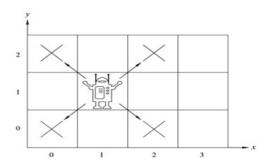
## Structural induction example robot grid



**Theorem**: A robot on an infinite 2-dimensional integer grid starts at (0,0) and at each step moves to diagonally adjacent grid point. This robot can / cannot (*circle one*) reach (1,0).

**Definition** The set of positions the robot can visit *Pos* is defined by:

Basis Step:  $(0,0) \in Pos$ 

Recursive Step: If  $(x, y) \in Pos$ , then

are also in *Pos* 

Example elements of Pos are:

**Lemma**:  $\forall (x,y) \in Pos \ (x+y \text{ is an even integer})$ 

Why are we calling this a lemma?

Proof of theorem using lemma: To show is  $(1,0) \notin Pos$ . Rewriting the lemma to explicitly restrict the domain of the universal, we have  $\forall (x,y) \ ((x,y) \in Pos \rightarrow (x+y) \text{ is an even integer})$ . Since the universal is true,  $((1,0) \in Pos \rightarrow (1+0) \text{ is an even integer})$  is a true statement. Evaluating the conclusion of this conditional statement: By definition of long division, since  $1 = 0 \cdot 2 + 1$  (where  $0 \in \mathbb{Z}$  and  $1 \in \mathbb{Z}$  and  $0 \le 1 < 2$  mean that 0 is the quotient and 1 is the remainder), 1  $\operatorname{mod} 2 = 1$  which is not 0 so the conclusion is false. A true conditional with a false conclusion must have a false hypothesis:  $(1,0) \notin Pos$ , QED.  $\square$ 

Proof of lemma by structural induction:

#### Basis Step:

**Recursive Step**: Consider arbitrary  $(x, y) \in Pos$ . To show is:

 $(x+y \text{ is an even integer}) \rightarrow (\text{sum of coordinates of next position is even integer})$ 

Assume as the induction hypothesis, IH that:

## Fundamental theorem proof

**Theorem**: Every positive integer *greater than 1* is a product of (one or more) primes.

Before we prove, let's try some examples:

```
20 = 100 = 5 = 5
```

**Proof by strong induction**, with b = 2 and j = 0.

Basis step: WTS property is true about 2.

Since 2 is itself prime, it is already written as a product of (one) prime.

Recursive step: Consider an arbitrary integer  $n \geq 2$ . Assume (as the strong induction hypothesis, IH) that the property is true about each of  $2, \ldots, n$ . WTS that the property is true about n + 1: We want to show that n + 1 can be written as a product of primes. Notice that n + 1 is itself prime or it is composite.

Case 1: assume n + 1 is prime and then immediately it is written as a product of (one) prime so we are done.

Case 2: assume that n+1 is composite so there are integers x and y where n+1=xy and each of them is between 2 and n (inclusive). Therefore, the induction hypothesis applies to each of x and y so each of these factors of n+1 can be written as a product of primes. Multiplying these products together, we get a product of primes that gives n+1, as required.

Since both cases give the necessary conclusion, the proof by cases for the recursive step is complete.

## Strong induction making change proof idea

Suppose we had postage stamps worth 5 cents and 3 cents. Which number of cents can we form using these stamps? In other words, which postage can we pay?

11?

15?

4?

$$CanPay(0) \land \neg CanPay(1) \land \neg CanPay(2) \land$$
  
 $CanPay(3) \land \neg CanPay(4) \land CanPay(5) \land CanPay(6)$   
 $\neg CanPay(7) \land \forall n \in \mathbb{Z}^{\geq 8} CanPay(n)$ 

where the predicate CanPay with domain  $\mathbb{N}$  is

$$CanPay(n) = \exists x \in \mathbb{N} \exists y \in \mathbb{N} (5x + 3y = n)$$

**Proof** (idea): First, explicitly give witnesses or general arguments for postages between 0 and 7. To prove the universal claim, we can use mathematical induction or strong induction.

Approach 1, mathematical induction: if we have stamps that add up to n cents, need to use them (and others) to give n + 1 cents. How do we get 1 cent with just 3-cent and 5-cent stamps?

Either take away a 5-cent stamps and add two 3-cent stamps,

or take away three 3-cent stamps and add two 5-cent stamps.

The details of this proof by mathematical induction are making sure we have enough stamps to use one of these approaches.

Approach 2, strong induction: assuming we know how to make postage for all smaller values (greater than or equal to 8), when we need to make n+1 cents, add one 3 cent stamp to however we make (n+1)-3 cents. The details of this proof by strong induction are making sure we stay in the domain of the universal when applying the induction hypothesis.

# Strong induction nim

Consid	er the following	ng game:	two players	start with	two (equal	l) piles of	jellybeans	in front	t of them.	. They
take tu	rns removing	any posi	tive integer	number of	jellybeans	at a time	from one	of two	piles in fi	ront of
them in	n turns.									

The player who removes the last jellybean wins the game.

Which player (if any) has a strategy to guarantee to win the game?

Try out some games, starting with 1 jellybean in each pile, then 2 jellybeans in each pile, then 3 jellybeans in each pile. Who wins in each game?

Notice that reasoning about the strategy for the 1 jellybean game is easier than about the strategy for the 2 jellybean game.

Formulate a winning strategy by working to transform the game to a simpler one we know we can win.

Player 2's Strategy: Take the same number of jellybeans that Player 1 did, but from the opposite pile.

Why is this a good idea: If Player 2 plays this strategy, at the next turn Player 1 faces a game with the same setup as the original, just with fewer jellybeans in the two piles. Then Player 2 can keep playing this strategy to win.

Claim: Player 2's strategy guarantees they will win the game.

**Proof**: By strong induction, we will prove that for all positive integers n, Player 2's strategy guarantees a win in the game that starts with n jellybeans in each pile.

Basis step: WTS Player 2's strategy guarantees a win when each pile starts with 1 jellybean.

In this case, Player 1 has to take the jellybean from one of the piles (because they can't take from both piles at once). Following the strategy, Player 2 takes the jellybean from the other pile, and wins because this is the last jellybean.

**Recursive step**: Let n be a positive integer. As the strong induction hypothesis, assume that Player 2's strategy guarantees a win in the games where there are 1, 2, ..., n many jellybeans in each pile at the start of the game.

WTS that Player 2's strategy guarantees a win in the game where there are n + 1 in the jellybeans in each pile at the start of the game.

In this game, the first move has Player 1 take some number, call it c (where  $1 \le c \le n+1$ ), of jellybeans from one of the piles. Playing according to their strategy, Player 2 then takes the same number of jellybeans from the other pile.

Notice that  $(c = n + 1) \lor (c \le n)$ .

Case 1: Assume c = n + 1, then in their first move, Player 2 wins because they take all of the second pile, which includes the last jellybean.

Case 2: Assume  $c \leq n$ . Then after Player 2's first move, the two piles have an equal number of jellybeans. The number of jellybeans in each pile is

$$(n+1) - c$$

and, since  $1 \le c \le n$ , this number is between 1 and n. Thus, at this stage of the game, the game appears identical to a new game where the two piles have an equal number of jellybeans between 1 and n. Thus, the strong induction hypothesis applies, and Player 2's strategy guarantees they win.