EGMO Solutions

Oly Geo

Tastes Nice

Problem 1.36

Let $ABCDE$ be a convex pentagon such that $BCDE$ is square with center O and $A = 90°$. Prove AO bisects $\angle BAE$.

Solution

Since $\angle BAE = \angle EOB = 90$, $BAEO$ is a cyclic quadrilateral. And because $\triangle EOB$ is an isosceles right triangle, $\angle BEO = 45°$. Since $BAEO$ is cyclic, we have $\angle BAO = \angle BEO = 45°$, and $\angle EAO = 90° - 45° = 45°$. Thus, we conclude that $\angle BAO = \angle EAO = 45°$, or that AO bisects $\angle BAE$. $\square$

Problem 1.37

Let $O = (0, 0), A = (0, a)$, and $B = (0, b)$, where $0 < a < b$ are reals. Let C be a circle with diameter AB and let P be any other point on C. Line PA meets the x-axis at Q. Prove that $\angle BQP = \angle BOP$.

Solution

Since AB is the diameter of C, $\angle BPA = 90°$. Also, the x-axis and y-axis meet at a right angle, so $\angle BOQ = 90°$ as well. Therefore, since $\angle BPA = \angle BOQ$, we know that $PBQO$ is a cyclic quadrilateral. And because it is cyclic, we have $\angle BQP = \angle BOP$. $\square$

Problem 1.38

In the cyclic quadrilateral $ABCD$, let I_1 and I_2 denote the incenters of $\triangle ABC$ and DBC, respectively. Prove that I_1I_2BC is cyclic.

Solution

Since $\angle BCA < \angle BCD$, we know $\angle BCI_2 > \angle BCI_1$. This makes sure that there are no configuration issues. Now note that

$$\angle I_1CI_2 = \angle BCI_2 - \angle BCI_1 = \frac{\angle BCD}{2} - \frac{\angle BCA}{2} = \frac{\angle BCD - \angle BCA}{2} = \frac{\angle ACD}{2}$$

and

$$\angle I_1BI_2 = \angle I_1BC - \angle I_2BC = \frac{\angle ABC}{2} - \frac{\angle DBC}{2} = \frac{\angle ABC - \angle DBC}{2} = \frac{\angle ABD}{2}.$$

Since quadrilateral $ABCD$ is cyclic, we have $\angle ABD = \angle ACD$, so $\angle I_1BI_2 = \angle I_1CI_2$, or BI_1I_2C is cyclic. $\square$

Problem 1.39

Let ΔABC be a triangle. The incircle of ΔABC is tangent to AB and AC at D and E respectively. Let the incenter of $\triangle ABC$ be I. Let O denote the circumcenter of ΔBCI. Prove that $\angle ODB = \angle OEC$.

Solution

By the Incenter-Excenter Lemma, we know that O lies on $\overline{AI}$. Since AI is the angle bisector of $\angle BAC$, or that $\angle DAO = \angle EAO$, and $AD = AE$, it follows that $\triangle ADO \cong \triangle AEO$. Thus, $\angle ADO = \angle AEO$, or

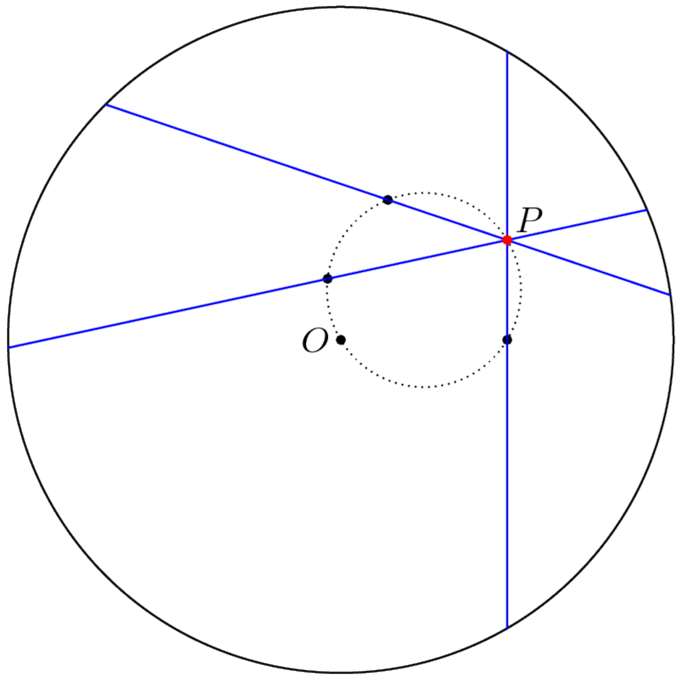
$$180^\circ - \angle ADO = \angle ODB = 180^\circ - AEO = \angle OEC$$

. $\square$

Problem 1.40

Let P be a point inside circle ω. Consider the set of chords ω that contain P. Prove that their midpoint all lie on a circle.

Solution



Let the circle have center O. Let the midpoint of some arbitrary chord passing through P be M. First, because we know that P is a midpoint of some chord passing through P itself, it will be contained in the locus of midpoints. Second, we know that $\angle \mathsf{OMP} = 90^\circ$ for all M (other than P), so M lies on the circle with diameter OP.

Problem 1.41

Points E and F are on side BC of convex quadrilateral ABCD (with E closer than F to B). It is known that $\angle \mathsf{BAE} = \angle \mathsf{CDF}$ and $\angle \mathsf{EAF} = \angle \mathsf{FDE}$. Prove that $\angle \mathsf{FAC} = \angle \mathsf{EDB}$.

Solution

Let $\angle \mathsf{BAE} = \angle \mathsf{CDF} = x$, $\angle \mathsf{EAF} = \angle \mathsf{FDE} = y$, and $\angle \mathsf{FAD} = z$. Since $\angle \mathsf{EAF} = \angle \mathsf{FDE}$, EFDA is cyclic, or $\angle \mathsf{FED} = z$. Now using $\triangle \mathsf{EFD}$, we have

$$\angle \mathsf{EFD} = 180 - (y + z) \to \angle \mathsf{CFD} = y + z \to \angle \mathsf{FCD} = \angle \mathsf{C} = 180 - (x + y + z).$$

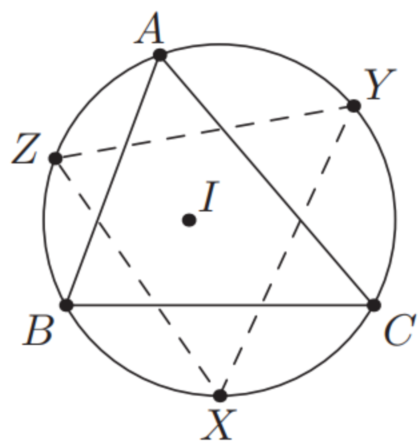
However, $\angle \mathsf{A} = x + y + z$, so $\angle \mathsf{A} + \angle \mathsf{C} = 180$, or that ABCD is cyclic. Thus $\angle \mathsf{BAC} = \angle \mathsf{CDB}$. But since $\angle \mathsf{BAF} = \angle \mathsf{CDE}$, we have

$$\angle \mathsf{BAC} - \angle \mathsf{BAF} = \angle \mathsf{FAC} = \angle \mathsf{CDB} - \angle \mathsf{CDE} = \angle \mathsf{EDB}. \square$$

Problem 1.42

Let ABC be an acute triangle inscribed in circle Ω. Let X be the midpoint of the arc BC not containing A and define Y, Z similarly. Show that the orthocenter of $\triangle XYZ$ is the incenter I of $\triangle ABC$,

Solution



Because of the Incenter-Excenter Lemma, A-I-X. Note that it suffices to prove $YZ \perp XA$, as the other altitudes similarly follow. To show this, note that $\angle AXZ = \frac{\angle C}{2}$ and $\angle YZX = \frac{\angle A + \angle B}{2}$. Adding gives

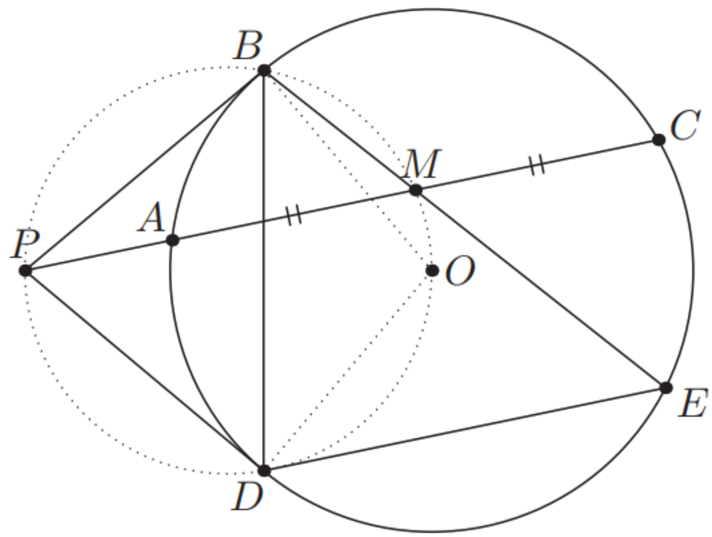
$$\angle AXZ + \angle YZX = \frac{\angle A + \angle B + \angle C}{2} = \frac{180}{2} = 90^\circ,$$

so $YZ \perp XA$. Thus, applying the same logic, we have that the three lines are perpendicular and I is thus the orthocenter of XYZ. $\square$

Problem 1.43

Points A, B, C, D, E lie on a circle ω and point P lies outside the circle. The given points are such that (i) lines PB and PD are tangent to ω, (ii) P, A, C are collinear, and (iii) $DE \parallel AC$. Prove that BE bisects AC.

Solution



Let $\mathsf{M} = \overline{\mathsf{BE}} \cup \overline{\mathsf{AC}}$. Note that it is sufficient to prove $\overline{\mathsf{OM}} \perp \overline{\mathsf{AC}}$. Since PB and PD are tangent to ω, $\measuredangle \mathsf{PBO} = \measuredangle \mathsf{PDO} = 90°$, so PBOD is cyclic.

Claim: M also lies on this circle.

Proof: Because $\overline{\mathsf{DE}} \parallel \overline{\mathsf{AC}}$,

$$\measuredangle \mathsf{BMP} = \measuredangle \mathsf{BMA} = \measuredangle \mathsf{BED} = \measuredangle \mathsf{PBD} = \measuredangle \mathsf{BDP},$$

which shows that M lies on the same circle as $\mathsf{P}, \mathsf{B}, \mathsf{O}$ and D do.

Therefore, we now conclude that $\angle \mathsf{OBP} = \angle \mathsf{OMP} = 90°$, as desired. $\square$

Problem 1.44

Let ABC be an acute triangle. Let BE and CF be the altitudes of $\triangle \mathsf{ABC}$, and denote by M the midpoint of BC. Prove that ME, MF, and the line through A parallel to BC are all tangents to (AEF).

Solution

Let H be the orthocenter of $\triangle \mathsf{ABC}$ and let the center of (AEF) be O. Since $\angle \mathsf{HFA} = \angle \mathsf{HEA} = 90°$, (AEF) has diameter AH. We know that $\mathsf{A}, \mathsf{O}, \mathsf{H}$ are collinear (definition of diameter), and because $\mathsf{AH} \perp \mathsf{BC}$, we have $\mathsf{AO} \perp \mathsf{BC}$, and because the line through A is parallel to BC, AO is perpendicular to that line, and thus it is tangent to the circle.

Now we prove that MF, ME are tangent to (AFE). Note that

$$MB = MF = ME = MC$$

by properties of right triangles. It suffices to show that $\angle FEM = \angle FAE$ by tangent-to-a-circle properties. So, we see that

$$\angle FEM = \angle FEB + \angle BEM = \angle FEB + \angle EBM = 90^\circ - \angle C + \angle FEB = 90^\circ - \angle C + \angle FAH = 90^\circ - \angle C + 90^\circ - \angle B = 180^\circ -$$

as desired. The proof that MF is also tangent follows since $ME = MF$. $\square$

Problem 1.45

The incircle of ABC is tangent to $\overline{BC}, \overline{CA}, \overline{AB}$ at D, E, F, respectively. Let M and N be the midpoints of $\overline{BC}$ and $\overline{AC}$, respectively. Ray BI meets line EF at K. Show that $\overline{BK} \perp \overline{CK}$. Then show K lies on line MN.

Solution

Let $K = BI \cap MN$. Now since $\triangle ABC \sim \triangle NMC$, we know that $BA \| MN$, or

$$\angle BKM = \angle KBA = \angle KBM = \frac{\angle B}{2}$$

so $MK = MB = MC$. This implies that $\triangle BKC$ is right, or that $BK \perp CK$.

Now we have to show that K lies on $\overline{MN}$. In the previous part, we showed that $\angle BKM = \angle KBM$, which means that $MK \parallel AB$. However, we already know that $MN \parallel AB$, so this implies that K, M, and N are collinear, or that K lies on line $\overline{MN}$. $\square$

Problem 1.46

The point O is situated inside the parallelogram $ABCD$ such that $\angle AOB + \angle COD = 180^\circ$. Prove that $\angle OBC = \angle ODC$.

Solution

We translate O to O' and D, C to A, B, respectively. Thus $AO'OD, O'BCO$ are parallelograms. Now $\angle AO'B = \angle DOC$ because translations preserve angles, and it's given that $\angle DOC + \angle AOB = 180^\circ$, so $AO'BO$ is cyclic. This implies that $\angle OBD = \angle O'AB = \angle O'OB = \angle OBC$, as desired. $\square$

note - motivation for translating O to O': the x and $180-x$ immediately remind of cyclic quadrilaterals, but we don't see any in the figure, so we attempt to make one. in order to do that, we needed another vertex and then it clicked to make it outside $ABCD$. then the rest clicks very easily.

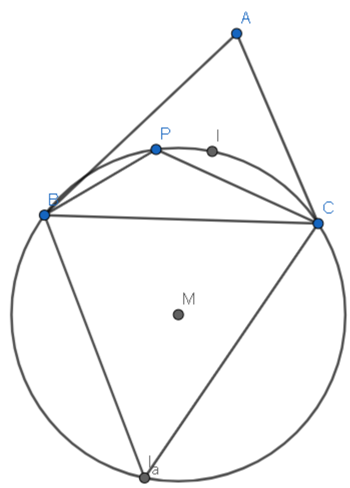
Problem 1.47

Let ABC be a triangle with incenter I. A point P in the interior of the triangle satisfies

$$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$

Show that $AP \geq AI$ and that equality holds if and only if $P = I$.

Solution



Notice that $(\angle PBA + \angle PCA) + (\angle PBC + \angle PCB) = 180 - \angle A$. This combined with the given condition yields $\angle PBC + \angle PCB = 90 - \frac{\angle A}{2}$, so $\angle BPC = 180 - (90 - \frac{\angle A}{2}) = 90 + \frac{\angle A}{2}$. However, it is well known that $\angle BIC = 90 + \frac{\angle A}{2}$ as well, so $\angle BPC = \angle BIC$, which implies that $BPIC$ is cyclic.

Now by the Incenter-Excenter lemma, the center of $\odot(BPIC)$ is $\odot(ABC) \cap AI = M \implies PM = IM$ and finally by the triangle inequality on $\triangle APM$ we get that

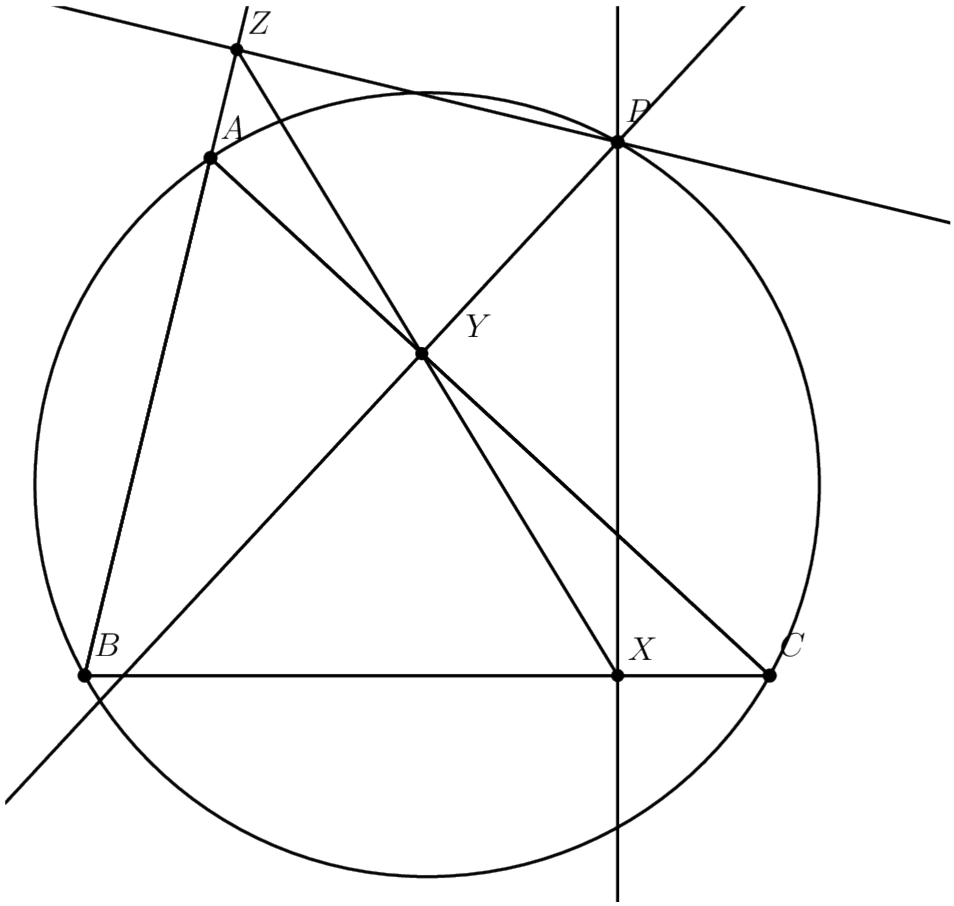
$$AP + PM \geq AI + IM \iff AP \geq AI(1)$$

Equality holds when A, I, P, M lie on a line, which is attained only when $P = I(2)$. We have proved both parts of the problem in (1) and (2). $\square$

Problem 1.48 (Simson Lines)

Let ABC be a triangle and P be any point on (ABC). Let X, Y, Z be the feet of the perpendiculars from P onto lines BC, CA, and AB. Prove that points X, Y, Z are collinear.

Solution



We know $ABCP, AYPZ, PYXC$ are cyclic. Hence,

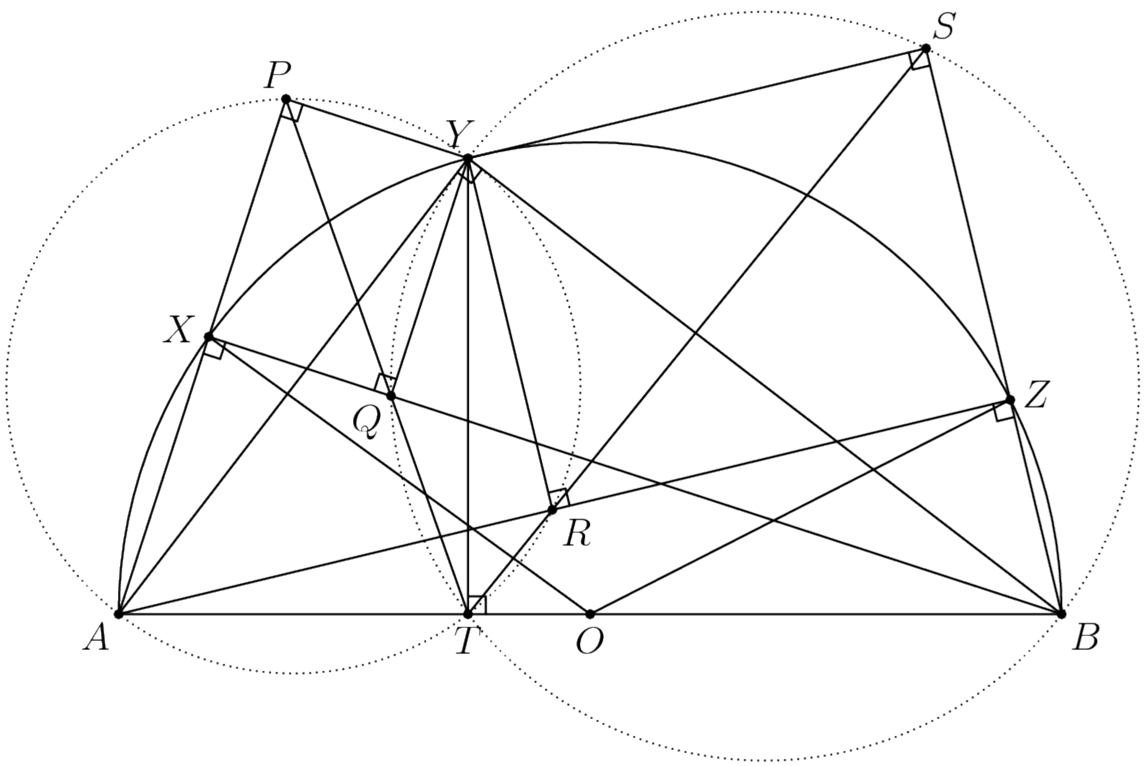
$$\measuredangle PYX = \measuredangle PCX = \measuredangle PAB = \measuredangle PAZ = \measuredangle PYZ = -\measuredangle ZYP.$$

Thus, $\measuredangle PYX + \measuredangle ZYP = 0$, which, in other words, means that X, Y, Z are collinear.$\square$

Problem 1.49

Let $AXYZB$ be a convex pentagon inscribed in a semicircle of diameter AB. Denote by P, Q, R, S the feet of the perpendiculars from Y onto lines AX, BX, AZ, BZ, respectively. Prove that the acute angle formed by lines PQ and RS is half the size of $\angle XOZ$, where O is the midpoint of segment AB.

Solution



Let T be the foot of the altitude form Y to AB. Notice that since Y lies on (AXB) and (AZB) and $\mathsf{P}, \mathsf{Q}, \mathsf{T}$ lie on the Simson line from Y to $\triangle\mathsf{AXB}$ and $\mathsf{S}, \mathsf{R}, \mathsf{T}$ lie on the Simson line from Y to $\triangle\mathsf{AZB}$. Thus, the acute angle formed by PQ and RS is $\angle\mathsf{PTS}$. Since $\angle\mathsf{YPA} = \angle\mathsf{YTA} = 90^\circ$ and $\angle\mathsf{YSB} = \angle\mathsf{YTB} = 90^\circ$, quadrilaterals ATYP and BTYS are cyclic. Hence,

$$\angle\mathsf{PTS} = \angle\mathsf{PTY} + \angle\mathsf{YTS} = \angle\mathsf{PAY} + \angle\mathsf{YBS} = \angle\mathsf{XAY} + \angle\mathsf{YBZ}$$

To manipulate this further, note that $\angle\mathsf{XAY}$ is the inscribed angle of minor arc $\widehat{\mathsf{PY}}$, and $\angle\mathsf{XOY}$ is the central angle of minor arc $\widehat{\mathsf{AB}}$, so $\angle\mathsf{XAY} = \frac{\widehat{\mathsf{PY}}}{2} = \frac{\angle\mathsf{XOY}}{2}$. Similarly, $\angle\mathsf{YBZ} = \frac{\widehat{\mathsf{YZ}}}{2} = \frac{\angle\mathsf{YOZ}}{2}$. Now we have

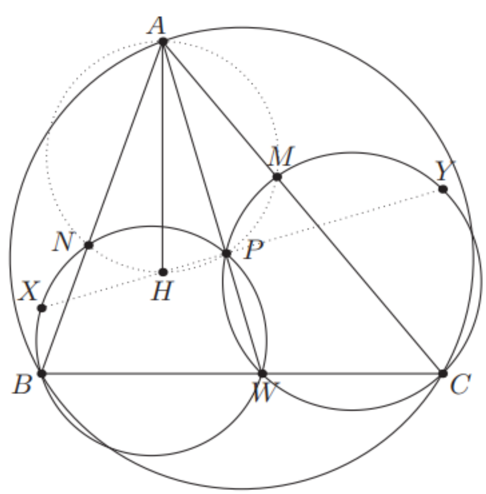
$$\angle\mathsf{XAY} + \angle\mathsf{YBZ} = \frac{1}{2}\angle\mathsf{XOY} + \frac{1}{2}\angle\mathsf{YOZ} = \frac{1}{2}\angle\mathsf{XOZ},$$

as desired. $\square$

Problem 1.50

Let ABC be an acute triangle with orthocenter H, and let W be a point on the side BC, lying strictly between B and C. The points M and N are the feet of the altitudes from B and C, respectively. Denote by ω_1 is [sic] the circumcircle of BWN, and let X be the point on ω_1 such that WX is a diameter of ω_1. Analogously, denote by ω_2 the circumcircle of triangle CWM, and let Y be the pointsaasaaaas such that WY is a diameter of ω_2. Prove that X, Y and H are collinear.

Solution



Denote the Miquel Point of $\triangle\mathsf{ABC}$ by P. Thus P lies on (AMN), and by orthic triangle properties, we know that (AMN) has diameter AH. This means $\angle\mathsf{APH} = 90$. Since WX, WY are diameters, we have $\angle\mathsf{XPW} = 90$ and $\angle\mathsf{YPW} = 90$. We also have $\angle\mathsf{APY} = 180 - \angle\mathsf{YPW} = 90$, so points $\mathsf{X}, \mathsf{H}, \mathsf{P}$ and $\mathsf{Y}, \mathsf{H}, \mathsf{P}$ are collinear. This means points $\mathsf{X}, \mathsf{Y}, \mathsf{H}$ are collinear, as desired.

Problem 1.51

A circle has center on the side $\overline{\mathsf{AB}}$ of the cyclic quadrilateral ABCD. The other three sides are tangent to the circle. Prove that $\mathsf{AD} + \mathsf{BC} = \mathsf{AB}$.

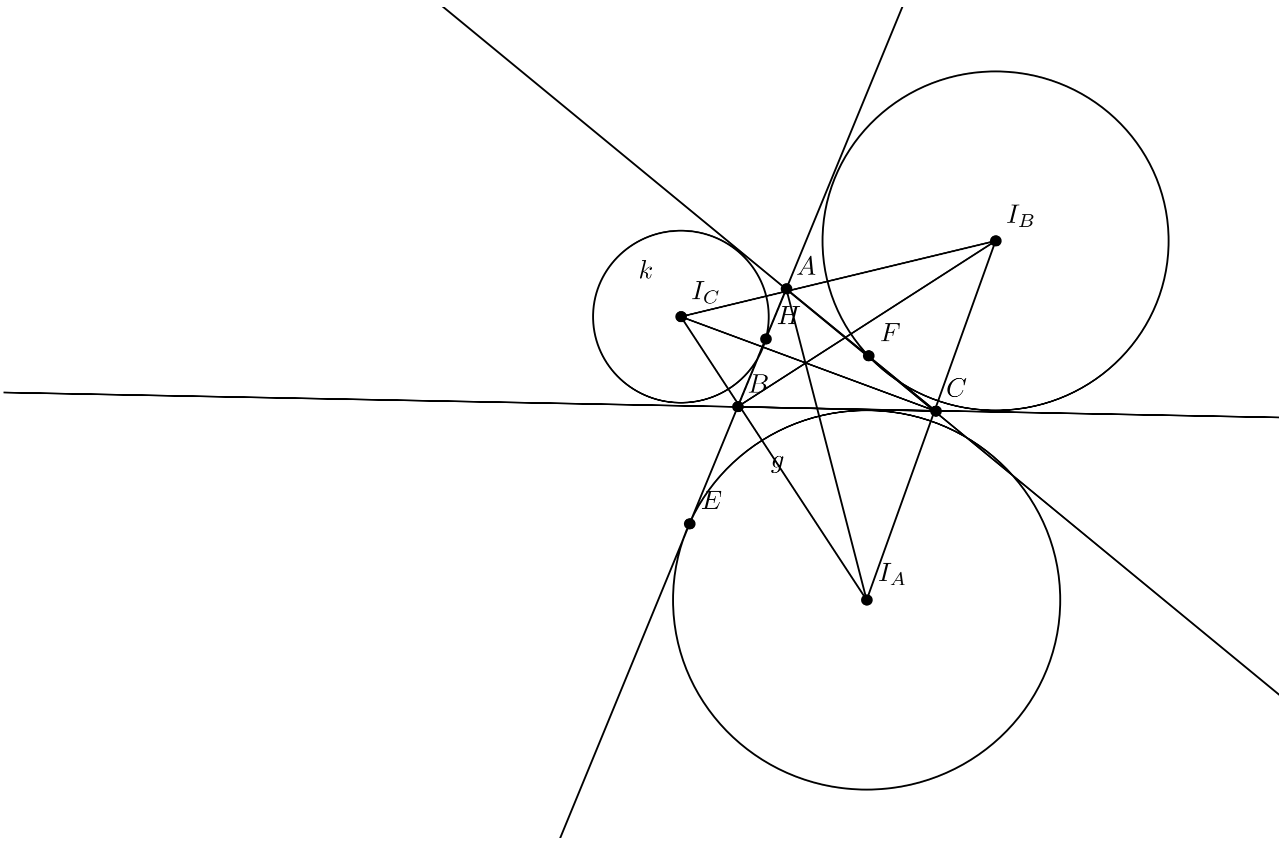
Solution

DO LATER IN FREE TIME

Problem 2.24

Let $\triangle ABC$ have excenters I_A, I_B and I_C, and let I be the incenter of $\triangle ABC$. Prove that $\triangle I_A I_B I_C$ has orthocenter I and that triangle $\triangle ABC$ is its orthic triangle.

Solution



Note I_C, A, I_B are collinear, since I_C, I_B both lie on the external bisector of $\angle A$. Now by the Incenter-Excenter Lemma, A, I, I_A are collinear. Now we prove that $\angle I_A A I_C = 90°$.

Let $\angle A = a$. Note that $\angle I_C AB = \dfrac{180 - a}{2}$ and $\angle I_A AB = \dfrac{a}{2}$. Summing those two angles up we see $\angle I_A A I_C = 90°$.

Doing the same for I_B and I_C yields that I is the orthocenter of $\triangle I_A I_B I_C$ (since they all pass through I). Since A, B, C are the feet of the altitudes $\triangle ABC$

is indeed the orthic triangle of $\triangle I_A I_B I_C$. $\square$

Problem 2.25

Let $ABCD$ be a quadrilateral. If a circle can be inscribed in it, prove that $AB + CD = BC + DA$.

Solution

Let the tangency points of AB, BC, CD, AD be W, X, Y, Z. Now we find $AB+CD$ by using the break-up of line segments and the two tangents to a circle theorem.

$$AB+CD = (AW+WB)+(CY+YD) = (AZ+BX)+(CX+DZ) = (AZ+DZ)+(BX+CX) = DA+BC,$$

as desired. $\square$

Problem 2.26

An acute-angled triangle ABC is given in the plane. The circle with diameter AB intersects altitude CC', and its extension at points M and N, and the circle with diameter AC intersects altitude BB' and its extensions at P and Q. Prove that the points M, N, P, Q lie on a common circle.

Solution

Let A' be the foot of the altitude from A to BC, and note that A' lies on both of the circles with diameter AB and with diameter AC, since $\angle AA'B = \angle AA'C = 90°$. By the Radical Axis of Intersecting Circles theorem (theorem 2.9 in book), we know that M, N, P, Q are concyclic. $\square$

1 Solution 2

Denote the circle with radius AB as ω_1, and the circle with radius AC as ω_2. Let A' be the foot of the altitude from A to BC. By the Radical Axis Theorem, we know that AA' is the radical axis of ω_1 and ω_2. Since $\angle AA'B = \angle AA'C = 90°$, H is the orthocenter of $\triangle ABC$, and therefore H lies on AA' (one of the altitudes). Thus, H is on the radical axis. We now have $\mathrm{Pow}_{\omega_1}(H) = \mathrm{Pow}_{\omega_2}(H)$. That means $HQ \cdot HP = HM \cdot HN$ so N, M, P, Q are indeed concyclic by the converse of power of a point theorem.

Problem 2.27

Given a segment AB in the plane, choose on it a point M different from A and B. Two equilateral triangles AMC and BMD in the plane are constructed on the same side of segment AB. The circumcircles of the two triangles intersect at point M and another point N.

(a) Prove that AD and BC pass through point N.

(b) Prove that no matter where one chooses the point M along segment AB, all lines MN will pass through some fixed point K in the plane.

Solution

a) **Proof 1:** Let AD intersect (BMD) at N'. Since quadrilateral $BMN'D$ is cyclic, $\angle MN'D = 180° - \angle MBD = 120°$. Since quadrilateral $BDN'M$ is cyclic, $\angle AN'M = 60° = \angle ACM$, so N' lies on (AMC), and because there are exactly two intersections points of (AMC) and (BMD) and M is already one, we know that $N' = N$. So, N lies on AD. Similarly, N lies on BC as well.

Proof 2: Note that

$$\angle CNA = \angle CMA = 60° = \angle DMB = \angle DNB$$

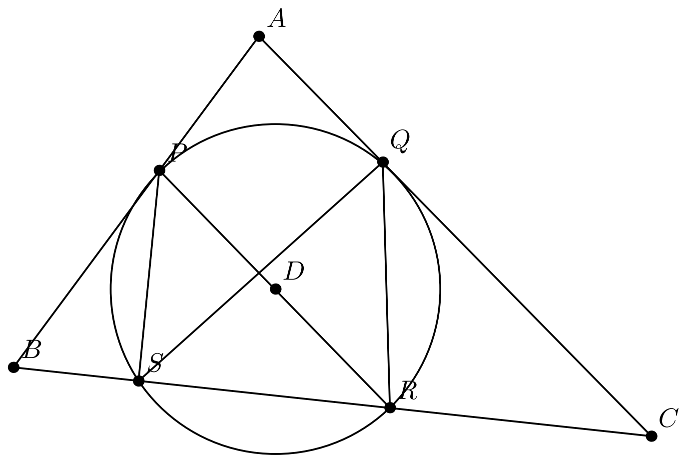
Thus, $N = BC \cap AD$.

b) Let K be the point on the opposite side of AB from C, D such that $\triangle AKB$ is equilateral. Let O_1 be the center of (ACM) and let O_2 be the center of (BMD). Notice that $\angle KAO_1 = \angle KAB + \angle BAO_1 = 90°$, and similarly $\angle KBO_2 = 90°$. Thus AK is tangent to (AMC) and BK is tangent to (BMD). Finally, $\mathbf{Pow}_{(AMC)}K = AK^2 = BK^2 = \mathbf{Pow}_{(BMD)}K$, so K lies on the radical axis of the two circles, which is MN. This proves that all lines MN, irrespective of where one chooses the point M on segment AB, will pass through K.

Problem 2.28

Given a triangle ABC, let P and Q be points on segments AB and AC, respectively, such that $AP = AQ$. Let S and R be distinct points on segment BC such that S lies between B and R, $\angle BPS = \angle PRS$, and $\angle CQR = \angle QSR$. Prove that P, Q, R and S are concyclic

Solution



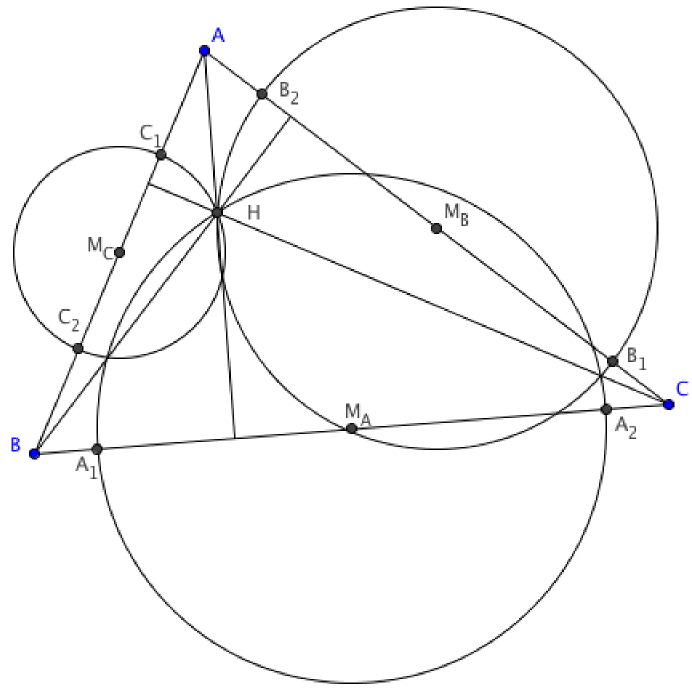
Notice that $\angle CQR = \angle QSC$ and $\angle QCR = \angle QCS$. Thus, $\triangle CSQ \sim \triangle CQR$ which means $\frac{CS}{CQ} = \frac{CQ}{CR}$, or $CQ^2 = CR \cdot CS$. Now consider (QRS). Notice that, since C, R, S are collinear, $CQ^2 = CR \cdot CS = \mathbf{Pow}_{(QRS)}C$, which means CQ is tangent to (QRS), and since C, Q, A are collinear, AQ is tangent to (QRS). Similarly, AP is tangent to (PRS).

Assume for the sake of contradiction that (QRS) and (PRS) are distinct. Then their radical axis is line RS, or line BC. Notice that $\mathbf{Pow}_{(QRS)}A = AQ^2 = AP^2 = \mathbf{Pow}_{(PRS)}A$, so A lies on BC, a contradiction with the definition of a triangle. Thus (QRS) and (PRS) are the same, so P, Q, R, S are concyclic. $\square$

Problem 2.29

An acute-angled triangle ABC has orthocentre H. The circle passing through H with centre the midpoint of BC intersects the line BC at A_1 and A_2. Similarly, the circle passing through H with centre the midpoint of CA intersects the line CA at B_1 and B_2, and the circle passing through H with centre the midpoint of AB intersects the line AB at C_1 and C_2. Show that A_1, A_2, B_1, B_2, C_1, C_2 lie on a circle.

Solution

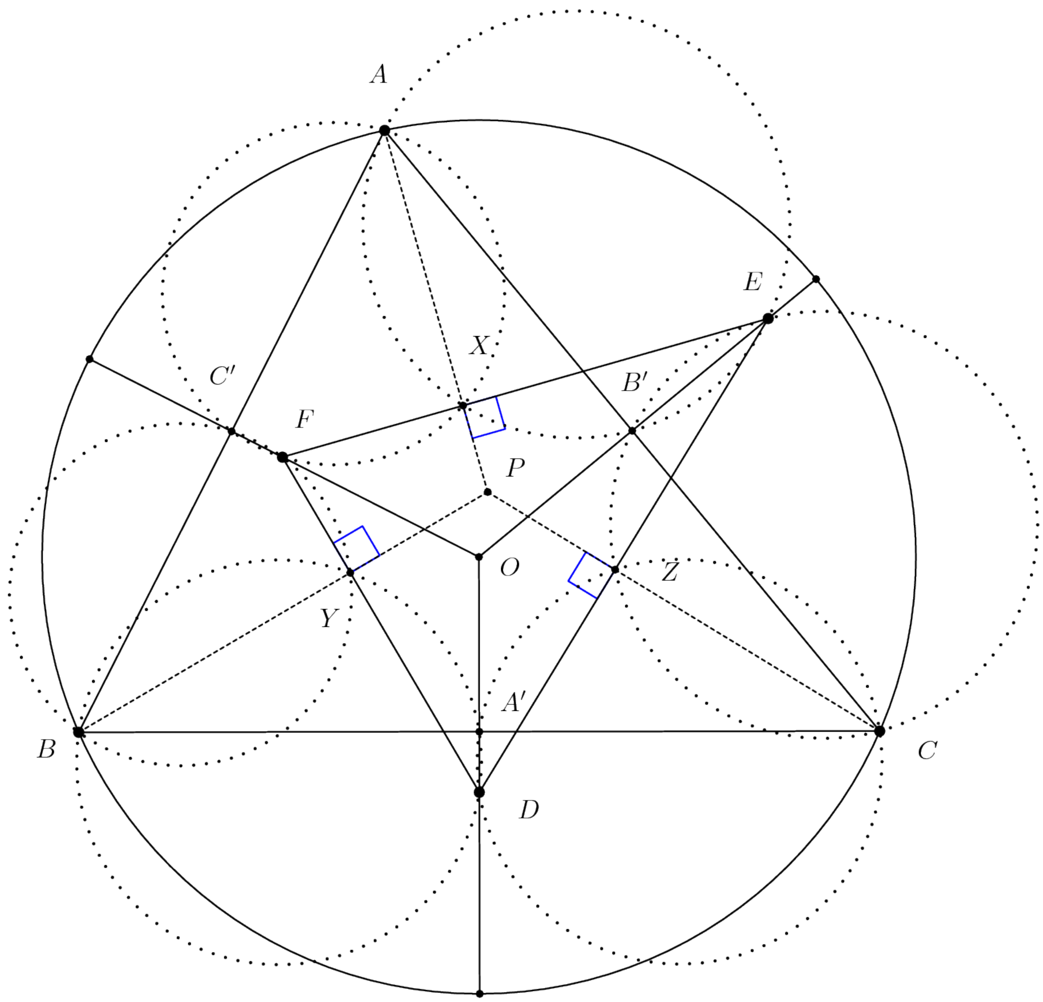


Let M_A, M_B, and M_C be the midpoints of sides BC, CA, and AB, respectively, and let the circles themselves be O_A, O_B, and O_C, respectively. Let the second intersection of O_B, O_C, other than H, be X. Note that $\triangle AM_CM_B \sim \triangle ABC$ by SAS similarity, so $M_BM_C \parallel BC$. We know that $AH \perp BC$, so $AH \perp M_BM_C$ as well, which shows that A, H, X are collinear. It follows that since $\overline{HX}$ is the radical axis of O_B, O_C by the Radical Axis theorem, A lies on that radical axis. We then have $AC_1 \cdot AC_2 = AB_2 \cdot AB_1 \Rightarrow \frac{AB_2}{AC_1} = \frac{AC_2}{AB_1}$. This implies that $\triangle AB_2C_1 \sim \triangle AC_2B_1$, so $\angle AB_2C_1 = \angle AC_2B_1$. Therefore $\angle C_1B_2B_1 = 180° - \angle AB_2C_1 = 180° - \angle AC_2B_1$. This shows that quadrilateral $C_1C_2B_1B_2$ is cyclic. Now note that $(C_1C_2B_1B_2) = (C_1C_2B_2) = (B_1C_2B_2)$, so the circumcenter of $(C_1C_2B_1B_2)$ is the intersection of the perpendicular bisectors of $\overline{C_1C_2}$ and $\overline{B_1B_2}$. However, these are just the perpendicular bisectors of AB and CA, which meet at the circumcenter of $\triangle ABC$, so the circumcenter of $C_1C_2B_1B_2$ is the circumcenter of triangle ABC. Similarly, the circumcenters of $A_1A_2B_1B_2$ and $C_1C_2A_1A_2$ are coincident with the circumcenter O of ABC, from which we know that the circumcenters of $C_1C_2B_1B_2, A_1A_2B_1B_2, C_1C_2A_1A_2$ are the same point, namely O. Furthermore, by the properties of a circumcenter, we know that $OB_1 = OB_2 = OC_1 = OC_2$ and $OA_1 = OA_2 = OB_1 = OB_2$, which shows that they also have the same radius. These two facts combined show that $A_1, A_2, B_1, B_2, C_1, C_2$ are concyclic, as desired. $\square$

Problem 2.30

Let ABC be a triangle. Take points D, E, F on the perpendicular bisectors of BC, CA, AB respectively. Show that the lines through A, B, C perpendicular to EF, FD, DE respectively are concurrent.

Solution

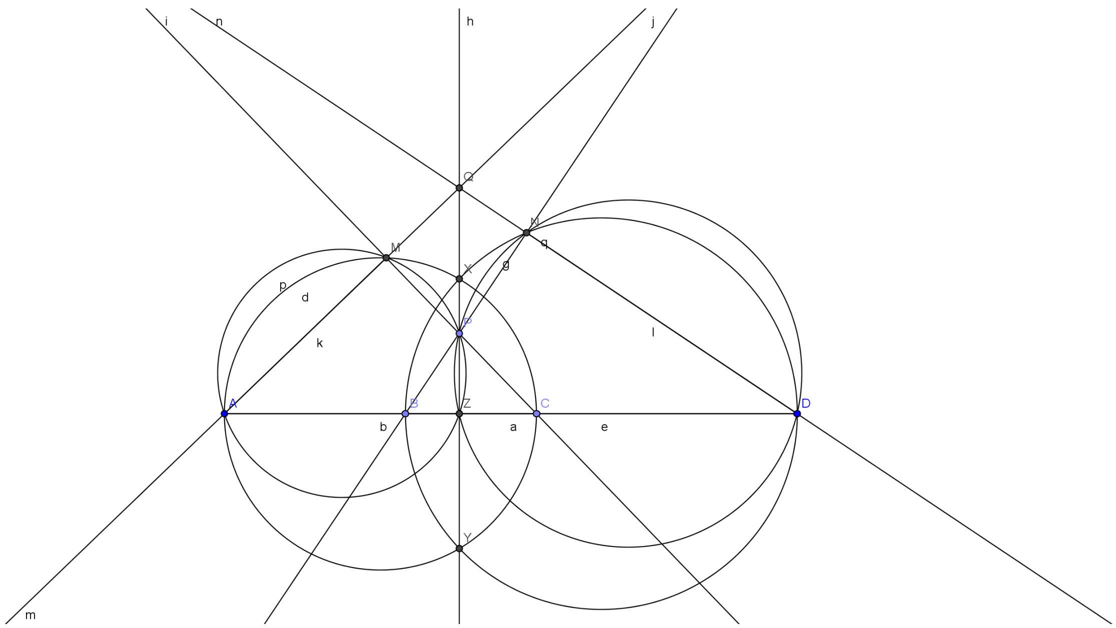


Either D, E, F are collinear, or they are not. If they are not, then consider the circles with centers X, Y, Z and radii BX, CY, and AZ, respectively. Their pairwise radical axes are the desired lines, which must concur at the radical center. If they are, then the perpendiculars from A, B, C to the single line DEF are parallel, meaning that they will meet at the point of infinity. □

Problem 2.31

Let A, B, C, D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at X and Y. The line XY meets BC at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, XY are concurrent.

Solution 1



Note that $\angle AMC = 90^\circ = \angle BNC$. Therefore, quadrilateral $AMND$ is cyclic. If the center of $(AMND)$ is on AD, then $\angle AMD = 90 = \angle AMC$, meaning $C = D$, a contradiction. Thus, by the Radical Axis Theorem (collinear case not possible) on $(AMND), (AMC), (BNC)$, we see that the pairwise radical axes, namely AM, DN, XY are concurrent. $\square$

Solution 2

Note that the line XY is the radical axis of the two circles with diameters AC and BD. Thus, since P is on XY, we have $PN \cdot PB = PM \cdot PC$ and so by the converse of Power of a Point, the quadrilateral $MNCB$ is cyclic. Thus, $90 - \angle MAC = \angle MCA = \angle BNM$. Next, note that

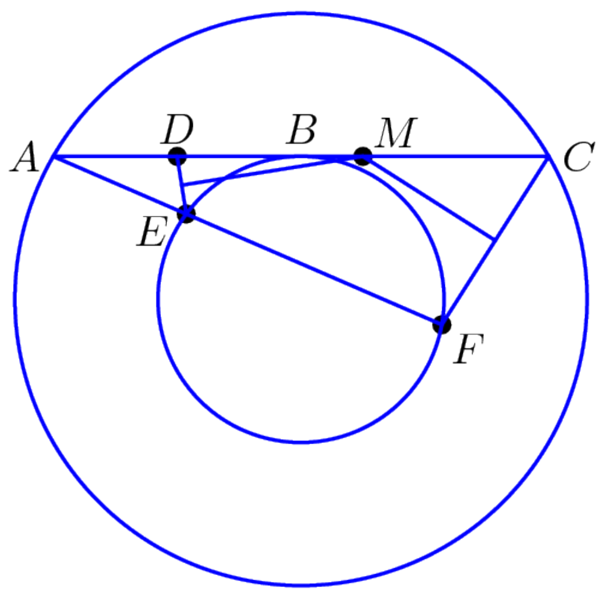
$$\angle MND = \angle MNB + \angle BND = 90 - \angle MAC + 90 = 180 - \angle MAC$$

, so quadrilateral $AMND$ is cyclic. Then, the radical axis of $(AMND)$ and the circle with diameter AC is line AM and the radical axis of O and the circle with diameter BD is line DN. Since the pairwise radical axes of 3 circles are concurrent, we have AM, DN, XY are concurrent as desired. $\square$

Problem 2.32

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be concentric circles, with $\mathcal{C}_2$ in the interior of $\mathcal{C}_1$. From a point A on $\mathcal{C}_1$ one draws the tangent AB to $\mathcal{C}_2$ ($B \in \mathcal{C}_2$). Let C be the second point of intersection of AB and $\mathcal{C}_1$, and let D be the midpoint of AB. A line passing through A intersects $\mathcal{C}_2$ at E and F in such a way that the perpendicular bisectors of DE and CF intersect at a point M on AB. Find, with proof, the ratio AM/MC.

Solution



Note that

$$AF \cdot AE = AB^2 = AB \times AB = 2AD \times \frac{AC}{2} = AD \cdot AC.$$

By the converse of power of a point, we know that C, E, F, D are concyclic and M is the circumcenter since it is the intersection of two of the perpendicular bisectors, and the third must pass through this common point as well. Thus, M is the midpoint of DC. So, now we can find AM, MC in terms of AC. Note that

$$\frac{AM}{MC} = \frac{AC - MC}{MC} = \frac{AC - \frac{CD}{2}}{\frac{CD}{2}} = \frac{AC - \frac{AC-AD}{2}}{\frac{AC-AD}{2}} = \frac{AC - \frac{3AC}{8}}{\frac{3AC}{8}} = \frac{\frac{5AC}{8}}{\frac{3AC}{8}} = \boxed{\frac{5}{3}}$$

□