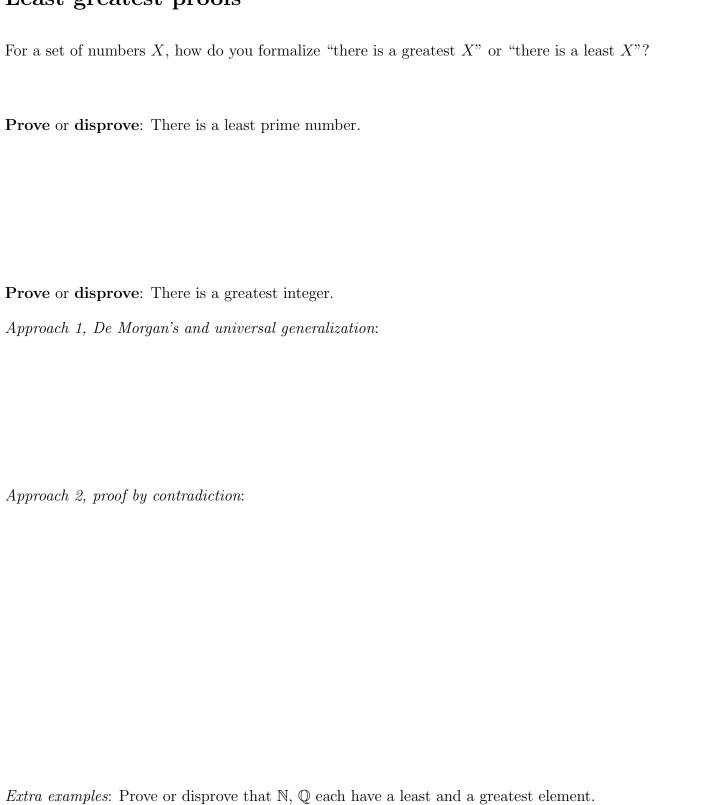
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Gcd definition



Gcd examples

Why do we restrict to the situation where a and b are not both zero?

Calculate gcd((10, 15))

Calculate gcd((10,20))

Gcd basic claims

Claim : For any integers a, b (not both zero), $gcd((a, b)) \ge 1$.

Proof: Show that 1 is a common factor of any two integers, so since the gcd is the greatest common factor it is greater than or equal to any common factor.

Claim: For any positive integers a,b, gcd((a,b) $) \leq a$ and gcd((a,b) $) \leq b$.

Proof Using the definition of gcd and the fact that factors of a positive integer are less than or equal to that integer.

Claim: For any positive integers a, b, if a divides b then gcd((a, b)) = a.

Proof Using previous claim and definition of gcd.

Claim: For any positive integers a, b, c, if there is some integer q such that a = bq + c,

$$\gcd(\ (a,b)\)=\gcd(\ (b,c)\)$$

Proof Prove that any common	$divisor\ of\ a,b\ divid$	es c and that any com	$mon\ divisor\ of\ b,c\ divide$	es a.

Gcd lemma relatively prime

Lemma: For any integers p,q (not both zero), $\gcd\left(\left(\frac{p}{\gcd((p,q))},\frac{q}{\gcd((p,q))}\right)\right)=1$. In other words, can reduce to relatively prime integers by dividing by \gcd .

Proof:

Let x be arbitrary positive integer and assume that x is a factor of each of $\frac{p}{\gcd((p,q))}$ and $\frac{q}{\gcd((p,q))}$. This gives integers α , β such that

$$\alpha x = \frac{p}{\gcd((p,q))}$$
 $\beta x = \frac{q}{\gcd((p,q))}$

Multiplying both sides by the denominator in the RHS:

$$\alpha x \cdot gcd((p,q)) = p$$
 $\beta x \cdot gcd((p,q)) = q$

In other words, $x \cdot gcd(p,q)$ is a common divisor of p,q. By definition of gcd, this means

$$x \cdot gcd((p,q)) \le gcd((p,q))$$

and since $\gcd(\ (p,q)\)$ is positive, this means, $x\leq 1.$

Sets numbers subsets

We have the following subset relationships between sets of numbers:

$$\mathbb{Z}^+ \subsetneq \mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R}$$

Which of the proper subset inclusions above can you prove?

Definitions set prereqs

Term	Notation Example(s)	We say in English
all reals	\mathbb{R}	The (set of all) real numbers (numbers on the number
		line)
all integers	\mathbb{Z}	The (set of all) integers (whole numbers including neg-
		atives, zero, and positives)
all positive integers	\mathbb{Z}^+	The (set of all) strictly positive integers
all natural numbers	N	The (set of all) natural numbers. Note : we use the
		convention that 0 is a natural number.

Defining sets

To define sets:

To define a set using **roster method**, explicitly list its elements. That is, start with { then list elements of the set separated by commas and close with }.

To define a set using **set builder definition**, either form "The set of all x from the universe U such that x is ..." by writing

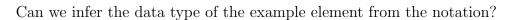
$$\{x \in U \mid ...x...\}$$

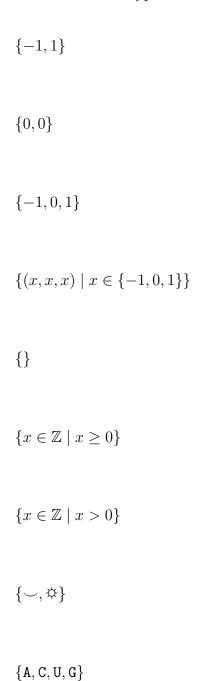
or form "the collection of all outputs of some operation when the input ranges over the universe U" by writing

$$\{...x...\mid x\in U\}$$

We use the symbol \in as "is an element of" to indicate membership in a set.

Example sets: For each of the following, identify whether it's defined using the roster method or set builder notation and give an example element.





{AUG, UAG, UGA, UAA}

Set operations

To define a set we can use the roster method, set builder notation, a recursive definition, and also we can apply a set operation to other sets.

New! Cartesian product of sets and set-wise concatenation of sets of strings

Definition: Let X and Y be sets. The **Cartesian product** of X and Y, denoted $X \times Y$, is the set of all ordered pairs (x, y) where $x \in X$ and $y \in Y$

$$X \times Y = \{(x, y) \mid x \in X \text{ and } y \in Y\}$$

Conventions: (1) Cartesian products can be chained together to result in sets of n-tuples and (2) When we form the Cartesian product of a set with itself $X \times X$ we can denote that set as X^2 , or X^n for the Cartesian product of a set with itself n times for a positive integer n.

Definition: Let X and Y be sets of strings over the same alphabet. The **set-wise concatenation** of X and Y, denoted $X \circ Y$, is the set of all results of string concatenation xy where $x \in X$ and $y \in Y$

$$X \circ Y = \{xy \mid x \in X \text{ and } y \in Y\}$$

Pro-tip: the meaning of writing one element next to another like xy depends on the data-types of x and y. When x and y are strings, the convention is that xy is the result of string concatenation. When x and y are numbers, the convention is that xy is the result of multiplication. This is (one of the many reasons) why is it very important to declare the data-type of variables before we use them.

Fill in the missing entries in the table:

Set	Example elements in this set and their data type:
B	A C G U
	(A,C) (U,U)
$B \times \{-1, 0, 1\}$	
$\{-1,0,1\} \times B$	
	(0, 0, 0)
$\{\mathtt{A},\mathtt{C},\mathtt{G},\mathtt{U}\}\circ\{\mathtt{A},\mathtt{C},\mathtt{G},\mathtt{U}\}$	
	GGGG

Definitions functions prereqs

Term	Notation Example(s)	We say in English
sequence	x_1, \ldots, x_n	A sequence x_1 to x_n
summation	$\sum_{i=1}^{n} x_i \text{ or } \sum_{i=1}^{n} x_i$	The sum of the terms of the sequence x_1 to x_n
piecewise rule definition	$f(x) = \begin{cases} \text{rule 1 for } x & \text{when COND 1} \\ \text{rule 2 for } x & \text{when COND 2} \end{cases}$	Define f of x to be the result of applying rule 1 to x when condition COND 1 is true and the result of applying rule 2 to x when condition COND 2 is true. This can be generalized to having more than two conditions (or cases).
function applica-	f(7)	f of 7 or f applied to 7 or the image of 7 under f
UOII	f(z)	f of z or f applied to z or the image of z under f
	f(g(z))	f of g of z or f applied to the result of g applied to z
absolute value	-3	The absolute value of -3
square root	$\sqrt{9}$	The non-negative square root of 9

Pro-tip: the meaning of two vertical lines | | depends on the data-types of what's between the lines. For example, when placed around a number, the two vertical lines represent absolute value. We've seen a single vertial line | used as part of set builder definitions to represent "such that". Again, this is (one of the many reasons) why is it very important to declare the data-type of variables before we use them.