CS2020
Data Structures and Algorithms

Dynamic Programming!

Semester Roadmap

Where are we?

- Searching
- Sorting
- Lists
- Trees
- Hash Tables
- Graphs
- **Advanced material**

You are here

The Plan

Today

- Dynamic Programming

More details when you take an Algorithms module (e.g., CS3230)

Next Friday, Next Wednesday

- Geometric Algorithms

More details when you take a module on Computational Geometry

Next Next Friday

- Parallel Algorithms

More details when you take a Parallel Computing module

Roadmap

Today: Dynamic Programming

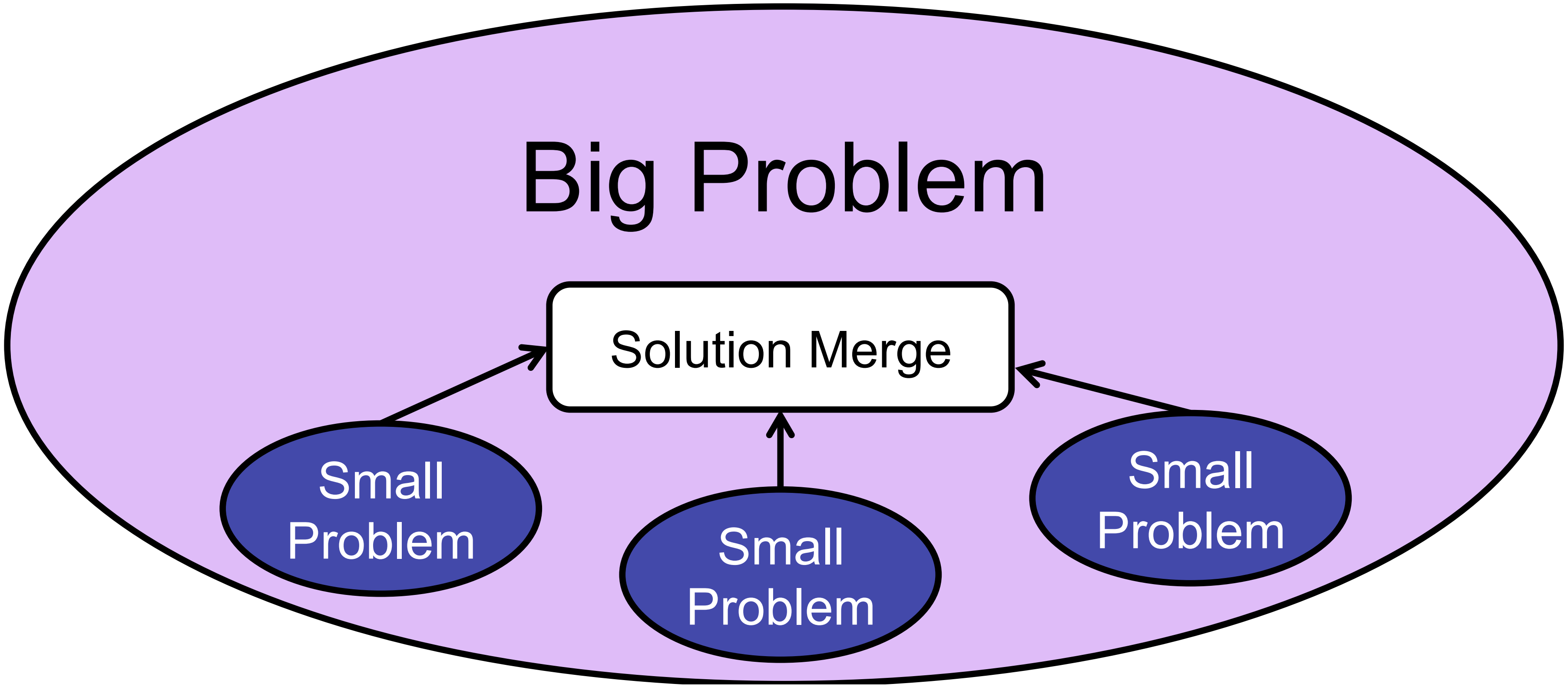
- Basics of DP
- Example: Longest Increasing Subsequence
- Example: Bounded Prize Collecting
- Example: Vertex Cover on a Tree
- Example: All-Pairs Shortest Paths

Dynamic Programming Basics

Dynamic Programming Basics

Optimal sub-structure:

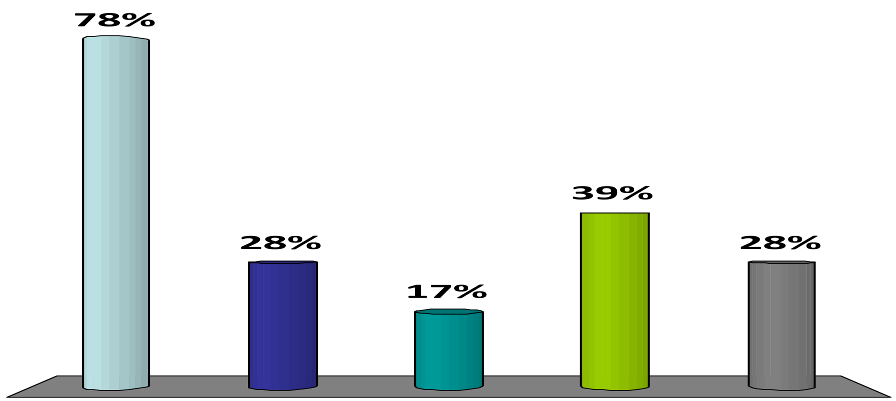
- Optimal solution can be constructed from optimal solutions to smaller sub-problems.



Which of these problems exhibit optimal sub-structure? (Choose all that apply.)

1. Sorting
2. Reversing a string
3. Calculating a hash function
4. Shortest paths
5. Minimum spanning tree

Response Counter

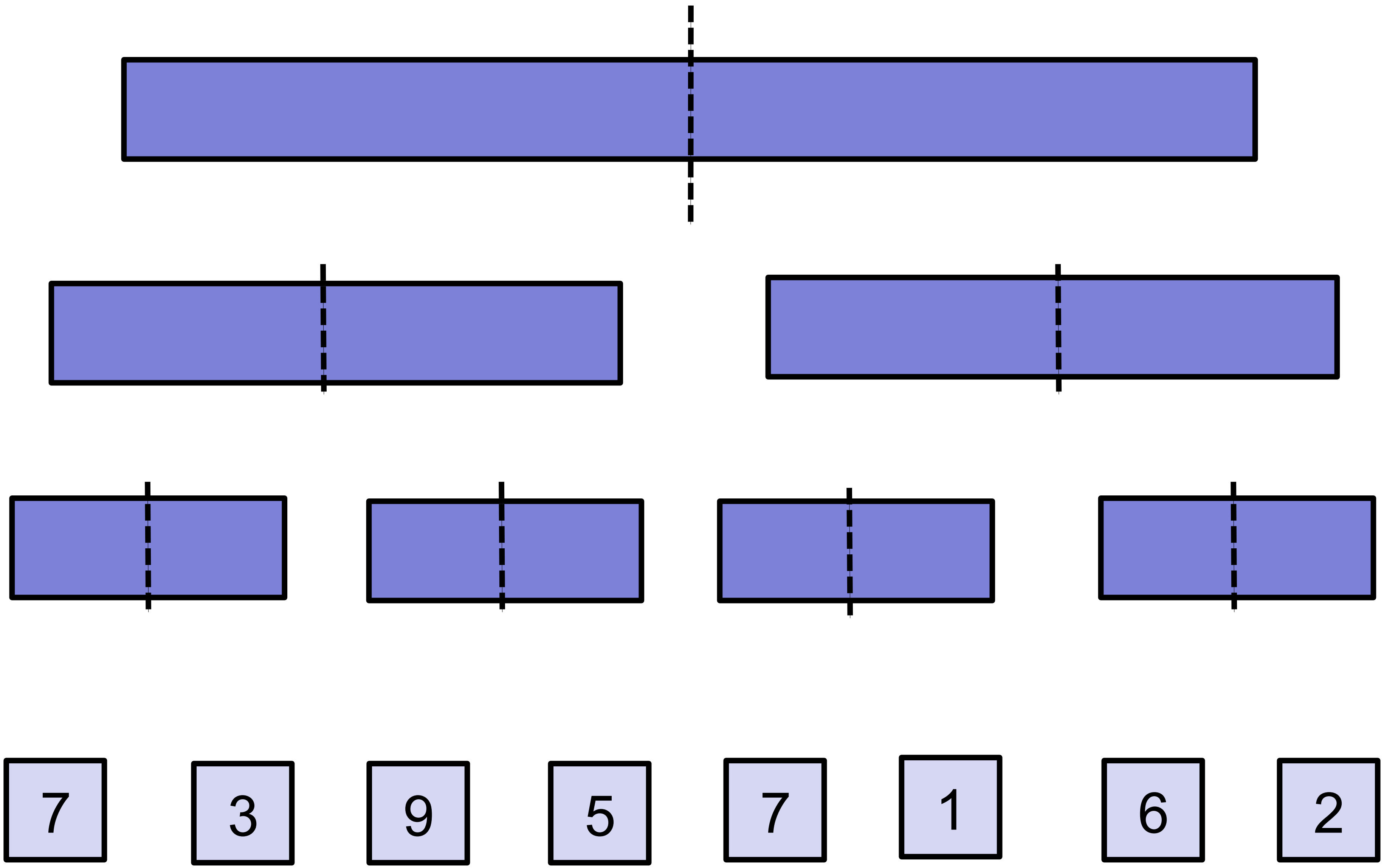


Optimal Sub-structure

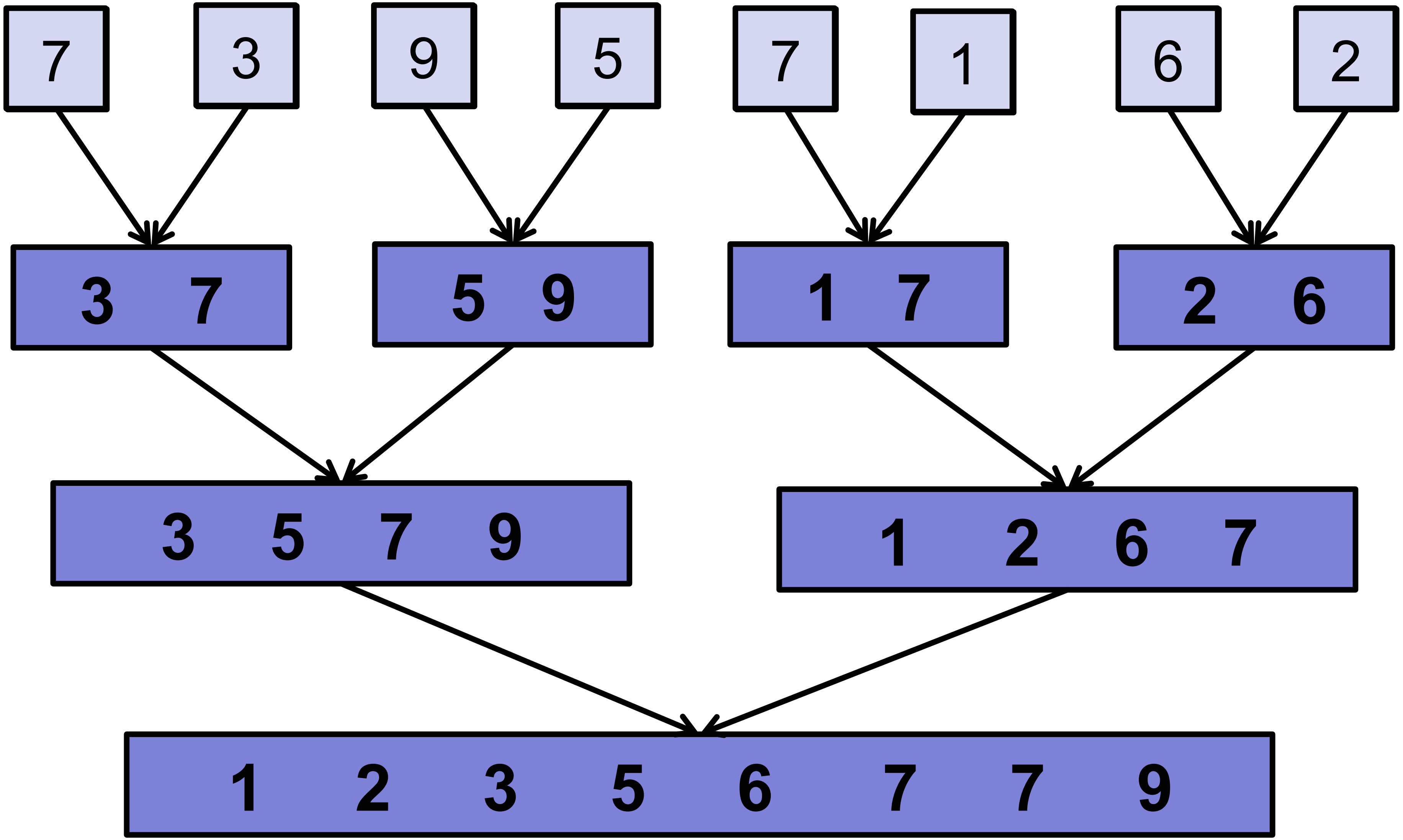
Property of (nearly) every problem we study:

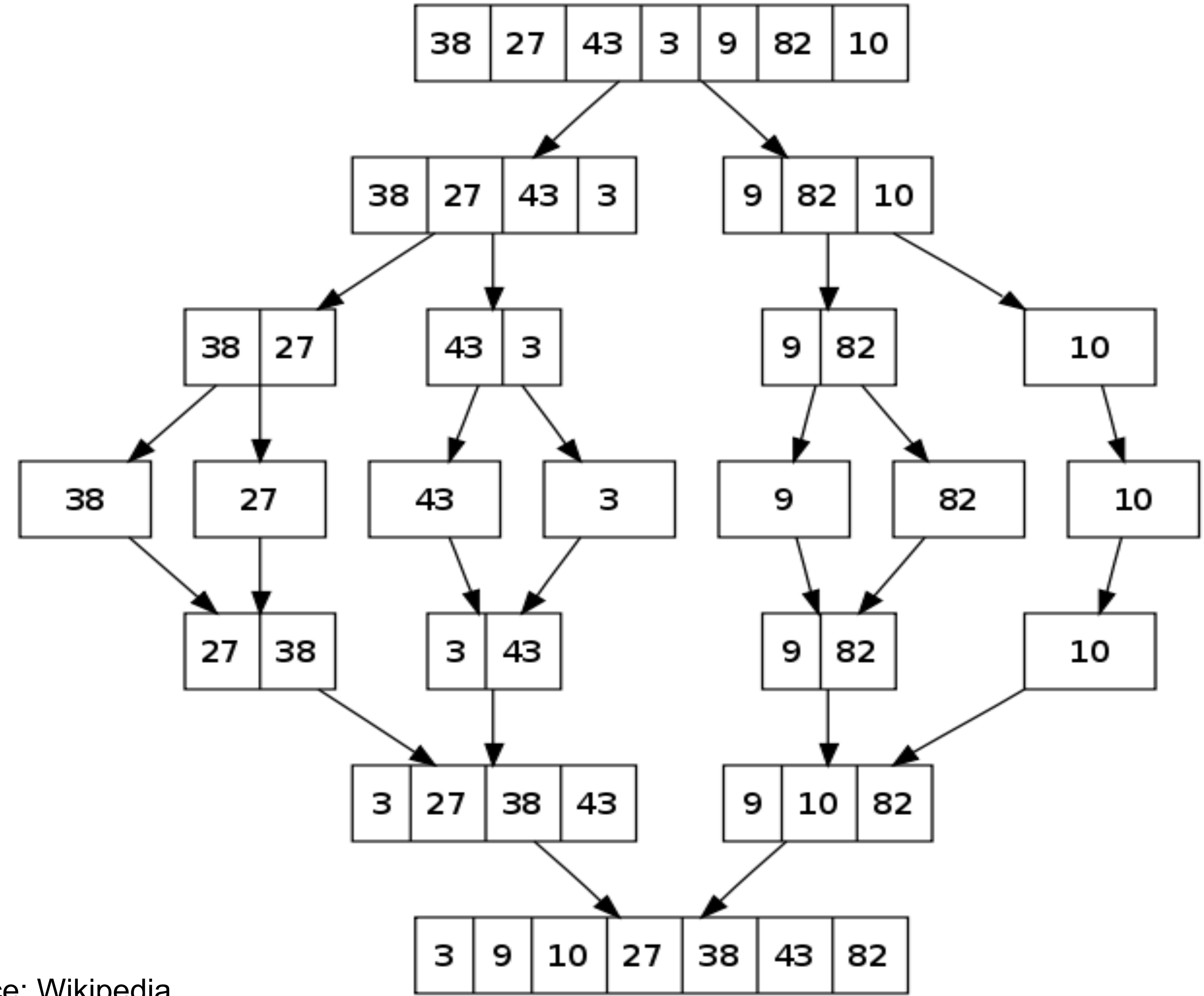
- Greedy algorithms
  - Dijkstra's Algorithm
  - Minimum Spanning Tree algorithms
- Divide-and-conquer algorithms
  - MergeSort
  - Fast Fourier Transform

Divide-and-Conquer



Merging





Source: Wikipedia

Optimal Sub-structure

Property of (nearly) every problem we study:

- Greedy algorithms
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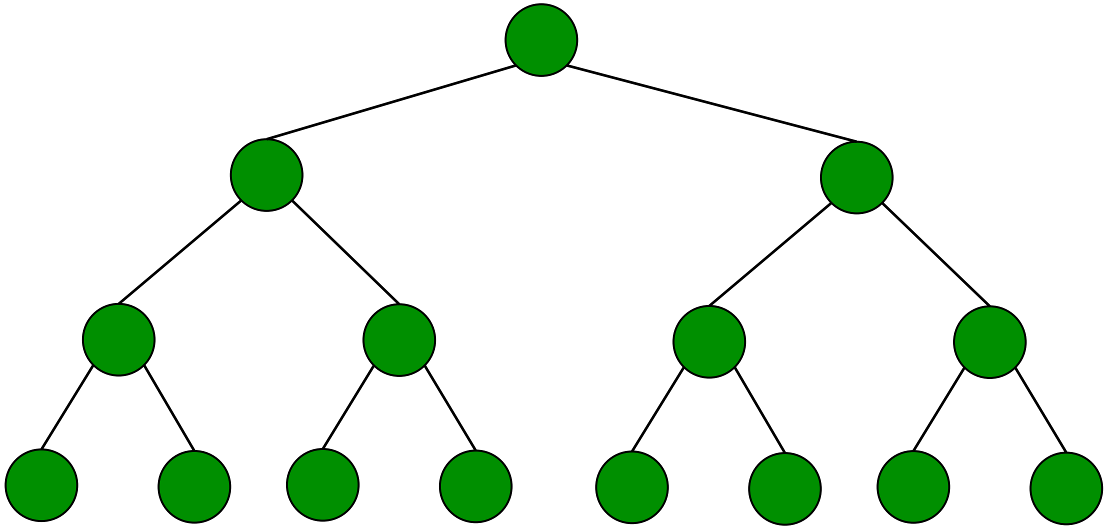
  - Minimum Spanning Tree algorithms
- Divide-and-conquer algorithms
  - MergeSort
  - Fast Fourier Transform

Dynamic Programming

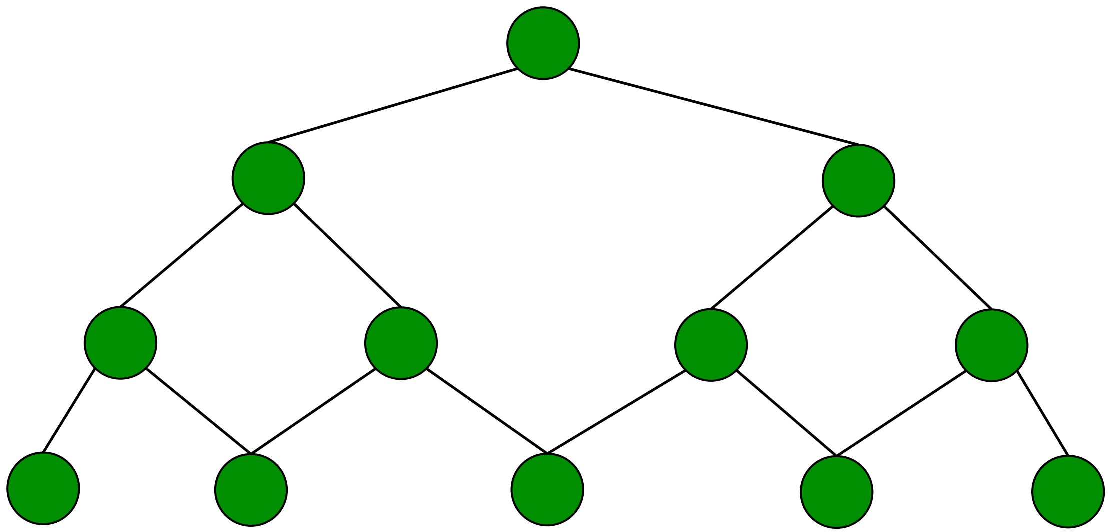
Optimal substructure:



Dynamic Programming

Overlapping sub-problems:

– The same smaller problem is used to solve multiple different bigger problems.



Dynamic Programming

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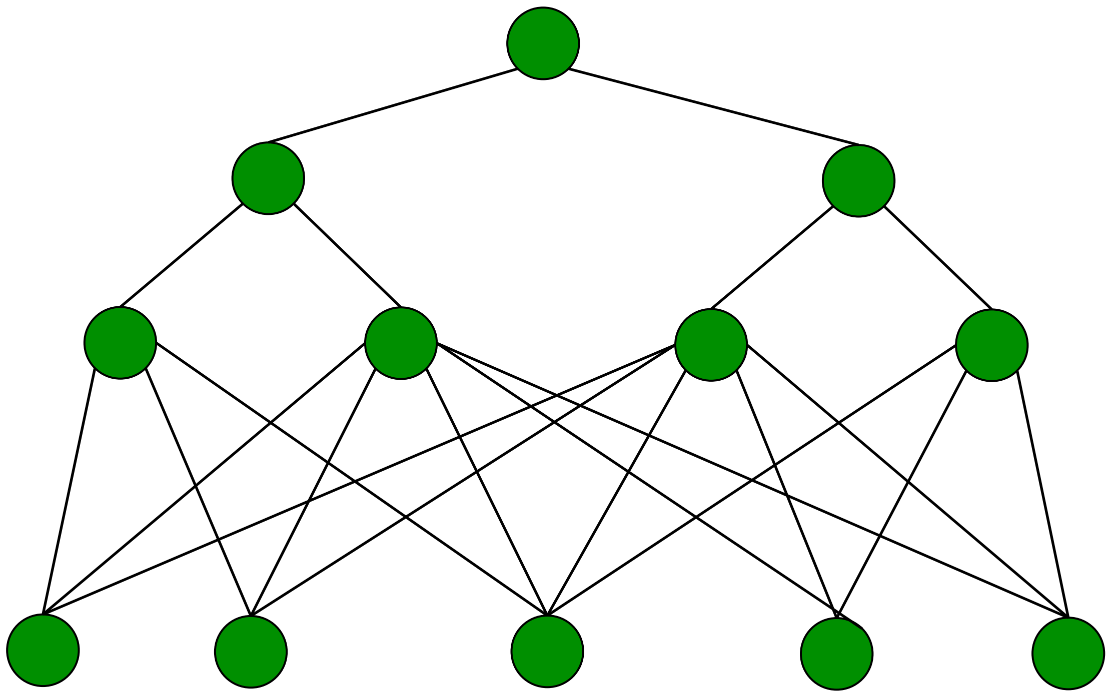
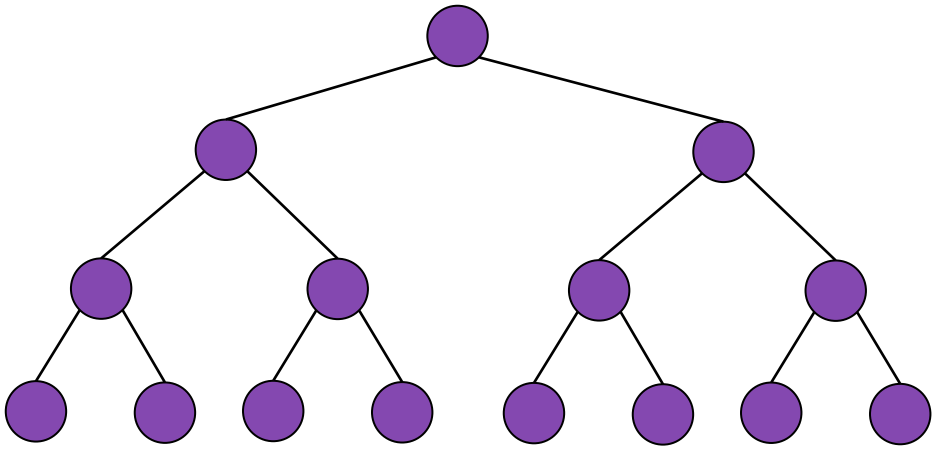
- The same smaller problem is used to solve multiple different bigger problems.



Dynamic Programming

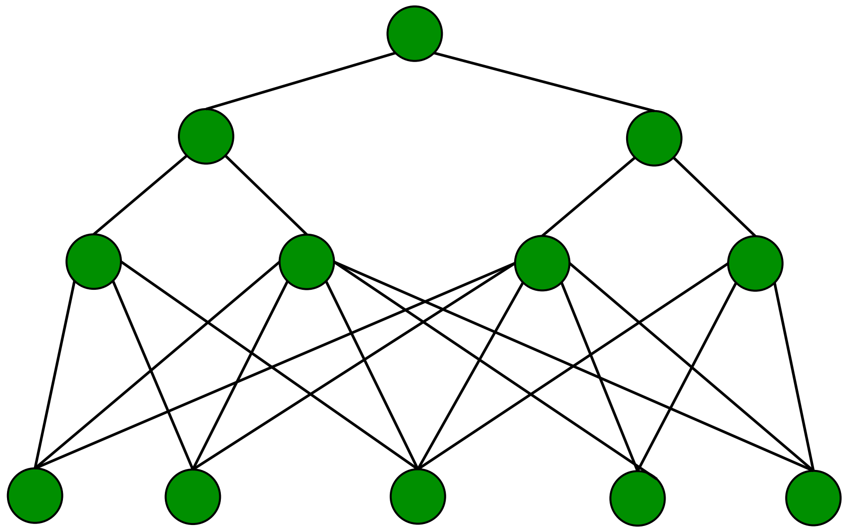
Contrast: Both have optimal substructure

No overlapping subproblems



Divide-and-Conquer

Overlapping subproblems



Dynamic Programming

Dynamic Programming

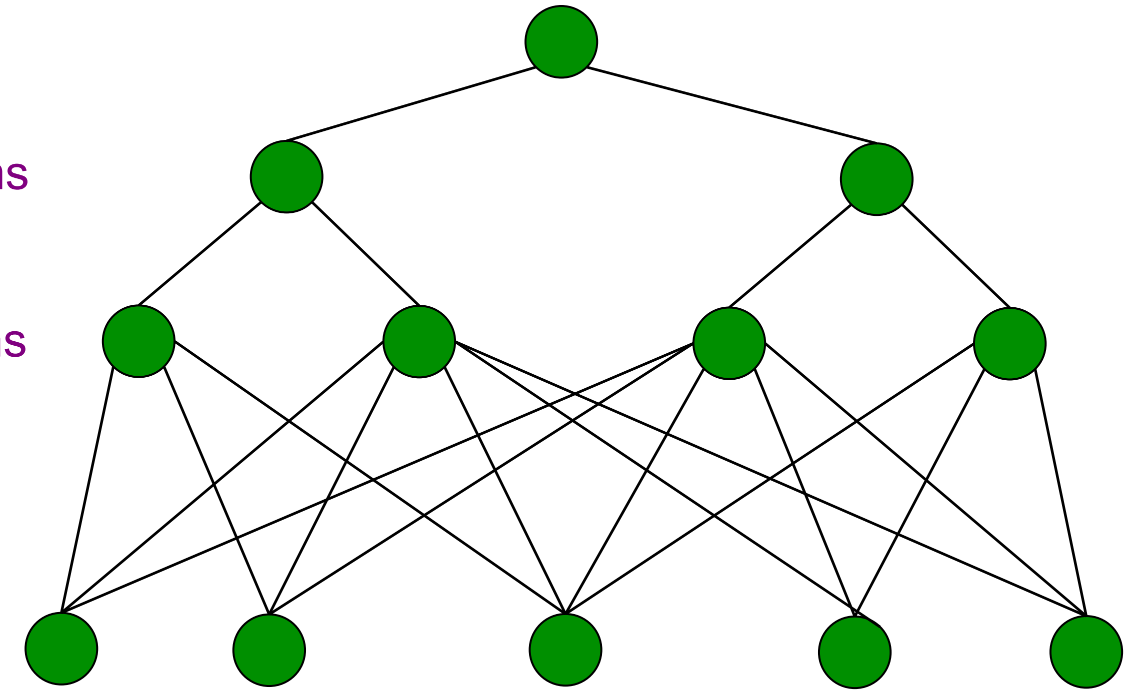
Basic strategy: *(bottom up dynamic programming)*

Step 4: solve root problem

Step 3: combine smaller problems

Step 2: combine smaller problems

Step 1: solve smallest problems

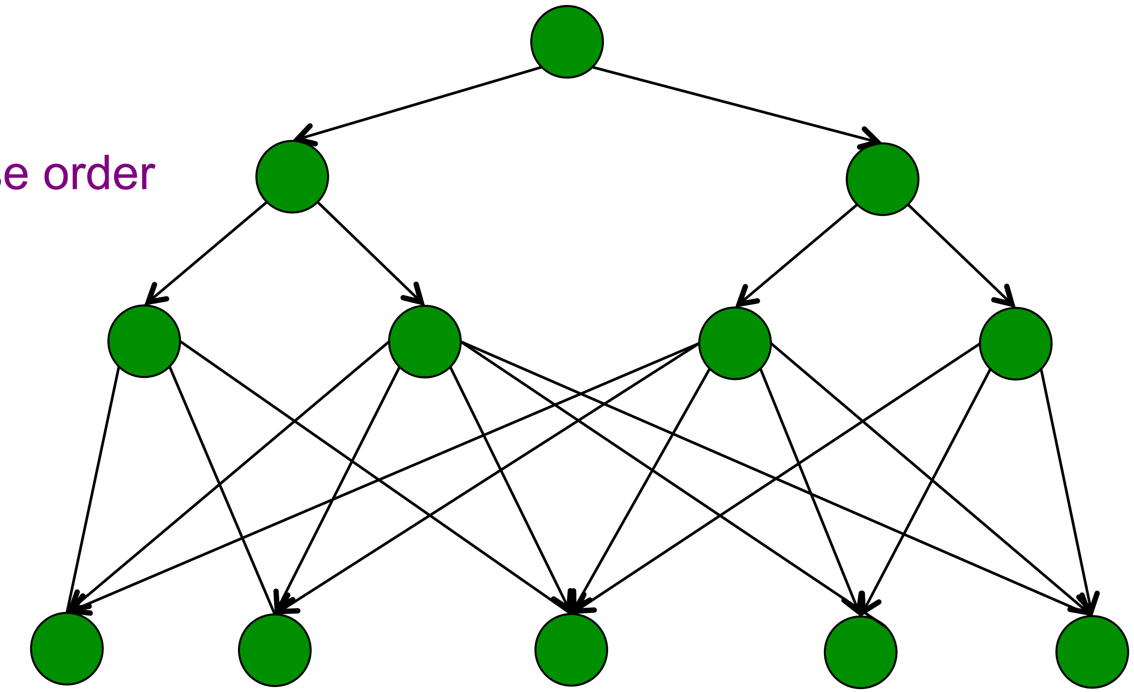


Dynamic Programming

Basic strategy: *(DAG + topological sort)*

Step 1: Topologically sort DAG

Step 2: Solve problems in reverse order



Dynamic Programming

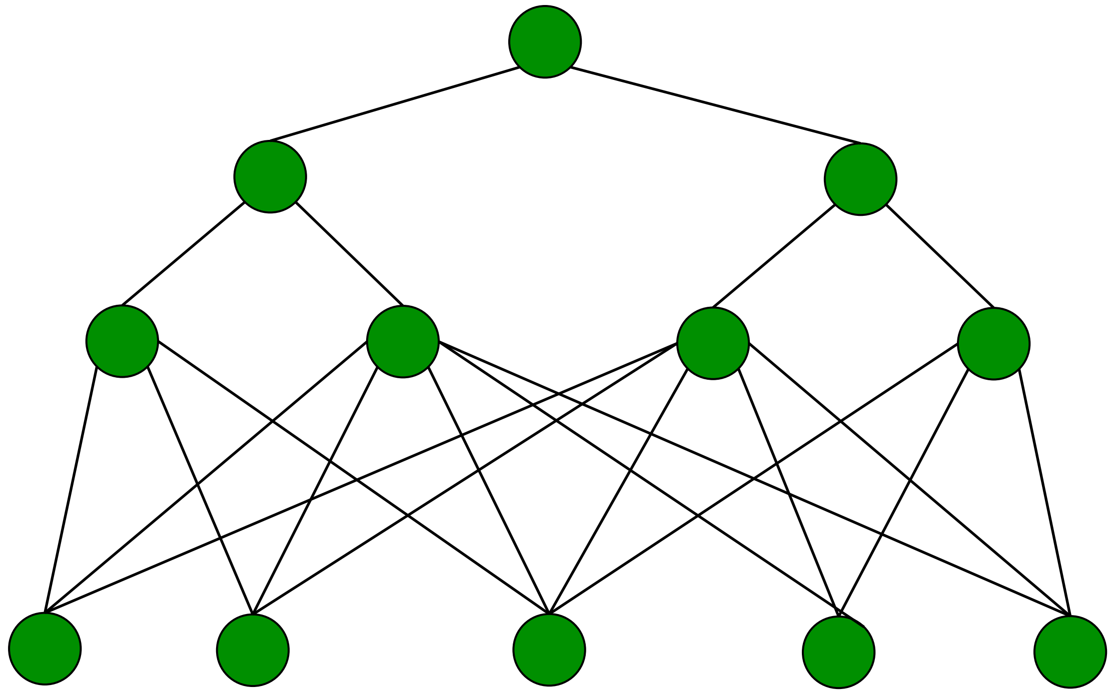
Basic strategy: *(top down dynamic programming)*

Step 1: Start at root and recurse.

Step 2: Recurse.

Step 3: Recurse.

Step 4: Solve and memoize.
Only compute each
solution once.



Dynamic Programming

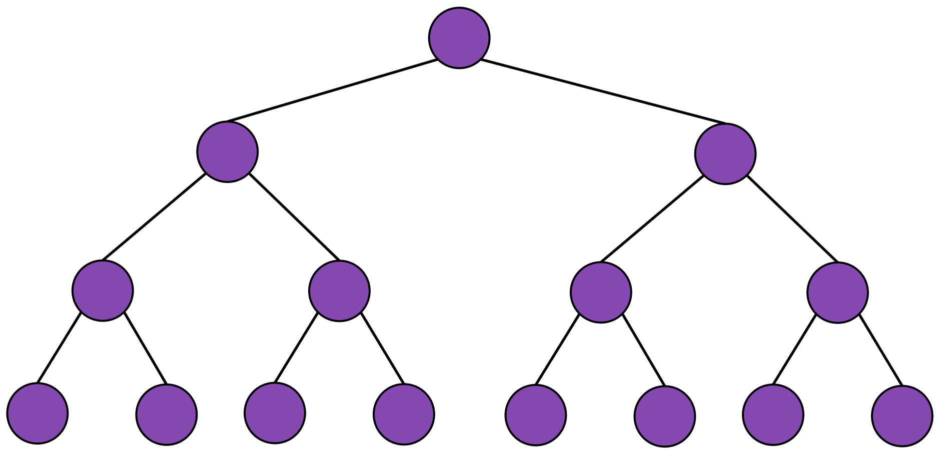
Table view:

	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p
1	17	22	14	19	8	4	9	12	15	7	5	9	13	14	18	4
2	15	12	13	13	7											
3																
4																
5																
6																
7																
8																
9																
10																
11																

Dynamic Programming

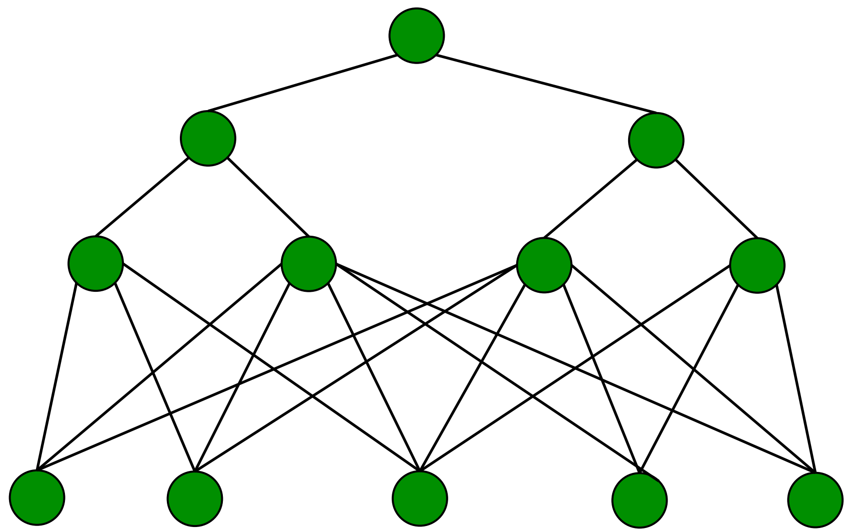
Contrast: Both have optimal substructure

No overlapping subprojects



Divide-and-Conquer

Overlapping subproblems



Dynamic Programming

Roadmap

Today: Dynamic Programming

- Basics of DP
- Example: Longest Increasing Subsequence
- Example: Bounded Prize Collecting
- Example: Vertex Cover on a Tree
- Example: All-Pairs Shortest Paths

Longest Increasing Subsequence

Input: Sequence of integers (or Comparable)

- Example: {8, 3, 6, 4, 5, 7, 7}

Output: Increasing subsequence

- Example: {8, 3, 6, 4, 5, 7, 7}

Goal: Output sequence of maximum length

- Example: {8, 3, 6, 4, 5, 7, 7}

Longest Increasing Subsequence

Input: Sequence of integers (or Comparable)

- Example: {8, 3, 6, 4, 5, 7, 7}

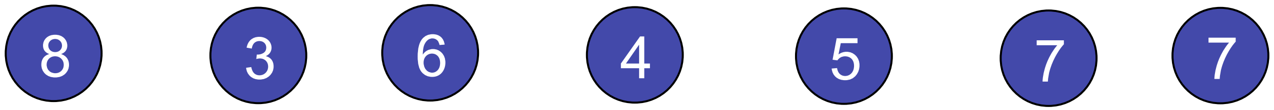
Output: <u>Length</u> of increasing subsequence

- Example: 3 → {8, 3, 6, 4, 5, 7, 7}

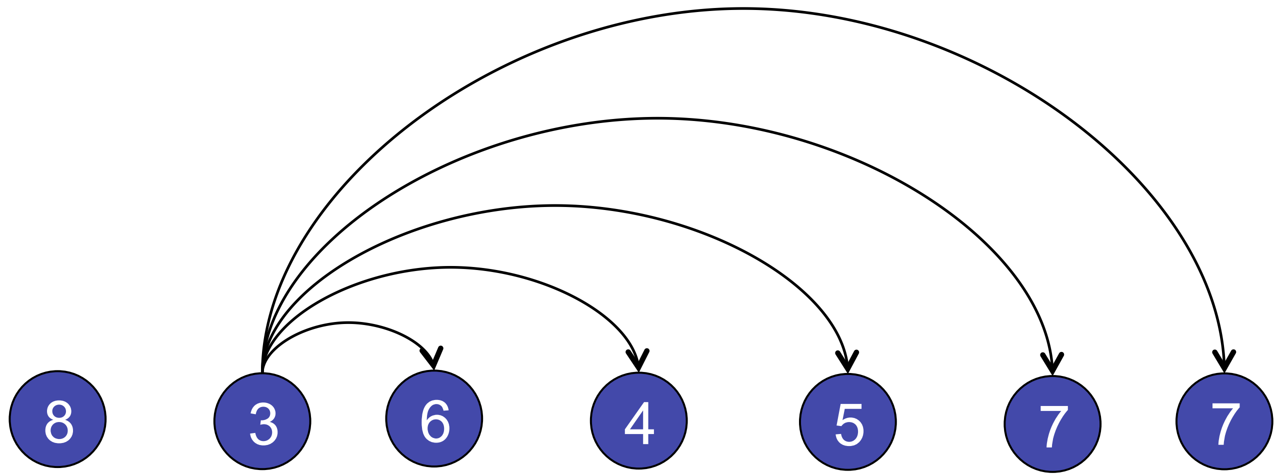
Goal: Output maximum length

- Example: 4 →{8, 3, 6, 4, 5, 7, 7}

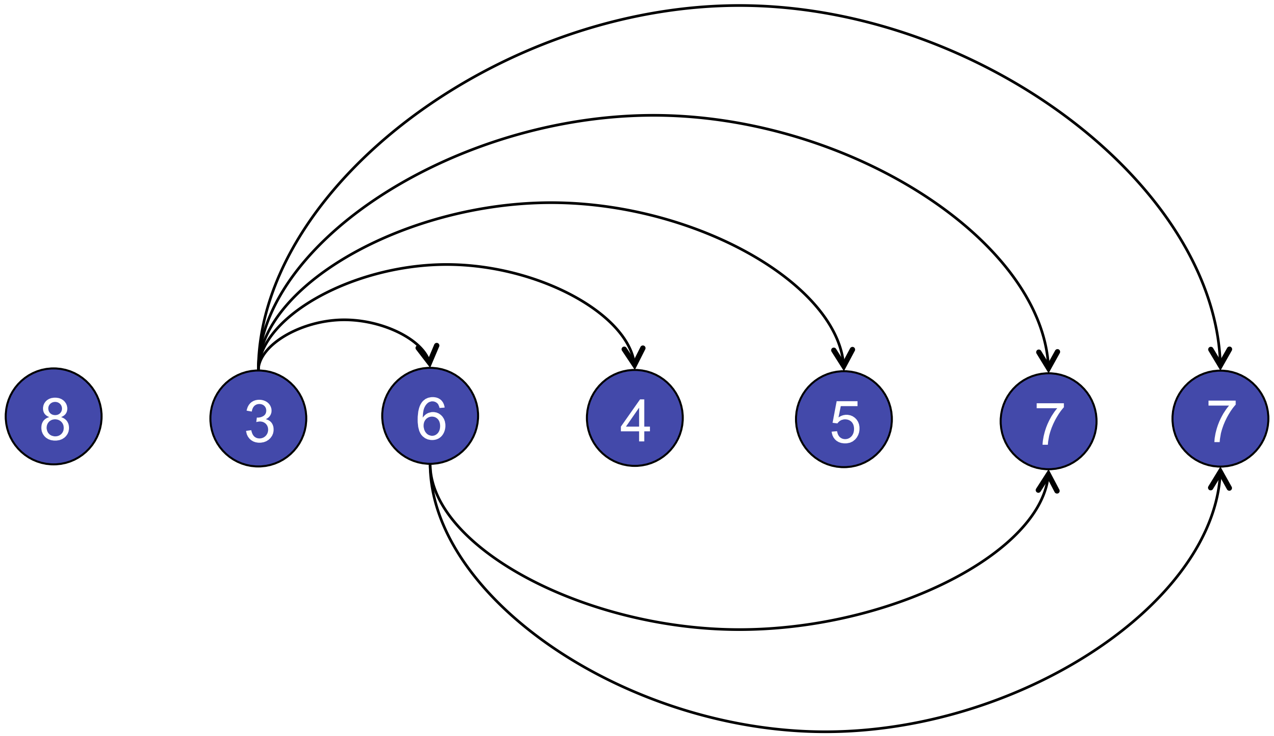
DAG Solution



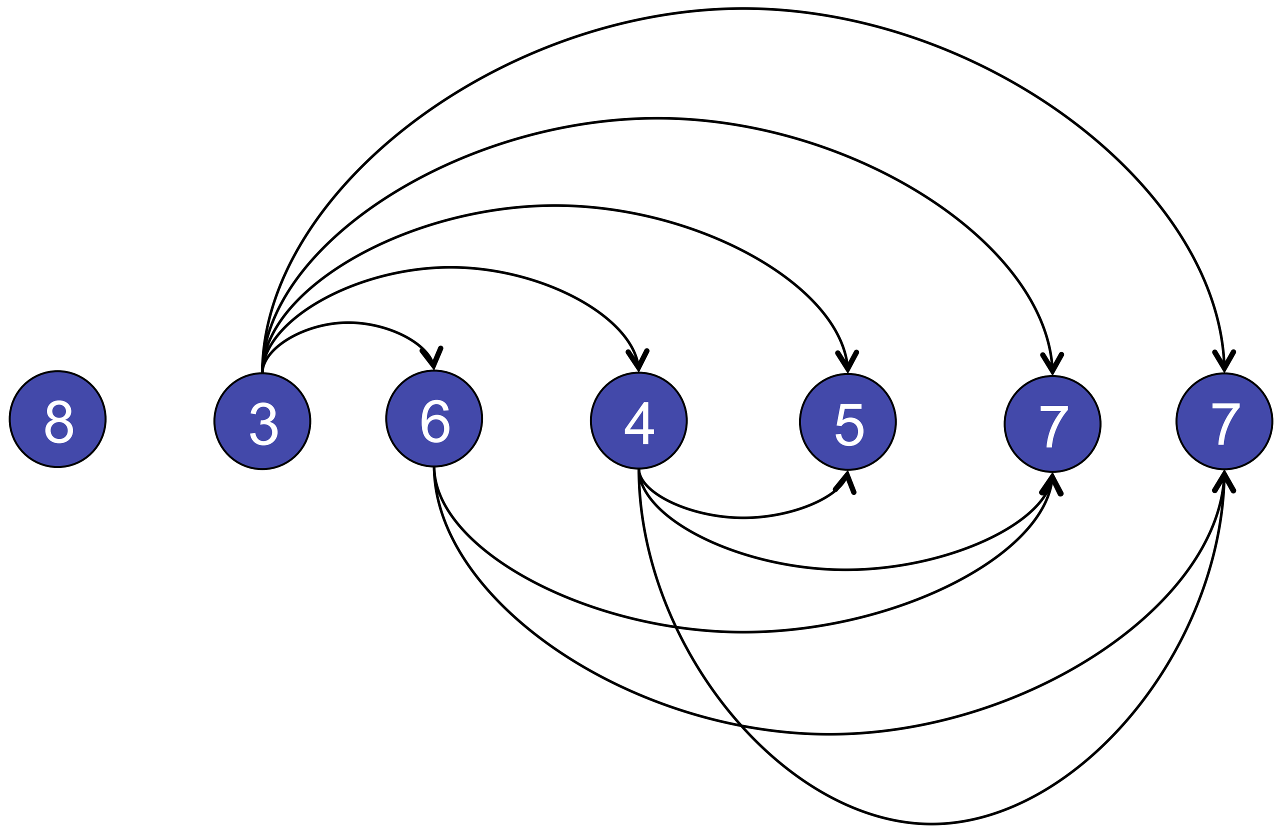
DAG Solution



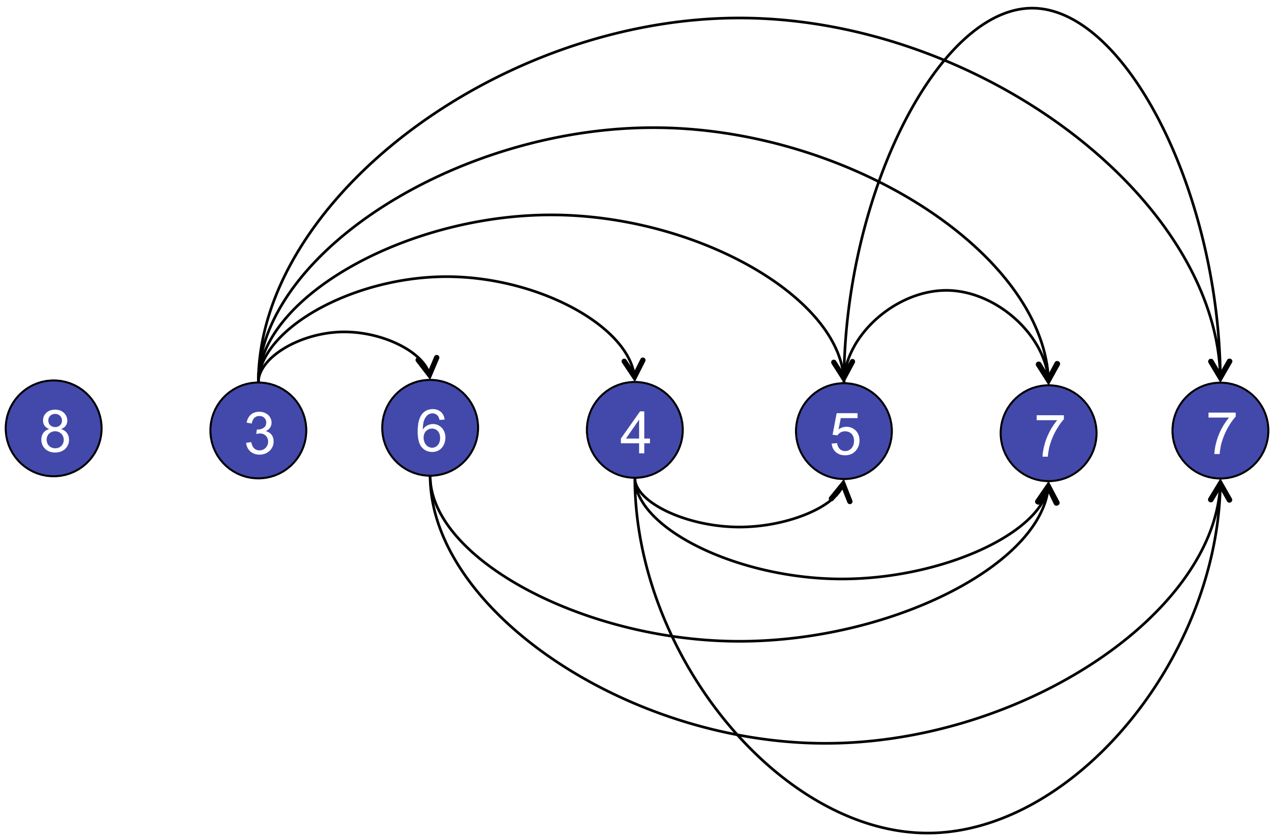
DAG Solution



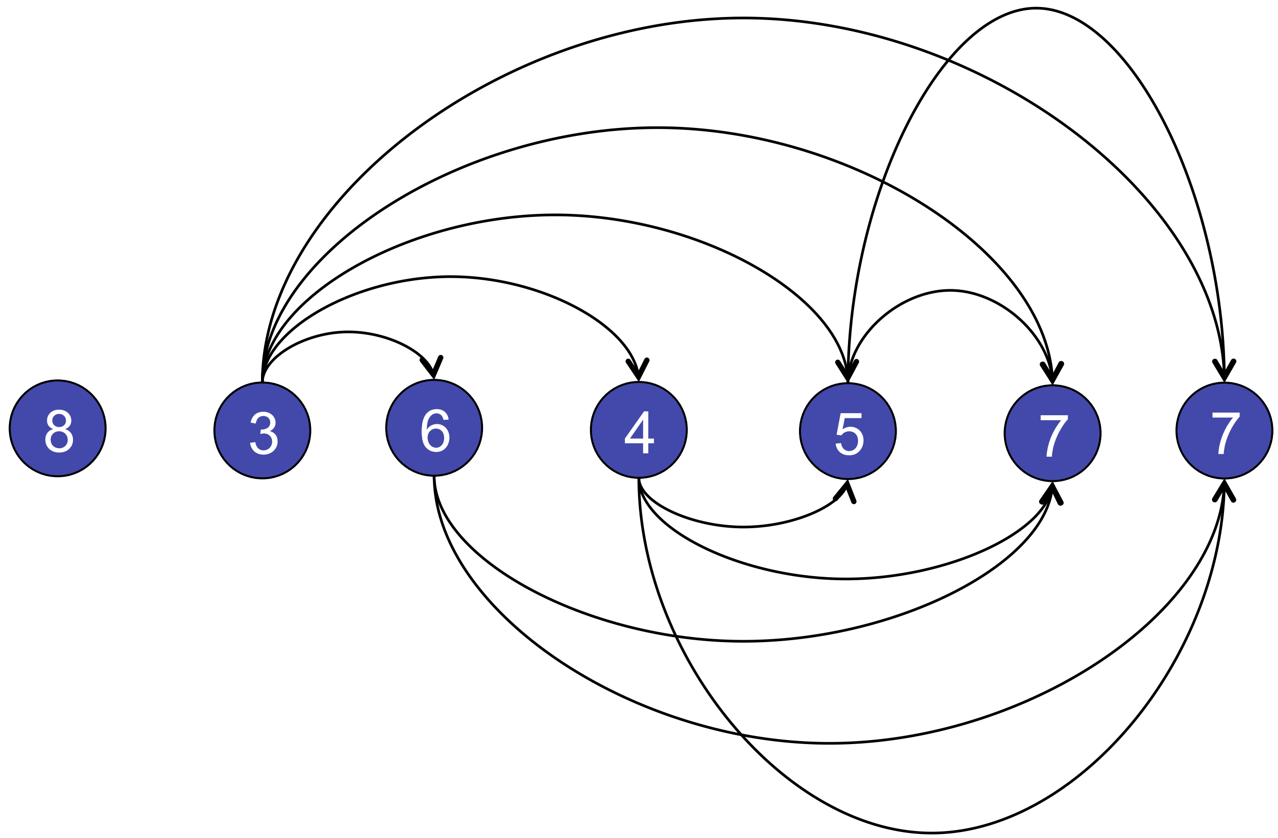
DAG Solution



DAG Solution

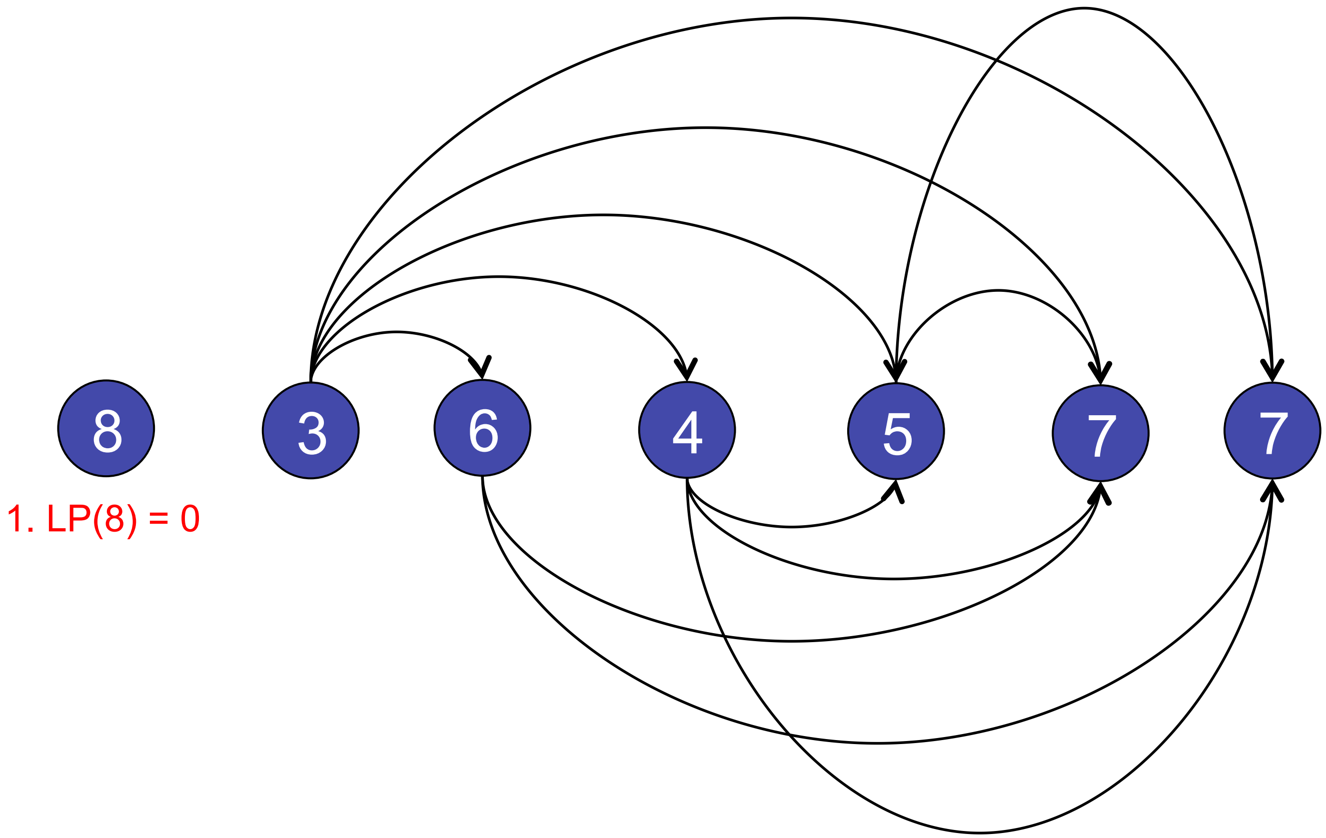


DAG Solution



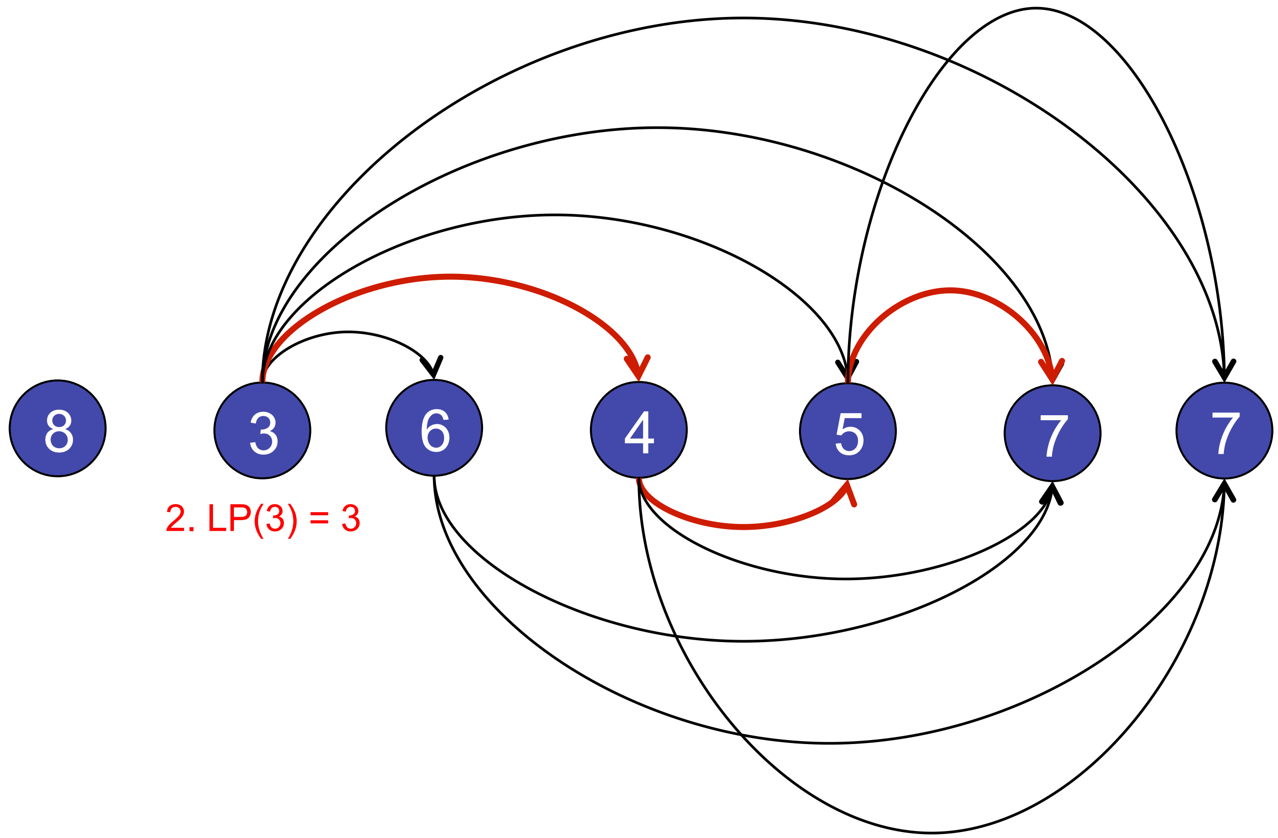
Step 1: Topological sort. (Oops, nothing to do.)

DAG Solution



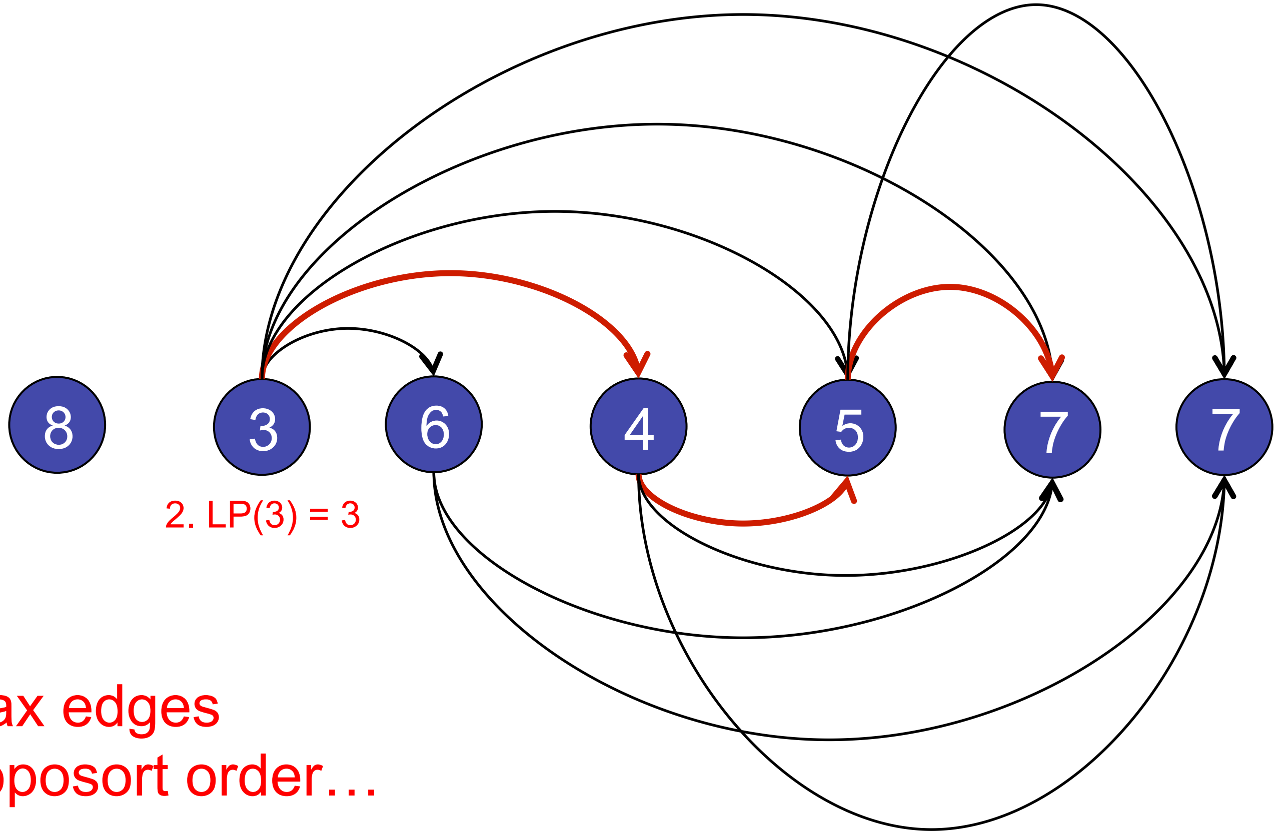
Step 2: Calculate longest paths.

DAG Solution



Step 2: Calculate longest paths.

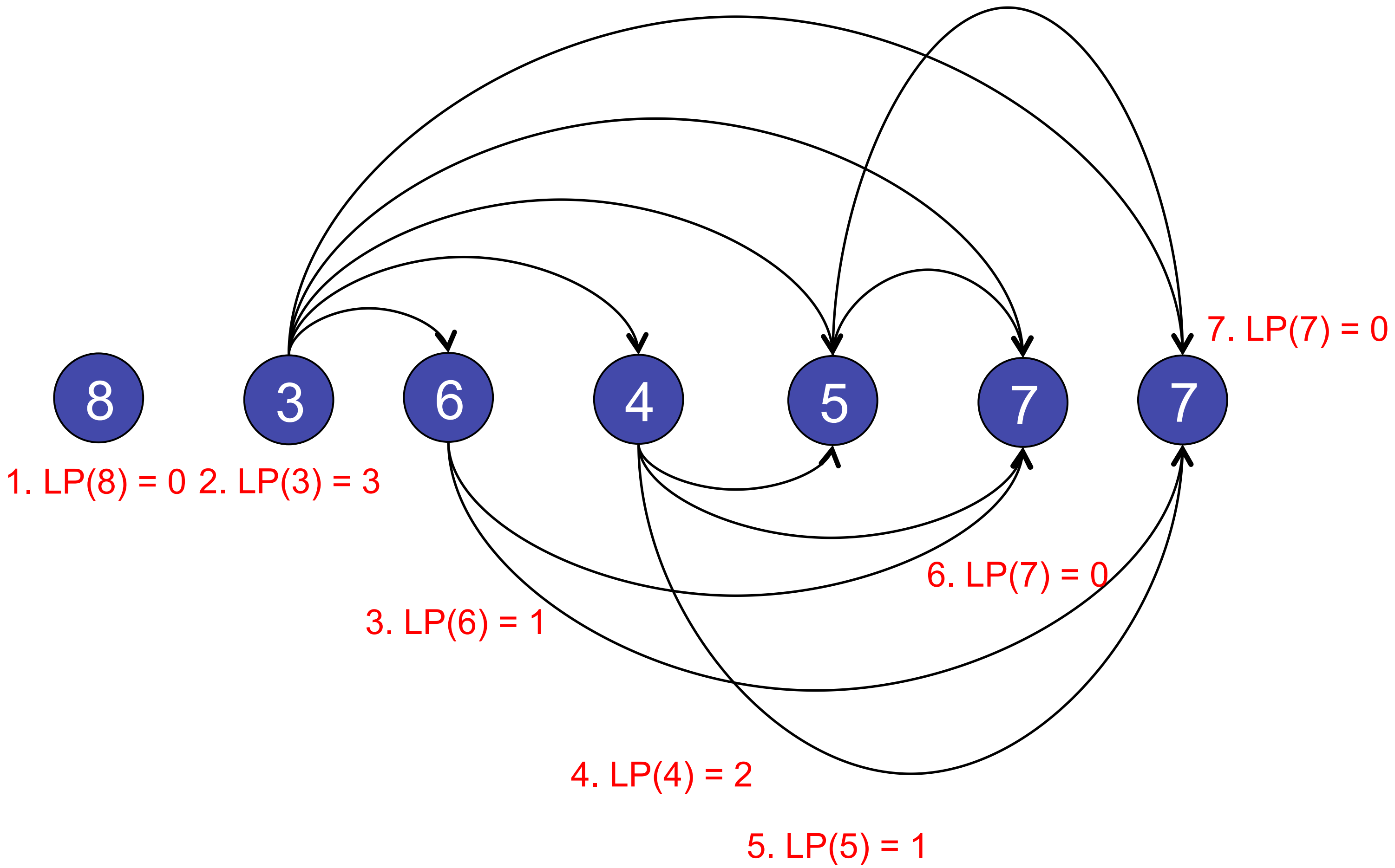
DAG Solution



Relax edges
in toposort order…

Step 2: Calculate longest paths.

DAG Solution

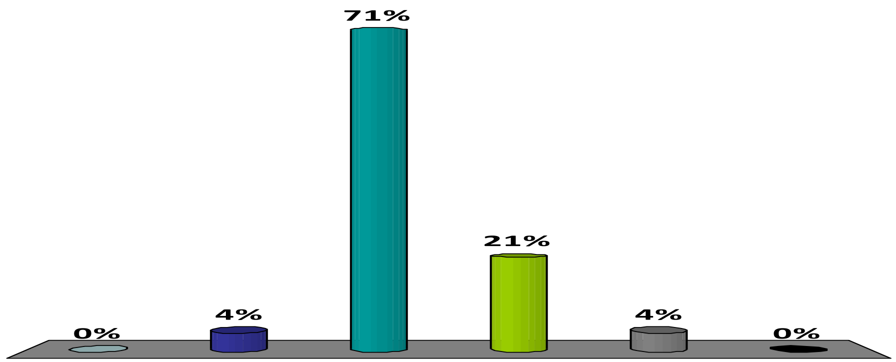


Step 2: Calculate longest paths. LIS = max(LP)+1

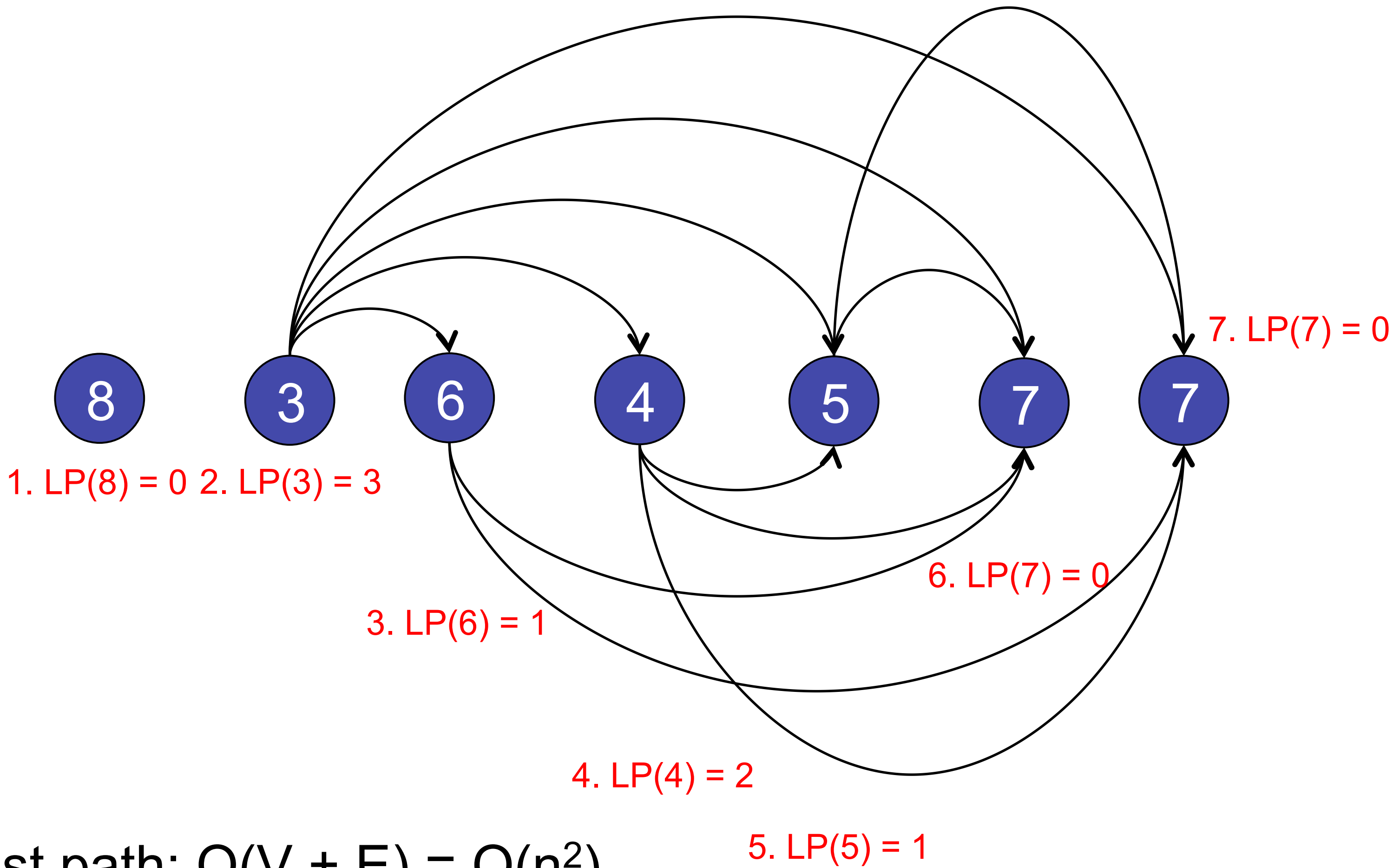
What is the running time of the LP-LIS alg for a sequence of n numbers?

1. $O(n)$
2. $O(n \log n)$
3. $O(n^2)$
4. $O(n^2 \log n)$
5. $O(n^3)$
6. None of the above.

Response Counter



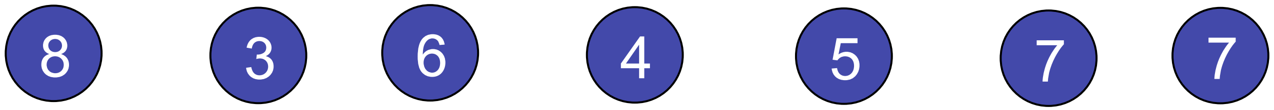
DAG Solution



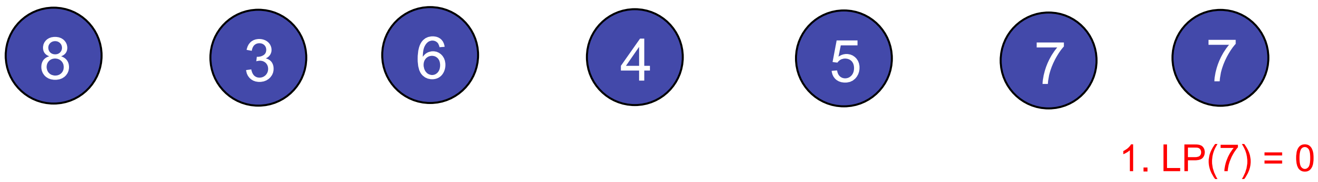
Longest path: $O(V + E) = O(n^2)$

Run longest path n times = $O(n^3)$

Overlapping Subproblems

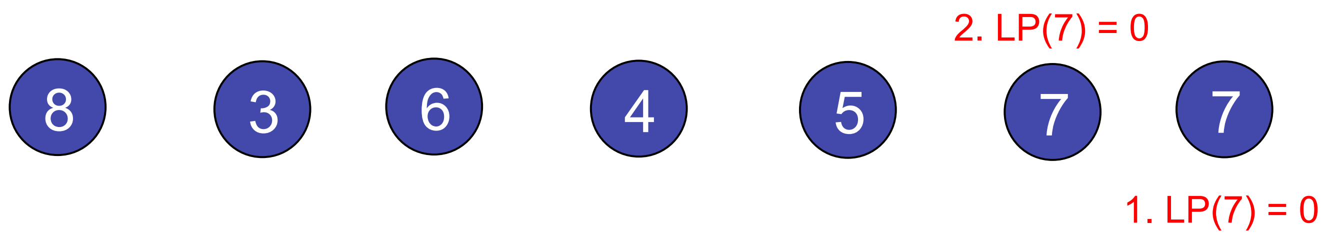


Overlapping Subproblems



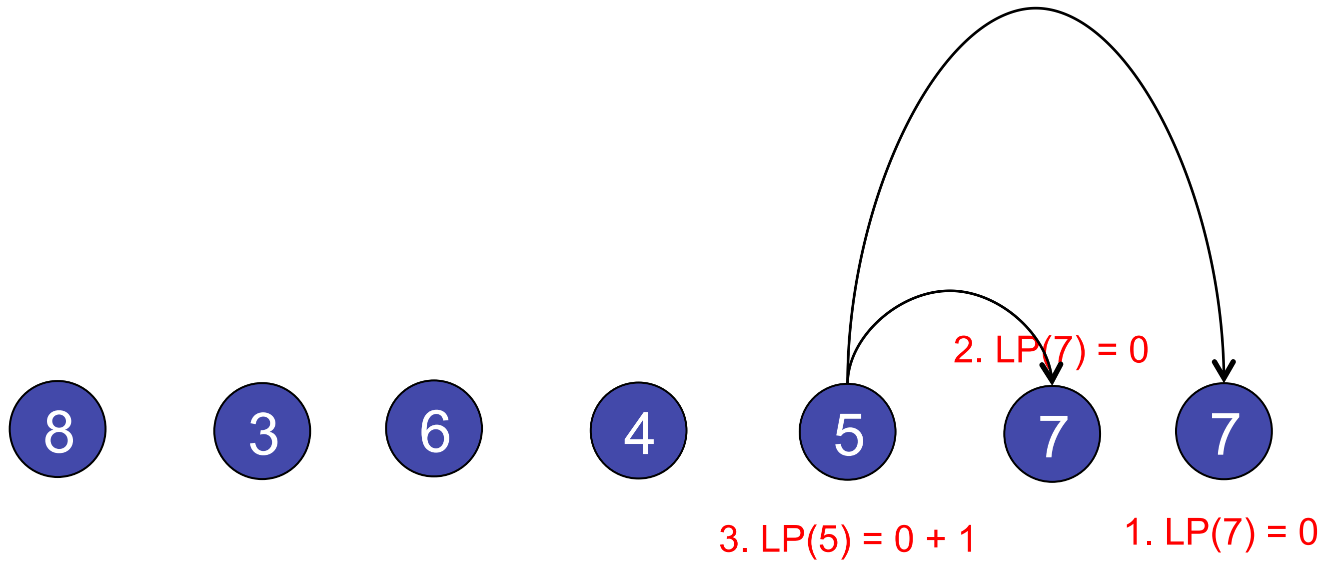
Start with the smallest sub-problem: LP(7)

Overlapping Subproblems



Start with the smallest sub-problem: LP(7)

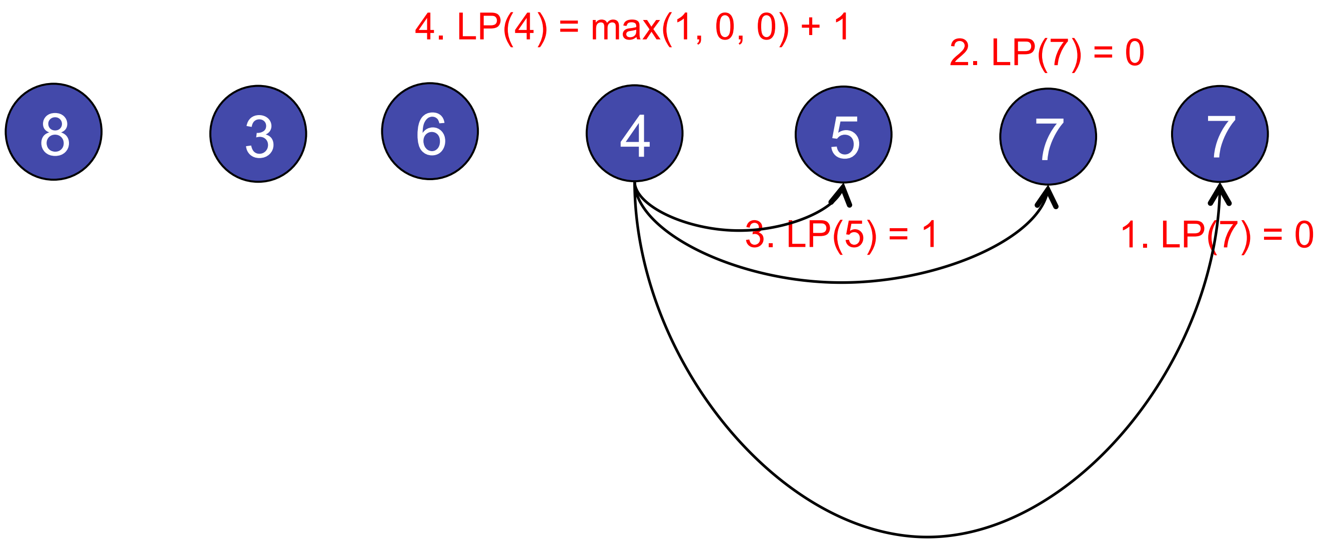
Overlapping Subproblems



Calculate LP(5):

- Examine each outgoing edge.
- Find the maximum.
- Add 1.

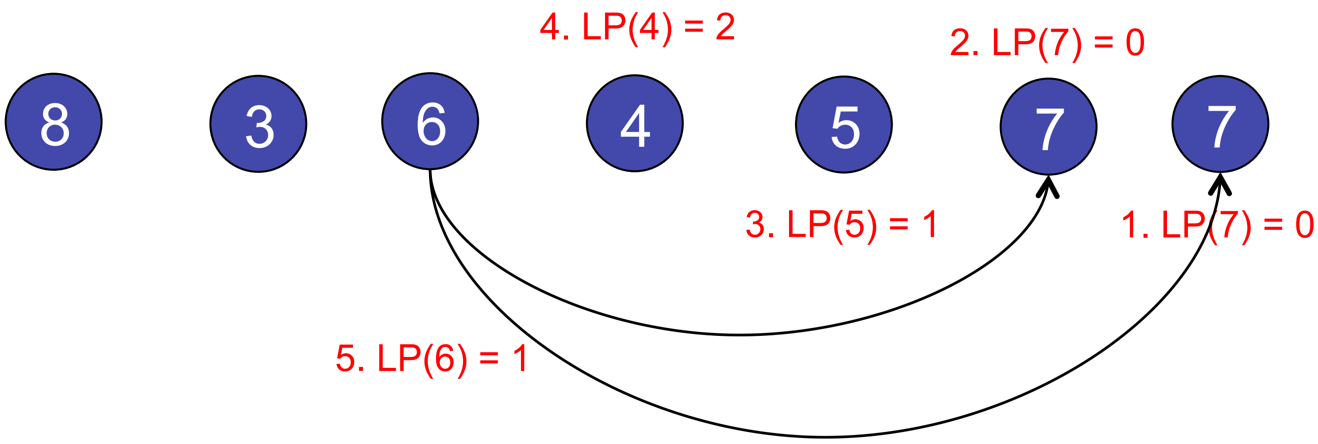
Overlapping Subproblems



Calculate LP(4):

- Examine each outgoing edge.
- Find the maximum.
- Add 1.

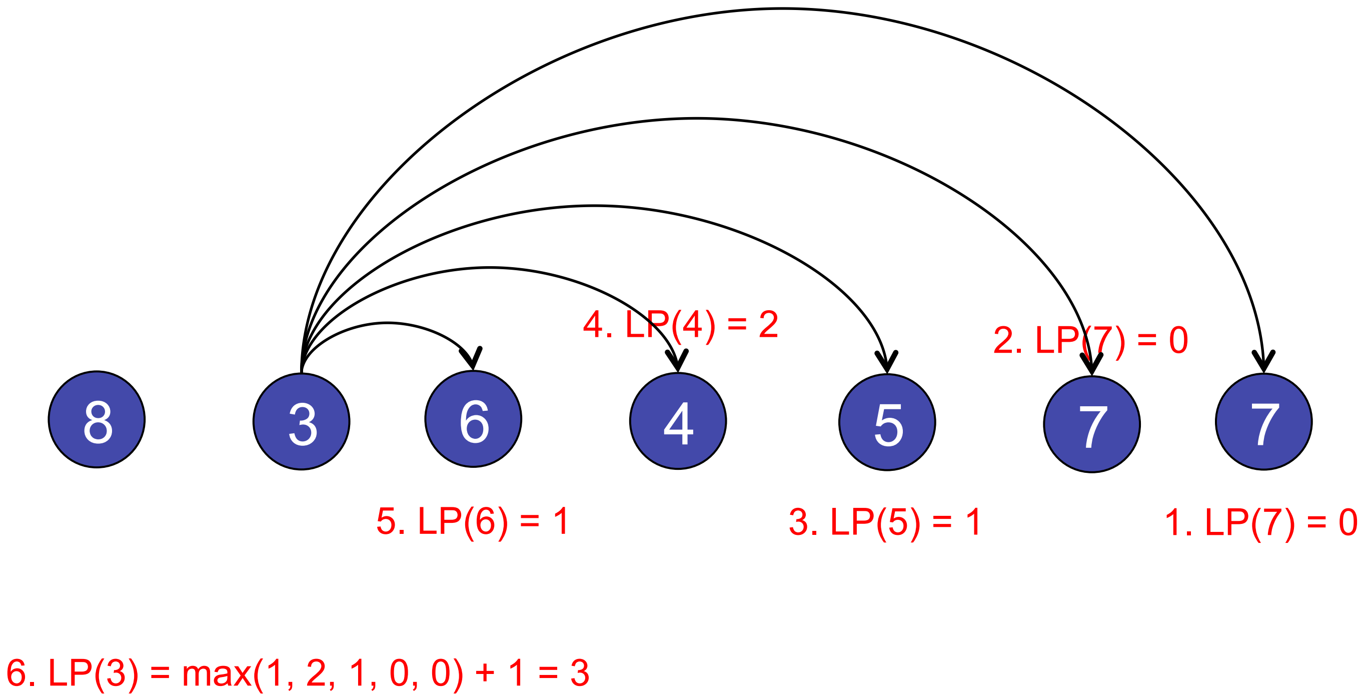
Overlapping Subproblems



Calculate LP(6):

- Examine each outgoing edge.
- Find the maximum.
- Add 1.

Overlapping Subproblems



Calculate LP(3):

- Examine each outgoing edge.
- Find the maximum.
- Add 1.

Longest Increasing Subsequence

Input:

– Array A[1..n]

Define sub-problems:

– S[i] = LIS(A[i..n]) starting at A[i]

Example: {8, 3, 6, 4, 5, 7, 7}

– S[5] = 2 → {8, 3, 6, 4, 5, 7, 7}

– S[2] = 4 → {8, 3, 6, 4, 5, 7, 7}

Dynamic Programming

Table view:

Node	Longest path that starts at node X
7	0
7	0
5	…
4	
6	
3	
8	

Longest Increasing Subsequence

Input:

- Array A[1..n]

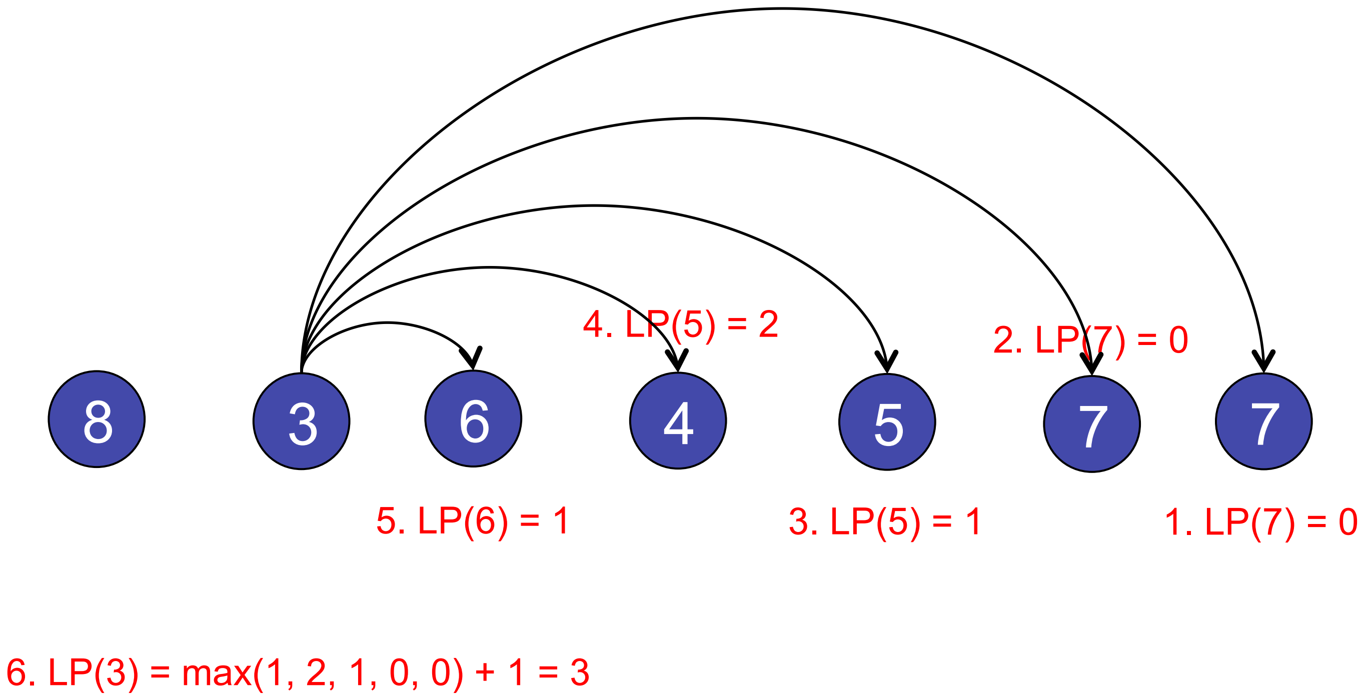
Define sub-problems:

- S[i] = LIS(A[i..n]) starting at A[i]

Solve using sub-problems:

- S[n] = 0
- $S[i] = (\max_{(i,j) \in E} S[j]) + 1$

Overlapping Subproblems



Calculate LP(3):

- Examine each outgoing edge.
- Find the maximum.
- Add 1.

Longest Increasing Subsequence

Input:

- Array A[1..n]

Define sub-problems:

- S[i] = LIS(A[1..i]) ending at A[i]

Example: {8, 3, 6, 4, 5, 7, 7}

- S[4] = 2 → {8, 3, 6, 4, 5, 7, 7}
- S[5] = 3 → {8, 3, 6, 4, 5, 7, 7}

Longest Increasing Subsequence

Input:

- Array A[1..n]

Define sub-problems:

- S[i] = LIS(A[1..i]) ending at A[i]

Solve using sub-problems:

- S[1] = 0
- $S[i] = (\max_{(j < i,\ A[j] < A[i])} S[j]) + 1$

Longest Increasing Subsequence

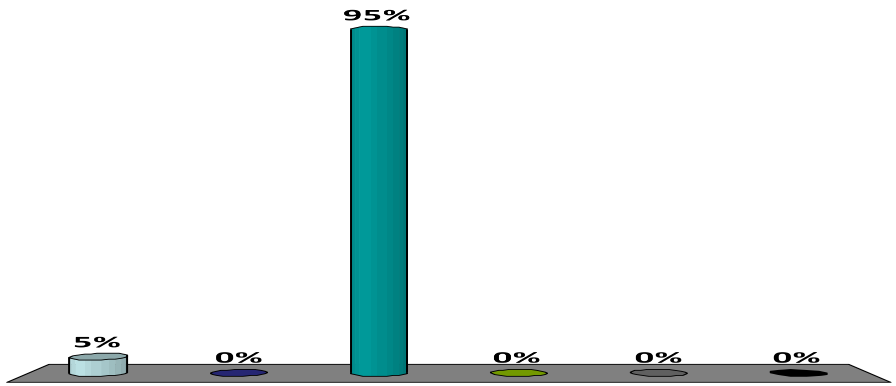
LIS(A):

```
int[] S = new int[A.length];  // Create memo array
for (i=0; i<A.length; i++) S[i] = 0; // Initialize array to zero
S[0] = 1; // Base case: length 1
for (int i = 0; i<A.length; i++) {
    int max = 0; // Find maximum S for any preceding node
    for (int j=0; j<i; j++) { // Examine each preceding element in the sequence
            if (A[j] < A[i]) // If A[i] is bigger than A[j]
                    if (S[j] > max)
                            max = S[j]; // If S[j] is longer sequence
    }
    S[i] = max + 1; // Calculate S[i] based on max of preceding elements.
}
```

What is the running time of the LP-LIS alg for a sequence of n numbers?

1. O(n)
2. O(n log n)
3. $O(n^2)$
4. $O(n^2 \log n)$
5. $O(n^3)$
6. None of the above.

Response Counter



Longest Increasing Subsequence

Summary:

- Greedy subproblems: S[i] = LIS(A[1..i])
  - n subproblems
  - Subproblem i takes takes times O(i)
- Total time: $O(n^2)$

NB Challenge of the Day:

How do you solve LIS in time O(n log n)

Hint: use binary search to solve subproblem faster.

Roadmap

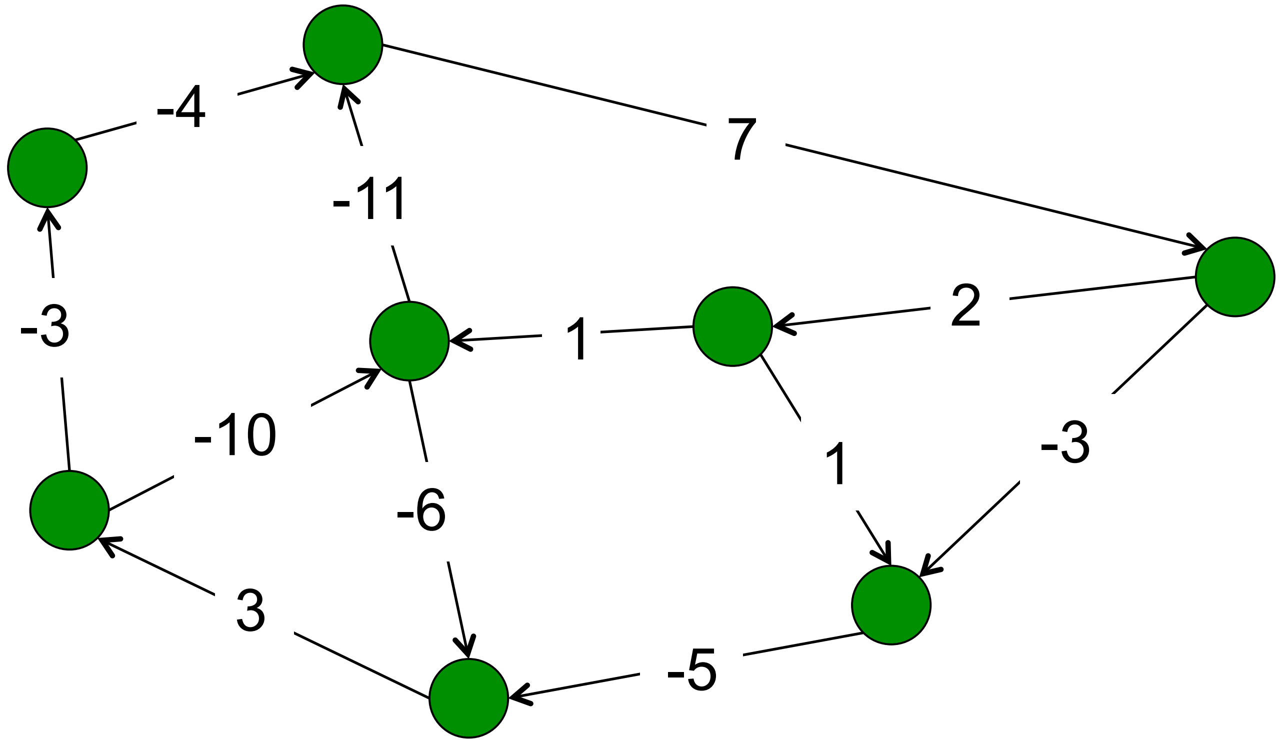
Today: Dynamic Programming

- DP Basics
- Longest Increasing Subsequence
- Prize Collecting
- Vertex Cover on a Tree
- All-Pairs-Shortest-Paths

Prize Collecting

Input:

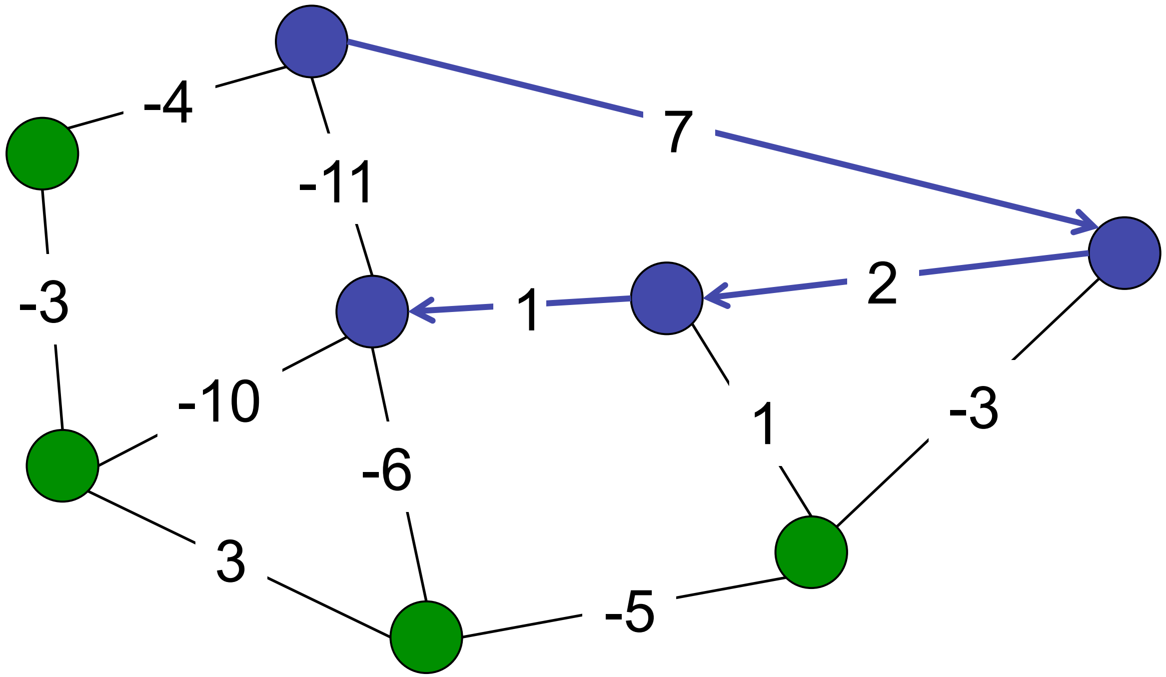
- Directed Graph G = (V,E)
- Edge weights **w** = prizes on each edge



Prize Collecting

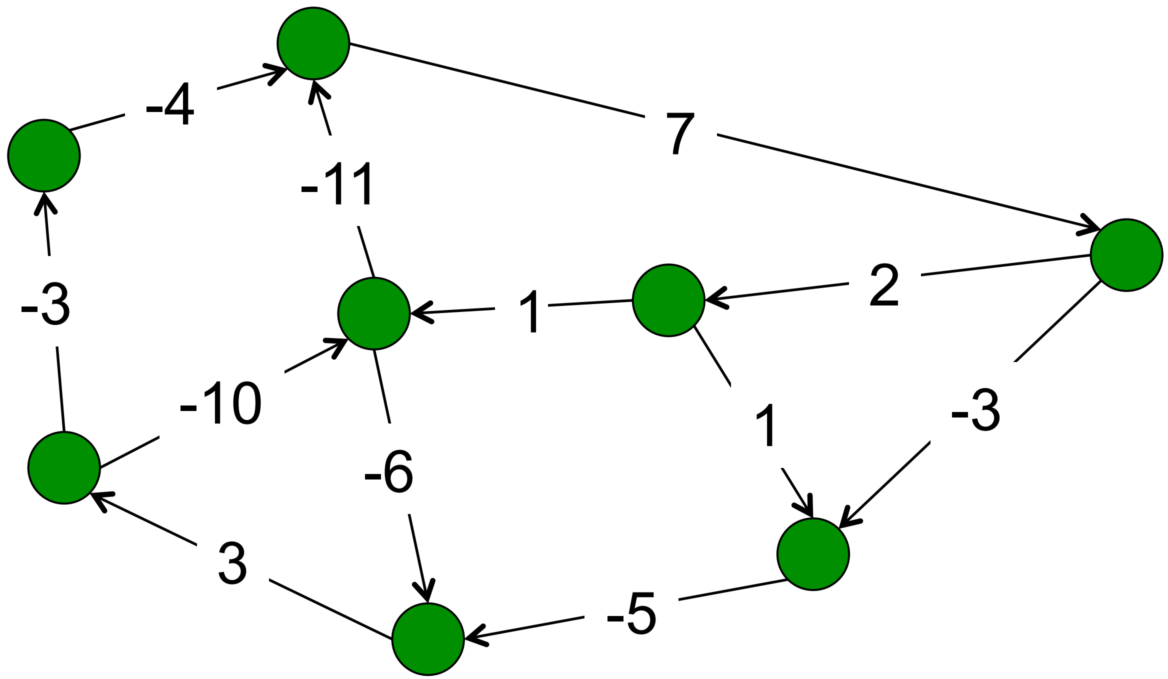
Output:

- Prize collecting path
- Example: 7 + 2 + 1 = 10

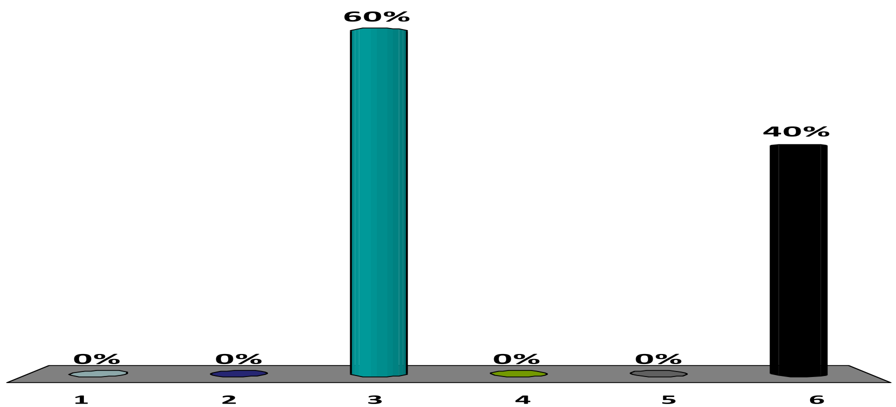


What is the maximum prize?

1. 1
2. 3
3. 10
4. 15
5. 17
6. Infinite



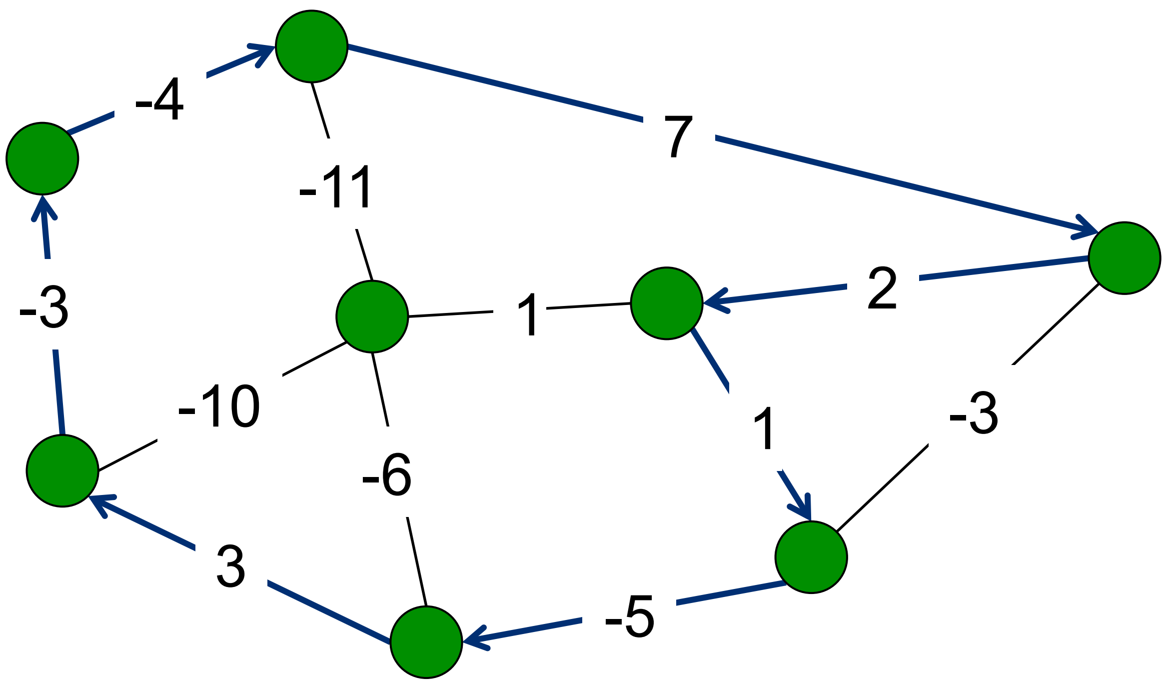
Response Counter



Prize Collecting

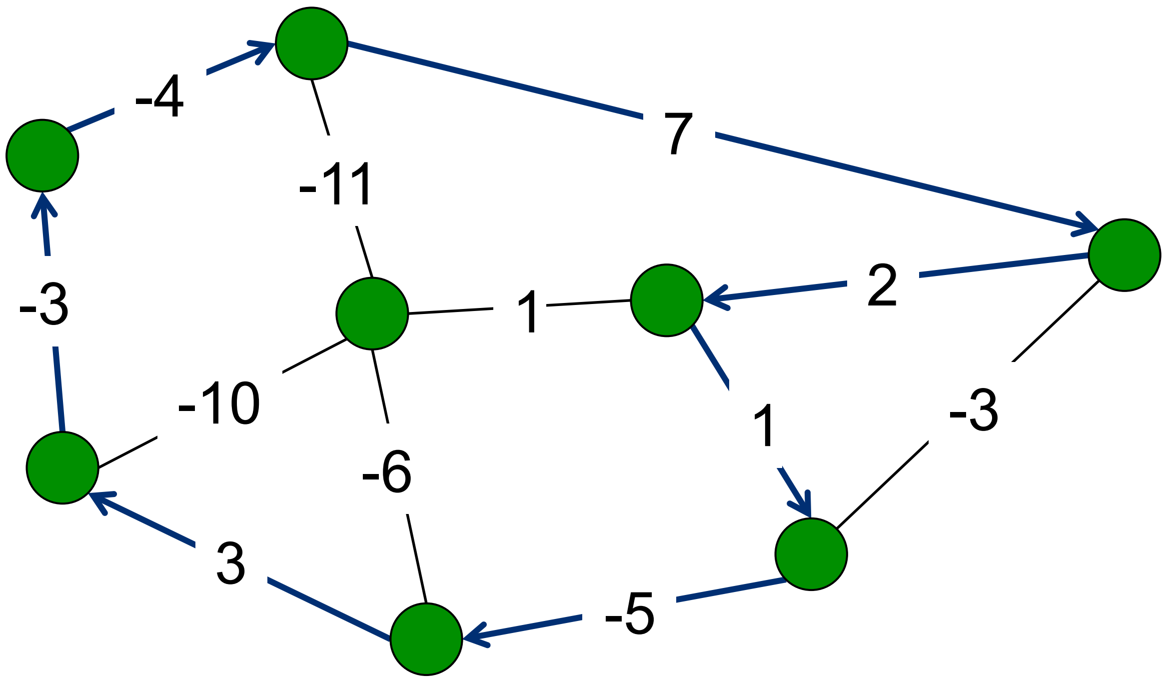
Output:

- Prize collecting path: 7 + 2 + 1 - 5 + 3 - 4 - 5 = 1
- Positive weight cycle → infinite prizes!



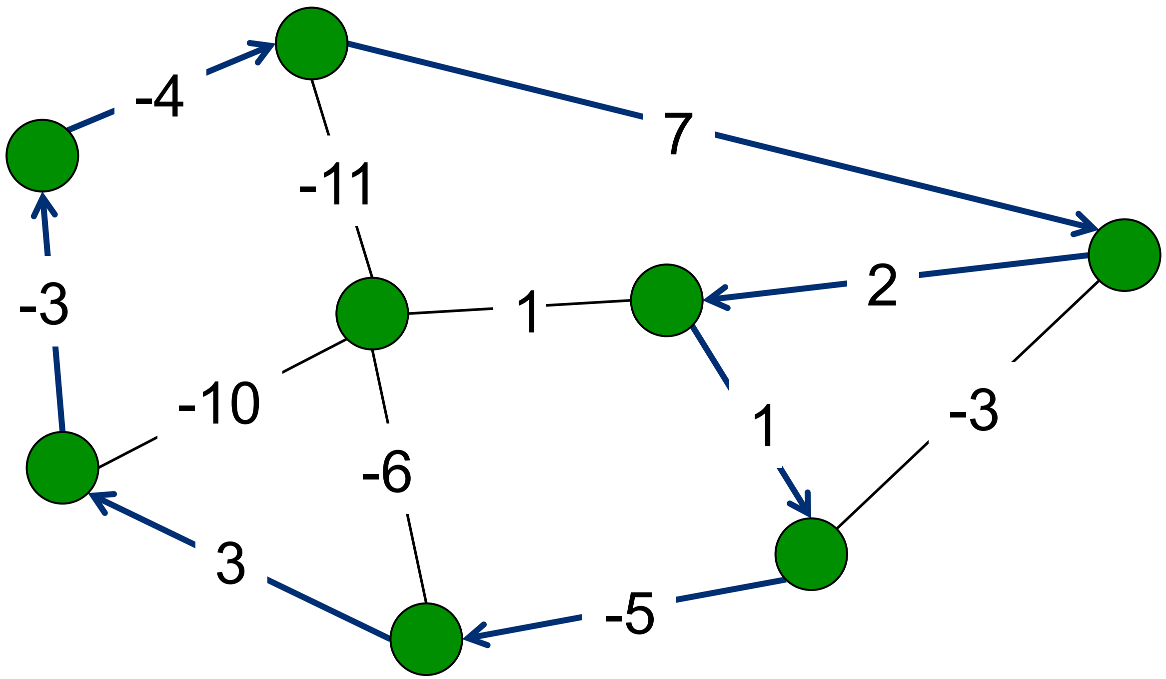
Prize Collecting

Aside: How could we determine if there is a positive weight cycle in a graph?



Prize Collecting

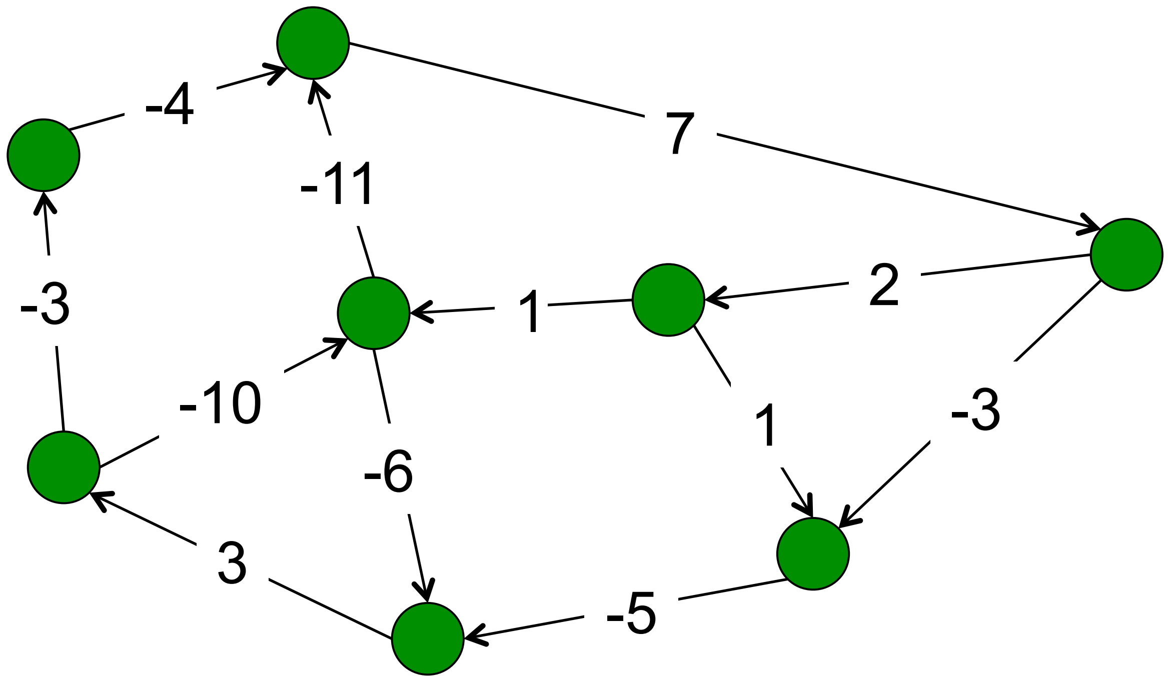
1. Check for positive weight cycles.
2. If not, negate the edges, run BF (or APSP).



Lazy Prize Collecting

Input:

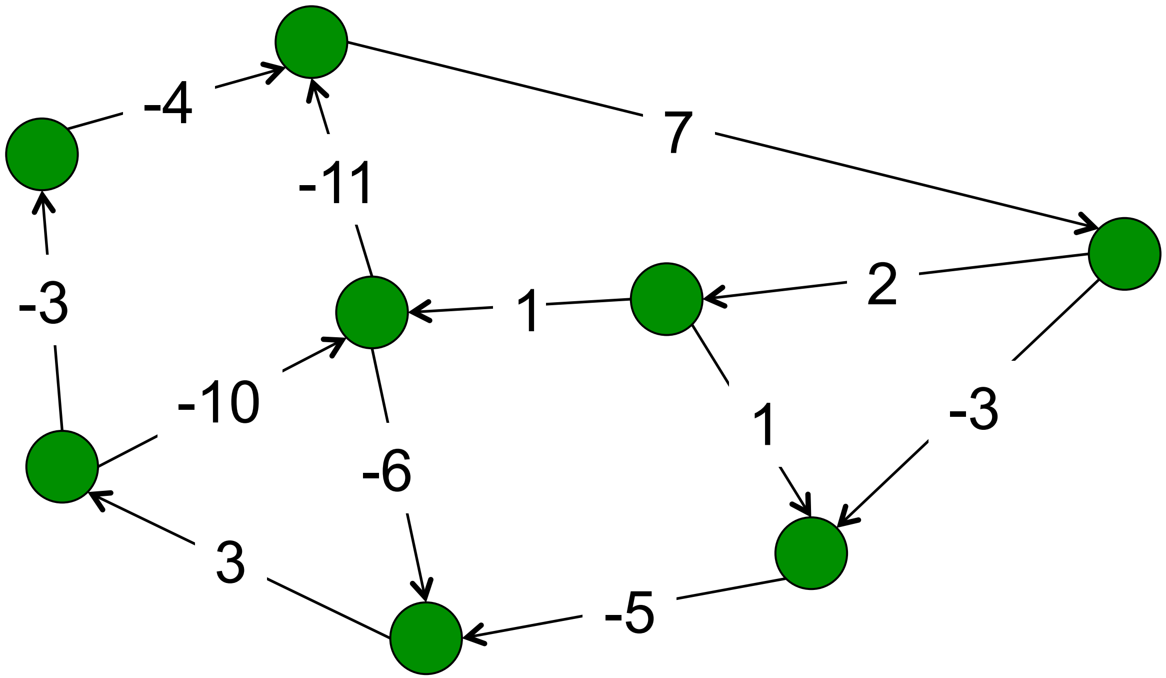
- Graph G = (V,E)
- Edge weights w = prizes on each edge
- Limit k: only cross at most k edges



Lazy Prize Collecting

Example:

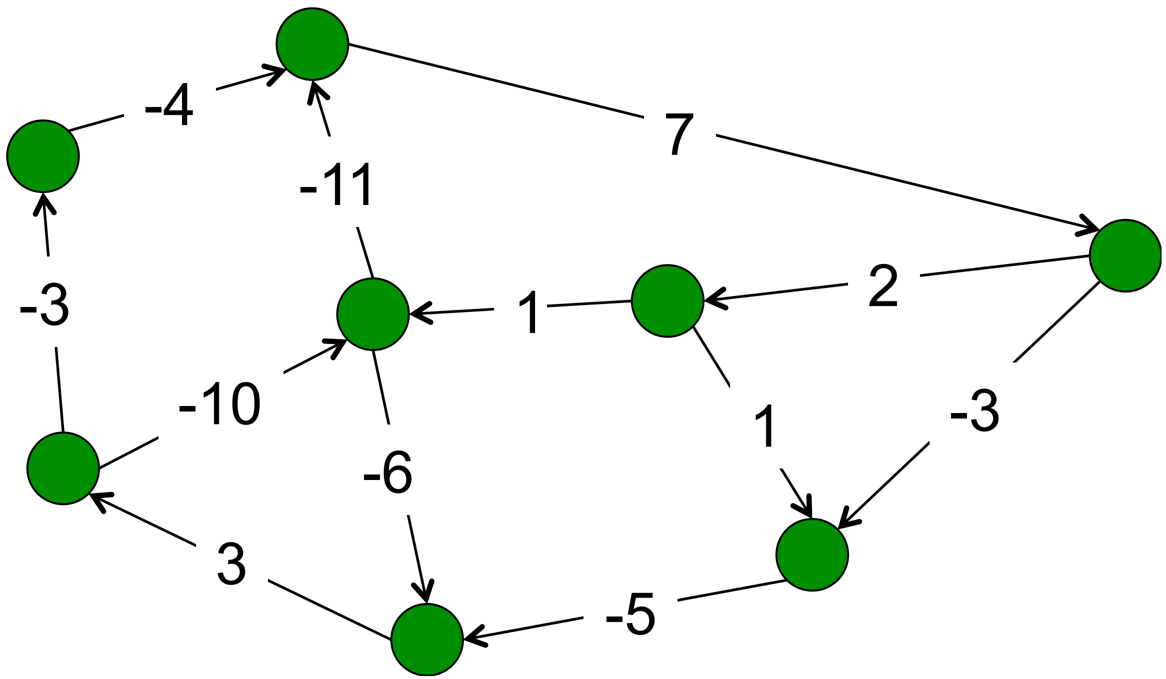
- $k = 1 \rightarrow 7$
- $k = 2 \rightarrow 9$
- $k = 3 \rightarrow 10$
- $k = 4 \rightarrow 10$
- $k = 5 \rightarrow 10$
- …
- $k = 71 \rightarrow 17$



Lazy Prize Collecting

Note: Not a shortest path problem

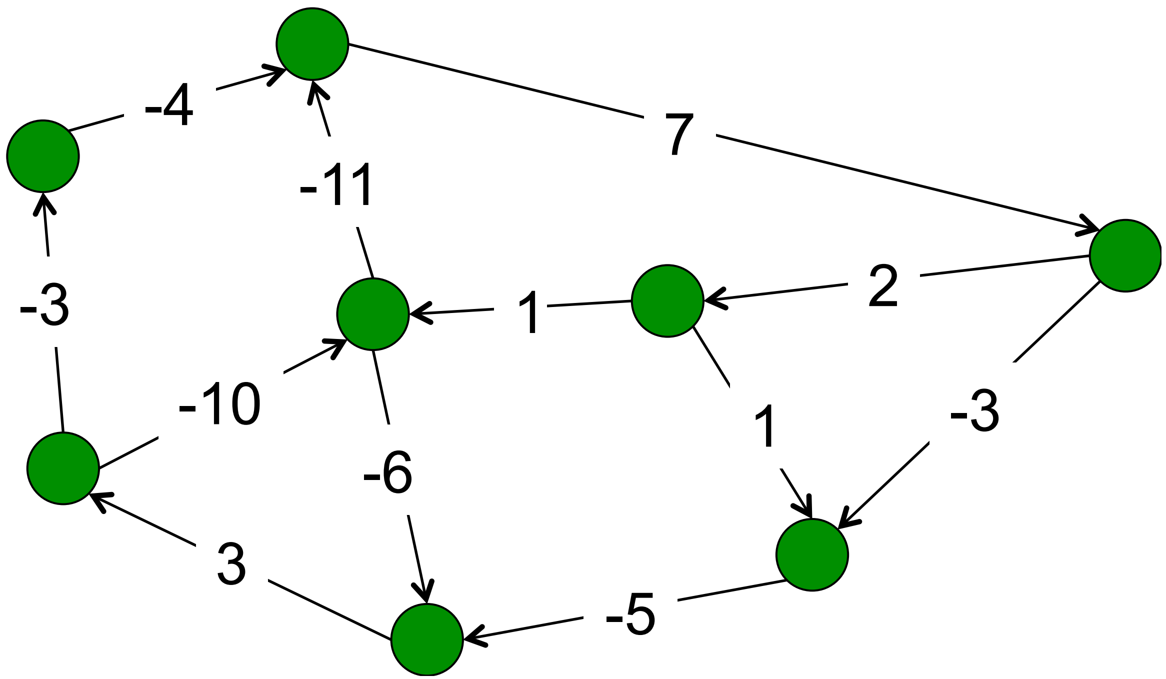
- Not a shortest path problem! Longest path…
- Negative weight cycles.
- Positive weight cycles.



Lazy Prize Collecting

Idea 1:

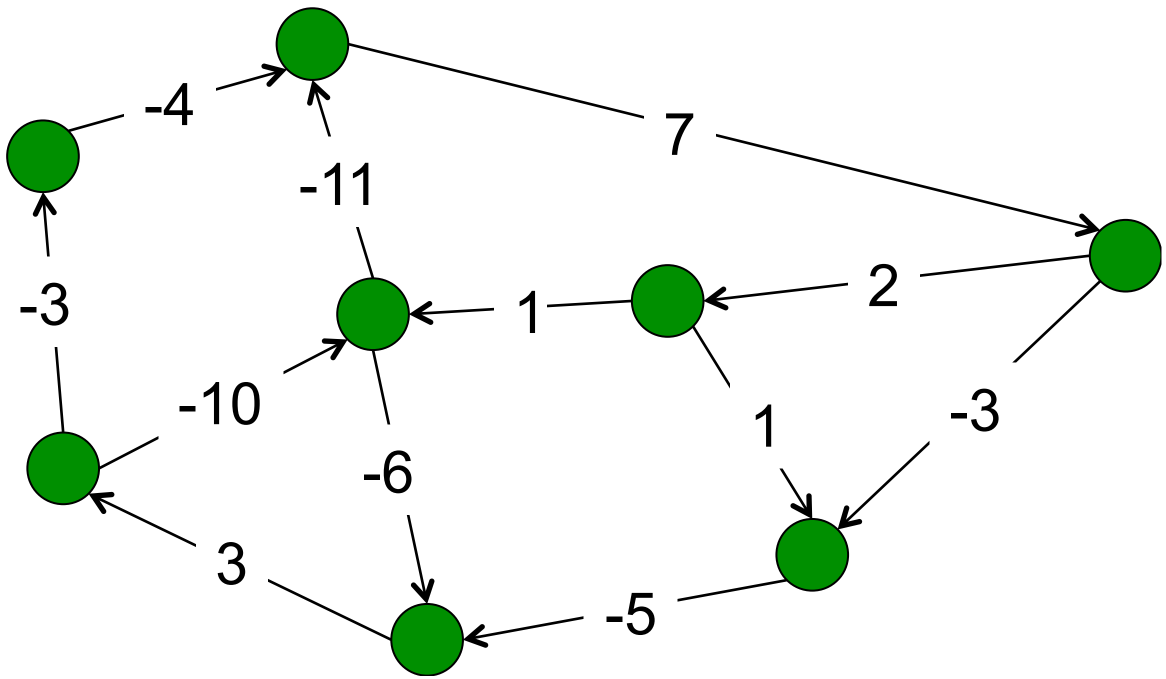
- Transform G into a DAG



Lazy Prize Collecting

Idea 1:

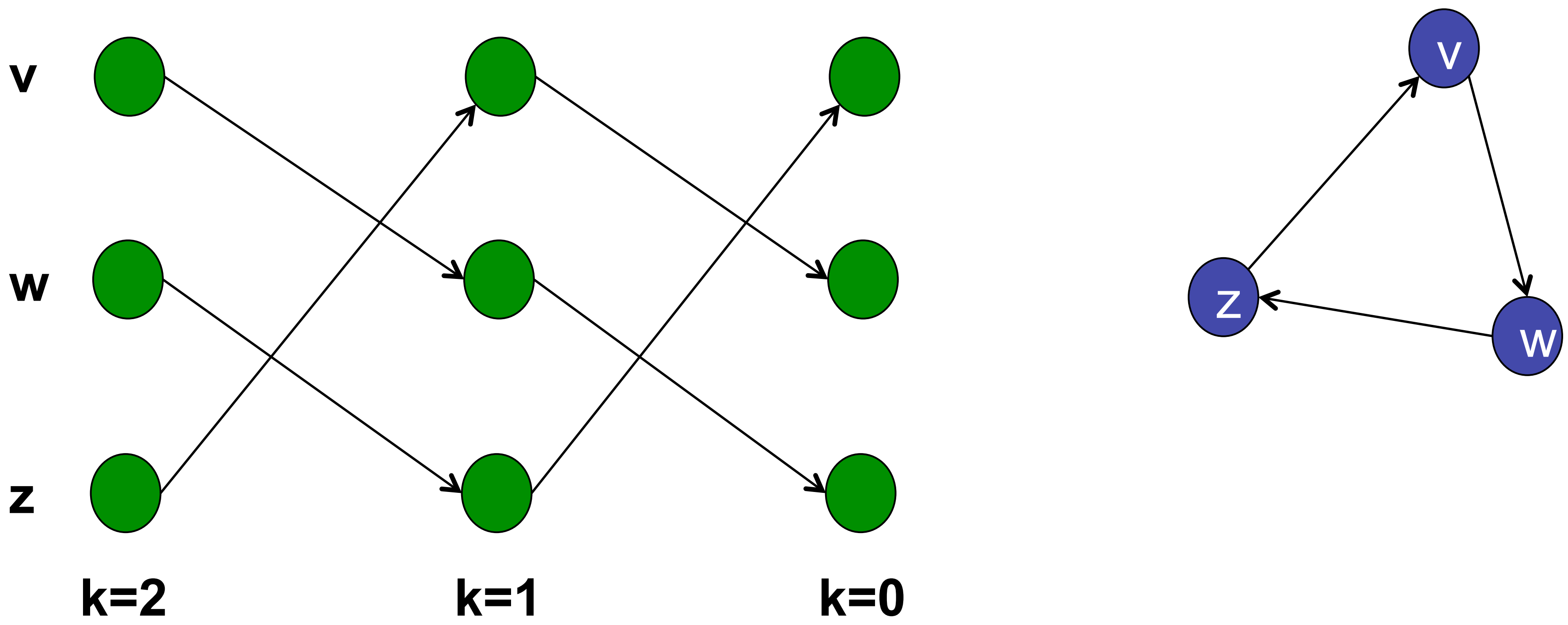
- Transform G into a DAG
- Make **k** copies of every node: (v,1), (v,2), (v,3), …



Lazy Prize Collecting

Idea 1:

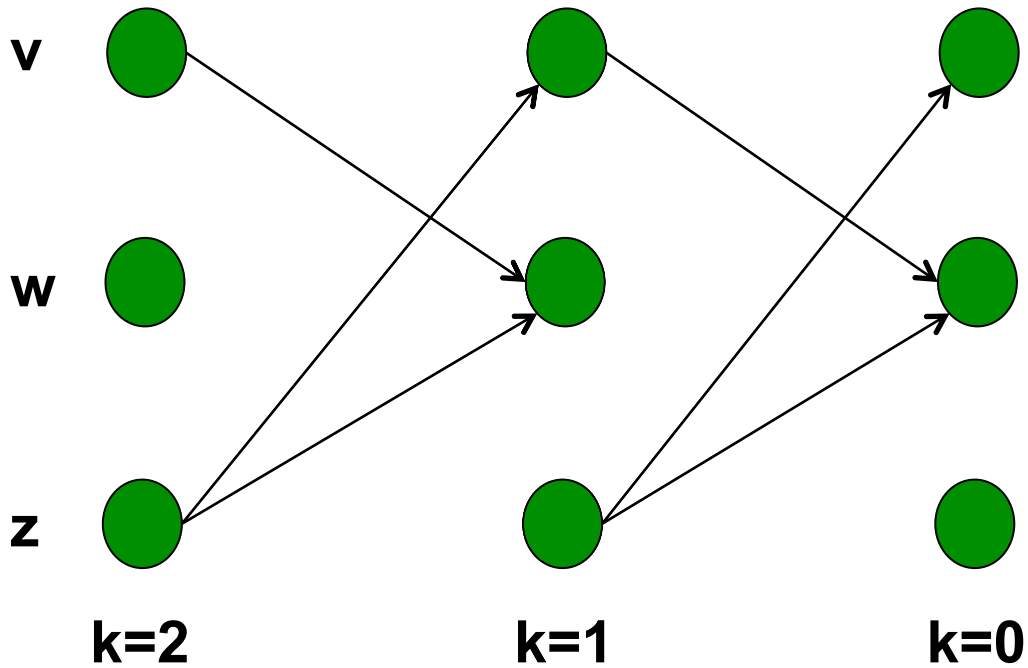
- Transform G into a DAG
- Make **k** copies of every node: (v,1), (v,2), (v,3), …

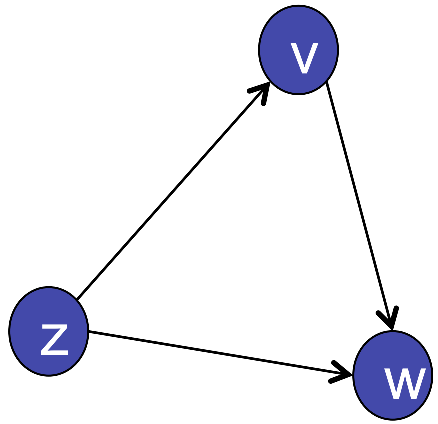


Lazy Prize Collecting

Idea 1:

- Transform G into a DAG
- Make **k** copies of every node: (v,1), (v,2), (v,3), …

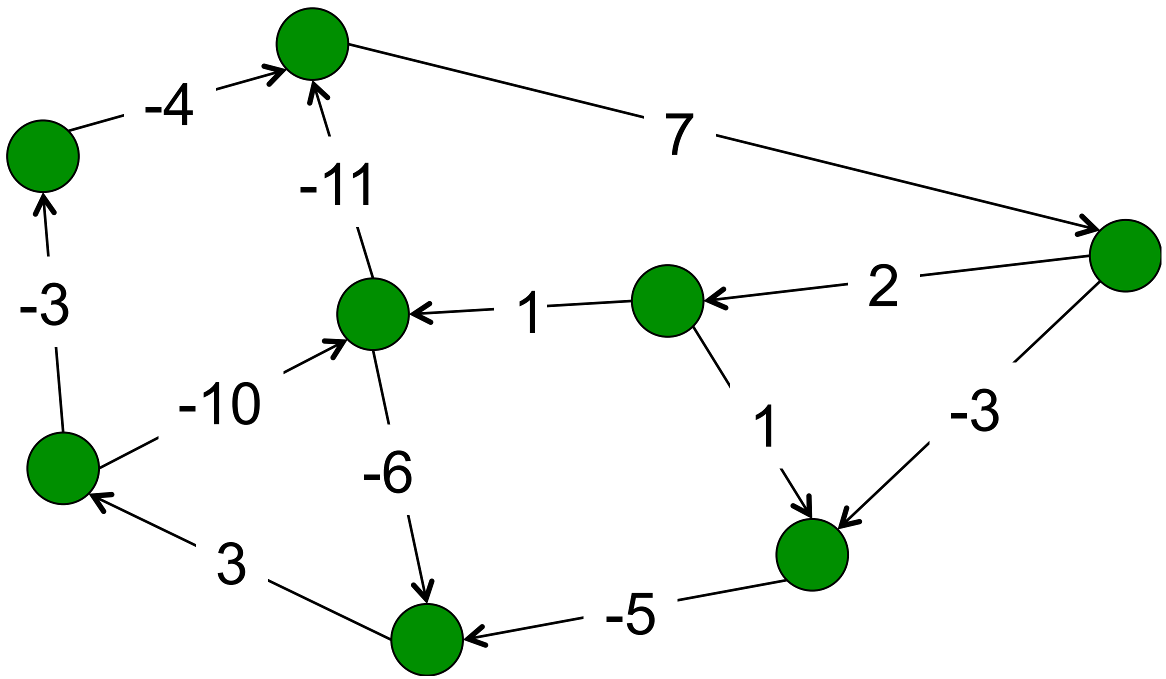




Lazy Prize Collecting

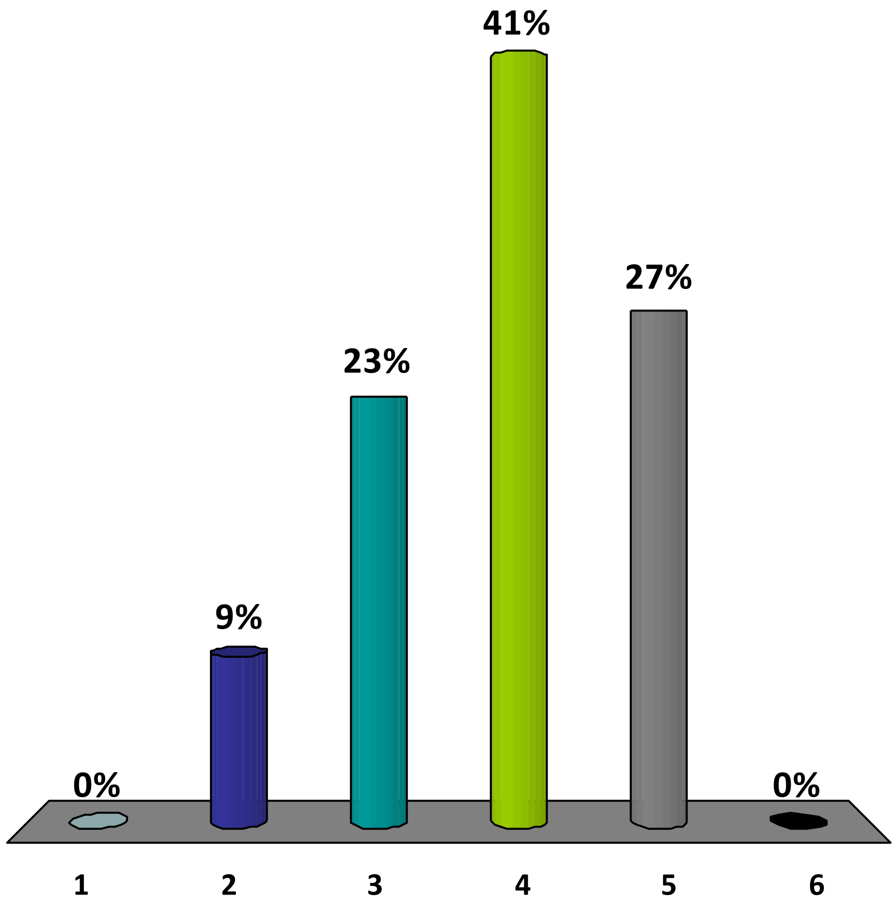
Idea 1:

- Transform G into a DAG
- Make **k** copies of every node: (v,1), (v,2), (v,3), …
- Solve longest-path problem for each source.



What is the running time of Idea 1?

1. O(E)
2. O(VE)
3. O(kE)
4. O(kVE)
5. $O(kV^2E)$
6. None of the above

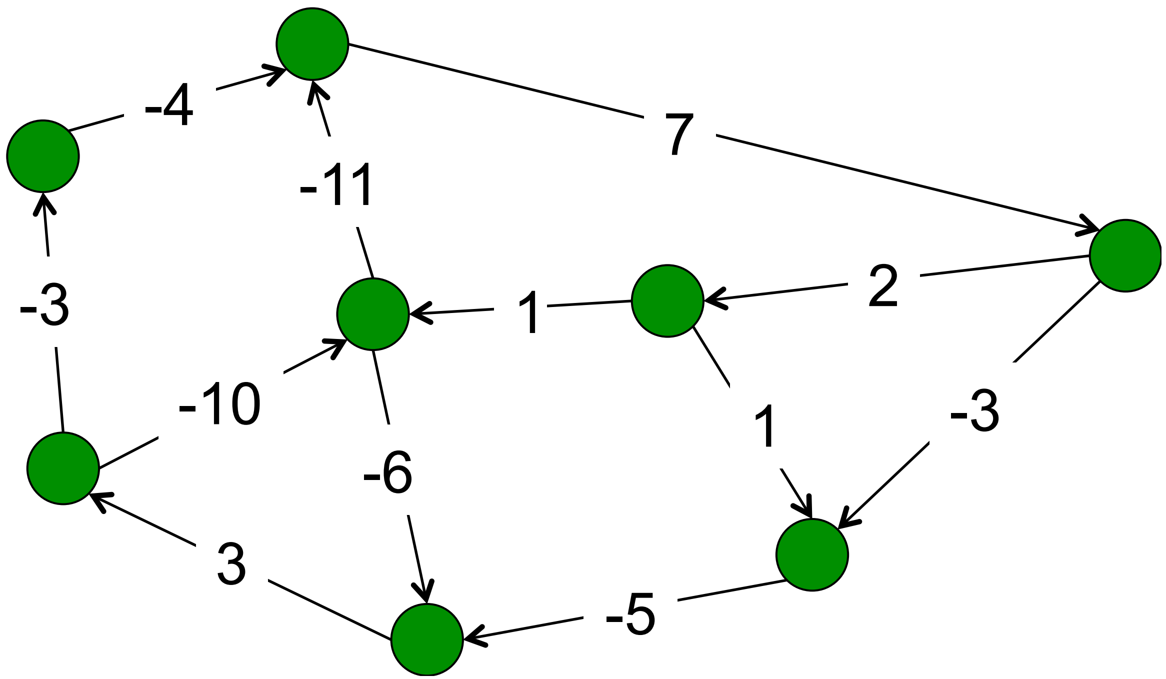


Response Counter

Lazy Prize Collecting

Running Time:

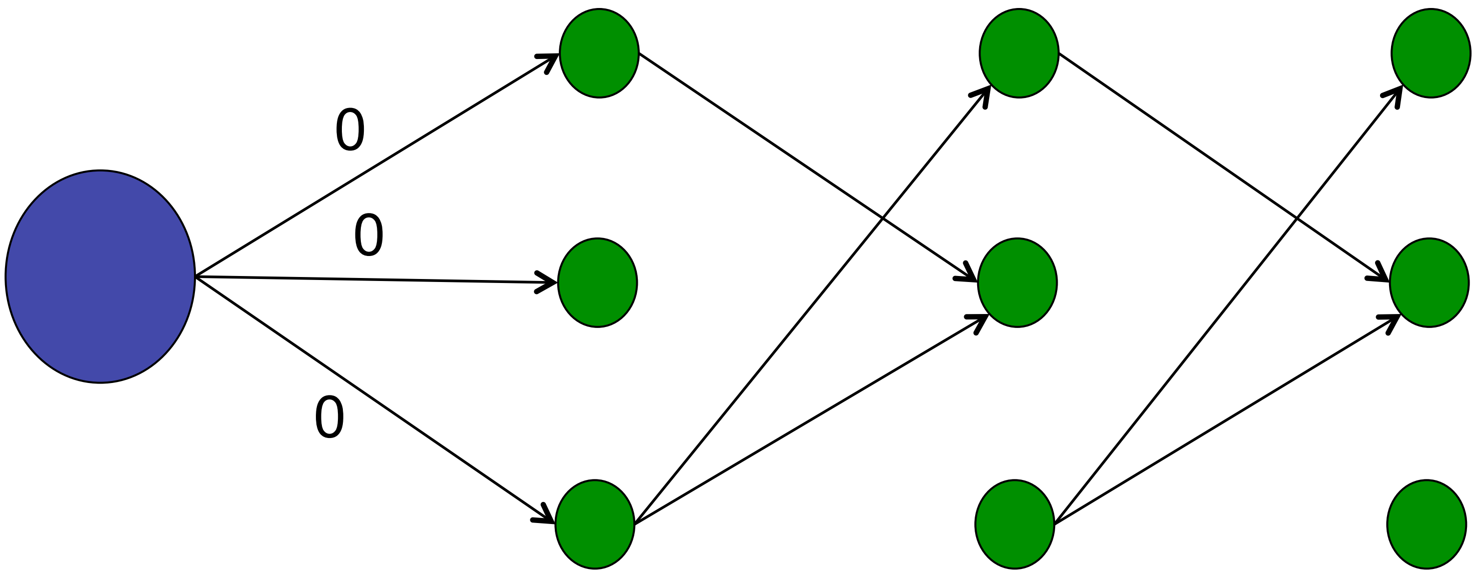
- Transformed graph: kV nodes, kE edges
- Topo-sort / Longest path: O(kV + kE)
- Once per source: repeat V times?



Lazy Prize Collecting

Running Time:

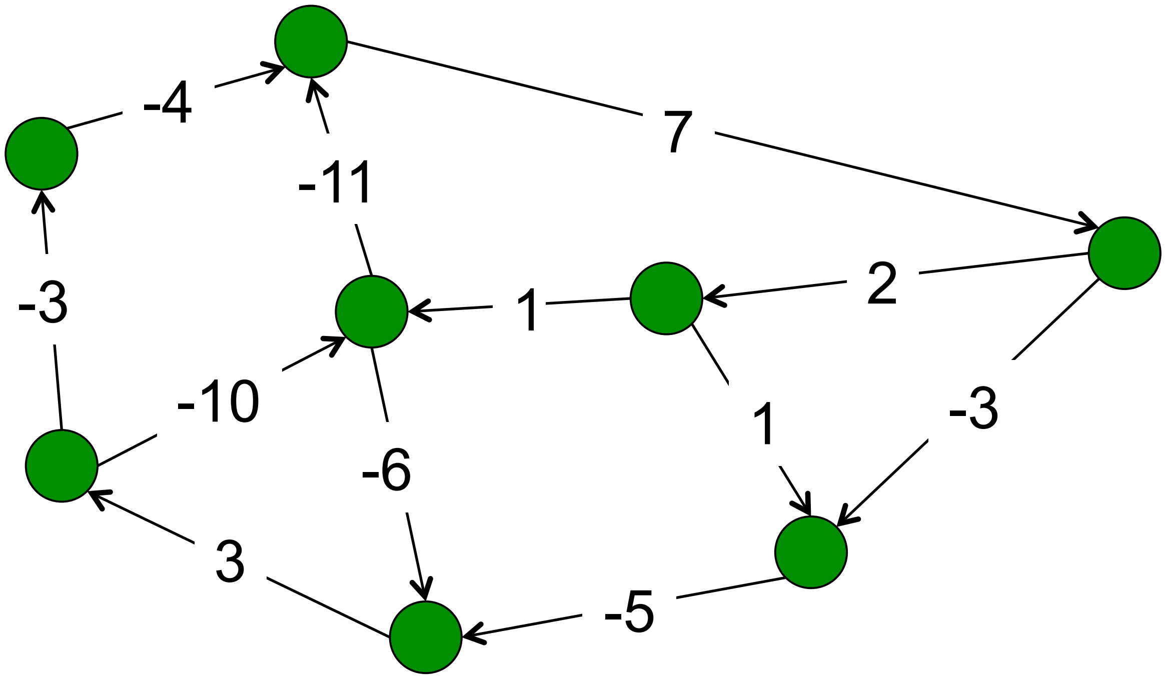
- Transformed graph: kV nodes, kE edges
- Topo-sort / Longest path: O(kV + kE)
- Create super-source….



Lazy Prize Collecting

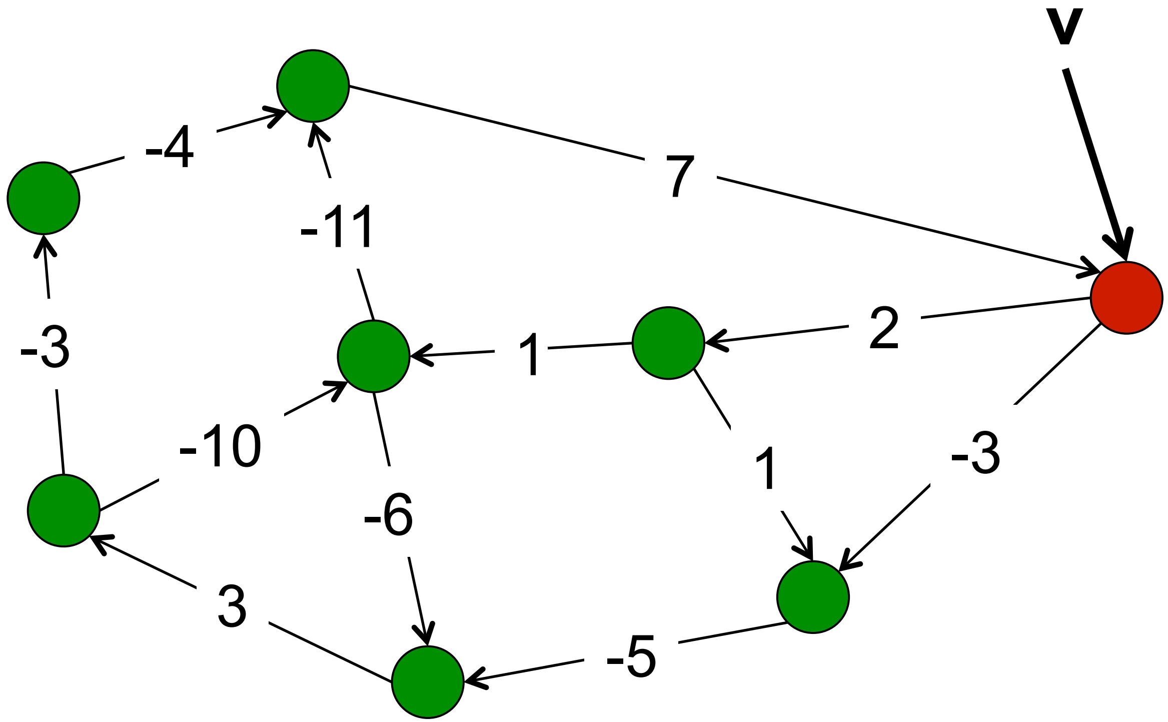
Idea 2: Dynamic Programming

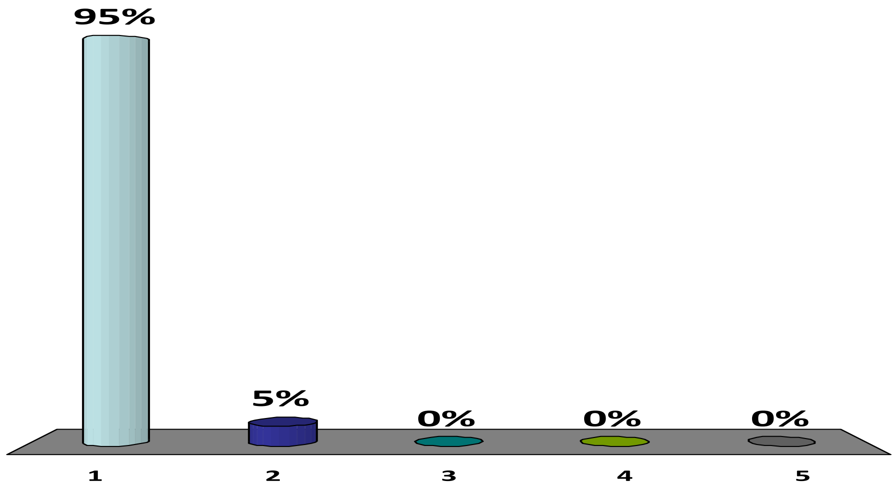
P[v, k] = maximum prize that you can collect starting at v and taking exactly k steps.



P(v, 0) = ??

1. 0
2. 2
3. -3
4. 4
5. 5





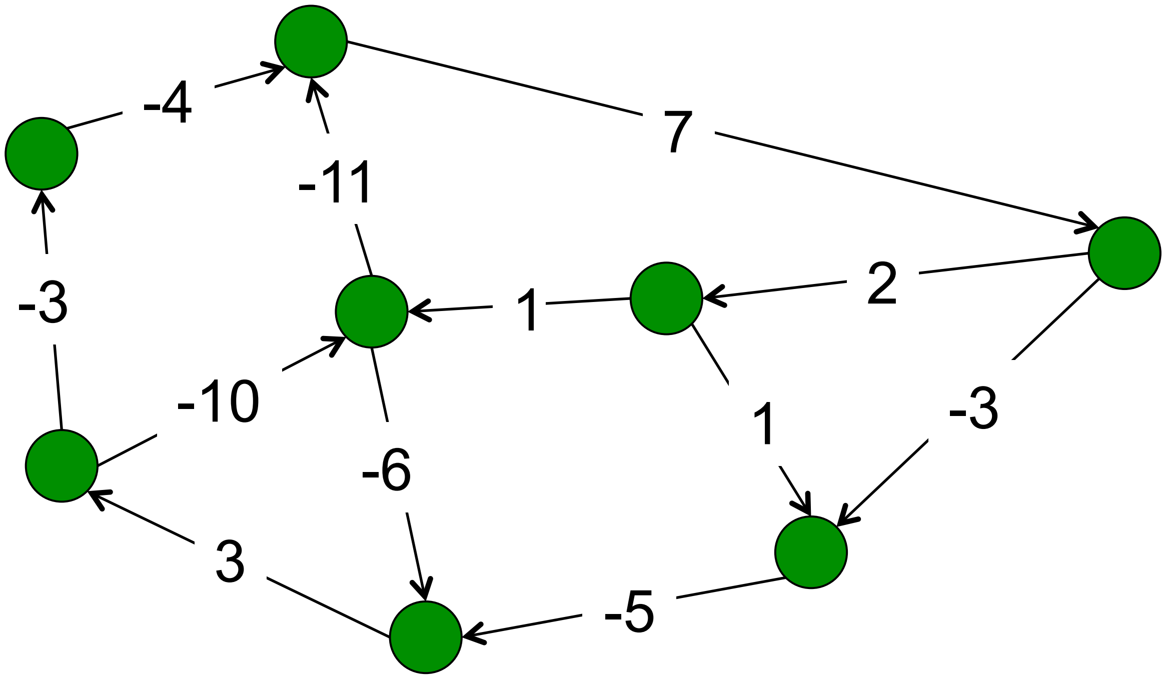
Response Counter

Lazy Prize Collecting

Idea 2: Dynamic Programming

P[v, k] = maximum prize that you can collect starting at v and taking exactly k steps.

P[v, 0] = 0



Lazy Prize Collecting

Idea 2: Dynamic Programming

$P[v, k]$ = maximum prize that you can collect starting at v and taking exactly k steps.

Solve P[v,k] using subproblems:

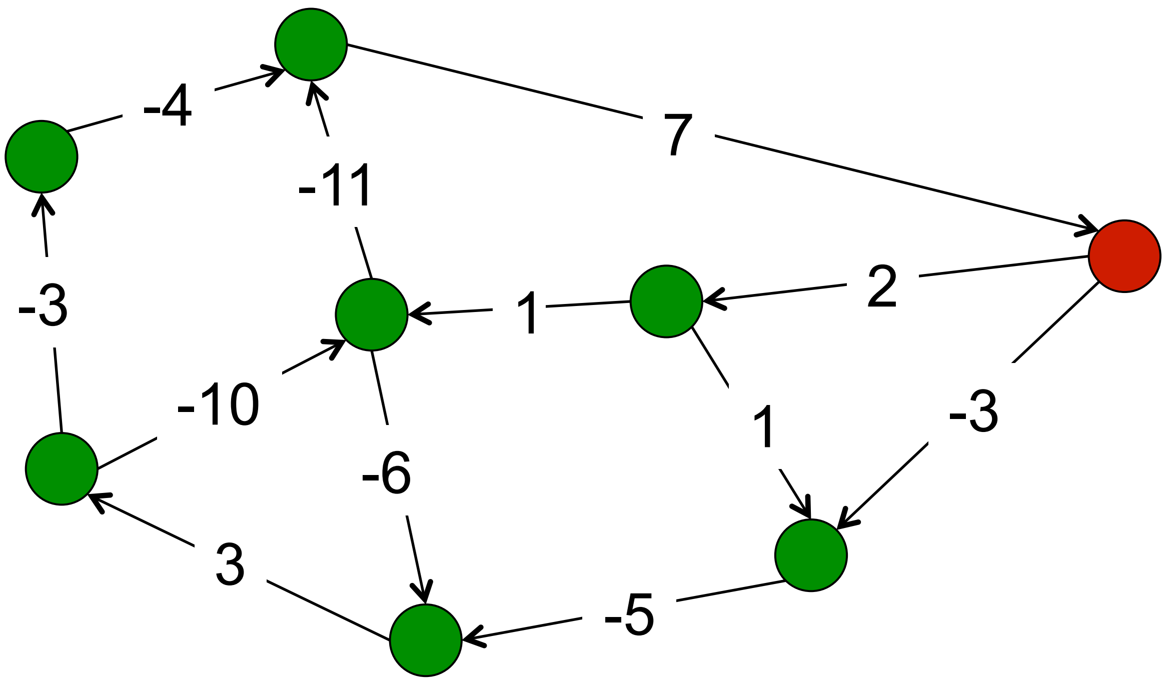
$$P[v, k] = \text{MAX} \{ \quad P[w_1, k-1] + w(v, w_1), \quad P[w_2, k-1] + w(v, w_2), \quad P[w_3, k-1] + w(v, w_3), \ldots \quad \}$$

where $v.\text{nbrList}() = \{w_1, w_2, w_3, \ldots\}$

Lazy Prize Collecting

Idea 2: Dynamic Programming

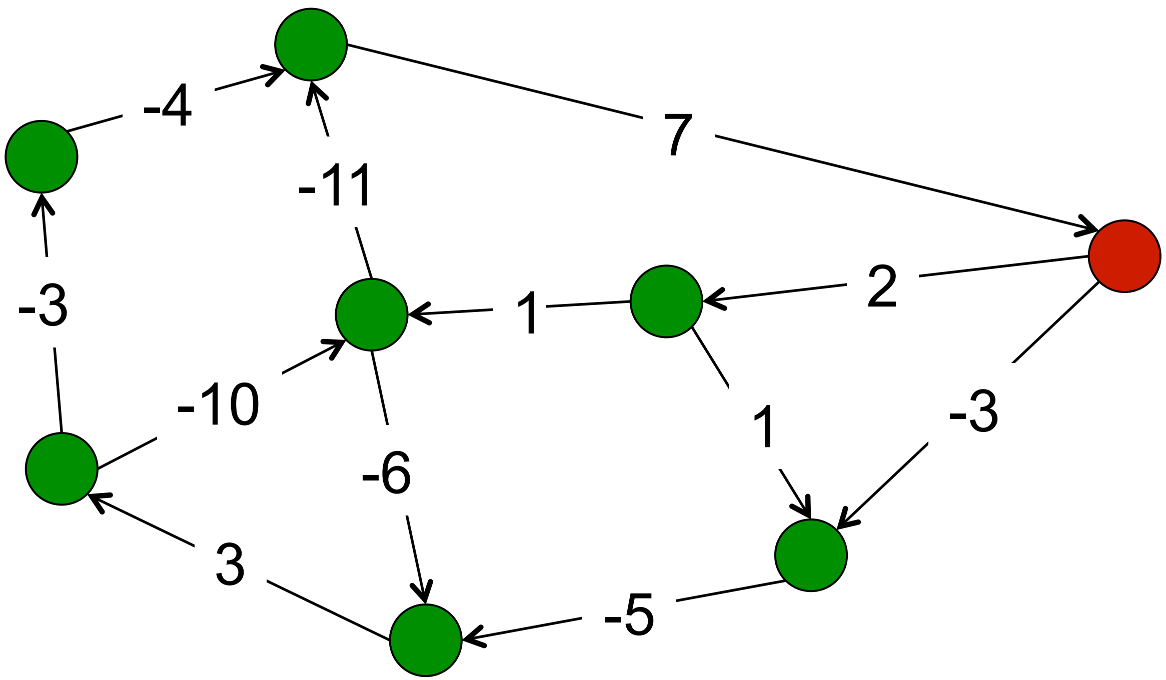
$P[v, 1] = \max(0+2, 0-3) = 2$



Lazy Prize Collecting

Idea 2: Dynamic Programming

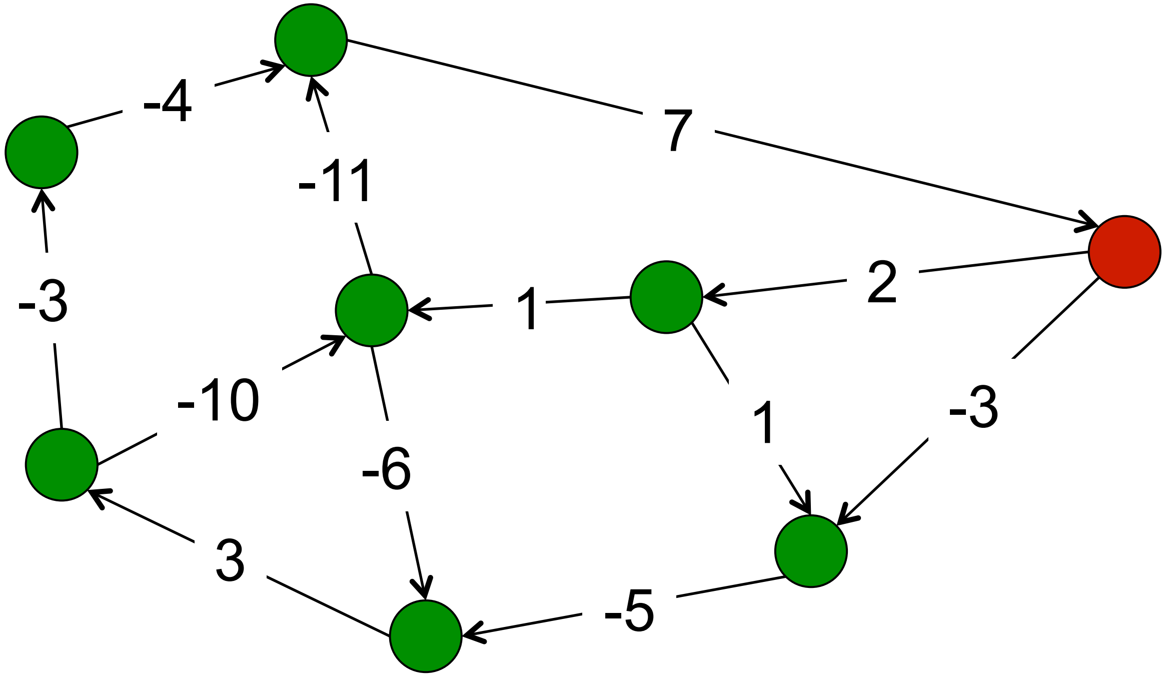
P[v, 2] = max(1+2, -5-3) = 3



Lazy Prize Collecting

Idea 2: Dynamic Programming

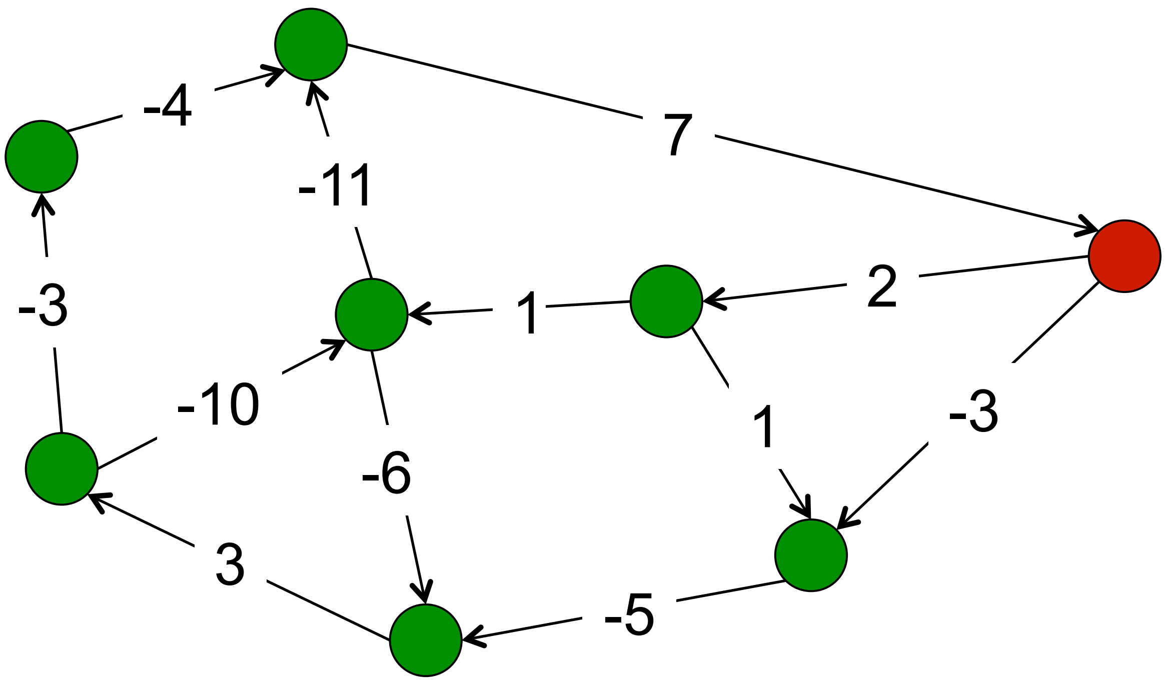
P[v, 3] = max(-4+2, -2-3) = -2



Lazy Prize Collecting

Idea 2: Dynamic Programming

When is it worth crossing a negative edge?



Dynamic Programming

Table view: P[k, v]

k	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}
1	17	22	14	19	8	4	9	12	15	7
2	15	12	13	13	7					
3										
4										
5										
6										
7										
8										
9										
10										
11										

```
int LazyPrizeCollecting(V, E, kMax) {

    int[][] P = new int[V.length][kMax+1]; // create memo table P
    for (int i=0; i<V.length; i++) // initialize P to zero
       for (int j=0; j<kMax+1; j++)
              P[i][j] = 0;

    for (int k=1; k<kMax+1; k++) { // Solve for every value of k
       for (int v = 0; v<V.length; v++) { // For every node…
              int max = -INFTY;
              // …find max prize in next step
              for (int w : V[v].nbrList()) {
                     if (P[w,k-1] + E[v,w] > max)
                            max = P[w,k-1] + E[v,w];
              }
              P[v, k] = max;
       }
    }
    return maxEntry(P); // returns largest entry in P
}
```

Lazy Prize Collecting

Idea 2: Dynamic Programming

P[v, k] = maximum prize that you can collect starting at v and taking exactly k steps.

Total Cost:

Two factors:

– Number of subproblems: kV

– Cost to solve each subproblem: |v.nbrList|

Total: $O(kV^2)$

Dynamic Programming

Table view: P[k, v]

k	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}
1	17	22	14	19	8	4	9	12	15	7
2	15	12	13	13	7					
3										
4										
5										
6										
7										
8										
9										
10										
11										

Lazy Prize Collecting

Idea 2: Dynamic Programming

P[v, k] = maximum prize that you can collect starting at v and taking exactly k steps.

Total Cost:

Two factors:

– Number of rows: k

– Cost to solve each row: E

Total: O(kE)

Roadmap

Today: Dynamic Programming

- DP Basics
- Longest Increasing Subsequence
- Prize Collecting
- Vertex Cover on a Tree
- All-Pairs-Shortest-Paths

Puzzle of the Day

Two players go into separate booths, and each presses a button:

- Each player gets a random number between zero and one.
- The number is chosen uniformly from the interval [0,1].

Each players makes a choice:

- They can keep their original number.
- They can discard their number and press the button again.
- If they press the button again, they get a new number, as before, randomly chosen from the interval [0,1].

Whichever player has the highest number in the end wins.

When should a player press the button again?

Roadmap

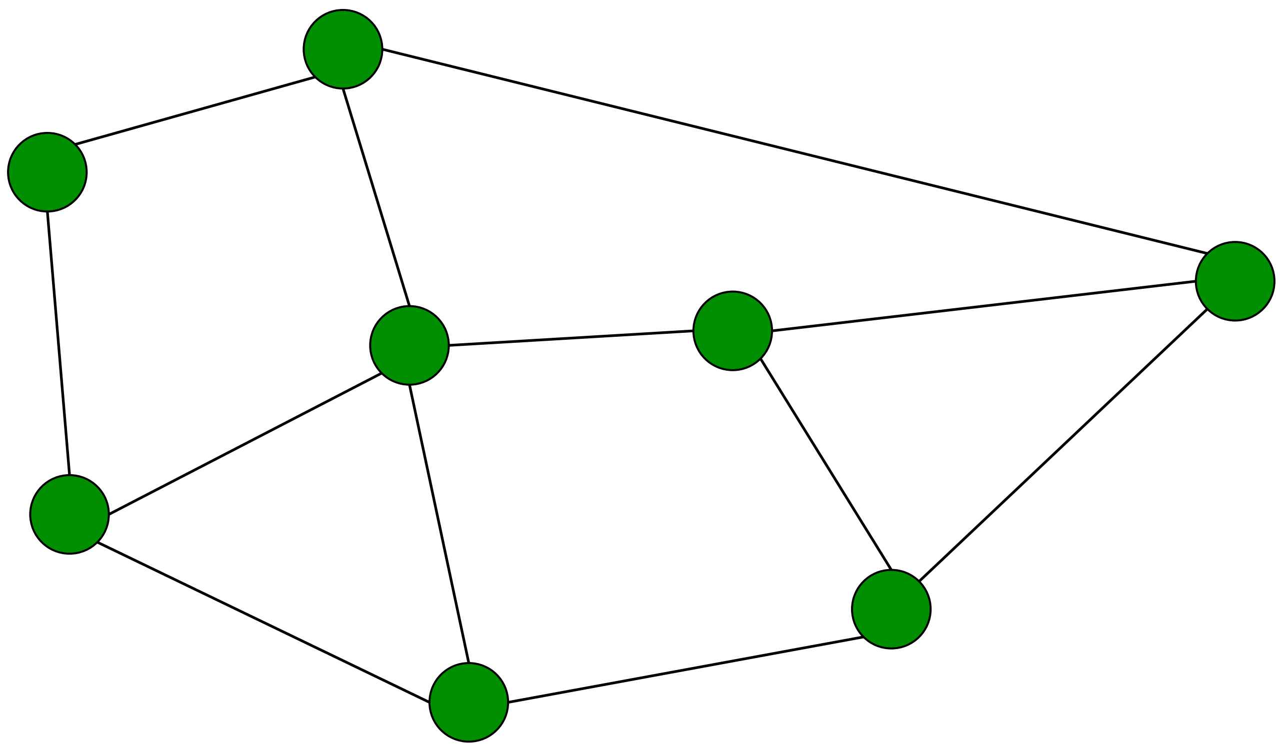
Today: Dynamic Programming

- DP Basics
- Longest Increasing Subsequence
- Prize Collecting
- Vertex Cover on a Tree
- All-Pairs-Shortest-Paths

Vertex Cover

Input:

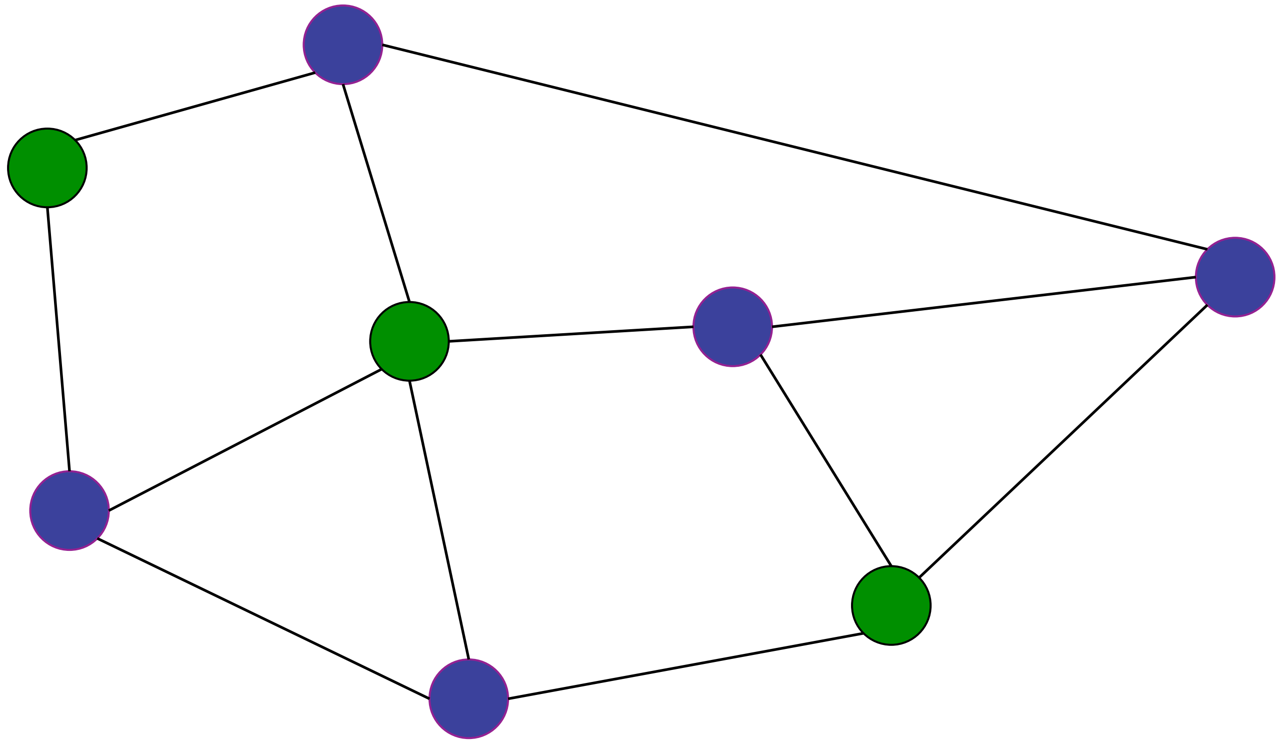
- Undirected, unweighted graph G = (V,E)



Vertex Cover

Output:

Set of nodes C where every edge is adjacent to at least one node in C.



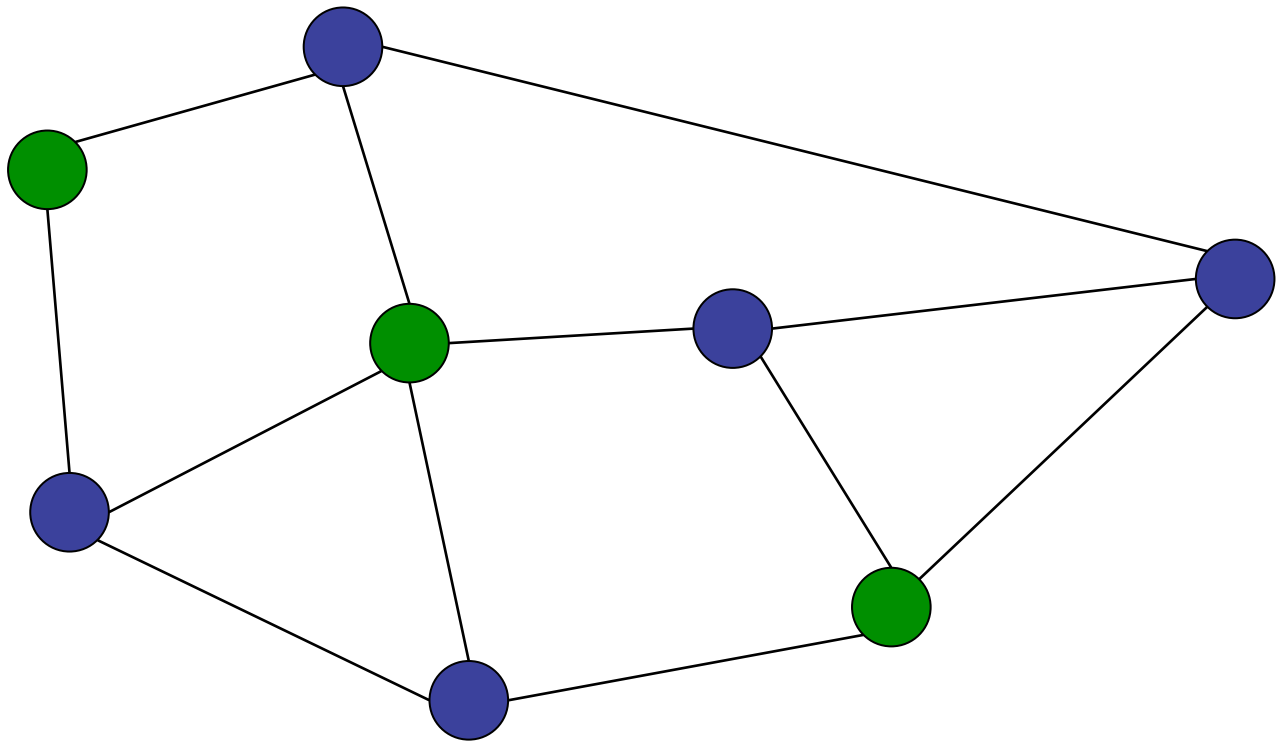
Minimum Vertex Cover

NP-complete:

No polynomial time algorithm (unless P=NP).

Easy 2-approximation (via matchings).

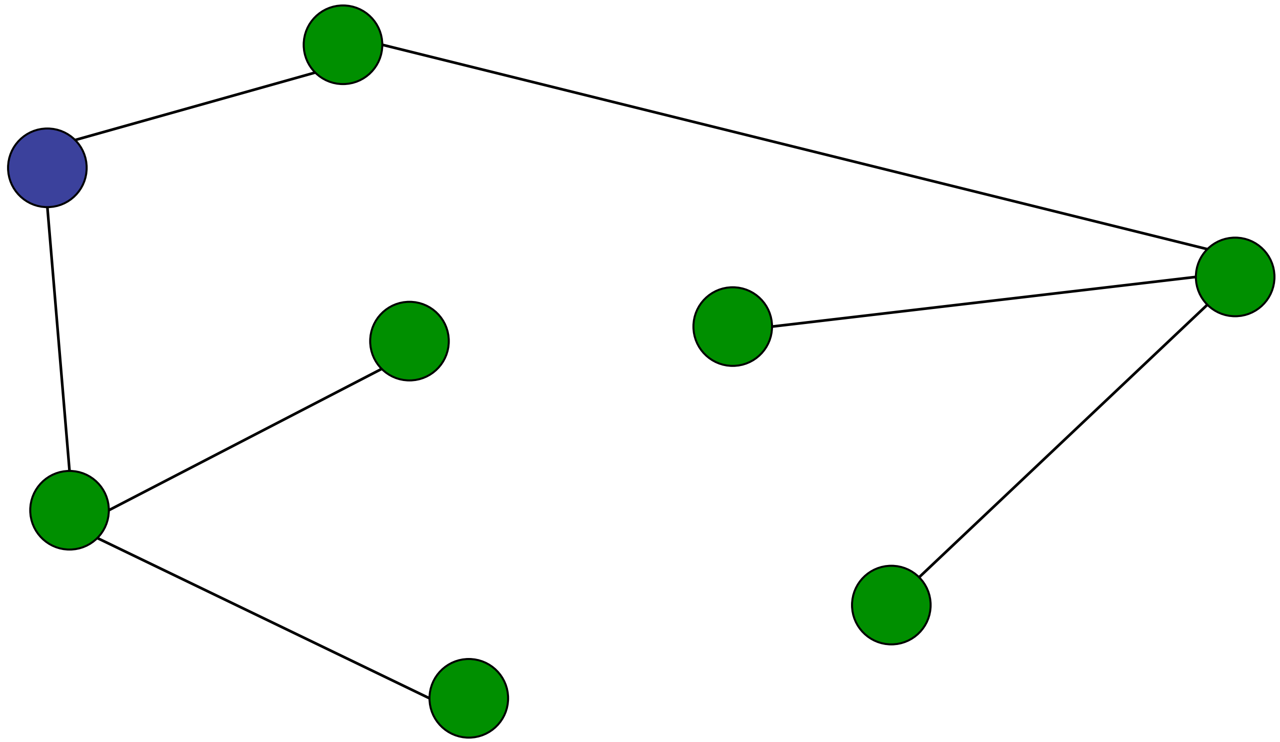
Nothing better known.



Vertex Cover on a Tree

Input:

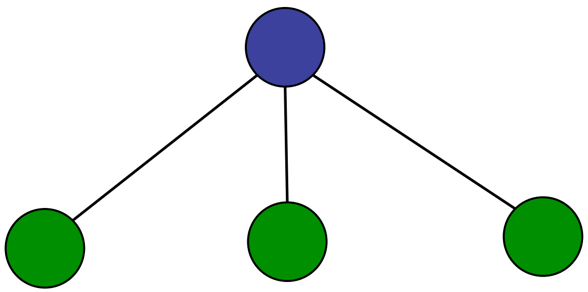
- Undirected, unweighted **tree** G = (V,E)
- Root of tree r

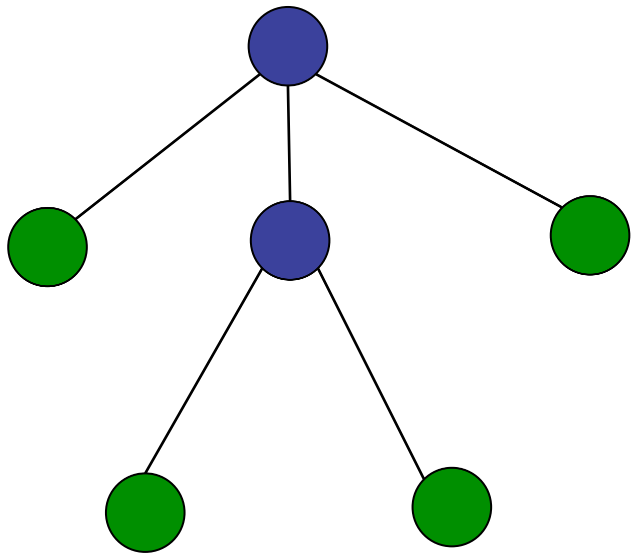


Vertex Cover on a Tree

Output:

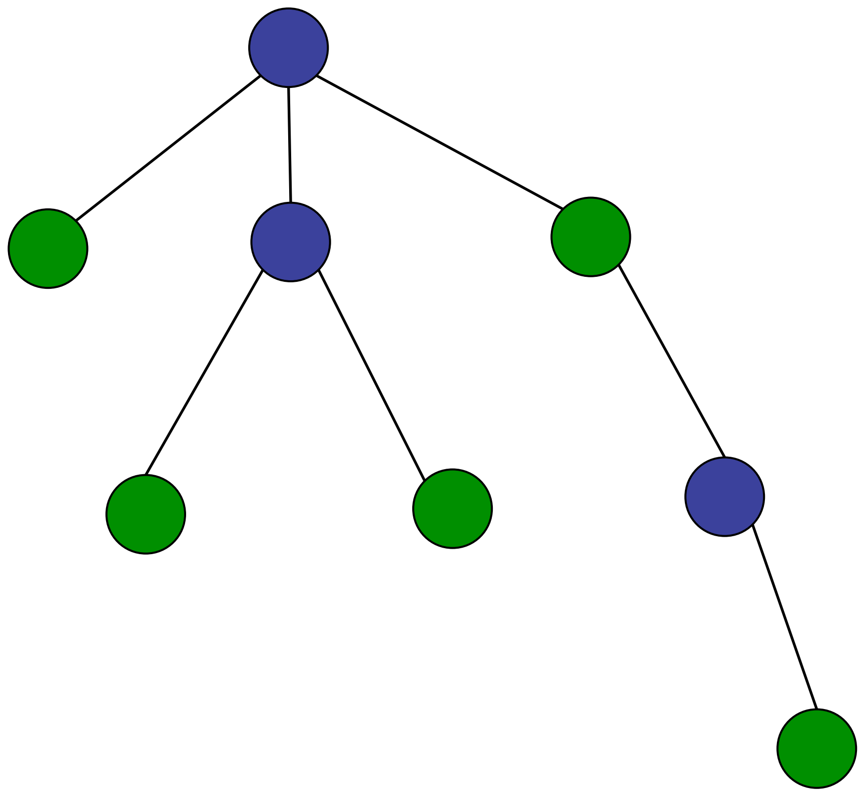
– size of the minimum vertex cover





Vertex Cover on a Tree

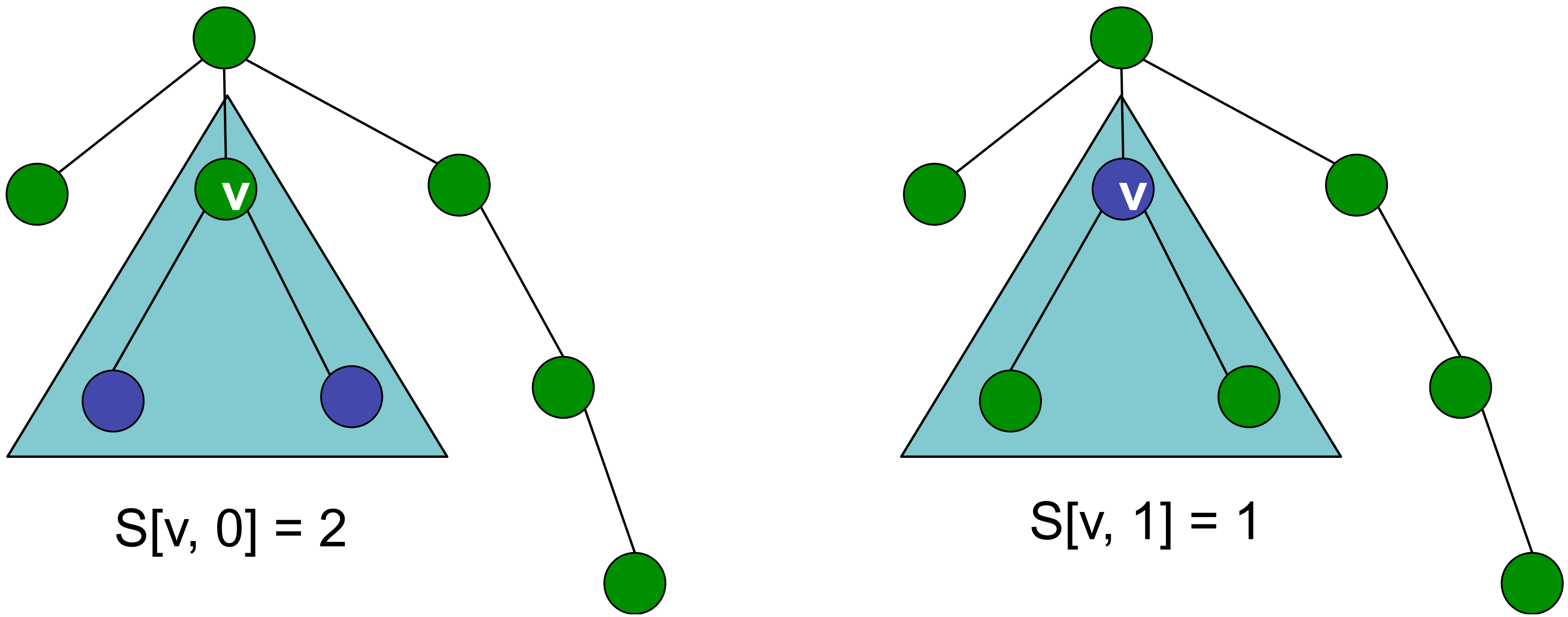
What are the subproblems?



Vertex Cover on a Tree

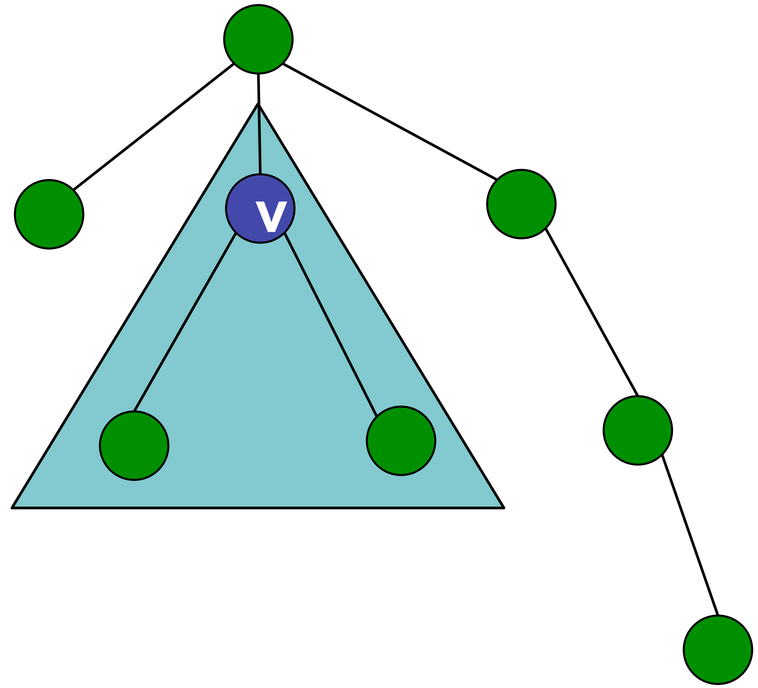
S[v, 0] = size of vertex cover in subtree rooted at node v, if v is NOT covered.

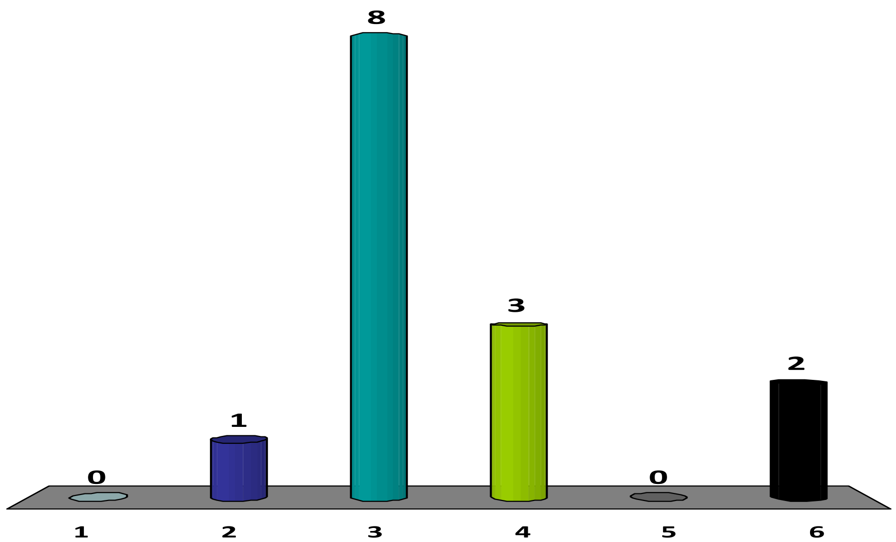
S[v, 1] = size of vertex cover in subtree rooted at node v, if v IS covered.



How many subproblems?

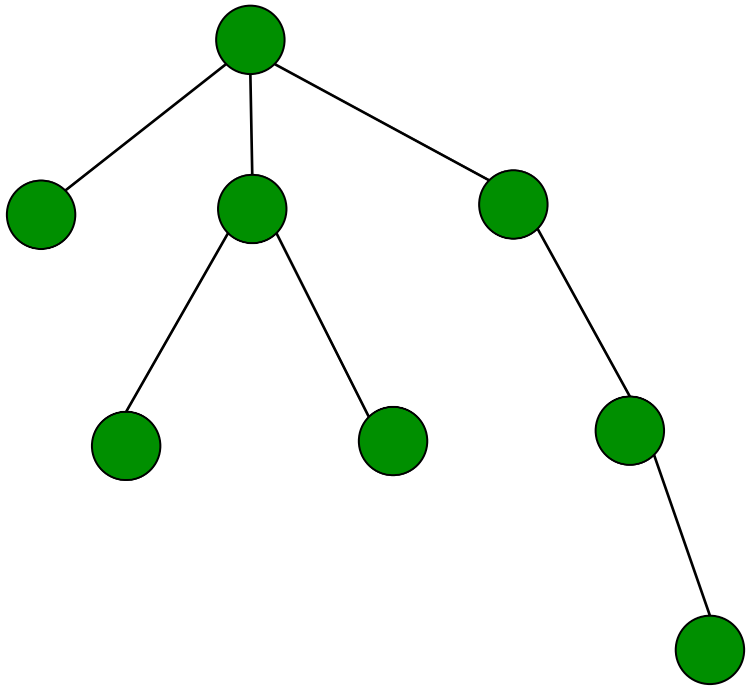
1. 2
2. V
3. 2V
4. E
5. 2E
6. VE





Vertex Cover on a Tree

What is the base case?



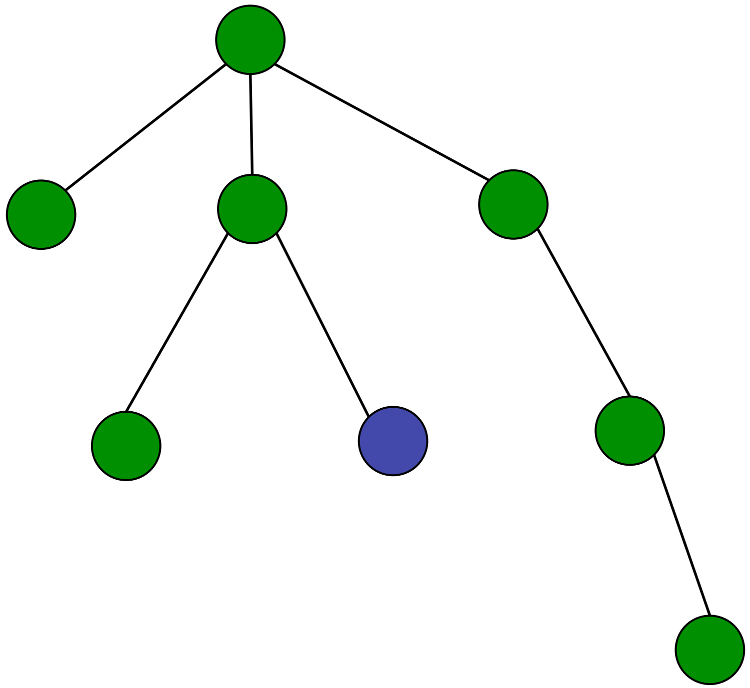
Vertex Cover on a Tree

What is the base case?

Start at the leaves!

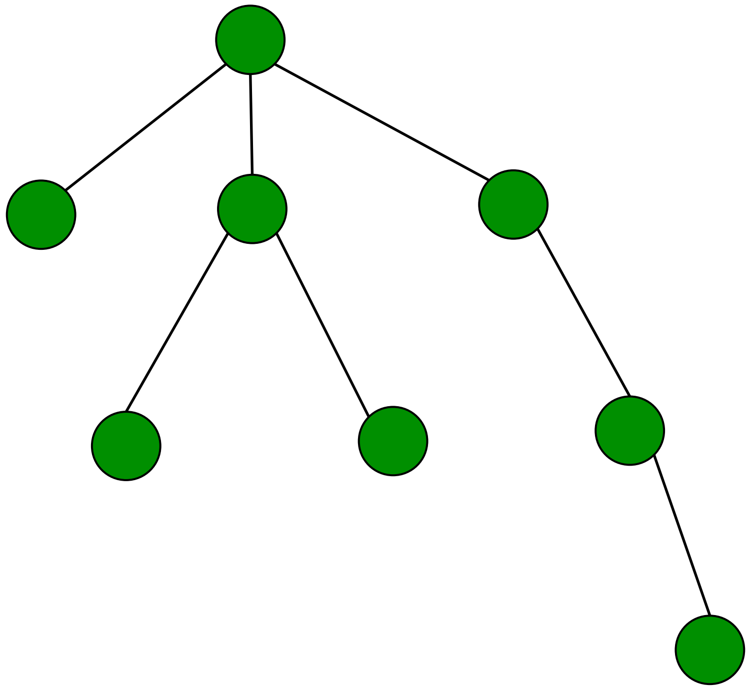
S[leaf, 0] = 0

S[leaf, 1] = 1



Vertex Cover on a Tree

How do we calculate S[v, 0]?

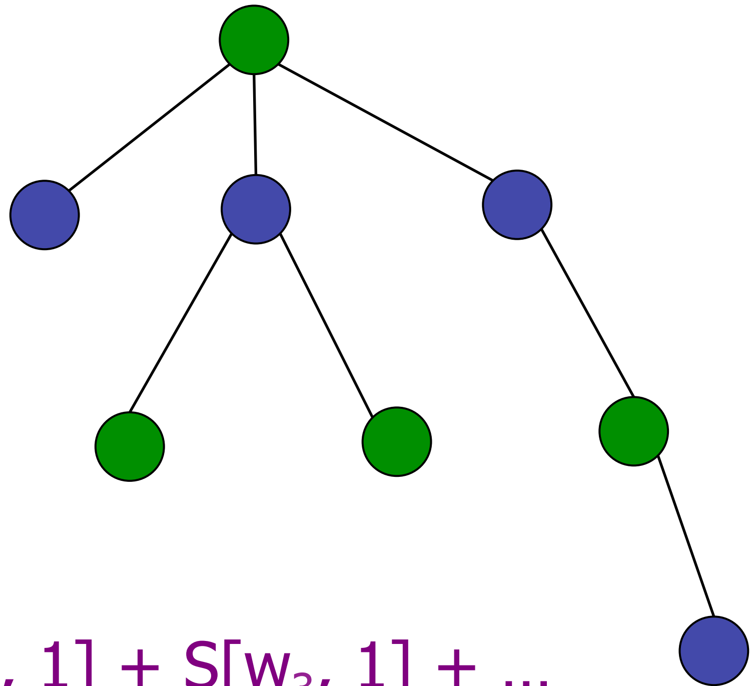


Vertex Cover on a Tree

How do we calculate S[v, 0]?

If we do not cover v, then we need to cover all of v's children.

Remember: we have already solved the subproblems!



$S[v, 0] = S[w_1, 1] + S[w_2, 1] + S[w_3, 1] + \ldots$

$v.nbrList() = \{w_1, w_2, w_3, \ldots \}$

Vertex Cover on a Tree

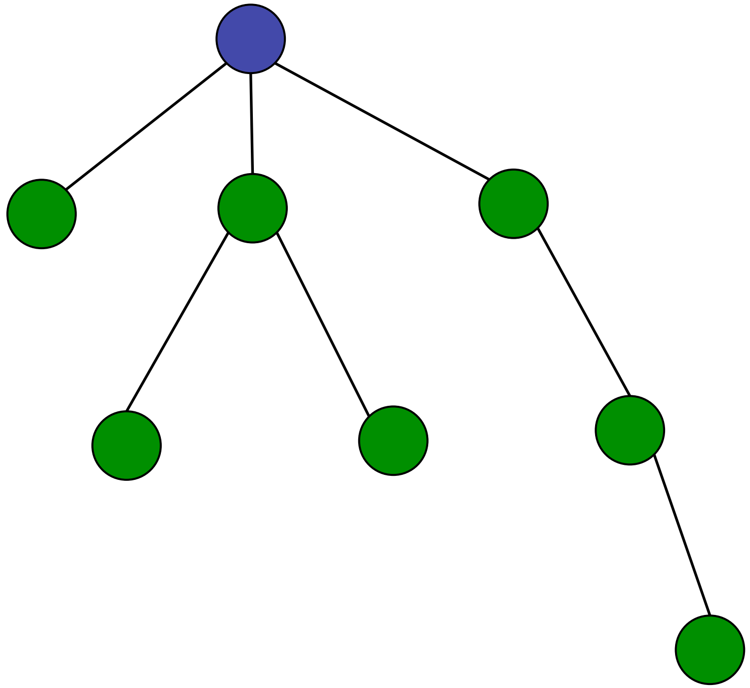
How do we calculate S[v, 1]?

We can either cover or uncover v's children.

$W_1 = \min(S[w_1, 0], S[w_1, 1])$

$W_2 = \min(S[w_2, 0], S[w_2, 1])$

$W_3 = \min(S[w_3, 0], S[w_3, 1])$



$S[v, 1] = 1 + W_1 + W_2 + W_3 + \ldots$

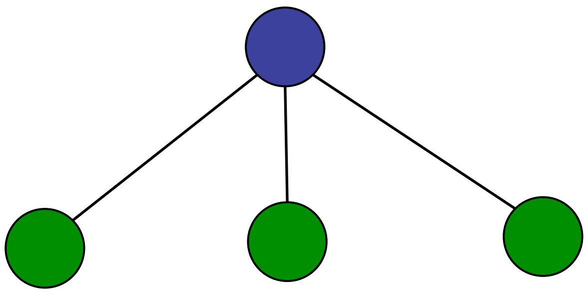
$v.\text{nbrList}() = \{w_1, w_2, w_3, \ldots\}$

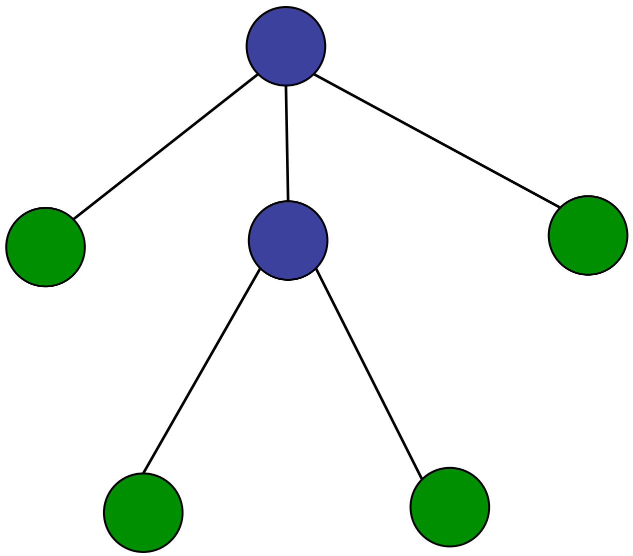
```
int treeVertexCover(V){//Assume tree is ordered from root-to-leaf

    int[][] S = new int[V.length][2]; // create memo table S

    for (int v=V.length-1; v>=0; v--){//From the leaf to the root
       if (v.childList().size()==0) { // If v is a leaf…
              S[v][0] = 0;
              S[v][1] = 1;
       }
       else{  // Calculate S from v's children.
              int S[v][0] = 0;
              int S[v][1] = 1;
              for (int w : V[v].childList()) {
                     S[v][0] += S[w][1];
                     S[v][1] += Math.min(S[w][0], S[w][1]);
              }
       }
    }
    return Math.min(S[0][0], S[0][1]); // returns min at root
}
```

Vertex Cover on a Tree

Running time:

- 2V sub-problems
- O(V) time to solve all sub-problems.
  - Each edge explored once.
  - Each sub-problem involves exploring children edges.





Roadmap

Today: Dynamic Programming

- DP Basics
- Longest Increasing Subsequence
- Prize Collecting
- Vertex Cover on a Tree
- All-Pairs-Shortest-Paths

All Pairs Shortest Path

Input:

- Directed, weighted graph G = (V,E)

Goal:

- Preprocess G
- Answer queries: min-distance(v, w)?

Example:

- On-line map service

All Pairs Shortest Path

Simple solution:

- Run Dijkstra's Algorithm on every query

Cost:

- Preprocessing: 0
- Responding to q queries: O(q*E*log V)

All Pairs Shortest Path

Simple solution++:

On query(v,w):

- Run Dijkstra's Algorithm from source v
- Set dist[v,*] = ...
- Next time, on query(v, ?) don't run Dijkstra's.

Cost:

- Preprocessing: 0
- Responding to q queries: O(VE*log V)

All Pairs Shortest Path

Preprocessing solution:

On preprocessing:

- For all (v,w): calculate distance(v,w)

On query:

- Return precalculated value.

Cost:

– Preprocessing: all-pairs-shortest-paths

– Responding to q queries: O(q)

Diameter of a Graph

Input:

Undirected, weighted graph G=(V, E)

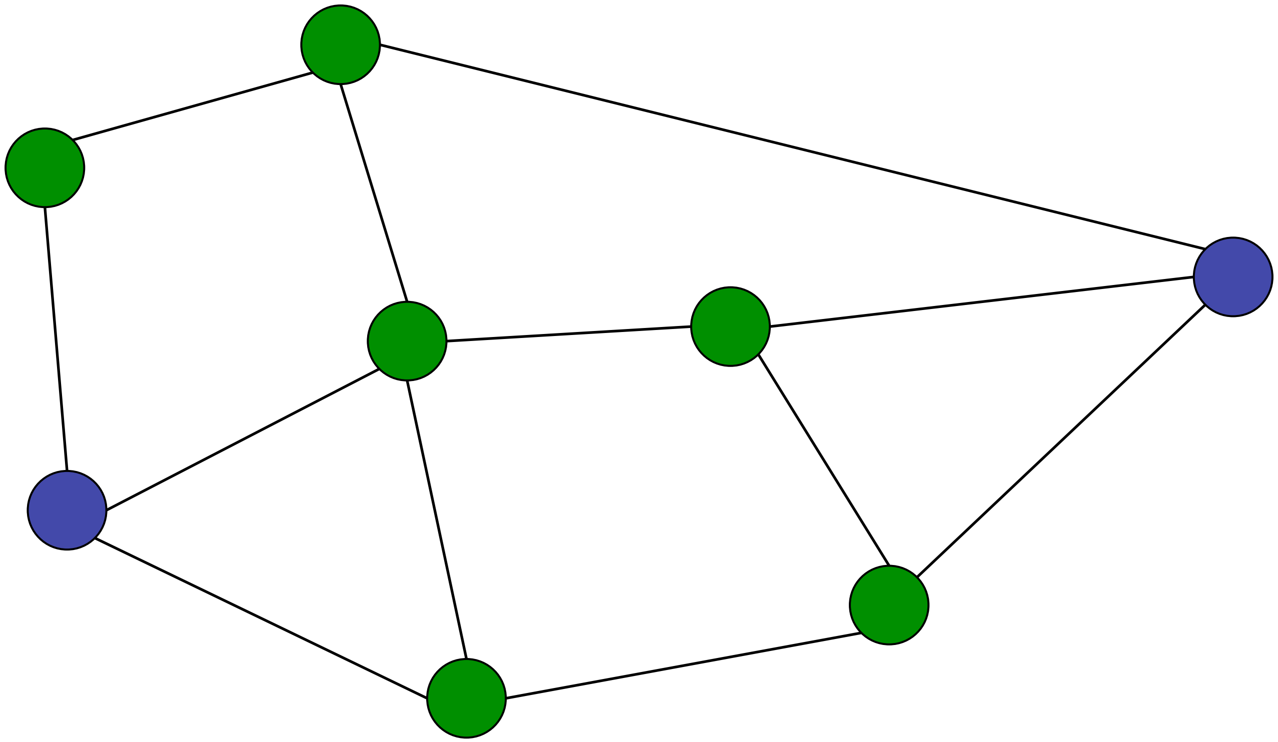
Output:

A pair of nodes (v,w) such that the shortest path from v to w is maximal.

Diameter of a Graph

Example:

diameter = 3



Diameter of a Graph

Examples:

In 1999, the diameter of the world-wide-web was (supposedly) 19.

Milgram claimed in the 1960’s that the diameter of the United Social social network was 6.

(“Six degrees of separation”)

Diameter of the Erdos collaboration graph is 23.

All Pairs Shortest Paths

Input:

- Weighted, directed graph G = (V,E)

Output:

- dist[v,w] : shortest distance from v to w, for all pairs of vertices (v,w)

All Pairs Shortest Paths

Input:

- Weighted, directed graph G = (V,E)

Output:

- dist[v,w] : shortest distance from v to w, for all pairs of vertices (v,w)

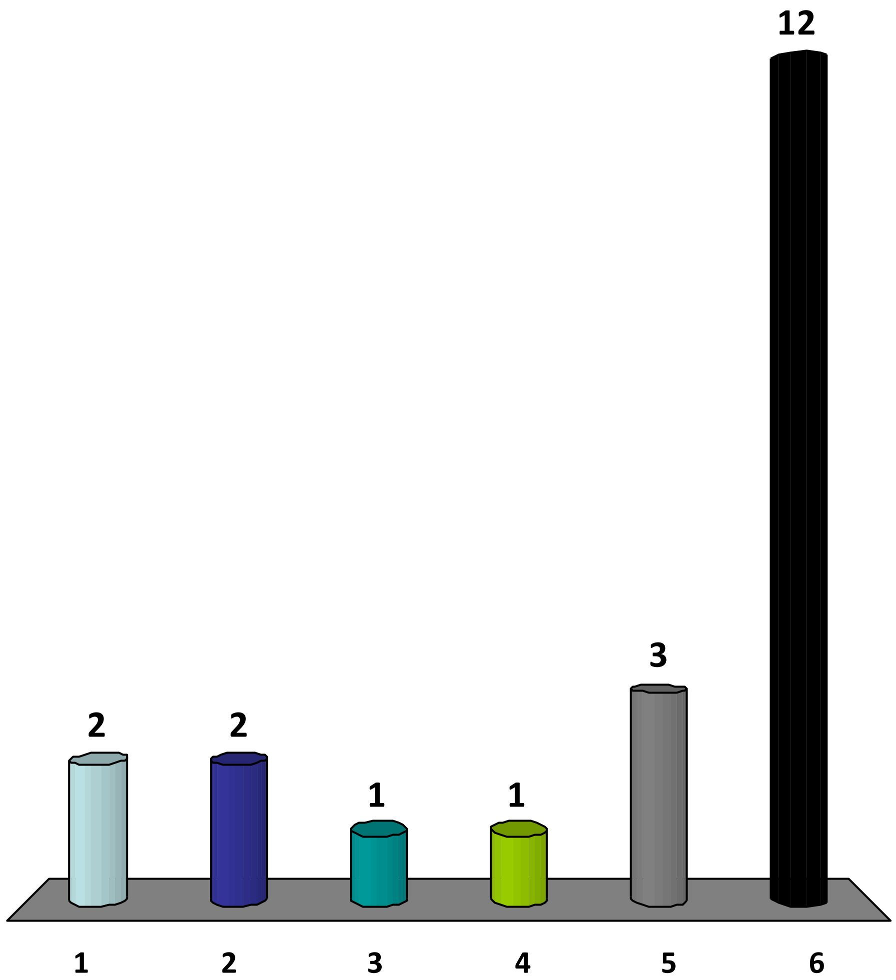
Solution:

- Run single-source-shortest paths once for every vertex v in the graph.

What is the running time of running SSSP for every vertex in V?

1. O(VE)
2. $O(V^2E)$
3. $O(V^2 + E^2)$
4. O(E log V)
5. $O(V^2 \log E)$
6. O(VE log V)

Response Counter



All Pairs Shortest Paths

Solution:

- Run single-source-shortest paths once for every vertex v in the graph .
- Assume weights are all positive…

Note:

- In a sparse graph where E = O(V): $O(V^2 \log V)$
  - We don't know how to do any better.
- In an unweighted graph, use BFS: $O(V(E+V))$
  - In dense graph: $O(V^3)$
  - In sparse graph: $O(V^2)$

Floyd-Warshall

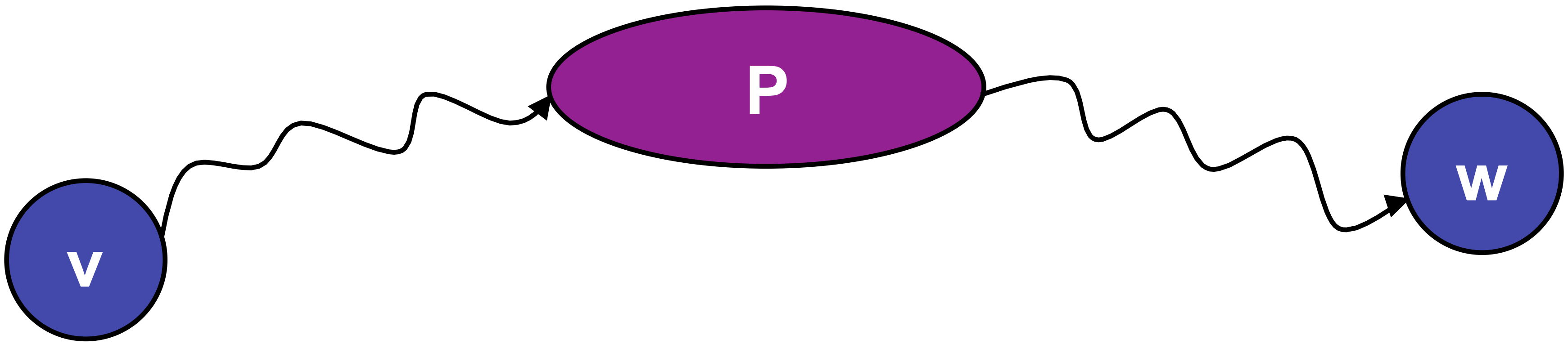
Dynamic programming:

- Shortest paths have optimal sub-structure:
  - If P is the shortest path (u→v→w), then P contains the shortest path from (u→v) and from (v→w).
- Shortest paths have overlapping subproblems
  - Many shortest path calculations depends on the same sub-pieces.
- Hard question: what are the right subproblems?

Floyd-Warshall

Dynamic programming:

Let S[v,w,P] be the shortest path from v to w that only uses intermediate nodes in the set P.



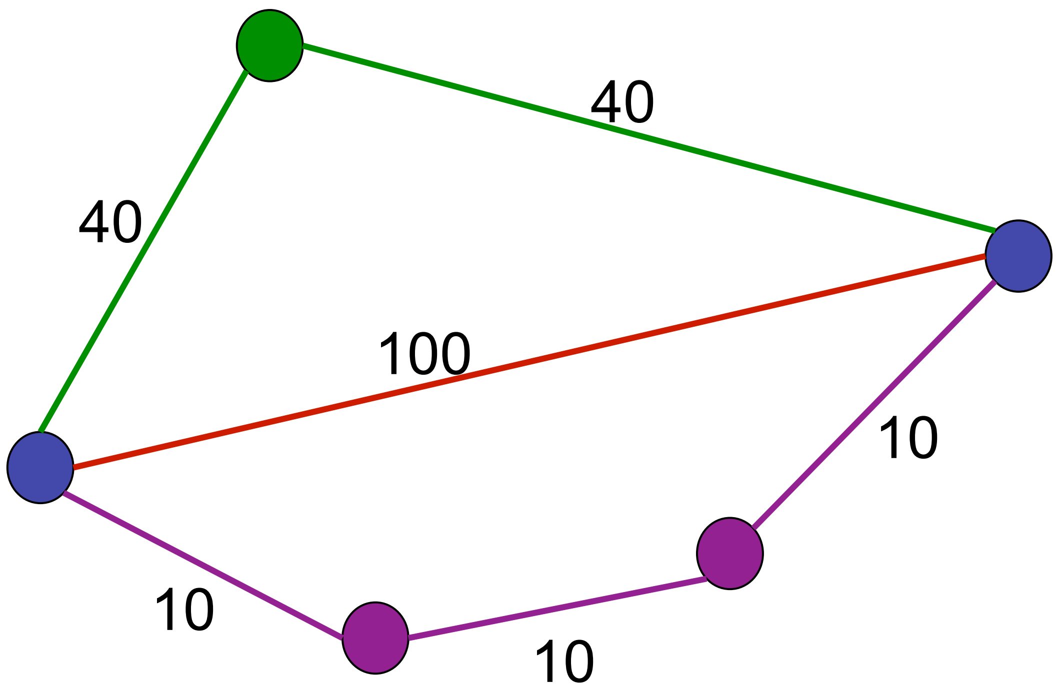
Floyd-Warshall

Let S[v,w,P] be the shortest path from v to w that only uses intermediate nodes only in the set P.

P1 = no nodes (empty set)

P2 = green nodes

P3 = purple nodes



Floyd-Warshall

Let S[v,w,P] be the shortest path from v to w that only uses intermediate nodes only in the set P.

P1 = no nodes (empty set)

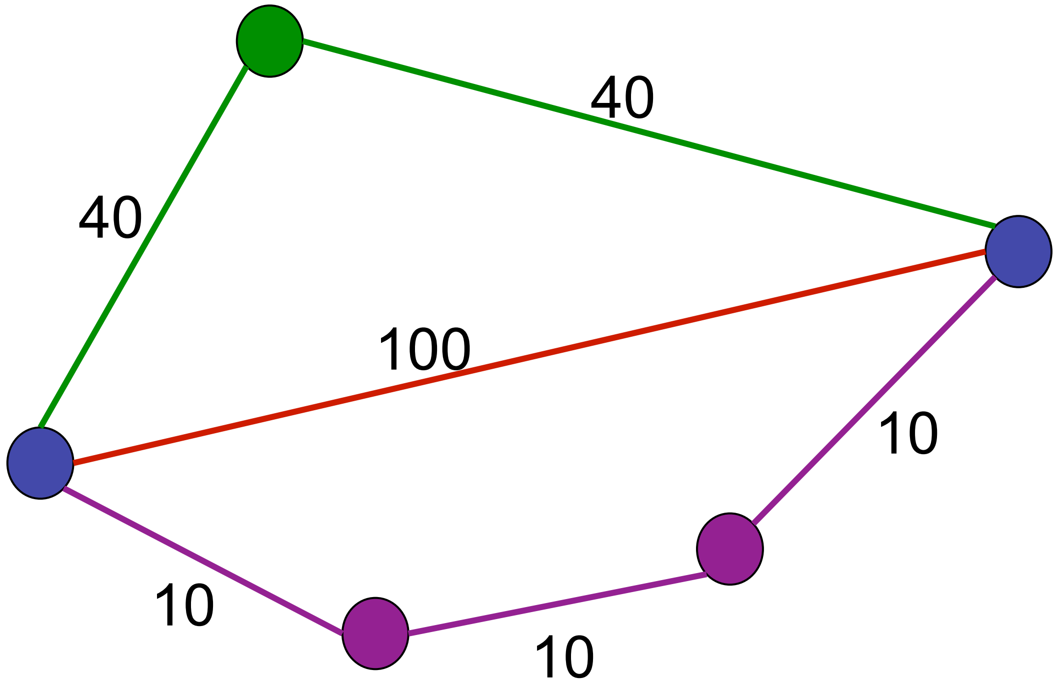
P2 = green nodes

P3 = purple nodes

S(v,w,P1) = 100

S(v,w,P2) = 80

S(v,w,P3) = 30

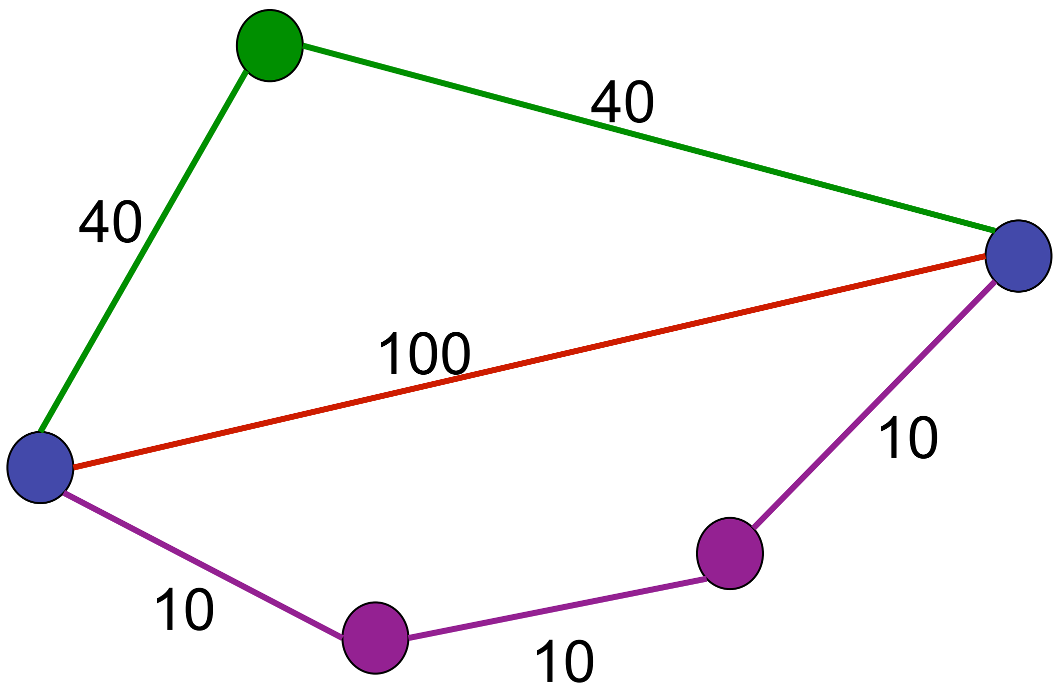


Floyd-Warshall

Let S[v,w,P] be the shortest path from v to w that only uses intermediate nodes only in the set P.

Base case:

S[v, w, ∅] = E[v,w]



E[v,w] = weight of edge from v to w.

Floyd-Warshall

Which sets P?

$P_0 = \emptyset$

$P_1 = \{1\}$

$P_2 = \{1, 2\}$

$P_3 = \{1, 2, 3\}$

$P_4 = \{1, 2, 3, 4\}$

…

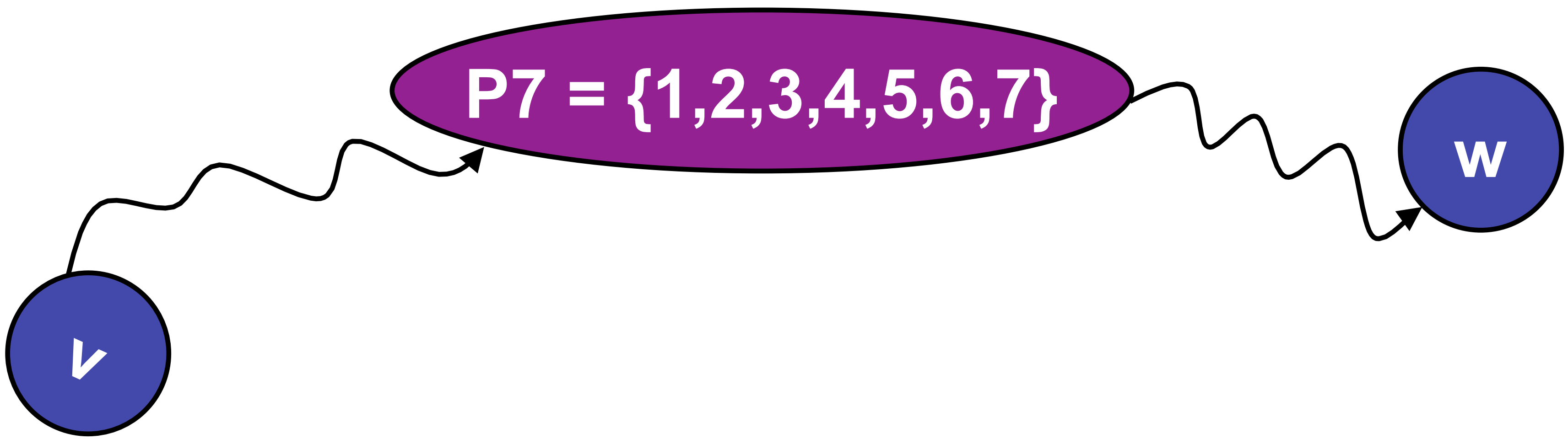
$P_n = \{1, 2, 3, 4, \ldots, n\}$

Floyd-Warshall

Use the precalculated subproblems:

Assume we have calculated $S[v,w,P_7] = 42$.

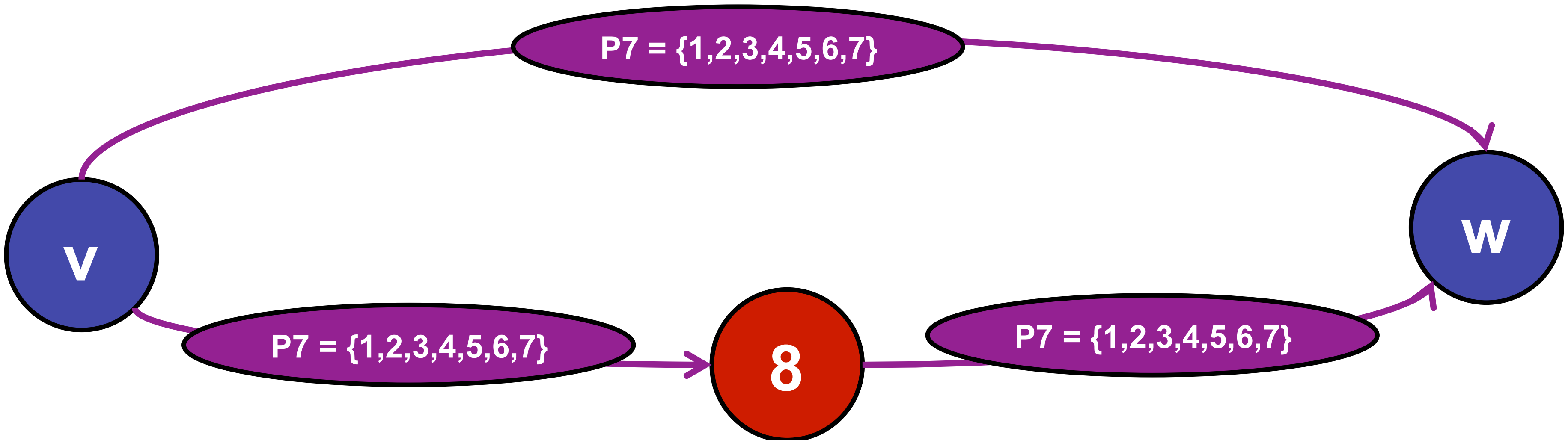
How do we calculate $S[v,w,P_8]$?



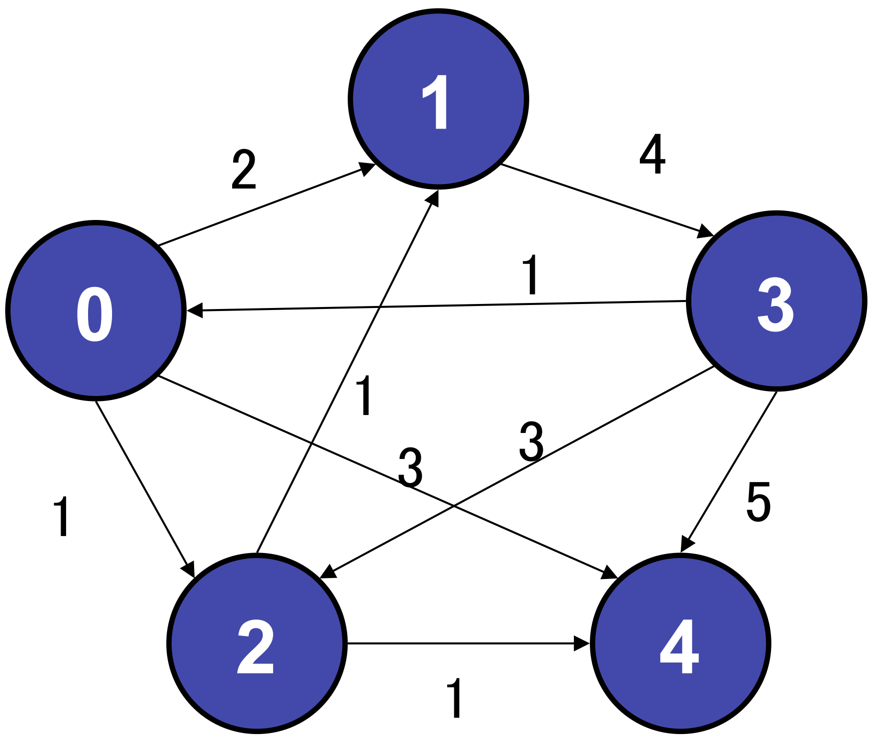
Floyd-Warshall

Use the precalculated subproblems:

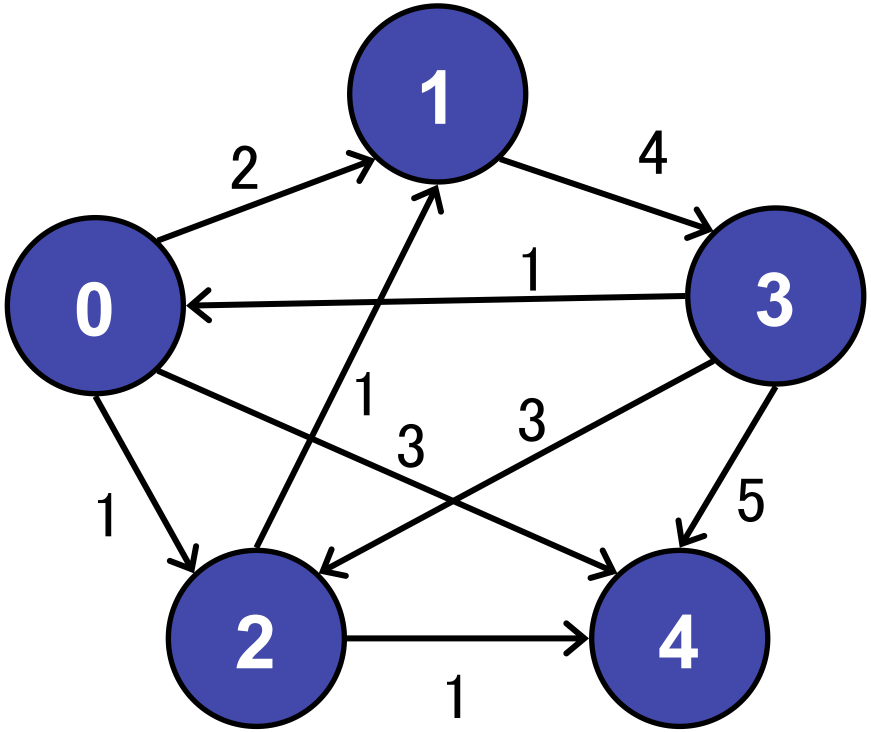
$$S[v,w,P_8] = \min(\, S[v, w, P_7],\ S[v, 8, P_7] + S[8, w, P_7])$$



Example:

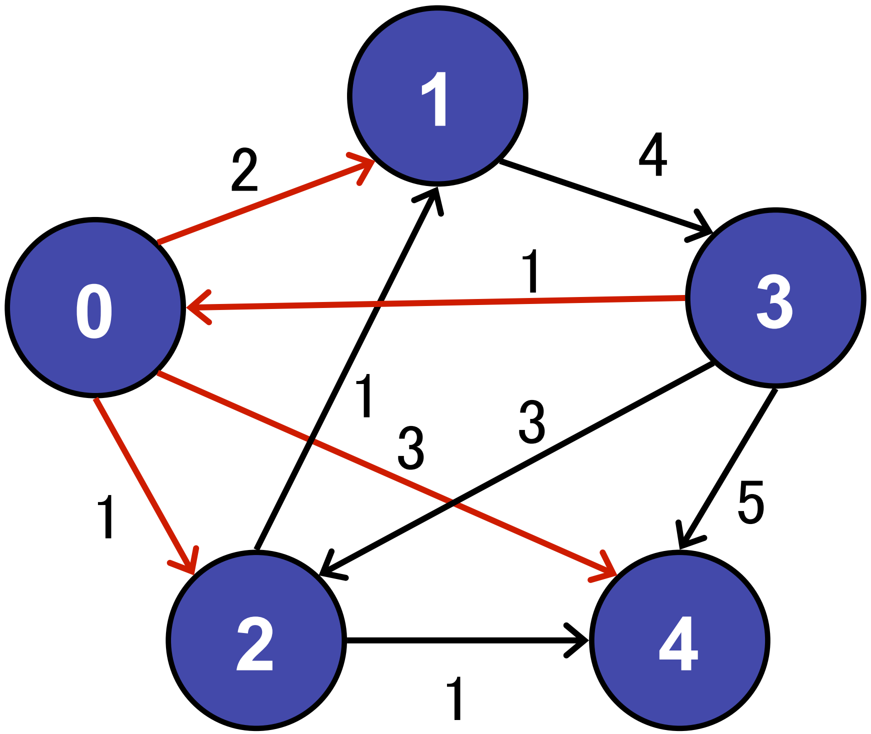


Initially:



	0	1	2	3	4
0	0	2	1	∞	3
1	∞	0	∞	4	∞
2	∞	1	0	∞	1
3	1	∞	3	0	5
4	∞	∞	∞	∞	0

Step: P = {0}

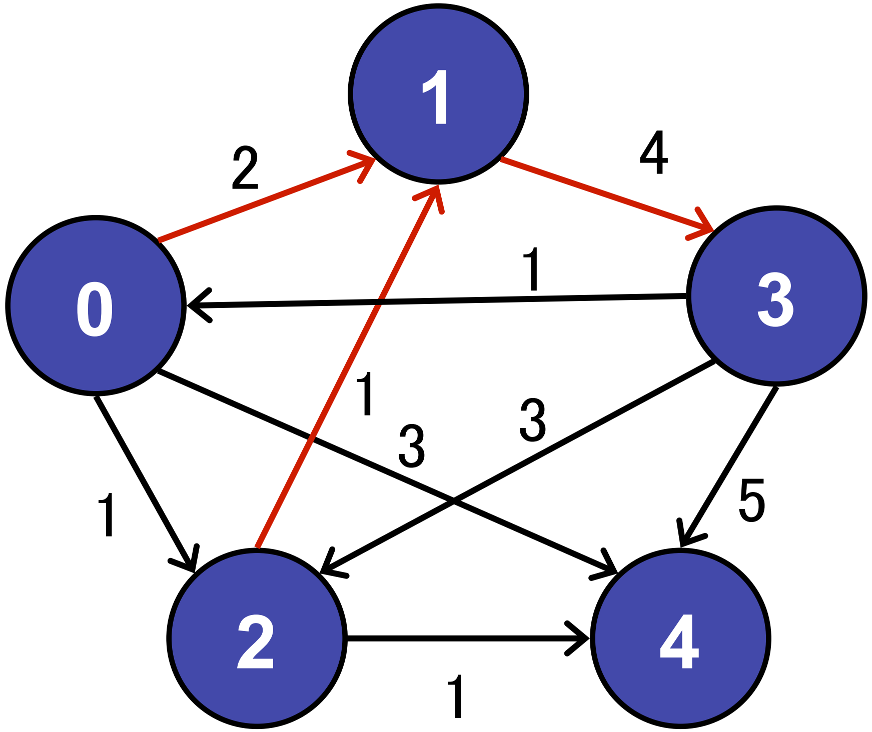


	0	1	2	3	4
0	0	**2**	**1**	∞	**3**
1	∞	0	∞	4	∞
2	∞	1	0	∞	1
3	**1**	∞	3	0	5
4	∞	∞	∞	∞	0

⇩

	0	1	2	3	4
0	0	2	1	∞	3
1	∞	0	∞	4	∞
2	∞	1	0	∞	1
3	1	**3**	**2**	0	**4**
4	∞	∞	∞	∞	0

Step: P = {0, 1}

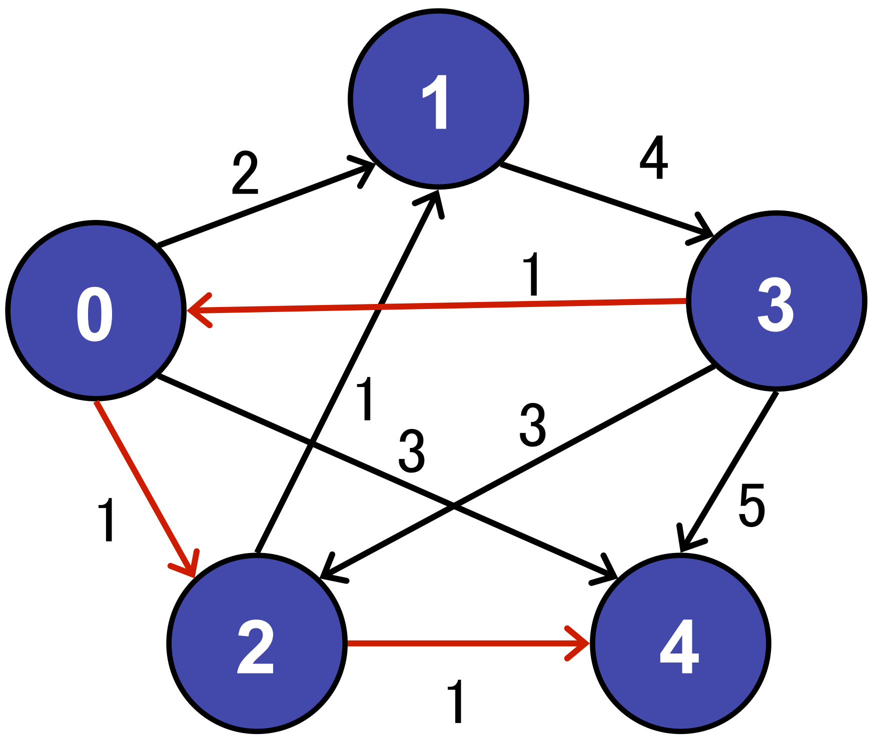


	0	1	2	3	4
0	0	**2**	1	∞	3
1	∞	0	∞	**4**	∞
2	∞	**1**	0	∞	1
3	1	3	2	0	4
4	∞	∞	∞	∞	0

⇩

	0	1	2	3	4
0	0	2	1	**6**	3
1	∞	0	∞	4	∞
2	∞	1	0	**5**	1
3	1	3	2	0	4
4	∞	∞	∞	∞	0

Step: P = {0, 1, 2}

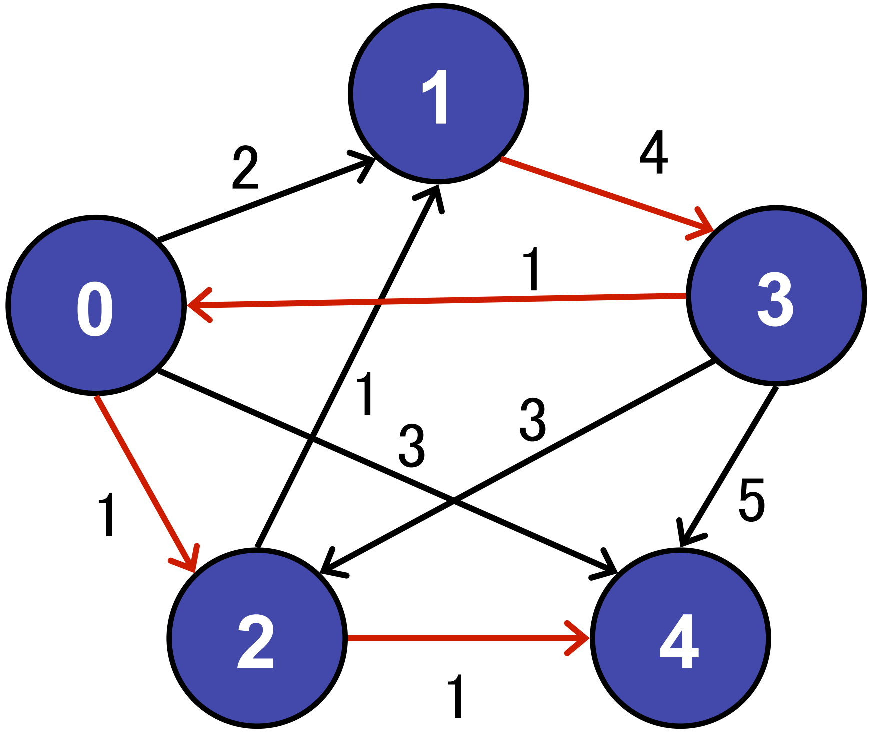


	0	**1**	**2**	**3**	**4**
0	0	2	**1**	6	3
1	∞	0	∞	4	∞
2	∞	1	0	5	**1**
3	1	3	**2**	0	4
4	∞	∞	∞	∞	0

⇩

	0	**1**	**2**	**3**	**4**
0	0	2	1	6	**2**
1	∞	0	∞	4	∞
2	∞	1	0	5	1
3	1	3	2	0	**3**
4	∞	∞	∞	∞	0

Step: P = {0, 1, 2, 3}

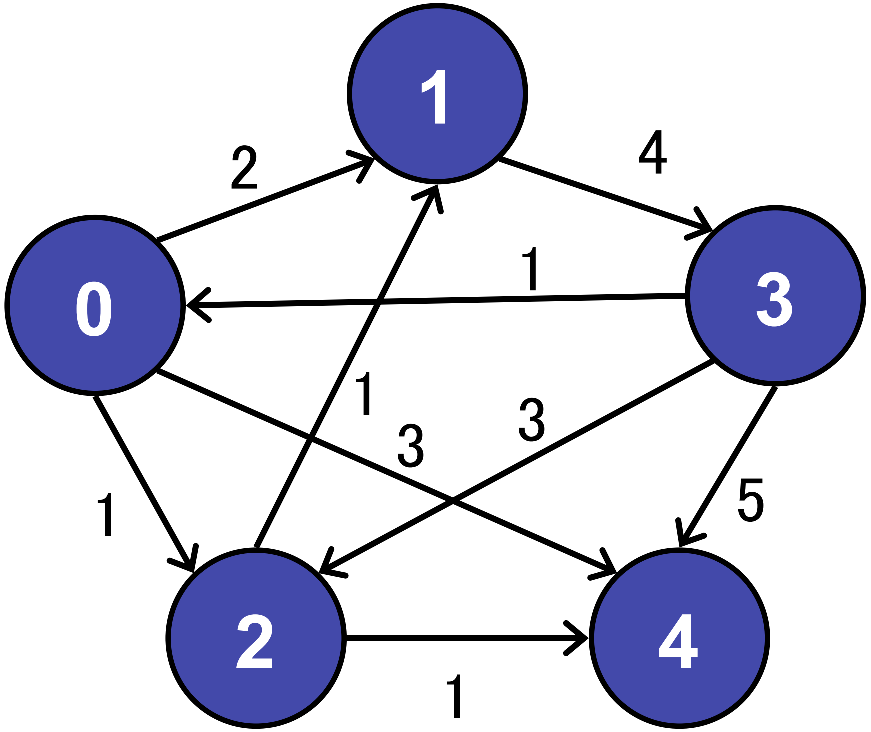


	0	1	2	3	4
0	0	2	1	6	2
1	∞	0	∞	**4**	∞
2	∞	1	0	**5**	1
3	**1**	3	**2**	0	**3**
4	∞	∞	∞	∞	0

⇩

	0	1	2	3	4
0	0	2	1	6	2
1	**5**	0	**6**	4	**7**
2	**6**	1	0	5	1
3	1	3	2	0	3
4	∞	∞	∞	∞	0

Done: P = {0, 1, 2, 3, 4}

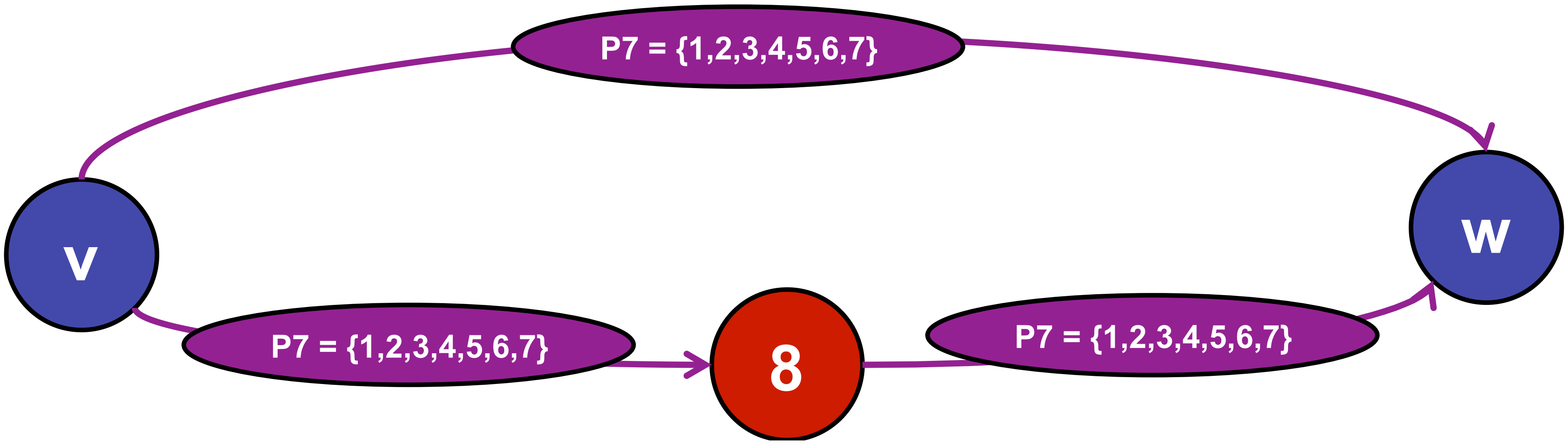


	0	1	2	3	4
0	0	2	1	6	2
1	5	0	6	4	7
2	6	1	0	5	1
3	1	3	2	0	3
4	∞	∞	∞	∞	0

Floyd-Warshall

Use the precalculated subproblems:

$$S[v,w,P_8] = \min(\ S[v, w, P_7],\ S[v, 8, P_7] + S[8, w, P_7]$$



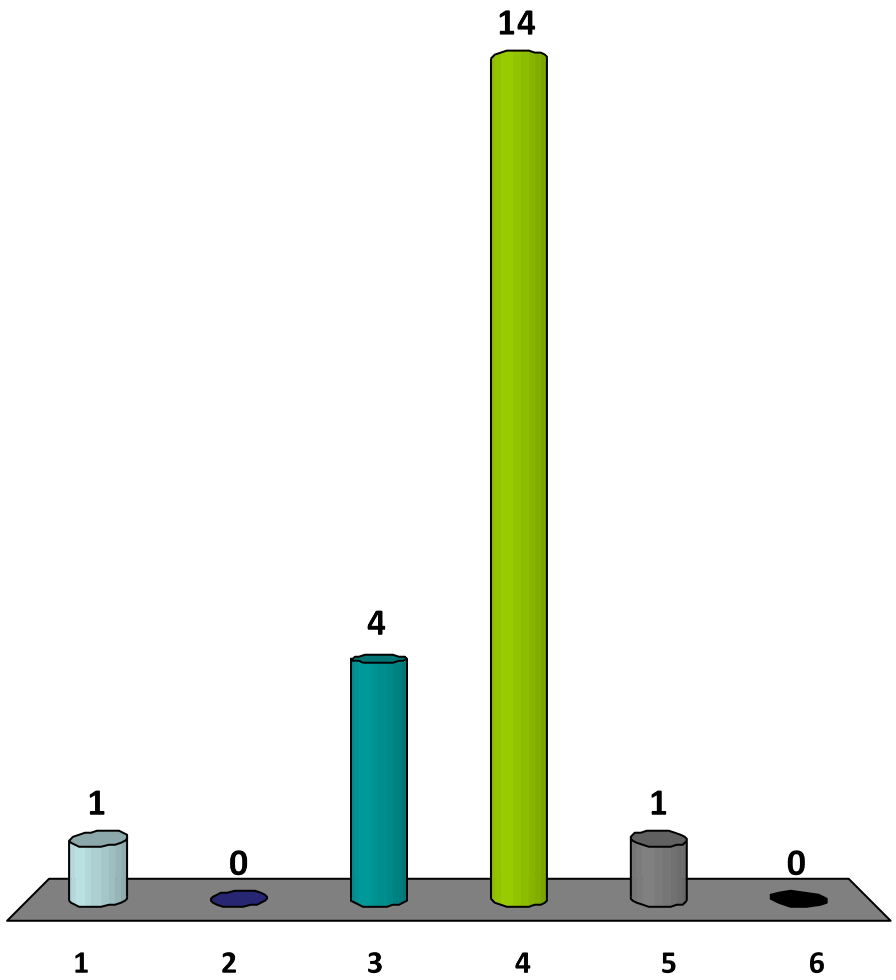
```
int[][] APSP(E){ // Adjacency matrix E
    int[][][] S = new int[V.length][V.length][V.length];

    // Initialize every pair of nodes for k=0
    for (int v=0; v<V.length; v++)
        for (int w=0; w<V.length; w++)
                S[0][v][w] = E[v][w];

    // For sets P0, P1, P2, P3, …
    for (int k=0; k<V.length; k++)
        // For every pair of nodes
        for (int v=0; v<V.length; v++)
                for (int w=0; w<V.length; w++) {
                        int currD = S[k][v][w];
                        int toK = S[k][v][k];
                        int fromK = S[k][k][w];
                        S[k+1][v][w] = Math.min(currD, toK+fromK);
                }
    return S;
}
```

```
int[][] APSP(E){ // Adjacency matrix E
    int[][] S = new int[V.length][V.length];//create memo table S

    // Initialize every pair of nodes
    for (int v=0; v<V.length; v++)
       for (int w=0; w<V.length; w++)
              S[v][w] = E[v][w];

    // For sets P0, P1, P2, P3, …
    for (int k=0; k<V.length; k++)
       // For every pair of nodes
       for (int v=0; v<V.length; v++)
              for (int w=0; w<V.length; w++) {
                     int currD = S[v][w];
                     int toK = S[v][k];
                     int fromK = S[k][w];
                     S[v][w] = Math.min(currD, toK+fromK);
              }
    return S;
}
```

```
int[][] APSP(E){ // Adjacency matrix E
    int[][] S = new int[V.length][V.length];//create memo table S

    // Initialize every pair of nodes
    for (int v=0; v<V.length; v++)
       for (int w=0; w<V.length; w++)
             S[v][w] = E[v][w];

    // For sets P0, P1, P2, P3, …, for every pair (v,w)
    for (int k=0; k<V.length; k++)
       for (int v=0; v<V.length; v++)
             for (int w=0; w<V.length; w++)
                    S[v][w] = Math.min(S[v][w], S[v][k]+S[k][w]);
    return S;
}
```

What is the running time of Floyd Warshall?

1. $O(VE)$
2. $O(VE^2)$
3. $O(V^2E)$
4. $O(V^3)$
5. $O(V^3 \log E)$
6. $O(V^4)$

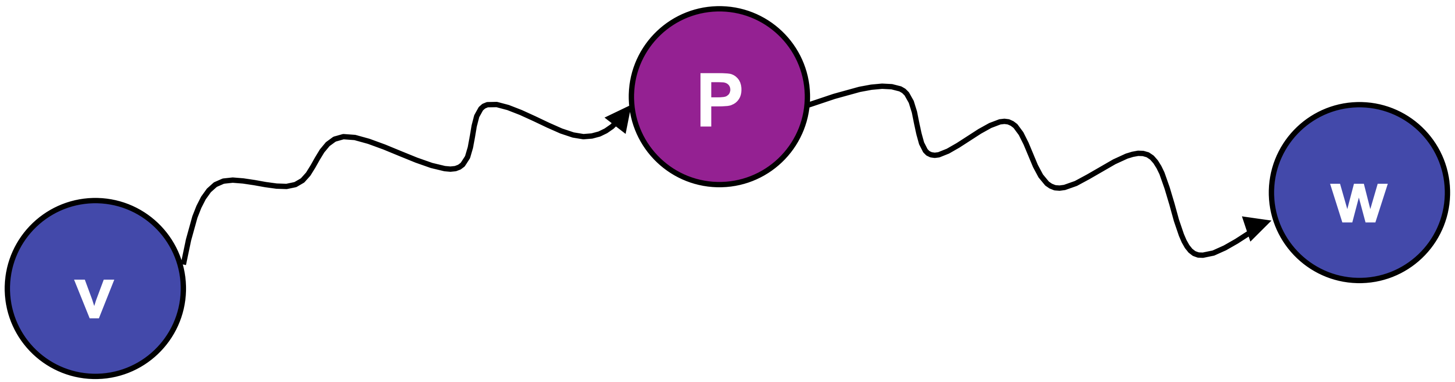


```
int[][] APSP(E){ // Adjacency matrix E
    int[][] S = new int[V.length][V.length];//create memo table S

    // Initialize every pair of nodes
    for (int v=0; v<V.length; v++)
       for (int w=0; w<V.length; w++)
              S[v][w] = E[v][w]

    // For sets P0, P1, P2, P3, …, for every pair (v,w)
    for (int k=0; k<V.length; k++)
       for (int v=0; v<V.length; v++)
              for (int w=0; w<V.length; w++)
                     S[v][w] = Math.min(S[v][w], S[v][k]+S[k][w]);
    return S;
}
```

Floyd-Warshall

Dynamic programming:

Let S[v,w,P] be the shortest path from v to w that only uses intermediate nodes only in the set P.



Floyd-Warshall Variants

Path Reconstruction:

- Return the **actual** path from (v,w).
- How to represent it succinctly?

Floyd-Warshall Variants

Transitive Closure:

- Return a matrix M where:
  - M[v,w] = 1 if there exists a path from v to w;
  - M[v,w] = 0, otherwise.

Roadmap

Today: Dynamic Programming

- DP Basics
- Longest Increasing Subsequence
- Prize Collecting
- Vertex Cover on a Tree
- All-Pairs-Shortest-Paths