CS2020
Data Structures and Algorithms

Welcome!

Semester Roadmap

Where are we?

- Searching
- Sorting
- Lists
- Trees
- Hash Tables
- **Graphs**
- Advanced material

Last week

You are here

Roadmap

Today: Graph Basics

- What is a graph?
- Modeling problems as graphs.
- Graph representations (list vs. matrix)
- Searching graphs (DFS / BFS)

Roadmap

Next time: Searching Graphs

- Searching graphs
- Shortest path problem
- Bellman-Ford Algorithm
- Dijkstra's Algorithm

Roadmap

Next week:

- Connected component problem
  - Union-Find data structure
- Priority Queues
  - Binary heaps
- The Minimum Spanning Tree Problem
  - Kruskal’s Algorithm
  - Prim’s Algorithm

Roadmap

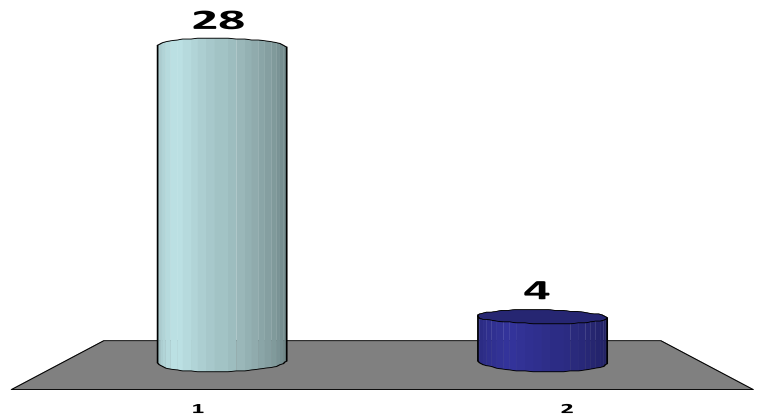
Today: Graph Basics

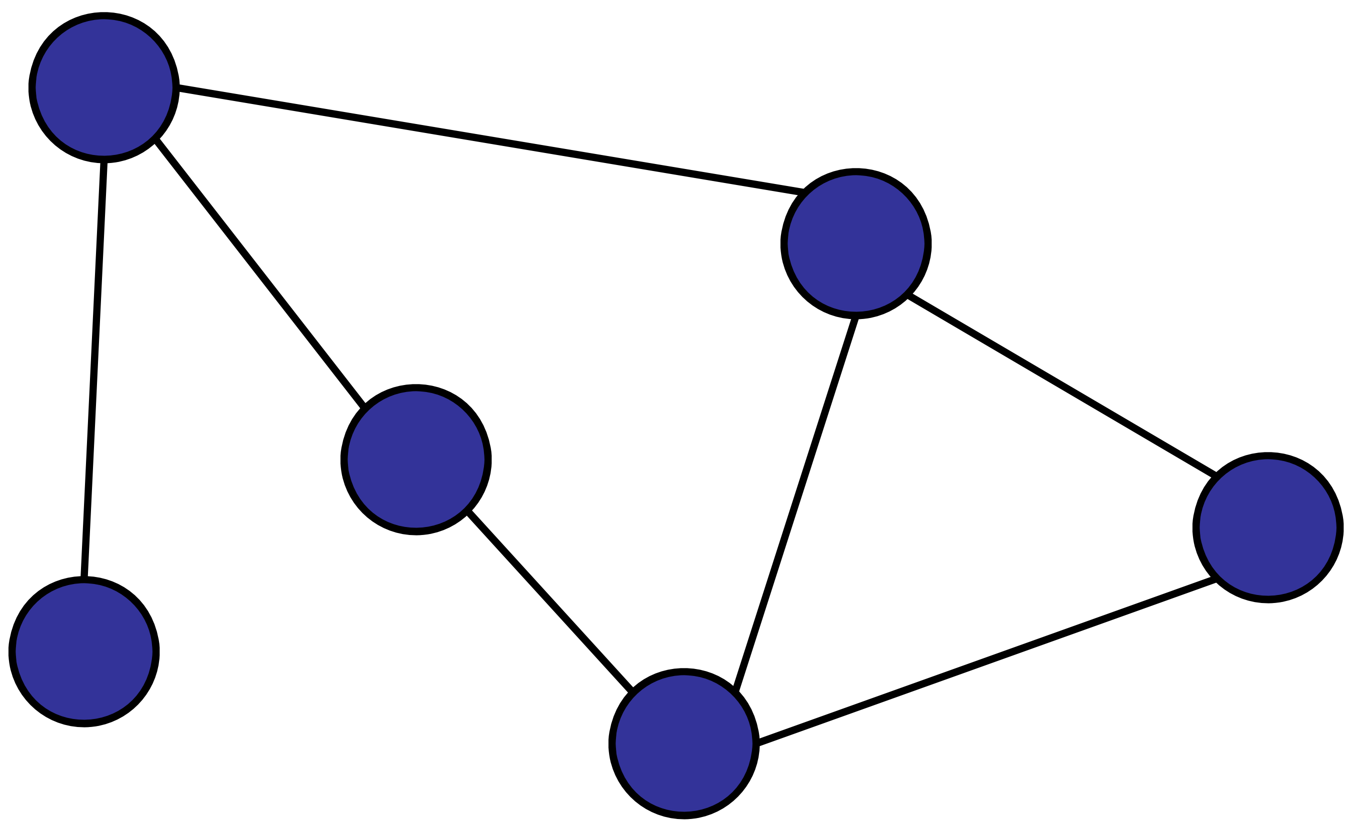
- What is a graph?
- Modeling problems as graphs.
- Graph representations (list vs. matrix)
- Searching graphs (DFS / BFS)

What is a graph?

Is it a graph?

1. Yes
2. No.





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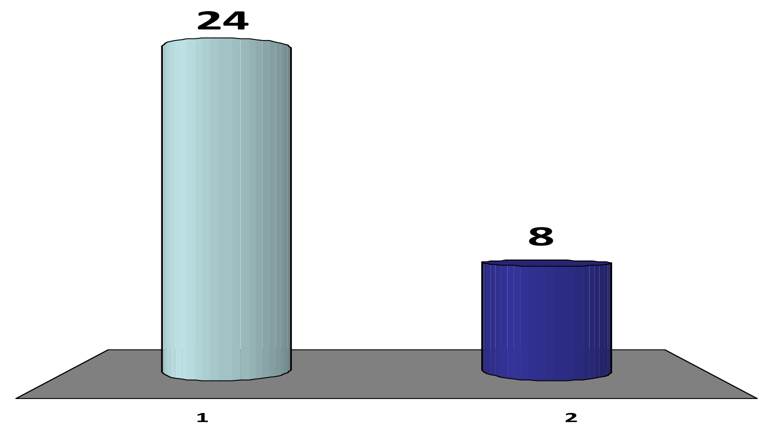

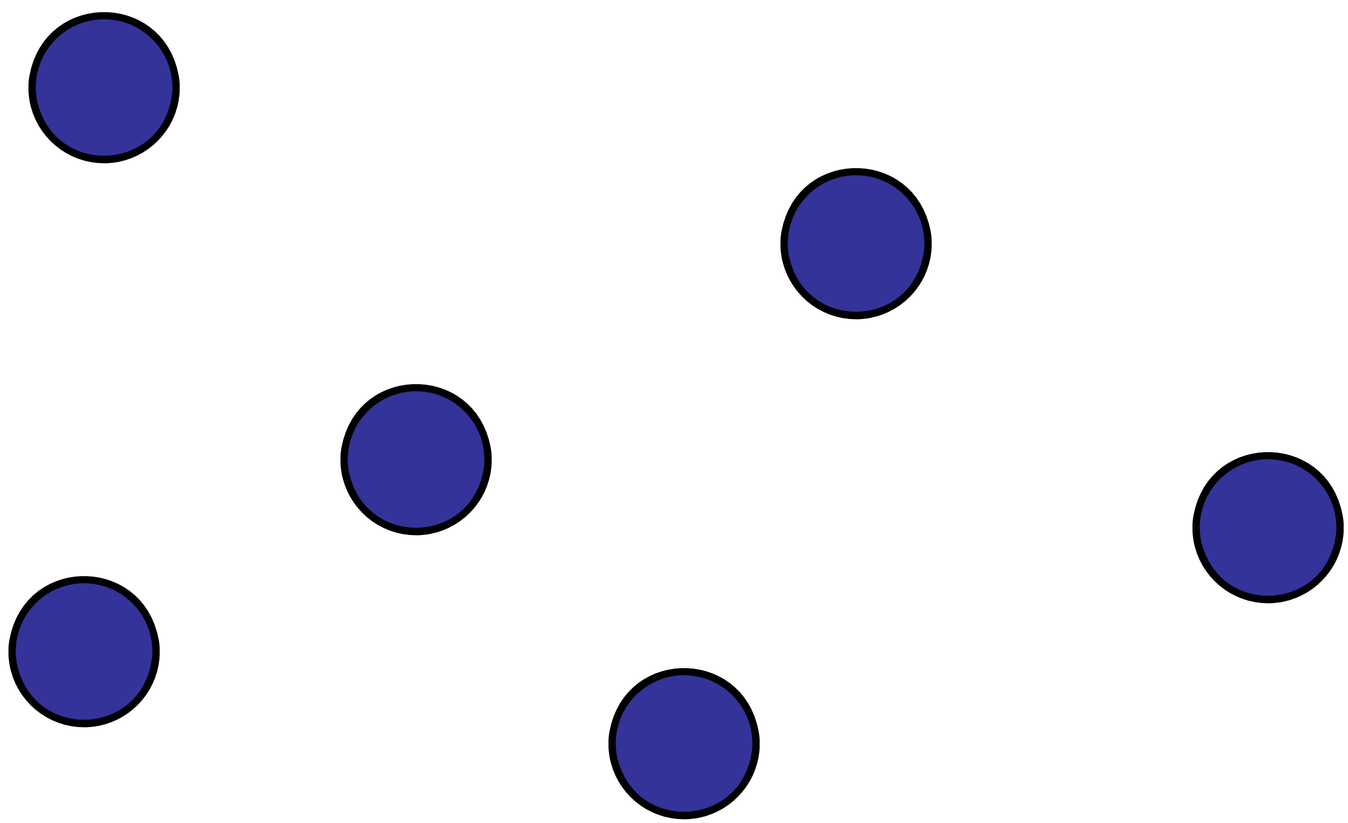

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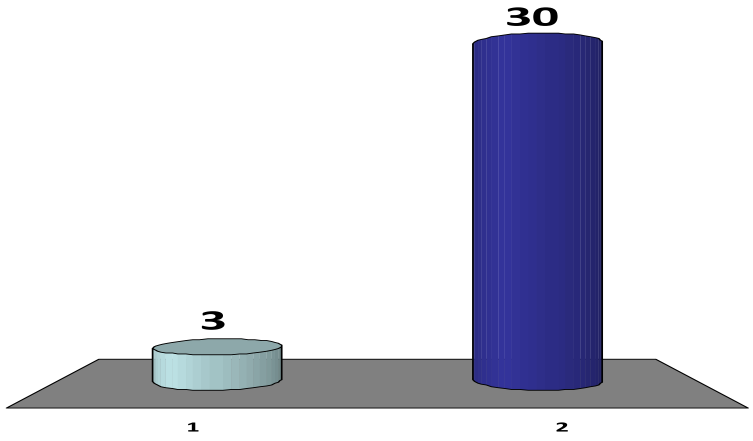
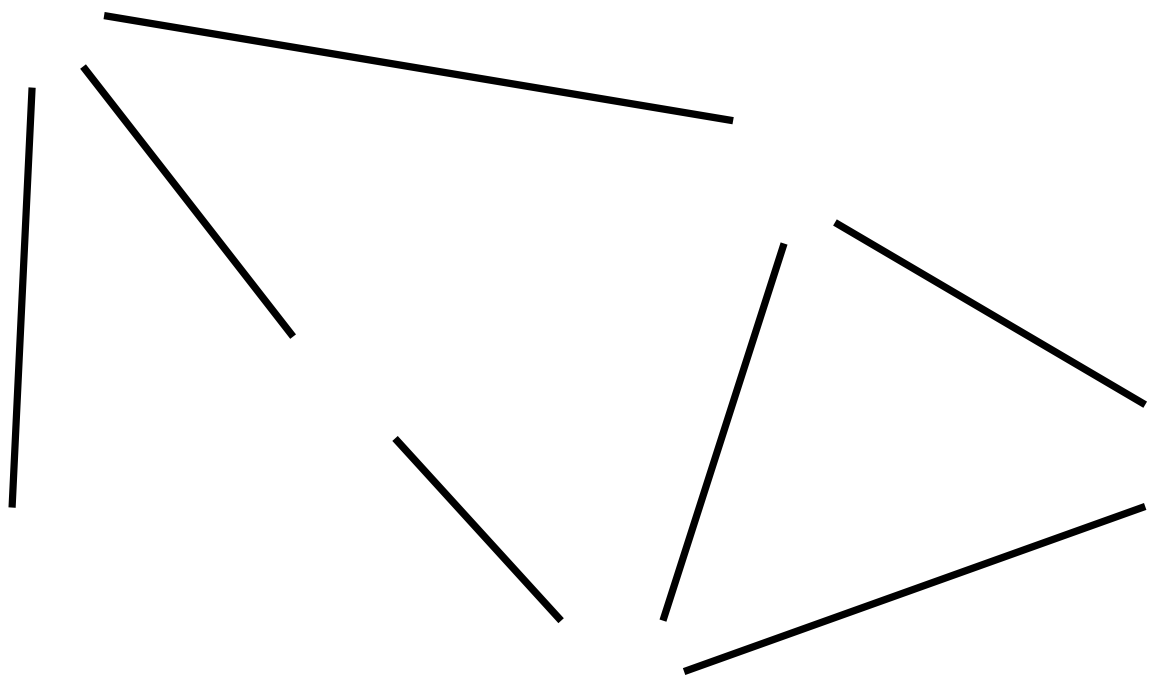
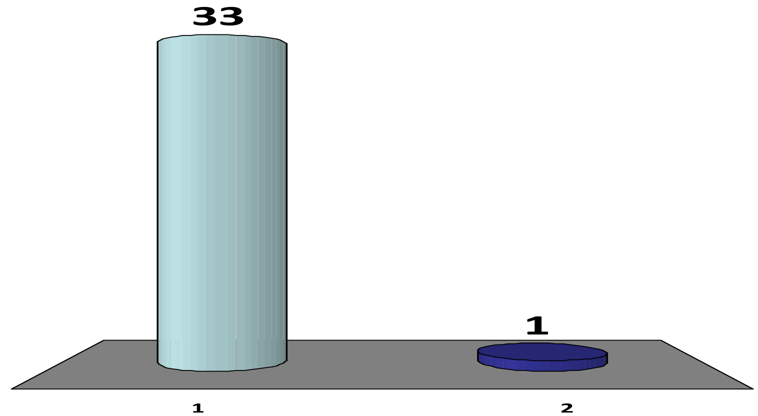
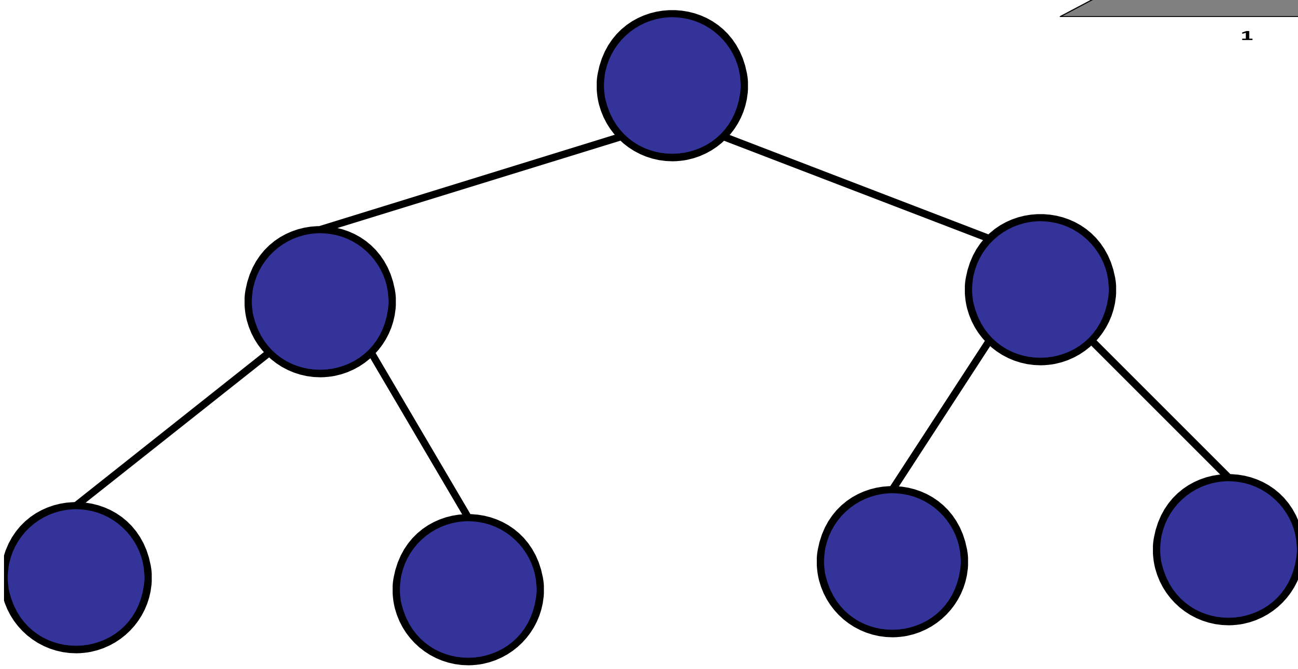




Is it a graph?

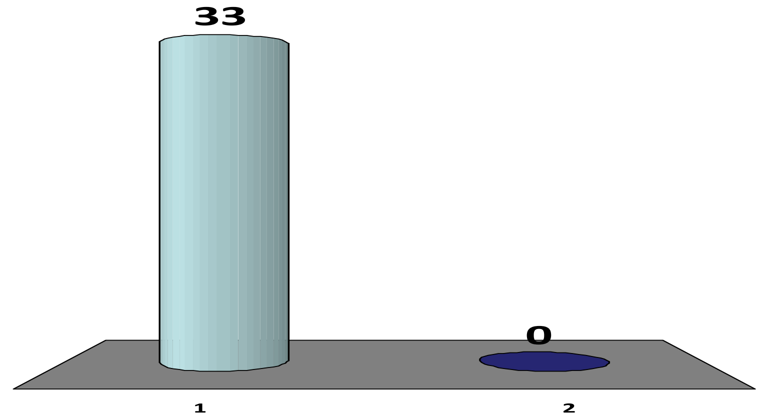
1. Yes
2. No.

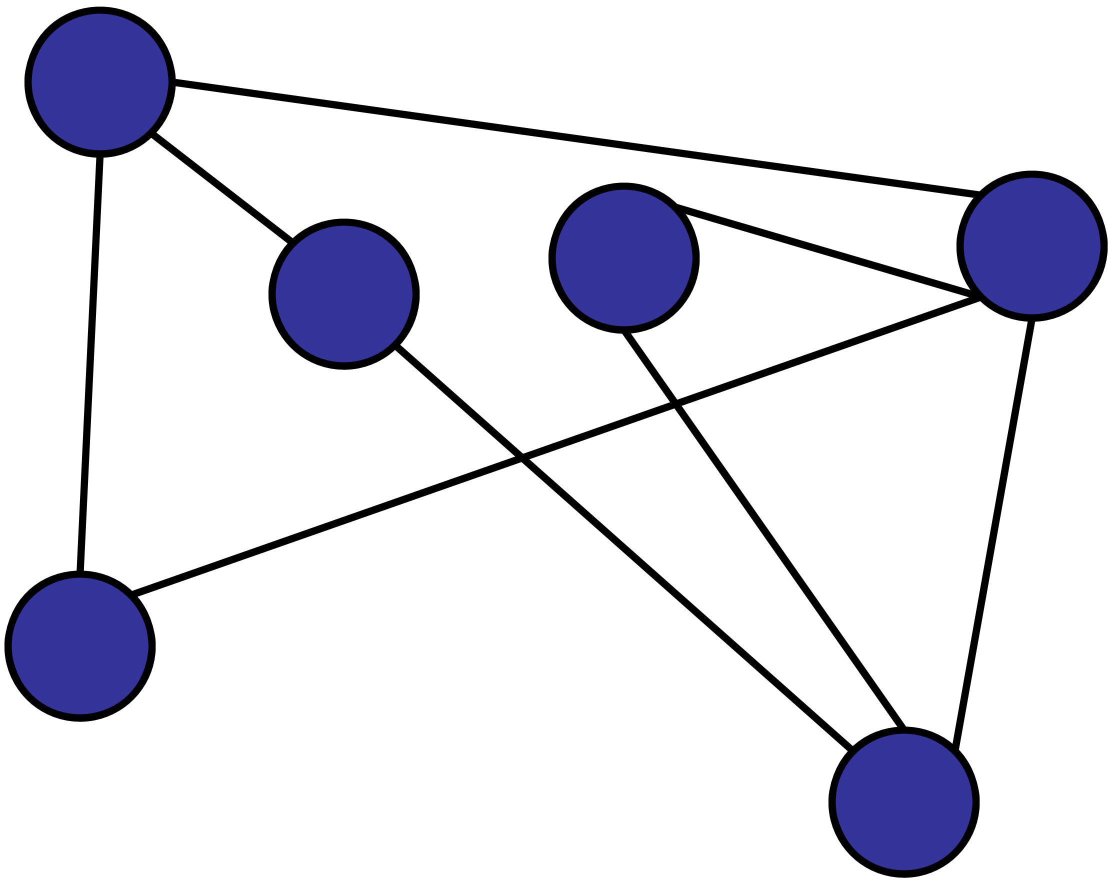




Is it a graph?

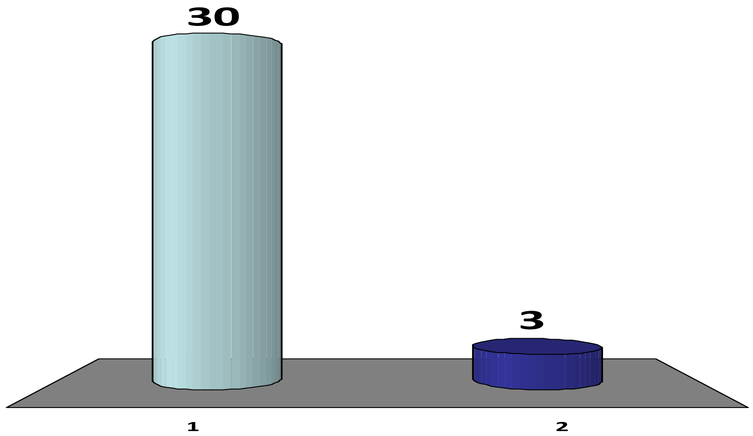
1. Yes
2. No.

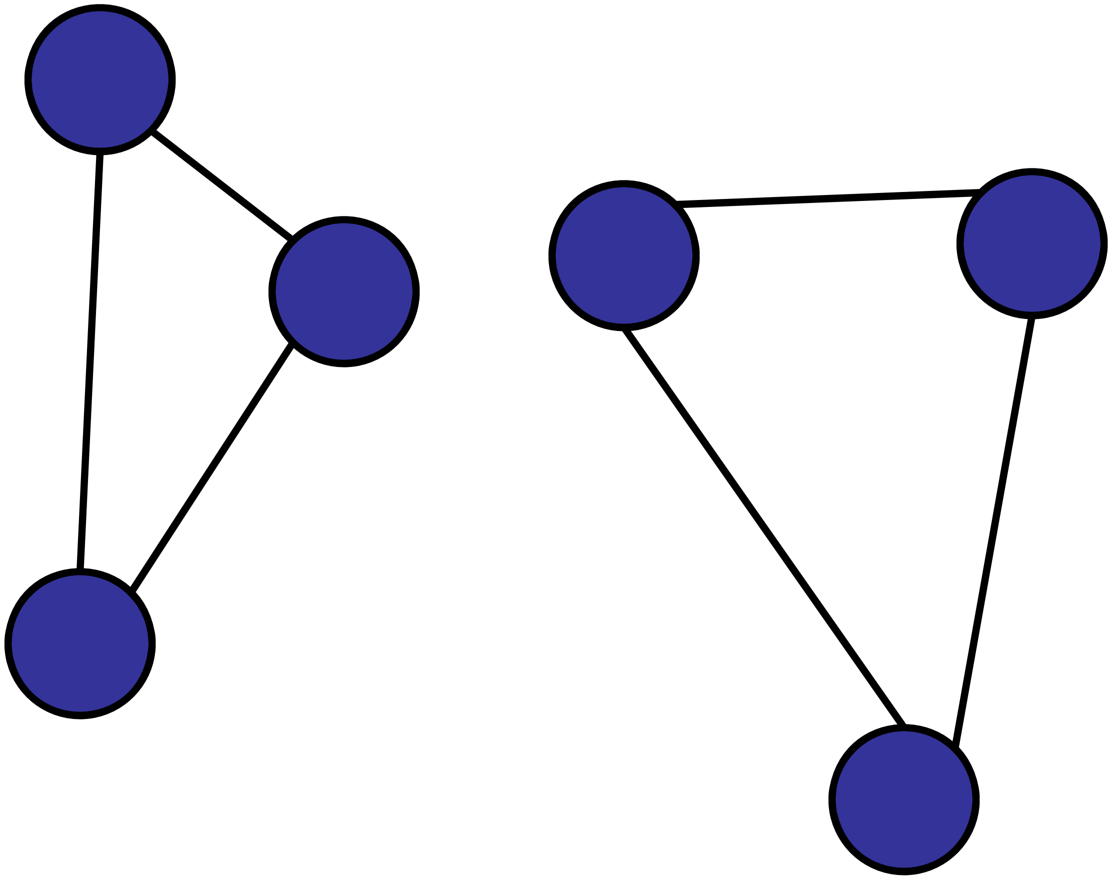




Is it a graph?

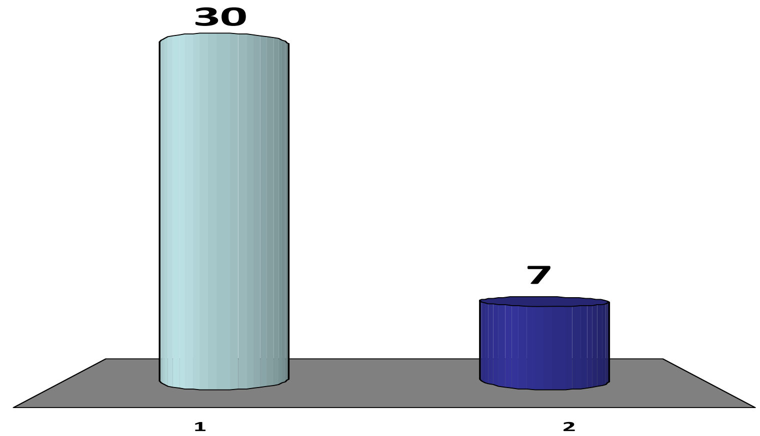
1. Yes ✓
2. No.

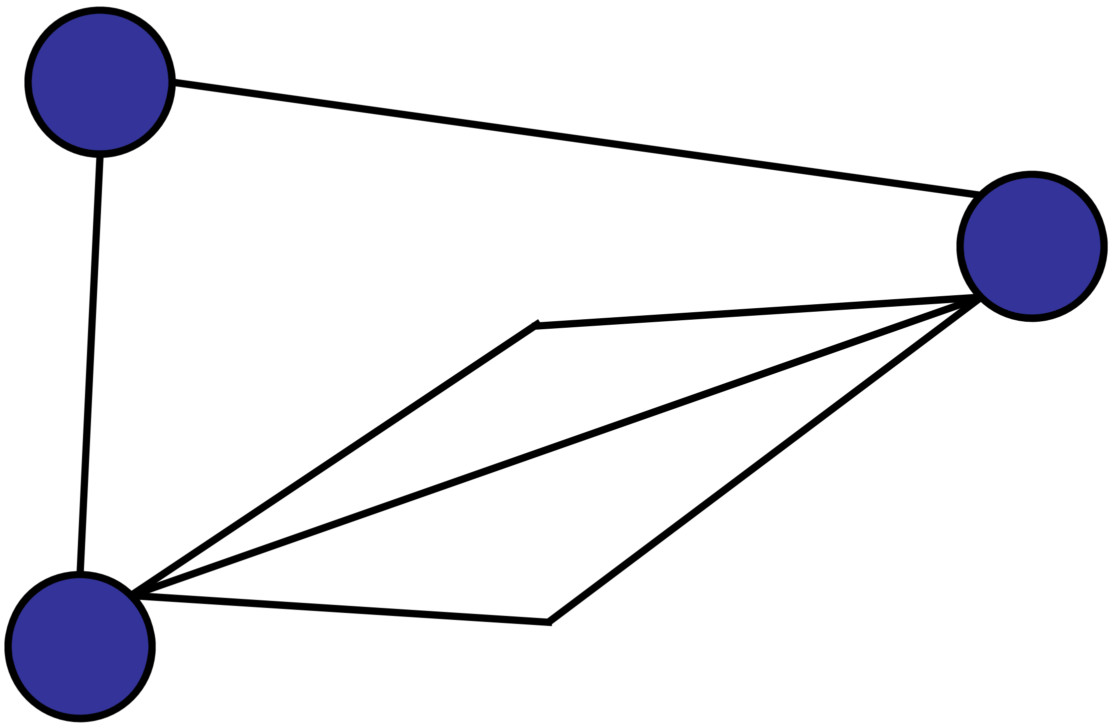




Is it a graph?

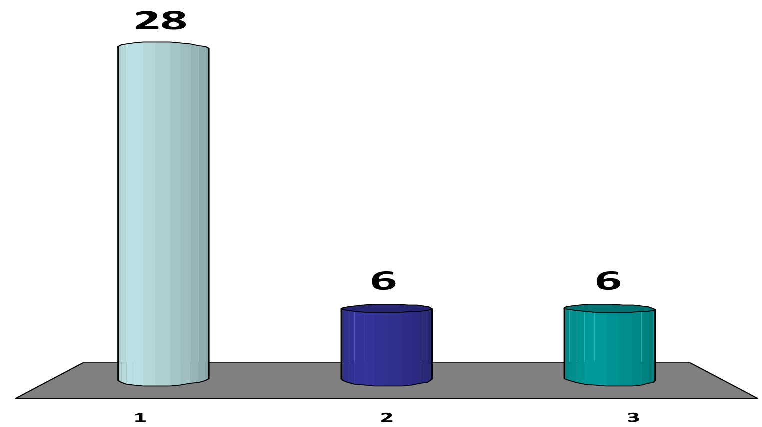
1. Yes
2. No.





Is it a graph?

1. Yes
2. No.
3. I am not quite sure



Harary, F. and Read, R. "Is the Null Graph a Pointless Concept?" In Graphs and Combinatorics Conference, George Washington University. New York: Springer-Verlag, 1973.

IS THE NULL-GRAPH A POINTLESS CONCEPT?

Frank Harary
University of Michigan
and Oxford University

Ronald C. Read
University of Waterloo

ABSTRACT

The graph with no points and no lines is discussed critically. Arguments for and against its official admittance as a graph are presented. This is accompanied by an extensive survey of the literature. Paradoxical properties of the null-graph are noted. No conclusion is reached.

What is a graph?

Graph consists of two types of elements:

- Nodes (or vertices)
  - At least one.

- Edges (or arcs)
  - Each edge connects two nodes in the graph
  - Each edge is unique.

What is a hypergraph?

Graph consists of two types of elements:

- Nodes (or vertices)
  - At least one.
- Edges (or arcs)
  - Each edge connects >= 2 nodes in the graph
  - Each edge is unique.

(Not in CS2020)

What is a multigraph?

Graph consists of two types of elements:

- Nodes (or vertices)
  - At least one.

- Edges (or arcs)
  - Each edge connects two nodes in the graph
  - Two nodes may be connected by more than one edge.

(Maybe in CS2020, maybe not.)

What is a graph?

Graph $G = \langle V, E \rangle$

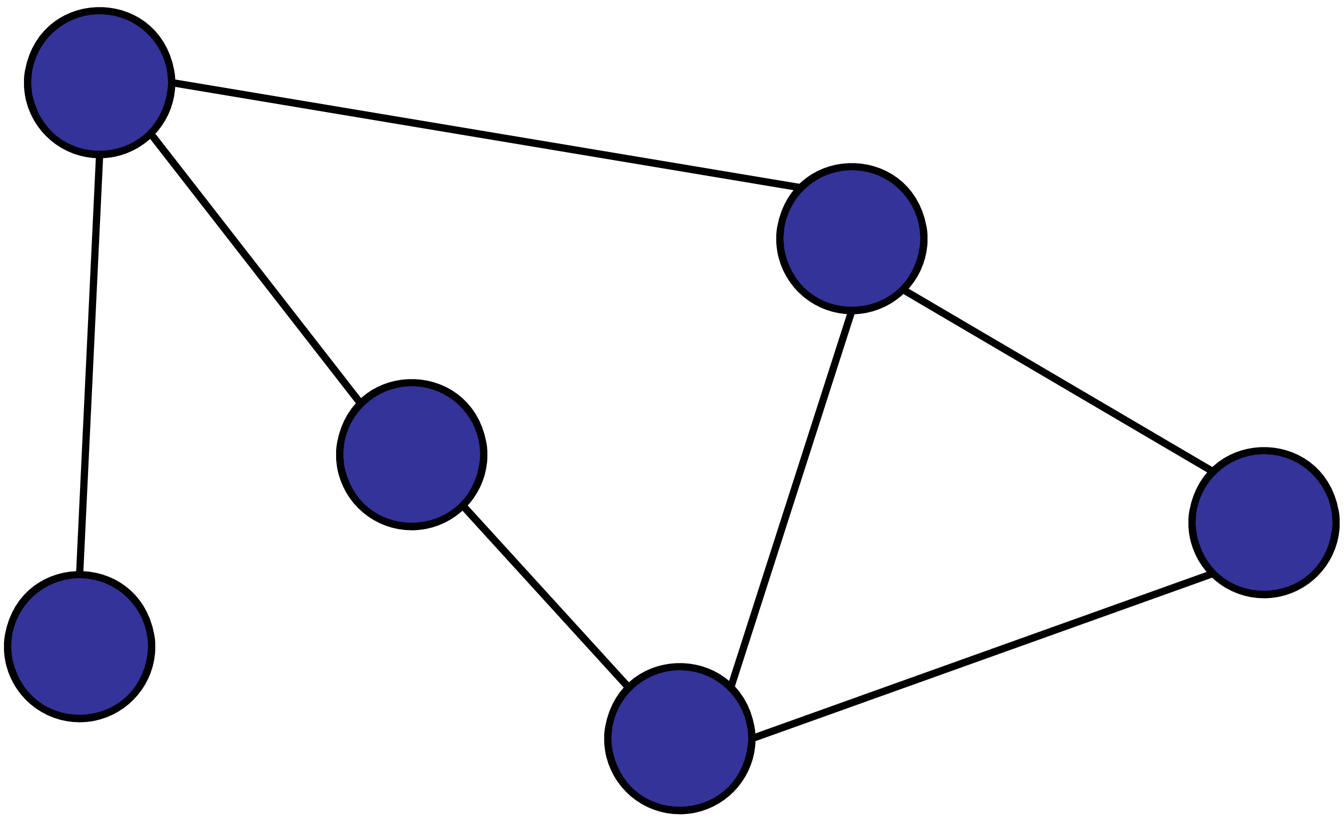
- V is a set of nodes
  - At least one: $|V| > 0$.
- E is a set of edges:
  - $E \subseteq \{ (v,w) : (v \in V), (w \in V) \}$
  - $e = (v,w)$
  - For all $e_1, e_2 \in E : e_1 \neq e_2$

Do not allow self-loops

Graph Terminology

Connected:

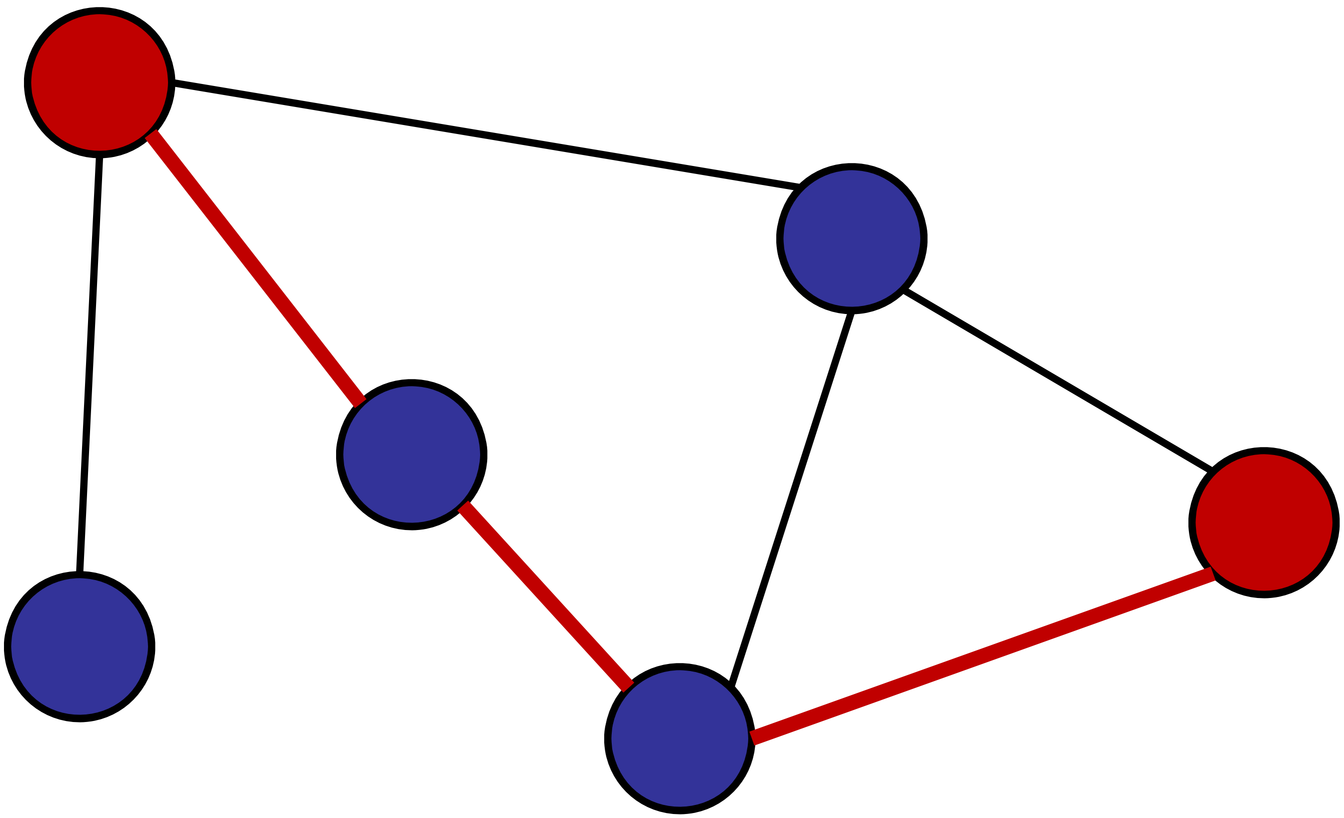
– Every pair of nodes is connected by a path.



Graph Terminology

Connected:

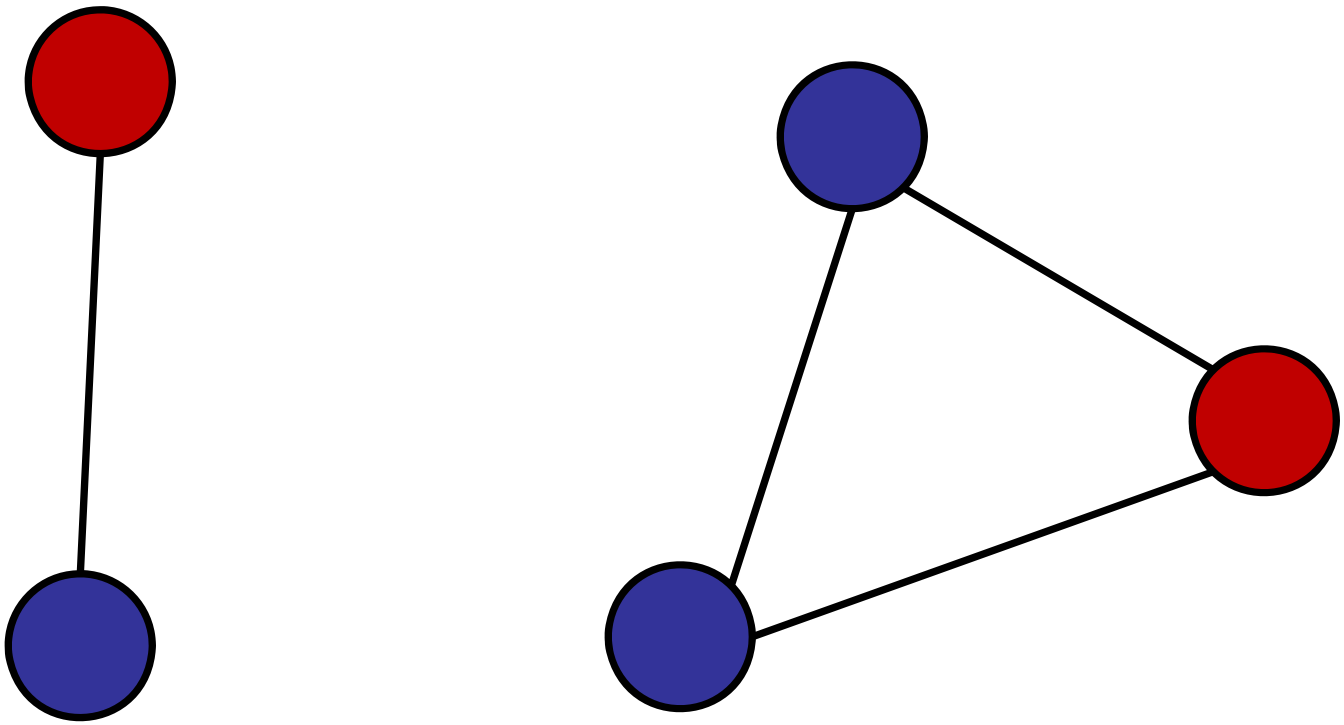
– Every pair of nodes is connected by a **path**.



Graph Terminology

Disconnected:

– Some pair of nodes is not connected by a path.

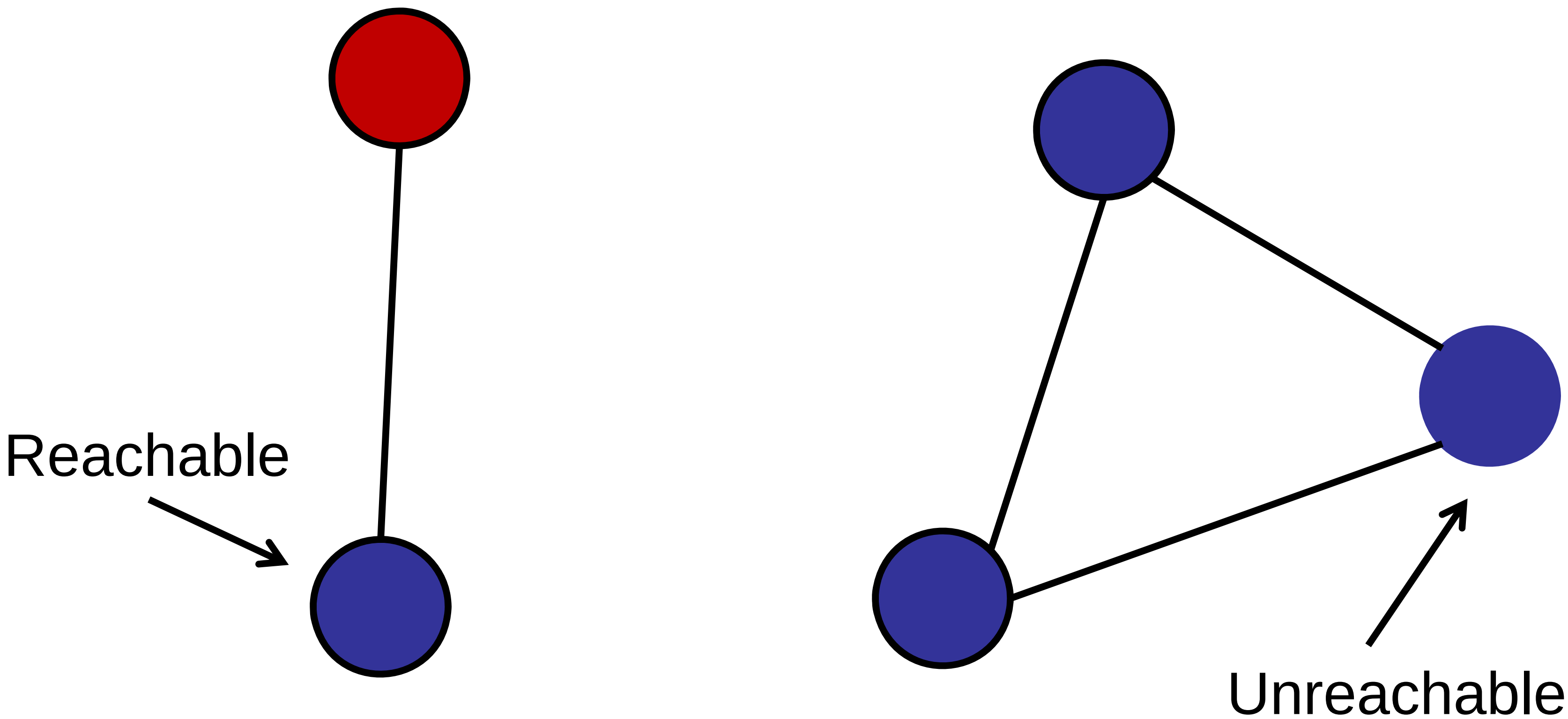


– Two **connected components.**

Graph Terminology

Disconnected:

- Some pair of nodes is not connected by a path.

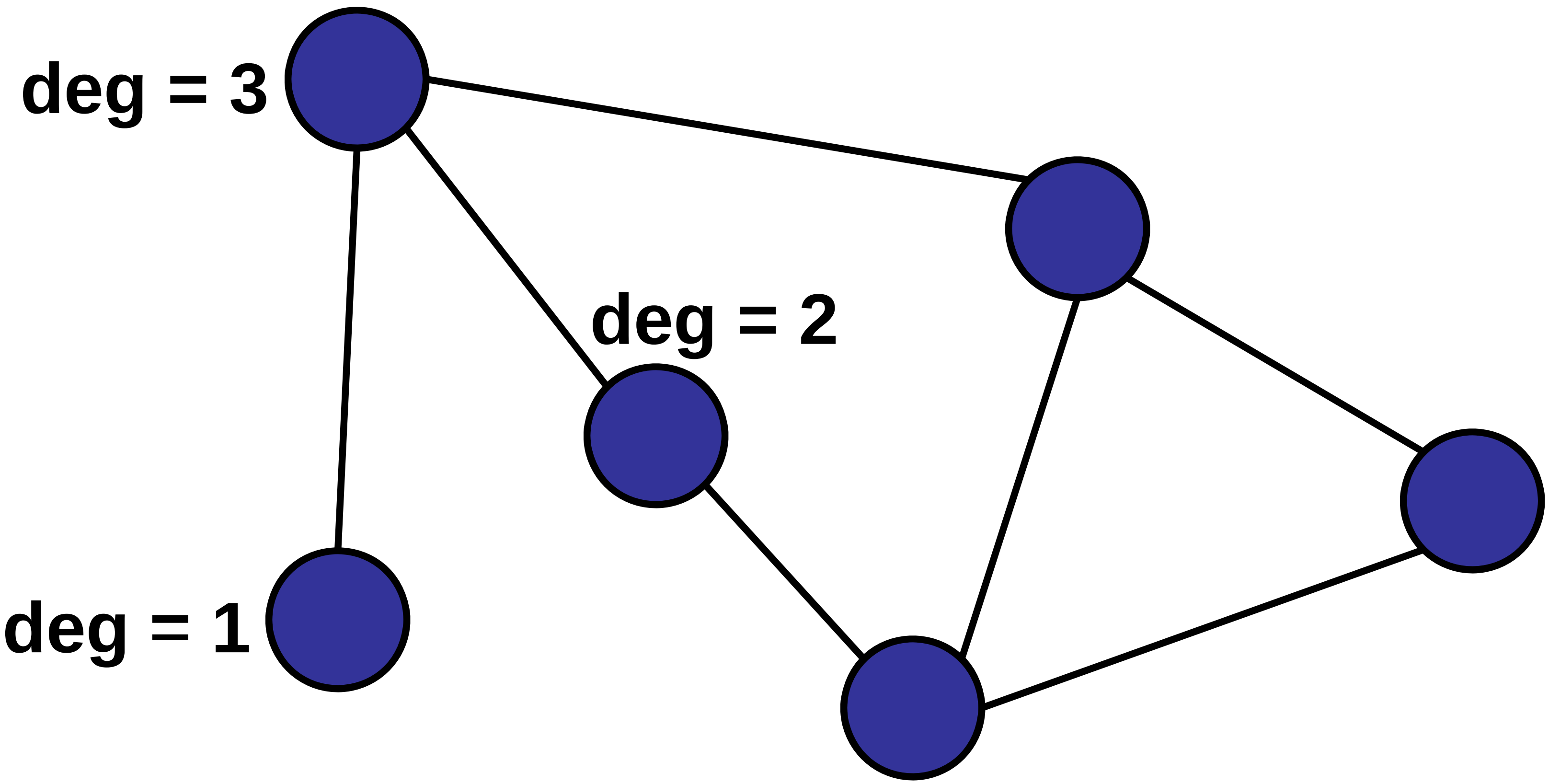


- Two **connected components.**

Graph Terminology

Degree of a node:

– Number of **adjacent** edges.

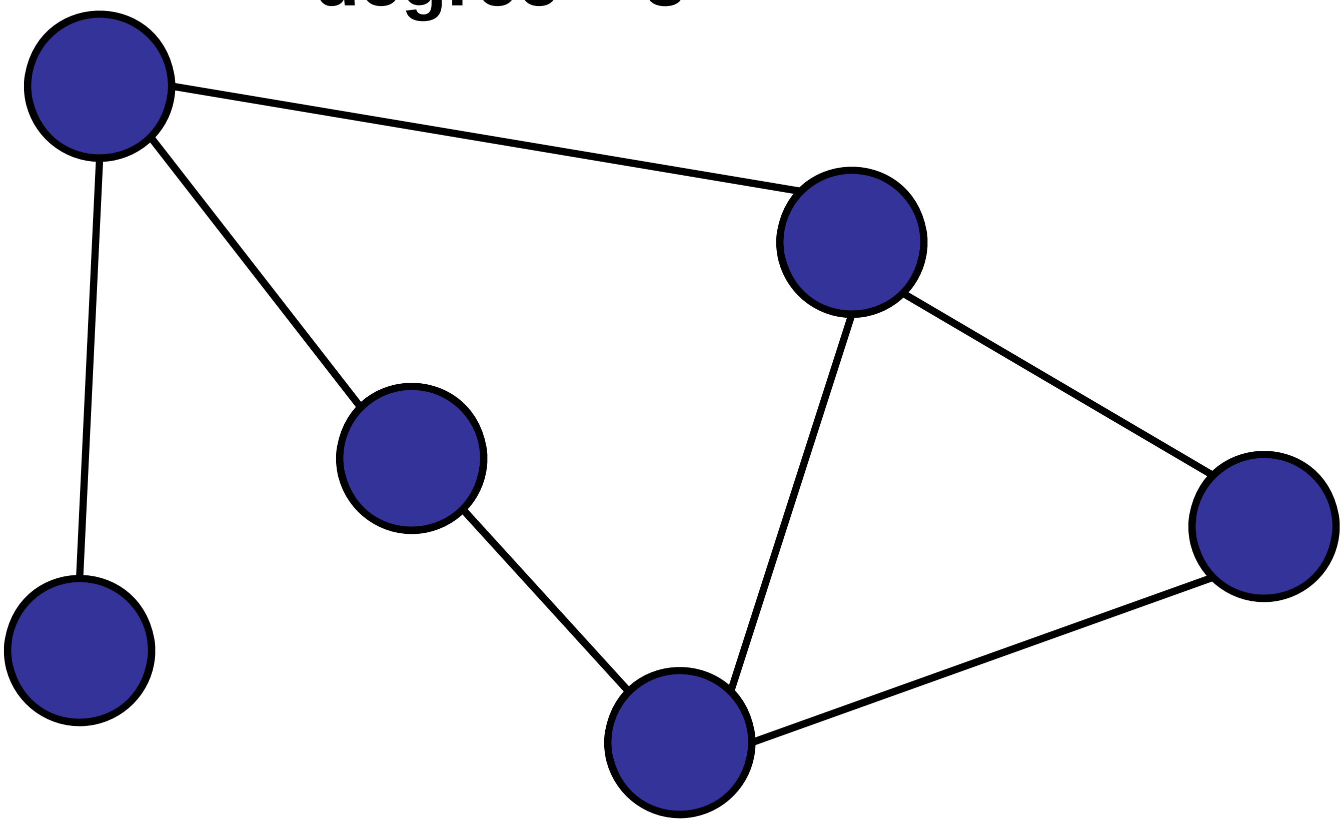


Graph Terminology

Degree of a graph:

– Maximum number of **adjacent** edges.

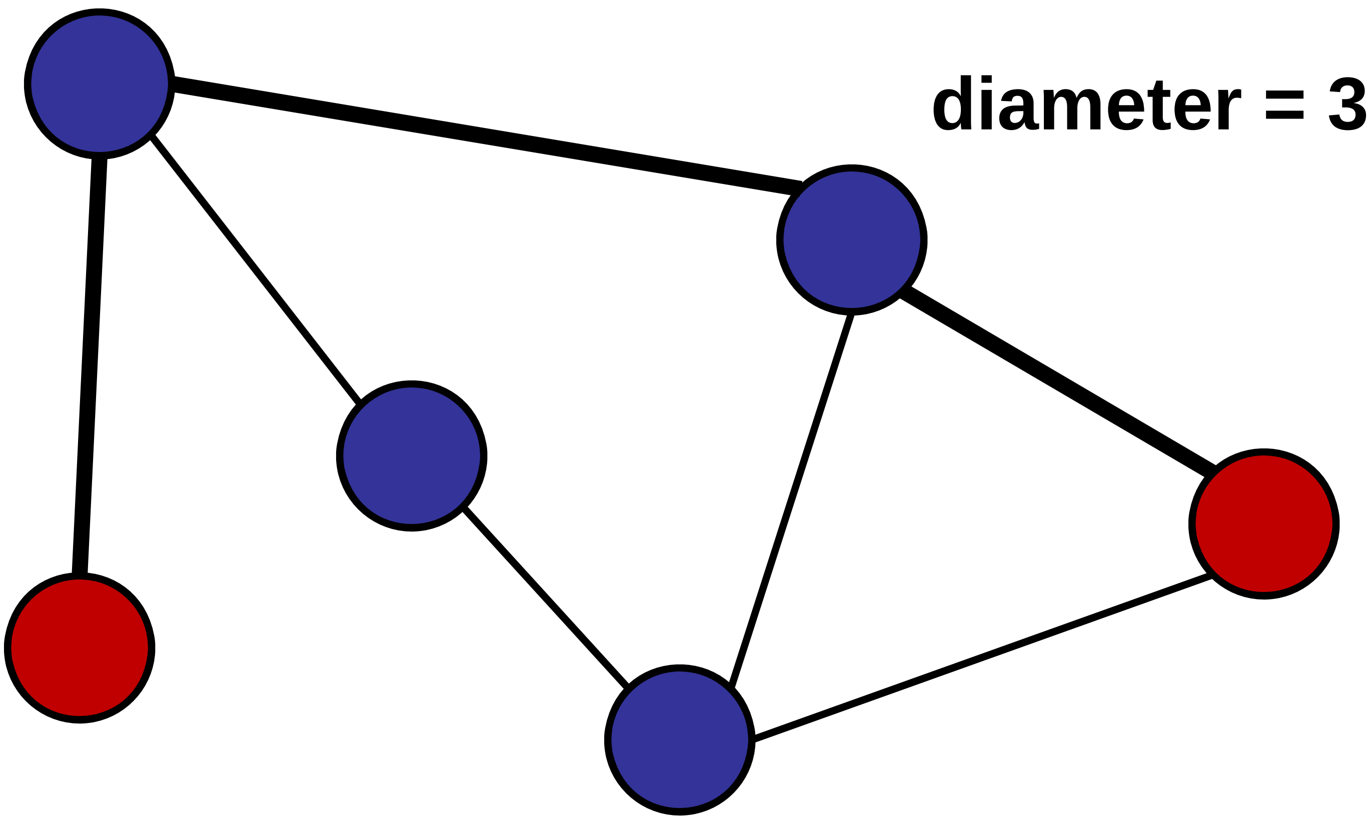
degree = 3



Graph Terminology

Diameter:

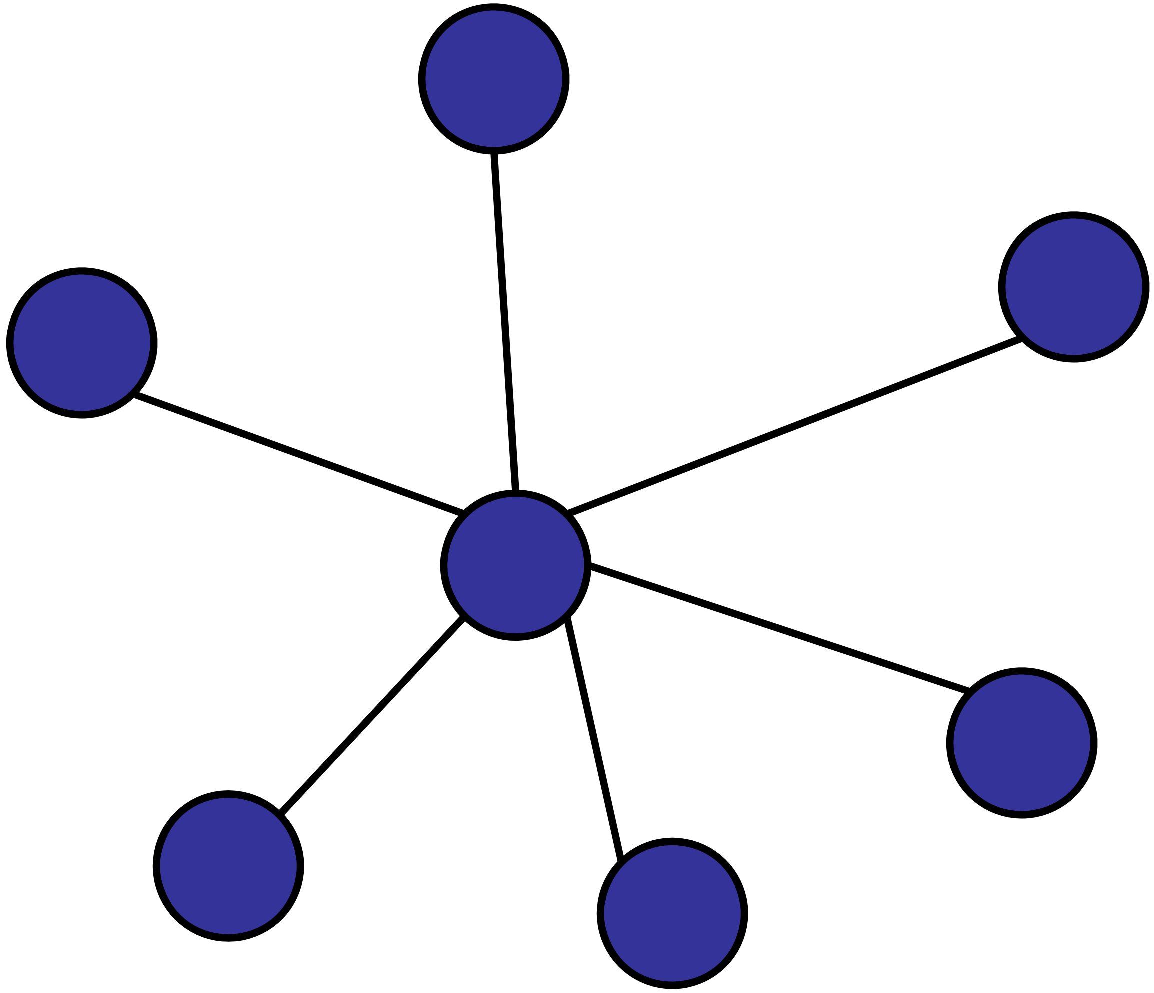
– Maximum distance between two nodes, following the shortest path.



Special Graphs

Special Graphs

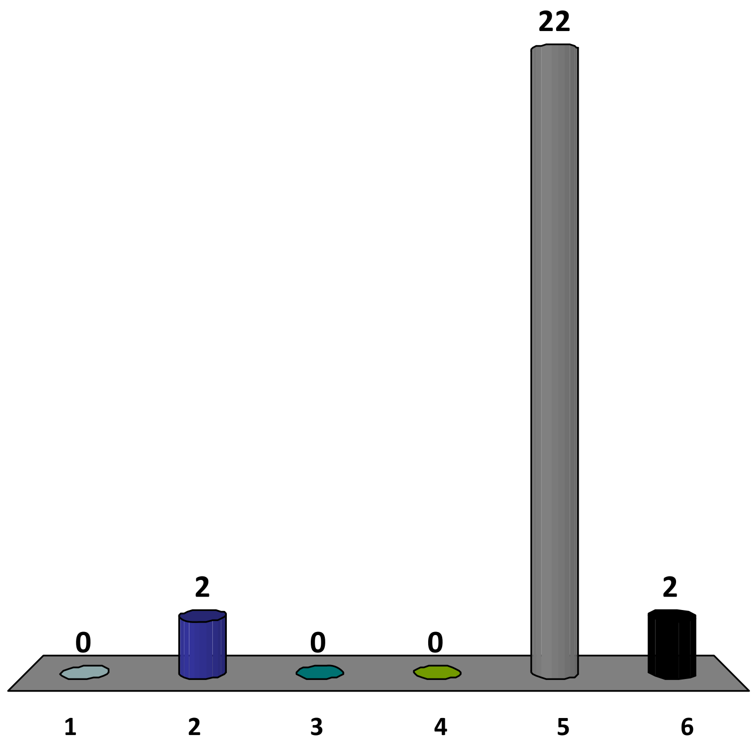
Star



One central node, all edges connect center to edges.

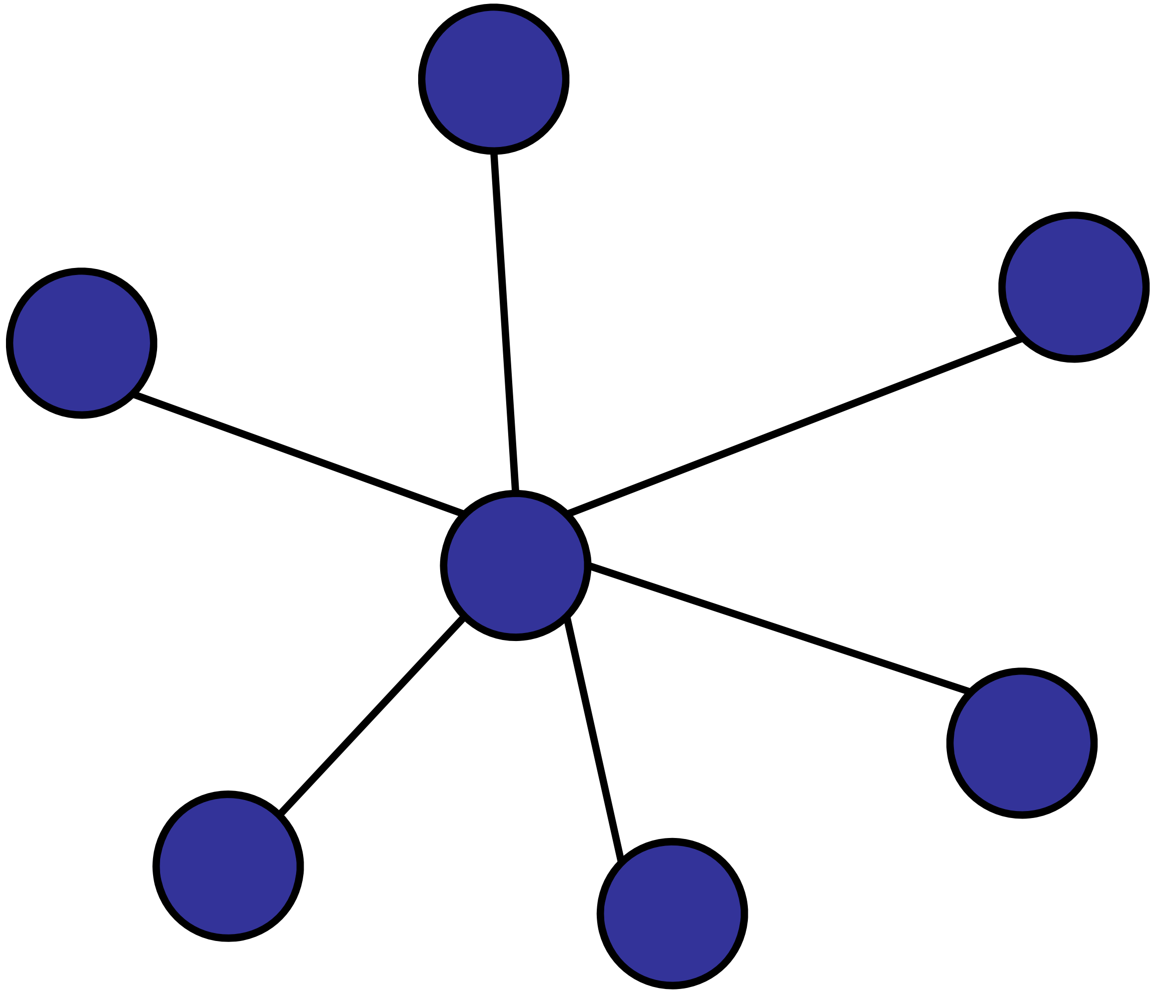
Degree of n-node star is:

1. 1
2. 2
3. n/2
4. n-2
5. n-1
6. n



Special Graphs

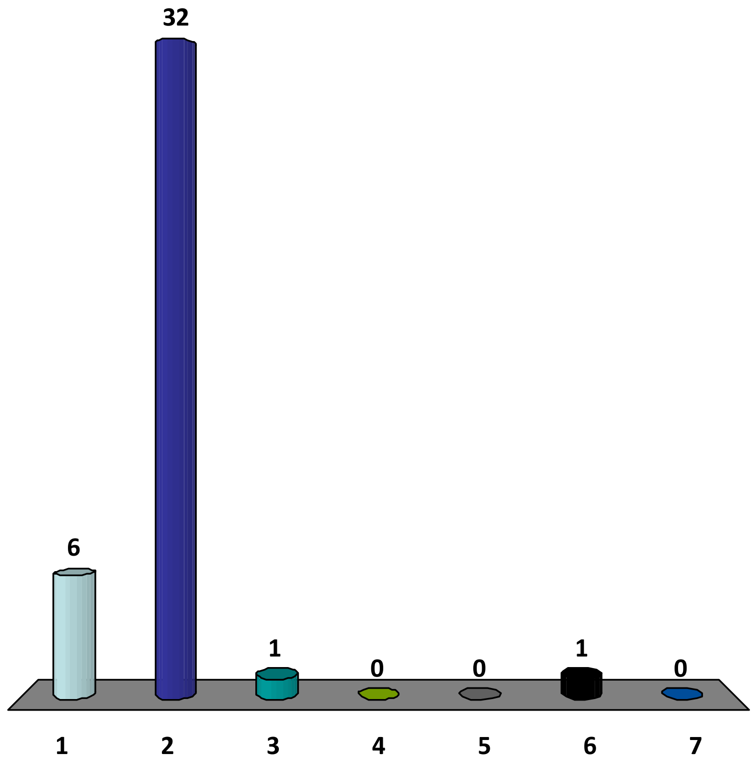
Star



One central node, all edges connect center to edges.

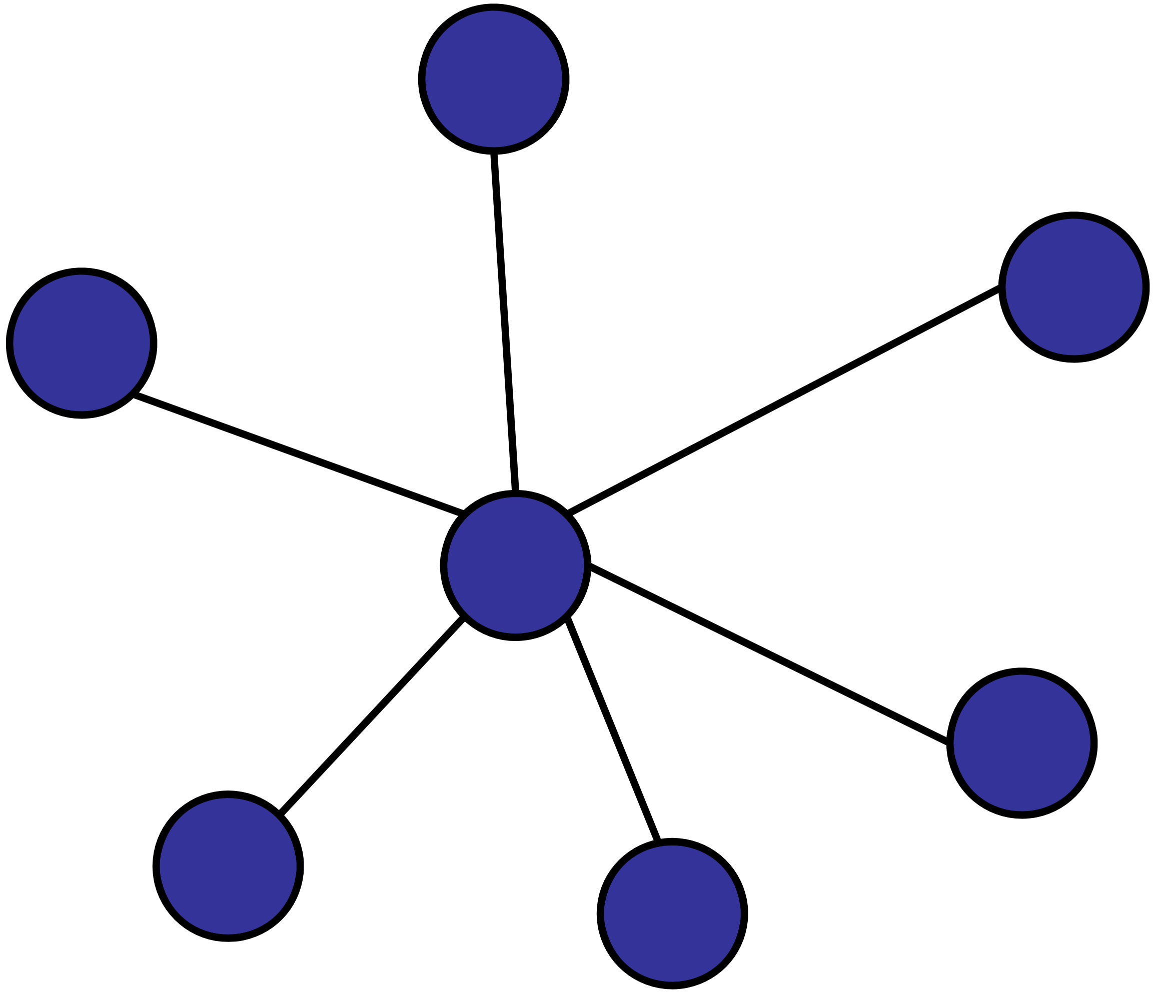
Diameter of n-node star:

1. 1
2. 2
3. 3
4. n/2
5. n-2
6. n-1
7. n



Special Graphs

Star



One central node, all edges connect center to edges.

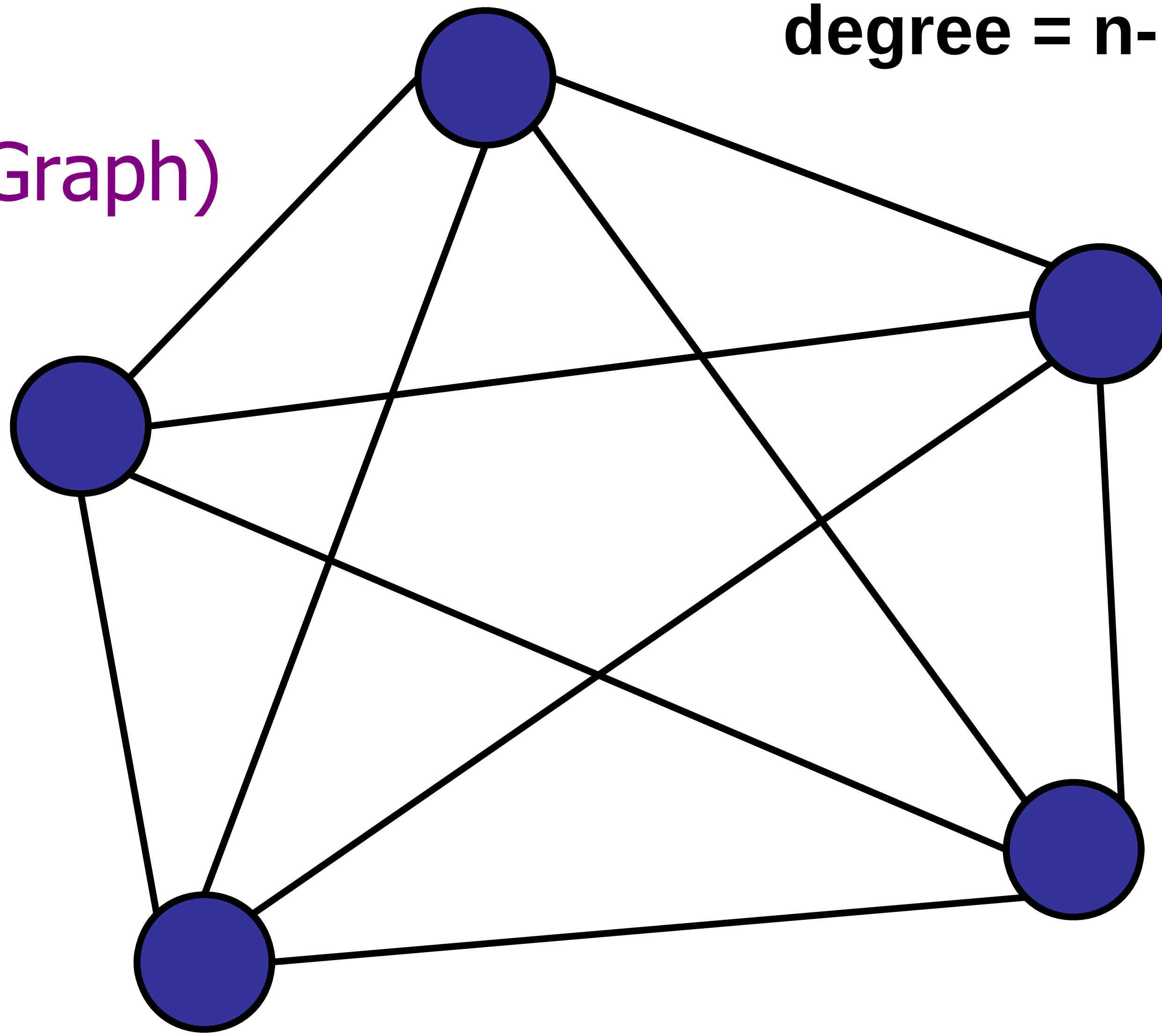
Special Graphs

Clique

(Complete Graph)

diameter = 1

degree = n-1



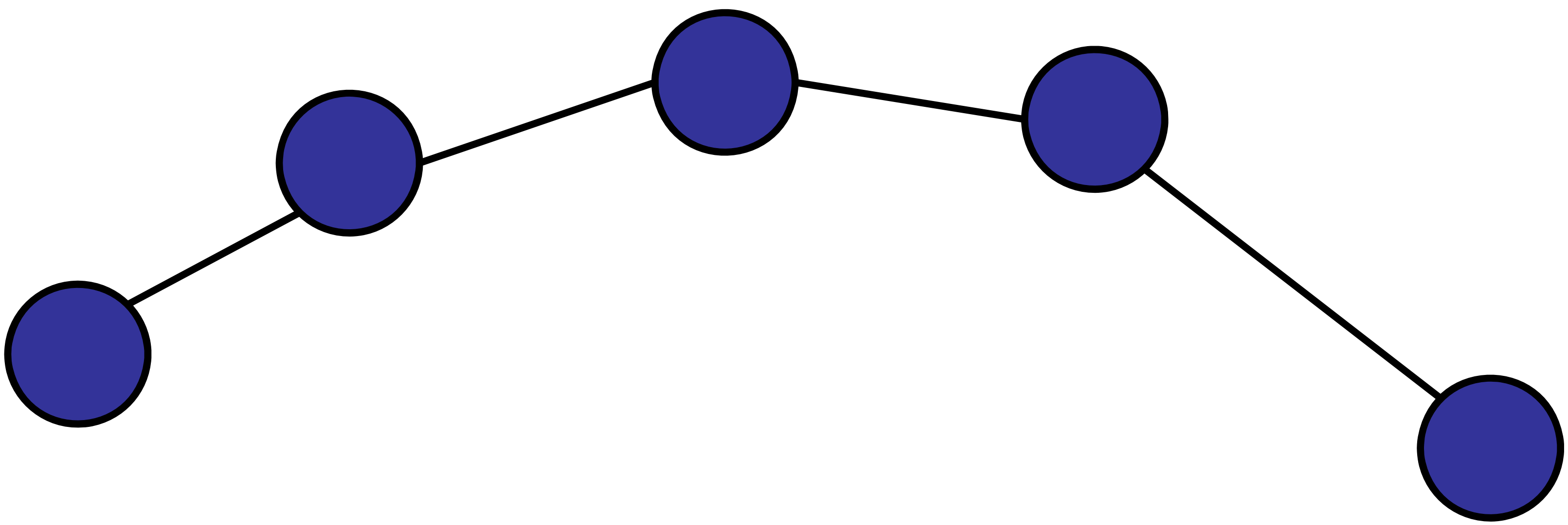
All pairs connected by edges.

Special Graphs

Line (or path)

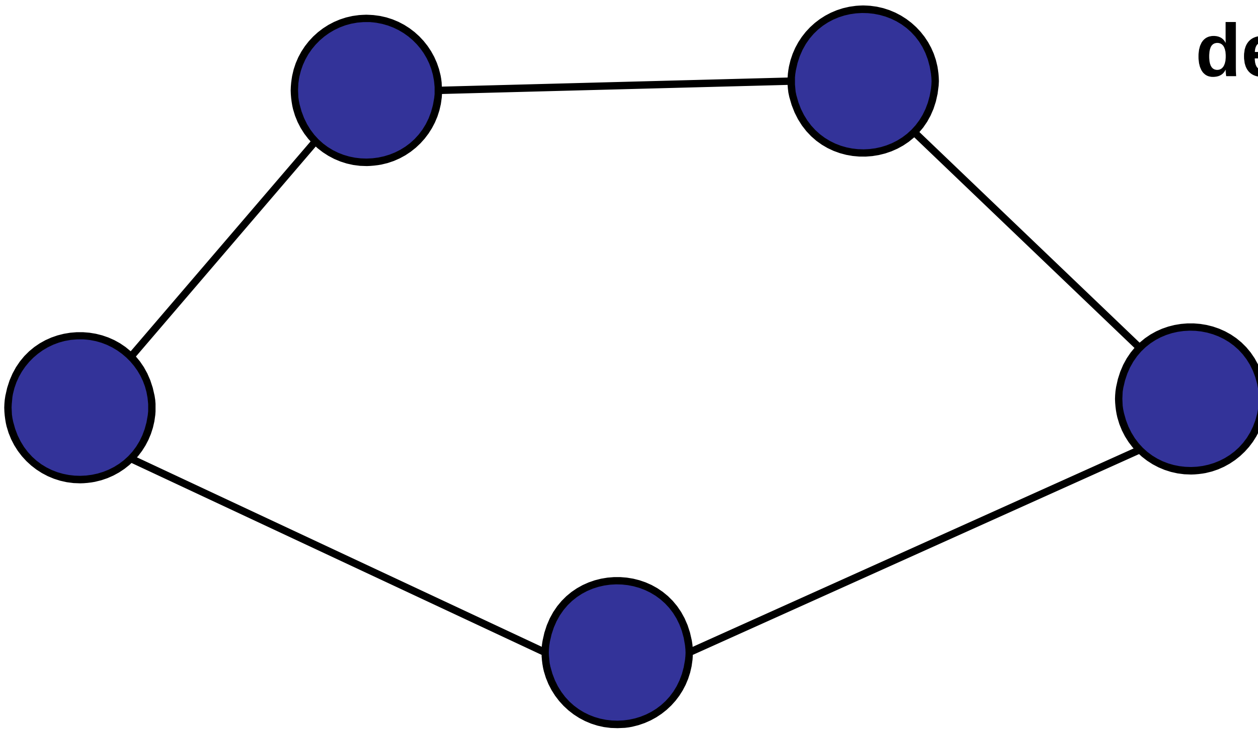
diameter = n-1

degree = 2



Special Graphs

Cycle



diameter = n/2

or

diameter = n/2-1

degree = 2

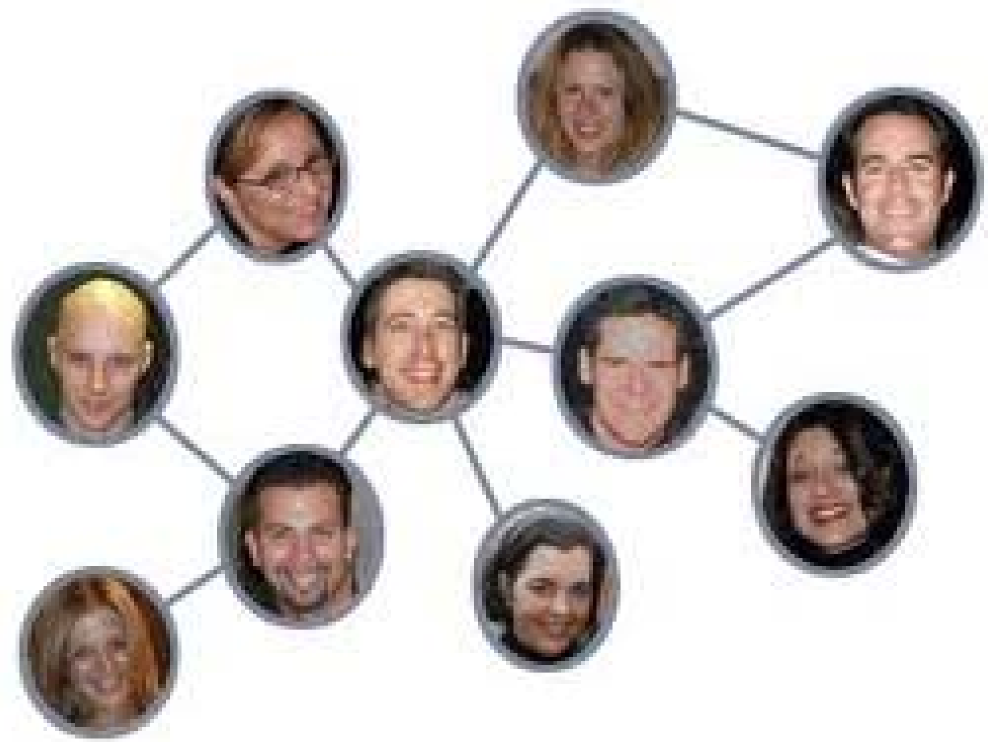
Where do we find graphs?

Where do we find graphs?

Social network:

- Nodes are people
- Edge = friendship

facebook



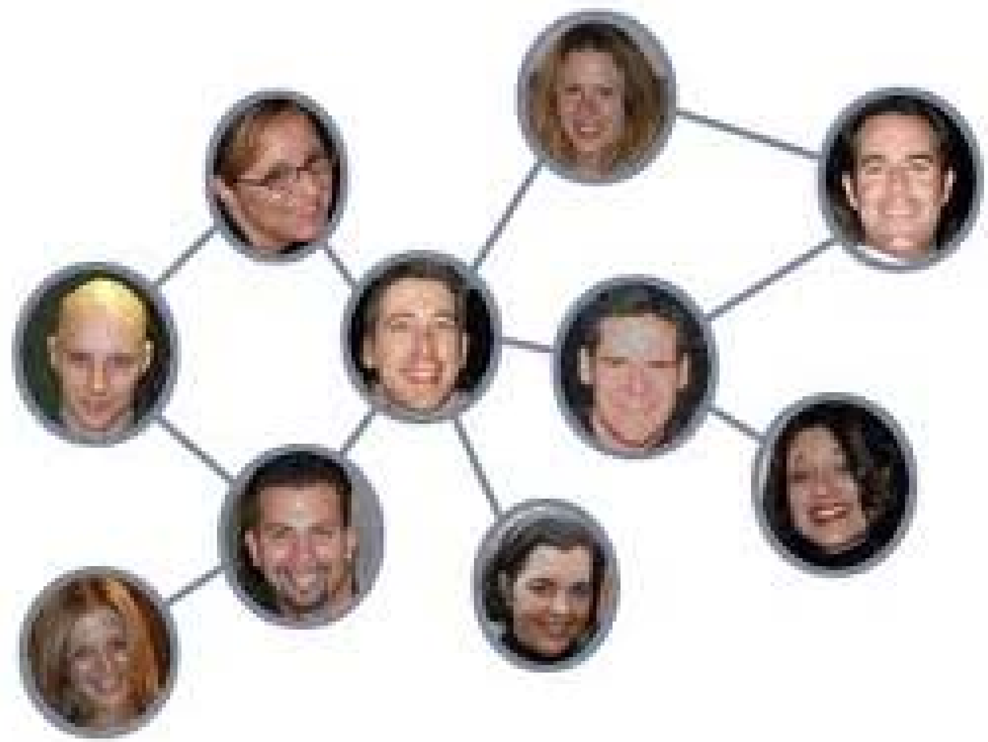
Where do we find graphs?

Social network:

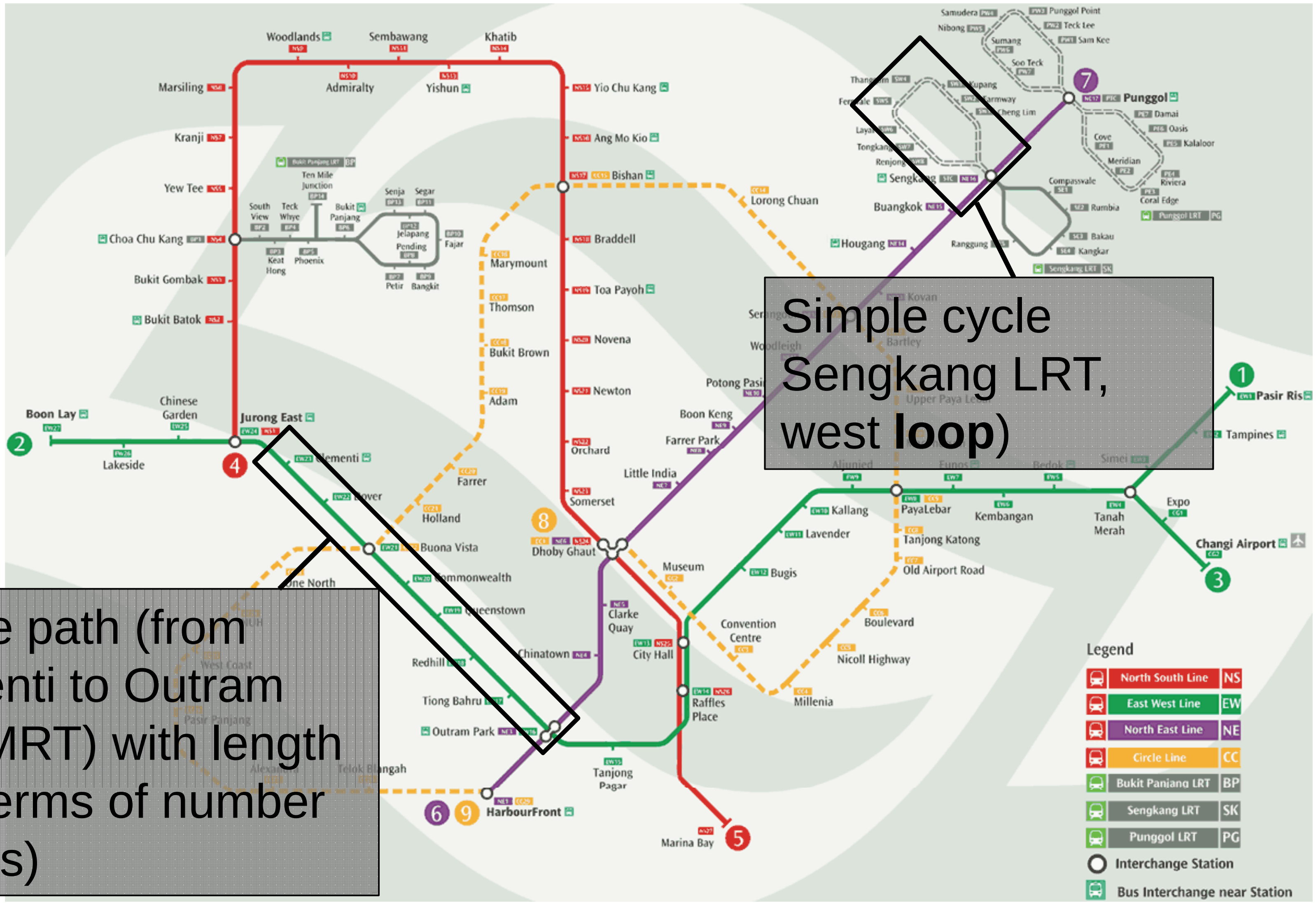
- Nodes are people
- Edge = friendship

Questions:

- Connected?
- Diameter?
- Degree?



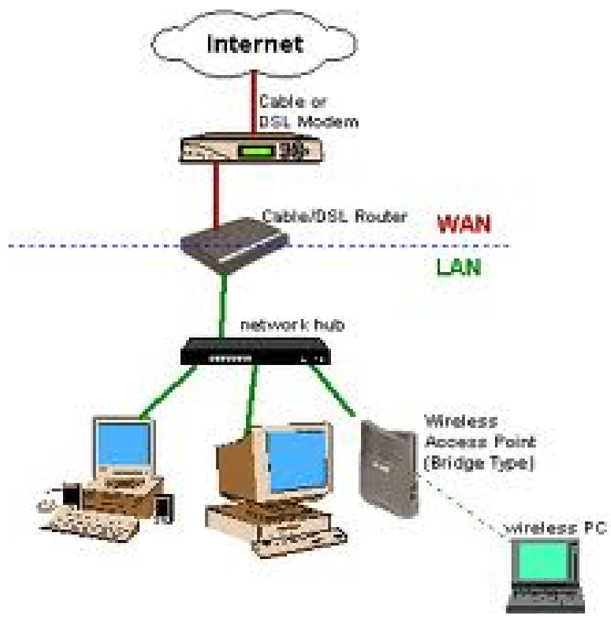
Transportation Network

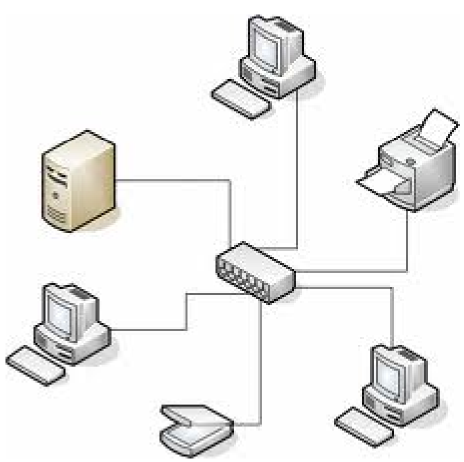


Simple cycle Sengkang LRT, west **loop**)

Simple path (from Clementi to Outram Park MRT) with length 7 (in terms of number of hops)

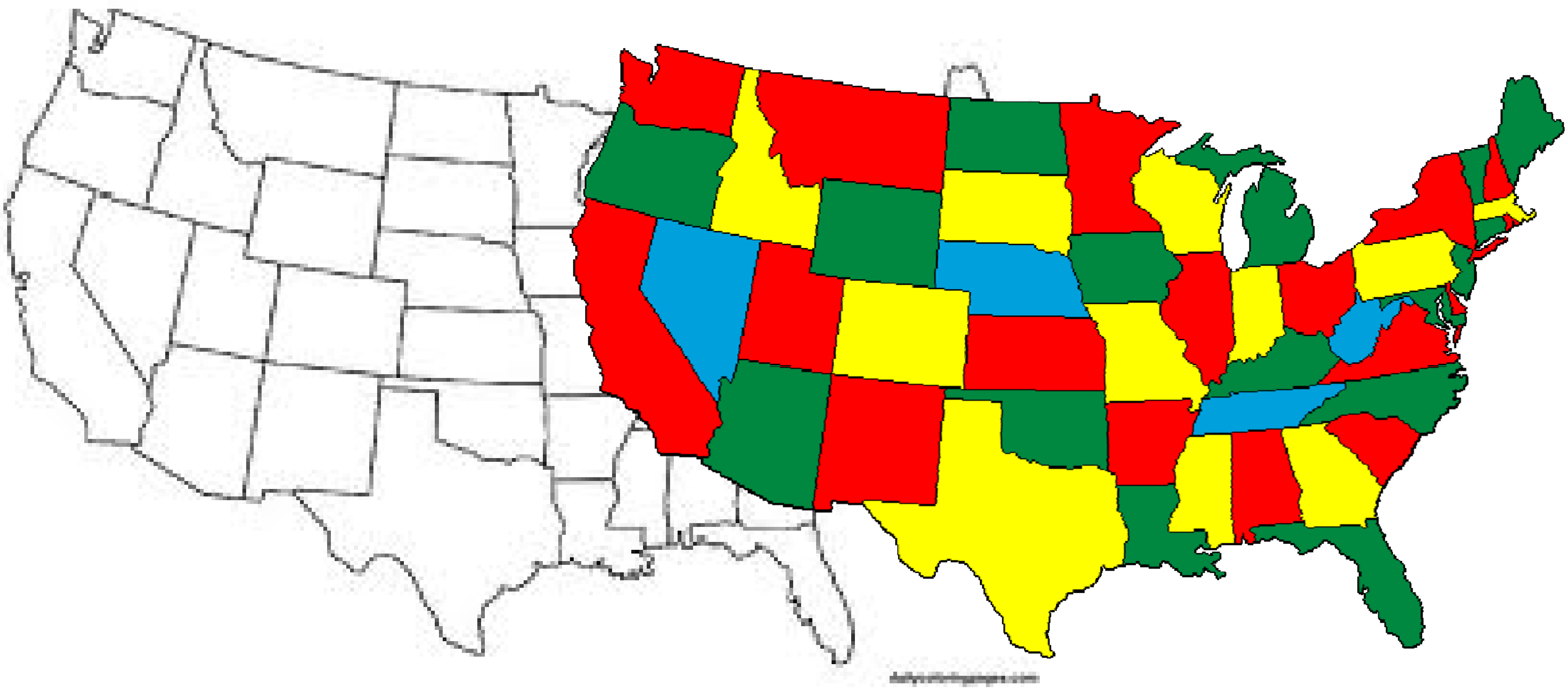
Internet / Computer Networks



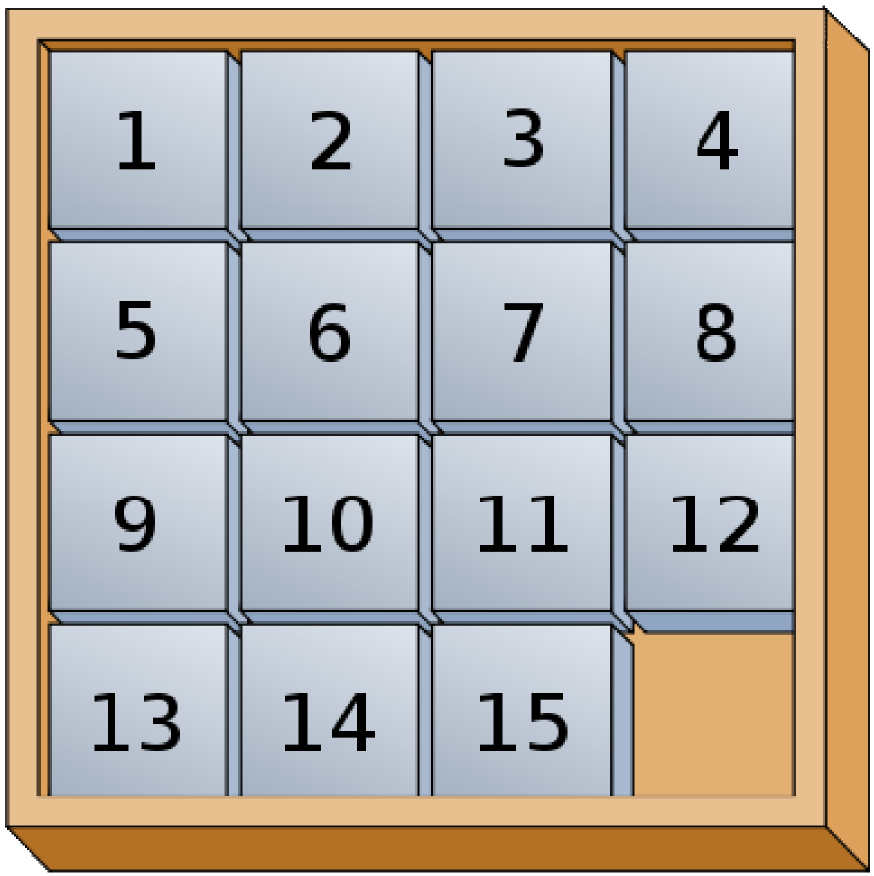


Communication Network

Optimization



Sliding Puzzle



Sliding Puzzle

4	5	7
3	1	6
8	2	

Sliding Puzzle

4	5	7
3	1	
8	2	6

Sliding Puzzle

4	5	
3	1	7
8	2	6

Sliding Puzzle

4		5
3	1	7
8	2	6

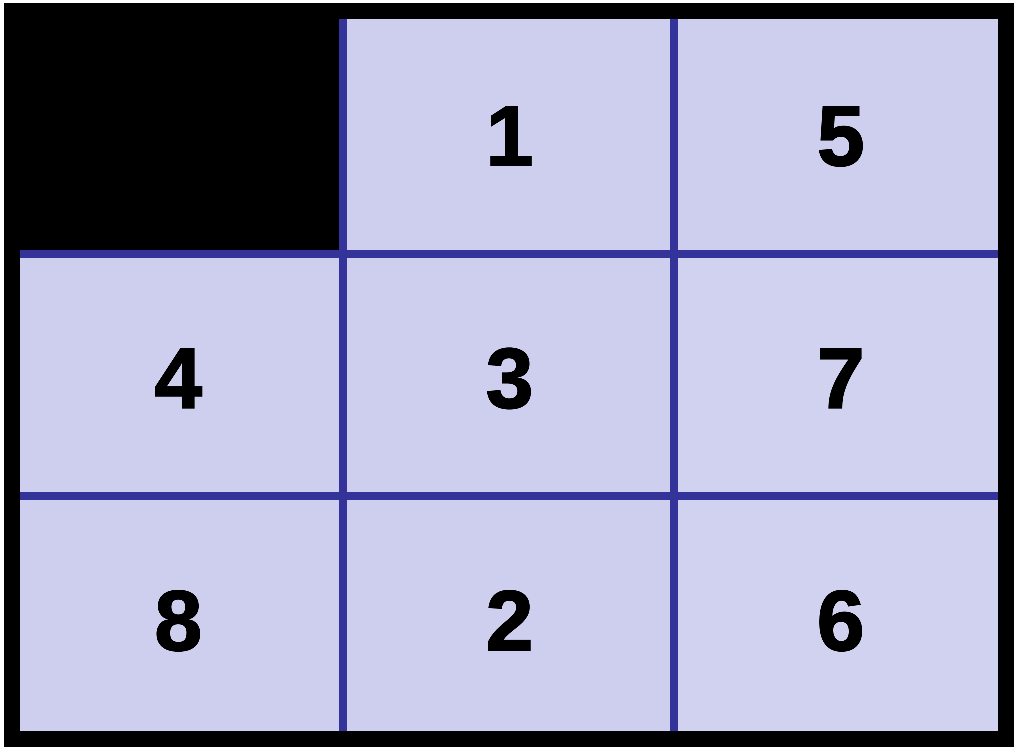
Sliding Puzzle

4	1	5
3		7
8	2	6

Sliding Puzzle

4	1	5
	3	7
8	2	6

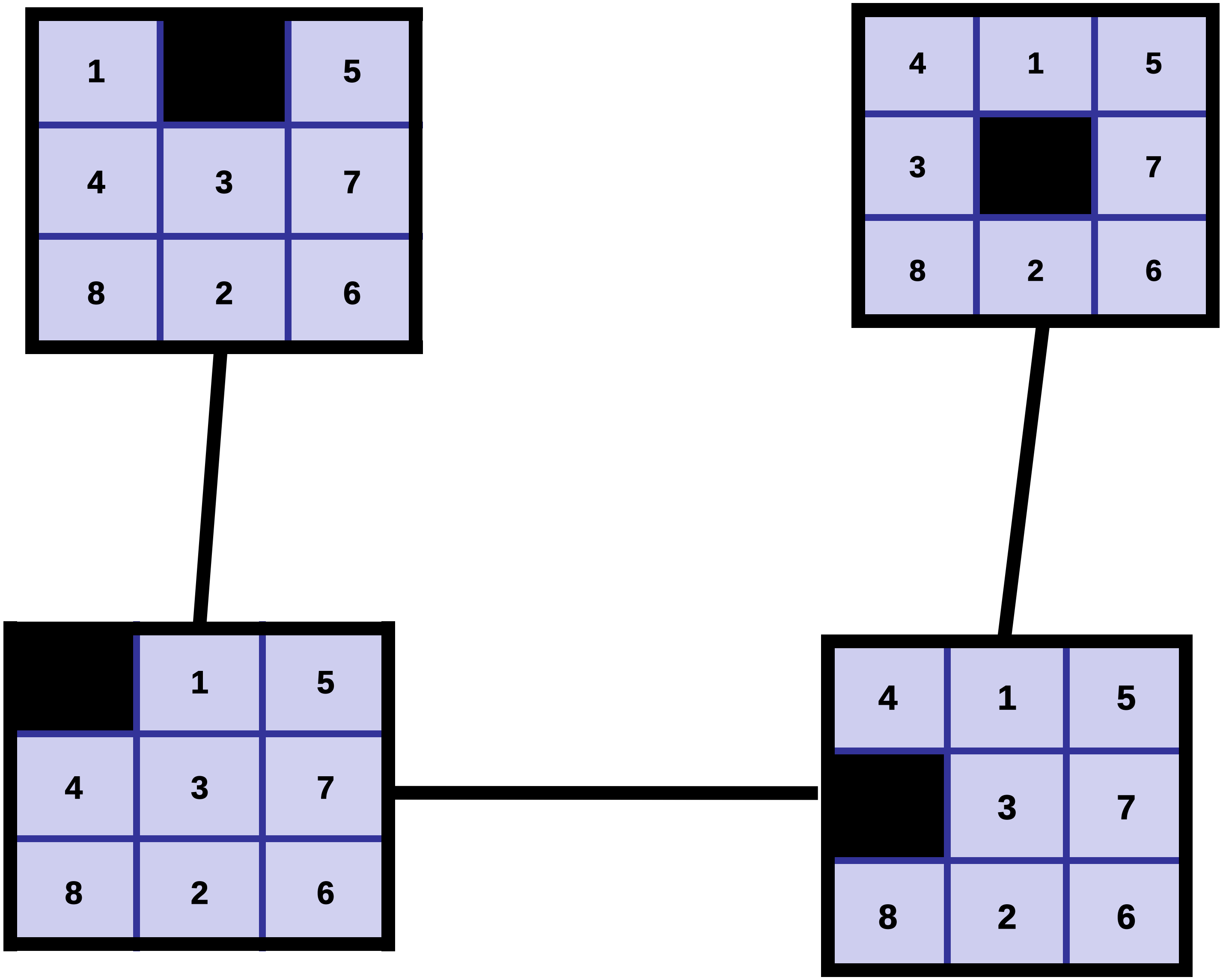
Sliding Puzzle



Sliding Puzzle

1		5
4	3	7
8	2	6

Sliding Puzzle is a Graph



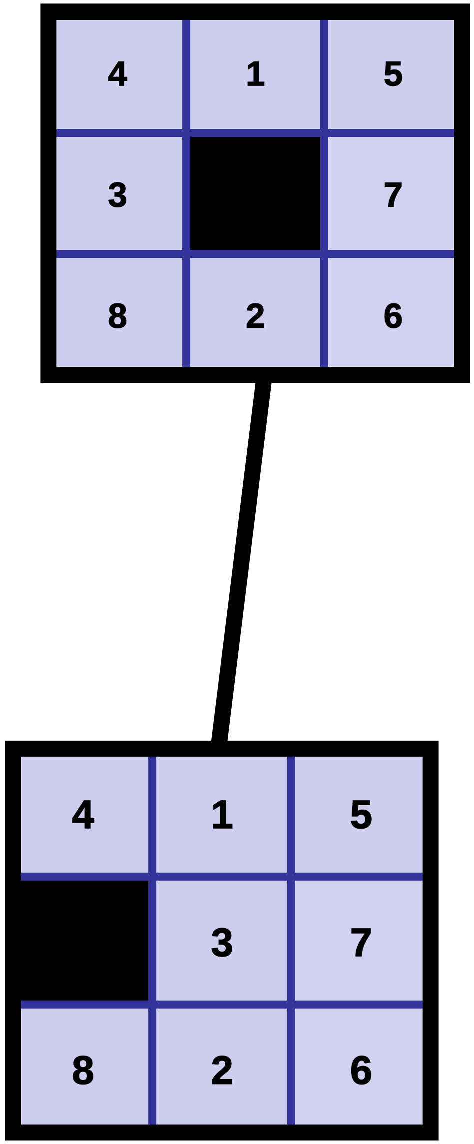
Sliding Puzzle

Nodes:

- State of the puzzle
- Permutation of nine tiles

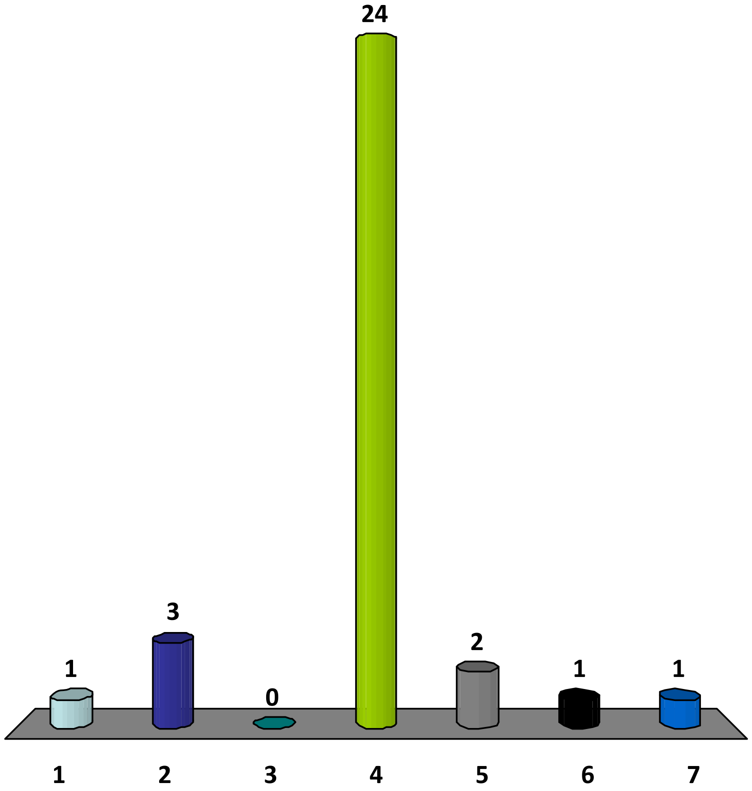
Edges:

- Two states are edges if they differ by only one move.



What is the maximum degree of the Sliding Puzzle graph?

1. 1
2. 2
3. 3
4. 4
5. n/2
6. n
7. n!



Sliding Puzzle

Nodes:

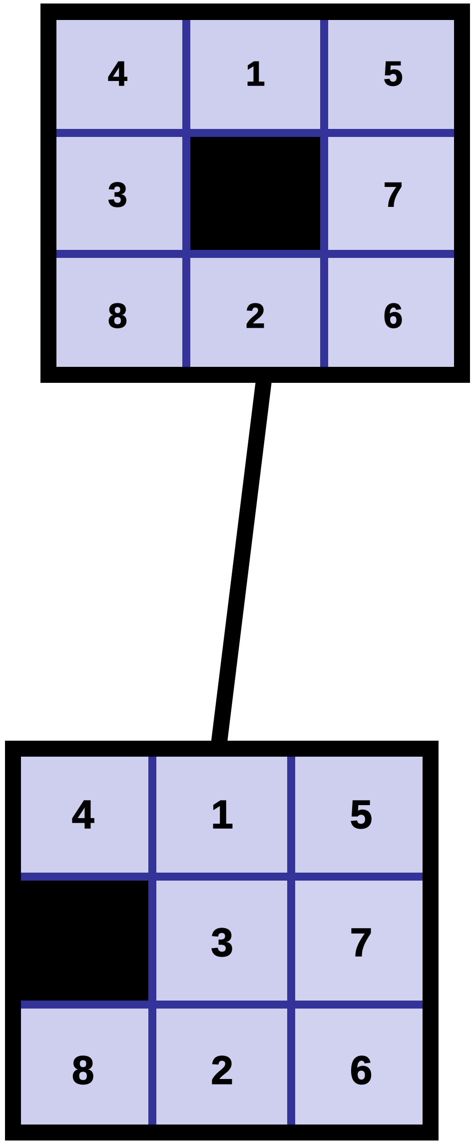
- State of the puzzle
- Permutation of nine tiles

Edges:

- Two states are edges if they differ by only one move.

Nodes = 9! = 362,880

Edges < 4*9! < 1,451,520

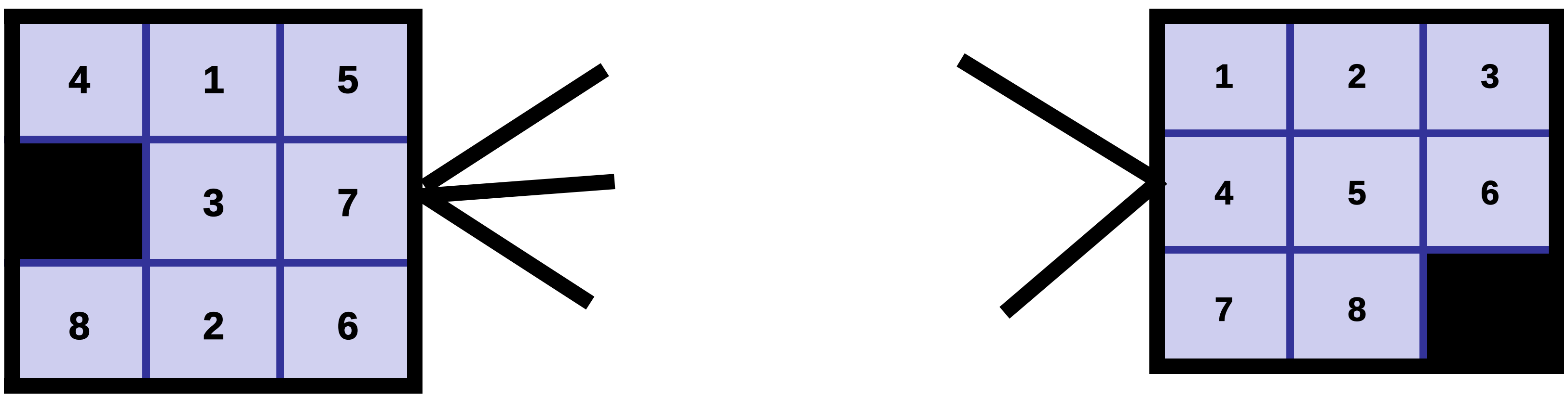


Sliding Puzzle

Number of moves to solve the puzzle?

Initial, scrambled state:

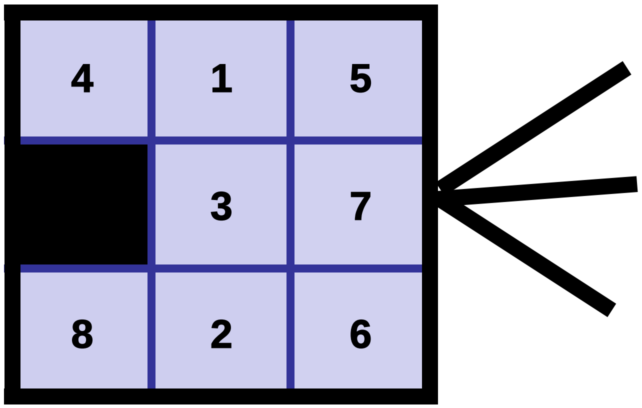
Final, unscrambled state:



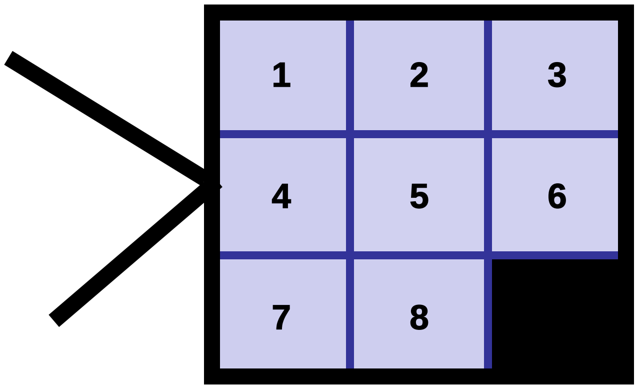
Sliding Puzzle

Number of moves <= Diameter

Initial, scrambled state:



Final, unscrambled state:



How many moves does it take to solve the following puzzle?

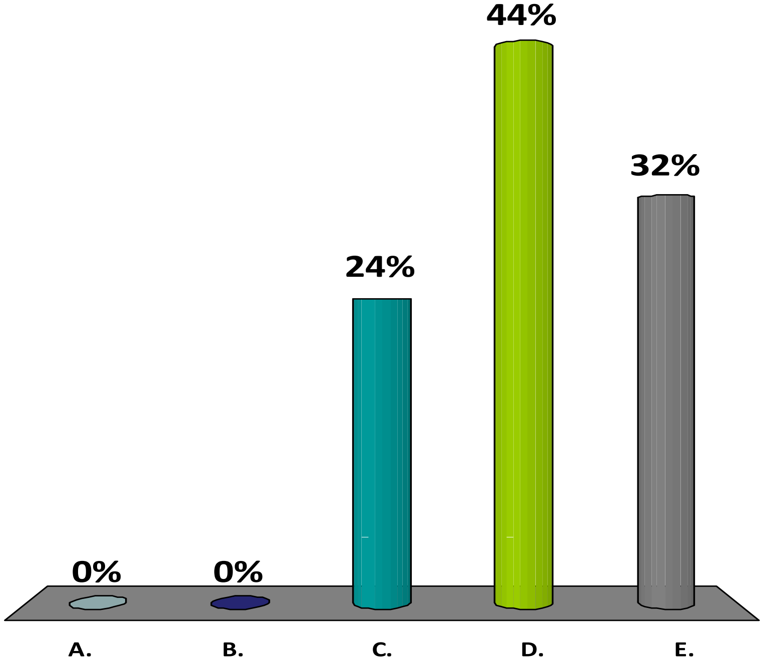
A. 1

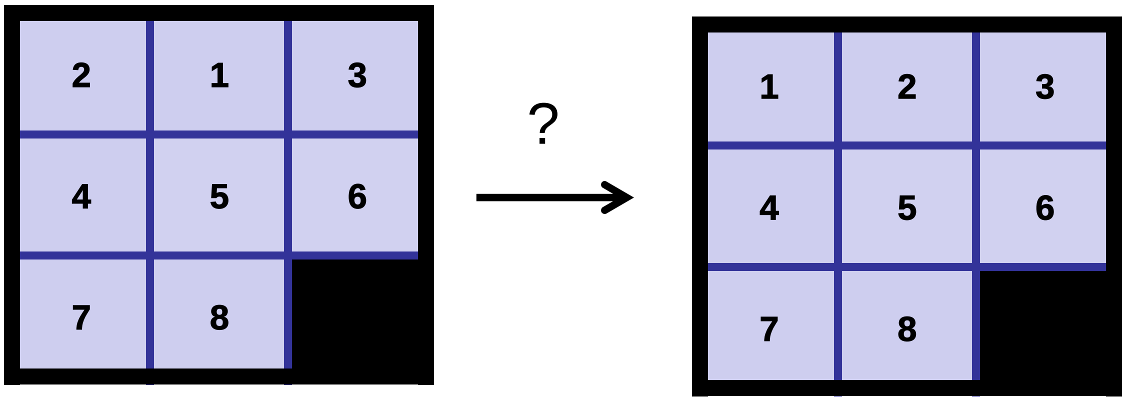
B. 2

C. Odd # of moves

D. Even # of moves

E. Just break it!



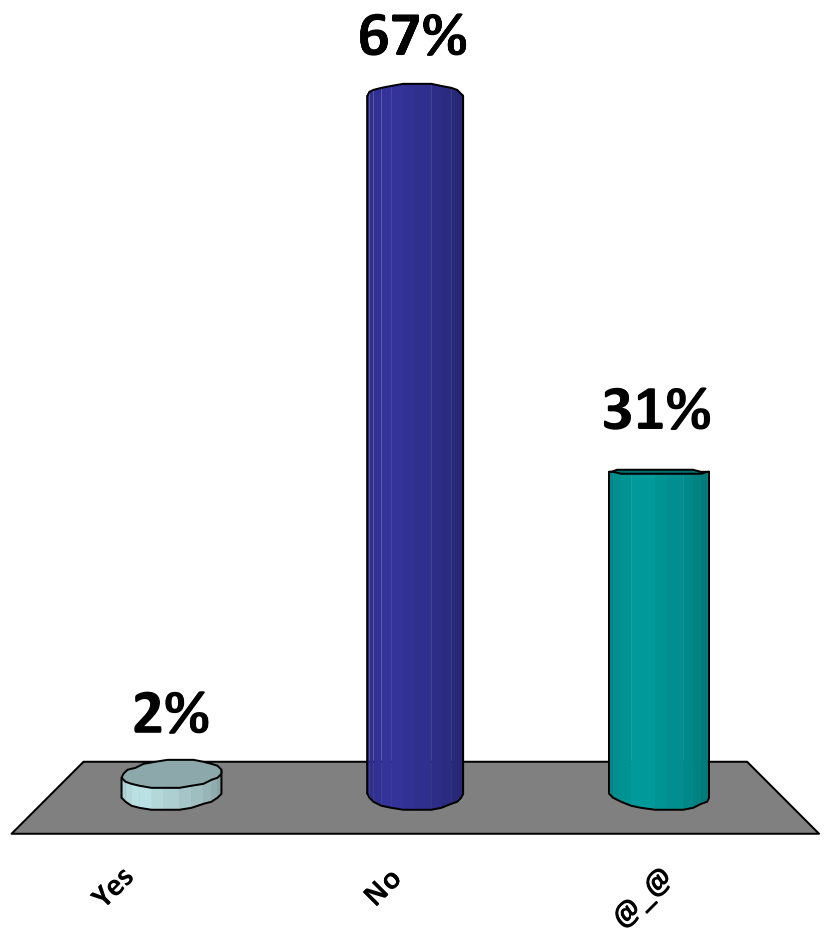


Puzzle of the Sliding Puzzle
Is the graph of all possible configurations connected?

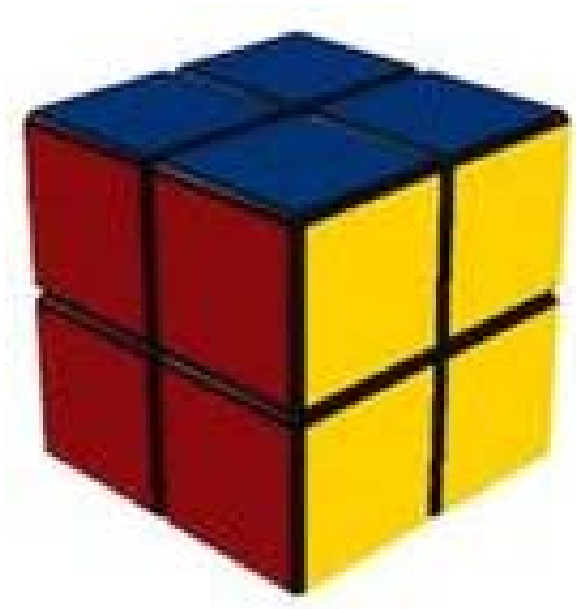
A. Yes

B. No

C. @_@

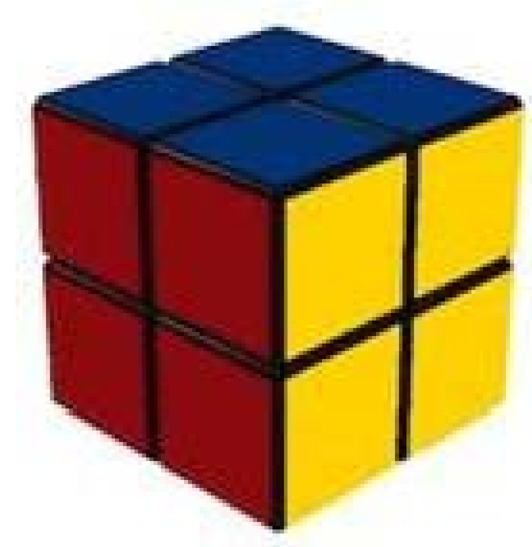


2 x 2 x 2 Rubik's Cube



Record solve time: 0.69 seconds

2 x 2 x 2 Rubik's Cube



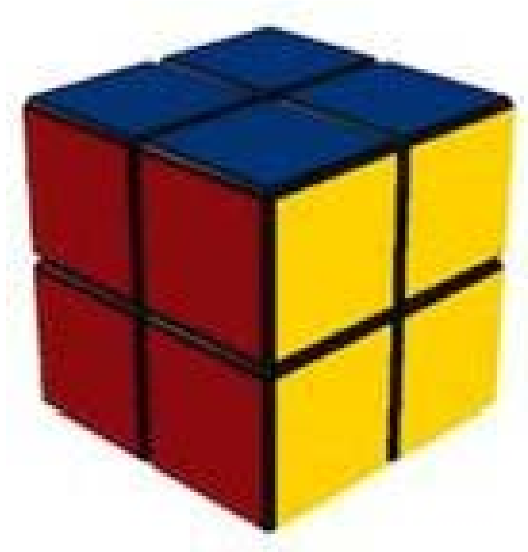
Configuration Graph

- Vertex for each possible state
- Edge for each basic move
  - 90 degree turn
  - 180 degree turn

Puzzle: given initial state, find a path to the solved state.

2 x 2 x 2 Rubik's Cube

How many vertices?



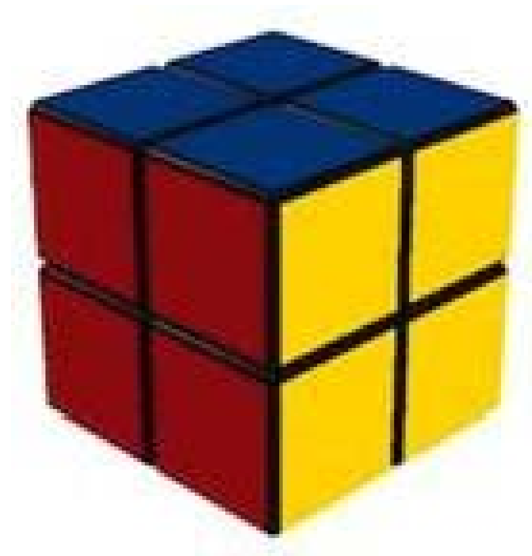
$$8! \cdot 3^8 = 264{,}539{,}520$$

cubelets

Each cubelet is
in one of 8 positions.

Each of the 8 cubelets
can be in one of three
orientations

2 x 2 x 2 Rubik's Cube



How many vertices?

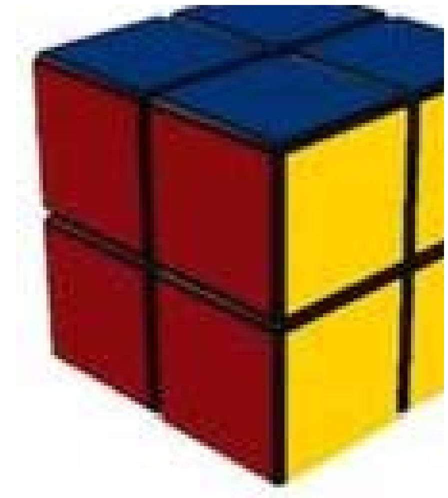
$$7! \cdot 3^7 = 11{,}022{,}480$$

Symmetry:

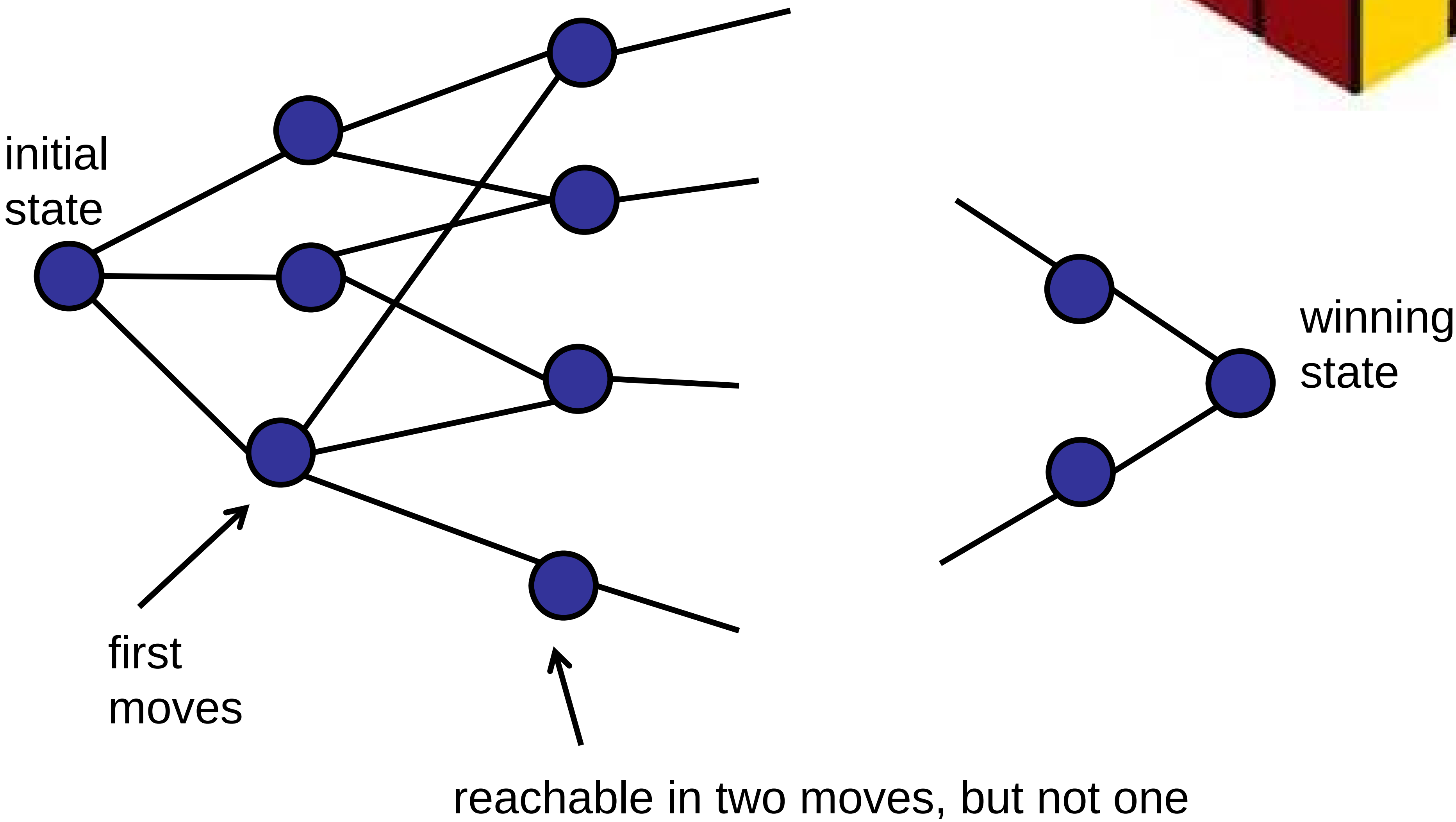
Fix one cubelet.

Each of the 8 cubelets can be in one of three orientations

2 x 2 x 2 Rubik's Cube

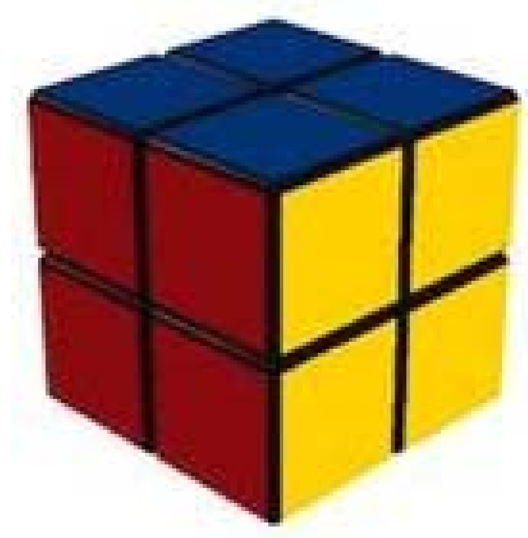


Geography of Rubik's configurations:



#configurations requires n turns

n	90 deg. Turns only	90/180 deg. turns
0	1	1
1	6	9
2	27	54
3	120	321
4	534	1,847
5	2,256	9,992
6	8,969	50,136
7	33,058	227,536
8	114,149	870,072
9	360,508	1,887,748
0	930,588	623,800
11	1,350,852	2,644
12	782,536	
13	90,280	
14	276	

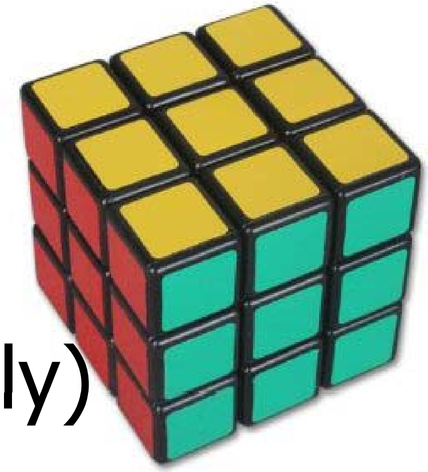


#configurations requires n turns

n	90 deg. turns	90/180 deg. turns
0	1	1
1	6	9
2	27	54
3		
4		
5		
6		
7		
8		
9	360,508	1,887,748
0	930,588	623,800
11	1,350,852	2,644
12	782,536	
13	90,280	
14	276	

Challenge:
How do you generate this table?

3 x 3 x 3 Rubik's Cube



Configuration Graph

- 43 quintillion vertices (approximately)
- Diameter: 20
  - 1995: require at least 20 moves.
  - 2010: 20 moves is enough from every position.
  - Using Google server farm.
  - 35 CPU-years of computation.
  - 20 seconds / set of 19.5 billion positions.
  - Lots of mathematical and programming tricks.

Date	Lower bound	Upper bound	Gap	Notes and Links
July, 1981	18	52	34	Morwen Thistlethwaite proves 52 moves suffice.
December, 1990	18	42	24	Hans Kloosterman improves this to 42 moves.
May, 1992	18	39	21	Michael Reid shows 39 moves is always sufficient.
May, 1992	18	37	19	Dik Winter lowers this to 37 moves just one day later!
January, 1995	18	29	11	Michael Reid cuts the upper bound to 29 moves by analyzing Kociemba's two-phase algorithm.
January, 1995	20	29	9	Michael Reid proves that the "superflip" position (corners correct, edges placed but flipped) requires 20 moves.
December, 2005	20	28	8	Silviu Radu shows that 28 moves is always enough.
April, 2006	20	27	7	Silviu Radu improves his bound to 27 moves.
May, 2007	20	26	6	Dan Kunkle and Gene Cooperman prove 26 moves suffice.
March, 2008	20	25	5	Tomas Rokicki cuts the upper bound to 25 moves.
April, 2008	20	23	3	Tomas Rokicki and John Welborn reduce it to only 23 moves.
August, 2008	20	22	2	Tomas Rokicki and John Welborn continue down to 22 moves.
July, 2010	20	20	0	Tomas Rokicki, Herbert Kociemba, Morley Davidson, and John Dethridge prove that God's Number for the Cube is exactly 20.

3 x 3 x 3 Rubik's Cube

What is the diameter of an (n x n x n) cube?

$\theta(n^2 / \log n)$

Roadmap

Today: Graph Basics

- What is a graph?
- Modeling problems as graphs.
- Graph representations (list vs. matrix)
- Searching graphs (DFS / BFS)

Representing a Graph

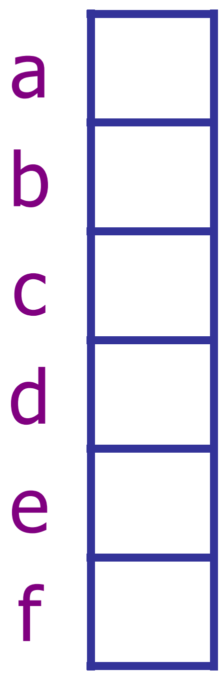
Graph consists of:

- Nodes
- Edges

Representing a Graph

Graph consists of:

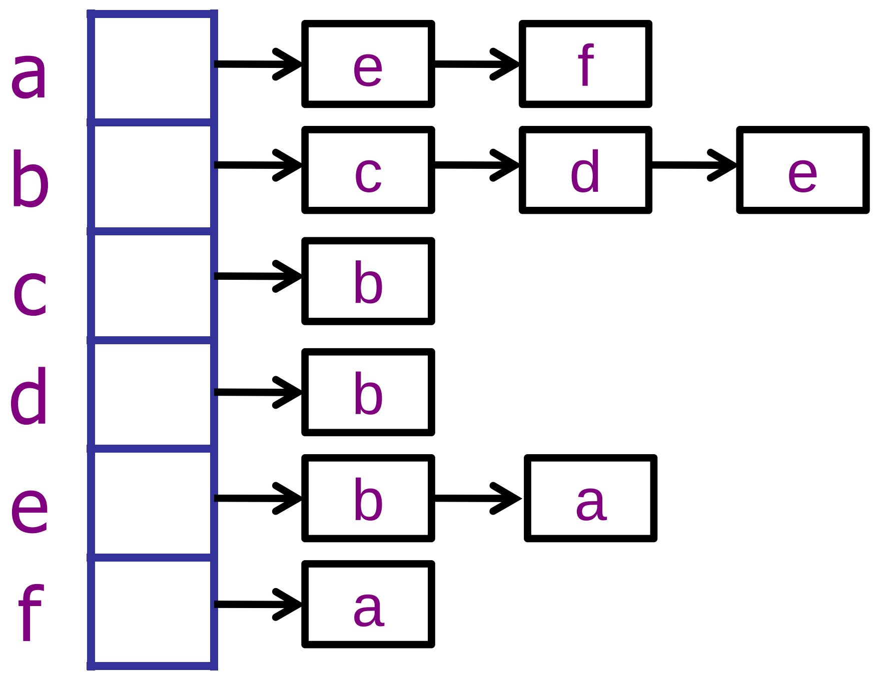
- Nodes: stored in an array
- Edges

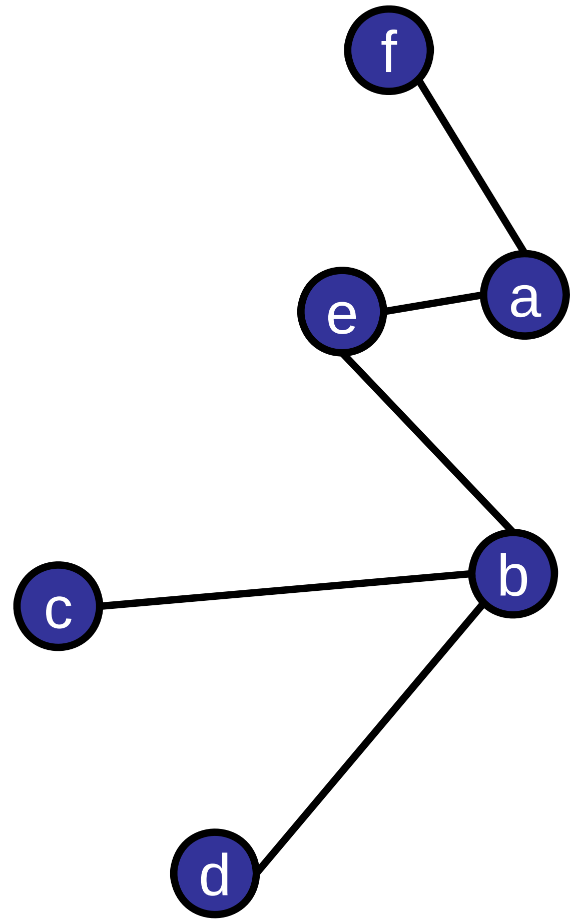


Adjacency List

Graph consists of:

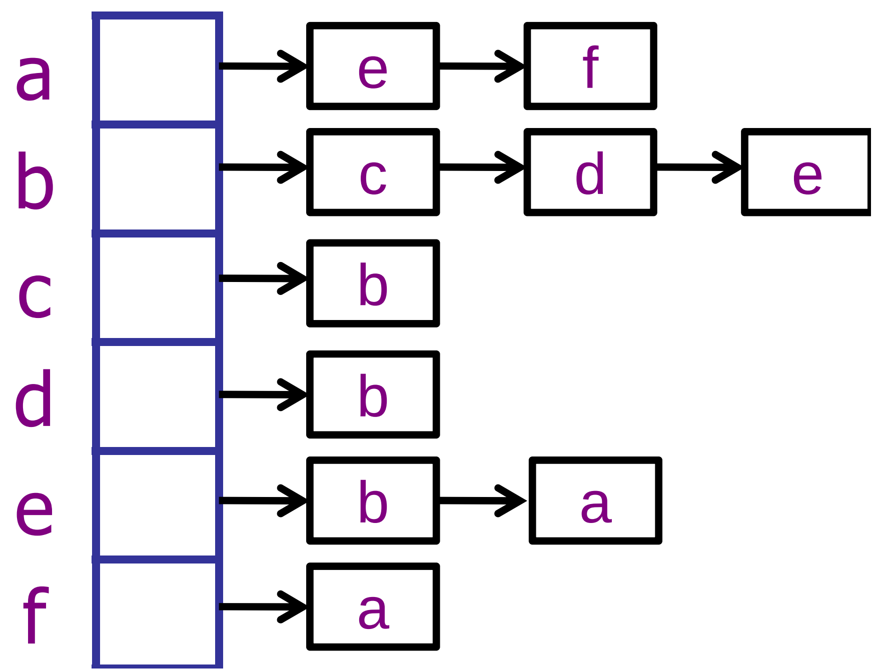
- Nodes: stored in an array
- Edges: linked list per node





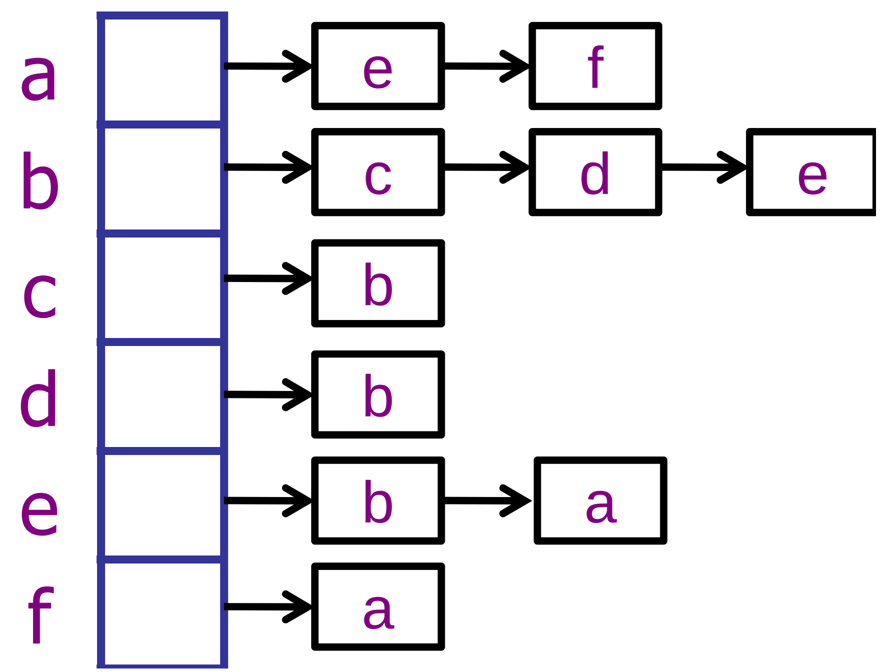
Adjacency List in Java

```
class NeighborList extends LinkedList<Integer> {
}
```



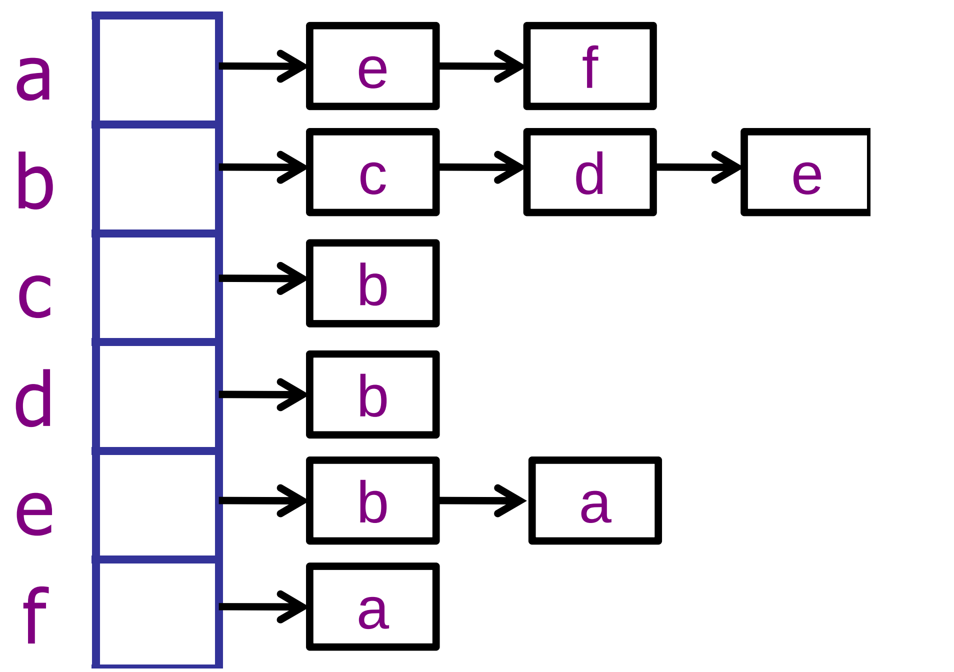
Adjacency List in Java

```
class NeighborList extends LinkedList<Integer> {
}

class Node {
  int key;
  NeighborList nbrs;
}
```



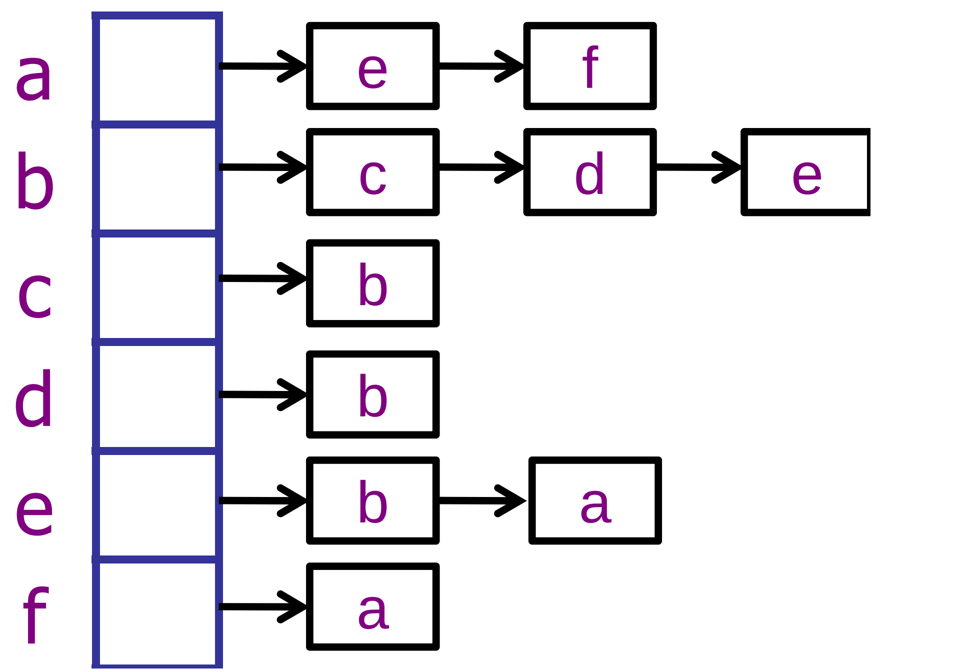
Adjacency List in Java

```
class NeighborList extends LinkedList<Integer> {
}

class Node {
  int key;
  NeighborList nbrs;
}

class Graph {
  Node[] nodeList;


}
```



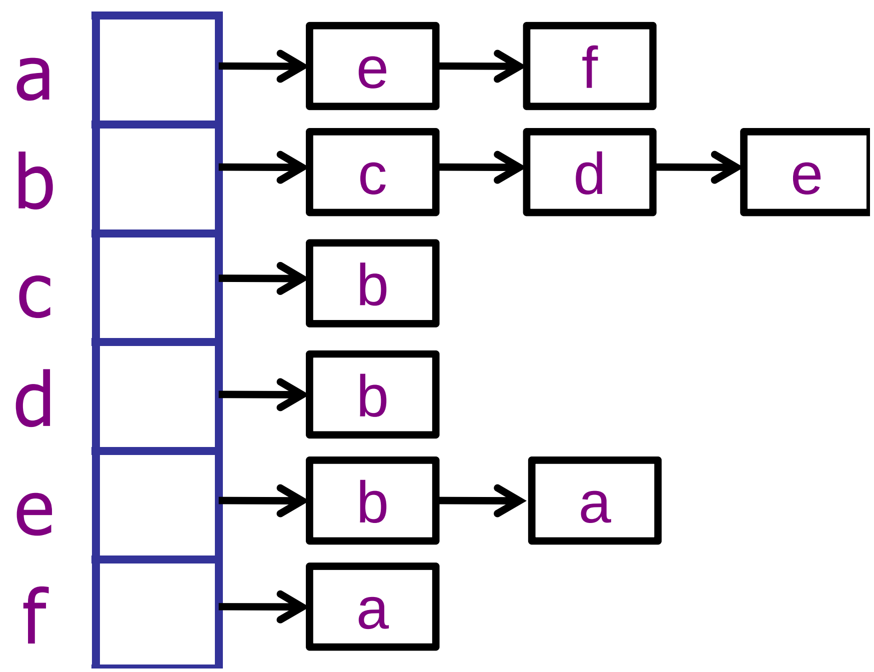
Adjacency List in Java

```
class NeighborList extends ArrayList<Integer> {
}

class Node {
  int key;
  NeighborList nbrs;
}

class Graph {
  List<Node> nodeList;


}
```



Adjacency List in Java

```
class Graph{
    List<List<Integer>> m_nodes;


}
```

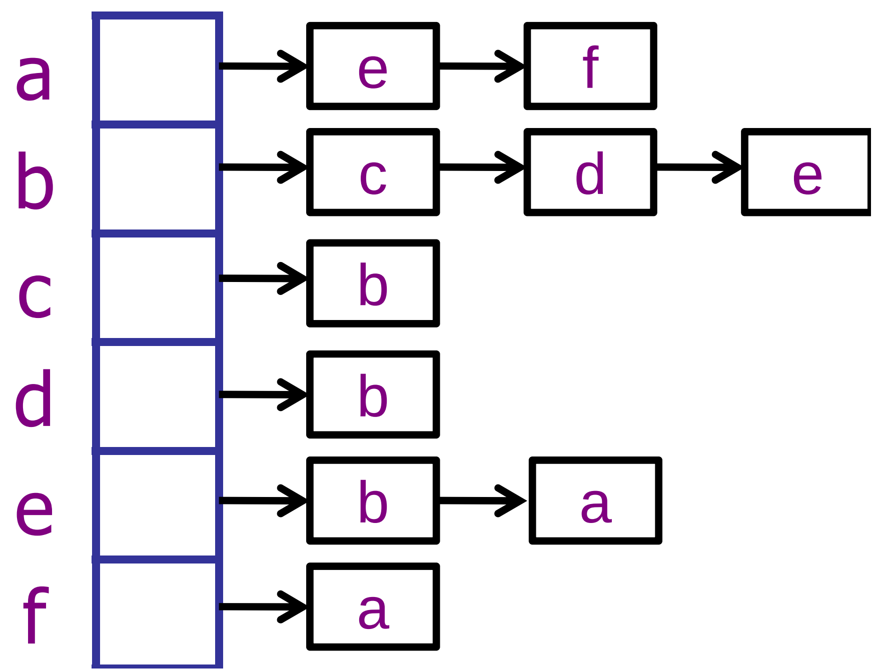


Adjacency List in Java

```
class Graph{
    List<List<Integer>> m_nodes;


}
```

More concise code is not *always* better…

- Harder to read
- Harder to debug
- Harder to extend



Representing a Graph

Graph consists of:

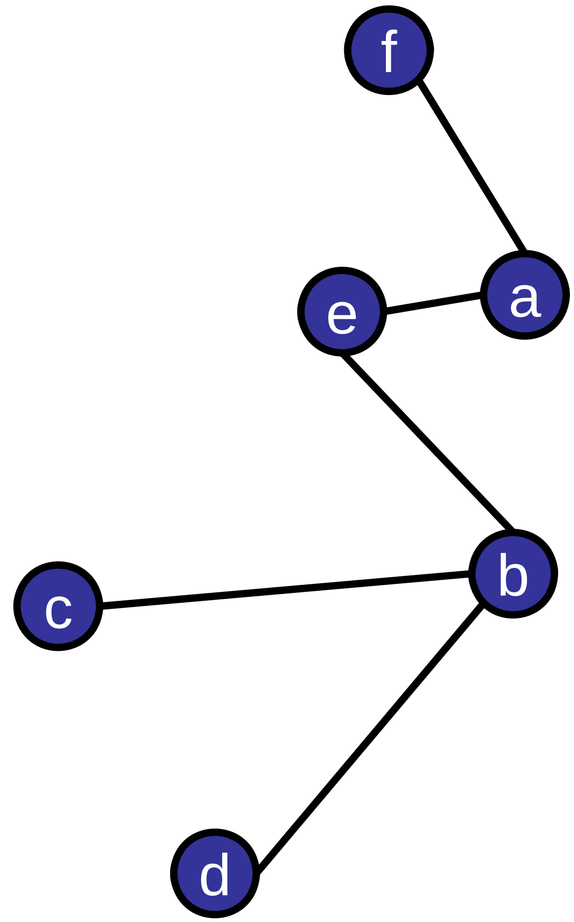
- Nodes
- Edges = pairs of nodes

Adjacency **Matrix**

Graph consists of:

- Nodes
- Edges = pairs of nodes

	a	b	c	d	e	f
a	0	0	0	0	**1**	**1**
b	0	0	**1**	**1**	**1**	0
c	0	**1**	0	0	0	0
d	0	**1**	0	0	0	0
e	**1**	**1**	0	0	0	0
f	**1**	0	0	0	0	0



Adjacency Matrix

Graph represented as:

$A[v][w] = 1$ iff $(v,w) \in E$

Neat property:

- A^2 = length 2 paths

	a	b	c	d	e	f
a	0	0	0	0	**1**	**1**
b	0	0	**1**	**1**	**1**	0
c	0	**1**	0	0	0	0
d	0	**1**	0	0	0	0
e	**1**	**1**	0	0	0	0
f	**1**	0	0	0	0	0

Adjacency Matrix

To find out if c and d are 2-hop neighbors:

- Let B = A^2.
- B[c, d] = A[c, .]•A[., d]

- B[c, d] = 1 iff

  A[c, x] == A[x, d]

  for some x.

	a	b	c	d	e	f
a	0	0	0	0	**1**	**1**
b	0	0	**1**	**1**	**1**	0
c	0	**1**	0	0	0	0
d	0	**1**	0	0	0	0
e	**1**	**1**	0	0	0	0
f	**1**	0	0	0	0	0

Adjacency Matrix

To find out if c and d are 2-hop neighbors:

- Let B = A^2.
- $B[c, d] = A[c, .] \bullet A[., d] > 0 \; ? \; 1 : 0$

- B[c, d] = 1 iff

  A[c, x] == A[x, d]

  for some x.

	a	b	c	d	e	f
a	0	0	0	0	**1**	**1**
b	0	0	**1**	**1**	**1**	0
c	0	**1**	0	0	0	0
d	0	**1**	0	0	0	0
e	**1**	**1**	0	0	0	0
f	**1**	0	0	0	0	0

Adjacency Matrix

Graph represented as:

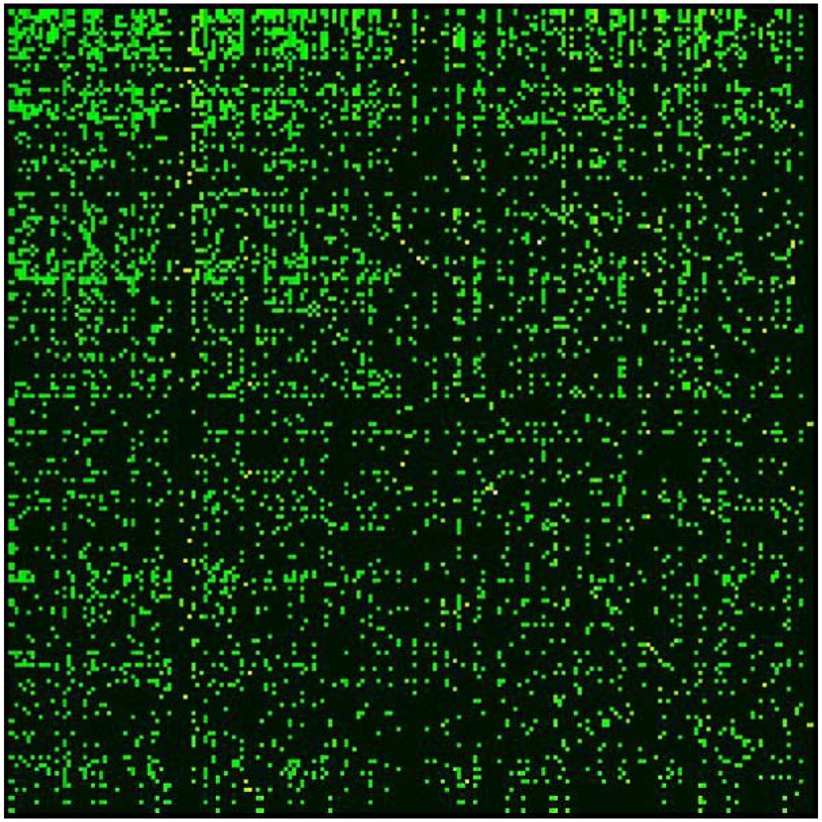
$A[v][w] = 1$ iff $(v,w) \in E$

Neat properties:

- A^2 = length 2 paths
- A^∞ = Google pagerank

	a	b	c	d	e	f
a	0	0	0	0	**1**	**1**
b	0	0	**1**	**1**	**1**	0
c	0	**1**	0	0	0	0
d	0	**1**	0	0	0	0
e	**1**	**1**	0	0	0	0
f	**1**	0	0	0	0	0

A Google matrix is a particular stochastic matrix that is used by Google's PageRank algorithm. The matrix represents a graph with edges representing links between pages. The rank of each page can be generated iteratively from the Google matrix using the power method. However, in order for the power method to converge, the matrix must be stochastic, irreducible and aperiodic.



Adjacency Matrix in Java

Graph represented as:

$A[v][w] = 1$ iff $(v,w) \in E$

```
class Graph {
  boolean[][] m_adjMatrix;

}
```

	a	b	c	d
a	0	0	0	0
b	0	0	**1**	**1**
c	0	**1**	0	0
d	0	**1**	0	0
e	**1**	**1**	0	0
f	**1**	0	0	0

Adjacency Matrix in Java

Graph represented as:

A[v][w] = 1 iff $(v,w) \in E$

```
class Graph {

  Node[][] m_adjMatrix;


}
```

	a	b	c	d
a	0	0	0	0
b	0	0	1	1
c	0	1	0	0
d	0	1	0	0
e	1	1	0	0
f	1	0	0	0

Adjacency Matrix in Java

Graph represented as:

$A[v][w] = 1$ iff $(v,w) \in E$

	a	b	c	d
a	0	0	0	0
b	0	0	1	1
c	0	1	0	0
d	0	1	0	0
e	1	1	0	0
f	1	0	0	0

```
class Graph {

  List<List<Boolean>> m_adjMatrix;


}
```

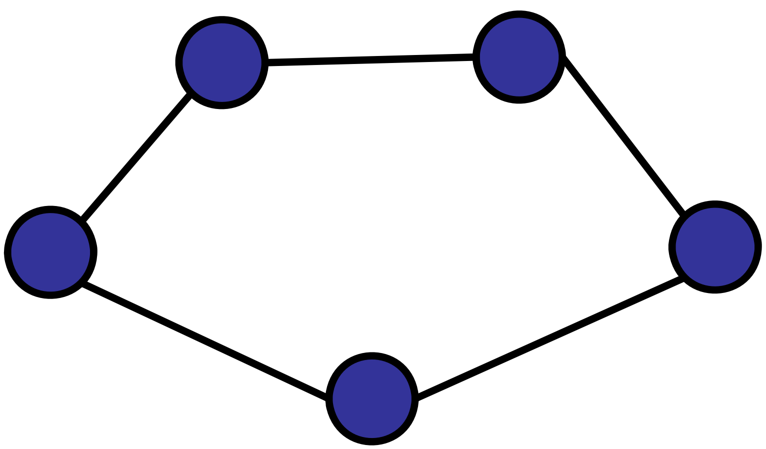
Resizable, but harder to use.

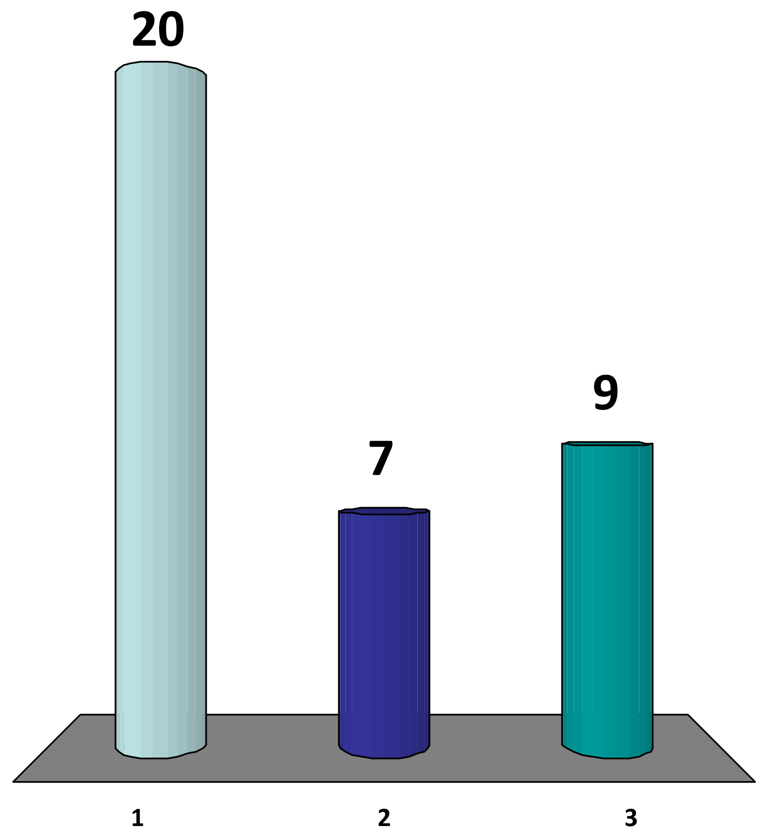
Trade-offs

Adjacency Matrix vs. Array?

For a cycle, which representation is better?

✓ 1. Adjacency list
2. Adjacency matrix
3. Equivalent





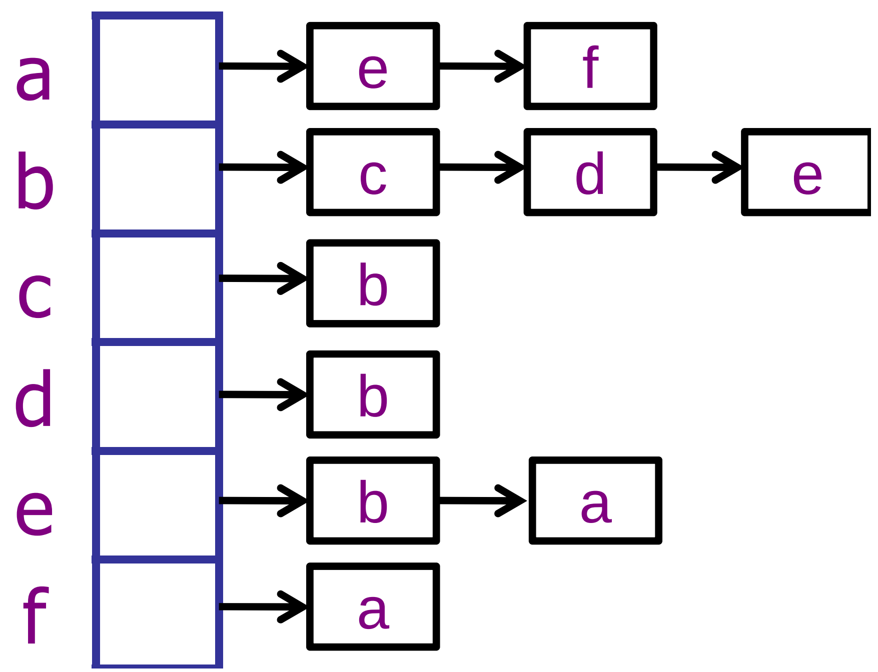
Adjacency List

Memory usage for graph G = (V, E):

- array of size |V|
- linked lists of size |E|

Total: O(V + E)

For a cycle: O(V)



Adjacency Matrix

Memory usage for graph G = (V, E):

- array of size |V|*|V|

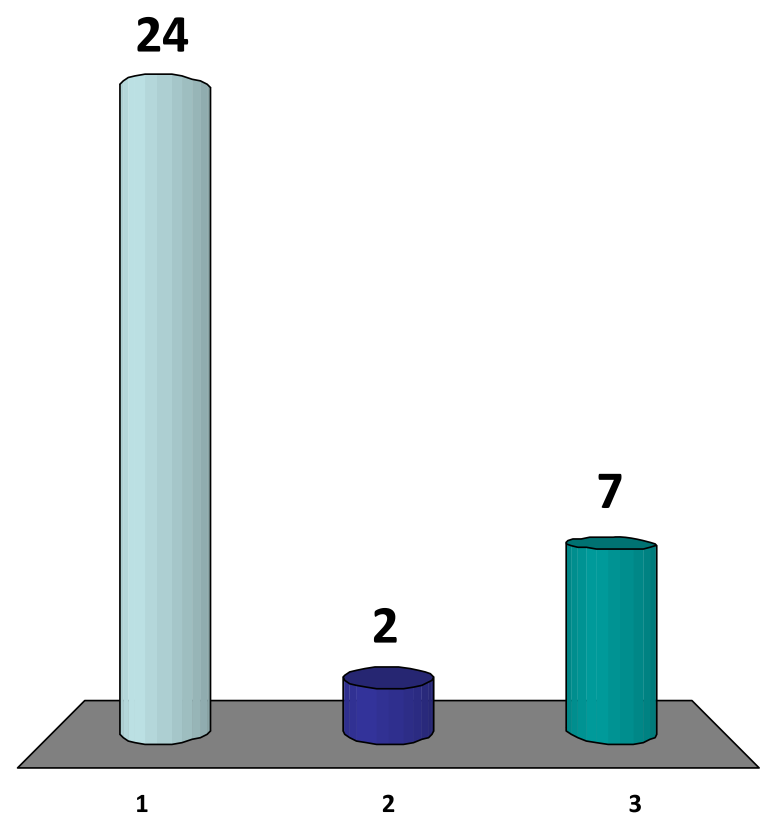
Total: $O(V^2)$

For a cycle: $O(V^2)$

	a	b	c	d	e	f
a	0	0	0	0	**1**	**1**
b	0	0	**1**	**1**	**1**	0
c	0	**1**	0	0	0	0
d	0	**1**	0	0	0	0
e	**1**	**1**	0	0	0	0
f	**1**	0	0	0	0	0

For a clique, which representation is better?

1. Adjacency matrix
2. Adjacency list
3. Equivalent



Adjacency List vs. Matrix

Memory usage for graph $G = (V, E)$:

- Adjacency List: $O(V + E)$
- Adjacency Matrix: $O(V^2)$

For a cycle: $O(V)$ vs. $O(V^2)$

For a clique: $O(V + E) = O(V^2)$ vs. $O(V^2)$

Adjacency List vs. Matrix

Memory usage for graph $G = (V, E)$:

- Adjacency List: $O(V + E)$
- Adjacency Matrix: $O(V^2)$

For a cycle: $O(V)$ vs. $O(V^2)$

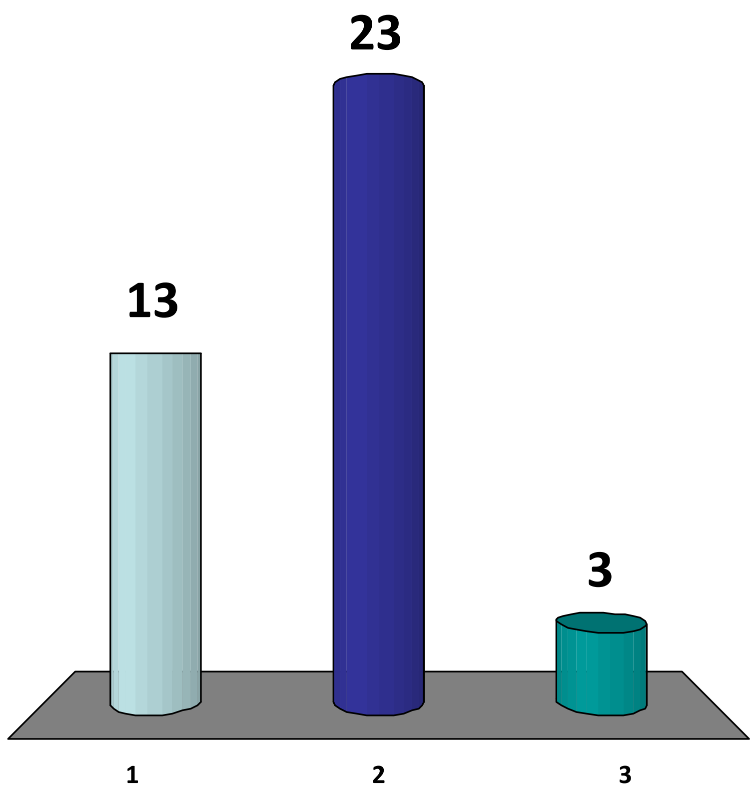
For a clique: $O(V + E) = O(V^2)$ vs. $O(V^2)$

Base rule: if graph is dense then use an adjacency matrix; else use an adjacency list.

dense: $|E| = \theta(V^2)$

Which representation for Facebook Graph? Query: Are Bob and Joe friends?

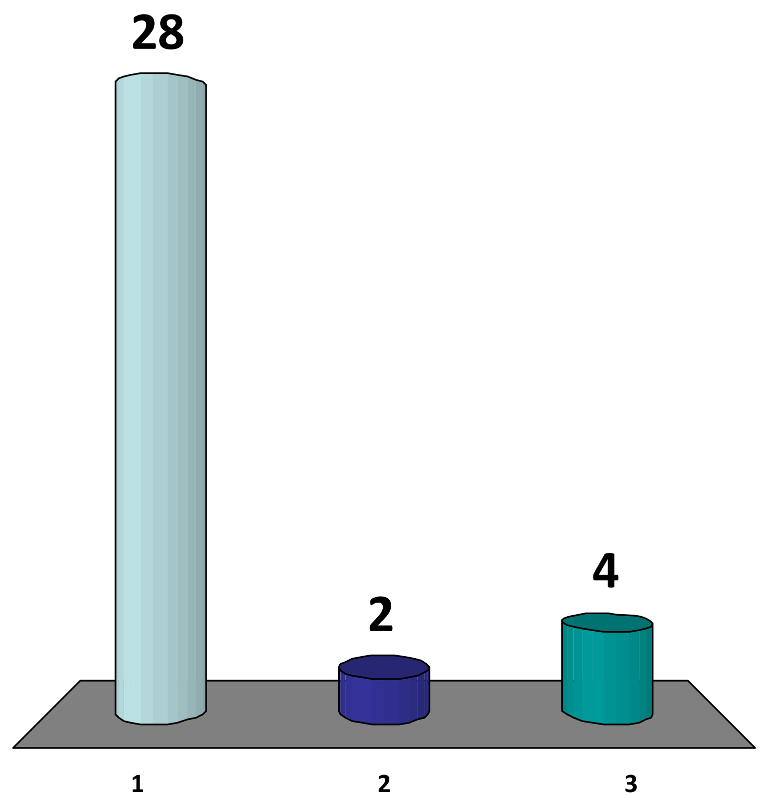
1. Adjacency List
2. ✓ Adjacency Matrix
3. Equivalent



List: (much) better space.
Matrix: somewhat faster

Which representation for Facebook Graph? Query: List all my friends?

1. Adjacency List
2. Adjacency Matrix
3. Equivalent



Trade-offs

Adjacency Matrix:

- Fast query: are v and w neighbors?
- Slow query: find me any neighbor of v.
- Slow query: enumerate all neighbors.

Adjacency List:

- Fast query: find me any neighbor.
- Fast query: enumerate all neighbors.
- Slower query: are v and w neighbors?

Graph Representations

Key questions to ask:

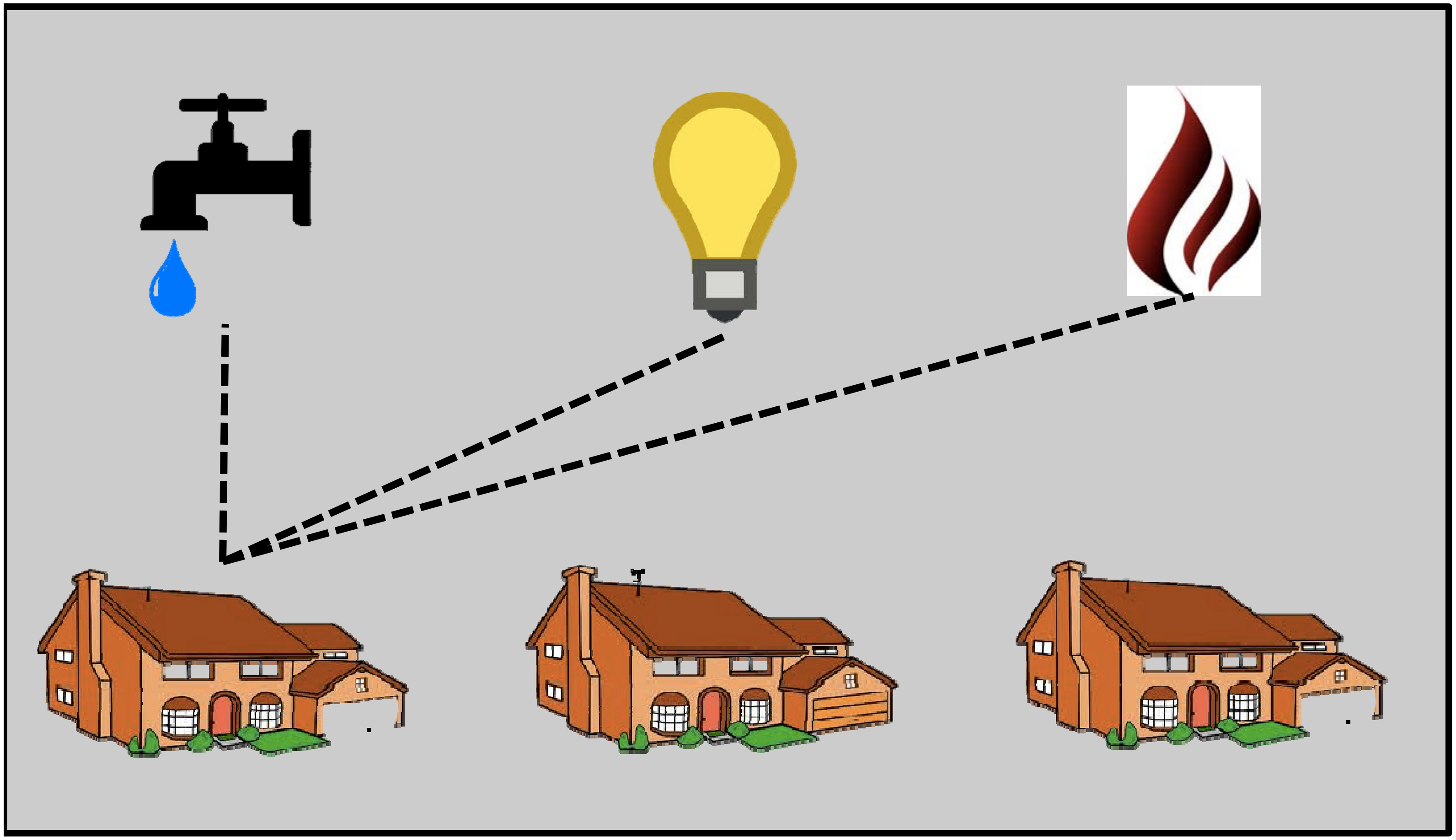
- Space usage: is graph dense or sparse?
- Queries: what type of queries do I need?
  - Enumerate neighbors?
  - Query relationship?

Roadmap

Today: Graph Basics

- What is a graph?
- Modeling problems as graphs.
- Graph representations (list vs. matrix)
- Searching graphs (DFS / BFS)

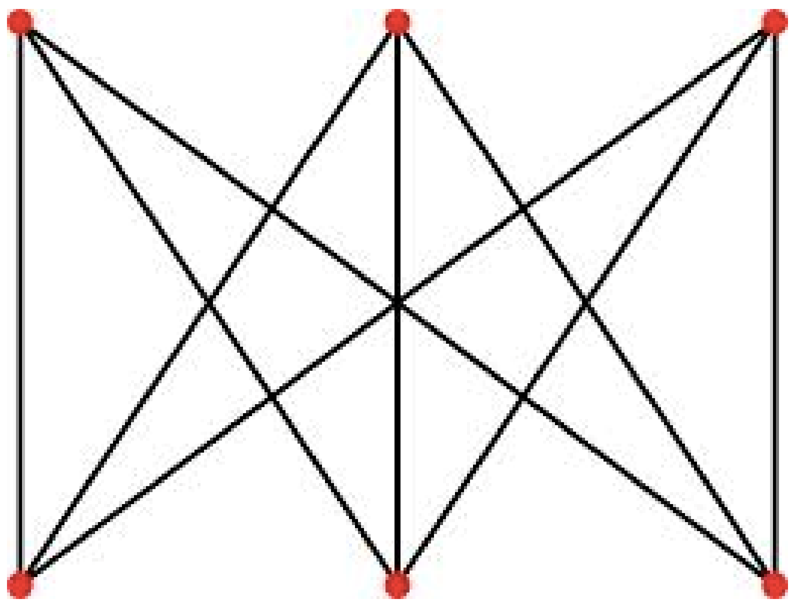
Puzzle



Connect each house to all three utilities (water, electricity, gas).
Do not let any of the cables or pipes cross.
(Or show that it is impossible.)

Puzzle Explanation

Can you draw this graph with no crossing lines?

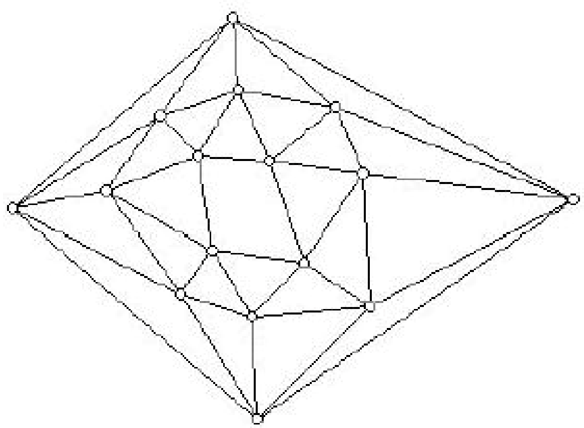


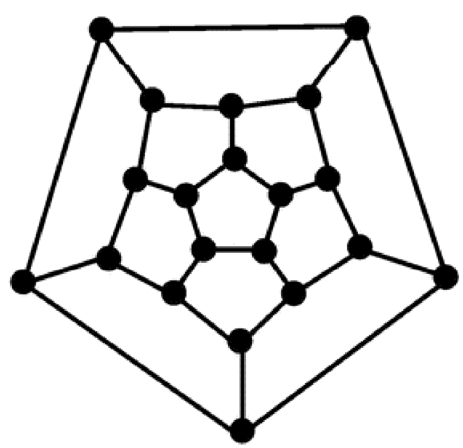
Bipartite Clique

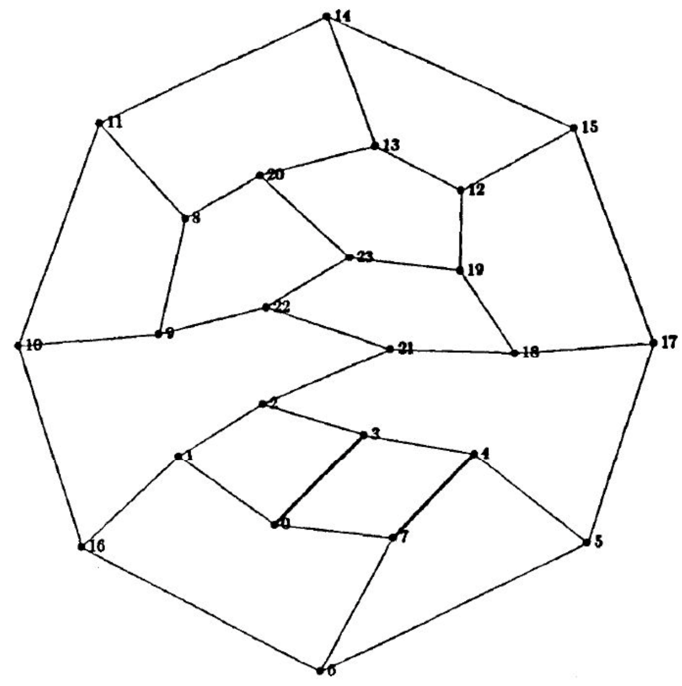
Puzzle Explanation

Planar Graph:

Any graph that can be drawn on a flat 2d piece of paper with no crossing lines.



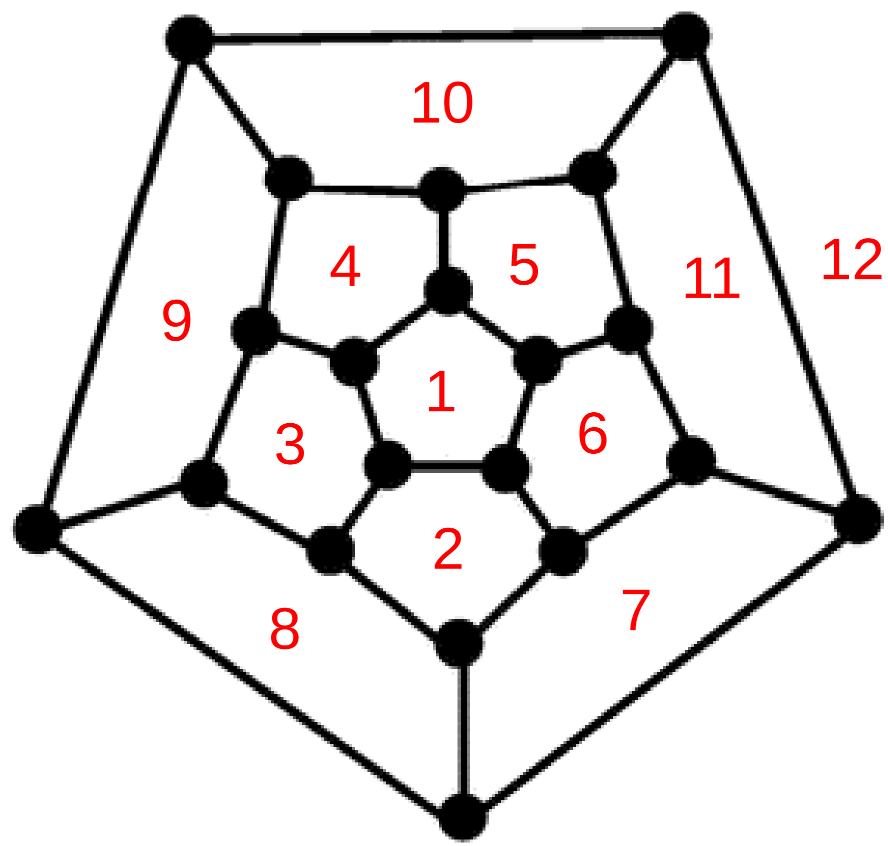




Puzzle Explanation

Terms:

- vertex
- edge
- face
  - area bounded by edges
  - outer (infinite) area



Puzzle Explanation

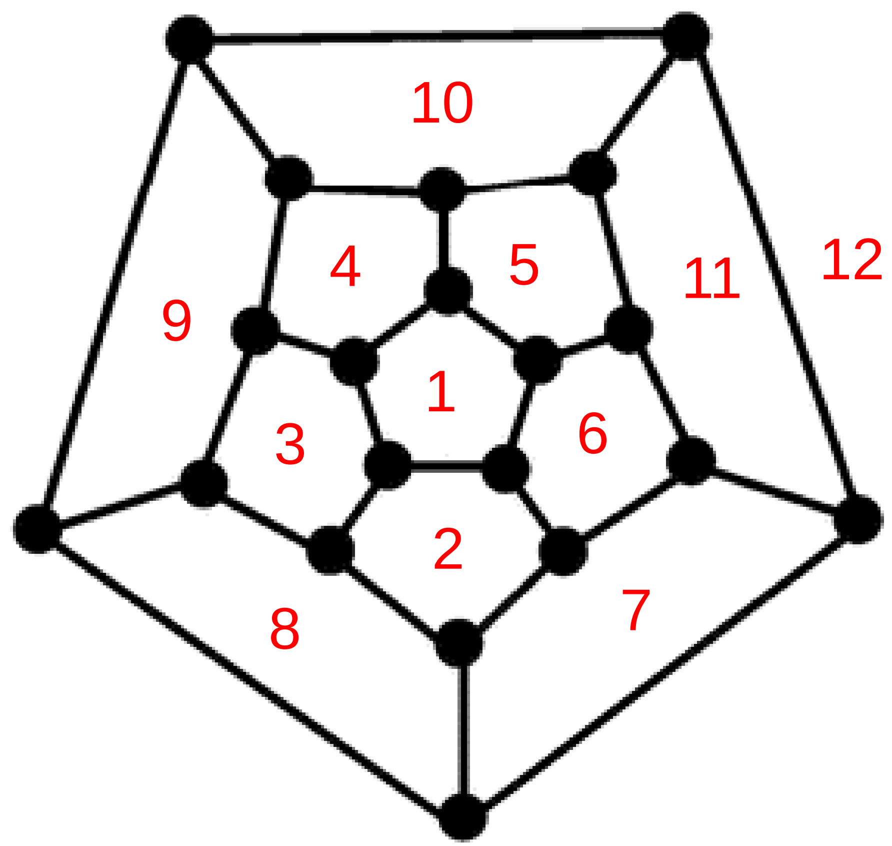
Euler's Formula:
(planar graphs)

$$V - E + F = 2$$

V = # vertices

E = # edges

F = # faces



Prove by induction.

Puzzle Explanation

Euler's Formula:
(planar graphs)

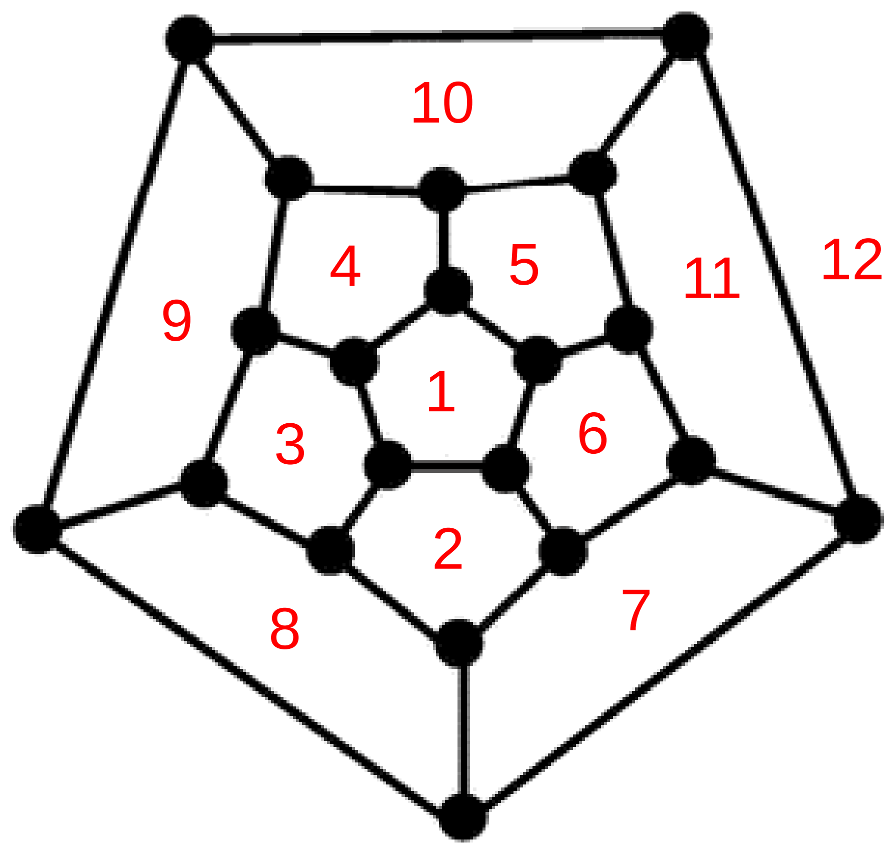
$V - E + F = 2$

$V = 20$

$E = 30$

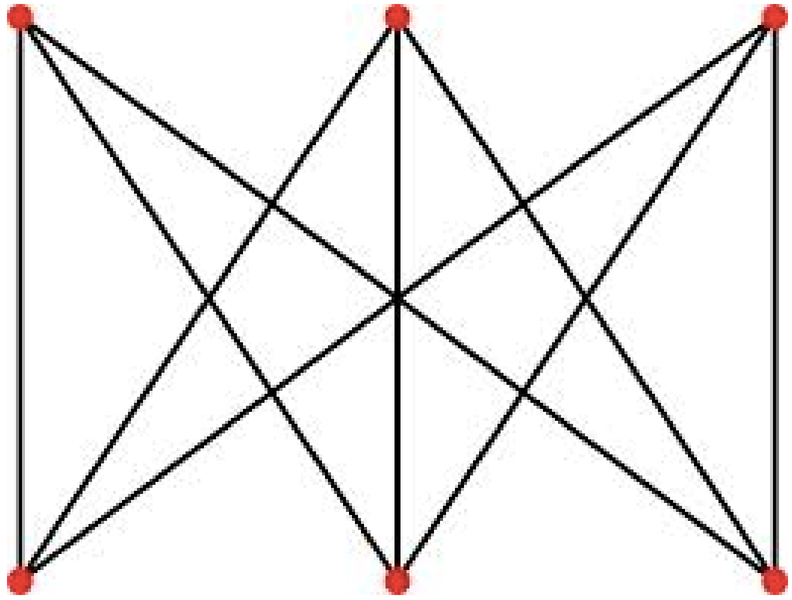
$F = 12$

$20 - 30 + 12 = 2$



Puzzle Explanation

Can you draw this graph with no crossing lines?



Bipartite Clique

Puzzle Explanation

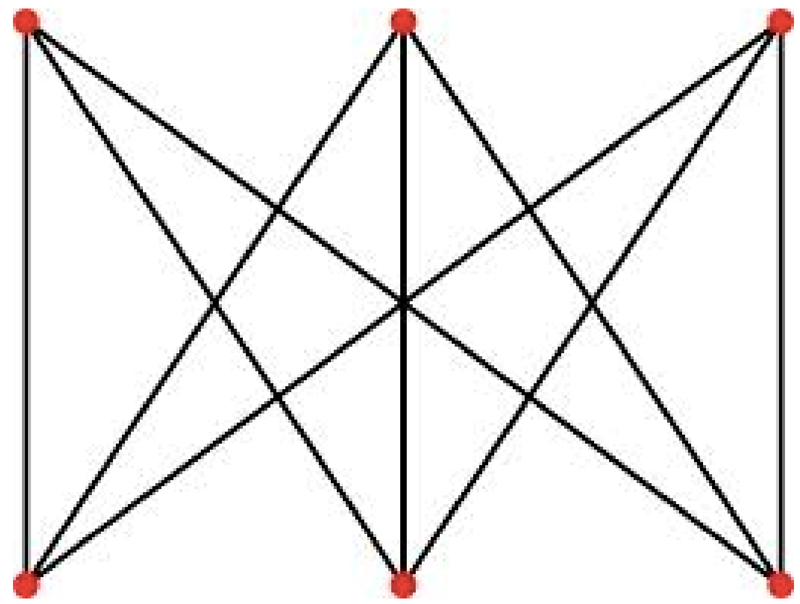
Euler’s Formula:
(planar graphs)

If G is planar then

$V - E + F = 2$

So, if $V - E + F \neq 2$

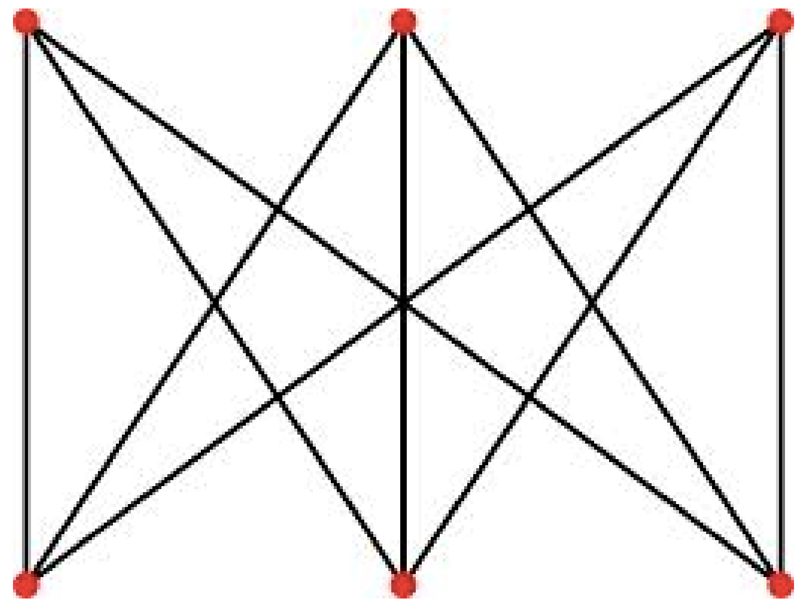
Then G is not planar



Puzzle Explanation

Euler's Formula:
(planar graphs)

$V - E + F = 2$



$V = 6$

$E = 9$

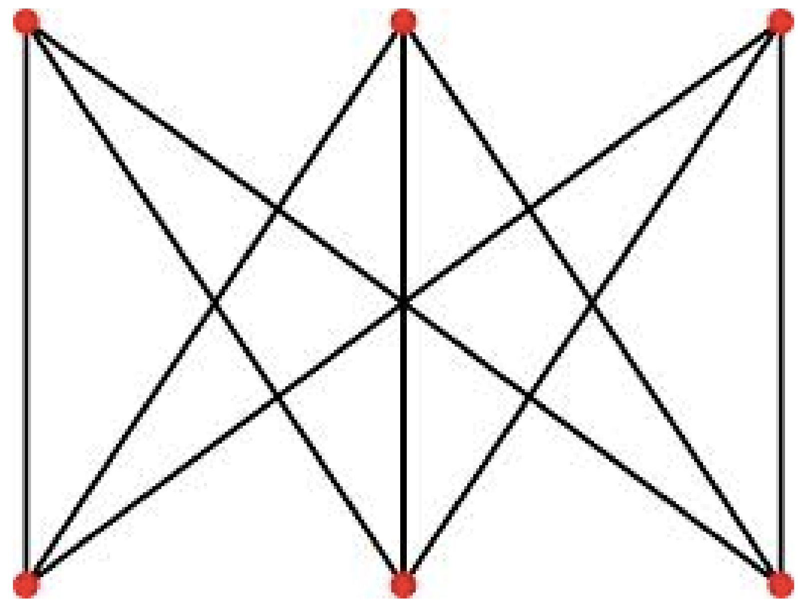
$F = ??$

$6 - 9 + F = 2$

Puzzle Explanation

Euler's Formula:
(planar graphs)

$V - E + F = 2$



$V = 6$
$E = 9$
$F = 5$

$6 - 9 + F = 2$

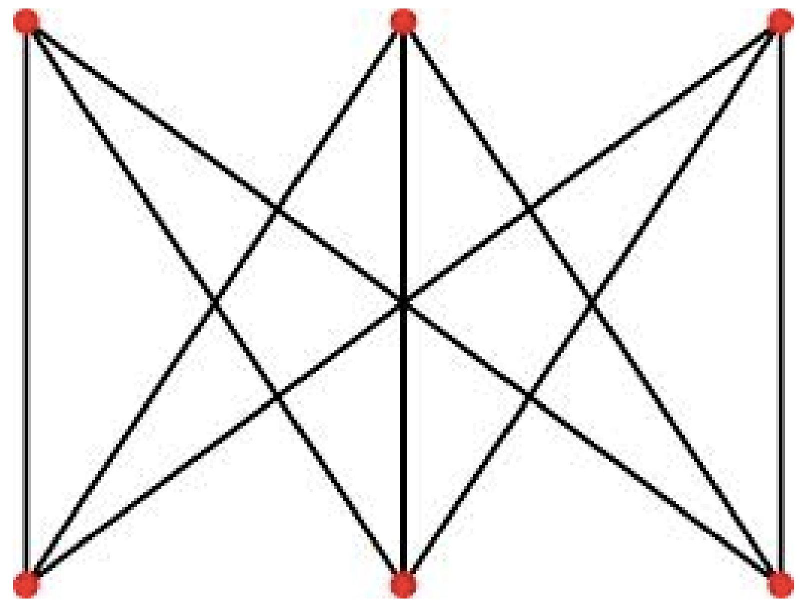
Puzzle Explanation

For bipartite clique:

Every face has at least 4 edges.

Every edge is used in at most 2 faces.

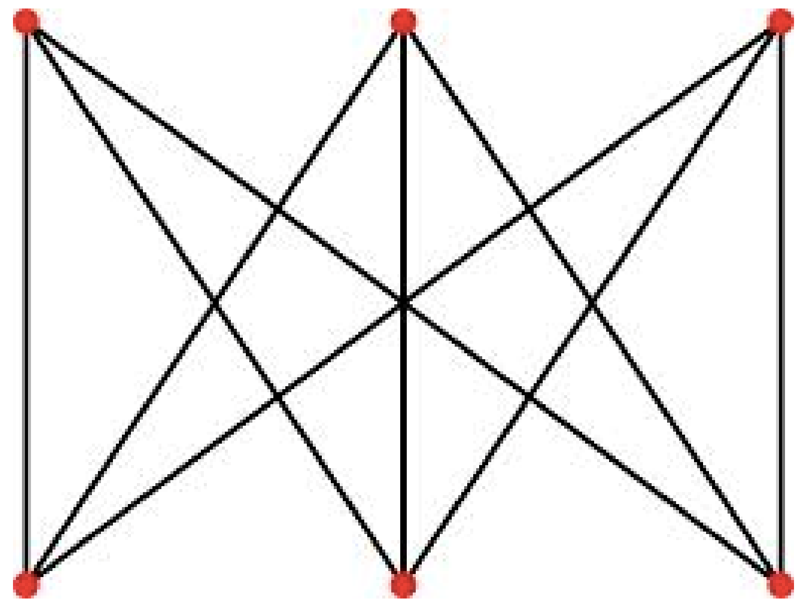
$F \leq (2E) / 4 \leq E/2$



Puzzle Explanation

Impossible!

$F \leq E/2$



$V = 6$

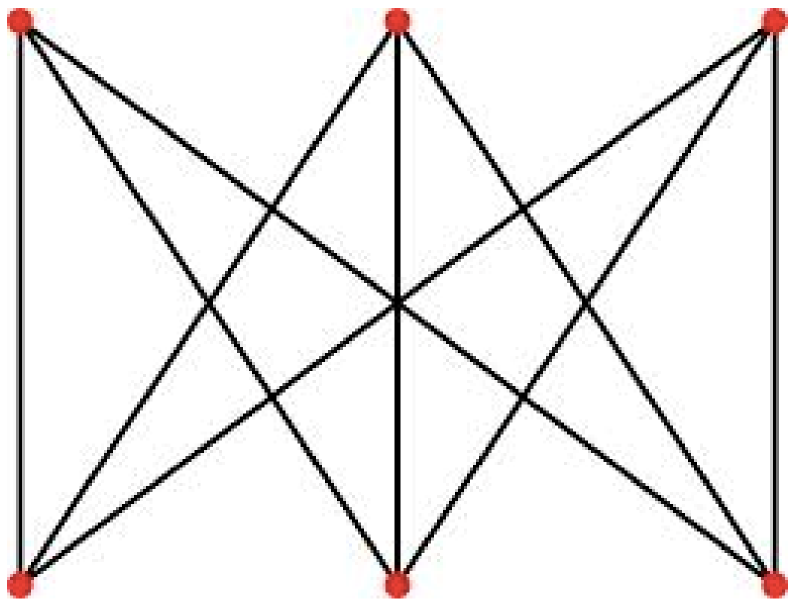
$E = 9$

$F = 5$

BUT: $5 > 9/2$

Puzzle Explanation

Impossible to draw bipartite clique without crossing lines.

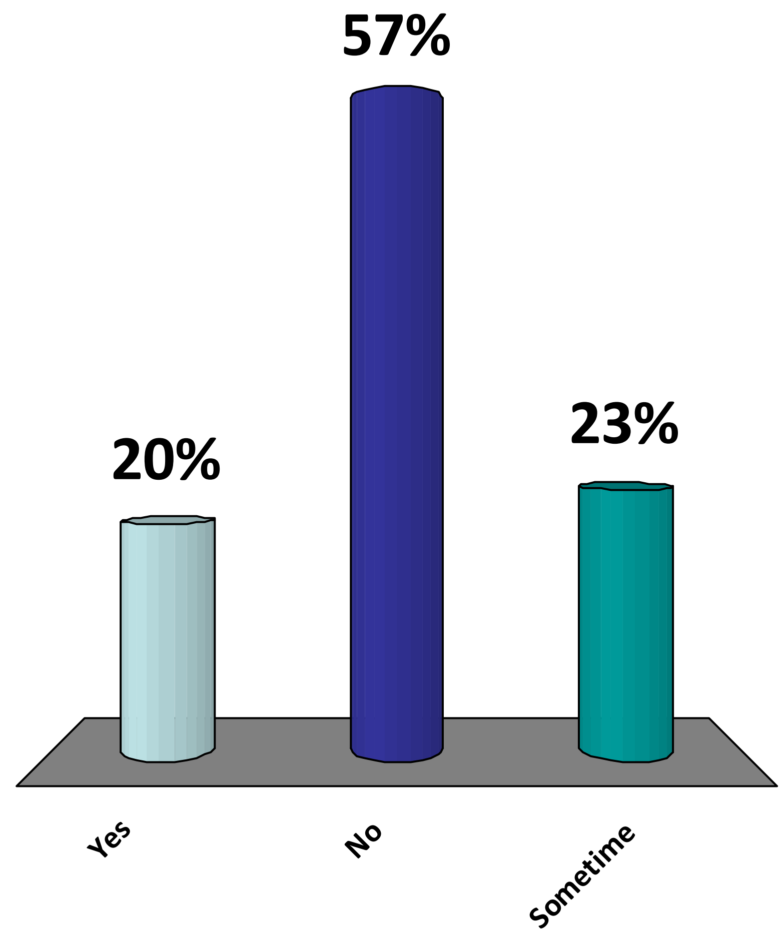


Bipartite Clique

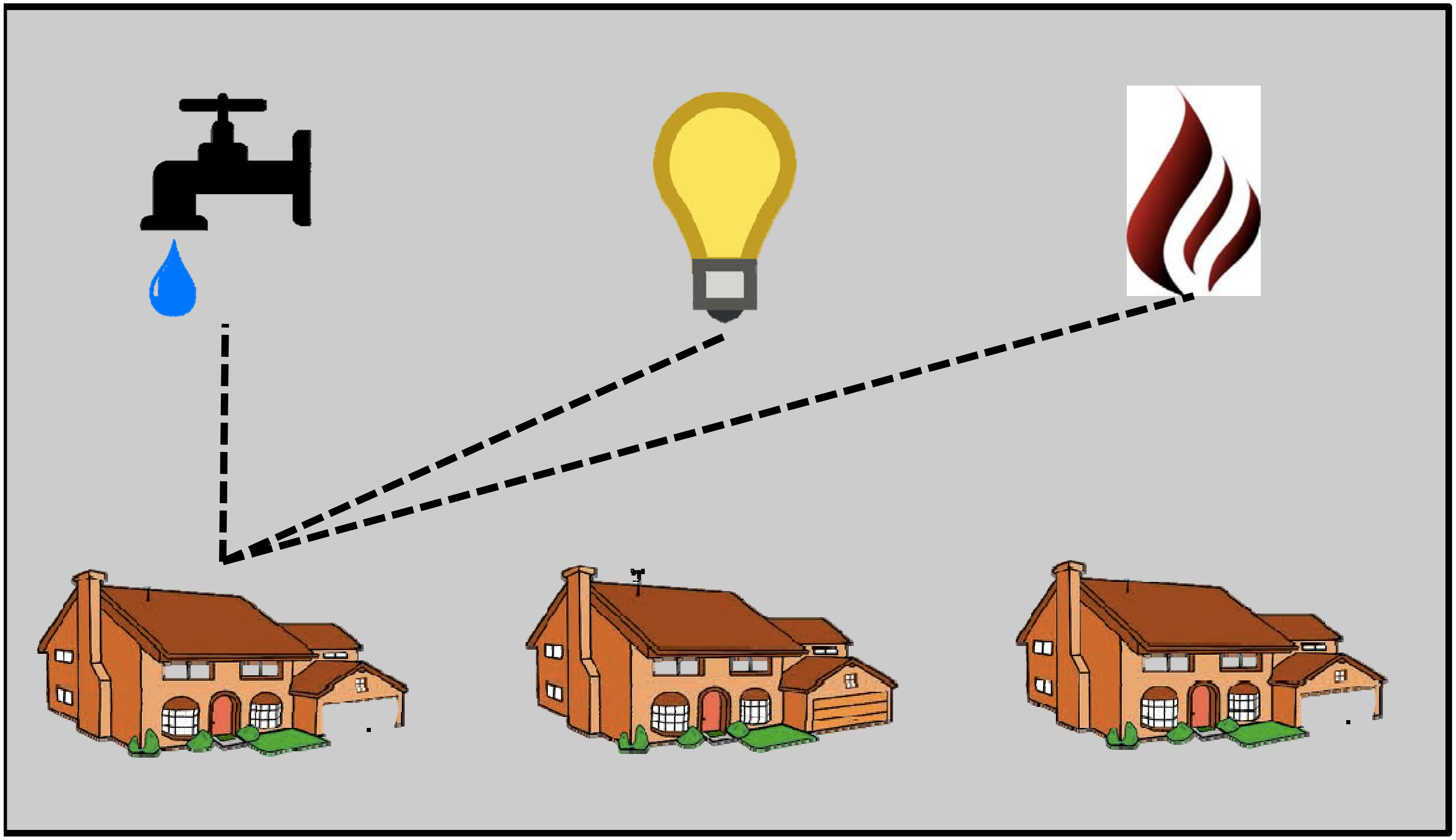
Can we say that:

If $V - E + F = 2$ Then G MUST BE planar?

A. Yes
B. No
C. Sometime



Puzzle



Connect each house to all three utilities (water, electricity, gas).
Do not let any of the cables or pipes cross.
(Or show that it is impossible.)

Puzzle

Can we draw K_5 (a clique with 5 vertices) without crossing?

Roadmap

Today: Graph Basics

- What is a graph?
- Modeling problems as graphs.
- Graph representations (list vs. matrix)
- Searching graphs (DFS / BFS)

Searching a Graph

Goal:

- Start at some vertex **s** = start.
- Find some other vertex **f** = finish.

  Or: visit **all** the nodes in the graph;

Two basic techniques:

- Breadth-First Search (BFS)
- Depth-First Search (DFS)

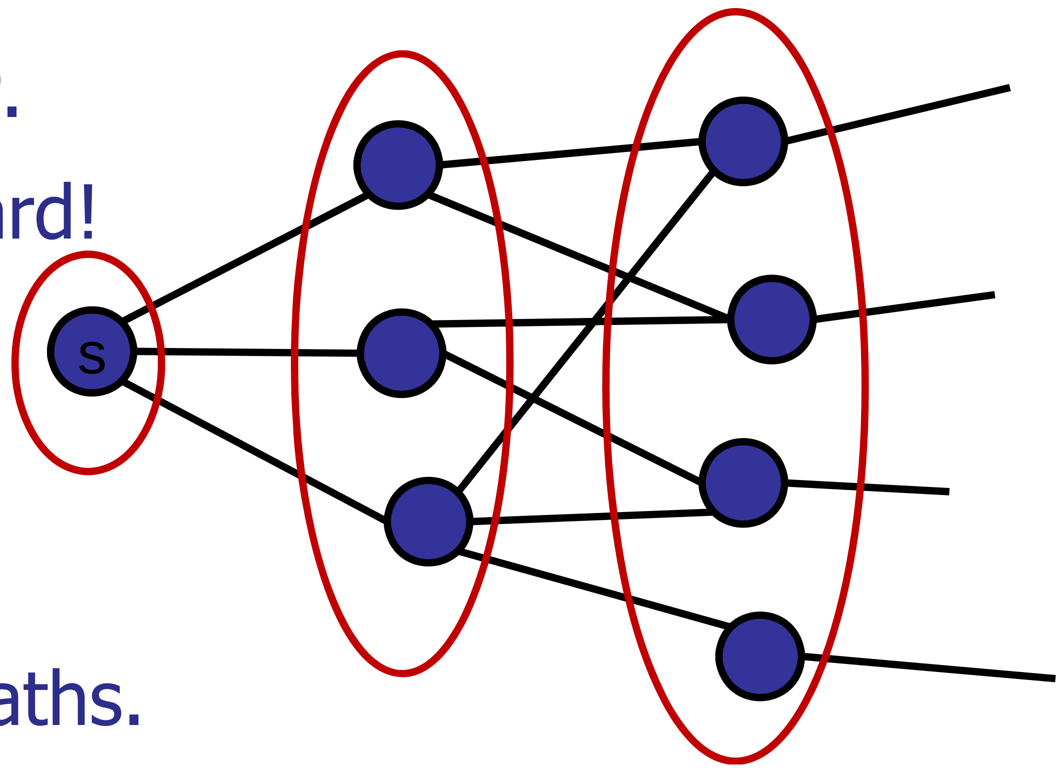
Graph representation:

- Adjacency list

Searching a graph

Breadth-First Search:

- Explore level by level
- Frontier: current level
- Initially: {s}
- Advance frontier.
- Don't go backward!

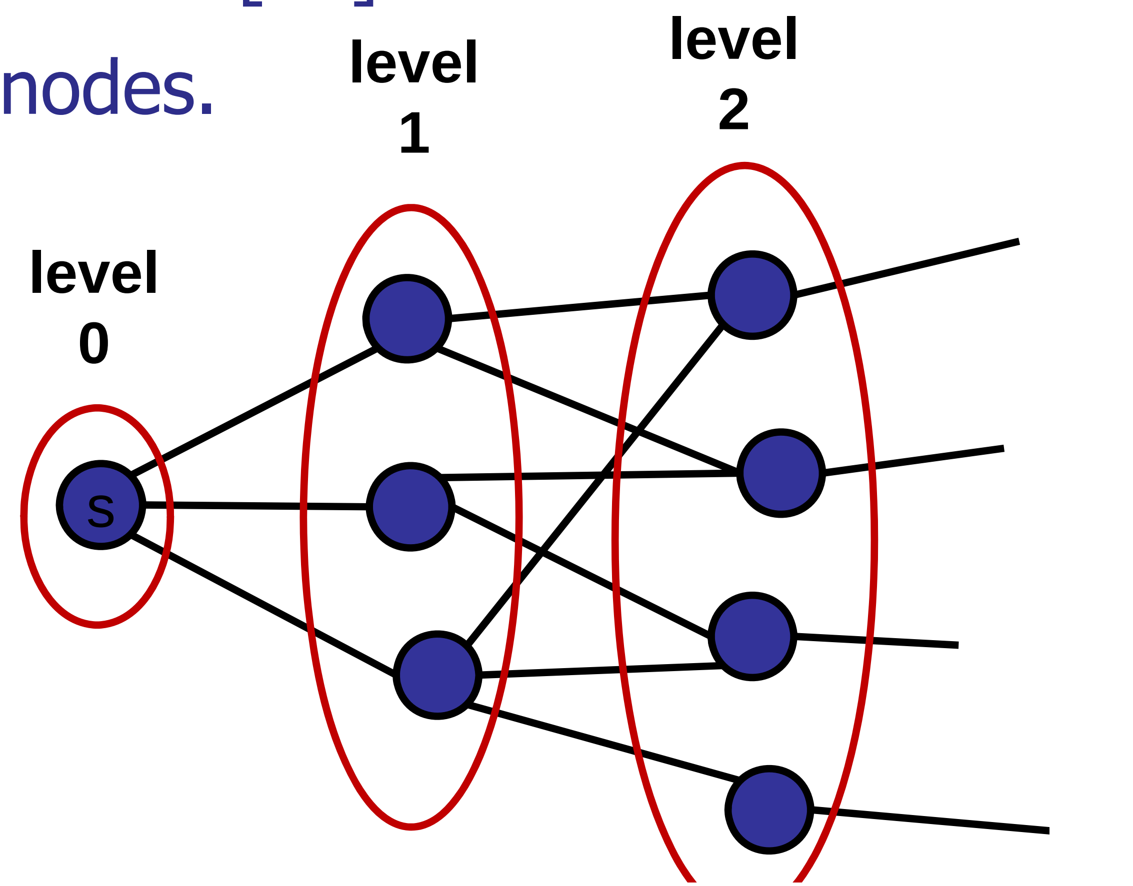


- Finds <u>shortest </u>paths.

Searching a graph

Breadth-First Search:

- Build levels.
- Calculate level[i] from level[i-1]
- Skip already visited nodes.



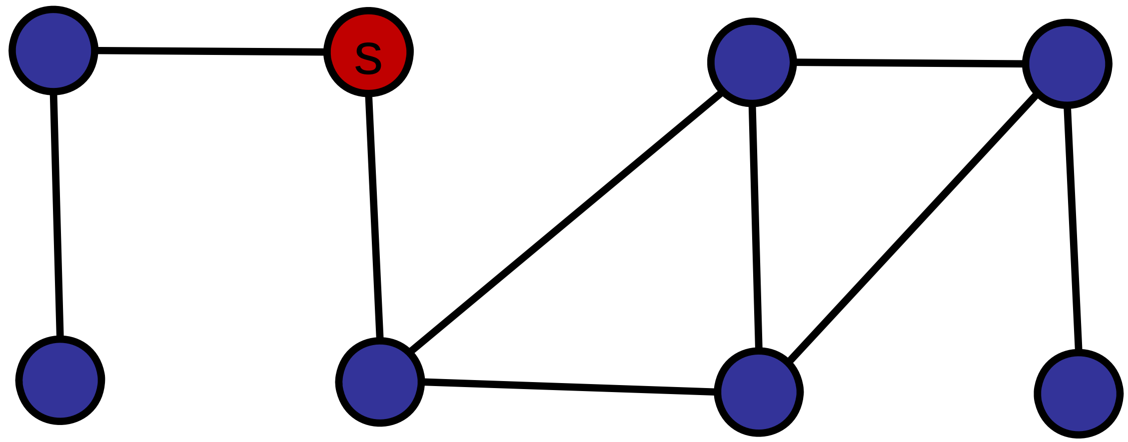
Breadth-First Search

```
BFS(Node[] nodeList, int startId) {
  boolean[] visited = new boolean[nodeList.length];
  Arrays.fill(visited, false);

  int[] parent = new int[nodelist.length];
  Arrays.fill(parent, -1);

  Bag<Integer> frontier = new Bag<Integer>;
  frontier.add(startId);

  // Main code goes here!

}
```

Breadth-First Search

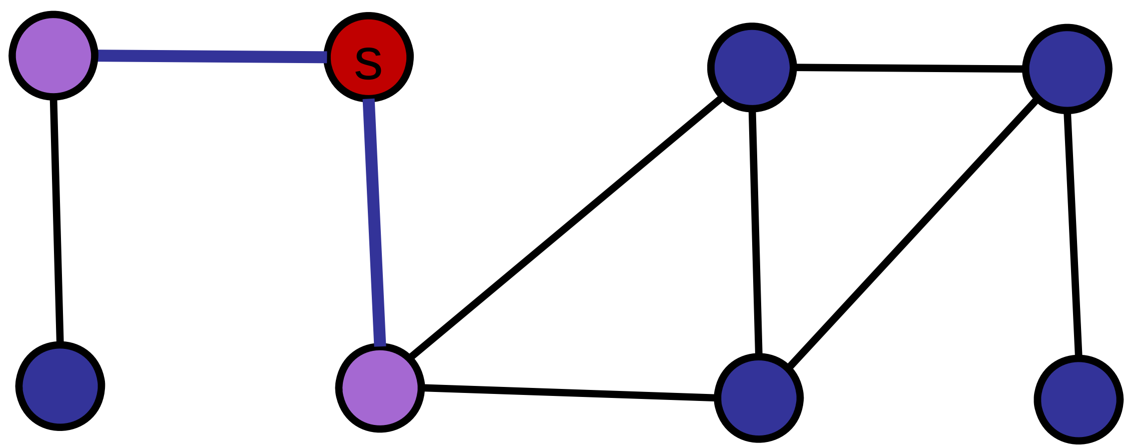
```
while (!frontier.isEmpty()){
   Bag<Integer> nextFrontier = new Bag<Integer>;
   for (Integer v : frontier) {
         for (Integer w : nodeList[v].nbrList) {
               if (!visited[w]) {
                     visited[w] = true;
                     parent[w] = v;
                     nextFrontier.add(w);
               }
         }
   }
   frontier = nextFrontier;
}
```

Breadth-First Search Example



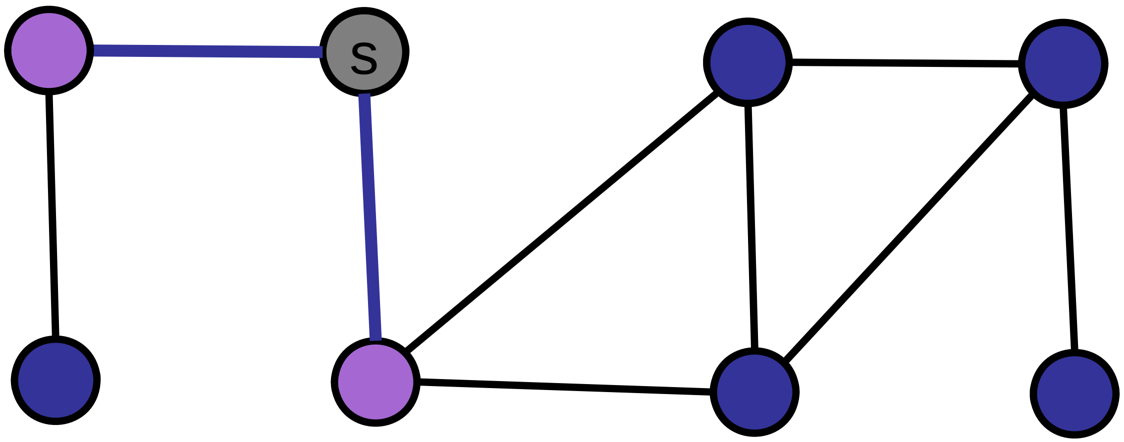
Red = active frontier
Purple = next
Gray = visited
Blue = unvisited

Breadth-First Search Example



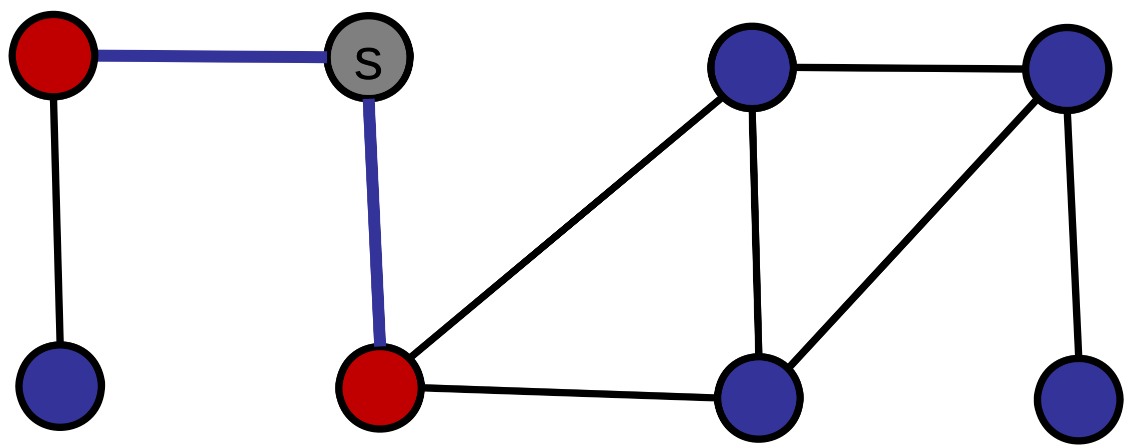
Red = active frontier
Purple = next
Gray = visited
Blue = unvisited

Breadth-First Search Example



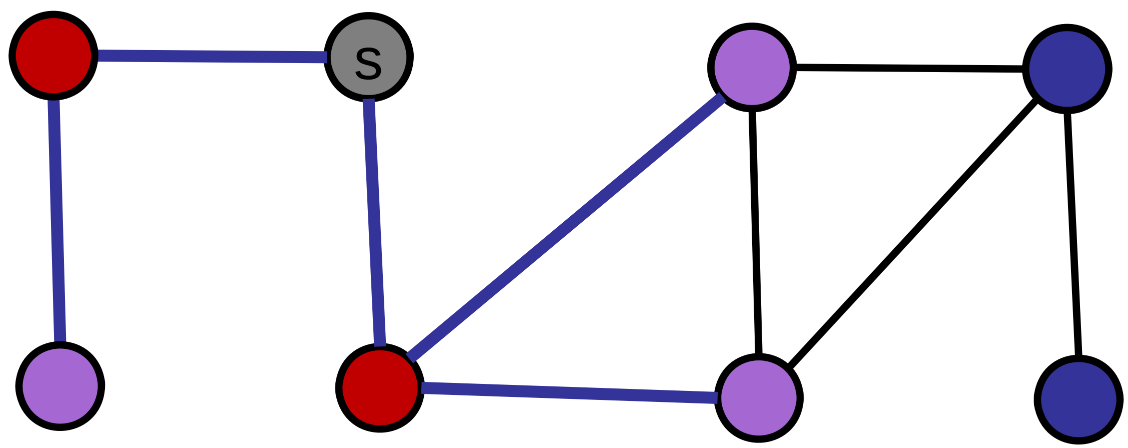
Red = active frontier
Purple = next
Gray = visited
Blue = unvisited

Breadth-First Search Example



Red = active frontier
Purple = next
Gray = visited
Blue = unvisited

Breadth-First Search Example



Red = active frontier
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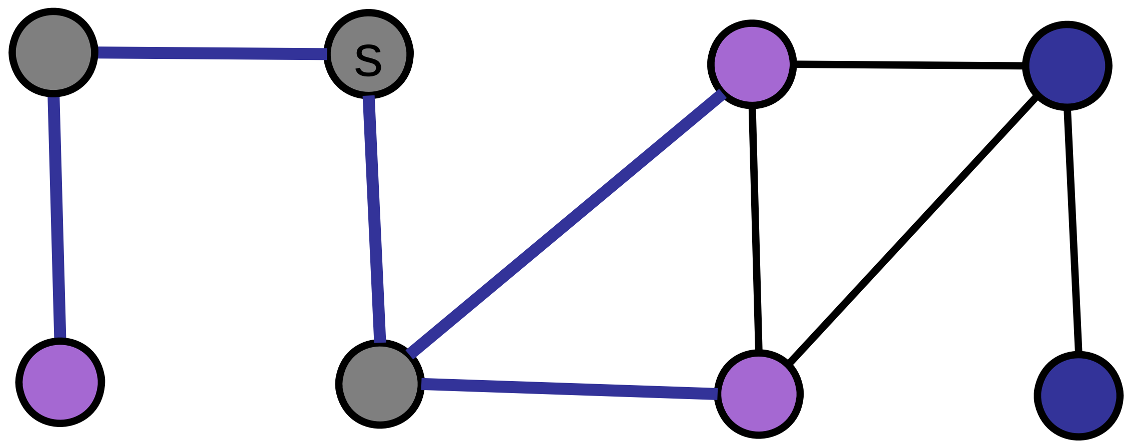
Blue = unvisited

Breadth-First Search Example



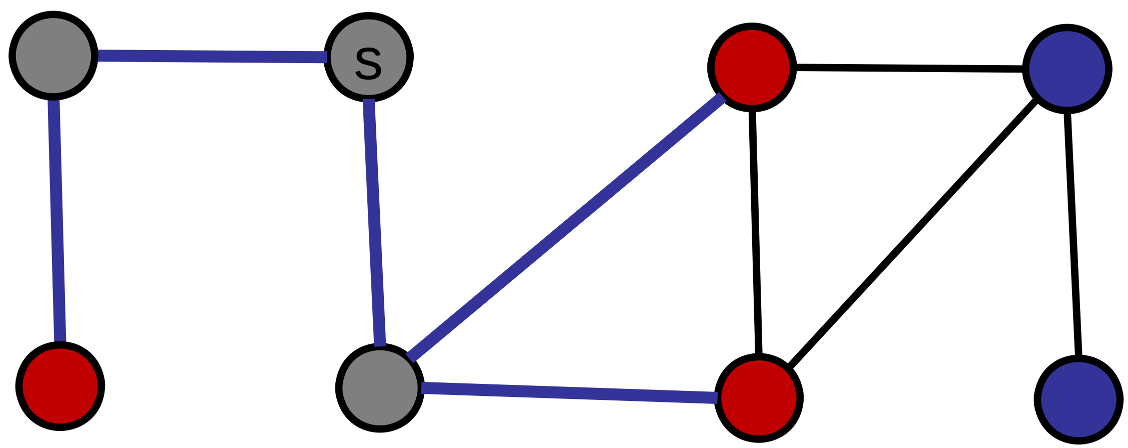
Red = active frontier

Purple = next

Gray = visited

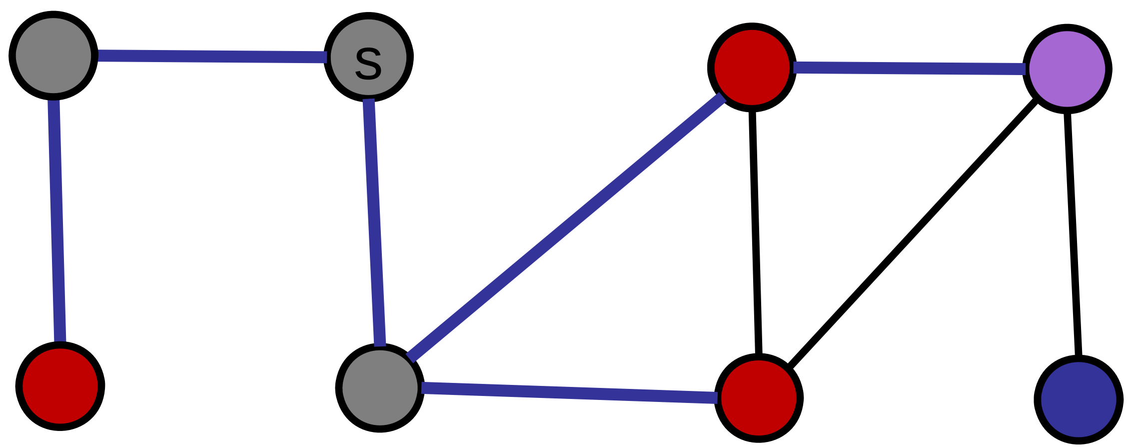
Blue = unvisited

Breadth-First Search Example



Red = active frontier
Purple = next
Gray = visited
Blue = unvisited

Breadth-First Search Example



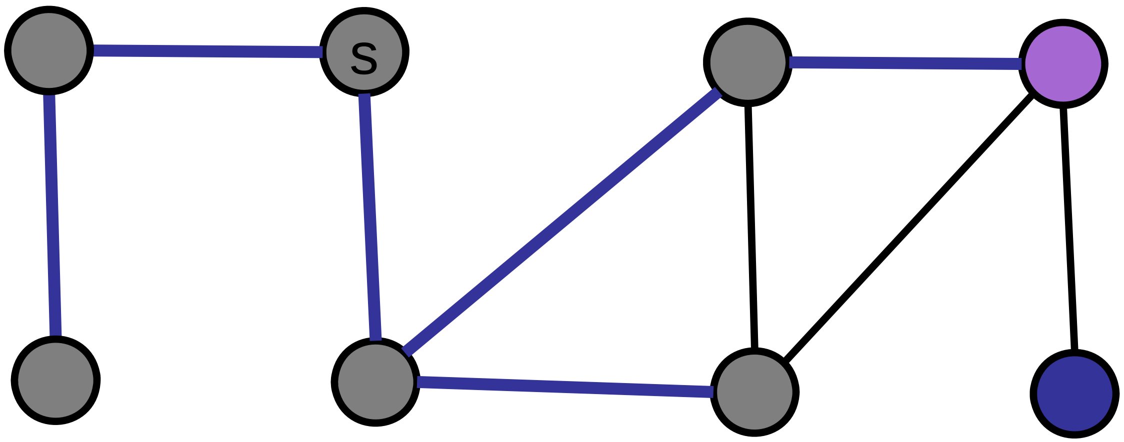
Red = active frontier

Purple = next

Gray = visited

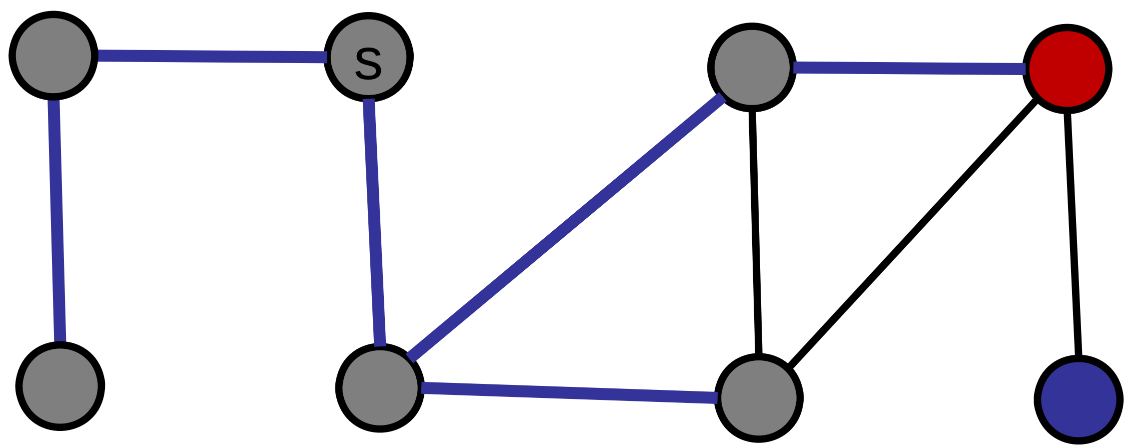
Blue = unvisited

Breadth-First Search Example



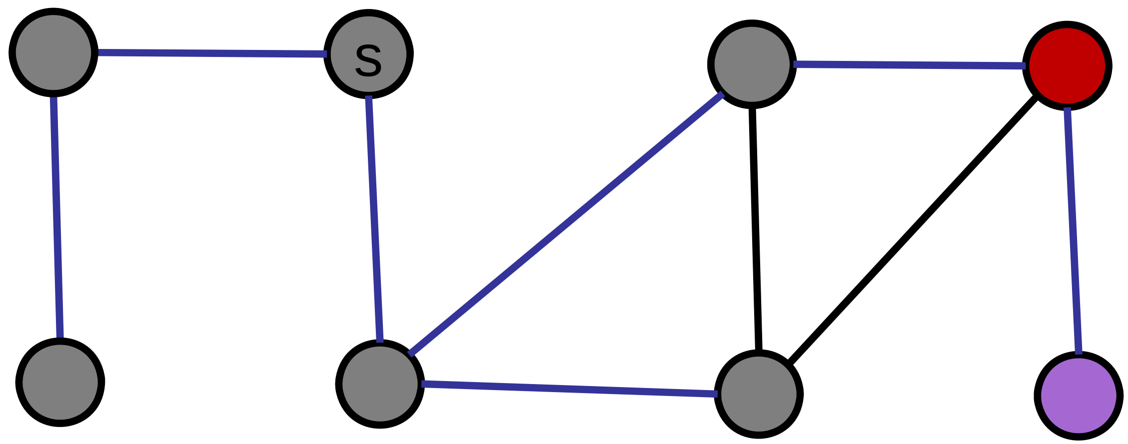
Red = active frontier
Purple = next
Gray = visited
Blue = unvisited

Breadth-First Search Example



Red = active frontier
Purple = next
Gray = visited
Blue = unvisited

Breadth-First Search Example



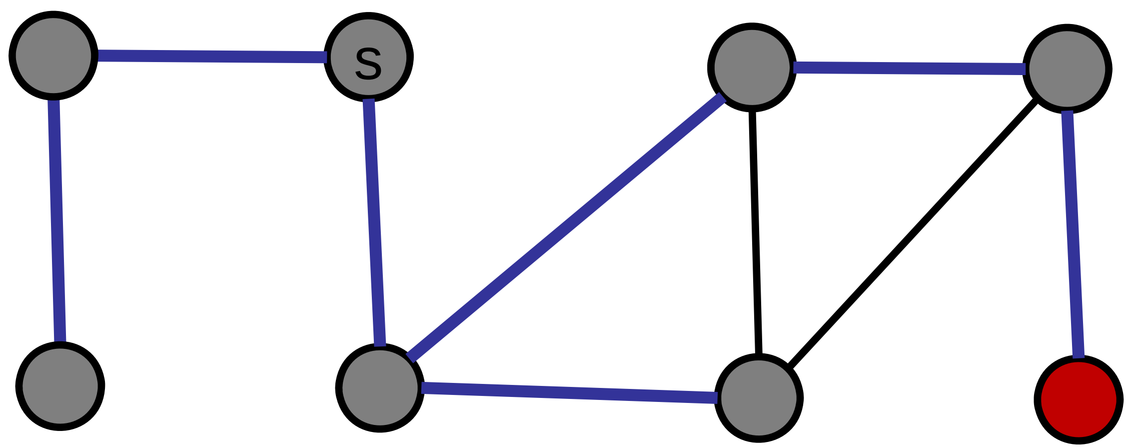
Red = active frontier

Purple = next

Gray = visited

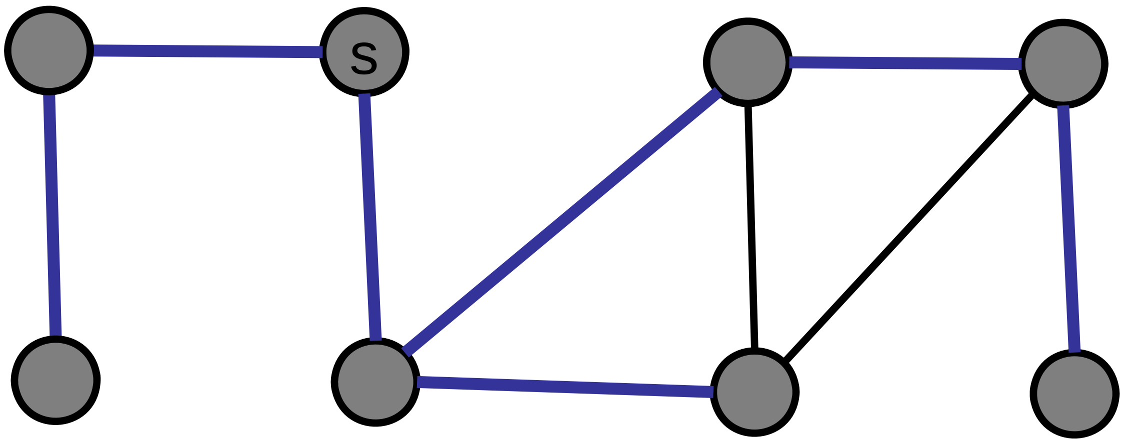
Blue = unvisited

Breadth-First Search Example



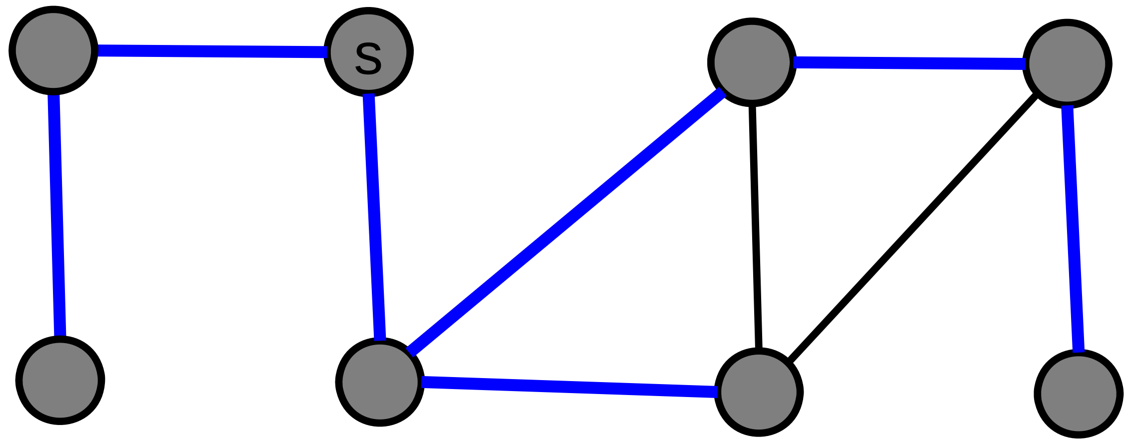
Red = active frontier
Purple = next
Gray = visited
Blue = unvisited

Breadth-First Search Example



Red = active frontier
Purple = next
Gray = visited
Blue = unvisited

Breadth-First Search Example



Red = active frontier

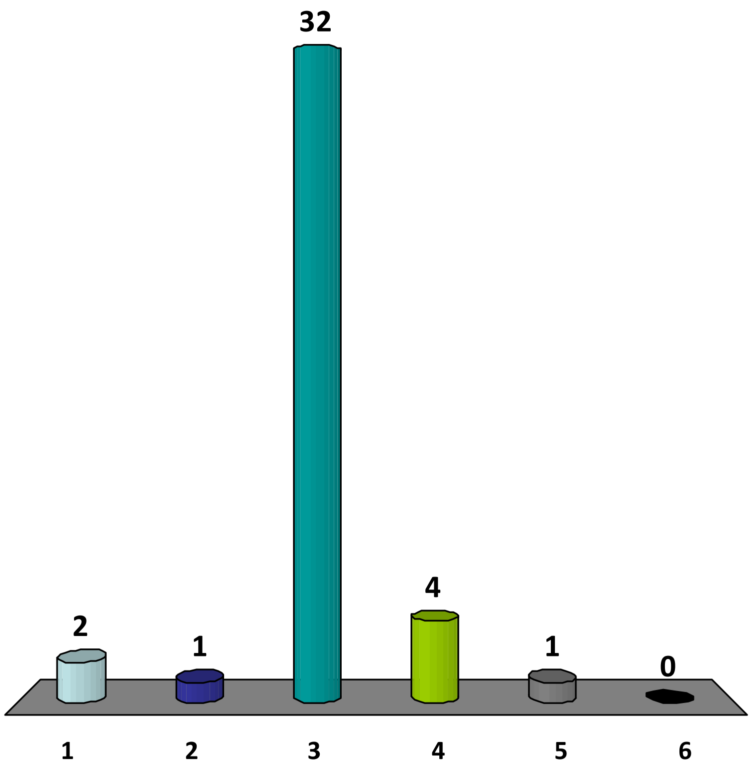
Purple = next

Gray = visited

Blue = unvisited

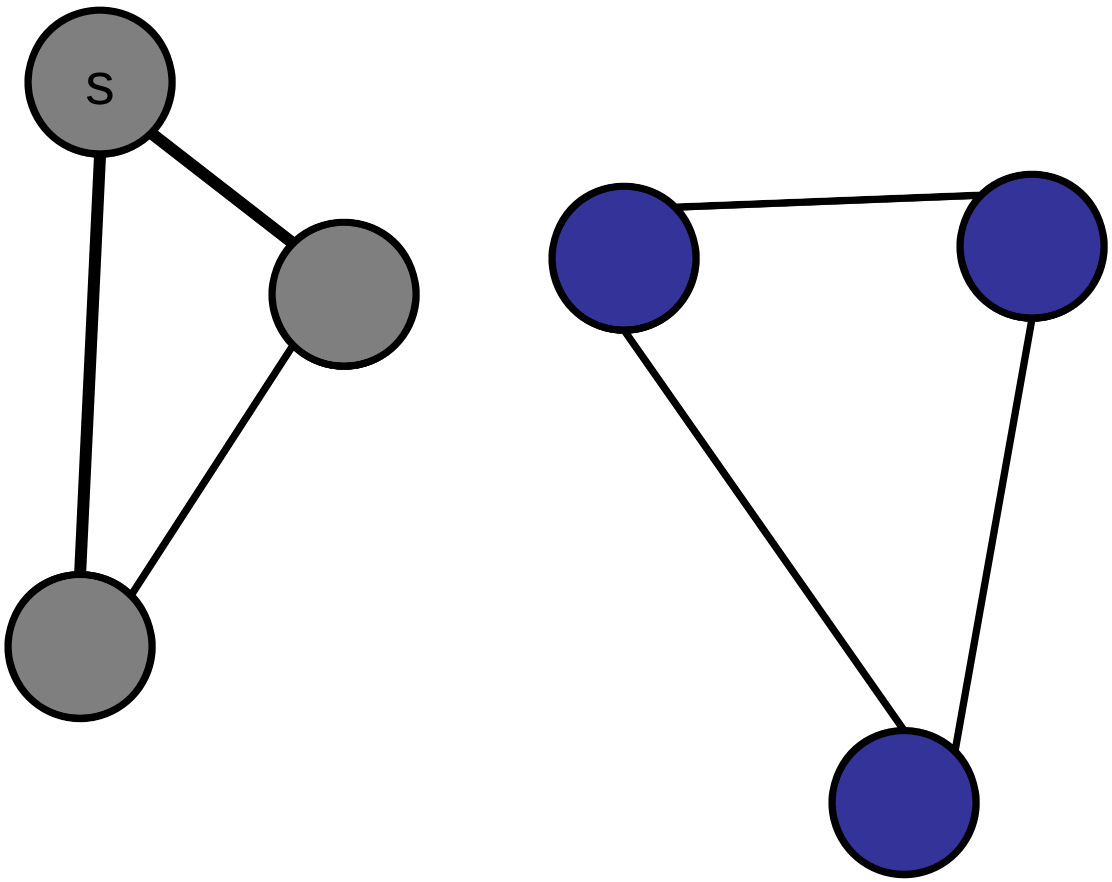
When does BFS fail to visit every node?

1. In a clique.
2. In a cycle.
3. In a graph with two components.
4. In a sparse graph.
5. In a dense graph.
6. Never.



BFS on Disconnected Graph

Example:

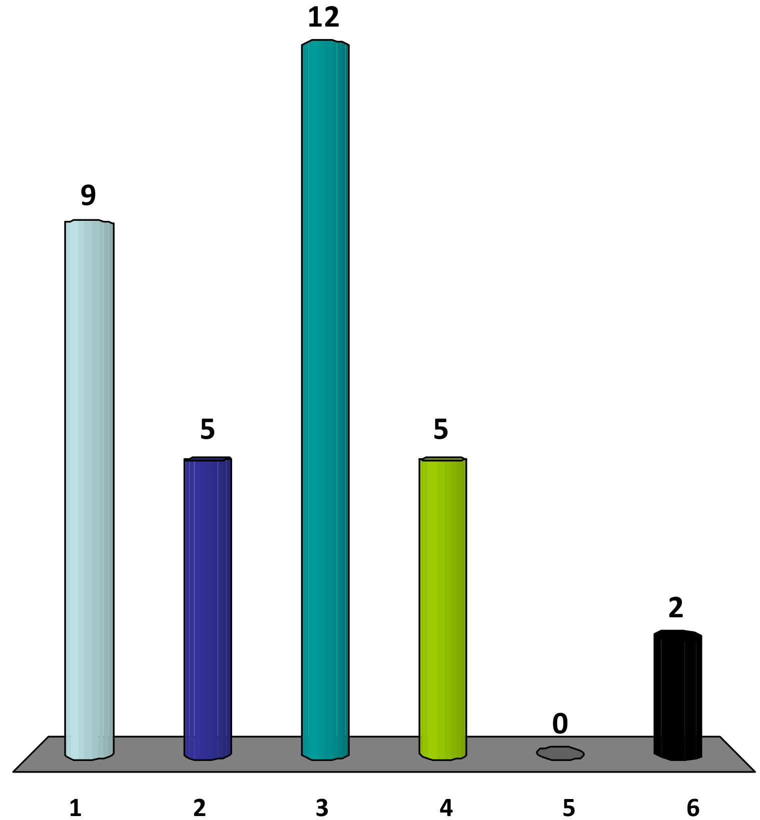


Breadth-First Search

```
BFS(Node[] nodeList) {
  boolean[] visited = new boolean[nodeList.length];
  Arrays.fill(visited, false);

  int[] parent = new int[nodelist.length];
  Arrays.fill(parent, -1);

  for (int start = 0; start < nodeList.length; start++) {
     if (!visited[start]){
           Bag<Integer> frontier = new Bag<Integer>;
           frontier.add(startId);

           // Main code goes here!
     }
  }
```

The running time of BFS is:

1. O(V)
2. O(E)
3. O(V+E)
4. O(VE)
5. (V^2)
6. I have no idea.



Breadth-First Search

Analysis:

- Vertex v = “start” once. ← $O(V)$

- Vertex v added to nextFrontier (and frontier) once.
  - After visited, never re-added.

- Each v.nbrlist is enumerated once.
  - When v is removed from frontier. ← $O(E)$

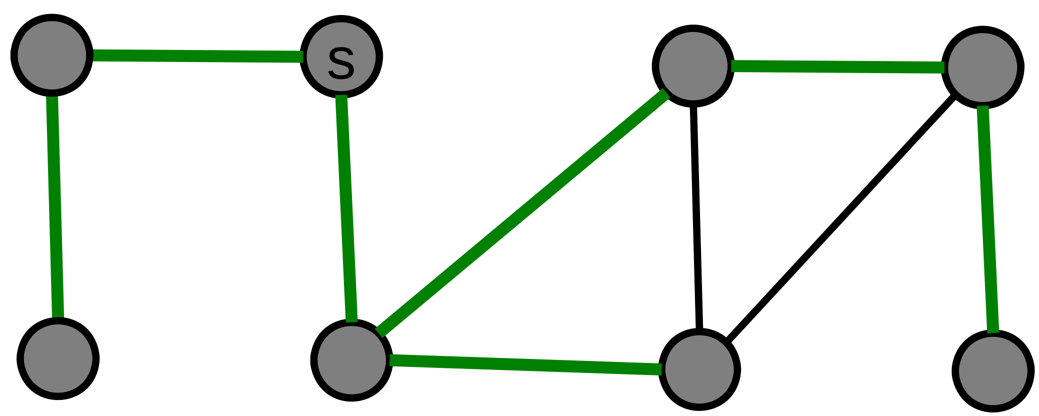
Breadth-First Search

```
while (!frontier.isEmpty()){
   Bag<Integer> next = new Bag<Integer>;
   for (Integer v : frontier) {
         for (Integer w : nodeList[v].nbrList) {
               if (!visited[w]) {
                     visited[w] = true;
                     parent[w] = v;
                     next.add(w);
               }
         }
   }
   frontier = next;
}
```

Breadth-First Search

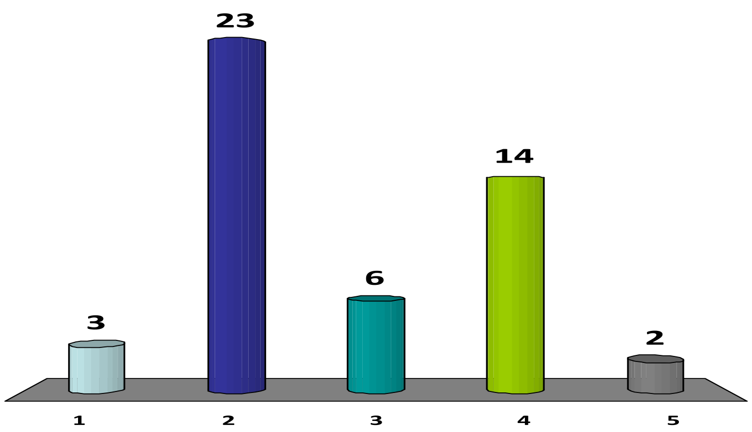
Shortest paths:

– Parent pointers store shortest path.



Which is true? (More than one may apply.)

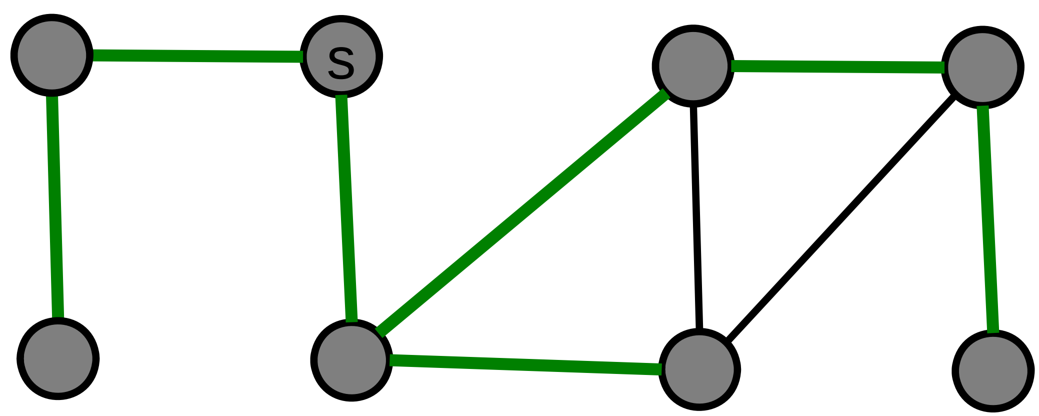
1. Shortest path graph is a cycle.
2. Shortest path graph is a tree.
3. Shortest path graph has low-degree.
4. Shortest path graph has low diameter.
5. None of the above.



Breadth-First Search

Shortest paths:

- Parent pointers store shortest path.
- Shortest path is a tree.
- (Possibly high degree; possibly high diameter.)



What if there are
two components?

Searching a Graph

Goal:

- Start at some vertex **s** = start.
- Find some other vertex **f** = finish.

  Or: visit **all** the nodes in the graph;

Two basic techniques:

- Breadth-First Search (BFS)
- Depth-First Search (DFS)

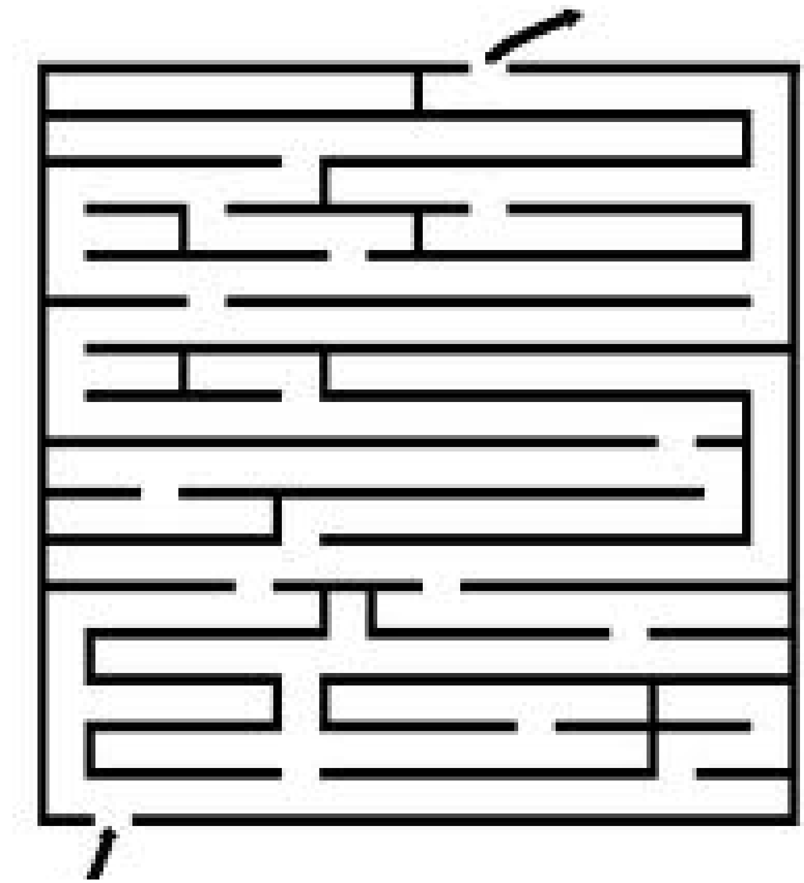
Graph representation:

- Adjacency list

Depth-First Search

Exploring a maze:

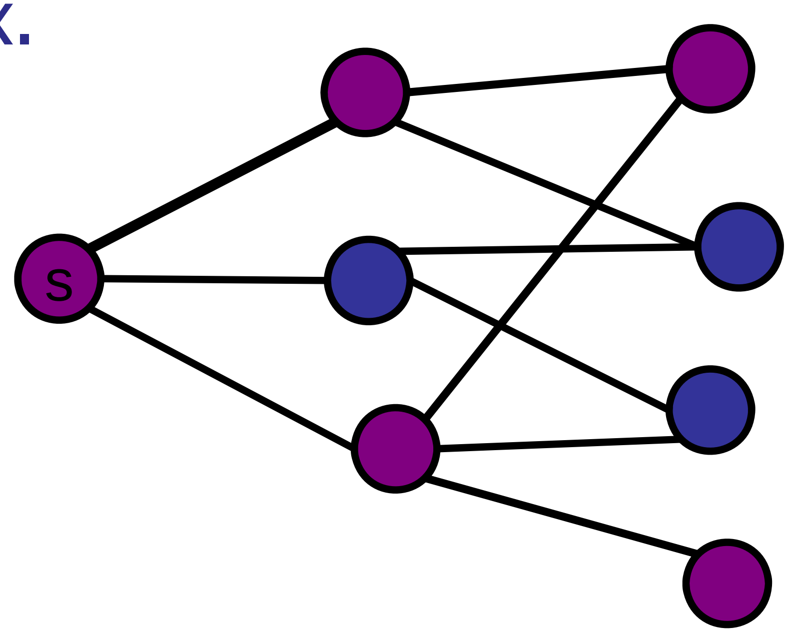
- Follow path until stuck.
- Backtrack along breadcrumbs until reach unexplored neighbor.
- Recursively explore.



Searching a graph

Depth-First Search:

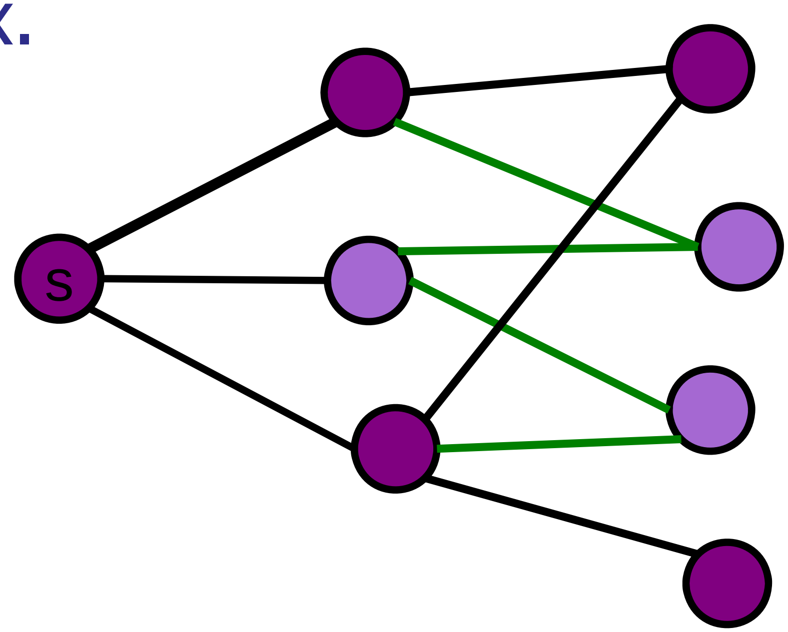
- Follow path until you get stuck
- Backtrack until you find a new edge
- Recursively explore it
- Don't repeat a vertex.



Searching a graph

Depth-First Search:

- Follow path until you get stuck
- Backtrack until you find a new edge
- Recursively explore it
- Don’t repeat a vertex.



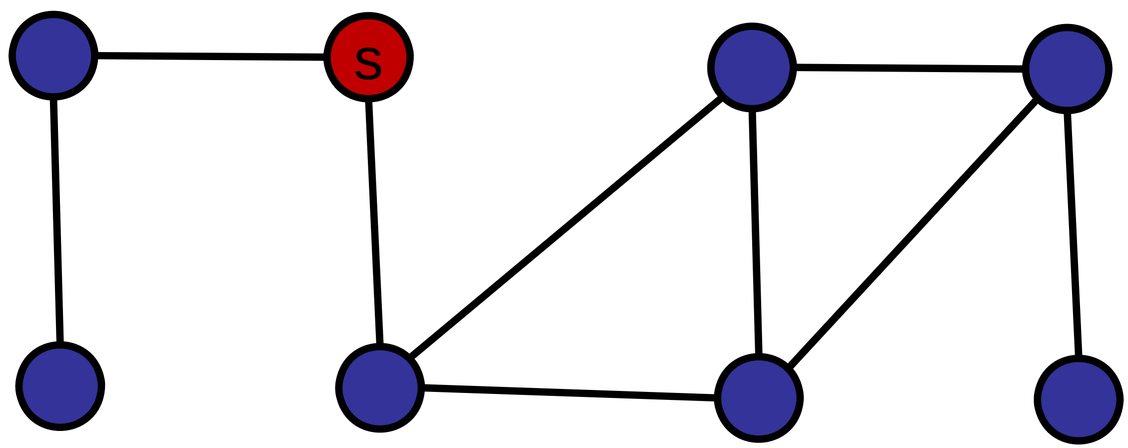
Depth-First Search

```
DFS-visit(Node[] nodeList, boolean[] visited, int startId){
  for (Integer v : nodeList[startId].nbrList) {
     if (!visited[v]){
           visited[v] = true;
           DFS-visit(nodeList, visited, v);
     }
  }
}
```

Depth-First Search

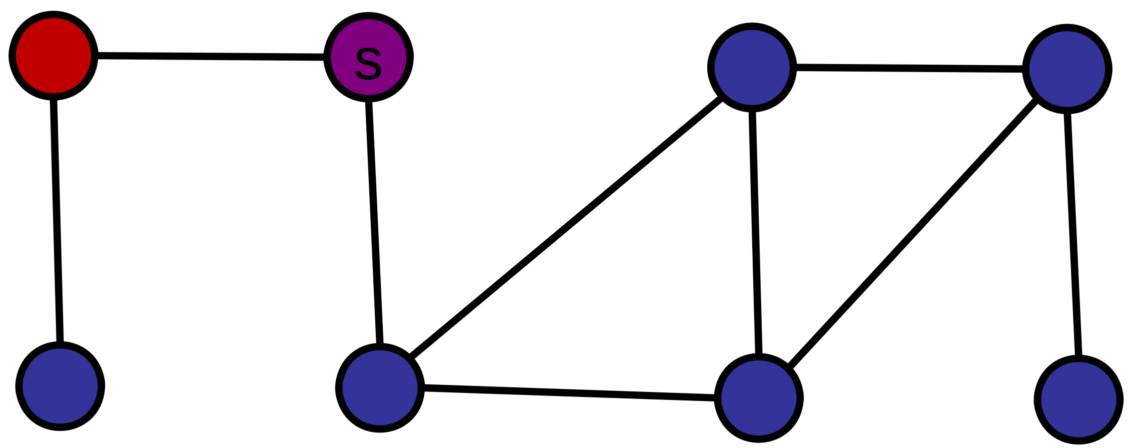
```
DFS(Node[] nodeList){
  boolean[] visited = new boolean[nodeList.length];
  Arrays.fill(visited, false);

  for (start = i; start<nodeList.length; start++) {
     if (!visited[start]){
           visited[start] = true;
           DFS-visit(nodeList, visited, start);
     }
  }
}
```

Depth-First Search Example



Red = active frontier
Purple = next
Gray = visited
Blue = unvisited

Depth-First Search Example



Red = active frontier
Purple = next
Gray = visited
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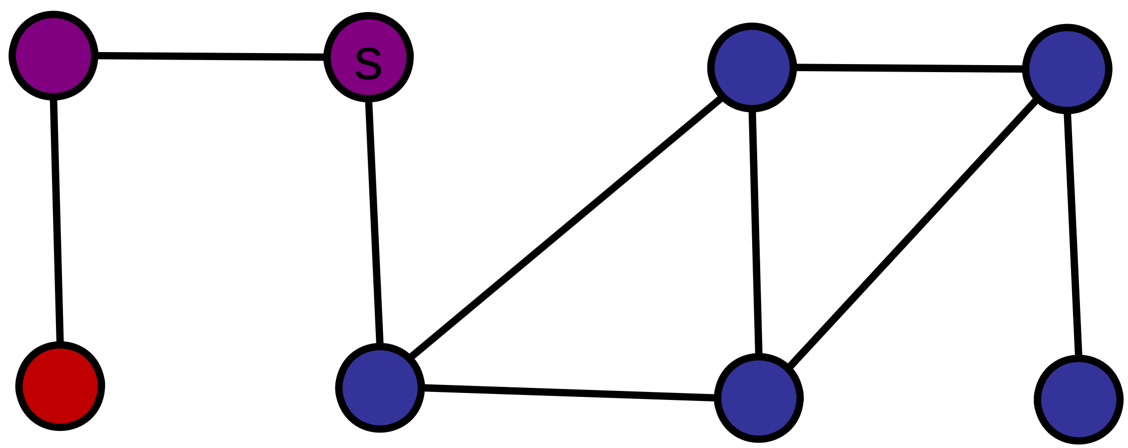
Depth-First Search Example



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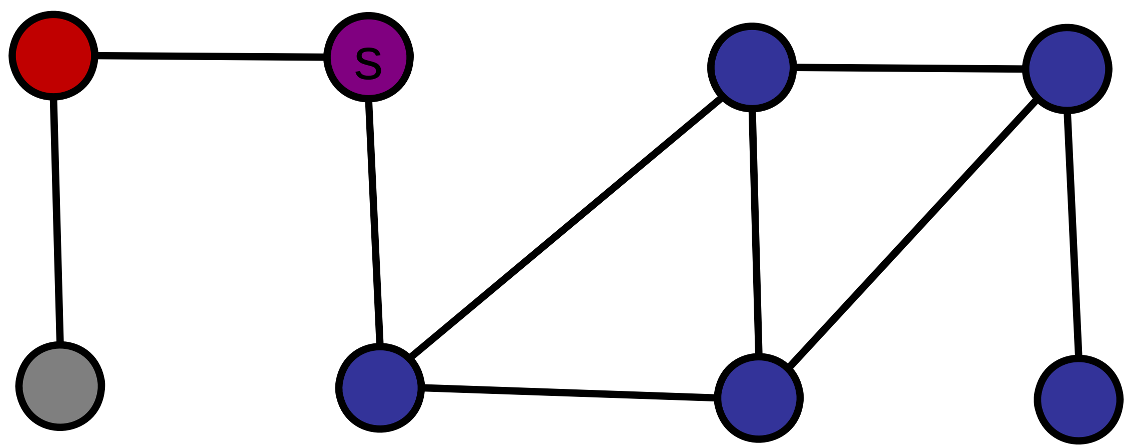
Depth-First Search Example



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Purple = next
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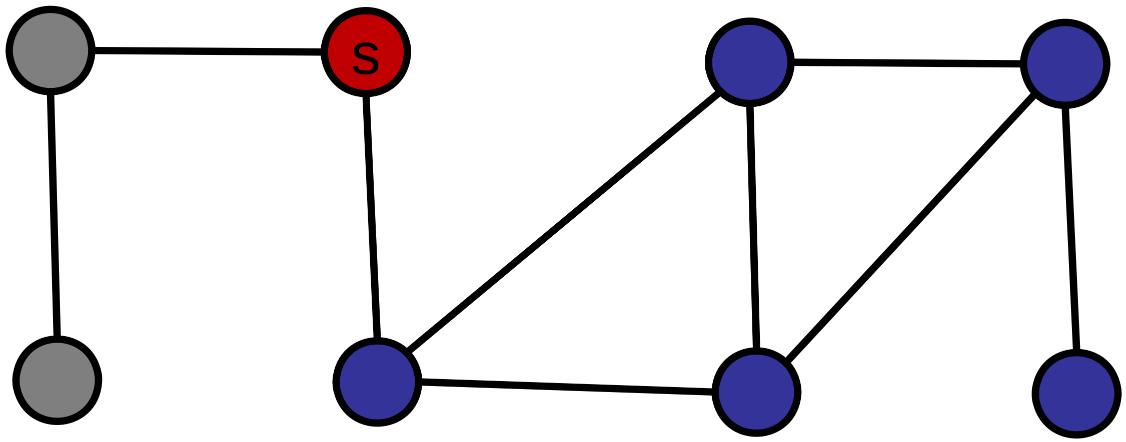
Blue = unvisited

Depth-First Search Example



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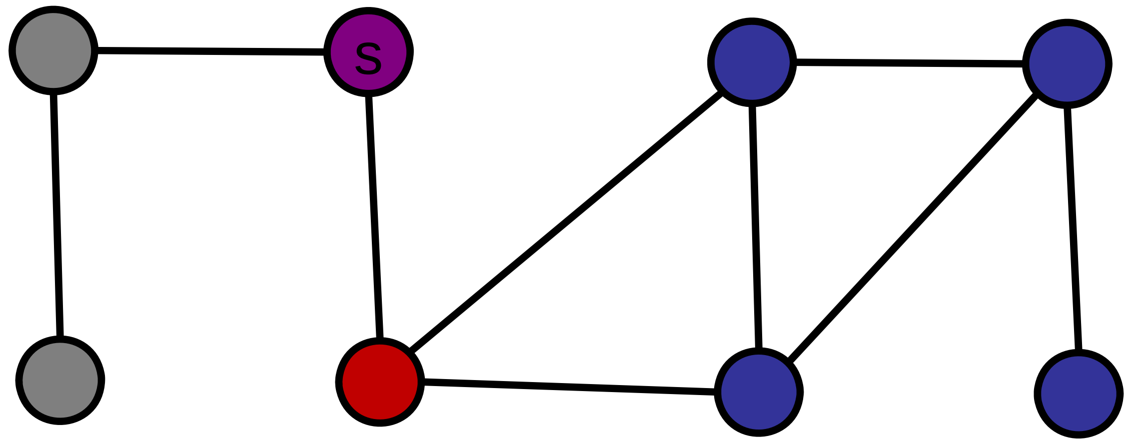
Purple = next
Gray = visited
Blue = unvisited

Depth-First Search Example



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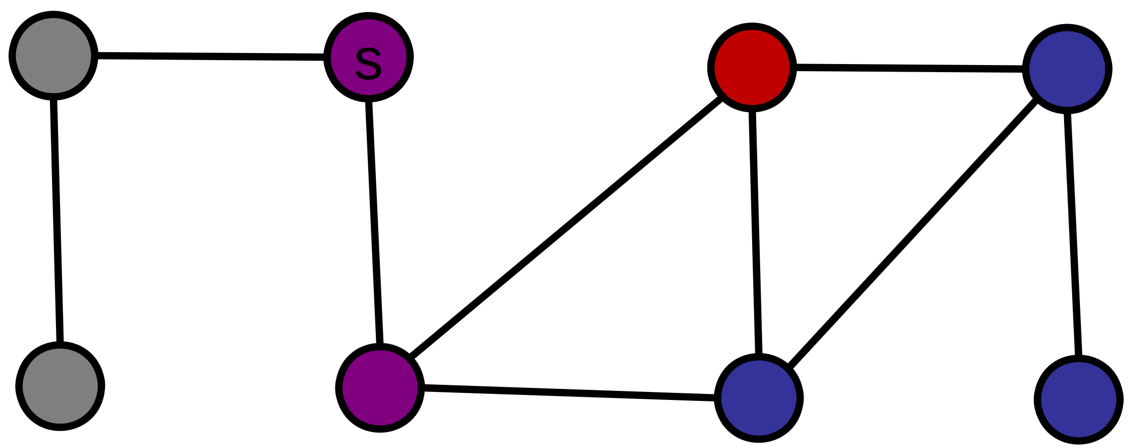
Depth-First Search Example



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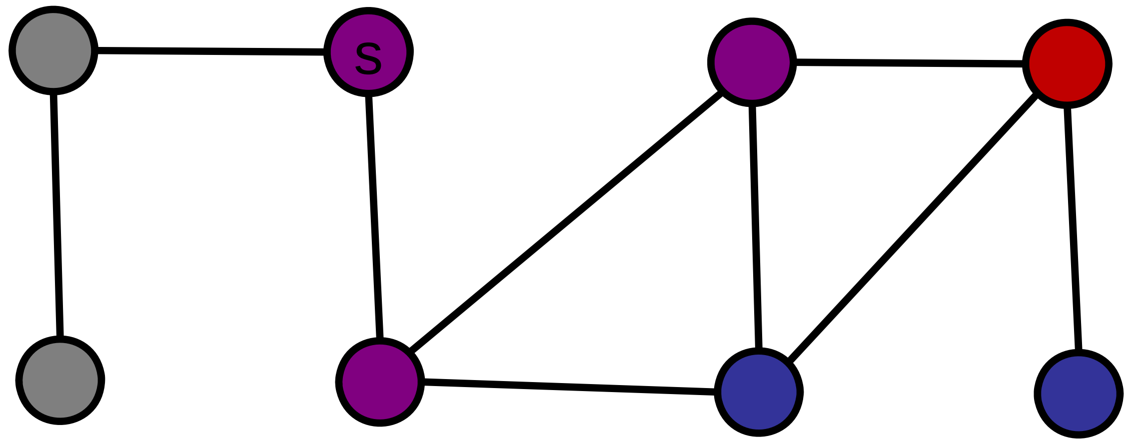
Depth-First Search Example



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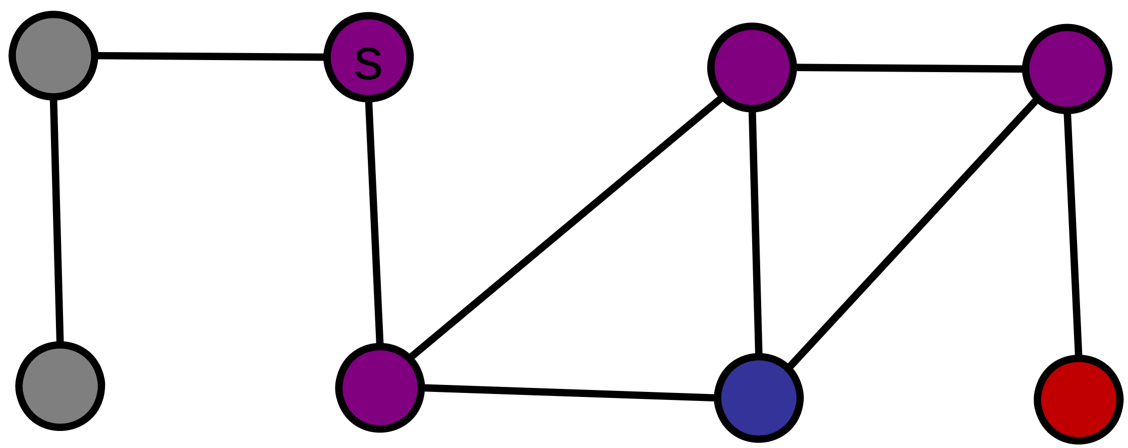
Depth-First Search Example



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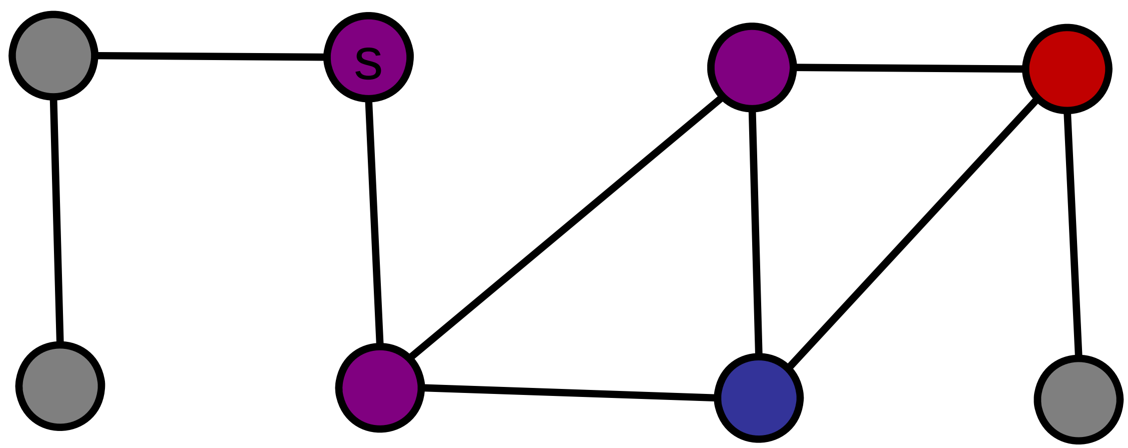
Depth-First Search Example



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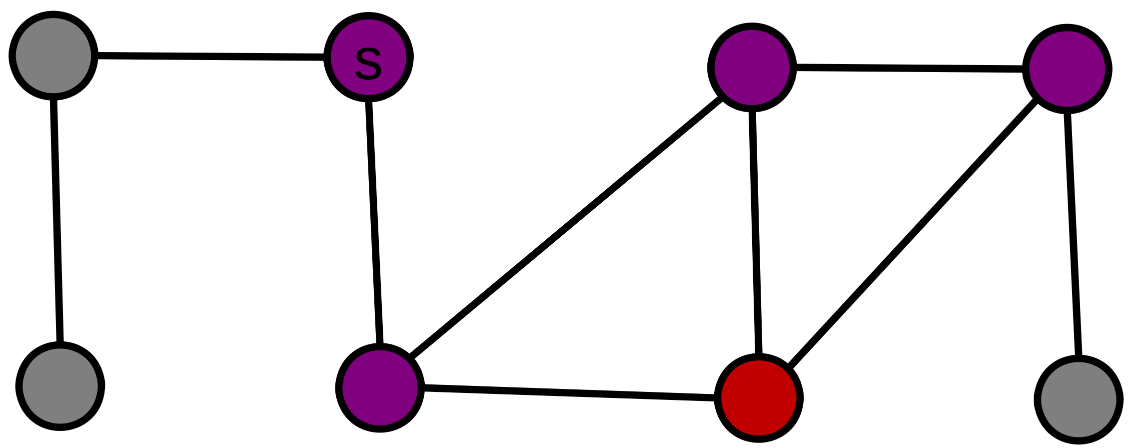
Depth-First Search Example



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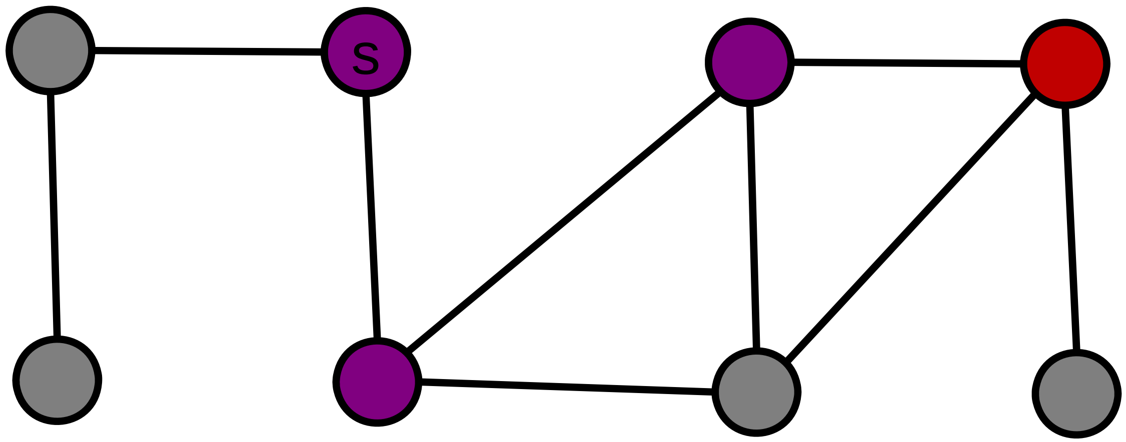
Depth-First Search Example



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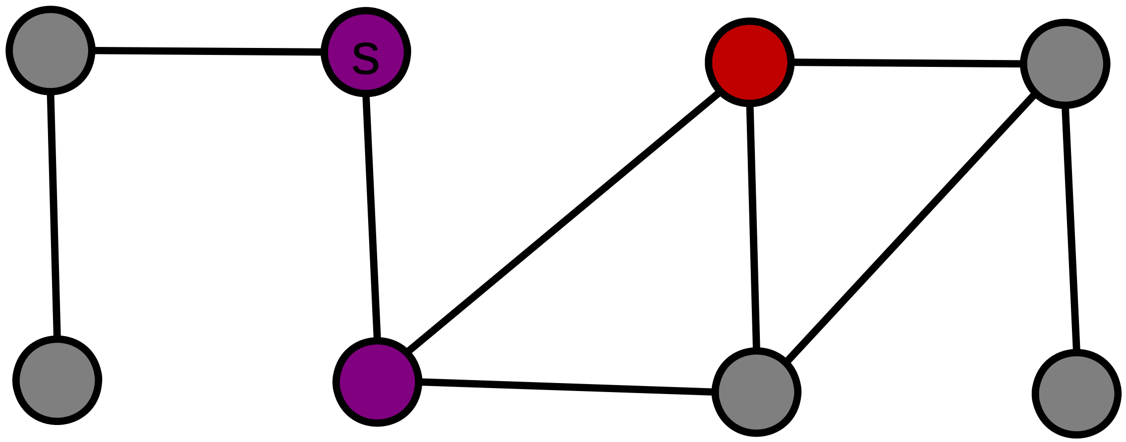
Depth-First Search Example



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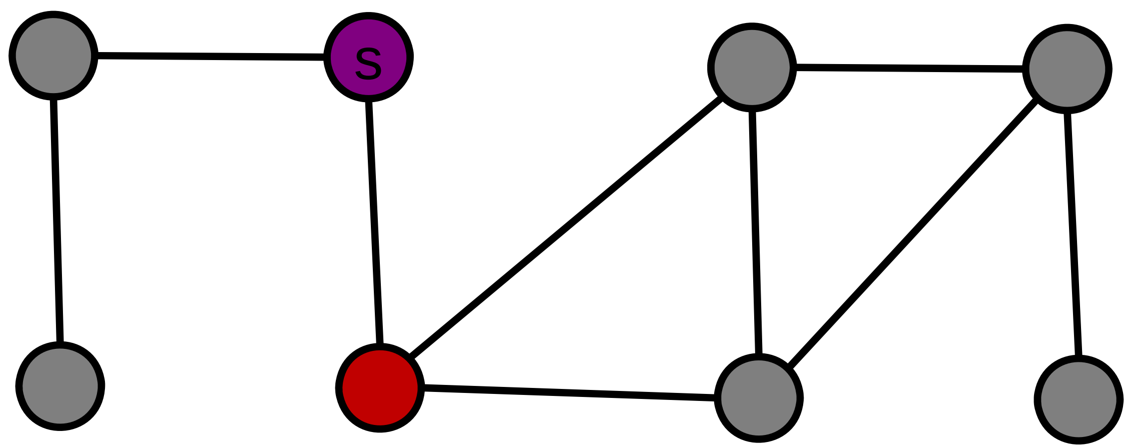
Depth-First Search Example



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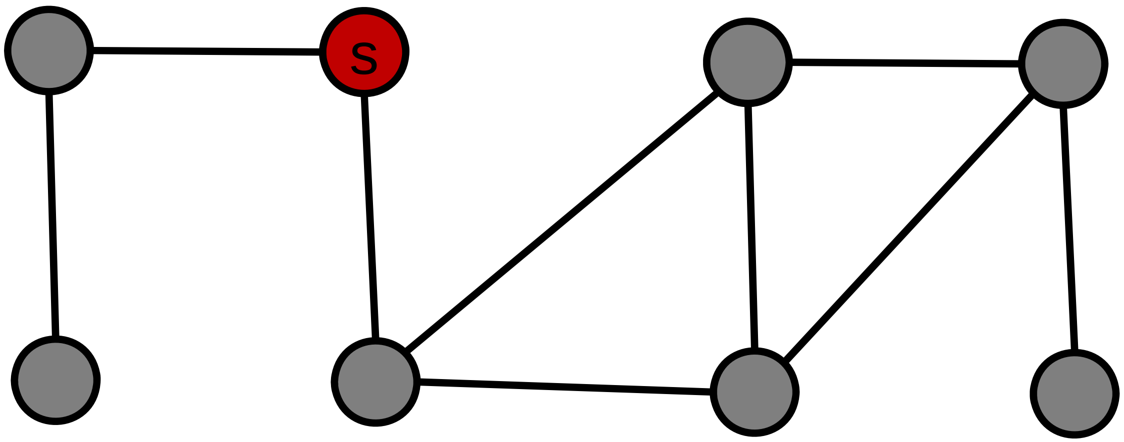
Depth-First Search Example



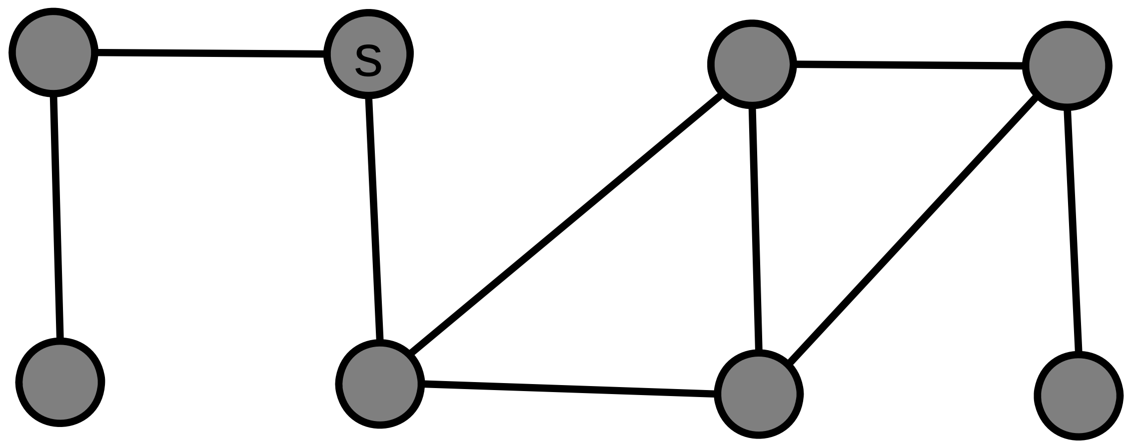
Red = active frontier

Purple = next

Gray = visited

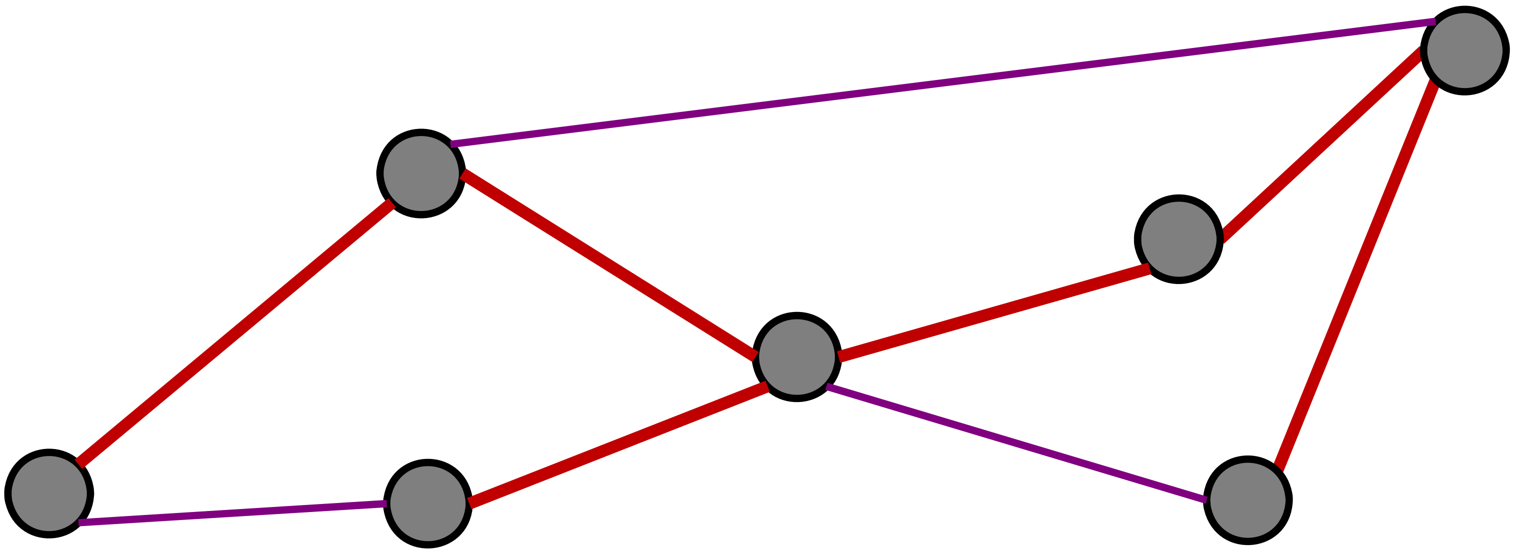
Blue = unvisited

Depth-First Search Example



Red = active frontier
Purple = next
Gray = visited
Blue = unvisited

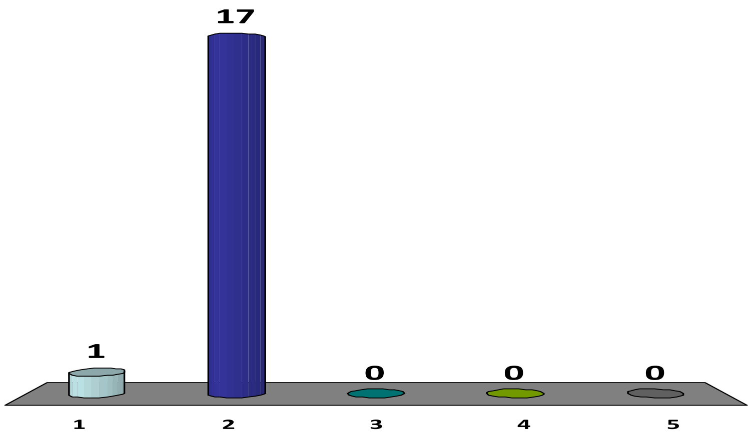
DFS parent edges



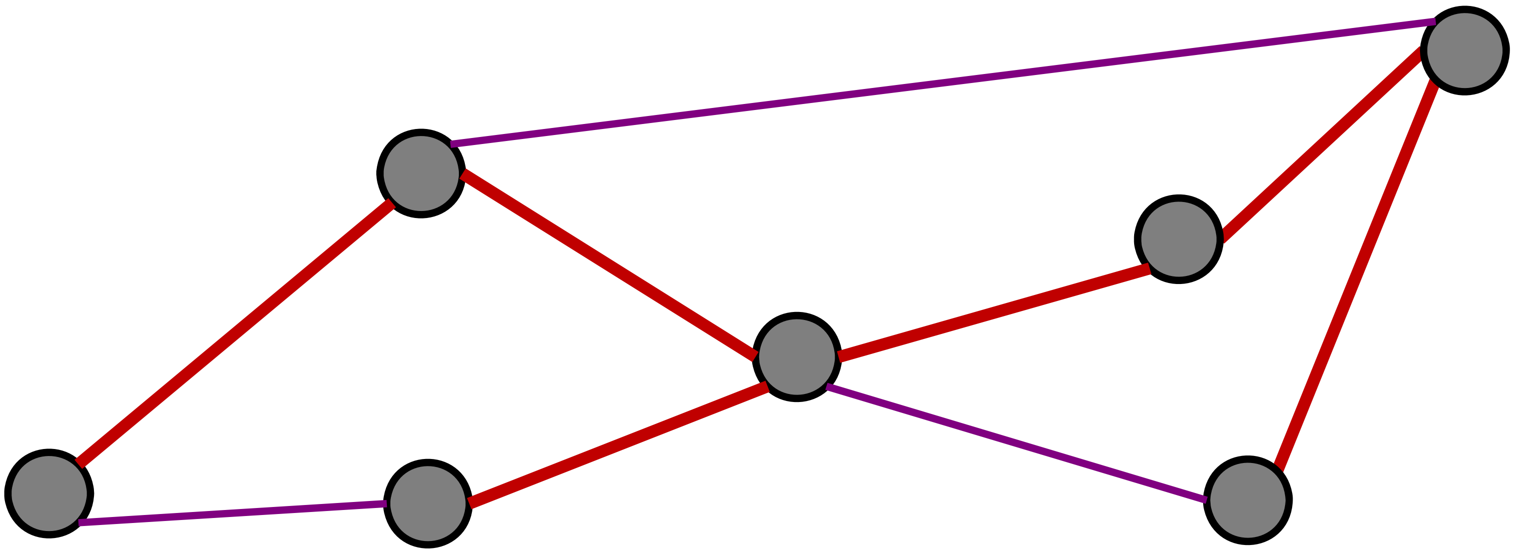
Red = Parent Edges
Purple = Non-parent edges

Which is true? (More than one may apply.)

1. DFS parent graph is a cycle.
2. ✓ DFS parent graph is a tree.
3. DFS parent graph has low-degree.
4. DFS parent graph has low diameter.
5. None of the above.



DFS parent edges = tree

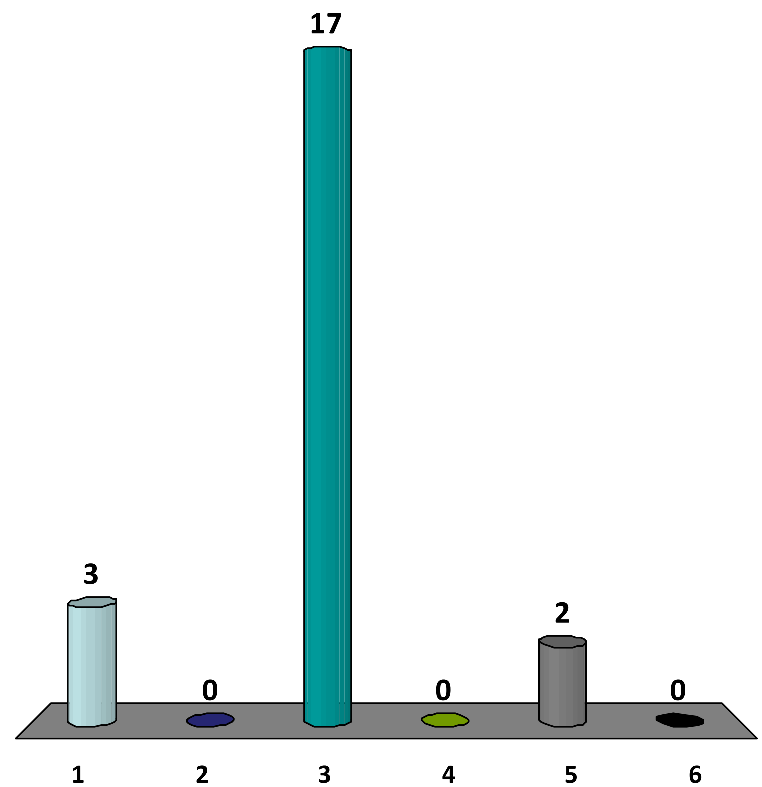


Red = Parent Edges
Purple = Non-parent edges

Note: not shortest paths!

The running time of DFS is:

1. O(V)
2. O(E)
3. O(V+E)
4. O(VE)
5. (V^2)
6. I have no idea.



Depth-First Search

Analysis:

- DFS-visit called only once per node.
  - After visited, never call DFS-visit again.

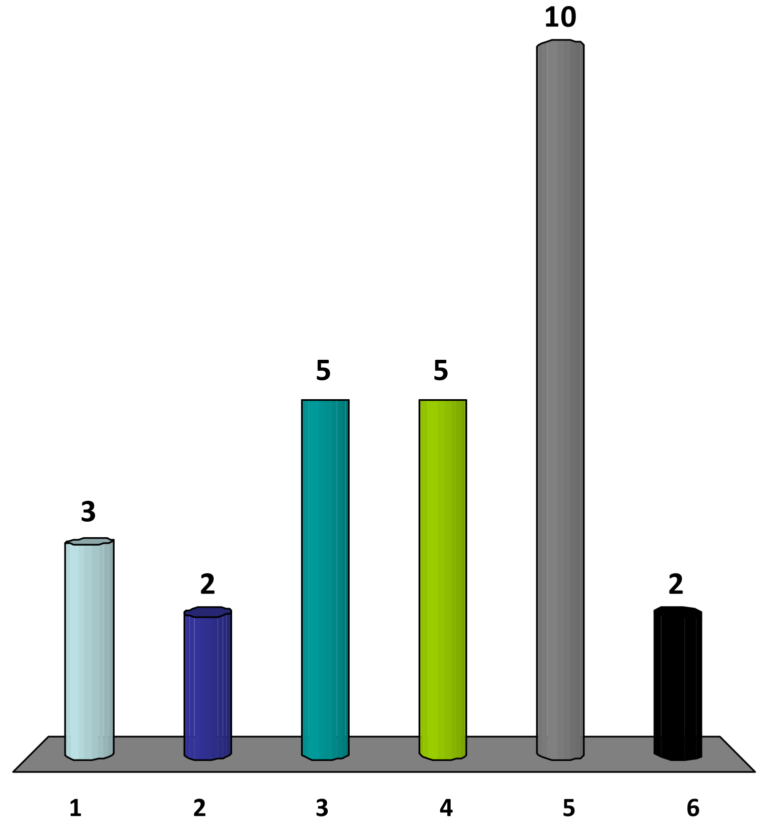
- In DFS-visit, each neighbor is enumerated.

O(V)

O(E)

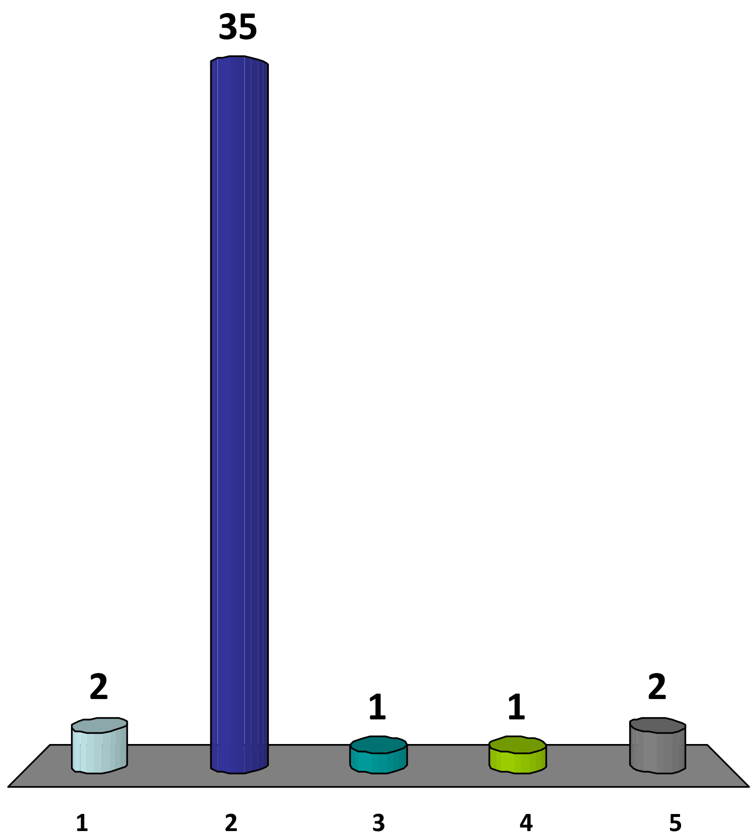
If the graph is stored as an adjacency matrix, what is the running time of DFS?

1. O(V)
2. O(E)
3. (V+E)
4. O(VE)
5. $O(V^2)$
6. $O(E^2)$



To implement an iterative version of DFS:

1. Use a queue.
2. ✓ Use a stack.
3. Use a bag.
4. Use a set.
5. Don't.



Graph Search

BFS and DFS are the same algorithm:

- BFS: use a queue
  - Every time you visit a node, add all unvisited neighbors to the queue.

- DFS: use a stack
  - Every time you visit a node, add all unvisited neighbors to the stack.

Graph Search

Breadth-first search:

Same algorithm, implemented with a queue:

Add start-node to queue.

Repeat until queue is empty:

- Remove node v from the front of the queue.
- Visit v.
- Explore all outgoing edges of v.
- Add all unvisited neighbors of v to the queue.

Graph Search

Depth-first search:

Same algorithm, implemented with a stack:

Add start-node to stack.

Repeat until stack is empty:

- Pop node v from the front of the stack.
- Visit v.
- Explore all outgoing edges of v.
- Push all unvisited neighbors of v on the front of the stack.

Review: Searching Graphs

BFS and DFS are the same algorithm:

- BFS: use a queue
  - Every time you visit a node, add all unvisited neighbors to the queue.
- DFS: use a stack
  - Every time you visit a node, add all unvisited neighbors to the stack.

Graph searching illustrations

See:

http://www.comp.nus.edu.sg/~stevenha/visualization/dfsbfs.html

Roadmap

Today: Graph Basics

- What is a graph?
- Modeling problems as graphs.
- Graph representations (list vs. matrix)
- Searching graphs (DFS / BFS)