

Cardinality rationale for functions

Key idea for cardinality: Counting distinct elements is a way of labelling elements with natural numbers. This is a function! In general, functions let us associate elements of one set with those of another. If the association is “*good*”, we get a correspondence between the elements of the subsets which can relate the sizes of the sets.

Musical chairs analogy

Analogy: Musical chairs



People try to sit down when the music stops

Person☼ sits in Chair 1, Person☺ sits in Chair 2,

Person☹ is left standing!

What does this say about the number of chairs and the number of people?

Injective functions visually

Informally, a function being one-to-one means “no duplicate images”.

Cardinality lower bound definition

Definition: For nonempty sets A, B , we say that **the cardinality of A is no bigger than the cardinality of B** , and write $|A| \leq |B|$, to mean there is a one-to-one function with domain A and codomain B . Also, we define $|\emptyset| \leq |B|$ for all sets B .

Injective cardinality musical chairs

In the analogy: The function $sitter : \{Chair1, Chair2\} \rightarrow \{Person\otimes, Person\ominus, Person\odot\}$ given by $sitter(Chair1) = Person\otimes, sitter(Chair2) = Person\odot$, is one-to-one and witnesses that

$$|\{Chair1, Chair2\}| \leq |\{Person\otimes, Person\ominus, Person\odot\}|$$

Cardinality upper bound definition

Definition: For nonempty sets A, B , we say that **the cardinality of A is no smaller than the cardinality of B** , and write $|A| \geq |B|$, to mean there is an onto function with domain A and codomain B . Also, we define $|A| \geq |\emptyset|$ for all sets A .

Surjective cardinality musical chairs

In the analogy: The function $triedToSit : \{Person\otimes, Person\ominus, Person\odot\} \rightarrow \{Chair1, Chair2\}$ given by $triedToSit(Person\otimes) = Chair1, triedToSit(Person\ominus) = Chair2, triedToSit(Person\odot) = Chair2$, is onto and witnesses that

$$|\{Person\otimes, Person\ominus, Person\odot\}| \geq |\{Chair1, Chair2\}|$$

Cardinality properties

Properties of cardinality

$$\begin{aligned} &\forall A (|A| = |A|) \\ &\forall A \forall B (|A| = |B| \rightarrow |B| = |A|) \\ &\forall A \forall B \forall C ((|A| = |B| \wedge |B| = |C|) \rightarrow |A| = |C|) \end{aligned}$$

Extra practice with proofs: Use the definitions of bijections to prove these properties.

Cardinality power sets

Recall: When U is a set, $\mathcal{P}(U) = \{X \mid X \subseteq U\}$

Key idea: For finite sets, the power set of a set has strictly greater size than the set itself. Does this extend to infinite sets?

Definition: For two sets A, B , we use the notation $|A| < |B|$ to denote $(|A| \leq |B|) \wedge \neg(|A| = |B|)$.

$\emptyset = \{\}$	$\mathcal{P}(\emptyset) = \{\emptyset\}$	$ \emptyset < \mathcal{P}(\emptyset) $
$\{1\}$	$\mathcal{P}(\{1\}) = \{\emptyset, \{1\}\}$	$ \{1\} < \mathcal{P}(\{1\}) $
$\{1, 2\}$	$\mathcal{P}(\{1, 2\}) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$	$ \{1, 2\} < \mathcal{P}(\{1, 2\}) $

\mathbb{N} and its power set

Example elements of \mathbb{N}

Example elements of $\mathcal{P}(\mathbb{N})$

Claim: $|\mathbb{N}| \leq |\mathcal{P}(\mathbb{N})|$

Claim: There is an uncountable set. Example: _____

Proof: By definition of countable, since _____ is not finite, **to show** is $|\mathbb{N}| \neq |\mathcal{P}(\mathbb{N})|$.

Rewriting using the definition of cardinality, **to show** is

Towards a proof by universal generalization, consider an arbitrary function $f : \mathbb{N} \rightarrow \mathcal{P}(\mathbb{N})$.

To show: f is not a bijection. It's enough to show that f is not onto.

Rewriting using the definition of onto, **to show:**

$$\neg \forall B \in \mathcal{P}(\mathbb{N}) \exists a \in \mathbb{N} (f(a) = B)$$

. By logical equivalence, we can write this as an existential statement:

In search of a witness, define the following collection of nonnegative integers:

$$D_f = \{n \in \mathbb{N} \mid n \notin f(n)\}$$

. By definition of power set, since all elements of D_f are in \mathbb{N} , $D_f \in \mathcal{P}(\mathbb{N})$. It's enough to prove the following Lemma:

Lemma: $\forall a \in \mathbb{N} (f(a) \neq D_f)$.

Proof of lemma:

By the Lemma, we have proved that f is not onto, and since f was arbitrary, there are no onto functions from \mathbb{N} to $\mathcal{P}(\mathbb{N})$. QED

Where does D_f come from? The idea is to build a set that would “disagree” with each of the images of f about some element.

$n \in \mathbb{N}$	$f(n) = X_n$	Is $0 \in X_n$?	Is $1 \in X_n$?	Is $2 \in X_n$?	Is $3 \in X_n$?	Is $4 \in X_n$?	...	Is $n \in D_f$?
0	$f(0) = X_0$	Y / N	Y / N	Y / N	Y / N	Y / N	...	N / Y
1	$f(1) = X_1$	Y / N	Y / N	Y / N	Y / N	Y / N	...	N / Y
2	$f(2) = X_2$	Y / N	Y / N	Y / N	Y / N	Y / N	...	N / Y
3	$f(3) = X_3$	Y / N	Y / N	Y / N	Y / N	Y / N	...	N / Y
4	$f(4) = X_4$	Y / N	Y / N	Y / N	Y / N	Y / N	...	N / Y
⋮								

Cardinality rationals reals

Comparing \mathbb{Q} and \mathbb{R}

Both \mathbb{Q} and \mathbb{R} have no greatest element.

Both \mathbb{Q} and \mathbb{R} have no least element.

The quantified statement

$$\forall x \forall y (x < y \rightarrow \exists z (x < z < y))$$

is true about both \mathbb{Q} and \mathbb{R} .

Both \mathbb{Q} and \mathbb{R} are infinite. But, \mathbb{Q} is countably infinite whereas \mathbb{R} is uncountable.

The set of real numbers

$$\mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R}$$

Order axioms (Rosen Appendix 1):

Reflexivity	$\forall a \in \mathbb{R} (a \leq a)$
Antisymmetry	$\forall a \in \mathbb{R} \forall b \in \mathbb{R} ((a \leq b \wedge b \leq a) \rightarrow (a = b))$
Transitivity	$\forall a \in \mathbb{R} \forall b \in \mathbb{R} \forall c \in \mathbb{R} ((a \leq b \wedge b \leq c) \rightarrow (a \leq c))$
Trichotomy	$\forall a \in \mathbb{R} \forall b \in \mathbb{R} ((a = b \vee b > a \vee a < b))$

Completeness axioms (Rosen Appendix 1):

Least upper bound	Every nonempty set of real numbers that is bounded above has a least upper bound
Nested intervals	For each sequence of intervals $[a_n, b_n]$ where, for each n , $a_n < a_{n+1} < b_{n+1} < b_n$, there is at least one real number x such that, for all n , $a_n \leq x \leq b_n$.

Each real number $r \in \mathbb{R}$ is described by a function to give better and better approximations

$$x_r : \mathbb{Z}^+ \rightarrow \{0, 1\} \quad \text{where } x_r(n) = n^{\text{th}} \text{ bit in binary expansion of } r$$

r	Binary expansion	x_r
0.1	0.00011001...	$x_{0.1}(n) = \begin{cases} 0 & \text{if } n > 1 \text{ and } (n \bmod 4) = 2 \\ 0 & \text{if } n = 1 \text{ or if } n > 1 \text{ and } (n \bmod 4) = 3 \\ 1 & \text{if } n > 1 \text{ and } (n \bmod 4) = 0 \\ 1 & \text{if } n > 1 \text{ and } (n \bmod 4) = 1 \end{cases}$
$\sqrt{2} - 1 = 0.4142135\dots$	0.01101010...	Use linear approximations (tangent lines from calculus) to get algorithm for bounding error of successive operations. Define $x_{\sqrt{2}-1}(n)$ to be n^{th} bit in approximation that has error less than $2^{-(n+1)}$.

Claim: $\{r \in \mathbb{R} \mid 0 \leq r \wedge r \leq 1\}$ is uncountable.

Approach 1: Mimic proof that $\mathcal{P}(\mathbb{Z}^+)$ is uncountable.

Proof: By definition of countable, since $\{r \in \mathbb{R} \mid 0 \leq r \wedge r \leq 1\}$ is not finite, **to show** is $|\mathbb{N}| \neq |\{r \in \mathbb{R} \mid 0 \leq r \wedge r \leq 1\}|$.

To show is $\forall f : \mathbb{Z}^+ \rightarrow \{r \in \mathbb{R} \mid 0 \leq r \wedge r \leq 1\}$ (f is not a bijection) . Towards a proof by universal generalization, consider an arbitrary function $f : \mathbb{Z}^+ \rightarrow \{r \in \mathbb{R} \mid 0 \leq r \wedge r \leq 1\}$. **To show:** f is not a bijection. It's enough to show that f is not onto. Rewriting using the definition of onto, **to show:**

$$\exists x \in \{r \in \mathbb{R} \mid 0 \leq r \wedge r \leq 1\} \forall a \in \mathbb{N} (f(a) \neq x)$$

In search of a witness, define the following real number by defining its binary expansion

$$d_f = 0.b_1b_2b_3 \dots$$

where $b_i = 1 - b_{ii}$ where b_{jk} is the coefficient of 2^{-k} in the binary expansion of $f(j)$. Since¹ $d_f \neq f(a)$ for any positive integer a , f is not onto.

Approach 2: Nested closed interval property

To show $f : \mathbb{N} \rightarrow \{r \in \mathbb{R} \mid 0 \leq r \wedge r \leq 1\}$ is not onto. **Strategy:** Build a sequence of nested closed intervals that each avoid some $f(n)$. Then the real number that is in all of the intervals can't be $f(n)$ for any n . Hence, f is not onto.

Consider the function $f : \mathbb{N} \rightarrow \{r \in \mathbb{R} \mid 0 \leq r \wedge r \leq 1\}$ with $f(n) = \frac{1+\sin(n)}{2}$

n	$f(n)$	Interval that avoids $f(n)$
0	0.5	
1	0.920735...	
2	0.954649...	
3	0.570560...	
4	0.121599...	
\vdots		

¹There's a subtle imprecision in this part of the proof as presented, but it can be fixed.