

## Strong induction making change proof idea

Suppose we had postage stamps worth 5 cents and 3 cents. Which number of cents can we form using these stamps? In other words, which postage can we pay?

11?

15?

4?

$$\begin{aligned} &CanPay(0) \wedge \neg CanPay(1) \wedge \neg CanPay(2) \wedge \\ &CanPay(3) \wedge \neg CanPay(4) \wedge CanPay(5) \wedge CanPay(6) \\ &\neg CanPay(7) \wedge \forall n \in \mathbb{Z}^{\geq 8} CanPay(n) \end{aligned}$$

where the predicate  $CanPay$  with domain  $\mathbb{N}$  is

$$CanPay(n) = \exists x \in \mathbb{N} \exists y \in \mathbb{N} (5x + 3y = n)$$

**Proof** (idea): First, explicitly give witnesses or general arguments for postages between 0 and 7. To prove the universal claim, we can use mathematical induction or strong induction.

*Approach 1, mathematical induction:* if we have stamps that add up to  $n$  cents, need to use them (and others) to give  $n + 1$  cents. How do we get 1 cent with just 3-cent and 5-cent stamps?

Either take away a 5-cent stamps and add two 3-cent stamps,

or take away three 3-cent stamps and add two 5-cent stamps.

The details of this proof by mathematical induction are making sure we have enough stamps to use one of these approaches.

*Approach 2, strong induction:* assuming we know how to make postage for **all** smaller values (greater than or equal to 8), when we need to make  $n + 1$  cents, add one 3 cent stamp to however we make  $(n + 1) - 3$  cents.

The details of this proof by strong induction are making sure we stay in the domain of the universal when applying the induction hypothesis.

## Strong induction nim

Consider the following game: two players start with two (equal) piles of jellybeans in front of them. They take turns removing any positive integer number of jellybeans at a time from one of two piles in front of them in turns.

The player who removes the last jellybean wins the game.

Which player (if any) has a strategy to guarantee to win the game?

Try out some games, starting with 1 jellybean in each pile, then 2 jellybeans in each pile, then 3 jellybeans in each pile. Who wins in each game?

Notice that reasoning about the strategy for the 1 jellybean game is easier than about the strategy for the 2 jellybean game.

*Formulate a winning strategy by working to transform the game to a simpler one we know we can win.*

*Player 2's Strategy:* Take the same number of jellybeans that Player 1 did, but from the opposite pile.

*Why is this a good idea:* If Player 2 plays this strategy, at the next turn Player 1 faces a game with the same setup as the original, just with fewer jellybeans in the two piles. Then Player 2 can keep playing this strategy to win.

**Claim:** Player 2's strategy guarantees they will win the game.

**Proof:** By strong induction, we will prove that for all positive integers  $n$ , Player 2's strategy guarantees a win in the game that starts with  $n$  jellybeans in each pile.

**Basis step:** WTS Player 2's strategy guarantees a win when each pile starts with 1 jellybean.

In this case, Player 1 has to take the jellybean from one of the piles (because they can't take from both piles at once). Following the strategy, Player 2 takes the jellybean from the other pile, and wins because this is the last jellybean.

**Recursive step:** Let  $n$  be a positive integer. As the strong induction hypothesis, assume that Player 2's strategy guarantees a win in the games where there are  $1, 2, \dots, n$  many jellybeans in each pile at the start of the game.

WTS that Player 2's strategy guarantees a win in the game where there are  $n + 1$  in the jellybeans in each pile at the start of the game.

In this game, the first move has Player 1 take some number, call it  $c$  (where  $1 \leq c \leq n + 1$ ), of jellybeans from one of the piles. Playing according to their strategy, Player 2 then takes the same number of jellybeans from the other pile.

Notice that  $(c = n + 1) \vee (c \leq n)$ .

*Case 1:* Assume  $c = n + 1$ , then in their first move, Player 2 wins because they take all of the second pile, which includes the last jellybean.

*Case 2:* Assume  $c \leq n$ . Then after Player 2's first move, the two piles have an equal number of jellybeans. The number of jellybeans in each pile is

$$(n + 1) - c$$

and, since  $1 \leq c \leq n$ , this number is between 1 and  $n$ . Thus, at this stage of the game, the game appears identical to a new game where the two piles have an equal number of jellybeans between 1 and  $n$ . Thus, the strong induction hypothesis applies, and Player 2's strategy guarantees they win.