Intersection Points in a Polygon

Saksham Sethi

1 Statement

In a n-sided convex polygon, draw all of the diagonals. The maximum possible intersection points is $\binom{n}{4}$.

2 Examples

Let's take some examples to get started.

In a triangle, there are 0 intersection points.

In every quadrilateral, there is only 1 intersection point.

In every pentagon, there are 5 intersection points.

In every hexagon, there are 15 intersection points.

In every septagon, there are 35 intersection points.

We'll stop there and make a table to get organized.

n	Points
3	0
4	1
5	5
6	15
7	35

Hmm.. It looks like $\binom{n}{4}$ does work. But how do we prove it?

3 Proof

Maybe we can form a one-to-one correspondence to something simpler, since something as simple as $\binom{n}{4}$ doesn't look a formula for a complex problem like this.

In any convex polygon, when we choose 4 random distinct points, we make a quadrilateral. And every quadrilateral has two diagonals, which make one intersection point.

Aha! Well, this does look like a one-to-one correspondence with the original problem. However, we have to prove the correspondence both ways.

If we have one intersection point, we can find the two diagonals whose intersection this is, and form a quadrilateral with those two diagonals. This completes our proof that the one-to-one correspondence exists.

Number of intersection points $\Longleftrightarrow$ Number of distinct sets of four points from the polygon

Thus, the number of quadrilaterals are the number of distinct pairs of two diagonals, which is the number of distinct intersection points.

However, we've skipped over one significant detail. Recall the original problem was about the *maximum* number of intersection points. What if more than two diagonals intersect at one point? Well that's not a problem, since the left side of the correspondence will now be strictly smaller than the right side. That's because there would be more intersection points if there was a distinct point for every pair of two diagonals, instead of only one point even after using up more than two diagonals.