

Defining functions more examples

Let's practice with functions related to some of our applications so far.

Recall: We model user ratings of the collection of the four movies Dune, Oppenheimer, Barbie, Nimona as the set $\{-1, 0, 1\}^4 \times \{-1, 0, 1\}^4$. One function that compares pairs of ratings is

$$d_0 : \{-1, 0, 1\}^4 \times \{-1, 0, 1\}^4 \rightarrow \mathbb{R}$$

given by

$$d_0((x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2 + (x_4 - y_4)^2}$$

Notice: any ordered pair of ratings is an okay input to d_0 .

Notice: there are (at most)

$$(3 \cdot 3 \cdot 3 \cdot 3) \cdot (3 \cdot 3 \cdot 3 \cdot 3) = 3^8 = 6561$$

many pairs of ratings. There are therefore lots and lots of real numbers that are not the output of d_0 .

Recall: RNA is made up of strands of four different bases that encode genomic information in specific ways. The bases are elements of the set $B = \{\mathbf{A}, \mathbf{C}, \mathbf{U}, \mathbf{G}\}$. The set of RNA strands S is defined (recursively) by:

$$\begin{array}{ll} \text{Basis Step:} & \mathbf{A} \in S, \mathbf{C} \in S, \mathbf{U} \in S, \mathbf{G} \in S \\ \text{Recursive Step:} & \text{If } s \in S \text{ and } b \in B, \text{ then } sb \in S \end{array}$$

where sb is string concatenation.

Pro-tip: informal definitions sometime use \dots to indicate “continue the pattern”. Often, to make this pattern precise we use recursive definitions.

Name	Domain	Codomain	Rule	Example
<i>rnalen</i>	S	\mathbb{Z}^+	<div> <div>Basis Step:</div> <div>If $b \in B$ then $rnalen(b) = 1$</div> <div>Recursive Step:</div> <div>If $s \in S$ and $b \in B$, then</div> <div>$rnalen(sb) = 1 + rnalen(s)$</div> </div>	<div> $rnalen(\mathbf{AC}) \overset{\text{rec step}}{=} 1 + rnalen(\mathbf{A})$ $\overset{\text{basis step}}{=} 1 + 1 = 2$ </div>
<i>basecount</i>	$S \times B$	\mathbb{N}	<div> <div>Basis Step:</div> <div>If $b_1 \in B, b_2 \in B$ then</div> <div>$basecount((b_1, b_2)) =$</div> <div> $\begin{cases} 1 & \text{when } b_1 = b_2 \\ 0 & \text{when } b_1 \neq b_2 \end{cases}$ </div> <div>Recursive Step:</div> <div>If $s \in S, b_1 \in B, b_2 \in B$</div> <div>$basecount((sb_1, b_2)) =$</div> <div> $\begin{cases} 1 + basecount((s, b_2)) & \text{when } b_1 = b_2 \\ basecount((s, b_2)) & \text{when } b_1 \neq b_2 \end{cases}$ </div> </div>	<div> $basecount((\mathbf{ACU}, \mathbf{C})) =$ </div>
“2 to the power of”	\mathbb{N}	\mathbb{N}	<div> <div>Basis Step:</div> <div>$2^0 = 1$</div> <div>Recursive Step:</div> <div>If $n \in \mathbb{N}, 2^{n+1} =$</div> </div>	
“ b to the power of i ”	$\mathbb{Z}^+ \times \mathbb{N}$	\mathbb{N}	<div> <div>Basis Step:</div> <div>$b^0 = 1$</div> <div>Recursive Step:</div> <div>If $i \in \mathbb{N}, b^{i+1} = b \cdot b^i$</div> </div>	

$2^0 = 1$
 $2^1 = 2$
 $2^2 = 4$
 $2^3 = 8$
 $2^4 = 16$
 $2^5 = 32$
 $2^6 = 64$
 $2^7 = 128$
 $2^8 = 256$
 $2^9 = 512$
 $2^{10} = 1024$

Definitions set prereqs

Term	Notation	Example(s)	We say in English ...
all reals	\mathbb{R}		The (set of all) real numbers (numbers on the number line)
all integers	\mathbb{Z}		The (set of all) integers (whole numbers including negatives, zero, and positives)
all positive integers	\mathbb{Z}^+		The (set of all) strictly positive integers
all natural numbers	\mathbb{N}		The (set of all) natural numbers. Note: we use the convention that 0 is a natural number.

Definitions functions prereqs

Term	Notation	Example(s)	We say in English ...
sequence	x_1, \dots, x_n		A sequence x_1 to x_n
summation	$\sum_{i=1}^n x_i$ or $\sum_{i=1}^n x_i$		The sum of the terms of the sequence x_1 to x_n
piecewise definition	rule	$f(x) = \begin{cases} \text{rule 1 for } x & \text{when COND 1} \\ \text{rule 2 for } x & \text{when COND 2} \end{cases}$	Define f of x to be the result of applying rule 1 to x when condition COND 1 is true and the result of applying rule 2 to x when condition COND 2 is true. This can be generalized to having more than two conditions (or cases).
function application		$f(7)$ $f(z)$ $f(g(z))$	f of 7 or f applied to 7 or the image of 7 under f f of z or f applied to z or the image of z under f f of g of z or f applied to the result of g applied to z
absolute value	$ -3 $		The absolute value of -3
square root	$\sqrt{9}$		The non-negative square root of 9

Pro-tip: the meaning of two vertical lines $| \quad |$ depends on the data-types of what's between the lines. For example, when placed around a number, the two vertical lines represent absolute value. We've seen a single vertical line $|$ used as part of set builder definitions to represent "such that". Again, this is (one of the many reasons) why is it very important to declare the data-type of variables before we use them.

Defining functions

New! Defining functions A function is defined by its (1) domain, (2) codomain, and (3) rule assigning each element in the domain exactly one element in the codomain.

The domain and codomain are nonempty sets.

The rule can be depicted as a table, formula, piecewise definition, or English description.

The notation is

“Let the function $\text{FUNCTION-NAME}: \text{DOMAIN} \rightarrow \text{CODOMAIN}$ be given by
 $\text{FUNCTION-NAME}(x) = \dots$ for every $x \in \text{DOMAIN}$ ”.

or

“Consider the function $\text{FUNCTION-NAME}: \text{DOMAIN} \rightarrow \text{CODOMAIN}$ defined as
 $\text{FUNCTION-NAME}(x) = \dots$ for every $x \in \text{DOMAIN}$ ”.

Example: The absolute value function

Domain

Codomain

Rule

Defining functions ratings

Recall our representation of Netflix users' ratings of movies as n -tuples, where n is the number of movies in the database. Each component of the n -tuple is -1 (didn't like the movie), 0 (neutral rating or didn't watch the movie), or 1 (liked the movie).

Consider the ratings $P_1 = (-1, 0, 1, 0)$, $P_2 = (1, 1, -1, 0)$, $P_3 = (1, 1, 1, 0)$, $P_4 = (0, -1, 1, 0)$

Which of P_1 , P_2 , P_3 has movie preferences most similar to P_4 ?

One approach to answer this question: use **functions** to quantify difference among user preferences.

For example, consider the function $d_0 : \{-1, 0, 1\}^4 \times \{-1, 0, 1\}^4 \rightarrow \mathbb{R}$ given by

$$d_0((x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2 + (x_4 - y_4)^2}$$

Defining functions recursively

When the domain of a function is a *recursively defined set*, the rule assigning images to domain elements (outputs) can also be defined recursively.

Recall: The set of RNA strands S is defined (recursively) by:

$$\begin{array}{ll} \text{Basis Step:} & \mathbf{A} \in S, \mathbf{C} \in S, \mathbf{U} \in S, \mathbf{G} \in S \\ \text{Recursive Step:} & \text{If } s \in S \text{ and } b \in B, \text{ then } sb \in S \end{array}$$

where sb is string concatenation.

Definition (Of a function, recursively) A function $rnalen$ that computes the length of RNA strands in S is defined by:

$$\begin{array}{llll} & & rnalen : S & \rightarrow \mathbb{Z}^+ \\ \text{Basis Step:} & \text{If } b \in B \text{ then} & rnalen(b) & = 1 \\ \text{Recursive Step:} & \text{If } s \in S \text{ and } b \in B, \text{ then} & rnalen(sb) & = 1 + rnalen(s) \end{array}$$

The domain of $rnalen$ is

The codomain of $rnalen$ is

Example function application:

$$rnalen(\mathbf{ACU}) =$$

Example: A function $basecount$ that computes the number of a given base b appearing in a RNA strand s is defined recursively:

$$\begin{array}{llll} & & basecount : S \times B & \rightarrow \mathbb{N} \\ \text{Basis Step:} & \text{If } b_1 \in B, b_2 \in B & basecount((b_1, b_2)) & = \begin{cases} 1 & \text{when } b_1 = b_2 \\ 0 & \text{when } b_1 \neq b_2 \end{cases} \\ \text{Recursive Step:} & \text{If } s \in S, b_1 \in B, b_2 \in B & basecount((sb_1, b_2)) & = \begin{cases} 1 + basecount((s, b_2)) & \text{when } b_1 = b_2 \\ basecount((s, b_2)) & \text{when } b_1 \neq b_2 \end{cases} \end{array}$$