Sum of Powers Theorem

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1 A case of the generalized form

1.1 Statement

$2^a + 2^b$ cannot be a power of 2 when a and b are positive integers and $a \neq b$.

1.2 Proof

Without loss of generality, let $b > a$. Let $b = a + n$ for a positive integer n. Then $2^a + 2^b = 2^a + 2^{a+n} = 2^a(1 + 2^n)$. 2^a is a power of 2, and $2^n + 1$ can never be a power of 2 since it's odd. So, since one of them is a power of 2 and the other isn't, in its prime factorization we will have a power of 2 multiplied by some other powers of prime numbers. Thus $2^a + 2^b$ isn't a power of 2.

2 Extended Form

2.1 Statement

$x^a + x^b$ cannot be a power of x when a and b are positive integers and $a \neq b$.

2.2 Proof

Without loss of generality, let $b > a$. Let $b = a + n$ for a positive integer n. Then $x^a + x^b = x^a + x^{a+n} = x^a(1 + x^n)$. x^a is a power of x.

Lemma: $x^n + 1$ is never a power of x: Let i be a positive integer. We note that the value of $x^{i+1} - x^i$ increases as i increases. In other words, it means that the difference between consecutive powers of x increases as the powers increase. And since the difference between the first two positive powers of 2 is $2^2 - 2^1 = 2$, we know that the minimum possible value of $x^{i+1} - x^i$ is 2.

We use the lemma. Since we know x^n is a power of n, by the lemma, at minimum $x^n + 2$ can be a power of x, and there is no way for $x^n + 1$ to be a power of x.

So, since one of them is a power of x and the other isn't, in its prime factorization we will have a power of x multiplied by some other powers of prime numbers. Because it has prime numbers in its factorization other than 2, $x^a + x^b$ cannot be a power of x for $b > a$ and $a \neq b$.

And we're done.