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The Traveling-Salesman Problem

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THE TRAVELING-SALESMAN PROBLEM

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THE TRAVELING-SALESMAN PROBLEM is that of finding a permutation $P=(1\ i_2\ i_3\ \cdots\ i_n)$ of the integers from 1 through n that minimizes the quantity

$$a_{1i_2}+a_{i_2i_3}+a_{i_3i_4}+\cdots+a_{i_n1},$$

where the $a_{\alpha\beta}$ are a given set of real numbers. More accurately, since there are only $(n-1)!$ possibilities to consider, the problem is to find an efficient method for choosing a minimizing permutation.

The problem takes its name from the fact that a salesman wishing to travel by shortest total distance from his home to each of $n-1$ specified cities, and then return home, could use such a method if he were given the distances $a_{\alpha\beta}$ between each pair of cities on his tour. Or, if the salesman desired the shortest total travel time, the $a_{\alpha\beta}$ would represent the individual travel times.

This problem was posed, in 1934, by Hassler Whitney in a seminar talk at Princeton University. There are as yet no acceptable computational methods, and surprisingly few mathematical results relative to the problem.

The problem is closely related to several considered by Hamilton, in which he was concerned with finding the number of different tours possible over a specified configuration. One of these problems, which Ball[1] termed the Hamiltonian Game, "consists in the determination of a route along the edges of a regular dodecahedron which will pass once and only once through every angular point." Ball also discusses the relationships between the Hamiltonian Game and other unicursal problems. König[2] discusses some of these same problems under the heading Hamiltonian Lines. I am indebted to A. W. Tucker for calling these connections to my attention, in 1937, when I was struggling with the problem in connection with a school-bus routing study in New Jersey.

The problem is also closely related to the *personnel-assignment problem*, which differs from the traveling-salesman problem only in that the allowable permutations P may also be noncyclic. In its most familiar application to personnel management,[3] the assignment problem is to assign N men optimally to N different jobs. In this application, it is supposed that a numerical performance rating is given for each of the N^2 man-job combinations and an optimal assignment is one that minimizes the sum of the N applicable ratings. For example, the ratings might be estimated times, or costs, for the various man-job combinations.

The personnel-assignment problem is mathematically equivalent to the so-called *transportation problem.*[3] In its most familiar application to transportation management, the transportation problem[4] is to redistribute a fleet of N identical carriers optimally in given proportions at a new set of stations. In this application, it is supposed that a numerical performance rating is given for moving a carrier from each old station to any new one and an optimal routing is one that minimizes the sum of the applicable ratings. For example, the ratings might be estimated times, or distances, for the various station-to-station movements and the carriers might be tankers, pallets, freight cars, or other such vehicles.

The assignment and transportation cases are special applications for the *distribution problem* as formulated originally by Hitchcock,[6] in 1941, and independently by Kantorovitch[7] at about the same time. The Hitchcock distribution problem is to find a set of values of the mn real variables x_{ij}, subject to the following conditions:

$$\sum_{i=1}^{m} x_{ij}=c_j, \quad \sum_{j=1}^{n} x_{ij}=r_i, \quad x_{ij}\geq 0, \quad \sum_{i,j} x_{ij}\, d_{ij}=\text{minimum},$$

where m, n, r_i, c_j, and d_{ij} are given positive integers with $\sum c_j=\sum r_i\equiv N$. The quantities d_{ij} represent the known performance ratings in the assignment and transportation problems.

In the transportation case, the quantity r_i represents the number of carriers initially at old station i and c_j represents the total number of carriers to be routed to new station j from all the m old stations. The solution for x_{ij} gives the number of carriers to be moved from old station i to new station j in the optimal routing.

In the assignment case, the men are separated into m categories such that each man has the same ratings as every other man in his category; the quantity r_i represents the number of men in category i. Similarly, the jobs are grouped into n classes, such that the ratings of all men are the same for every job in a class; the quantity c_j represents the number of jobs in class j. The solution for x_{ij} gives the number of men of category i that are placed on jobs of class j in the optimal assignment.

In another version of the distribution problem (commonly also called the transportation problem), r_i represents the (monthly) production capacity of plant i, c_j represents the (monthly) demand of warehouse or customer j, and d_{ij} represents the combined unit cost of production in plant i and shipment to j. The problem is to supply the warehouses at minimum cost. If production capacity exceeds demand, c_n is used to represent the total excess and the x_{in} take up the slack in a fictitious warehouse that can be supplied at no cost ($d_{in}=0$). Then the solution determines the percentage of capacity at which each plant can be most economically operated as well as the optimum routing to customers.

Both the assignment and transportation problems are stated equally well, as is the distribution problem also, as that of finding a doubly stochastic square matrix $\|x_{ij}\|$ of order N such that $\sum x_{ij}\, d_{ij}=$minimum. This formulation merely omits taking explicit notice of identical rows, or columns, of the rating matrix $\|d_{ij}\|$. Algebraically, this becomes the problem of finding real x_{ij} subject to the conditions

$$\sum_{i=1}^{N} x_{ij}=\sum_{j=1}^{N} x_{ij}=1, \quad x_{ij}\geq 0, \quad \sum_{i,j} x_{ij}\, d_{ij}=\text{minimum}.$$

It is also easily shown that there is a solution of this problem such that $x_{ij}^2=x_{ij}$, which means that $X\equiv\|x_{ij}\|$ is a permutation matrix—that is, X is a square matrix whose elements are all null or unity and with exactly one unit element in each row and in each column. A permutation matrix X that solves this distribution problem also solves the traveling-salesman problem, with distance matrix $\|d_{ij}\|$, if and only if no power of X less than the Nth is the unit matrix—in other words X must be cyclic. These considerations provide us with a useful alternative statement of the traveling-salesman problem, as follows: *Find a cyclic permutation matrix X such that trace$(XD)=$minimum, where D is a given square matrix.* The distribution problem is similarly stated as: *Find a permutation matrix X such that trace$(XD)=$minimum, where D is a given square matrix.* (The trace of a matrix is the sum of the elements on the main diagonal.)

The distribution problem is, of course, a special case of the linear-programming problem. The linear-programming problem is, in turn, a special case of that of finding a nonnegative solution for a system of linear algebraic equations. There is now considerable literature dealing with all these problems, and other variants, including equivalent problems arising in the theory of matrix games. Some of the techniques from this area have been used by Dantzig, Fulkerson, and Johnson[8] in their paper presenting a solution of the traveling-salesman problem for an actual case including Washington, D. C. and one city from each of the 48 states.

Tjalling Koopmans first brought to my attention the possibility of a connection between the traveling-salesman problem and the distribution problem, at the time of the 1948 meeting of the International Statistical Institute, in connection with his pioneering paper[4] on the distribution (transportation) problem. The present author[9] solved the degenerate case of the distribution problem, and hence also the assignment problem as a special case of degeneracy, as an extension of the graph-theoretic methods employed by Koopmans and Reiter[10] for the nondegenerate case. Julia Robinson[11] solved the assignment problem while searching for a solution to the traveling-salesman problem, and first made clear the nature of the relation between the two problems.

Very recent mathematical work on the traveling-salesman problem by

I. Heller,[12] H. W. Kuhn,[13] and others[8] indicates that the problem is fundamentally complex. It seems very likely that quite a different approach from any yet used may be required for successful treatment of the problem. In fact, there may well be no general method for treating the problem and impossibility results would also be valuable.

There is one useful general theorem, which is quickly discovered by each one who considers the traveling-salesman problem. In the euclidean plane it states simply that the minimal tour does not intersect itself, and this intersection condition generalizes easily for arbitrary $a_{\alpha\beta}$. The Rand Corporation offered a prize for anyone who could contribute another significant theorem pertaining to the traveling-salesman problem, but, to the best of my knowledge, no awards were made. There have been countless conjectures, but each has fallen victim to a counterexample. In brief, the problem may be fairly considered open.

The present paper gives a method for solving the assignment (distribution) problem and shows how this method may be used effectively in the initial preparation of a traveling-salesman problem for subsequent computations. Some techniques that are useful in seeking good approximate solutions, especially in the symmetric case, are given. The computational techniques due to Dantzig, Fulkerson, and Johnson[8] are discussed, and applied to an example. The various mathematical points are illustrated by numerical examples, including an analysis of the results of application of the approximate methods to the 49-city case treated by Dantzig *et al.* Connections are pointed out between the traveling-salesman problem and various other mathematical programming problems, and several of the more important industrial and management applications of the results are cited.

APPLICATIONS

The salesman's problem of choosing a short travel route is typical of one class of practical situations represented by the traveling-salesman problem. It is easy to think of other routing applications, and that for a school bus making specified stops each trip is one example.

Another familiar situation, in which a solution of the traveling-salesman problem would be useful, is that of scheduling a machine over a given set of repeated operations. For example, a machine tool may be required to perform a specified set of operations repetitively, and the operator's task is to choose a sequential order for the operations that minimizes the cycle time for the set.

The machine-scheduling example is very similar to an important design problem, also represented by the traveling-salesman conditions. The designer's task, say in laying out an automatic television-assembly line, is to so route the path of chassis among the assembly stations that travel time is minimized—or, equivalently, so that production rate is maximized. In actual practice, in this and in other applications, other factors are almost sure to enter into consideration. For example, there will often be precedence conditions that prohibit certain sequences of assembly operations from occurring. Sometimes these extra factors make the solution easier to find, by eliminating troublesome cases, and sometimes the solution is made more difficult by ruling out alternatives that would otherwise easily be shown to be optimal. These design applications are particularly attractive, however, because even small but real percentage improvements are well worth-while because of the highly repetitive yet essentially permanent character of the operation.

George Feeney, in a seminar talk at Columbia University in 1954, reported on one interesting case of machine-tool operation in which the operator appeared to stay rather closely with the rule that the next operation on the machine should be the one requiring the least setup time. This is analogous to the salesman always going next to the closest city not already visited. The rule is not optimal, of course, but it may well be the most practical one for an operator who does not even have good explicit estimates of average setup times—especially so when there is any uncertainty as to the precise set of operations to be performed during the cycle.

It may be of some interest to compare the length of optimal tour found among the 49 cities by Dantzig *et al.* with that which would be followed by a salesman living in Washington, D. C., who always went next to the closest city not already visited. This turns out to be 904 units against 699 units, or an increase of nearly 30 per cent. It seems likely that a considerably better route than that produced by the operator's rule could usually be found quite easily, and it also seems likely that rather simple methods could be found to yield a tour much nearer optimal then 30 per cent. However, even a few per cent gain would be well worth-while in some cases, so the problem does seem to have practical importance as well as mathematical interest.

MATHEMATICAL BACKGROUND

IT WILL BE HELPFUL to start with a solution of the assignment problem, as an introduction to the notation and methods to be used in discussing the traveling-salesman problem. The approach used is essentially the same as that of Kuhn,[14] and rests on the following fundamental theorem of König[2] as stated by Ergerváry:[15]

KÖNIG'S THEOREM: *If the elements of a matrix are partly zeros, and partly numbers different from zero, then the minimum number of lines that contain all the nonzero elements of the matrix is equal to the maximum number of nonzero elements that can be chosen with no two on the same line.*

In this paper, the word 'line' applies both to the rows and the columns of a matrix.

The assignment problem requires that a set of n elements be chosen from the square matrix $A=\|a_{ij}\|$ of order n, with no two elements in the same line, such that their sum is minimal. This can be written algebraically as follows: Find real x_{ij} such that

$$x_{ij}^2=x_{ij}, \quad \text{for} \quad i, j=1, 2, \cdots, n\geq 3, \tag{1}$$

$$\sum_{i=1}^{n} x_{ij}=\sum_{j=1}^{n} x_{ij}=1, \tag{2}$$

$$\sum_{ij} x_{ij}a_{ij}=\text{minimum}, \tag{3}$$

where the a_{ij} are given real numbers. Actually, we shall be interested only in those cases where the a_{ij} are nonnegative integers; but it can be shown[12] by continuity considerations that this restriction leads to no real loss in generality. It is also well known that (1) can be replaced, in the assignment problem, by the less restrictive condition

$$x_{ij}\geq 0. \tag{4}$$

The traveling-salesman problem can be represented by conditions (1), (2), (3), (5), and (6), where

$$x_{i_1 i_2}+x_{i_2 i_3}+\cdots+x_{i_{r-1} i_r}+x_{i_r i_1}\leq r-1 \quad \text{for} \quad r=2, \cdots, \tfrac{1}{2} n, \tag{5}$$

and where $(i_1\, i_2 \cdots i_r)$ is a permutation of the integers 1 through r, and

$$x_{ii}=0 \quad \text{for} \quad i=1, 2, \cdots, n. \tag{6}$$

Unfortunately, the condition (4) can not be used to replace (1) for the traveling-salesman problem, and this is the most obvious evidence of the substantial difference between the two problems.

It is an important fact that the solution of neither problem is changed by replacing a_{ij} by d_{ij}, where

$$d_{ij}\equiv a_{ij}-u_i-v_j, \tag{7}$$

and where u_i and v_j are arbitrary constants. This fact, together with König's theorem, is enough to establish a reasonably simple algorithm for solving the assignment problem—as has been shown by Kuhn.[14] In brief outline, the algorithm has the following steps:

Step 1. Subtract the smallest element in A from each element, obtaining a matrix A_1 with nonnegative elements and at least one null element.

Step 2. Find a minimal set S_1 of lines, n_1 in number, that includes all null elements of A_1. If $n_1=n$ there is a set of n null elements, no two of which are in the same line, and the elements of A in these n positions constitute the required solution.

Step 3. If $n_1 < n$, let h_1 denote the smallest element of A_1 not in any line of S_1. Add h_1 to each element of A_1 that is also in a line of S_1 and subtract h_1 from every element of A_1. Call the resulting matrix A_2.

Step 4. Repeat Steps 2 and 3, starting with A_2, until at some stage $n_k = n$. This process must terminate since the elements of A_k are always nonnegative but decreasing in total sum after each application of Step 3.

There are rather simple systematic graphical procedures for finding the set S_k at each stage, and also for finding the final set of n null elements, but these will not be discussed here because they are not important for present purposes.

We shall now direct our attention to the traveling-salesman problem. We may clearly replace condition (6) by an equivalent condition:

$$a_{ii} = \max_{i,j} a_{ij} \equiv \infty. \tag{8}$$

In other words, if we assume a_{ii} to be arbitrarily large this will require that $x_{ii} = 0$ in this solution.

The end result of the application of the assignment algorithm is a transformed matrix

$$D \equiv \|D_{\alpha\beta}\| \quad \text{for} \quad \alpha, \beta = 1, 2, \cdots, p \leq n, \tag{9}$$

which, after suitable symmetric permutations of rows and columns, is such that

(a) every element of each submatrix $D_{\alpha\beta}$ is positive if $\alpha \neq \beta$,

(b) every diagonal element of the submatrix $D_{\alpha\alpha}$ is simply a transform of some diagonal infinite element of A, and

(c) every element immediately above the diagonal of $D_{\alpha\alpha}$ is zero, and the element in the last row and first column of $D_{\alpha\alpha}$ is also zero.

Since there is no loss in generality in restricting our attention to matrices satisfying (9), because every matrix A can be reduced to this form by means of a transformation (7) that leaves the solutions of both the problems unaffected, we will suppose henceforth that $A = D$ is in this form.

It is obvious that one solution of the assignment problem is obtained by choosing the elements in the 'slants' of all the submatrices $D_{\alpha\alpha}$, where a slant is defined to consist of all the elements immediately above the diagonal together with the one in the lower left-hand corner of a matrix.

If $p = 1$, so that the slant of D solves the assignment problem, then the slant of D also solves the traveling-salesman problem. In fact, there is a solution of the assignment problem that also solves the traveling-salesman problem if and only if there is a symmetric permutation of the rows and columns of D such that the slant elements are all zero. In any event, the sum of the elements on the slant of D constitutes an upper bound on the length of the optimal tour.

PRELIMINARY EXAMPLES

It may help to clarify some of the mathematical points if a few simple examples are considered. As our first examples, we take some used by Dantzig *et al.*[8] Since their analysis was confined to the case of symmetric A, we can use their examples to illustrate an additional mathematical point that is valid only in the symmetric case. It is easily seen that the solution of the traveling-salesman problem is unchanged, *in the symmetric case*, if we set

$$d_{i_1 i_2} = d_{i_2 i_3} = d_{i_3 i_1} = \infty, \tag{10}$$

for any one set of distinct values for i_1, i_2, and i_3. However, no further elements may be changed to infinity without possibility of altering the solution, and these few changes are not usually of much help.

Example A

$A =$

	∞				
	5	∞			
	6	5	∞		
	10	12	8	∞	
	8	12	10	6	∞
$u_j = v_j =$	3	2	3	3	3

$D =$

∞				
0	∞			
0	0	∞		
4	7	2	∞	
2	7	4	0	∞

The sum of the slant elements of D is 4 and $p=2$. Using (10), we may set $d_{54}=\infty$ and subtract 2 from the fourth column and fifth row of D, whence

$$\begin{vmatrix} \infty & 0 & 0 & 4 & 2 \\ 0 & \infty & 0 & 7 & 7 \\ 0 & 0 & \infty & 2 & 4 \\ 4 & 7 & 2 & \infty & 0 \\ 2 & 7 & 4 & \infty & \infty \end{vmatrix} \longrightarrow \begin{vmatrix} \infty & 0 & 0 & 2 & 2 \\ 0 & \infty & 0 & 5 & 7 \\ 0 & 0 & \infty & 0 & 4 \\ 4 & 7 & 2 & \infty & 0 \\ 0 & 5 & 2 & \infty & \infty \end{vmatrix}$$

Since the slant of the transformed D is zero it follows that the slant of A solves the traveling-salesman problem. In this example,

$$D_{21} = \left\| \begin{matrix} 4 & 7 & 2 \\ 2 & 7 & 4 \end{matrix} \right\|, \quad D_{11} = \left\| \begin{matrix} \infty & 0 & 0 \\ 0 & \infty & 0 \\ 0 & 0 & \infty \end{matrix} \right\|, \quad D_{22} = \left\| \begin{matrix} \infty & 0 \\ 0 & \infty \end{matrix} \right\|.$$

Example B

$$A=\begin{array}{cccccc}
\infty & & & & & \\
4 & \infty & & & & \\
3 & 2 & \infty & & & \\
7 & 5 & 5 & \infty & & \\
7 & 7 & 6 & 3 & \infty & \\
6 & 7 & 6 & 5 & 3 & \infty \\
\hline
2\tfrac{1}{2} & 1\tfrac{1}{2} & \tfrac{1}{2} & 2\tfrac{1}{2} & \tfrac{1}{2} & 2\tfrac{1}{2}
\end{array}
\qquad
D=\begin{array}{cccccc}
\infty & & & & & \\
0 & \infty & & & & \\
0 & 0 & \infty & & & \\
2 & 1 & 2 & \infty & & \\
4 & 5 & 5 & 0 & \infty & \\
1 & 3 & 3 & 0 & 0 & \infty
\end{array}$$

$u_j=v_j=$ $2\tfrac{1}{2}$ | $1\tfrac{1}{2}$ | $\tfrac{1}{2}$ | $2\tfrac{1}{2}$ | $\tfrac{1}{2}$ | $2\tfrac{1}{2}$

The sum of the slant elements is 3, and $p=2$. Furthermore, no choice of three elements to be changed to infinity under (10) will alter the solution to this assignment problem, so stronger methods must be used. It is fairly obvious, however, that a traveling-salesman solution is given by the cycle (1 6 5 4 2 3), with sum 2, since at least two positive elements must enter and the only way to achieve a sum smaller than 3 from the slant is to use two of the unit entries. These observations are of little help since our task is to find a fairly systematic method of solution rather than one dependent too heavily upon discerning inspection. This example also shows how the traveling-salesman problem becomes fairly intractable even for n as small as 6.

If we consider the relations (1) to (6) for this particular example, of order 6, they become

$$x_{ij}^2=x_{ij} \quad \text{for} \quad i,j=1,2,\cdots,6, \tag{1'}$$

$$\sum_{i=1}^{6} x_{ij}=\sum_{j=1}^{6} x_{ij}=1, \tag{2'}$$

$$\lambda\equiv 2x_{41}+x_{42}+2x_{43}+4x_{51}+5x_{52}+5x_{53}+x_{61}+3x_{62}+3x_{63}=\min, \tag{3'}$$

$$x_{ij}\geq 0, \tag{4'}$$

$$x_{ij}+x_{ji}\leq 1, \qquad x_{ij}+x_{jk}+x_{ki}\leq 2 \quad \text{for} \quad i,j,k \text{ distinct.} \tag{5'}$$

The condition (6) is not needed because $d_{ii}=\infty$. It is easily shown that the ordinary linear-programming problem represented by (2′) to (5′) does have a solution which does not also satisfy (1′) since, for any ϵ such that $\tfrac{1}{3}\leq\epsilon\leq\tfrac{2}{3}$, if

$$||x_{ij}||\equiv\left\|\begin{array}{ccc|ccc}
0 & 1-\epsilon & \epsilon & 0 & 0 & 0\\
\epsilon & 0 & 1-\epsilon & 0 & 0 & 0\\
1-\epsilon & \epsilon & 0 & 0 & 0 & 0\\
\hline
0 & 0 & 0 & 0 & 1-\epsilon & \epsilon\\
0 & 0 & 0 & \epsilon & 0 & 1-\epsilon\\
0 & 0 & 0 & 1-\epsilon & \epsilon & 0
\end{array}\right\|,$$

then (2′), (4′), and (5′) are satisfied and $\lambda=0$. This example illustrates the fact that (1) can not be replaced by (4) for the traveling-salesman problem, and also represents a case in which the assignment problem has both integral and nonintegral solutions.

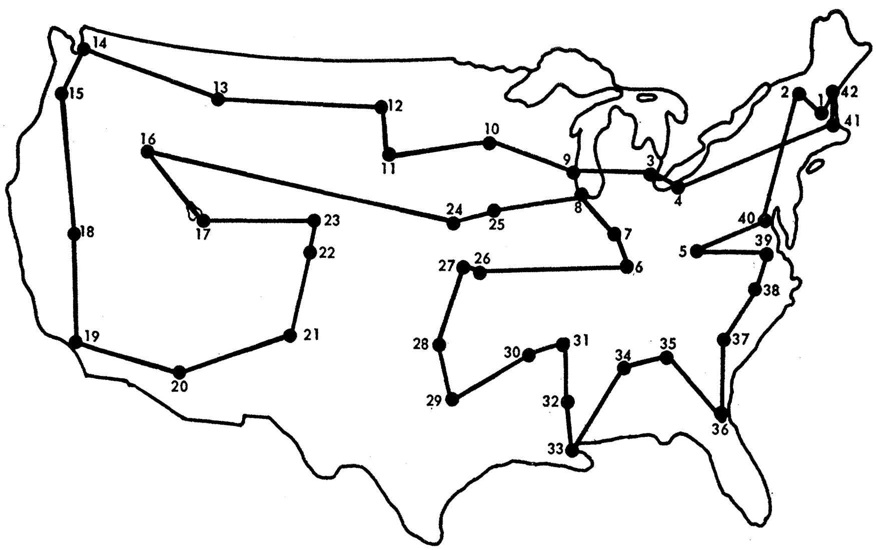

Fig. 1. Tour Comparison: intersectionless tour—follow the lined path; optimal tour—proceed between cities in numerical order.

The general intersection condition requires that an optimal tour $(i_1\, i_2\, \cdots\, i_n)$ satisfy

$$d_{i_1 i_2}+d_{i_2 i_3}+\cdots+d_{i_n i_1}$$
$$\leq d_{i_1 i_2}+\cdots+d_{i_{p-1} i_p}+d_{i_q i_{q-1}}+\cdots+d_{i_{p+1} i_p}+d_{i_q i_{q+1}}+\cdots+d_{i_n i_1} \quad (11)$$

for $1\leq p<q\leq n$, $i_0\equiv n$, and $i_{n+1}\equiv i_1$. When $d_{ij}=d_{ji}$ it is easily seen that (11) reduces to

$$d_{i_{p-1} i_p}+d_{i_q i_{q+1}}\leq d_{i_{p-1} i_q}+d_{i_p i_{q+1}}. \quad (12)$$

For instance, the slant (1 6 5 4 3 2) of D in Example B does not satisfy the intersection condition since $d_{43}+d_{21}=2>d_{42}+d_{31}=1$. Consequently, in this case, the tour (1 6 5 4 2 3) is better than the slant. Unfortunately, it is not always true that the removal of all intersections by this process will lead to an optimal tour.

In the case of the 49-city problem considered by Dantzig, *et al.*, or actually for the 42 cities entering into their calculations, the present author selected an initial trial tour by the next closest city method from the

TABLE I

Road Distances between Cities in Reduced Units

Each entry shown in italics should be increased by ½; thus *93*=93½, and *0*=½. The entries in this table are obtained from those of Dantzig, *et al.*,[8] by subtracting (u_i+u_j) from their entries; thus the entry *109* in row 16 and column 4 is 124−(13+1½)=109½. The values given here for u_j were computed by application of the algorithm for the assignment problem, as given in the present paper.

1																																										
2	0																																									
3	*30*	*30*																																								
4	*34*	*38*	0																																							
5	*35*	*28*	0	0																																						
6	*57*	*52*	11	16	1																																					
7	*53*	*49*	5	12	1	0																																				
8	55	50	*4*	*15*	*9*	*11*	*3*																																			
9	59	57	*10*	*21*	*15*	*17*	*9*	0																																		
10	66	60	*18*	*28*	*22*	*24*	*17*	7	4																																	
11	95	93	*47*	*58*	*51*	*53*	*46*	36	32	2																																
12	103	106	*54*	*65*	*59*	*61*	*53*	44	40	8	0																															
13	123	121	*75*	*85*	*79*	*82*	*74*	64	61	28	21	15																														
14	176	174	*128*	*138*	*132*	*135*	*127*	117	114	81	74	68	10																													
15	180	178	*132*	*142*	*136*	*133*	*127*	121	117	85	77	71	14	0																												
16	147	150	*99*	*109*	*103*	*99*	*93*	88	90	63	52	47	0	14	8																											
17	133	131	*85*	*94*	*89*	*86*	*79*	74	76	53	36	47	0	41	34	0																										
18	160	158	*112*	*123*	*116*	*113*	*106*	101	103	80	63	67	20	29	16	0	10																									
19	171	166	*121*	*128*	*113*	*114*	*109*	108	113	91	73	84	38	52	39	32	22	0																								
20	154	149	*103*	*112*	*101*	*94*	*93*	93	99	81	70	84	41	80	67	40	25	23	0																							
21	115	111	*65*	*73*	*59*	*56*	*53*	53	61	42	28	39	23	65	60	24	7	34	16	0																						
22	116	115	*69*	*78*	*69*	*65*	*58*	57	59	36	27	38	28	78	71	47	22	49	57	40	0																					
23	108	106	*60*	*71*	*65*	*61*	*54*	49	52	29	16	27	17	68	61	27	14	41	51	41	1	0																				
24	76	74	*28*	*38*	*31*	*30*	*22*	17	19	0	8	23	40	93	88	53	40	67	78	64	25	24	16																			
25	76	73	*28*	*38*	*32*	*31*	*23*	16	19	0	22	36	56	110	105	71	56	83	94	78	39	40	32	0																		
26	84	80	*34*	*42*	*30*	*25*	*22*	24	28	11	27	41	55	110	104	69	56	83	80	64	24	34	32	2	9																	
27	89	85	*39*	*47*	*33*	*30*	*27*	29	33	15	26	40	55	110	103	69	54	83	77	62	22	31	30	0	14	0																
28	94	89	*44*	*51*	*40*	*34*	*32*	36	42	24	29	45	54	105	99	65	48	75	56	40	0	26	27	10	23	9	9															
29	108	104	*59*	*67*	*50*	*46*	*47*	51	57	41	50	65	73	124	118	83	65	83	70	51	15	45	46	29	40	25	27	0														
30	84	79	*36*	*43*	*26*	*21*	*24*	29	35	29	47	61	76	131	124	90	76	96	80	66	26	55	51	22	28	16	21	4	12													
31	80	76	*32*	*39*	*22*	*17*	*20*	27	32	35	54	69	83	136	132	97	83	111	93	77	37	62	59	29	34	23	28	16	24	0												
32	83	79	*42*	*48*	*31*	*26*	*30*	36	42	44	64	78	94	146	139	105	87	105	91	72	36	66	68	39	44	32	38	20	17	4	5											
33	88	84	*50*	*55*	*36*	*33*	*39*	47	52	55	73	87	99	150	143	109	90	109	94	75	40	70	71	48	53	41	46	24	21	13	15	0										
34	70	71	*33*	*38*	*18*	*17*	*23*	33	39	45	68	82	97	152	146	111	97	120	108	89	51	76	73	43	47	37	42	30	34	14	9	6	12									
35	60	56	*28*	*33*	*11*	*16*	*22*	32	38	44	70	80	103	154	148	114	100	127	115	96	58	78	75	45	46	39	44	37	41	20	16	13	18	0								
36	61	57	*41*	*46*	*16*	*29*	*35*	45	52	57	83	94	114	169	161	130	113	134	121	101	65	93	90	59	59	52	57	48	46	30	26	19	15	10	0							
37	51	47	*33*	*34*	*6*	*22*	*27*	39	45	52	81	89	110	163	161	130	113	140	129	110	95	92	89	58	60	52	57	52	55	35	31	27	32	15	2	0						
38	*37*	*32*	27	26	0	25	28	*38*	*44*	*52*	*82*	*88*	*109*	*165*	*165*	*131*	*116*	*143*	*139*	*119*	*80*	*95*	*91*	*58*	*60*	*55*	*61*	*58*	*63*	*41*	*37*	*37*	*41*	*24*	*11*	*10*	*0*					
39	*31*	*27*	25	22	2	29	30	*40*	*46*	*53*	*83*	*90*	*110*	*164*	*167*	*135*	*118*	*145*	*147*	*127*	*94*	*99*	*93*	*60*	*62*	*60*	*66*	*71*	*73*	*49*	*45*	*48*	*53*	*35*	*23*	*23*	*13*	0				
40	*24*	*22*	19	16	1	29	26	*33*	*39*	*47*	*76*	*83*	*92*	*158*	*161*	*127*	*113*	*140*	*139*	*126*	*88*	*91*	*88*	*55*	*56*	*56*	*62*	*65*	*76*	*52*	*47*	*50*	*56*	*38*	*28*	*29*	*18*	5	0			
41	0	2	*31*	*33*	*31*	*52*	*49*	53	59	67	96	103	124	180	180	149	134	161	167	150	111	117	109	70	76	80	85	89	104	80	76	79	84	66	56	57	47	*32*	*27*	*19*		
42	0	1	*43*	*35*	*35*	*57*	*53*	54	60	66	100	105	125	128	182	149	135	163	171	154	115	120	110	78	83	84	89	93	108	67	80	83	88	73	64	61	51	*37*	*31*	*24*	0	
$j=$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**	**31**	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**	**41**	**42**
$u_j=$	1	7	*7*	*1*	*13*	*2*	*3*	3	2	14	7	4	21	4	6	13	8	13	13	9	21	0	5	8	0	2	1	10	2	6	2	5	6	3	6	12	5	*6*	*2*	*3*	2	4

matrix D (after reduction) and this trial tour was then improved by removal of successive intersections found by application of (12). The initial trial tour was of length 510, the improved tour without intersections was of length $266\frac{1}{2}$, and the optimal tour is of length $166\frac{1}{2}$. The next closest city method, when applied to the original unreduced matrix A, gave a length of 904 compared to minimal length of 699 in the original distance units; this is equivalent to $371\frac{1}{2}$ as compared with $166\frac{1}{2}$ in the reduced units. In other words, *in reduced units*, the two applications of the next closest city method led to tours about 2 or 3 times the length of the minimal tour but the intersectionless tour was only 60 per cent longer than minimal. The minimal tour is contrasted with this intersectionless tour in Fig. 1. The cities A–F were excluded from consideration in choosing the intersectionless tour, and the apparent intersection between lines 4–41 and 40–2 simply indicates that the actual distances are not accurately reflected in this planar map. The elements of the reduced matrix D, for the 42-city example, are shown in Table I, although not here rearranged so that the diagonal submatrices have null slants.

SUBTOUR RESTRICTIONS

DANTZIG ET AL. have discussed a technique that they have found effective in several numerical cases, including the one with 42 cities and symmetric distance matrix $\|d_{ij}\|$. The discussion here is not confined entirely to the symmetric case, since the nonsymmetric case presents little added formal difficulty.

The method starts with some trial tour $(1\, i_2 \cdots i_n)$. Let $\bar{x}_{ij}=1$ if (ij) is on the tour, and $\bar{x}_{ij}=0$ otherwise. Let S denote the set of all (ij) on the trial tour. Let T denote a set of nondiagonal positions (ij) not in S, but such that for each element $t_\theta \equiv (i_\theta\, j_\theta)$ in T there is a subset S_θ of S such that all members but t_θ of some proper subtour $(i_\theta\, j_\theta\, k_1\, k_2 \cdots k_s)$ are in S. In other words, the members of T are selected so that each can be grouped with a subset of members of S in such a way as to form a proper subtour. For example, if $\bar{x}_{45}=1$ then T could include (54); or, if $\bar{x}_{23}=\bar{x}_{34}=1$ then T could include (42). Finally let I denote the set of all (ij) for $i \neq j$.

The conditions (5) serve to prohibit subtours from appearing in a solution. Let some subset of these conditions be represented by

$$\sum_{\alpha,\beta=1}^{n} a_{\alpha\beta}{}^{\theta}\, x_{\alpha\beta} \leq k^{\theta} \quad \text{for} \quad \theta = n+1, \cdots, N. \tag{13}$$

Now if this subset is chosen, for each value of θ, so that $a_{\alpha\beta}{}^{\theta}=0$ unless $(\alpha\beta)$ is either in S or T then it follows that

$$\sum a_{\alpha\beta}{}^{\theta} = k^{\theta} \quad \text{if} \quad (\alpha\beta) \text{ in } S. \tag{14}$$

For convenience, we rewrite (13) as:

$$\sum_{\alpha,\beta=1}^{n} a_{\alpha\beta}{}^{\theta}\, x_{\alpha\beta}+y^{\theta}=k^{\theta} \quad\text{and}\quad y^{\theta}\geq 0. \tag{13'}$$

Also, we introduce variables σ_i and set

$$\Delta_{ij}(\sigma)\equiv d_{ij}-\sigma_i-\sigma_j+\sum_{\theta=n+1}^{N} a_{ij}{}^{\theta}\,\sigma_\theta. \qquad (i,j=1,2,\cdots,n) \tag{15}$$

Suppose, further, that S and T have been chosen so that $\bar{\sigma}$ can be found such that

$$\Delta_{ij}(\bar{\sigma})=0 \quad\text{if}\quad (ij) \text{ in } S \text{ or } T. \tag{16}$$

Now, consider the quantity

$$\begin{aligned}
D(x)&\equiv\sum_{i,j=1}^{n} d_{ij}\,x_{ij}=\sum_{I} x_{ij}\,[\bar{\sigma}_i+\bar{\sigma}_j-\sum_{\theta} a_{ij}{}^{\theta}\,\bar{\sigma}_\theta+\Delta_{ij}(\bar{\sigma})]\\
&=\sum_{I}\bar{x}_{ij}\,(\bar{\sigma}_i+\bar{\sigma}_j)-\sum_{I} x_{ij}\sum_{\theta} a_{ij}{}^{\theta}\bar{\sigma}_\theta+\sum_{I} x_{ij}\,\Delta_{ij}(\bar{\sigma})\\
&=\sum_{I}\bar{x}_{ij}\,[d_{ij}-\Delta_{ij}(\bar{\sigma})+\sum_{\theta} a_{ij}{}^{\theta}\,\bar{\sigma}_\theta]-\sum_{I} x_{ij}\sum_{\theta} a_{ij}{}^{\theta}\,\bar{\sigma}_\theta+\sum_{I} x_{ij}\,\Delta_{ij}(\bar{\sigma})\\
&=\sum_{I}\bar{x}_{ij}\,d_{ij}+\sum_{I} x_{ij}\,\Delta_{ij}(\bar{\sigma})+\sum_{I}(\bar{x}_{ij}-x_{ij})\sum_{\theta} a_{ij}{}^{\theta}\,\bar{\sigma}_\theta.
\end{aligned}$$

Finally,
$$D(x)=D(\bar{x})+\sum_{I-S-T} x_{ij}\,\Delta_{ij}(\bar{\sigma})+\sum_{\theta}\sigma_\theta y^{\theta}, \tag{17}$$

and it follows that $D(x)$ is necessarily minimal if

$$\Delta_{ij}[\bar{\sigma}]\geq 0 \text{ for } (ij) \text{ not in } S \text{ or } T, \text{ and } \bar{\sigma}_\theta\geq 0 \text{ for } \theta=n+1,\cdots,N. \tag{18}$$

Exactly the same argument holds, leading to (17) and (18), also if $T\&S$ is defined to include *all* positions of some rth ordered principal minor of D containing exactly $r-1$ consecutive members of S. Thus, if we denote the set of $r-1$ consecutive members by $\bar{S}\equiv\{i_{a+\alpha}\, i_{a+1+\alpha}\}$ for $\alpha=1,2,\cdots,r-1$, then T contains as members $\{i_{a+\alpha}\, i_{a+\beta}\}$ for $\alpha,\beta=1,2,\cdots,r$; $\beta\neq\alpha+1$. The truth of the argument hinges upon the fact that (14) is satisfied if T is defined in either of the two ways discussed, and the latter choice for T simply imposes a more stringent limitation on the x_{ij} under (13) than that represented by (5), but one satisfied whenever x represents a tour.

We next illustrate this technique by application to Example B, where we have selected a subset of the conditions (5) adequate to show that the tour (1 3 2 4 5 6) is optimal by the test of (18). For this we choose $T=\{12, 21, 23, 31; 46, 54, 64, 65\}$, so

$$\begin{aligned}
&x_{12}+x_{13}+x_{21}+x_{23}+x_{31}+x_{32}+y^{7}=2,\\
&x_{45}+x_{46}+x_{54}+x_{56}+x_{64}+x_{65}+y^{8}=2.
\end{aligned} \tag{13'}$$

Then
$$\begin{aligned}&\sigma_1+\sigma_3=\sigma_7, \quad \sigma_2+\sigma_4=1, \quad \sigma_5=\sigma_6+\sigma_8, \quad \sigma_1+\sigma_2=\sigma_7,\\ &\sigma_3+\sigma_2=\sigma_7, \quad \sigma_4+\sigma_5=\sigma_8, \quad \sigma_6+\sigma_1=1, \quad \sigma_4+\sigma_6=\sigma_8.\end{aligned} \tag{16'}$$

The solution of (16′) may be written $\bar\sigma_1=\bar\sigma_2=\bar\sigma_3=\frac{1}{2}\bar\sigma_7=1-\lambda$, and $\bar\sigma_4=\bar\sigma_5=\bar\sigma_6=\frac{1}{2}\bar\sigma_8=\lambda$, for λ arbitrary. Consequently

$$\|\Delta_{ij}(\bar\sigma)\|=\left\|\begin{array}{ccc|ccc}\infty&0&0&1&3&0\\0&\infty&0&0&4&2\\0&0&\infty&1&4&2\\\hline 1&0&1&\infty&0&0\\3&4&4&0&\infty&0\\0&2&2&0&0&\infty\end{array}\right\|,$$

which demonstrates, by the test of condition (18) with $0\leq\lambda\leq1$, that the tour (1 3 2 4 5 6) is optimal.

It should be noted that this technique also does not constitute a satisfactory systematic procedure since no algorithm has been given for the choice of T, and since the test represented by (18) is a sufficient condition for tour optimality but not a necessary one.

Example C

$$D=\left\|\begin{array}{ccc|ccc}\infty&0&0&1&4&1\\0&\infty&0&3&5&3\\0&0&\infty&2&5&3\\\hline 1&3&2&\infty&0&0\\4&5&5&0&\infty&0\\1&3&3&0&0&\infty\end{array}\right\|$$

Optimal tour is (1 2 3 4 5 6) of length 3.

Try $T_1=\{13, 21, 31, 32; 46, 54, 64, 65\}$. Then $\sigma_1+\sigma_2=\sigma_7$, $\sigma_2+\sigma_3=\sigma_7$, $\sigma_3+\sigma_4=2$, $\sigma_4+\sigma_5=\sigma_8$, $\sigma_5+\sigma_6=\sigma_8$, $\sigma_6+\sigma_1=1$, $\sigma_1+\sigma_3=\sigma_7$, $\sigma_4+\sigma_6=\sigma_8$, which equations are incompatible.

Try $T_2=T_1\&\{16\}$. Thus $T_2=\{13, 21, 31, 32; 46, 54, 64, 65; 16\}$ uses three principal minors, the third defined by rows (or columns) 1 and 6. From $x_{16}+x_{61}+y^9=1$, it follows that $0=\nabla_{16}=\nabla_{61}=1-\sigma_1-\sigma_6+\sigma_9$. Then the equations $\sigma_1+\sigma_2=\sigma_2+\sigma_3=\sigma_1+\sigma_3=\sigma_7$, $\sigma_4+\sigma_5=\sigma_5+\sigma_6=\sigma_6+\sigma_4=\sigma_8$, $\sigma_3+\sigma_4=2$, $\sigma_6+\sigma_1=1+\sigma_9$ have the solution: $\bar\sigma_1=\bar\sigma_2=\bar\sigma_3=\frac{1}{2}\bar\sigma_7=2-\lambda$, $\bar\sigma_4=\bar\sigma_5=\bar\sigma_6=\frac{1}{2}\bar\sigma_8=\lambda$, $\bar\sigma_9=1$.

Now $$\|\Delta_{ij}(\bar{\sigma})\| = \left\| \begin{array}{ccc|ccc} \infty & 0 & 0 & -1 & 2 & 0 \\ 0 & \infty & 0 & 1 & 3 & 1 \\ 0 & 0 & \infty & 0 & 3 & 1 \\ \hline -1 & 1 & 0 & \infty & 0 & 0 \\ 2 & 3 & 3 & 0 & \infty & 0 \\ 0 & 1 & 1 & 0 & 0 & \infty \end{array} \right\|$$

and the condition (18) is not fulfilled since $\Delta_{41} = -1$. Furthermore, no optimal tour can include $x_{41} = 1$ and this illustrates the fact that (18) is not a necessary condition for optimality.

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