Finite sets definition

Definition: A finite set is one whose distinct elements can be counted by a natural number.

Cardinality motivation

Motivating question: when can we say one set is bigger than another?

Which is bigger?

- The set $\{1, 2, 3\}$ or the set $\{0, 1, 2, 3\}$?
- The set $\{0, \pi, \sqrt{2}\}$ or the set $\{\mathbb{N}, \mathbb{R}, \emptyset\}$?
- The set \mathbb{N} or the set \mathbb{R}^+ ?

Which of the sets above are finite? infinite?

Cardinality rationale for functions

Key idea for cardinality: Counting distinct elements is a way of labelling elements with natural numbers. This is a function! In general, functions let us associate elements of one set with those of another. If the association is "good", we get a correspondence between the elements of the subsets which can relate the sizes of the sets.

Cardinality power sets

Recall: When U is a set, $\mathcal{P}(U) = \{X \mid X \subseteq U\}$

Key idea: For finite sets, the power set of a set has strictly greater size than the set itself. Does this extend to infinite sets?

Definition: For two sets A, B, we use the notation |A| < |B| to denote $(|A| \le |B|) \land \neg (|A| = |B|)$.

 $\mathbb N$ and its power set

Example elements of \mathbb{N}

Example elements of $\mathcal{P}(\mathbb{N})$

Claim: $|\mathbb{N}| \leq |\mathcal{P}(\mathbb{N})|$

Claim: There is an uncountable set. Example:

Proof: By definition of countable, since is not finite, to show is $|\mathbb{N}| \neq |\mathcal{P}(\mathbb{N})|$.

Rewriting using the definition of cardinality, to show is

Towards a proof by universal generalization, consider an arbitrary function $f: \mathbb{N} \to \mathcal{P}(\mathbb{N})$.

To show: f is not a bijection. It's enough to show that f is not onto.

Rewriting using the definition of onto, to show:

$$\neg \forall B \in \mathcal{P}(\mathbb{N}) \ \exists a \in \mathbb{N} \ (f(a) = B)$$

. By logical equivalence, we can write this as an existential statement:

In search of a witness, define the following collection of nonnegative integers:

$$D_f = \{ n \in \mathbb{N} \mid n \notin f(n) \}$$

. By definition of power set, since all elements of D_f are in \mathbb{N} , $D_f \in \mathcal{P}(\mathbb{N})$. It's enough to prove the following Lemma:

Lemma: $\forall a \in \mathbb{N} \ (f(a) \neq D_f)$.

Proof of lemma:

By the Lemma, we have proved that f is not onto, and since f was arbitrary, there are no onto functions from \mathbb{N} to $\mathcal{P}(\mathbb{N})$. QED

Where does D_f come from? The idea is to build a set that would "disagree" with each of the images of f about some element.

$n \in \mathbb{N}$	$f(n) = X_n$	Is $0 \in X_n$?	Is $1 \in X_n$?	Is $2 \in X_n$?	Is $3 \in X_n$?	Is $4 \in X_n$?	 Is $n \in D_f$?
0	$f(0) = X_0$	Y / N	Y / N	Y / N	Y / N	Y / N	 N / Y
1	$f(1) = X_1$	Y / N	Y / N	Y / N	Y / N	Y / N	 N / Y
2	$f(2) = X_2$	Y / N	Y / N	Y / N	Y / N	Y / N	 N / Y
3	$f(3) = X_3$	Y/N	Y/N	Y/N	Y/N	Y/N	 N/Y
4	$f(4) = X_4$	Y/N	Y/N	Y/N	Y/N	Y/N	 N/Y
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Cardinality rationals reals

Comparing \mathbb{Q} and \mathbb{R}

Both \mathbb{Q} and \mathbb{R} have no greatest element.

Both \mathbb{Q} and \mathbb{R} have no least element.

The quantified statement

$$\forall x \forall y (x < y \rightarrow \exists z (x < z < y))$$

is true about both \mathbb{Q} and \mathbb{R} .

Both \mathbb{Q} and \mathbb{R} are infinite. But, \mathbb{Q} is countably infinite whereas \mathbb{R} is uncountable.

The set of real numbers

 $\mathbb{Z}\subsetneq\mathbb{Q}\subsetneq\mathbb{R}$

Order axioms (Rosen Appendix 1):

Reflexivity $\forall a \in \mathbb{R} (a \leq a)$ Antisymmetry $\forall a \in \mathbb{R} \ \forall b \in \mathbb{R} \ (\ (a \leq b \ \land \ b \leq a) \rightarrow (a = b)\)$ Transitivity $\forall a \in \mathbb{R} \ \forall b \in \mathbb{R} \ \forall c \in \mathbb{R} \ (\ (a \leq b \land b \leq c) \ \rightarrow \ (a \leq c)\)$ Trichotomy $\forall a \in \mathbb{R} \ \forall b \in \mathbb{R} \ (\ (a = b \lor b > a \lor a < b)$

Completeness axioms (Rosen Appendix 1):

Least upper bound Nested intervals Every nonempty set of real numbers that is bounded above has a least upper bound For each sequence of intervals $[a_n, b_n]$ where, for each n, $a_n < a_{n+1} < b_{n+1} < b_n$, there is at least one real number x such that, for all n, $a_n \le x \le b_n$.

Each real number $r \in \mathbb{R}$ is described by a function to give better and better approximations

$$x_r: \mathbb{Z}^+ \to \{0,1\}$$
 where $x_r(n) = n^{th}$ bit in binary expansion of r

r	Binary expansion	x_r		
0.1	0.00011001	$x_{0.1}(n) = \begin{cases} 0 & \text{if } n > 1 \text{ and } (n \text{ mod } 4) = 2\\ 0 & \text{if } n = 1 \text{ or if } n > 1 \text{ and } (n \text{ mod } 4) = 3\\ 1 & \text{if } n > 1 \text{ and } (n \text{ mod } 4) = 0\\ 1 & \text{if } n > 1 \text{ and } (n \text{ mod } 4) = 1 \end{cases}$		
		Use linear approximations (tangent lines from calculus) to get algorithm for bounding error of successive operations. Define $x_{\sqrt{2}-1}(n)$ to be n^{th} bit in approximation that has error less than $2^{-(n+1)}$.		

Claim: $\{r \in \mathbb{R} \mid 0 \le r \land r \le 1\}$ is uncountable.

Approach 1: Mimic proof that $\mathcal{P}(\mathbb{Z}^+)$ is uncountable.

Proof: By definition of countable, since $\{r \in \mathbb{R} \mid 0 \le r \land r \le 1\}$ is not finite, **to show** is $|\mathbb{N}| \ne |\{r \in \mathbb{R} \mid 0 \le r \land r \le 1\}|$.

To show is $\forall f: \mathbb{Z}^+ \to \{r \in \mathbb{R} \mid 0 \le r \land r \le 1\}$ (f is not a bijection). Towards a proof by universal generalization, consider an arbitrary function $f: \mathbb{Z}^+ \to \{r \in \mathbb{R} \mid 0 \le r \land r \le 1\}$. **To show**: f is not a bijection. It's enough to show that f is not onto. Rewriting using the definition of onto, **to show**:

$$\exists x \in \{r \in \mathbb{R} \mid 0 \le r \land r \le 1\} \ \forall a \in \mathbb{N} \ (f(a) \ne x)$$

In search of a witness, define the following real number by defining its binary expansion

$$d_f = 0.b_1b_2b_3\cdots$$

where $b_i = 1 - b_{ii}$ where b_{jk} is the coefficient of 2^{-k} in the binary expansion of f(j). Since $d_f \neq f(a)$ for any positive integer a, f is not onto.

Approach 2: Nested closed interval property

To show $f: \mathbb{N} \to \{r \in \mathbb{R} \mid 0 \le r \land r \le 1\}$ is not onto. **Strategy**: Build a sequence of nested closed intervals that each avoid some f(n). Then the real number that is in all of the intervals can't be f(n) for any n. Hence, f is not onto.

Consider the function $f: \mathbb{N} \to \{r \in \mathbb{R} \mid 0 \le r \land r \le 1\}$ with $f(n) = \frac{1+\sin(n)}{2}$

$n \mid$	f(n)	Interval that avoids $f(n)$
0	0.5	
1	0.920735	
2	0.954649	
3	0.570560	
4	0.121599	
:		

Cardinality uncountable examples

• The power set of any countably infinite set is uncountable. For example:

$$\mathcal{P}(\mathbb{N}), \mathcal{P}(\mathbb{Z}^+), \mathcal{P}(\mathbb{Z})$$

are each uncountable.

- The closed interval $\{x \in \mathbb{R} \mid 0 \le x \le 1\}$, any other nonempty closed interval of real numbers whose endpoints are unequal, as well as the related intervals that exclude one or both of the endpoints.
- The set of all real numbers $\mathbb R$ is uncountable and the set of irrational real numbers $\overline{\mathbb Q}$ is uncountable.

¹There's a subtle imprecision in this part of the proof as presented, but it can be fixed.