3 divisors proof

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October 2021

1 Statement

A positive integer has 3 divisors if and only if it is the square of a prime number.

2 Proof

We need to prove the if and the only if condition separately.

The "if" condition is trivial: p^2 for some prime p does have $2+1=3$ divisors.

Now we prove the "only if" condition by contradiction. Consider some arbitrary positive integer $k = p_1^{e_1} p_2^{e_2} p_3^{e_3} \dots p_n^{e_n}$ such that it is **not** the square of a prime number. It will have $(e_1+1)(e_2+1)...(e_k+1)$ divisors, and we want $(e_1+1)(e_2+1)...(e_k+1) = 3$. Since 3 is prime, every term of $(e_1+1), (e_2+1), \dots (e_k+1)$, except one, must be equal to 1. The remaining one term has to be $3-1=2$. In other words, $e_1, e_2, e_3, \dots, e_k = 0$ and $e_a = 2$ for some $1 \leq a \leq k$. Therefore, we have shown that $k = p_1^0 p_2^0 p_3^0 \dots p_a^2 p_{a+1}^0 \dots p_n^0 = p_a^2$. Thus, it is proven that k, the original positive integer, is the square of a prime, as desired. $\square$