

The Exponential Inequality

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1 A case of the generalized form

$n^4 \leq 4^n$ for all $n \geq 4$.

2 Proof of case

We try to prove this using induction.

Base Case: $n = 4$: $4^4 \leq 4^4$. This works.

Inductive step: Let k satisfy this statement, making $k^4 \leq 4^k$. We have to prove that $(k+1)^4 \leq 4^{k+1}$. Multiplying both sides of the first equation by 4, we get $4k^4 \leq 4^{k+1}$. So it suffices to show that $(k+1)^4 \leq 4k^4$. Since both sides are positive, quad-root and get $(k+1) \leq \sqrt{2}k$. This holds if and only if $1 \leq (\sqrt{2}-1)k$, which holds true if and only if $k \geq \frac{1}{\sqrt{2}-1} = \frac{1}{\sqrt{2}-1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1} = \sqrt{2}+1$. Because $k \geq 4$, we know that $k \geq \sqrt{2}+1$ as well since $\sqrt{2}+1 < 4$. All our steps are reversible, so we can just walk backwards to conclude that $(k+1)^4 \leq 4^{k+1}$.

Our induction is complete.

3 Extension

3.1 Statement excluding one case (has a missing constraint)

$n^k \leq k^n$ for all $n \geq k$, where n and k are positive integers.

3.2 Proof

We use induction again, except this time, the induction is on the varying values of k . The base case is going to be 2. This uses "double induction".

Base Case on the values of n : $n = 3$: $n^3 \leq 3^n$ for all $n \geq 3$

Base Case on the values of k when $n = 3$: $3^3 \leq 3^3$ works.

Induction on the values of k when $n = 3$: We can trace our path from the case we took of the generalized form.

We've already proved that the general form works when $n = 3$, so now we just have to prove using induction that if it works for a positive integer m , it will work for $m + 1$.

Induction on the values of n : Let n work, which means $n^k \leq k^n$. We now have to prove $(n+1)^k \leq k^{n+1}$. Multiply both sides of the first inequality by k to get $kn^k \leq k^{n+1}$. So it suffices to show that $(n+1)^k \leq kn^k$. Since both sides are positive, k -root both sides to get $(n+1) \leq kn$. This holds if and only if $1 \leq n(k-1)$ is true, which is true if and only if $\frac{1}{n} \leq k-1$. If we recall, $n \geq k$ and n and k are positive integers. This makes $0 \leq \frac{1}{n} \leq 1$ and $k-1 = 0, 1, 2, 3, 4, \dots$. If $k-1 = 1$, and $\frac{1}{n}$ is at maximum 1, the constraint is satisfied. If $k-1 > 1$, the constraint is satisfied as well. Now, if $k-1 = 0$, then $k = 1$ and the original inequality becomes $n^1 \leq 1^n$ which is $n \leq 1$. This isn't necessarily true; in fact, it is never true. Thus we have to fix our statement and make $k > 1$. Other than that, we can reverse all our steps from bottom to up and complete the induction.

4 Final fixed statement

$n^k \leq k^n$ for all $n \geq k$, where n and k are positive integers, and $k \geq 1$.