

Algorithm redundancy

Real-life representations are often prone to corruption. Biological codes, like RNA, may mutate naturally¹ and during measurement; cosmic radiation and other ambient noise can flip bits in computer storage². One way to recover from corrupted data is to introduce or exploit redundancy.

Consider the following algorithm to introduce redundancy in a string of 0s and 1s.

Create redundancy by repeating each bit three times

```

1 procedure redun3( $a_{k-1} \cdots a_0$ : a nonempty bitstring)
2 for  $i := 0$  to  $k-1$ 
3    $c_{3i} := a_i$ 
4    $c_{3i+1} := a_i$ 
5    $c_{3i+2} := a_i$ 
6 return  $c_{3k-1} \cdots c_0$ 

```

Decode sequence of bits using majority rule on consecutive three bit sequences

```

1 procedure decode3( $c_{3k-1} \cdots c_0$ : a nonempty bitstring whose length is an integer multiple of 3)
2 for  $i := 0$  to  $k-1$ 
3   if exactly two or three of  $c_{3i}, c_{3i+1}, c_{3i+2}$  are set to 1
4      $a_i := 1$ 
5   else
6      $a_i := 0$ 
7 return  $a_{k-1} \cdots a_0$ 

```

Give a recursive definition of the set of outputs of the *redun3* procedure, *Out*,

Consider the message $m = 0001$ so that the sender calculates $\text{redun3}(m) = \text{redun3}(0001) = 000000000111$.

Introduce ____ errors into the message so that the signal received by the receiver is _____ but the receiver is still able to decode the original message.

Challenge: what is the biggest number of errors you can introduce?

Building a circuit for lines 3-6 in *decode* procedure: given three input bits, we need to determine whether the majority is a 0 or a 1.

c_{3i}	c_{3i+1}	c_{3i+2}	a_i
1	1	1	
1	1	0	
1	0	1	
1	0	0	
0	1	1	
0	1	0	
0	0	1	
0	0	0	

Circuit

¹Mutations of specific RNA codons have been linked to many disorders and cancers.

²This RadioLab podcast episode goes into more detail on bit flips: <https://www.wnycstudios.org/story/bit-flip>

Algorithm rna mutation insertion deletion

Recall that S is defined as the set of all RNA strands, nonempty strings made of the bases in $B = \{\text{A}, \text{U}, \text{G}, \text{C}\}$. We define the functions

$$\text{mutation} : S \times \mathbb{Z}^+ \times B \rightarrow S \qquad \text{insertion} : S \times \mathbb{Z}^+ \times B \rightarrow S$$

$$\text{deletion} : \{s \in S \mid \text{rinalen}(s) > 1\} \times \mathbb{Z}^+ \rightarrow S \qquad \text{with rules}$$

```

1 procedure mutation( $b_1 \cdots b_n$ : a RNA strand,  $k$ : a positive integer,  $b$ : an element of  $B$ )
2 for  $i := 1$  to  $n$ 
3   if  $i = k$ 
4      $c_i := b$ 
5   else
6      $c_i := b_i$ 
7 return  $c_1 \cdots c_n$  {The return value is a RNA strand made of the  $c_i$  values}

```

```

1 procedure insertion( $b_1 \cdots b_n$ : a RNA strand,  $k$ : a positive integer,  $b$ : an element of  $B$ )
2 if  $k > n$ 
3   for  $i := 1$  to  $n$ 
4      $c_i := b_i$ 
5    $c_{n+1} := b$ 
6 else
7   for  $i := 1$  to  $k-1$ 
8      $c_i := b_i$ 
9    $c_k := b$ 
10  for  $i := k+1$  to  $n+1$ 
11     $c_i := b_{i-1}$ 
12 return  $c_1 \cdots c_{n+1}$  {The return value is a RNA strand made of the  $c_i$  values}

```

```

1 procedure deletion( $b_1 \cdots b_n$ : a RNA strand with  $n > 1$ ,  $k$ : a positive integer)
2 if  $k > n$ 
3    $m := n$ 
4   for  $i := 1$  to  $n$ 
5      $c_i := b_i$ 
6 else
7    $m := n-1$ 
8   for  $i := 1$  to  $k-1$ 
9      $c_i := b_i$ 
10  for  $i := k$  to  $n-1$ 
11     $c_i := b_{i+1}$ 
12 return  $c_1 \cdots c_m$  {The return value is a RNA strand made of the  $c_i$  values}

```

Algorithm definition

New! An algorithm is a finite sequence of precise instructions for solving a problem.

Algorithms can be expressed in English or in more formalized descriptions like pseudocode or fully executable programs.

Sometimes, we can define algorithms whose output matches the rule for a function we already care about. Consider the (integer) logarithm function

$$\text{log}_b : \{b \in \mathbb{Z} \mid b > 1\} \times \mathbb{Z}^+ \rightarrow \mathbb{N}$$

defined by

$$\text{log}_b((b,n)) = \text{greatest integer } y \text{ so that } b^y \text{ is less than or equal to } n$$

Calculating integer part of base b logarithm

1

2

3

4

5

6

procedure $\text{log}_b(b,n$: positive integers with $b > 1$)
 $i := 0$
while $n > b - 1$
 $i := i + 1$
 $n := n \text{ div } b$
return i { i holds the integer part of the base b logarithm of n }

Trace this algorithm with inputs $b = 3$ and $n = 17$

	b	n	i	$n > b - 1?$
Initial value	3	17		
After 1 iteration				
After 2 iterations				
After 3 iterations				

Compare: does the output match the rule for the (integer) logarithm function?

Base expansion algorithms

Two algorithms for constructing base b expansion from decimal representation

Most significant first: Start with left-most coefficient of expansion (highest value)

Informally: Build up to the value we need to represent in “greedy” approach, using units determined by base.

Calculating base b expansion, from left

```
1 procedure baseb1( $n, b$ : positive integers with  $b > 1$ )
2    $v := n$ 
3    $k := 1 + \text{output of } \log b \text{ algorithm with inputs } b \text{ and } n$ 
4   for  $i := 1$  to  $k$ 
5      $a_{k-i} := 0$ 
6     while  $v \geq b^{k-i}$ 
7        $a_{k-i} := a_{k-i} + 1$ 
8        $v := v - b^{k-i}$ 
9   return  $(a_{k-1}, \dots, a_0) \{(a_{k-1} \dots a_0)_b \text{ is the base } b \text{ expansion of } n\}$ 
```

Least significant first: Start with right-most coefficient of expansion (lowest value)

Idea: (when $k > 1$)

$$\begin{aligned} n &= a_{k-1}b^{k-1} + \cdots + a_1b + a_0 \\ &= b(a_{k-1}b^{k-2} + \cdots + a_1) + a_0 \end{aligned}$$

so $a_0 = n \bmod b$ and $a_{k-1}b^{k-2} + \cdots + a_1 = n \operatorname{div} b$.

Calculating base b expansion, from right

```
1 procedure baseb2( $n, b$ : positive integers with  $b > 1$ )
2    $q := n$ 
3    $k := 0$ 
4   while  $q \neq 0$ 
5      $a_k := q \bmod b$ 
6      $q := q \operatorname{div} b$ 
7      $k := k + 1$ 
8   return  $(a_{k-1}, \dots, a_0)\{(a_{k-1} \dots a_0)_b \text{ is the base } b \text{ expansion of } n\}$ 
```

Base conversion algorithm

Practice: write an algorithm for converting from base b_1 expansion to base b_2 expansion: