(\omega,z)^T(x-\omega)+\frac{1}{2}(x-\omega)^T\nabla^2 f(\omega,z)(x-\omega)-\frac{4\delta^3}{3(1-2\delta)}\\
&\geqq\frac{\delta^2}{2}-\frac{4\delta^3}{3(1-2\delta)}.
\end{aligned}$$

The last inequality follows by using the fact that $f(\omega,z)=0$, $\nabla f(\omega,z)=0$ and x is on the boundary of $E(\omega,\delta)$. □

LEMMA 3.9. *Let* $x\in E(\omega,\delta)$, *and let* $\delta<0.243$. *Then*

$$\nabla f(x,z)^T(x-\omega)\geqq[(1-\tfrac{5}{2}\delta-2\delta^2)f(x,z)(x-\omega)^T\nabla^2 f(x,z)(x-\omega)]^{1/2}. \tag{3.9.1}$$

Proof. We first establish the following results that are used in proving this lemma:

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$$\nabla f(x, z)^T(x-\omega) \geqq f(x, z) \geqq 0, \tag{3.9.2}$$

$$\nabla f(x, z)^T(x-\omega) \geqq (1-\tfrac{5}{2}\delta-2\delta^2)(x-\omega)^T\nabla^2 f(x, z)(x-\omega). \tag{3.9.3}$$

Proof of (3.9.2). Since $f(\omega, z)=0$ and $f(x, z)$ is convex, we have

$$0 \leqq f(x, z) = f(x, z) - f(\omega, z) \leqq \nabla f(x, z)^T(x-\omega). \qquad \square$$

Proof of (3.9.3). Lemmas 3.5 and 3.6, and the relationships

$$\nabla q_i(x) = \nabla q_i(\omega) + Q_i(x-\omega),$$

$$q_i(x) = q_i(\omega) + \nabla q_i(\omega)^T(x-\omega) + \tfrac{1}{2}(x-\omega)^T Q_i(x-\omega)$$

are used frequently in the following proof. Note that

$$\begin{aligned}\nabla f(x, z)^T(x-\omega) &= -\frac{m\nabla q(x)^T(x-\omega)}{q(x)-z} - \sum_{i=1}^m \frac{\nabla q_i(x)^T(x-\omega)}{q_i(x)-b_i} \\ &= -\frac{m\nabla q(\omega)^T(x-\omega)}{q(x)-z} - \frac{m(x-\omega)^T Q(x-\omega)}{q(x)-z} \\ &\quad -\sum_{i=1}^m \left[\frac{\nabla q_i(\omega)^T(x-\omega)}{q_i(x)-b_i} + \frac{(x-\omega)^T Q_i(x-\omega)}{q_i(x)-b_i}\right].\end{aligned} \tag{3.9.4}$$

Now by using $\nabla f(\omega, z)^T(x-\omega)=0$ in (3.9.4) we have

$$\begin{aligned}&\nabla f(x, z)^T(x-\omega) \\ &= -\frac{m(x-\omega)^T Q(x-\omega)}{q(x)-z} - \sum_{i=1}^m \frac{(x-\omega)^T Q_i(x-\omega)}{q_i(x)-b_i} \\ &\quad + m\nabla q(\omega)^T(x-\omega)\left[\frac{1}{q(\omega)-z} - \frac{1}{q(x)-z}\right] \\ &\quad + \sum_{i=1}^m \nabla q_i(\omega)^T(x-\omega)\left[\frac{1}{q_i(\omega)-b_i} - \frac{1}{q_i(x)-b_i}\right] \\ &= -\frac{m(x-\omega)^T Q(x-\omega)}{q(x)-z} - \sum_{i=1}^m \frac{(x-\omega)^T Q_i(x-\omega)}{q_i(x)-b_i} \\ &\quad + m\frac{[q(x)-q(\omega)][\nabla q(\omega)^T(x-\omega)]}{[q(x)-z][q(\omega)-z]} \\ &\quad + \sum_{i=1}^m \frac{[q_i(x)-q_i(\omega)][\nabla q_i(\omega)^T(x-\omega)]}{[q_i(x)-b_i][q_i(\omega)-b_i]}.\end{aligned} \tag{3.9.5}$$

For simplicity of notations, we define $q_0(x)=q(x)$, $Q_0=Q$, and $b_0=z$. Notice that for $i=0, \cdots, m$ we have

$$\begin{aligned}&\frac{[q_i(x)-q_i(\omega)][\nabla q_i(\omega)^T(x-\omega)]}{[q_i(x)-b_i][q_i(\omega)-b_i]} \\ &= \frac{q_i(x)-b_i}{q_i(\omega)-b_i}\,\frac{q_i(x)-q_i(\omega)}{\nabla q_i(\omega)^T(x-\omega)}\left[\frac{\nabla q_i(\omega)^T(x-\omega)}{q_i(x)-b_i}\right]^2 \\ &= \frac{q_i(x)-b_i}{q_i(\omega)-b_i}\left[1+\frac{(x-\omega)^T Q_i(x-\omega)}{2\nabla q_i(\omega)^T(x-\omega)}\right]\left[\frac{\nabla q_i(\omega)^T(x-\omega)}{q_i(x)-b_i}\right]^2 \\ &= \frac{q_i(x)-b_i}{q_i(\omega)-b_i}\left[\frac{\nabla q_i(\omega)^T(x-\omega)}{q_i(x)-b_i}\right]^2 + \frac{1}{2}\,\frac{\nabla q_i(\omega)^T(x-\omega)}{q_i(\omega)-b_i}\,\frac{(x-\omega)^T Q_i(x-\omega)}{q_i(x)-b_i}\end{aligned}$$

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$$
\begin{aligned}
&\geqq \frac{q_i(x)-b_i}{q_i(\omega)-b_i}\left[\left(\frac{\nabla q_i(x)^T(x-\omega)}{q_i(x)-b_i}\right)^2 - 2\frac{\nabla q_i(x)^T(x-\omega)}{q_i(x)-b_i}\frac{(x-\omega)^TQ_i(x-\omega)}{q_i(x)-b_i}\right] \\
&\quad +\frac{\delta}{2}\frac{(x-\omega)^TQ_i(x-\omega)}{q_i(x)-b_i} \qquad (3.9.6)\\
&\geqq \left(1-\delta-\frac{\delta^2}{2}\right)\left[\frac{\nabla q_i(x)^T(x-\omega)}{q_i(x)-z}\right]^2 \\
&\quad -2\left[\frac{\nabla q_i(\omega)^T(x-\omega)}{q_i(\omega)-b_i}+\frac{(x-\omega)^TQ_i(x-\omega)}{q_i(\omega)-b_i}-\frac{\delta}{4}\right]\frac{(x-\omega)^TQ_i(x-\omega)}{q_i(x)-b_i} \\
&\geqq \left(1-\delta-\frac{\delta^2}{2}\right)\left[\frac{\nabla q_i(x)^T(x-\omega)}{q_i(x)-z}\right]^2 + 2\left(\delta+\delta^2+\frac{\delta}{4}\right)\frac{(x-\omega)^TQ_i(x-\omega)}{q_i(x)-b_i} \\
&= \left(1-\delta-\frac{\delta^2}{2}\right)\left[\frac{\nabla q_i(x)^T(x-\omega)}{q_i(x)-z}\right]^2 + \left(\frac{5\delta}{2}+2\delta^2\right)\frac{(x-\omega)^TQ_i(x-\omega)}{q_i(x)-b_i}.
\end{aligned}
$$

Thus from (3.9.5) and (3.9.6) we have

$$
\begin{aligned}
&\nabla f(x,z)^T(x-\omega) \\
&= -\frac{m(x-\omega)^TQ(x-\omega)}{q(x)-z} - \sum_{i=1}^m \frac{(x-\omega)^TQ_i(x-\omega)}{q_i(x)-b_i} + m\frac{[q(x)-q(\omega)][\nabla q(\omega)^T(x-\omega)]}{[q(x)-z][q(\omega)-z]} \\
&\quad + \sum_{i=1}^m \frac{[q_i(x)-q_i(\omega)][\nabla q_i(\omega)^T(x-\omega)]}{[q_i(x)-b_i][q_i(\omega)-b_i]} \\
&\geqq -\frac{m(x-\omega)^TQ(x-\omega)}{q(x)-z} - \sum_{i=1}^m \frac{(x-\omega)^TQ_i(x-\omega)}{q_i(x)-b_i} \\
&\quad + \left[\left(1-\delta-\frac{\delta^2}{2}\right)\left(\frac{\nabla q(x)^T(x-\omega)}{q(x)-z}\right)^2 + \left(\frac{5\delta}{2}+2\delta^2\right)\frac{(x-\omega)^TQ(x-\omega)}{q(x)-z}\right] \\
&\quad + \sum_{i=1}^m \left[\left(1-\delta-\frac{\delta^2}{2}\right)\left(\frac{\nabla q_i(x)^T(x-\omega)}{q_i(x)-b_i}\right)^2 + \left(\frac{5\delta}{2}+2\delta^2\right)\frac{(x-\omega)^TQ_i(x-\omega)}{q_i(x)-b_i}\right] \\
&\geqq \left(1-\frac{5\delta}{2}-2\delta^2\right)\left[-\frac{m(x-\omega)^TQ(x-\omega)}{q(x)-z} + m\left(\frac{\nabla q(x)^T(x-\omega)}{q(x)-z}\right)^2\right] \\
&\quad + \left(1-\frac{5\delta}{2}-2\delta^2\right)\sum_{i=1}^m \left[-\frac{(x-\omega)^TQ_i(x-\omega)}{q_i(x)-z} + \left(\frac{\nabla q_i(x)^T(x-\omega)}{q_i(x)-z}\right)^2\right] \\
&\geqq \left(1-\frac{5\delta}{2}-2\delta^2\right)(x-\omega)^T\nabla^2 f(x,z)(x-\omega). \qquad \square
\end{aligned}
$$

Proof of (3.9.1). For $0 \leqq \delta < 0.243$, we have $1-(5\delta/2)-2\delta^2 > 0$. Thus by multiplying (3.9.2) with (3.9.3), we get (3.9.1). $\square$

Let us fix iteration k in Algorithm 2.1 and represent x^k, x^{k+1}, z^k, z^{k+1} by x, x^+, z, z^+, respectively. In the following, Lemma 3.10 shows that moving to x^+ by taking a Newton step at x reduces the (normalized) potential function value by a "sufficient" amount, whenever x is "close to" ω. Lemma 3.12 shows that if x^+ is "sufficiently close" to ω then it remains "close to" the analytic center ω^+ that is defined for the convex set P_{z^+}. The improved lower bound z^+ is obtained by adding a fraction of $q(x^+)-z$

to z. Lemma 3.11 is preparatory for Lemma 3.12, and also indicates the monotone property of the analytic centers.

LEMMA 3.10. *Let* $x \in E(\omega, \delta)$, $0 \leqq \delta < 0.243$ *and* β *be a parameter such that* $0 \leqq \beta < 0.5$. *The point* x^+ *that minimizes* $\nabla f(x, z)^T y$ *over* $E(x, \beta) \equiv \{y \mid (y-x)^T \nabla^2 f(x, z)(y-x) \leqq \beta^2\}$ *satisfies*

$$f(x^+, z) \leqq f(x, z) - \beta \sqrt{f(x, z)\left(1 - \frac{5\delta}{2} - 2\delta^2\right)} + \frac{\beta^2}{2} + \frac{4\beta^3}{3(1-2\beta)}.$$

Proof. The Taylor expansion of $f(x^+, z)$ at x and Lemma 3.7 gives

$$\begin{aligned} f(x^+, z) &\leqq f(x, z) + \nabla f(x, z)^T (x^+ - x) + \frac{1}{2}(x^+ - x)^T \nabla^2 f(x, z)(x^+ - x) + \frac{4\beta^3}{3(1-2\beta)} \\ &\leqq f(x, z) + \nabla f(x, z)^T (x^+ - x) + \frac{\varepsilon^2}{2} + \frac{4\beta^3}{3(1-2\beta)}. \end{aligned} \tag{3.10.1}$$

Let $\bar{x}$ be the point where the straight line joining x to the analytic center ω intersects with the boundary of the ellipsoid $E(x, \beta)$. Since $\nabla f(x, z)^T x^+ \leqq \nabla f(x, z)^T \bar{x}$, from (3.10.1) we have

$$\begin{aligned} f(x^+, z) &\leqq f(x, z) + \nabla f(x, z)^T (\bar{x} - x) + \frac{\beta^2}{2} + \frac{4\beta^3}{3(1-2\beta)} \\ &= f(x, z) + \frac{\beta \nabla f(x, z)^T (\omega - x)}{\sqrt{(\omega - x)^T \nabla^2 f(x, z)(\omega - x)}} + \frac{\beta^2}{2} + \frac{4\beta^3}{3(1-2\beta)} \\ &\leqq f(x, z) - \beta \sqrt{f(x, z)\left(1 - \frac{5\delta}{2} - 2\delta^2\right)} + \frac{\beta^2}{2} + \frac{4\beta^3}{3(1-2\beta)}. \end{aligned}$$

The last inequality follows by using Lemma 3.9. □

LEMMA 3.11. *Let* $z^+ \geqq z$ *and let* $f(x, z^+) = F(\omega^+, z^+) - F(x, z^+)$, *where*

$$F(x, z^+) = \sum_{i=1}^{m} \ln(q_i(x) - b_i) + m \ln(q(x) - z^+),$$

and ω^+ *is the unique maximum of* $F(x, z^+)$ *over the convex set*

$$P_{z^+} = \{x \in R^n \mid q_i(x) \geqq b_i \ \forall i \text{ and } q(x) \geqq z^+\}.$$

Then,

$$0 \leqq q(\omega^+) - q(\omega) \leqq z^+ - z. \tag{3.11.1}$$

Proof. Let $z(t) = z + t(z^+ - z)$ and let $\omega(t)$ be the point that maximizes the function

$$F(x, z(t)) = \sum_{i=1}^{m} \ln(q_i(x) - b_i) + m \ln(q(x) - z(t)).$$

Since the gradient of $F(x, z(t))$ vanishes at $\omega(t)$, we have

$$\begin{aligned} 0 &= \frac{d}{dt}\left[\sum_{i=1}^{m} \frac{\nabla q_i(\omega(t))}{q_i(\omega(t)) - b_i} + m \frac{\nabla q(\omega(t))}{q(\omega(t)) - z(t)}\right] \\ &= \sum_{i=1}^{m} \frac{1}{(q_i(\omega(t)) - b_i)^2}\left[\frac{d\nabla q_i(\omega(t))}{d\omega(t)} \frac{d\omega(t)}{dt}(q(\omega(t)) - b_i) - \nabla q_i(\omega(t))^T \frac{dq_i(\omega(t))}{dt}\right] \\ &\quad + \frac{m}{(q(\omega(t)) - z(t))^2}\left[\frac{d\nabla q(\omega(t))}{d\omega(t)} \frac{d\omega(t)}{dt}(q(\omega(t)) - z(t)) \right. \\ &\qquad \left. - \nabla q(\omega(t))^T \left(\frac{dq(\omega(t))}{dt} - \frac{dz(t)}{dt}\right)\right] \end{aligned}$$

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$$= \sum_{i=1}^{m} \frac{1}{(q_i(\omega(t)) - b_i)^2} \left[(q_i(\omega(t)) - b_i) Q_i \frac{d\omega(t)}{dt} - \nabla q_i(\omega(t))^T \frac{dq_i(\omega(t))}{dt} \right]$$

$$+ \frac{m}{(q(\omega(t)) - z(t))^2} \left[(q(\omega(t)) - z(t)) Q \frac{d\omega(t)}{dt} - \nabla q(\omega(t))^T \left(\frac{dq(\omega(t))}{dt} - \frac{dz(t)}{dt} \right) \right].$$

Since

$$\nabla q_i(\omega(t))^T \frac{d\omega(t)}{dt} = \frac{dq_i(\omega(t))}{dt},$$

Q_i is negative semidefinite, and $q_i(\omega(t)) - b_i > 0$, for $i = 0, 1, 2, \cdots, m$, we have

$$0 \geqq \left(\frac{d\omega(t)}{dt} \right)^T \nabla q(\omega(t)) \left(\frac{dq(\omega(t))}{dt} - \frac{dz(t)}{dt} \right) = \frac{dq(\omega(t))}{dt} \left(\frac{dq(\omega(t))}{dt} - \frac{dz(t)}{dt} \right),$$

which implies

$$\left(\frac{dq(\omega(t))}{dt} \right)^2 \leqq \frac{dq(\omega(t))}{dt} \frac{dz(t)}{dt} = \frac{dq(\omega(t))}{dt} (z^+ - z).$$

Hence,

$$0 \leqq \frac{dq(\omega(t))}{dt} \leqq z^+ - z,$$

and therefore,

$$0 \leqq q(\omega^+) - q(\omega) = \int_0^1 \frac{dq(\omega(t))}{dt} \, dt \leqq z^+ - z. \qquad \square$$

LEMMA 3.12. *Let* $z^+ = z + (\alpha/\sqrt{m})(q(x^+) - z)$, $\sqrt{m} > \alpha > 0$, *and* $x^+ \in \operatorname{int} P_{z^+}$. *If* $x^+ \in E(\omega, \delta)$, *where* $0 \leqq \delta < 0.5$, *then we have*

$$f(x^+, z^+) \leqq f(x^+, z) + \frac{m\alpha}{\sqrt{m} - \alpha} \left(\frac{\delta}{\sqrt{m}} + \frac{\delta^2}{2m} \right) + \frac{\alpha^2 (1 + \delta/\sqrt{m} + \delta^2/2m)^2}{1 - (\alpha/\sqrt{m})(1 + \delta/\sqrt{m} + \delta^2/2m)}.$$

Proof. We may write

$$f(x^+, z^+) = f(x^+, z) + f(\omega, z^+) + m \ln \frac{(q(x^+) - z)(q(\omega) - z^+)}{(q(x^+) - z^+)(q(\omega) - z)}. \tag{3.12.1}$$

Since

$$\frac{(q(x^+) - z)(q(\omega) - z^+)}{(q(x^+) - z^+)(q(\omega) - z)} = 1 + \frac{(z^+ - z)(q(\omega) - q(x^+))}{(q(x^+) - z^+)(q(\omega) - z)}$$

$$= 1 + \frac{\alpha(q(x^+) - z)(q(\omega) - q(x^+))}{\sqrt{m}(q(x^+) - z^+)(q(\omega) - z)}$$

$$= 1 + \frac{\alpha}{\sqrt{m} - \alpha} \frac{q(\omega) - q(x^+)}{q(\omega) - z},$$

by using Lemma 3.6 we have

$$\frac{(q(x^+) - z)(q(\omega) - z^+)}{(q(x^+) - z^+)(q(\omega) - z)} \leqq 1 + \frac{\alpha}{\sqrt{m} - \alpha} \left(\frac{\delta}{\sqrt{m}} + \frac{\delta^2}{2m} \right). \tag{3.12.2}$$

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We now upper bound $(q(\omega^+)-z^+)/(q(\omega)-z^+)$ to get a bound on the value of $f(\omega, z^+)$.

$$\frac{q(\omega^+)-z^+}{q(\omega)-z^+}=\frac{q(\omega^+)-z}{q(\omega)-z}\left(1+\frac{(z^+-z)(q(\omega^+)-q(\omega))}{(q(\omega^+)-z)(q(\omega)-z^+)}\right).$$

Now by using Lemma 3.11 in the above equation we have

$$\begin{aligned}
&\frac{q(\omega^+)-z^+}{q(\omega)-z^+}\\
&\quad\leqq\frac{q(\omega^+)-z}{q(\omega)-z}+\frac{(z^+-z)^2}{(q(\omega)-z)(q(\omega)-z^+)}\\
&\quad=\frac{q(\omega^+)-z}{q(\omega)-z}+\frac{\alpha^2(q(x^+)-z)^2}{m(q(\omega)-z)(q(\omega)-z^+)}\\
&\quad=\frac{q(\omega^+)-z}{q(\omega)-z}+\frac{\alpha^2}{m}\left(\frac{q(x^+)-q(\omega)}{q(\omega)-z}+1\right)\left(\frac{q(x^+)-z}{q(\omega)-z-(\alpha/\sqrt{m})(q(x^+)-z)}\right)\\
&\quad=\frac{q(\omega^+)-z}{q(\omega)-z}+\frac{\alpha^2}{m}\left(\frac{q(x)-q(\omega)}{q(\omega)-z}+1\right)^2\left(\frac{1}{1-(\alpha(q(x^+)-z))/(\sqrt{m}(q(\omega)-z))}\right)\\
&\quad\leqq\frac{q(\omega^+)-z}{q(\omega)-z}+\frac{(\alpha^2/m)(1+\delta/\sqrt{m}+\delta^2/2m)^2}{1-(\alpha/\sqrt{m})(1+\delta/\sqrt{m}+\delta^2/2m)}.
\end{aligned}\tag{3.12.3}$$

The inequality (3.12.4) below follows by using (3.12.3), and (3.12.5) follows from $\nabla f(\omega, z)=0$. Now

$$\begin{aligned}
&f(\omega, z^+)\\
&\quad=\sum_{i=1}^m \ln\frac{q_i(\omega^+)-b_i}{q_i(\omega)-b_i}+m\ln\frac{q(\omega^+)-z^+}{q(\omega)-z^+}\\
&\quad\leqq\sum_{i=1}^m\left(\frac{q_i(\omega^+)-b_i}{q_i(\omega)-b_i}-1\right)+m\left(\frac{q(\omega^+)-z^+}{q(\omega)-z^+}-1\right)\\
&\quad\leqq\sum_{i=1}^m\frac{q_i(\omega^+)-q_i(\omega)}{q_i(\omega)-b_i}+m\left[\frac{q(\omega^+)-z}{q(\omega)-z}+\frac{(\alpha^2/m)(1+\delta/\sqrt{m}+\delta^2/2m)^2}{1-(\alpha/\sqrt{m})(1+\delta/\sqrt{m}+\delta^2/2m)}-1\right]
\end{aligned}\tag{3.12.4}$$

$$\begin{aligned}
&\quad\leqq\frac{1}{2}\sum_{i=1}^m\frac{(\omega^+-\omega)^TQ_i(\omega^+-\omega)}{q_i(\omega)-z}\\
&\qquad+\frac{m}{2}\frac{(\omega^+-\omega)^TQ(\omega^+-\omega)}{q(\omega)-z}+\frac{\alpha^2(1+\delta/\sqrt{m}+\delta^2/2m)^2}{1-(\alpha/\sqrt{m})(1+\delta/\sqrt{m}+\delta^2/2m)}
\end{aligned}\tag{3.12.5}$$

$$\leqq\frac{\alpha^2(1+\delta/\sqrt{m}+\delta^2/2m)^2}{1-(\alpha/\sqrt{m})(1+\delta/\sqrt{m}+\delta^2/2m)}.\tag{3.12.6}$$

The proof of Lemma 3.12 is complete by combining (3.12.1), (3.12.2), and (3.12.6). □

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. We first show that $f(x^k, z^k)\leqq .003$ by induction. This will imply via Lemma 3.5 and Lemma 3.8 that $x^k\in E(\omega^k, 0.2)\subset P_{z^k}$ and therefore the algorithm is well defined. The inequality is valid for $k=0$. Now we assume that $f(x^k, z^k)\leqq .003$ and show that $f(x^{k+1}, z^{k+1})\leqq .003$. Since $x^k\in E(\omega^k, 0.2)$, for the choice of $\beta=.03$, from Lemma 3.10 we can show that $f(x^{k+1}, z^k)\leqq .00244$. Hence $x^{k+1}\in E(\omega^k, 0.2)$ due to Lemma 3.8. Because $\alpha=.0024$ and $m\geqq 1$, Lemma 3.12 implies that

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$f(x^{k+1}, z^{k+1}) \leqq .003$. This finishes the induction. Now, by Lemma 3.5 and Lemma 3.6,

$$q(x^{k+1}) - z^k \geqq \left(1 - \frac{\delta}{\sqrt{m}} - \frac{\delta^2}{2m}\right)(q(\omega^k) - z^k) \geqq .84(q(\omega^k) - z^k) \geqq .42(z^* - z^k),$$

we have

$$z^* - z^{k+1} = z^* - z^k - \frac{\alpha}{\sqrt{m}}(q(x^{k+1}) - z^k) \leqq \left(1 - \frac{.0024 * 0.42}{\sqrt{m}}\right)(z^* - z^k).$$

The proof of Theorem 3.1 is now complete. □

4. Initialization of the algorithm. In this section we outline an approach that could be used to satisfy the initial assumption in our algorithm. The assumption is:

A solution $x^0 \in \text{int } P_{z^0}$ is known and $f(x^0, z^0) \leqq .003$.

It can be shown that the so-called "big-M" method works for (QCQP) as well; namely, there exists a large number M such that any optimal solution (x^*, t^*) to the problem

$$(\text{QCQP})': \begin{cases} \text{maximize} & q(x) - Mt \\ \text{subject to} & q_i(x) + t \geqq b_i, \; i = 1, 2, \cdots, m, \quad t \geqq 0, \end{cases}$$

will have $t^* = 0$, and thus x^* is an optimal solution to (QCQP). An interior feasible solution $(\bar{x}, \bar{t})$ to (QCQP)$'$ is easy to find, for instance, we may choose $t > 0$ such that $q_i(0) - b_i - t > 1$ for $i = 1, \cdots, m$. In addition, we may assume that t is a bounded variable, so the feasible set of (QCQP)$'$ is bounded. Therefore, without loss of generality, we assume that an interior feasible solution to (QCQP) is known and this solution satisfies $q_i(\bar{x}) - b_i \geqq 1$ for $i = 1, \cdots, m$.

Let

$$\bar{F}(x, \xi) \equiv 2m \ln [a^T x - \xi] + \sum_{i=1}^{m} \ln [q_i(x) - b_i] + m \ln [q(x) - b_0],$$

where $b_0 = q(\bar{x}) - 1$ and $a = \nabla f(\bar{x}, b_0)$. For the corresponding normalized potential function $\bar{f}(x, \xi)$ we have

$$\nabla \bar{f}(\bar{x}, \xi) = -\frac{2ma}{a^T \bar{x} - \xi} - \sum_{i=1}^{m} \frac{\nabla q_i(\bar{x})}{q_i(\bar{x}) - b_i} - \frac{m \nabla q(\bar{x})}{q(\bar{x}) - b_0} = -\frac{2ma}{a^T \bar{x} - \xi} + a.$$

For $\xi^0 = a^T \bar{x} - 2m$, $\nabla \bar{f}(\bar{x}, \xi^0) = 0$ and $\bar{f}(\bar{x}, \xi^0) = 0$. Starting with $\bar{x}$, we use the following algorithm.

ALGORITHM 4.1.

For $k = 0, 1, \cdots$ **until** the relationship (4.2.1) (see below) holds **do**:

Determine a step direction p by solving

$$\nabla^2 \bar{f}(\bar{x}^k, \xi^k) p = -\nabla \bar{f}(\bar{x}^k, \xi^k).$$

Let

$$\bar{x}^{k+1} \leftarrow \bar{x}^k + \frac{\beta}{\sqrt{p^T \nabla^2 \bar{f}(\bar{x}^k, \xi^k) p}} p.$$

$$\xi^{k+1} \leftarrow \xi^k - \frac{\alpha}{\sqrt{m}}(a^T \bar{x}^{k+1} - \xi^k).$$

End

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Compared with Algorithm 2.1, the effect of changing α into $-\alpha$ is an expanding sequence of convex sets associated with $\bar{f}$ (decreasing sequence of $\{\xi^k\}$) and a sequence of approximate centers $\bar{x}^k$ of these sets. An analysis similar to Theorem 3.1 indicates that by choosing suitable α and β, Algorithm 4.1 will maintain $\bar{f}(\bar{x}^k, \xi^k) \leqq .002$ and $\xi^k \downarrow -\infty$ as $k \to \infty$. It also indicates that in $O(\sqrt{m} \ln K)$ iterations we will get a $\xi^k < -K$ such that

$$(4.2.1) \qquad 2m \ln \frac{K - \xi^k}{-K - \xi^k} \leqq 0.001,$$

where K is a number such that $|a^T x| \leqq K$ for any x feasible to (QCQP). Thus,

$$\begin{aligned} f(\bar{x}^k, b_0) &= F(\omega_0, b_0) - F(\bar{x}^k, b_0) \leqq F(\omega_0, b_0) - F(\bar{x}^k, b_0) + \bar{F}(\bar{\omega}^k, \xi^k) - \bar{F}(\omega_0, \xi^k) \\ &= \bar{f}(\bar{x}^k, \xi^k) + 2m \ln \frac{a^T \bar{x}^k - \xi^k}{a^T \bar{\omega}^k - \xi^k} \leqq 0.002 + 2m \ln \frac{K - \xi^k}{-K - \xi^k} \leqq 0.003, \end{aligned}$$

where ω_0 and $\bar{\omega}^k$ maximize $F(x, b_0)$ and $\bar{F}(\bar{x}, \xi^k)$, respectively. Now Algorithm 2.1 can start with $\bar{x}^k$ and $z^0 = b_0$. We point out that, since the feasible region of (QCQP) is bounded, there is a fixed constant R such that $x^T x \leqq R^2$ for all feasible x. Thus, the norm of a is bounded by

$$\|a\| = \|\nabla f(\bar{x}, b_0)\| \leqq 2m \max\{\|\nabla q_i(\bar{x})\|, i = 0, \cdots, m\} = O(mR)$$

and therefore, $K = O(mR^2)$. This idea to find an initial point that satisfies all the assumptions for Algorithm 2.1 was conceptualized during a discussion with Jarre [14].

It should be noted that the number R could be thought of as a measure of the "size" of the feasible region. In practice, it is unlikely that R and M could be known accurately in advance. However, they can be guessed based on some knowledge on the feasible set and the objective function. The two values can be suitably modified, in the event that these guesses go wrong.

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