The Exponential Inequality

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1 A case of the generalized form

$n^4 \leq 4^n$ for all $n \geq 4$.

2 Proof of case

We try to prove this using induction.

Base Case: $n = 4$**:** $4^4 \leq 4^4$. This works.

Inductive step: Let k satisfy this statement, making $k^4 \leq 4^k$. We have to prove that $(k+1)^4 \leq 4^{k+1}$. Multiplying both sides of the first equation by 4, we get $4k^4 \leq 4^{k+1}$. So it suffices to show that $(k+1)^4 \leq 4k^4$. Since both sides are positive, quad-root and get $(k+1) \leq \sqrt{2}k$. This holds if and only if $1 \leq (\sqrt{2}-1)k$, which holds true if and only if$k \geq \frac{1}{\sqrt{2}-1} = \frac{1}{\sqrt{2}-1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1} = \sqrt{2}+1$. Because $k \geq 4$, we know that $k \geq \sqrt{2}+1$ as well since $\sqrt{2}+1 < 4$. All our steps are reversible, so we can just walk backwards to conclude that $(k+1)^4 \leq 4^{k+1}$.

Our induction is complete.

3 Extension

3.1 Statement excluding one case (has a missing constraint)

$n^k \leq k^n$ for all $n \geq k$, where n and k are positive integers.

3.2 Proof

We use induction again, except this time, the induction is on the varying values of k. The base case is going to be 2. This uses "double induction".

Base Case on the values of n: $n = 3$: $n^3 \leq 3^n$ for all $n \geq 3$

Base Case on the values of k when $n = 3$: $3^3 \leq 3^3$ works.

Induction on the values of k when $n = 3$: We can trace our path from the case we took of the generalized form.

We've already proved that it the general form works when $n = 3$, so now we just have to prove using induction that if it works for an positive integer m, it will work for $m + 1$.

Induction on the values of n: Let n work, which means $n^k \leq k^n$. We now have to prove $(n+1)^k \leq k^{n+1}$. Multiply both sides of the first inequality by k to get $kn^k \leq k^{n+1}$. So it suffices to show that $(n+1)^k \leq kn^k$. Since both sides are positive, $k-root$ both sides to get $(n + 1) \leq kn$. This holds if and only if $1 \leq n(k-1)$ is true, which is true if and only if $\frac{1}{n} \leq k-1$. If we recall, $n \geq k$ and n and k are positive integers. This makes $0 \leq \frac{1}{n} \leq 1$ and $k - 1 = 0, 1, 2, 3, 4, ...$. If $k - 1 = 1$, and $\frac{1}{n}$ is at maximum 1, the constraint is satisfied. If $k - 1 > 1$, the constraint is satisfied as well. Now, if $k - 1 = 0$, then $k = 1$ and the original inequality becomes $n^1 \leq 1^n$ which is $n \leq 1$. This isn't necessarily true; in fact, it is never true. Thus we have to fix our statement and make $k > 1$. Other than that, we can reverse all our steps from bottom to up and complete the induction.

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4 Final fixed statement

$n^k \leq k^n$ for all $n \geq k$, where n and k are positive integers, and $k \geq 1$.
