Sum of combinations' squares

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1 Statement

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

2 A rephrase

The problem basically wants us to prove that

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n-1}^2 + \binom{n}{n}^2 = \binom{2n}{n}$$

.

3 Proof

We will use a combinatorial argument instead of an algebraic one.

Say we have set $X = \{a_1, \ldots a_n, b_1, \ldots, b_n\}$ and it has $2n$ elements. We will find the number of n-element subsets in two different ways.

In the first way, we see that there are simply $\binom{2n}{n}$ subsets of X with n elements, because we have to choose n out of the total number of elements $2n$.

Now we look at the second way. When we choose n out of $2n$, k of those n are from $A = \{a_1, \ldots a_n\}$ and $n - k$ of those are from $B = \{b_1, \ldots b_n\}$ for some $0 \leq k \leq n$. Using this method, we see that we can divide up picking our n elements from the whole set X into picking from two different sets A and B which are disjoint and where $A \cup B = X$. We can pick k elements from A in $\binom{n}{k}$ ways and pick $n - k$ from B in $\binom{n}{n-k}$ ways. So, the total number of ways using this second method is

$$\sum_{k=0}^{n} \binom{n}{k}^2$$

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Therefore, we can find the number of n-element subsets in two different ways, namely

$$\binom{2n}{n}$$

and

$$\sum_{k=0}^{n} \binom{n}{k}^2$$

. Hence, we know both of them are equal. $\square$