CS2040C Data Structures and Algorithms

Trees

Outline

- Binary trees
- Implementation
- Binary Tree Traversal
- Binary Search Trees
- STL search algorithms

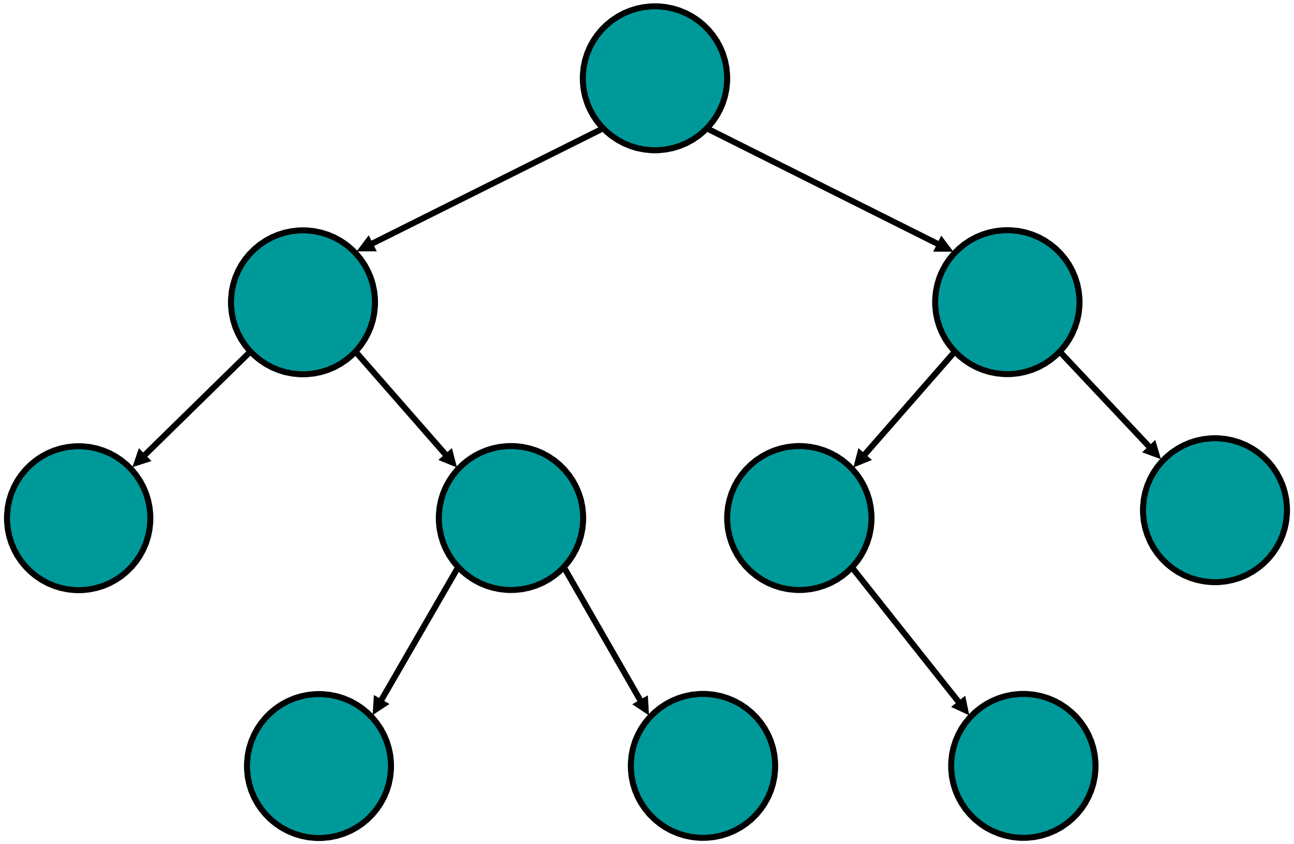
Binary Trees

Each node has at most **2** ordered children

Binary Tree

Each node has at most 2 ordered children

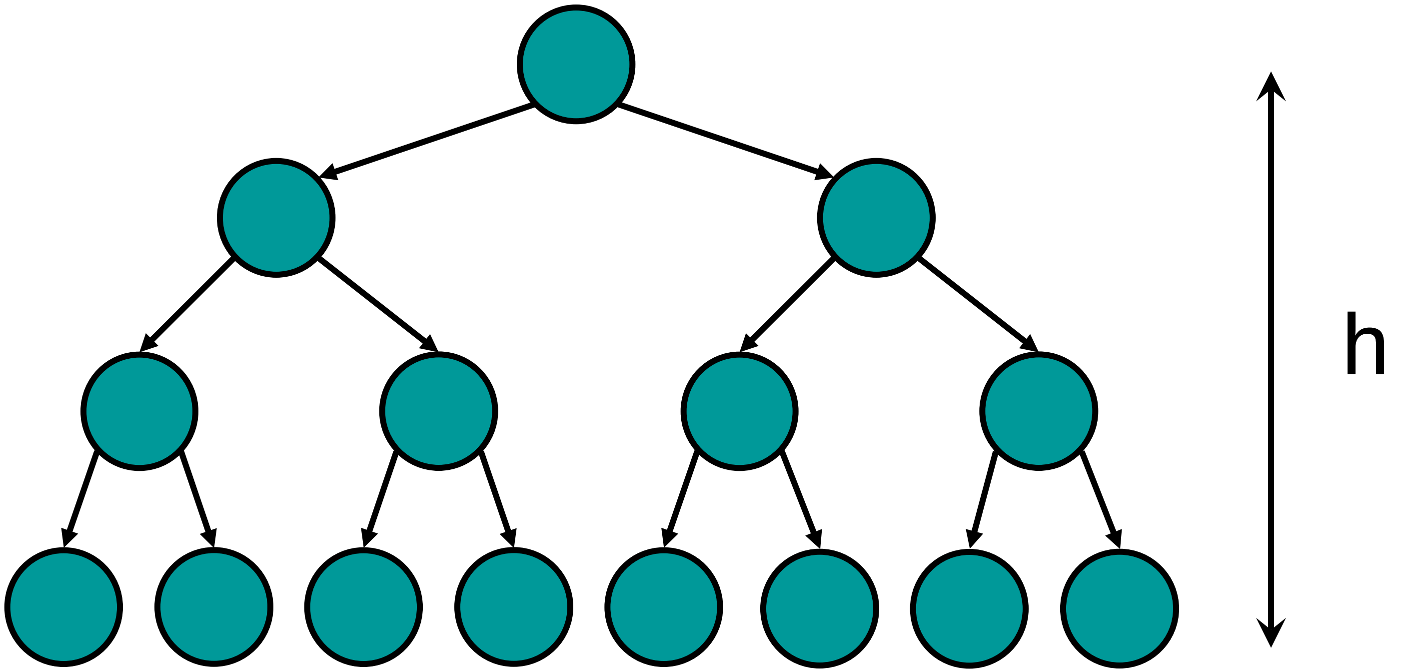
Binary Tree has a recursive structure



Note: a degree of a node is the number of subtrees it has

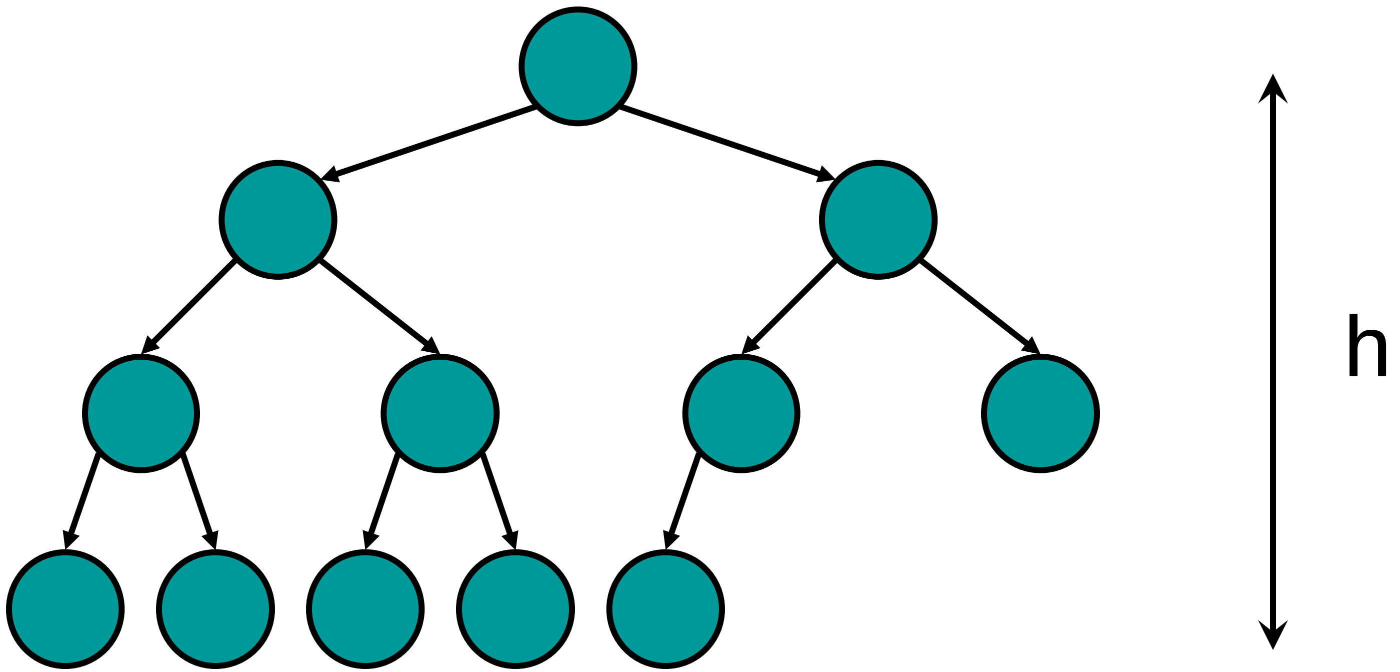
Full Binary Tree

- All nodes at a level < h have two children (where h is the height of the tree)



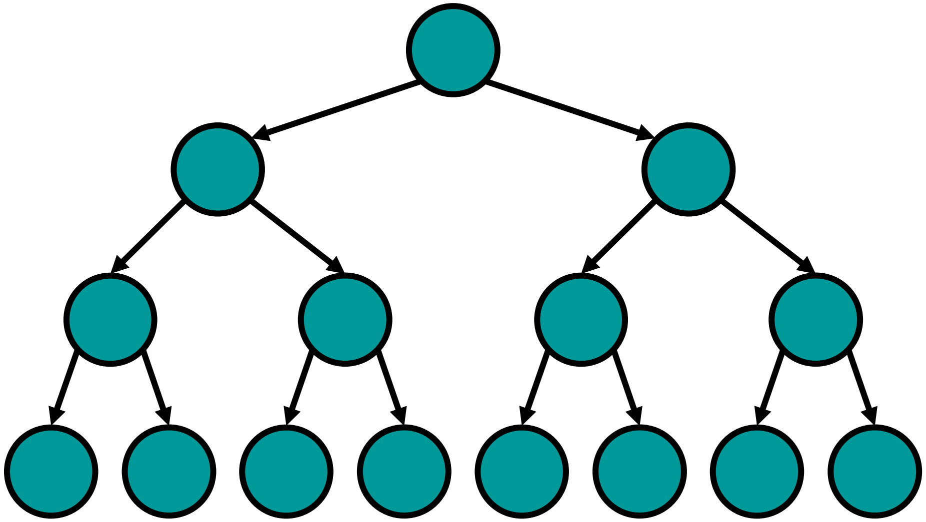
Complete Binary Tree

- Full down to level h-1
- level h filled in from left to right



Full Binary Tree Property

- Number of nodes in a full binary tree of height h is **$2^h - 1$**
- Therefore the height of a full binary tree is O(log N)



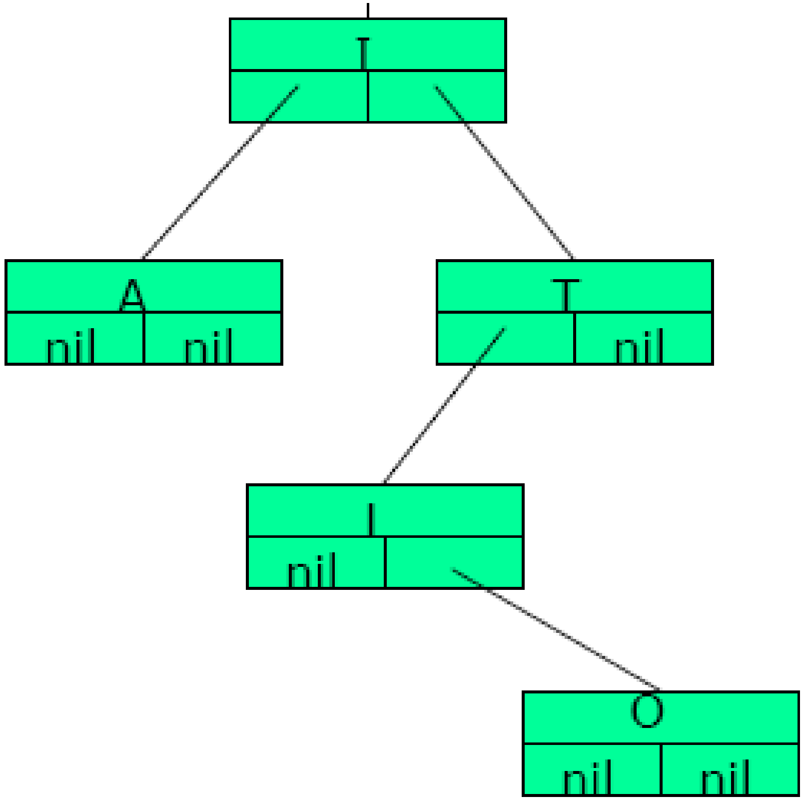
Q: How many nodes in a complete binary tree of height h?

Implementation

A tree can be implemented using **reference based** representation or **array based** representation

Reference Based

```
class TreeNode {
private:
    TreeItemType item;
    TreeNode *left;
    TreeNode *right;
    // More definitions…
    friend class BinaryTree;
};
class BinaryTree {
private:
    TreeNode root;
    // More definitions
};
```



(nil means “does not point to node”)

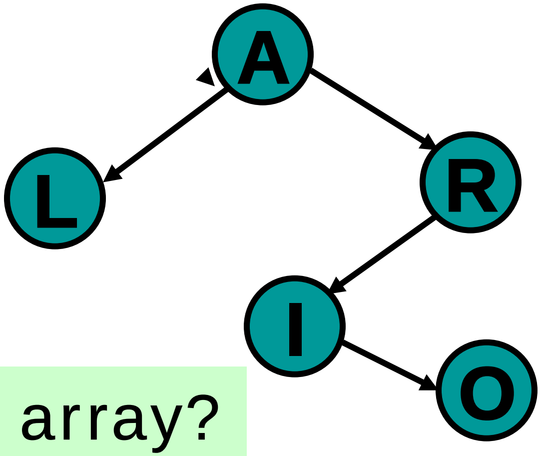
Array Based

```
class TreeNode {
private:
   TreeItemType item;
   int left;
   int right;
   // More definitions…
   friend class BinaryTree;
};
class BinaryTree {
private:
   TreeNode[…] tree;
   int root;
   int free;
};
```

tree[0] tree[1]

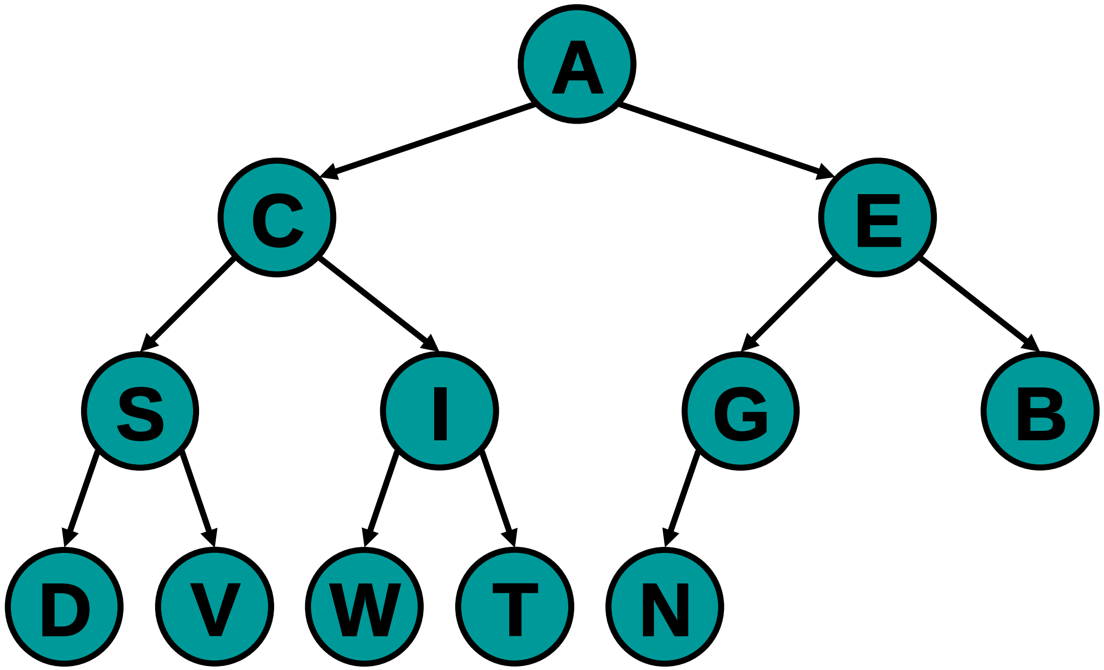
item	L	I	A	R	?	O
left	-1	-1	0	1	-1	-1
right	-1	5	3	-1	-1	-1

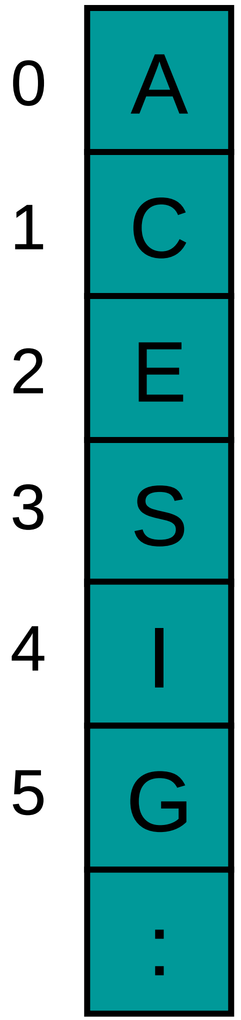
root = 2 free = 4



Q: How to handle free space in the array?

Representing a Complete Tree (using array)

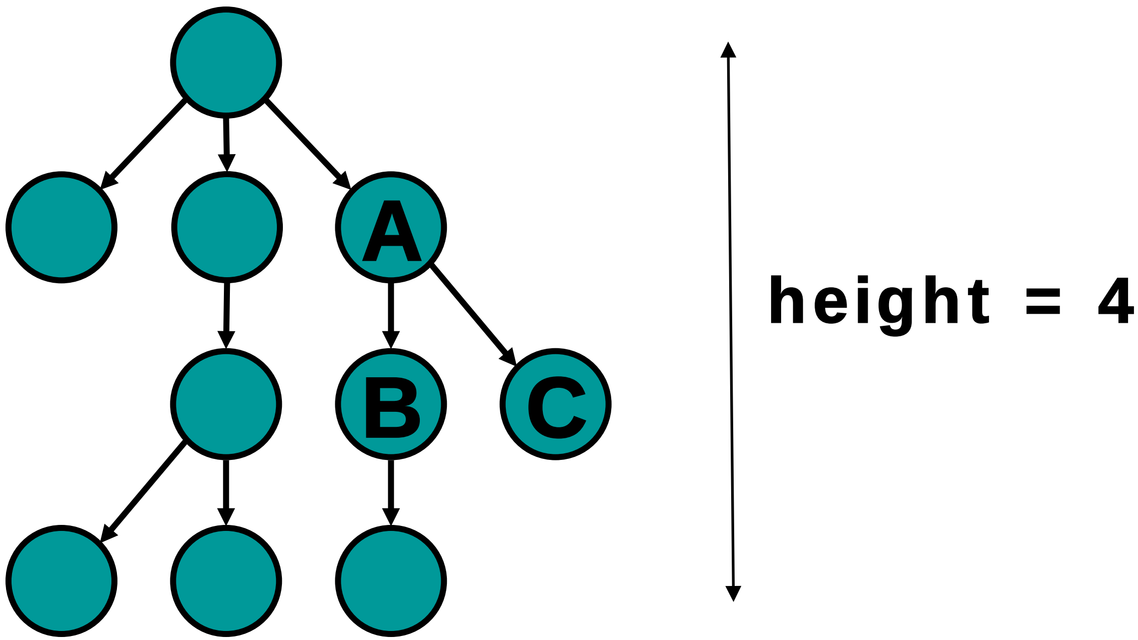




Q: Given that a node is stored in index position i, what is the position of its **parent**? **left child**? **right child**?

Height of a tree

- **Maximum level** of the nodes in the tree



Height of a tree (cont'd)

```
height(T)
    if T is empty
        return 0
    else
        return 1 + max (height(T.left), height(T.right))
```

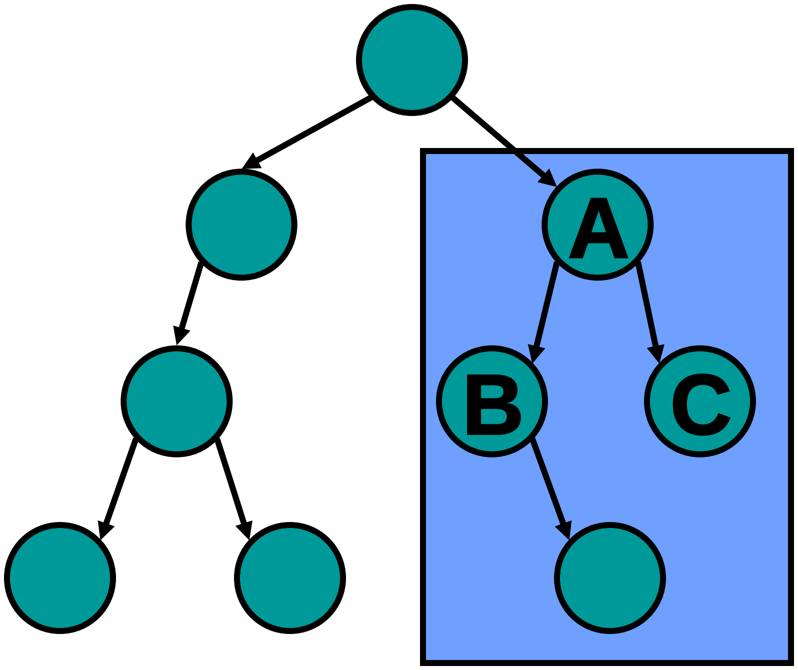
T.left and T.right represent the left and right subtrees of the node T respectively

Balanced Binary Trees

- Balanced binary trees
  - A binary tree is balanced if the height of any node's right subtree differs from the height of the node's left subtree by no more than 1
- Full binary trees are complete
- Complete binary trees are balanced

Size of a tree

- **Number of nodes in the tree**
  - The size of the subtree rooted at A is 4.



Size of a Tree (cont'd)

```
size(T)
    if T is empty
        return 0
    else
        return  1 + size(T.left) + size(T.right)
```

Binary Tree Traversal

Traversing a Tree

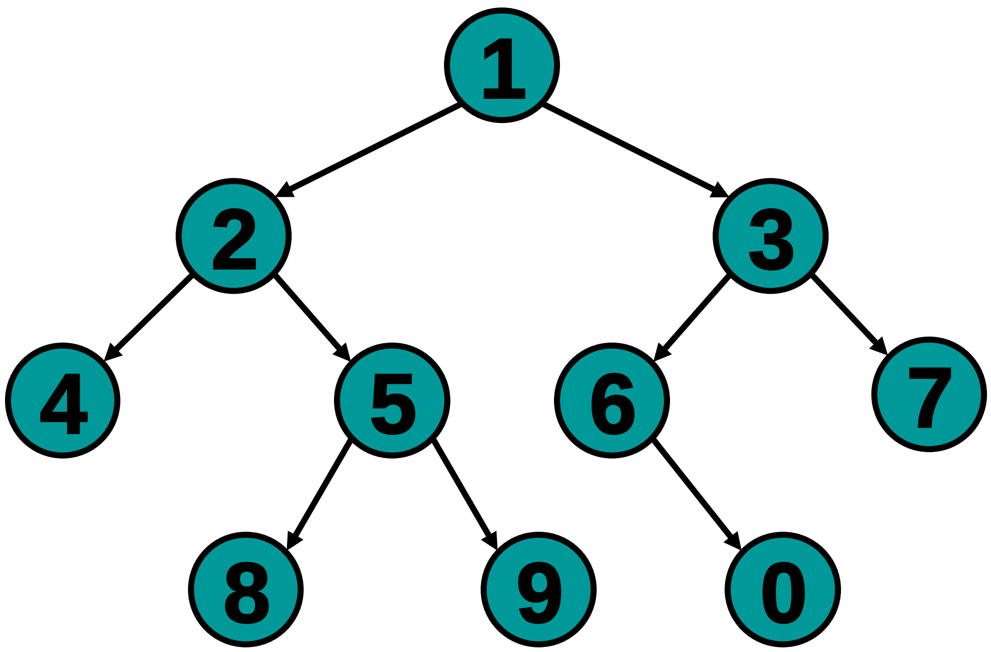
- Post-order traversal
- Pre-order traversal
- In-order traversal
- Level-order Traversal

Post-order Traversal

Traverse the subtrees first before processing the node

```
postorder(T)
    if T is not empty then
        postorder(T.left)
        postorder(T.right)
        process T.item
```

Traversal Example



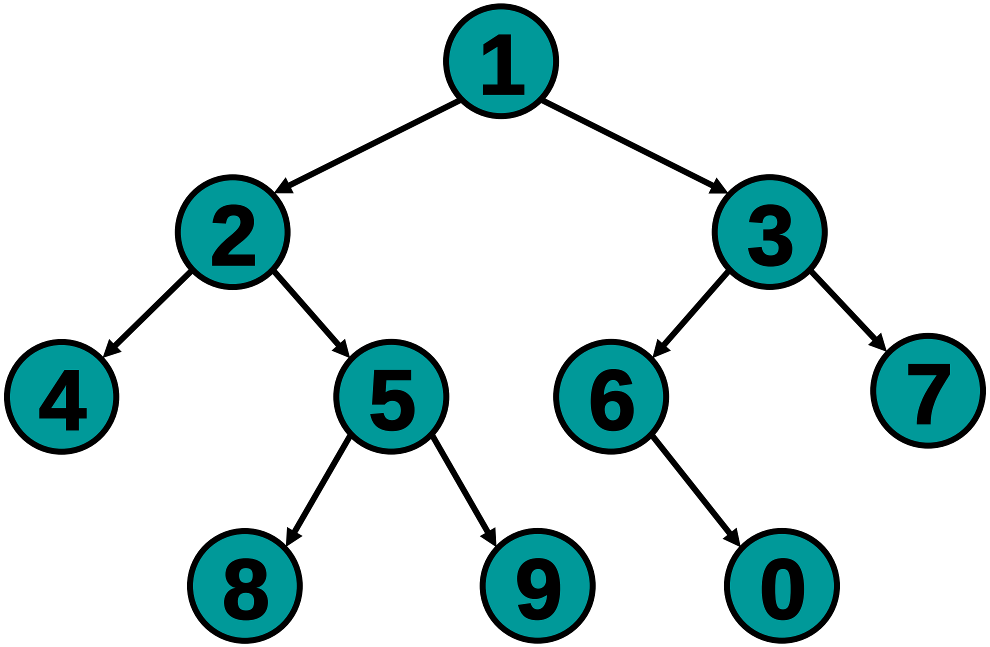
Post-order: 4 8 9 5 2 0 6 7 3 1

Pre-order traversal

Process the node first before traversing the subtrees

```
preorder(T)
    if T is not empty then
        process T.item
        preorder(T.left)
        preorder(T.right)
```

Traversal Example

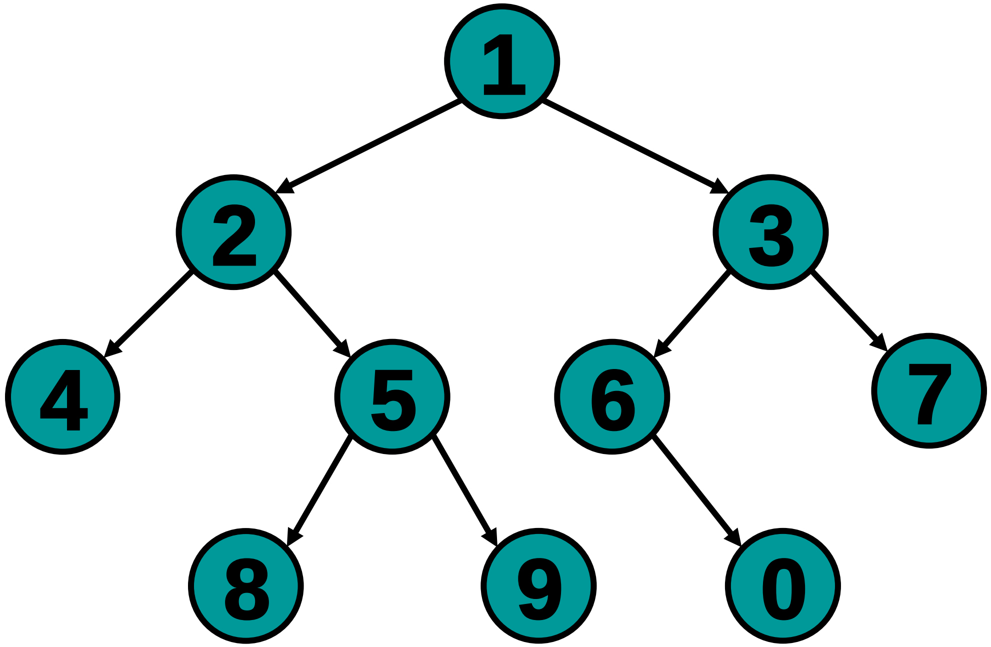


Pre-order: 1 2 4 5 8 9 3 6 0 7

In-order Traversal

```
inorder(T)
   if T is not empty then
        inorder(T.left)
        process T.item
        inorder(T.right)
```

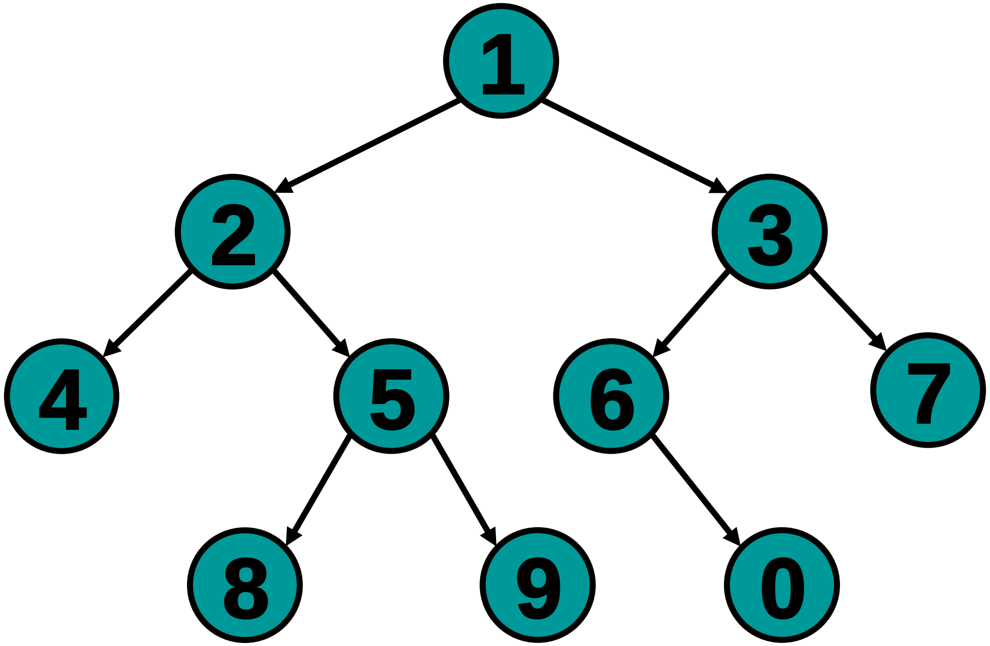
Traversal Example



In-order: 4 2 8 5 9 1 6 0 3 7

Level-order Traversal

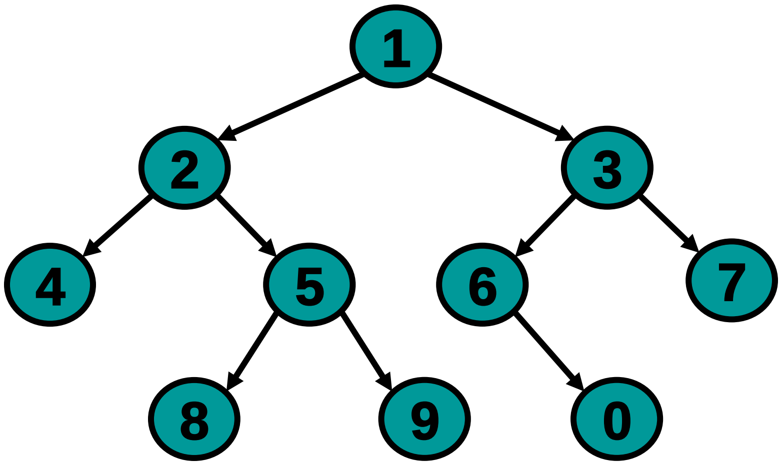
Traverse the tree level by level and from left to right



Level-order: 1 2 3 4 5 6 7 8 9 0

levelOrder(T) (using a queue)

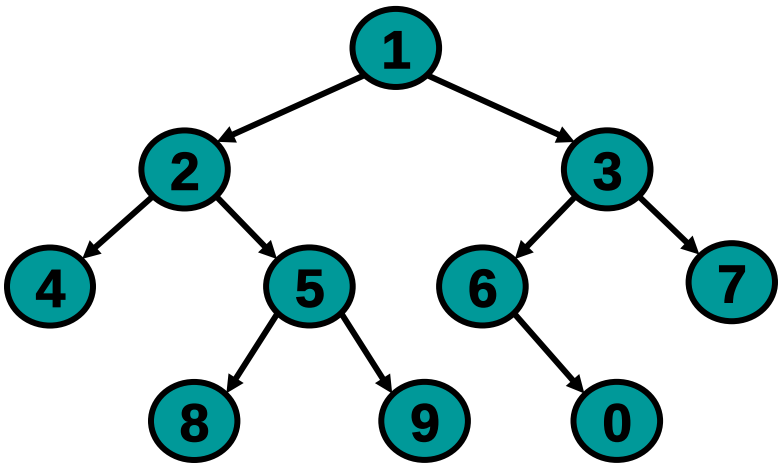
```
if T is empty return
Q = new Queue
Q.enqueue(T)

while Q is not empty
    curr = Q.dequeue()
    process curr.item
    if curr.left is not empty
        Q.enqueue(curr.left)
    if curr.right is not empty
        Q.enqueue(curr.right)
```



levelOrder(T) (using a queue, cont'd)

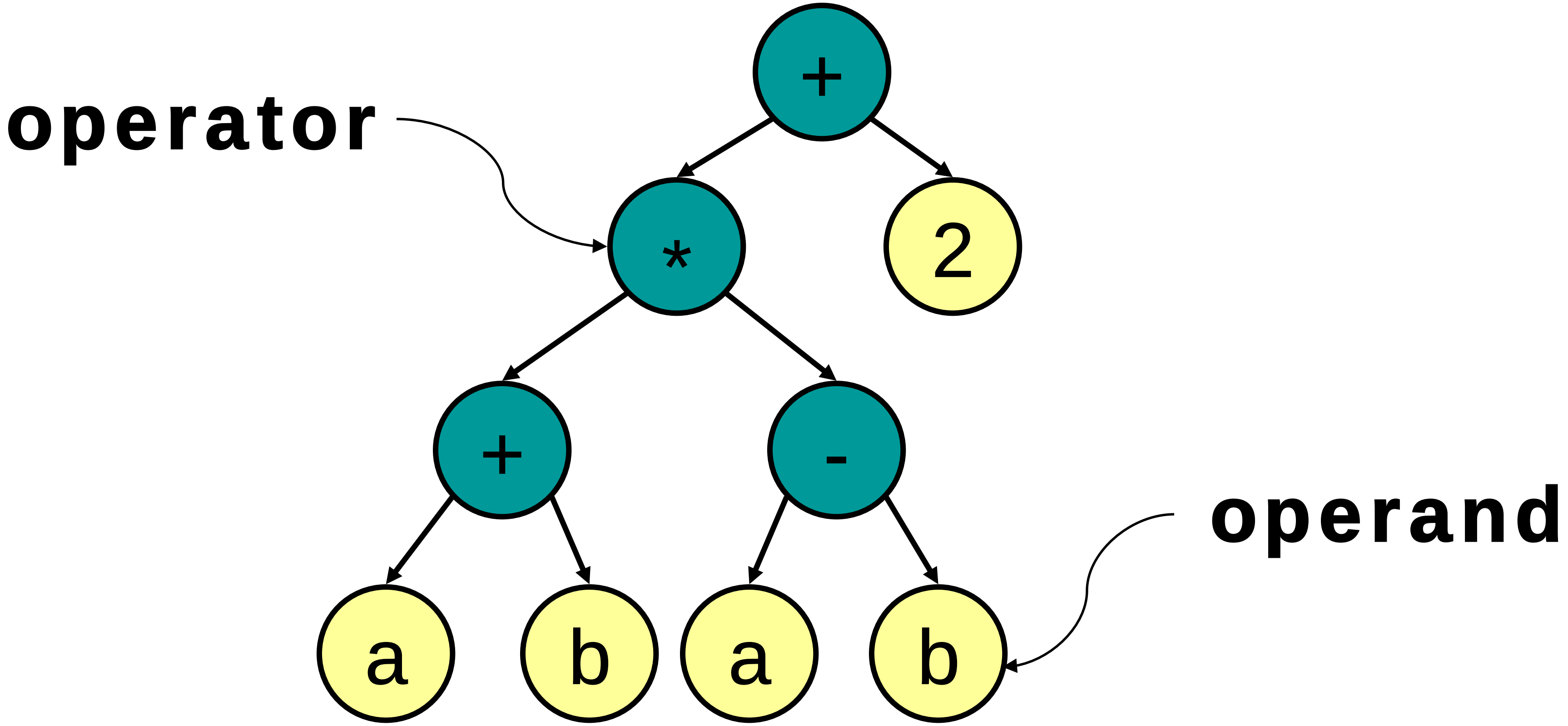
Queue	curr	print
1		
empty	*1*	*1*
2,3		
3	*2*	*2*
3,4,5		
4,5	*3*	*3*
4,5,6,7		
5,6,7	*4*	*4*
5,6,7		
6,7	*5*	*5*
6,7,8,9		
7,8,9	*6*	*6*
7,8,9,0		
8,9,0	*7*	*7*
8,9,0		
9,0	*8*	*8*
9,0		
0	*9*	*9*
0		
empty	*0*	*0*
empty	end	

Note: The numbers are references to the nodes



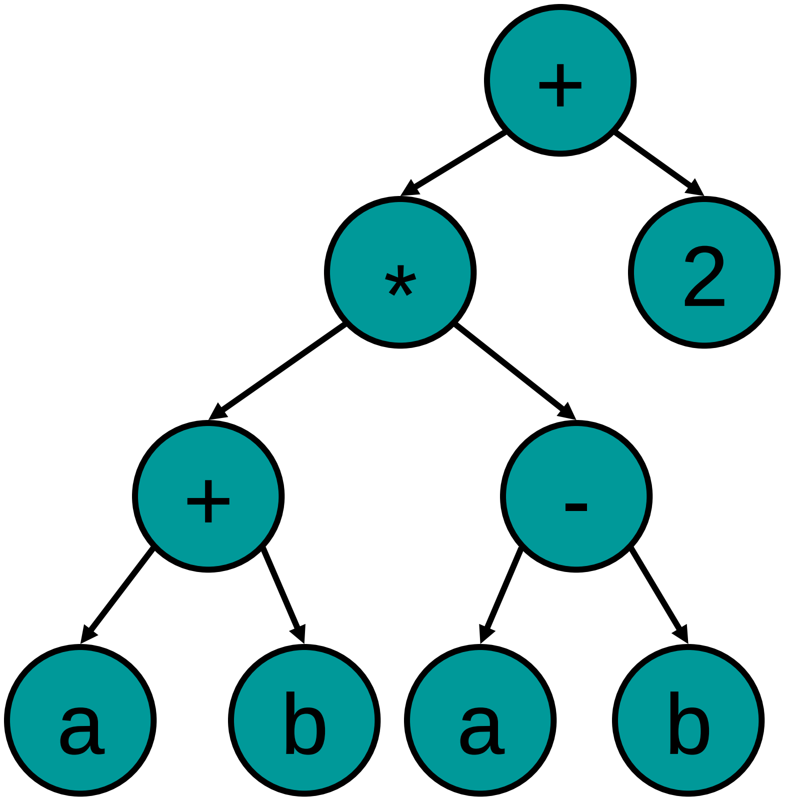
Expression Trees

Evaluating Expression Tree



Leaf nodes (or **leaves**) store operands.
Internal nodes and **root** store operators

Traversing Expression Tree



Post-order: **(((a b +) (a b -) *) 2 +)**

Note: Brackets can be omitted, i.e. **a b + a b - * 2 +**

Evaluation of Expression Tree

```
eval(T)
  if T is empty
     return 0
  if T is a leaf
    return value of T
  else if T.item is "+"
          return eval(T.left) + eval(T.right)
        else if T.item is "*"
                return eval(T.left) * eval(T.right)
```

Q: How to handle operators **/**, **-**, and **unary -** ?
Q: Do you need to consider the **priorities** of the operators?

Binary Search Tree (BST)

Tables

- Phone books
- Street directories
- Dictionaries
- Class schedule
- …

Key	Data
Carl	3849-3843
Alice	9493-9349
John	8934-3784

Table ADT operations

A table ADT provides operations to maintain a set of data, each can be uniquely identified by a **key**.

- **insert** (key, data)
- **delete** (key)
- data = **search** (key)

Running Time of operations

	Unsorted Array/List	Sorted Array	Sorted LinkedList
insert	O(1)	O(N)	O(1)
delete	O(N)	O(N)	O(1)
search	O(N)	$O(\log_2 N)$	O(N)

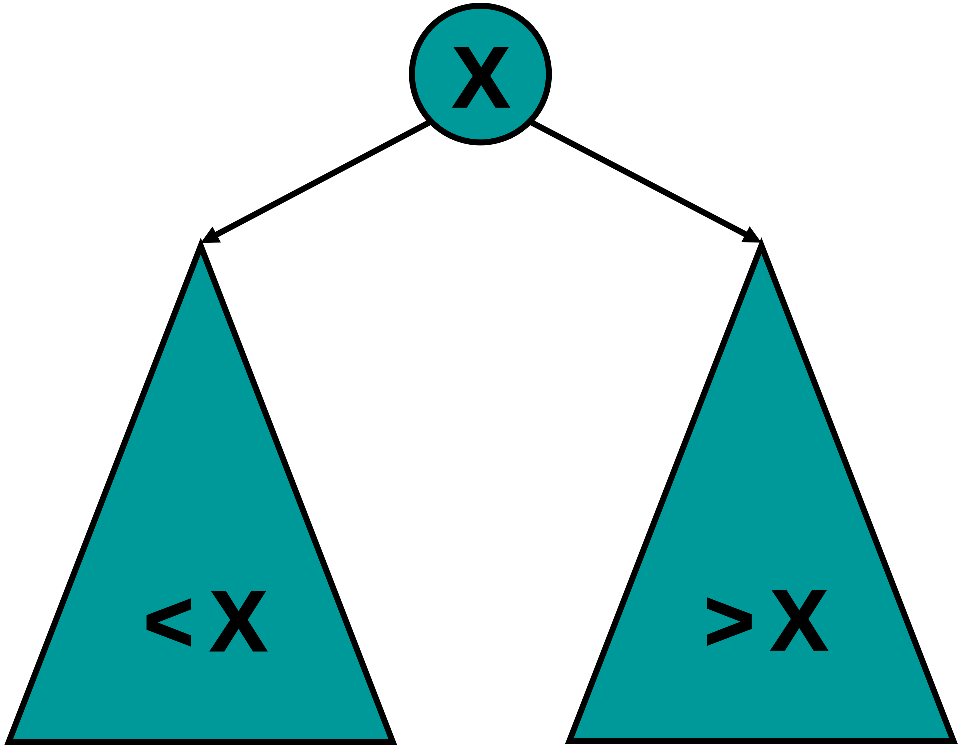
Binary Search Tree (BST)

- insert, delete, and search can usually be done in

$$O(\log_2 N)$$

Q: So, are the update operations' performances of BST better than unsorted and sorted array?

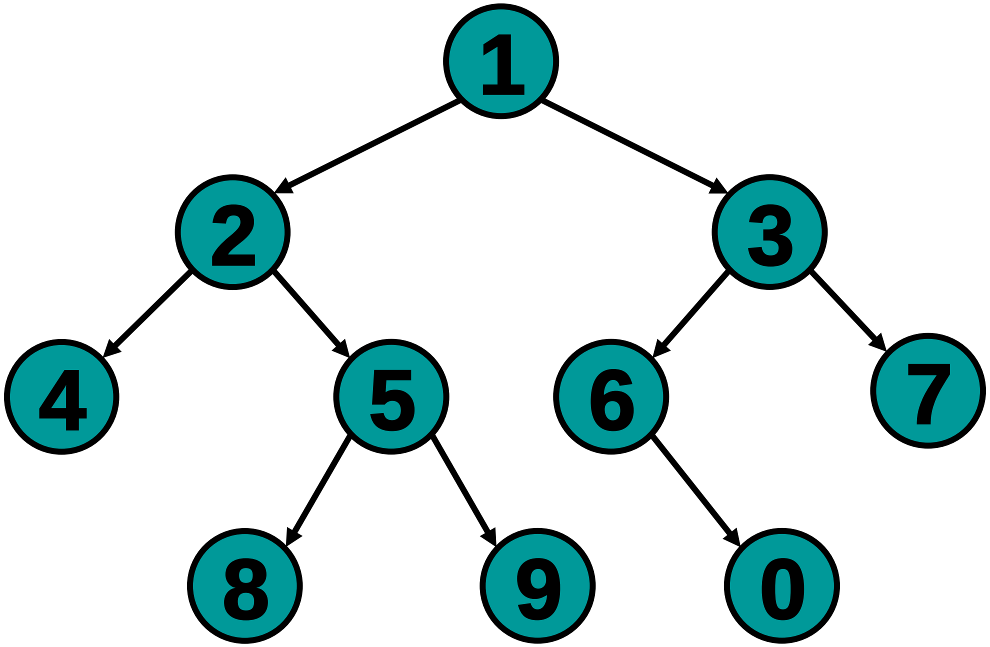
BST Property



BST organizes data in a binary tree such that
all keys **smaller** than the root are stored in the **left** subtree, and
all keys **larger** than the root are stored in the **right** subtree.

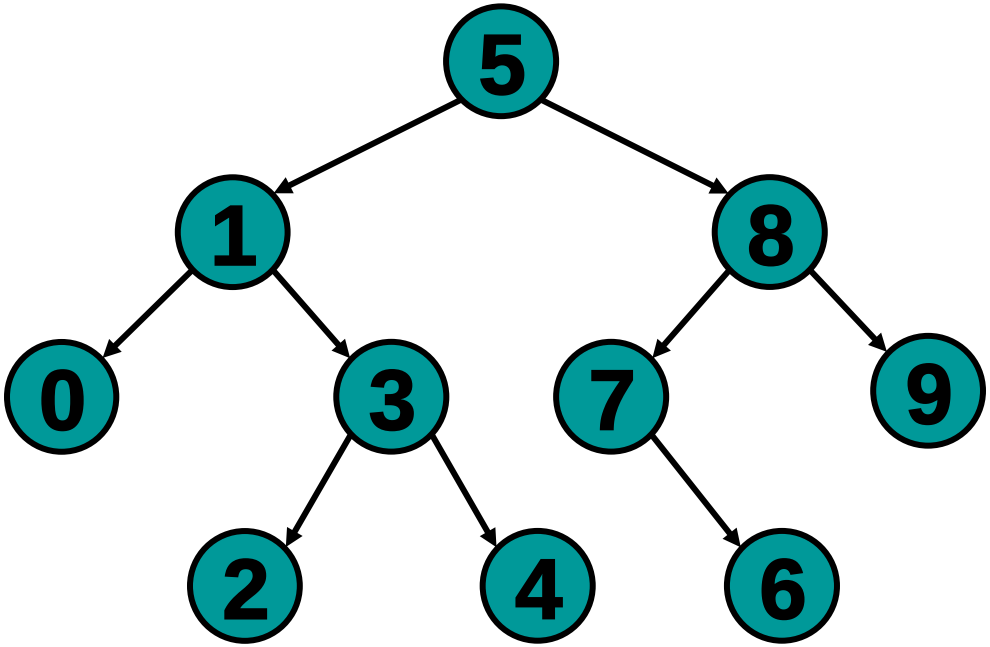
Q: Can we have the same key values in a BST?

Example



NOT a BST. Why?

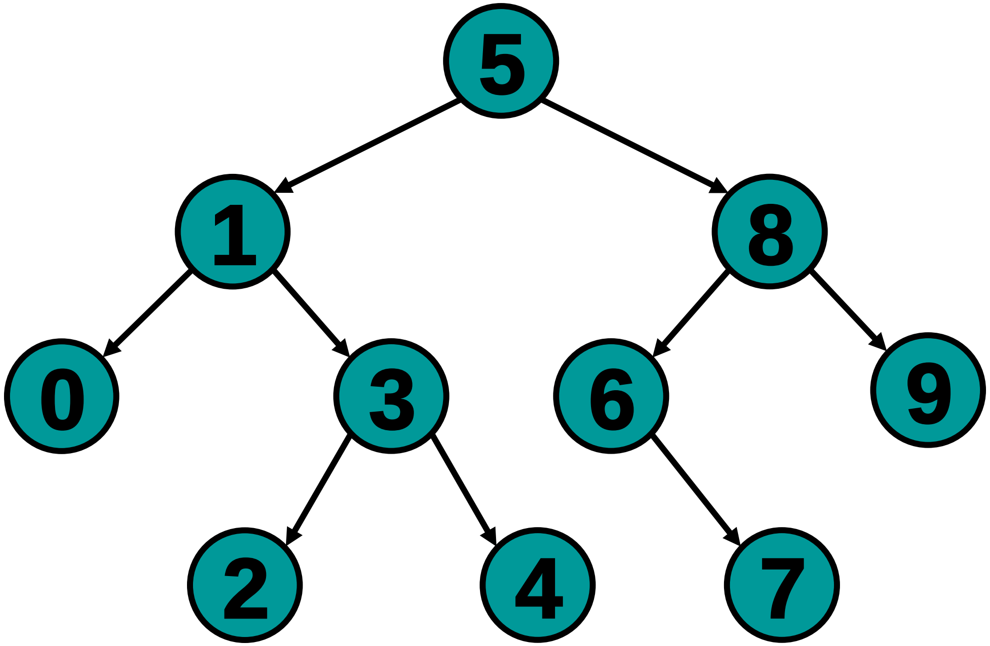
Example



A BST ?

NO. Why?

Example



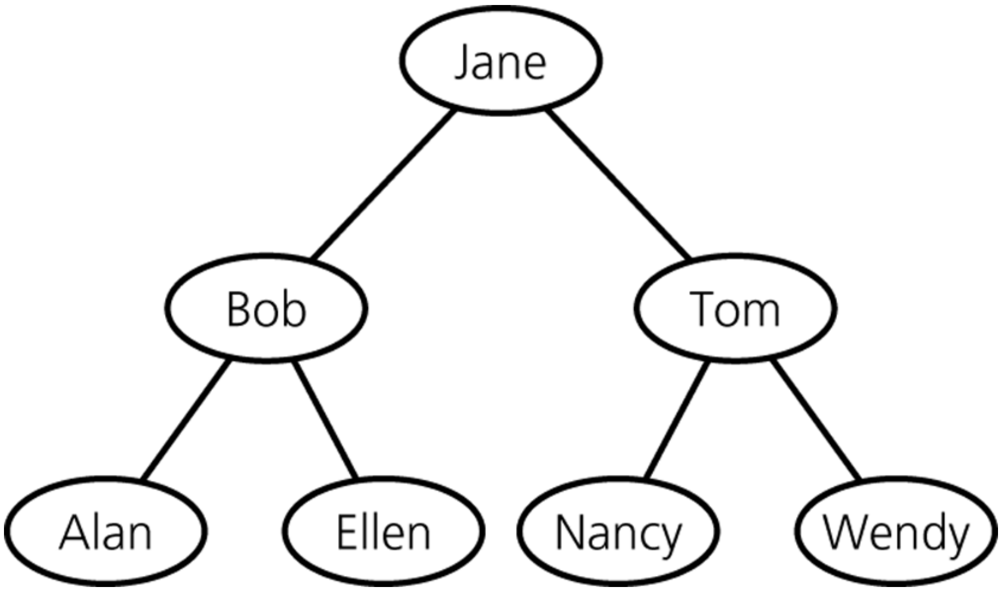
A BST

Q: What do you get when you traverse a BST in **in-order**?

Ans: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 (in increasing order).

Example

BST of names:



Compare heap with BST

- Both are binary trees
- Difference
  - Heap maintains **heap property**
    - It is not a search tree
  - BST maintains **BST property**
    - It is a search tree

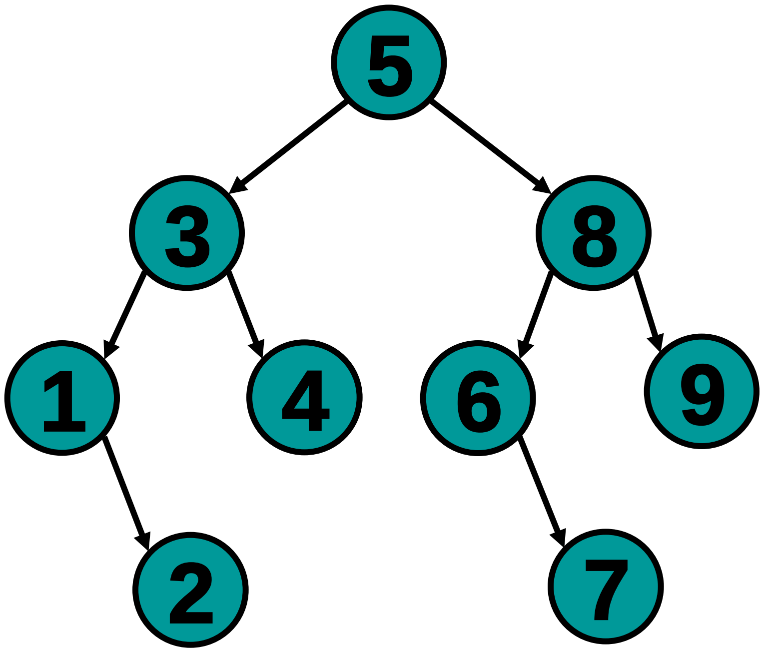
Operations on BST

Finding Minimum Element

while T.left is not empty

T = T.left

return T.item



Q: How to find maximum values?

Q: How to find top-k (or bottom-k) values?

e.g. find top-3 values.

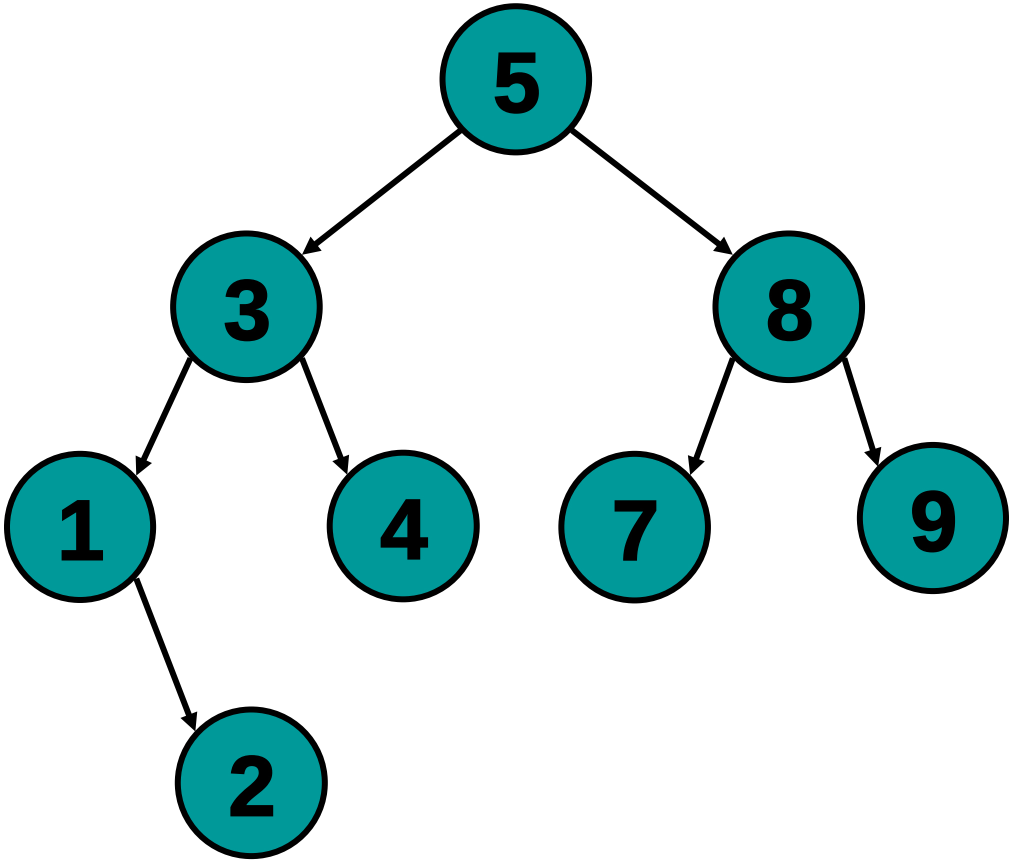
Searching x in T (iterative solution)

```
while T is not empty
    if T.item == x then
        return T
    else if T.item > x then
            T = T.left
        else
            T = T.right
return null   // T is empty, so X is not in T
```

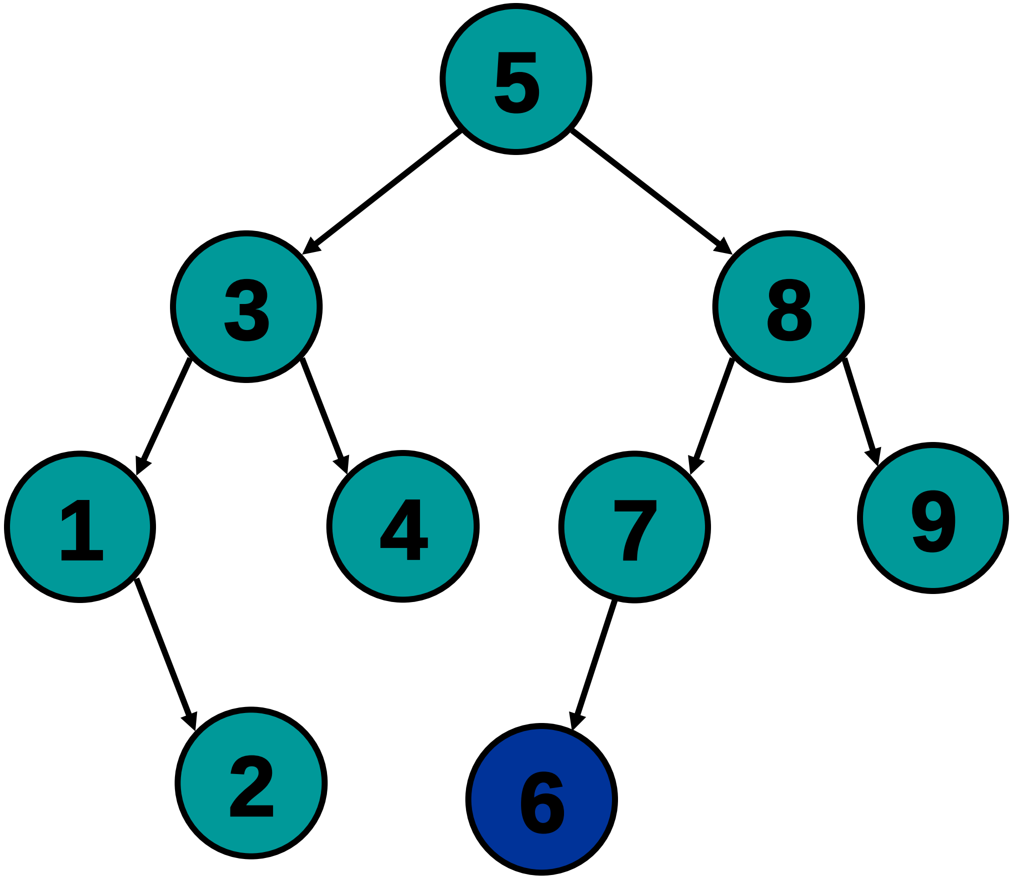
Searching x in T (recursive solution)

```
function search(x, T)
if T is empty
  return null
if x == T.item then
  return T
else if x < T.item
        return search(x, T.left)
      else
        return search(x, T.right)
```

How to Insert 6?



After Inserting 6



insert(x,T)

```
if T is empty
   return new TreeNode(x) // a tree with only node x
else if x < T.item
         T.left = insert(x,T.left)
      else if x > T.item
              T.right = insert(x, T.right)
           else
              ERROR!  // X already in T
return T   // return the new tree T
```

How to delete?

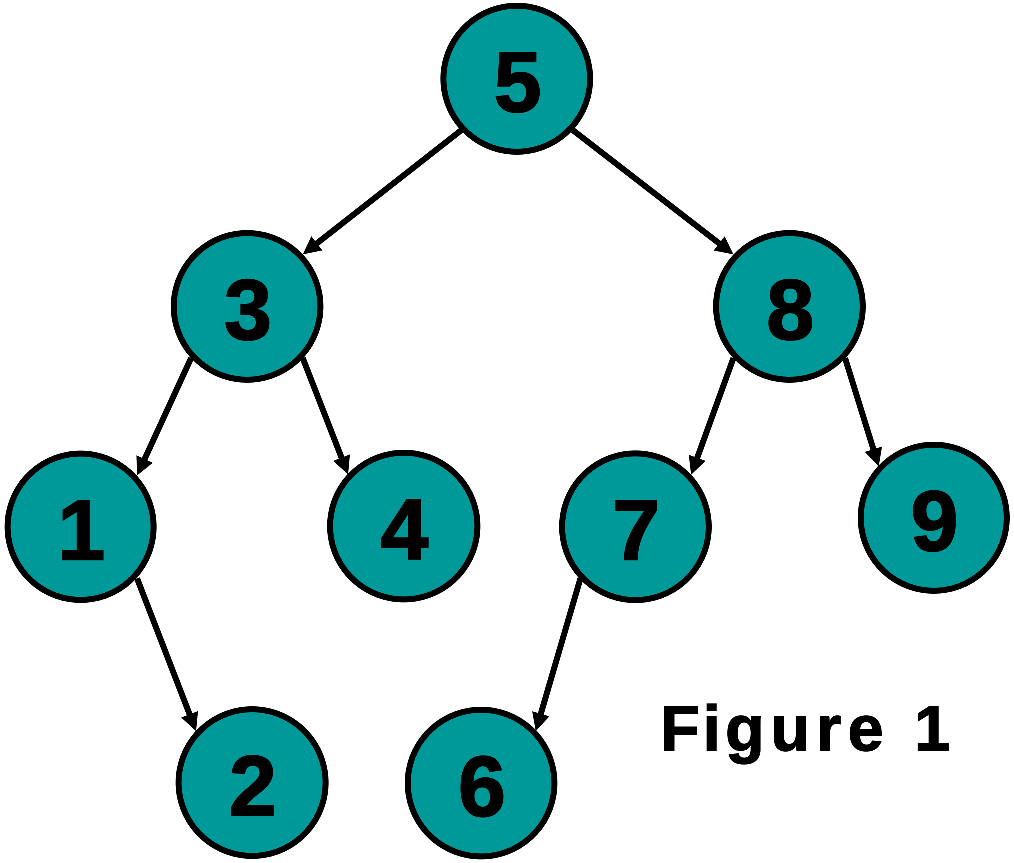


Figure 1

delete(x,T): Case 1

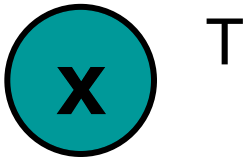
if T has no children

 if x == T.item

  return empty tree

 else

  NOT FOUND



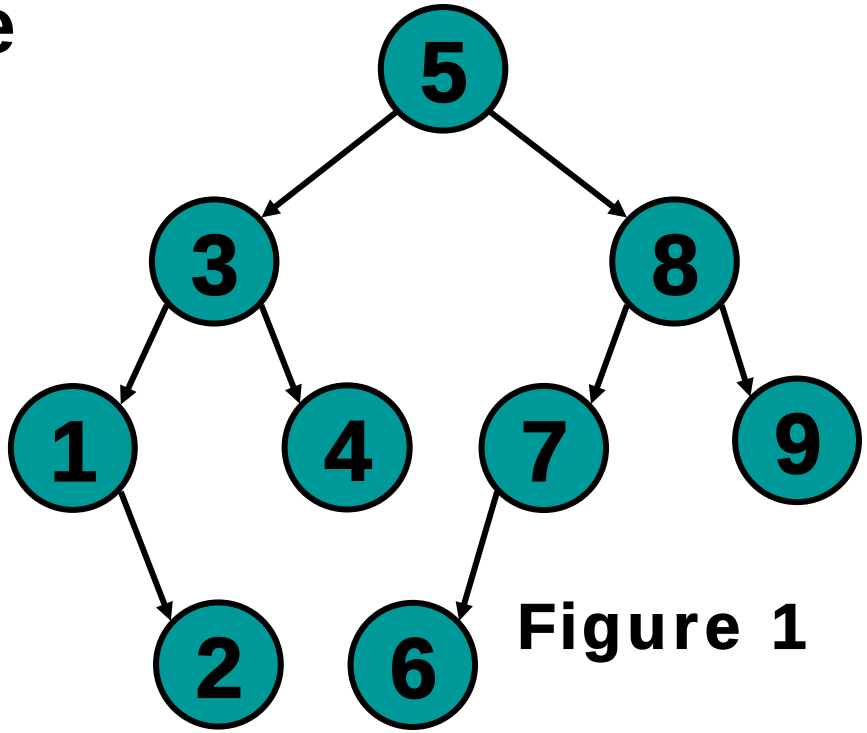
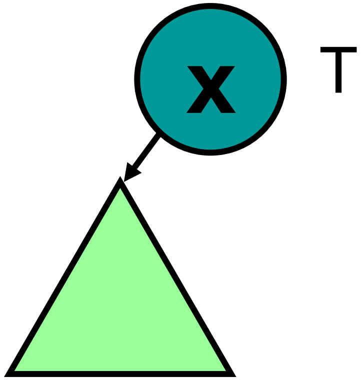


Figure 1

e.g. Delete 4 in Figure 1

delete(x,T): Case 2 (A)



if T has only 1 child (left)

if x == T.item

return T.left

else

T.left = delete(x,T.left)

return T

e.g. delete 7 in Figure 1

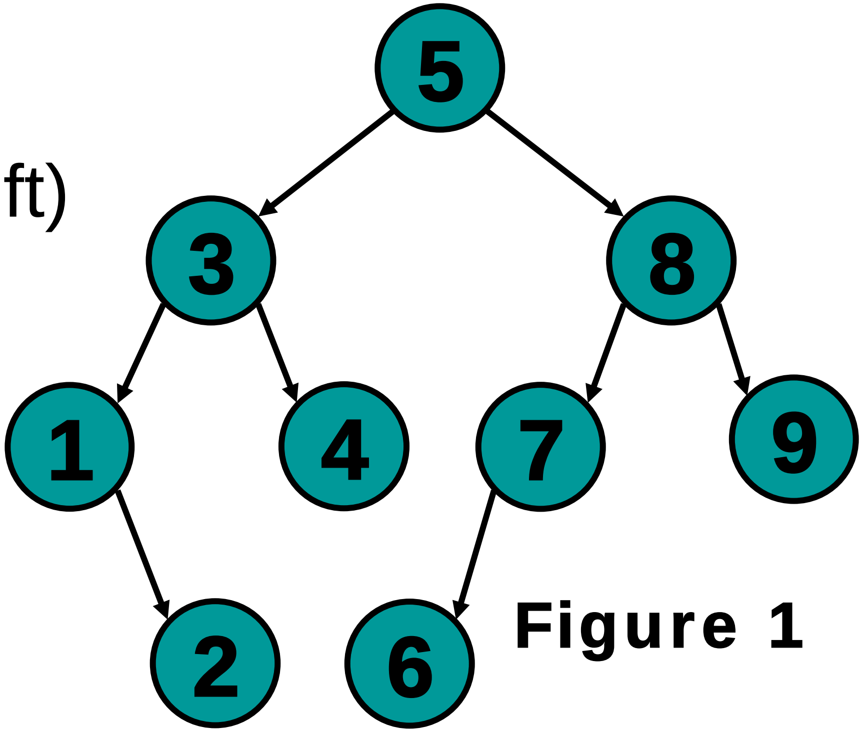
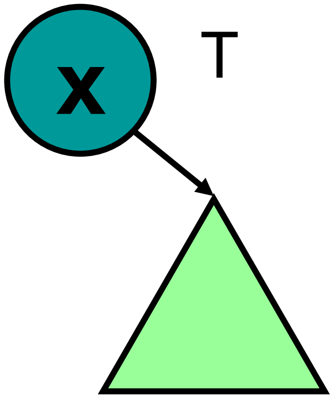


Figure 1

delete(x,T): Case 2 (B)



if T has only 1 child (right)

    if x == T.item

        return T.right

    else

        T.right = delete(x, T.right)

  return T

e.g. delete 1 in Figure 1

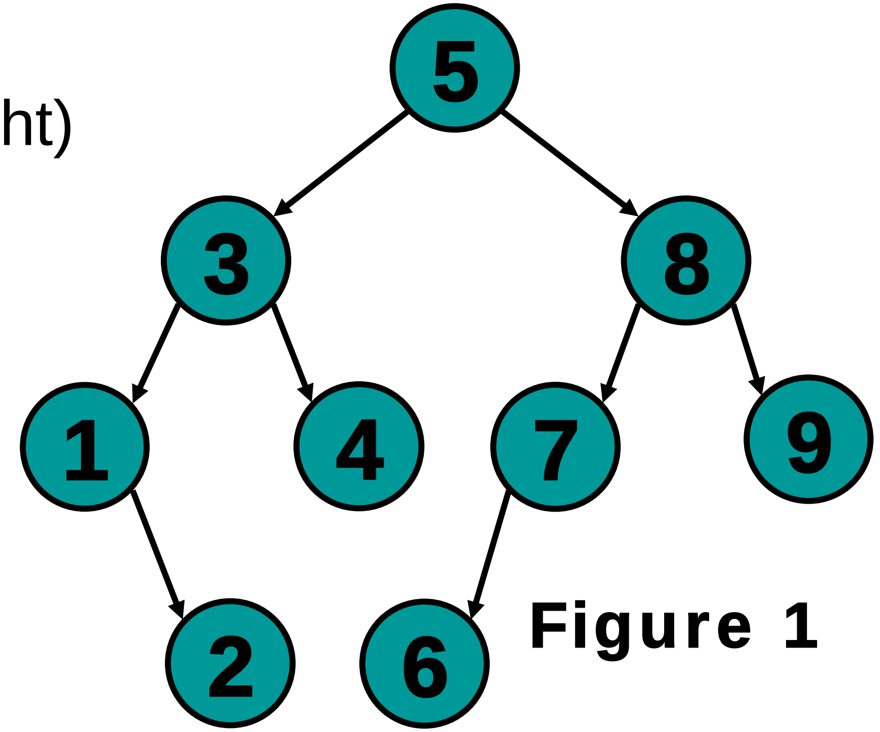
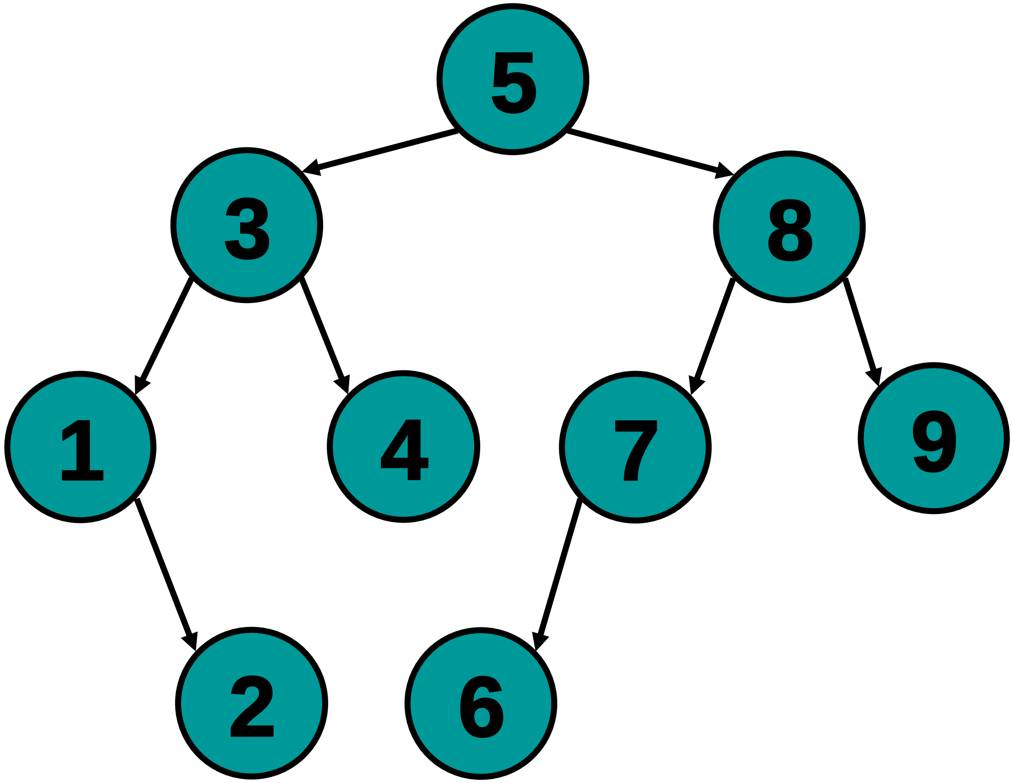


Figure 1

delete(x,T): Case 3

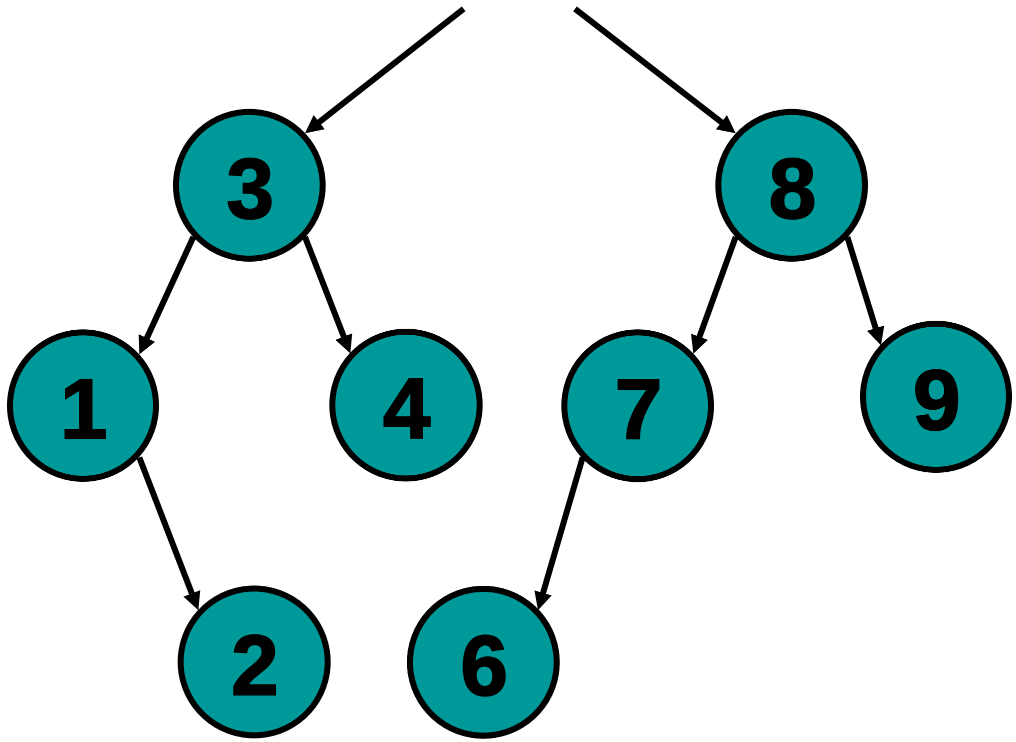
Node to be deleted has 2 children

e.g. delete 5



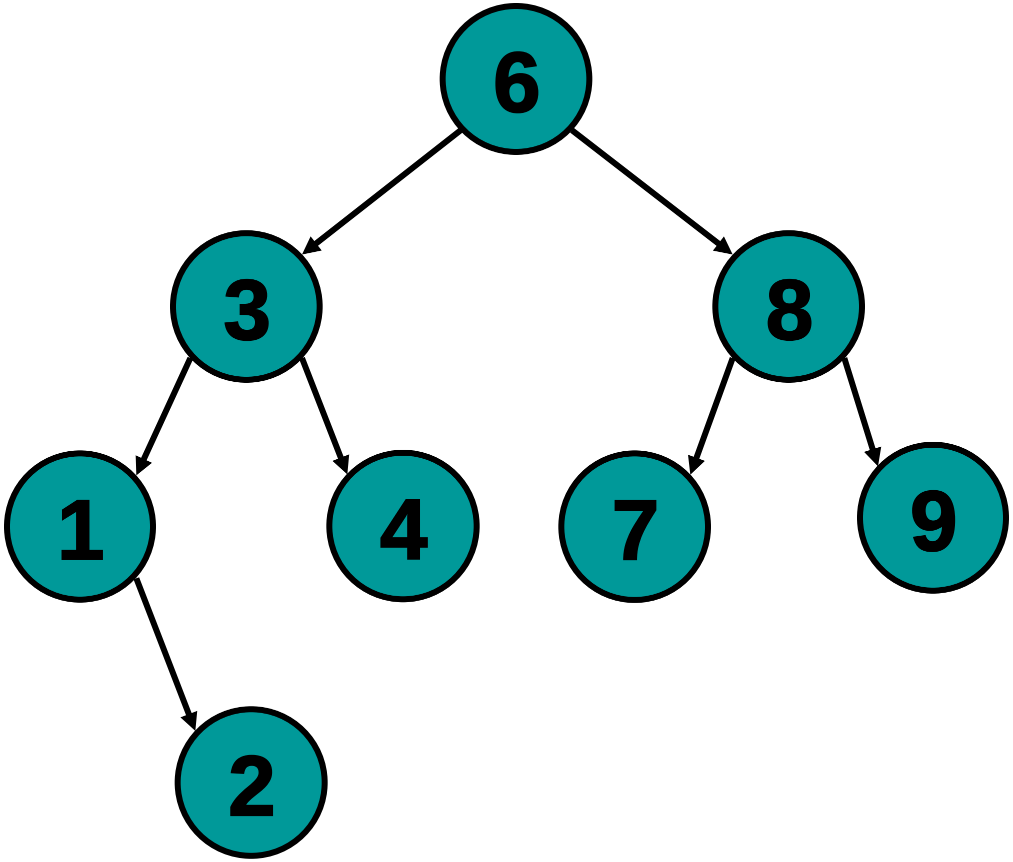
delete(x,T): Case 3

e.g. delete 5



delete(x,T): Case 3

5 deleted!



delete(x,T): Case 3

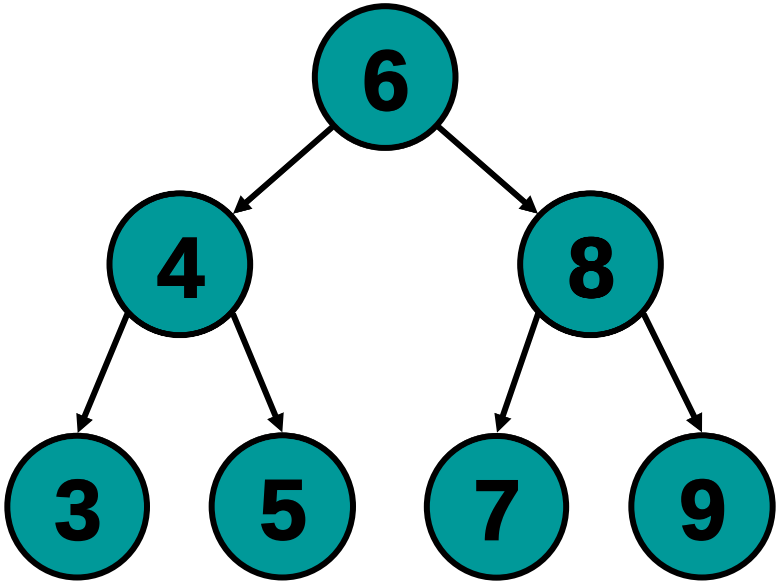
```
if T has two children
    if x == T.item
      T.item = findMin(T.right) // replace T.item by
                 // the min. item of the right subtree

      T.right = delete(T.item, T.right)
      // delete x (i.e. T.item) from the right substree
    else if x < T.item
            T.left = delete(x, T.left)
        else
            T.right = delete(x, T.right)
return T
```

Running time of BST

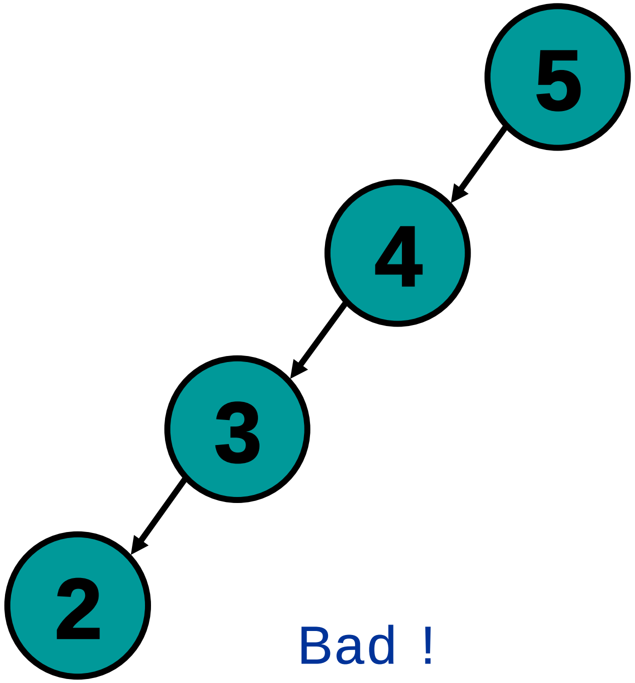
- findMin O(h) where h is the height of the BST
- search O(h)
- insert O(h)
- delete O(h)

Running time of BST (cont'd)

- But h is not always $O(\log_2 N)$!
  Where **N** is the total number of nodes in the BST.



Good !
$h = O(\log_2 N)$



Bad !
$h = O(N)$

When you insert nodes in increasing or decreasing order, you get a **skewed** tree

Applications of BST

- Treesort
  - Uses binary search tree to sort an array of records into search-key order
    - Average case: O(n * log n)
    - Worst case: O(n^2)

Applications of BST

- Algorithms for saving a binary search tree
  - Saving a binary search tree and then restoring it to its original shape
    - Uses preorder traversal to save the tree to a file
  - Saving a binary tree and then restoring it to a balanced shape
    - Uses inorder traversal to save the tree to a file
    - Can be used if the data is sorted and the number of nodes in the tree is known

The STL Search Algorithms for Sorted Ranges

- *binary_search*
  - Returns true if a specified value appears in the sorted range
- *lower_bound*; *upper_bound*
  - Returns an iterator to the first occurrence; or to one past the last occurrence of a value
- *equal_range*
  - Returns a pair of iterators that indicate the first and one past the last occurrence of a value

binary_search

```
#include <iostream>
#include <algorithm>
using namespace std;

int main() {
   int nums[] = { -242, -1, 0, 5, 8, 9, 11 };
   int start = 0;
   int end = 7;
   for( int i = 0; i < 10; i++ ) {
      if( binary_search( nums+start, nums+end, i ) ) {
         cout << "nums[] contains " << i << endl;
      } else {
         cout << "nums[] DOES NOT contain " << i  << endl;
      }
   }
   return 0;
}
```

lower_bound – (1)

```
#include <algorithm>
#include <vector>
using namespace std;

int main() {
   vector<int> nums;
   nums.push_back( -242 );
   nums.push_back( -1 );
   nums.push_back( 0 );
   nums.push_back( 5 );
   nums.push_back( 8 );
   nums.push_back( 8 );
   nums.push_back( 11 );
```

lower_bound – (2)

```
    cout << "Before nums is: ";
    for( unsigned int i = 0; i < nums.size(); i++ ) {
        cout << nums[i] << " ";
    }
    cout << endl;
    vector<int>::iterator result;
    int new_val = 7;
    result = lower_bound( nums.begin(), nums.end(), new_val );
    nums.insert( result, new_val );
    for( unsigned int i = 0; i < nums.size(); i++ ) {
        cout << nums[i] << " ";
    }
    return 0;
}
```

equal_range

```
#include <algorithm>
using namespace std;
int main() {
  // data declared in the previous example
  pair<vector<int>::iterator, vector<int>::iterator> result;
  int new_val = 8;
  result = equal_range( nums.begin(), nums.end(), new_val );
  cout << "The first place that "
   << new_val
   << " could be inserted is before "
   << *result.first
   << ", and the last place that it could be inserted is before "
   << *result.second << endl;
  return 0;
}
```