The Prime Square Theorem

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1 Statement

Let p_n denote the n^{th} prime number. $(p_1p_2p_3...p_k) + 1$ can never be a perfect square.

2 Proof

Let $q = (p_1p_2p_3...p_k) + 1$.

We use contradiction. Assume q IS a perfect square.

Then, $(p_1p_2p_3...p_k)+1 = m^2$, which simplifies to $p_1p_2p_3...p_k = (m+1)(m-1)$. Note $p_1 = 2$, which is even. This means $p_1p_2p_3...p_k$ is even and q is odd. So, m must be odd too since an even number squared can't be odd.

Now, since m is odd, $(m + 1)$ and $(m - 1)$ are both even, making $(m + 1)(m - 1) = p_1p_2p_3...p_k$ a multiple of 4. Now if we recall, 2 is the only prime and $p_1p_2p_3...p_k$ only has one factor of 2. This contradicts!

And we're done.