

Strong induction making change proof idea

Suppose we had postage stamps worth 5 cents and 3 cents. Which number of cents can we form using these stamps? In other words, which postage can we pay?

11?

15?

4?

$$\begin{aligned} &CanPay(0) \wedge \neg CanPay(1) \wedge \neg CanPay(2) \wedge \\ &CanPay(3) \wedge \neg CanPay(4) \wedge CanPay(5) \wedge CanPay(6) \\ &\neg CanPay(7) \wedge \forall n \in \mathbb{Z}^{\geq 8} CanPay(n) \end{aligned}$$

where the predicate $CanPay$ with domain \mathbb{N} is

$$CanPay(n) = \exists x \in \mathbb{N} \exists y \in \mathbb{N} (5x + 3y = n)$$

Proof (idea): First, explicitly give witnesses or general arguments for postages between 0 and 7. To prove the universal claim, we can use mathematical induction or strong induction.

Approach 1, mathematical induction: if we have stamps that add up to n cents, need to use them (and others) to give $n + 1$ cents. How do we get 1 cent with just 3-cent and 5-cent stamps?

Either take away a 5-cent stamp and add two 3-cent stamps,

or take away three 3-cent stamps and add two 5-cent stamps.

The details of this proof by mathematical induction are making sure we have enough stamps to use one of these approaches.

Approach 2, strong induction: assuming we know how to make postage for **all** smaller values (greater than or equal to 8), when we need to make $n + 1$ cents, add one 3 cent stamp to however we make $(n + 1) - 3$ cents.

The details of this proof by strong induction are making sure we stay in the domain of the universal when applying the induction hypothesis.