DYNAMIC PROGRAMMING AND LAGRANGE MULTIPLIERS

By Richard Bellman

RAND CORPORATION, SANTA MONICA, CALIFORNIA

Communicated by Einar Hille, August 13, 1956

1. *Introduction.*—The purpose of this note is to indicate how a suitable combination of the classical method of the Lagrange multiplier and the functional-equation method of the theory of dynamic programming[1] can be used to solve numerically, and treat analytically, a variety of variational problems that cannot readily be treated by either method alone.

A series of applications of the method presented here will appear in further publications.

2. *Functional Equation Approach.*—Consider the problem of maximizing the function

$$F(x_1, x_2, \ldots, x_N) = \sum_{i=1}^{N} g_i(x_i), \tag{2.1}$$

subject to the constraints

$$\begin{aligned} &(a) \quad \sum_{j=1}^{N} a_{ij}(x_j) \leq c_i, \qquad i = 1, 2, \ldots, M, \\ &(b) \quad x_i \geq 0, \end{aligned} \tag{2.2}$$

where the functions $a_{ij}(x)$, $g_i(x)$ are taken to be continuous for $x \geq 0$, and monotone increasing. For $c_i \geq 0$, define the sequence of functions

$$f_N(c_1, c_2, \ldots, c_M) = \operatorname*{Max}_{\{x\}} F(x_1, x_2, \ldots, x_N) \tag{2.3}$$

for $N \geq 1$.

Then $f_1(c_1, c_2, \ldots, c_M)$ is determined immediately, and, employing the principle of optimality,[1] we obtain the recurrence relation

$$f_{k+1}(c_1, c_2, \ldots, c_M) = \operatorname*{Max}_{0 \leq a_{i,\,k+1}(x) \leq c_i} [g_{k+1}(x) + f_k(c_1 - a_{1,\,k+1}(x), \ldots, c_M - a_{M,\,k+1}(x))], \tag{2.4}$$

for $k = 1, 2, \ldots, N - 1$.

Due to the limited memory of present-day digital computers, this method founders on the reef of dimensionality when $M \geq 4$. If we wish to treat applied problems of greater and greater realism, we must develop methods capable of handling problems involving higher dimensions.

In this paper we shall present one method of overcoming these dimensionality difficulties.

3. *Functional Equations and Lagrange Multipliers.*—The method of the Lagrange multiplier in classical variational theory consists of forming the function

$$\phi(x_1, x_2, \ldots, x_N) = \sum_{i=1}^{N} g_i(x_i) - \sum_{i=1}^{M} \lambda_i\left(\sum_{j=1}^{N} a_{ij}(x_j)\right), \tag{3.1}$$

where the λ_i are parameters determined subsequently by means of relations (2.2), and then utilizing a direct variational approach on this new function.

We shall employ an approach intermediate between this method and the method sketched in section 2.

Consider the function

$$F(x_1, x_2, \ldots, x_N; \lambda_1, \lambda_2, \ldots, \lambda_K) = \sum_{i=1}^{N} g_i(x_i) - \sum_{i=1}^{K} \lambda_i \left(\sum_{j=1}^{N} a_{ij}(x_j) \right), \quad (3.2)$$

where $1 \leq K \leq M - 1$. We wish to maximize this function over the region defined by the constraints

$$\begin{aligned} &(a) \quad \sum_{j=1}^{N} a_{ij}(x_j) \leq c_i, \qquad i = K + 1, \ldots, M, \\ &(b) \quad x_i \geq 0. \end{aligned} \quad (3.3)$$

For fixed values of the λ_i, we have a problem of precisely the type discussed in section 2, with the advantage that we now require functions of dimension $M - K$ for a computational solution.

Once the sequence $\{\phi_N(c_{K+1}, \ldots, c_N; \lambda_1, \lambda_2, \ldots, \lambda_K)\}$, $N = 1, 2, \ldots$,

$$\phi_N(c_{K+1}, \ldots, c_N; \lambda_1, \lambda_2, \ldots, \lambda_K) = \operatorname*{Max}_{\{x\}} F(x_1, x_2, \ldots, x_N; \lambda_1, \lambda_2, \ldots, \lambda_K) \quad (3.4)$$

has been computed, we vary the parameters λ_i to determine the range of the parameters $c_1, c_2, \ldots, c_K$.

We have thus partitioned the computation of the original sequence of functions of M variables into the computation of a sequence of functions of K variables, followed by the computation of a sequence of functions of $M - K$ variables. The choice of K will depend upon the process.

There are a number of rigorous details which we will discuss elsewhere.

4. *Successive Approximations.*—Since the Lagrange multipliers are intimately connected with "marginal returns," or "prices," in a number of applied problems, we begin with a certain hold on the process as far as approximate solutions are concerned. Iterative techniques based upon this observation, and the connection with optimal search procedures, will be discussed elsewhere.

5. *Application to the Calculus of Variations.*—In the theory of control processes,[3] one encounters a problem such as that of minimizing a nonanalytic functional such as

$$J_1(u) = \int_0^T |1 - u| \, dt \quad (5.1)$$

over all functions $v(t)$ satisfying the constraints

$$\begin{aligned} &(a) \quad -k_1 \leq v(t) \leq k_2, \qquad 0 \leq t \leq T, \\ &(b) \quad \int_0^T |v(t)| \, dt \leq c_1, \end{aligned} \quad (5.2)$$

where u and v are connected by a relation

$$\frac{du}{dt} = g(u, t, v), \qquad u(0) = a. \quad (5.3)$$

Replacing $J_1(u)$ by

$$J_2(u) = \int_0^T |1 - u| \, dt + \lambda \int_0^T |v(t)| \, dt, \tag{5.4}$$

$\lambda \geq 0$, we can employ functions of two variables in determining the analytic or numerical solution, rather than functions of three variables (cf. an earlier paper[2]). This reduction by one in dimensionality results simultaneously in a tremendous saving in computing time and in a great increase in accuracy of the numerical results.

[1] R. Bellman, "The Theory of Dynamic Programming," *Bull. Am. Math. Soc.*, **60**, 503–516, 1954.

[2] R. Bellman, "Dynamic Programming and a New Formalism in the Calculus of Variations," these PROCEEDINGS, **40**, 231–235, 1954.

[3] R. Bellman, I. Glicksberg, and O. Gross, "On Some Variational Problems Occurring in the Theory of Dynamic Programming," *Rend. Palermo*, Ser. II, **3**, 1–35, 1954.

EIGENFUNCTION EXPANSIONS FOR FORMALLY SELF-ADJOINT PARTIAL DIFFERENTIAL OPERATORS. I

BY FELIX E. BROWDER

DEPARTMENT OF MATHEMATICS, YALE UNIVERSITY

Communicated by Marshall H. Stone, August 16, 1956

Let A be a formally self-adjoint differential operator defined on a domain G of Euclidean n-space. In 1953, Gårding and the author,[1] following a general line of argument due to Mautner,[2] independently proved the existence of expansions of the Weyl type in eigenfunctions of $(A - \lambda I)$ for elliptic A, corresponding to every self-adjoint realization of A in $L_2(G)$.[3] An extension to self-adjoint realizations of C^{∞}-elliptic differential operators with constant coefficients was given by L. Hörmander in 1955.[4] A more general result on expansions in eigenfunctions of $(A - \lambda B)$ for A merely formally self-adjoint and B positive has been announced recently by Gelfand and Kostyucenko,[5] who sketched a proof based upon a theorem of Gelfand on the differentiation and integration of Banach space-valued functions on the line. In their result it is assumed that $B^{-1}A$ has a self-adjoint realization in a special Hilbert space with inner product (Bf, g), and the eigenfunctions obtained are distributions in the sense of L. Schwartz.

It is the purpose of this note to present a proof, using the Gelfand differentiation theorem, of the eigenfunction expansion theorem for A formally self-adjoint and B positive without any assumption on the existence of self-adjoint realizations (which may not exist for A and B with complex coefficients). In the following note we establish existence and regularity of the eigenfunction expansion for $(A - \lambda B)$ with elliptic A under weaker differentiability conditions on the coefficients by use of the Radon-Nikodym theorem. The proofs go over to the case of systems with purely formal alterations.

Let A and B be differential operators with complex coefficients on G of orders s and $2r$, $(,)$ the inner product in $L_2(G)$, $C^j(G)$ the j-times continuously differentiable