Physics 2415 Homework

Hanzhi Zhou

School of Engineering and Applied Science
University of Virginia

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Homework 0

Problem 0.1

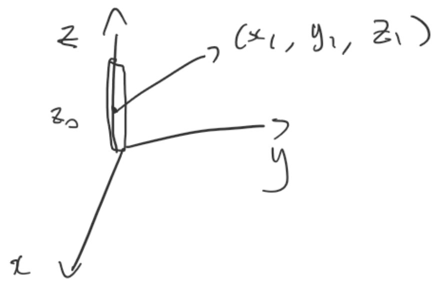
a)

$$m = \rho V = \rho \pi R^2 L$$

b)

$$\Delta m = \rho \Delta V = \rho \pi R^2 \Delta z$$

c)



$$\vec{r} = x_1 \hat{i} + y_1 \hat{j} + (z_1 - z_0)\hat{k}$$

d)

$$\hat{L} = \sin\theta \hat{i} + 0\hat{j} + \cos\theta \hat{k}$$

Problem 0.2

a)

$$\begin{aligned} a(t) &= v_0 \frac{d}{dt}\left(e^{-\gamma t} \sin \omega t\right) \\ &= v_0 \left(-e^{-\gamma t} \sin \omega t + e^{-\gamma t} \cos \omega t\right) \end{aligned}$$

b)

$$
\begin{aligned}
y(t) &= v_0 \int e^{-\gamma t} \sin \omega t \; dt \\
&= -v_0 \frac{1}{\gamma} \left(e^{-\gamma t} \sin \omega t - \omega \int e^{-\gamma t} \cos \omega t \; dt \right) \\
&= -v_0 \frac{1}{\gamma} \left(e^{-\gamma t} \sin \omega t + \omega \frac{1}{\gamma} \left(e^{-\gamma t} \cos \omega t - \omega \int e^{-\gamma t} (-\sin \omega t) \; dt \right) \right) \\
&= -\frac{v_0}{\gamma} \left(e^{-\gamma t} \sin \omega t + \frac{\omega}{\gamma} e^{-\gamma t} \cos \omega t + \frac{\omega^2}{\gamma} \int e^{-\gamma t} (\sin \omega t) \; dt \right)
\end{aligned}
$$

It can be observed that the integral $\int e^{-\gamma t} \sin \omega t \; dt$ appears at the both side of the equations. We can rearrange the terms to solve for it

$$
\begin{aligned}
v_0 \int e^{-\gamma t} \sin \omega t \; dt + v_0 \frac{\omega^2}{\gamma^2} \int e^{-\gamma t} (\sin \omega t) \; dt &= -\frac{v_0}{\gamma} e^{-\gamma t} \left(\sin \omega t + \frac{\omega}{\gamma} \cos \omega t \right) \\
v_0 \left(1 + \frac{\omega^2}{\gamma^2} \right) \int e^{-\gamma t} \sin \omega t \; dt &= -\frac{v_0}{\gamma} e^{-\gamma t} \left(\sin \omega t + \frac{\omega}{\gamma} \cos \omega t \right)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
y(t) &= v_0 \int e^{-\gamma t} \sin \omega t \; dt \\
&= -\frac{v_0}{\gamma \left(1 + \frac{\omega^2}{\gamma^2} \right)} e^{-\gamma t} \left(\sin \omega t + \frac{\omega}{\gamma} \cos \omega t \right) \\
&= -\frac{v_0 \gamma}{\omega^2 + \gamma^2} e^{-\gamma t} \left(\sin \omega t + \frac{\omega}{\gamma} \cos \omega t \right) \\
&= -\frac{v_0}{\omega^2 + \gamma^2} e^{-\gamma t} \left(\gamma \sin \omega t + \omega \cos \omega t \right)
\end{aligned}
$$

Homework 1

Problem 1.1

a)

$$\sigma = \frac{Q}{\pi R^2}$$

b)

$$dq = \sigma 2\pi r' dr' = \frac{2Qr'dr'}{R^2}$$

c)

$$dE = \frac{1}{4\pi\epsilon_0} d \frac{dQ}{(d^2 + r^2)^{\frac{3}{2}}} = \frac{1}{4R^2\pi\epsilon_0} d \frac{2Qrdr}{(d^2 + r^2)^{\frac{3}{2}}} = \frac{d}{2R^2\pi\epsilon_0} \frac{Qrdr}{(d^2 + r^2)^{\frac{3}{2}}}$$

d)

$$\begin{aligned}E &= \int_0^R \frac{d}{2R^2\pi\epsilon_0} \frac{Qr}{(d^2 + r^2)^{\frac{3}{2}}} dr \\ &= \frac{Qd}{2R^2\pi\epsilon_0} \int_0^R \frac{r}{(d^2 + r^2)^{\frac{3}{2}}} dr\end{aligned}$$

e)

$$\begin{aligned}E &= \frac{Qd}{2R^2\pi\epsilon_0} \left[-\frac{1}{\sqrt{d^2 + r^2}} \right]_0^R \\ &= \frac{Q}{2\pi R^2\epsilon_0} \left(1 - \frac{d}{\sqrt{d^2 + R^2}} \right)\end{aligned}$$

Problem 1.2

a)

$$\begin{aligned}Q &= \int dQ \\ &= \int \lambda(\theta) dl \\ &= \int_0^\pi \lambda_0 \cos\theta R d\theta \\ &= 0\end{aligned}$$

b)

$$\begin{aligned}
dE_x &= \frac{1}{4\pi\epsilon_0}\frac{dQ}{R^2}\cos\theta \\
&= \frac{1}{4\pi\epsilon_0}\frac{\lambda_0\cos\theta R d\theta}{R^2}\cos\theta \\
&= \frac{\lambda_0}{4\pi\epsilon_0 R}\cos^2\theta d\theta \\
dE_y &= \frac{\lambda_0}{4\pi\epsilon_0 R}\cos\theta\sin\theta d\theta \\
\vec{E} &= \frac{\lambda_0}{4\pi\epsilon_0 R}\left(\int_0^\pi \cos^2\theta d\theta\hat{i} + \int_0^\pi \cos\theta\sin\theta d\theta\hat{j}\right) \\
&= \frac{\lambda_0}{4\pi\epsilon_0 R}\left(\frac{\pi}{2}\hat{i} + 0\hat{j}\right) \\
&= \frac{\lambda_0}{8\epsilon_0 R}\hat{i}
\end{aligned}$$

Problem 1.3

a)

i)

Upward if $z_0 > \frac{h}{2}$ since there're more charges at the bottom, downward if $z_0 < \frac{h}{2}$ since there're more charges at the top

ii)

When $z_0 = \frac{h}{2}$. This point is the center of the cylindrical shell, and the electrical forces should be zero by symmetry.

iii)

When $z_0 >> h$, the cylindrical shell is a ring with radius R. Therefore,

$$E = \frac{1}{4\pi\epsilon_0}\frac{Qz_0}{(z_0^2+R^2)^{\frac{3}{2}}} = \frac{1}{4\pi\epsilon_0}\frac{\sigma 2\pi R h z_0}{(z_0^2+R^2)^{\frac{3}{2}}} = \frac{\sigma R h}{2\epsilon_0}\frac{z_0}{(z_0^2+R^2)^{\frac{3}{2}}}$$

When $z_0 >> h$ and $z_0 >> R$, the cylindrical shell acts like a point charge.

$$E = \frac{1}{4\pi\epsilon_0}\frac{\sigma 2\pi R h}{z_0^2} = \frac{\sigma R h}{2\epsilon_0}\frac{1}{z_0^2}$$

b)

$$dQ = \sigma 2\pi R dz$$

c)

$$d = |z_0 - z|$$

d)

$$dE = \frac{1}{4\pi\epsilon_0}\frac{d(dQ)}{(d^2+R^2)^{\frac{3}{2}}} = \frac{1}{4\pi\epsilon_0}\frac{(z_0-z)\sigma 2\pi R}{((z_0-z)^2+R^2)^{\frac{3}{2}}}dz = \frac{\sigma R}{2\epsilon_0}\frac{z_0-z}{((z_0-z)^2+R^2)^{\frac{3}{2}}}dz$$

e)

$$E = \frac{\sigma R}{2\epsilon_0}\int_0^h \frac{z_0-z}{((z_0-z)^2+R^2)^{\frac{3}{2}}}dz$$

f)

By calculator,

$$E = \frac{\sigma R}{2\epsilon_0}\left(\frac{1}{\sqrt{R^2+(z_0-h)^2}} - \frac{1}{\sqrt{R^2+z_0^2}}\right)$$

The expectations **i)** and **ii)** are all met.

g)

Rewriting the formula in **f)**

$$E = \frac{\sigma R}{2\epsilon_0}\left[\frac{1}{z_0-h}\left(1+\frac{R^2}{(z_0-h)^2}\right)^{-\frac{1}{2}} - \frac{1}{z_0}\left(1+\frac{R^2}{z_0^2}\right)^{-\frac{1}{2}}\right]$$

By first order binomial approximation, $(1+x)^n \approx 1+nx$

$$E = \frac{\sigma R}{2\epsilon_0}\left[\frac{1}{z_0-h}\left(1-\frac{1}{2}\frac{R^2}{(z_0-h)^2}\right) - \frac{1}{z_0}\left(1-\frac{1}{2}\frac{R^2}{z_0^2}\right)\right]$$

$$= \frac{\sigma R}{2\epsilon_0}\left[\frac{1}{z_0-h} - \frac{1}{z_0} - \frac{1}{2}\frac{R^2}{(z_0-h)^3} + \frac{1}{2}\frac{R^2}{z_0^3}\right]$$

Because z_0 is large, we ignore the terms containing the cube of z_0 in the denominator.

$$E = \frac{\sigma R}{2\epsilon_0}\left[\frac{1}{z_0-h} - \frac{1}{z_0}\right]$$

$$= \frac{\sigma R}{2\epsilon_0}\frac{h}{(z_0-h)z_0}$$

$$\approx \frac{\sigma R}{2\epsilon_0}\frac{h}{z_0^2}$$

Homework 2

Problem 2.1

a)

$$E_r = \frac{1}{4\pi\epsilon_0}\frac{Q}{d^2} = \frac{1}{4\pi\epsilon_0}\frac{Q}{z_0^2 + r^2}$$

b)

$$E_\perp = E_r\frac{z_0}{\sqrt{z_0^2 + r^2}} = \frac{1}{4\pi\epsilon_0}\frac{Qz_0}{(z_0^2 + r^2)^{\frac{3}{2}}}$$

c)

$$d\Phi = E_\perp dA = \frac{1}{4\pi\epsilon_0}\frac{Qz_0}{(z_0^2 + r^2)^{\frac{3}{2}}}2\pi r dr = \frac{1}{2\epsilon_0}\frac{Qz_0}{(z_0^2 + r^2)^{\frac{3}{2}}}r dr$$

d)

$$\begin{aligned}
\Phi &= \int d\Phi \\
&= \int_0^{r_0} \frac{1}{2\epsilon_0}\frac{Qz_0}{(z_0^2 + r^2)^{\frac{3}{2}}} r dr \\
&= \frac{Qz_0}{2\epsilon_0}\int_0^{r_0} \frac{r}{(z_0^2 + r^2)^{\frac{3}{2}}} dr \\
&= \frac{Qz_0}{2\epsilon_0}\left(\frac{1}{z_0} - \frac{1}{\sqrt{r_0^2 + z_0^2}}\right) \\
&= \frac{Q}{2\epsilon_0}\left(1 - \frac{z_0}{\sqrt{r_0^2 + z_0^2}}\right)
\end{aligned}$$

e)

$$\Phi = \frac{Q}{2\epsilon_0}(1 - \cos\theta_0)$$

Problem 2.2

a)

i)

First calculate the charge as a function of radius for $r \le r_0$

$$Q(r) = \int \rho \, dV = \int_0^r \frac{C}{r'} 4\pi r'^2 dr' = 4C\pi \int_0^r r' dr' = 4\pi C \frac{1}{2} r^2 = 2\pi C r^2$$

For $r \le r_0$

$$\Phi(r) = \frac{Q}{\epsilon_0} = \frac{2\pi C r^2}{\epsilon_0} = E(r) \times 4\pi r^2$$

$$E(r) = \frac{C}{2\epsilon_0}$$

For $r > r_0$

$$\Phi(r) = \frac{Q}{\epsilon_0} = \frac{2\pi C r_0^2}{\epsilon_0} = E(r) \times 4\pi r^2$$

$$E(r) = \frac{C r_0^2}{2\epsilon_0 r^2}$$

ii)

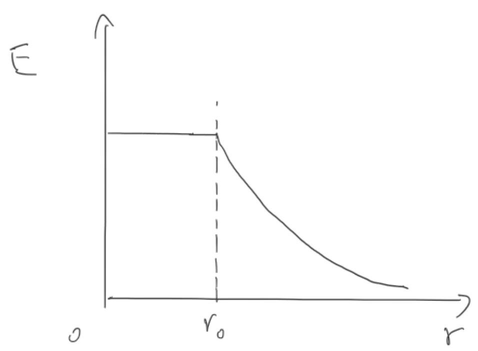
For $r > r_0$

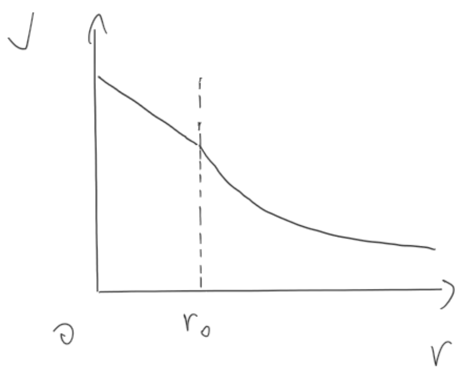
$$V(r) = -\frac{C r_0^2}{2\epsilon_0} \int_\infty^r \frac{1}{r'^2} dr' = \frac{C r_0^2}{2\epsilon_0} \frac{1}{r}$$

For $r \le r_0$

$$\begin{aligned} V(r) &= V(r_0) - \int_{r_0}^r \frac{C}{2\epsilon_0} dr' \\ &= \frac{C r_0}{2\epsilon_0} - \frac{C}{2\epsilon_0} [r - r_0] \\ &= \frac{C}{\epsilon_0} \left(r_0 - \frac{r}{2} \right) \end{aligned}$$

iii)





b)

i)

First find the charge enclosed as a function of radius.

For $r < a$, $Q(r) = 0$, because charges reside on the surface of a conductor.
For $a < r < b$, $Q(r) = +2Q$.

For $b < r < c$, $Q(r) = 0$, because the $+2Q$ inside attracts $-2Q$ charges on the inner surface of the conducting shell, and the net charge enclosed is zero.

For $r > c$, $Q(r) = +Q$.

Then, use Gauss's law to find the electric field

$$\Phi(r) = \frac{Q(r)}{\epsilon_0} = E(r)\ 4\pi r^2$$

$$E(r) = \frac{Q(r)}{4\pi\epsilon_0 r^2}$$

For $r < a$,

$$E(r) = \frac{0}{4\pi\epsilon_0 r^2} = 0$$

For $a < r < b$,

$$E(r) = \frac{2Q}{4\pi\epsilon_0 r^2} = \frac{Q}{2\pi\epsilon_0 r^2}$$

For $b < r < c$,

$$E(r) = \frac{0}{4\pi\epsilon_0 r^2} = 0$$

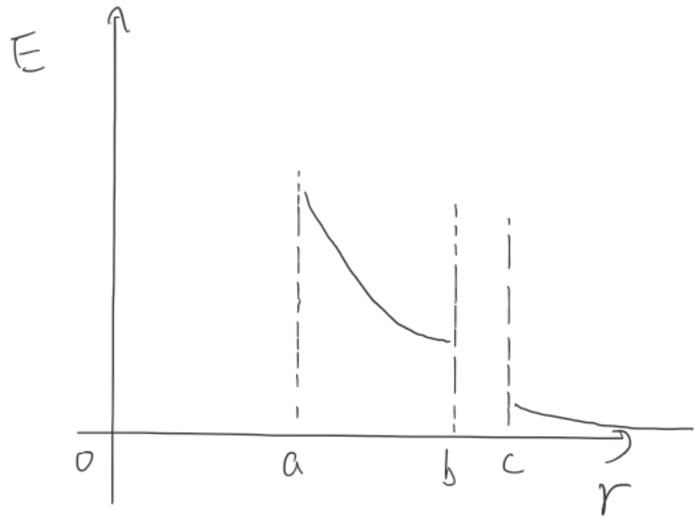
For $r > c$,

$$E(r) = \frac{Q}{4\pi\epsilon_0 r^2}$$

ii)

-2Q amount of charges appear on the inner surface of the shell. +Q amount of charges appear on the outer surface of the shell.

iii)



Problem 2.3

a)

$$dV = \frac{dQ}{4\pi\epsilon_0}\frac{1}{r} = \frac{\lambda dz}{4\pi\epsilon_0}\frac{1}{\sqrt{x^2+z^2}}$$

b)

$$\begin{aligned} V(x) &= \int dV \\ &= \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{\lambda}{4\pi\epsilon_0}\frac{1}{\sqrt{x^2+z^2}}dz \\ &= \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{\sqrt{4x^2+L^2}+L}{2|x|}\right) \end{aligned}$$

c)

$$\begin{aligned} dV &= \frac{\lambda dz}{4\pi\epsilon_0}\frac{1}{z-z'} \\ V(z) &= \frac{\lambda}{4\pi\epsilon_0}\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{1}{z-z'}dz' \\ &= -\frac{\lambda}{4\pi\epsilon_0}\ln\left|\frac{2z-L}{2z+L}\right| \end{aligned}$$

d)

$$W_P = q_e V_x(0.25) = -1.60\times 10^{-19}C \times -51923.6V = 8.308\times 10^{-15}J$$

e)

$V_z(0.75) = -28943.5V > V_x(0.25)$.

Therefore, point Q has a higher potential.

f)

$W = q_e(V_z(0.75) - V_x(0.25)) = -1.60 \times 10^{-19}C(-28943.5V - -51923.6V) = -22980\ eV$

Therefore, the potential energy of the electron decreases by 22980 eV

Homework 3

Problem 3.1

a)

$$V_{load} = \varepsilon \frac{R_{load}}{R_{load} + r}$$
$$P_{load} = \frac{V_{load}^2}{R_{load}} = \varepsilon^2 \frac{R_{load}}{(R_{load} + r)^2}$$

b)

$$\frac{d}{dR}\left(\frac{R}{(R+r)^2}\right) = \frac{r-R}{(R+r)^3} = 0$$
$$R = r$$

Therefore the power will be largest when $R_{load} = r$

c)

$$P_{total} = I^2 r + I^2 R = 2I^2 r$$

Therefore 50% of the power is dissipated by the battery

Problem 3.2

a)

$$C = K\varepsilon_0 \frac{A}{d}$$

b)

$$r = \rho \frac{d}{A}$$

c)

$$V(t) = \frac{Q(t)}{C}$$
$$I(t) = -\frac{dQ}{dt} = -\frac{V(t)}{r} = -\frac{Q(t)}{Cr}$$

Solve the separable differential equation:

$$\int \frac{1}{Q} dQ = \int \frac{1}{Cr} dt$$
$$\ln |Q| = -\frac{t}{Cr} + D$$
$$Q = De^{\frac{-t}{Cr}}$$

Given the initial value $Q(0) = Q_0$, $D = Q_0$.

$$\frac{1}{4} Q_0 = Q_0 e^{\frac{-t}{Cr}}$$
$$\ln \frac{1}{4} = \frac{-t}{Cr}$$
$$t = Cr \ln 4$$

Therefore $Cr \ln 4$ amount of time is required for the charge to decrease from Q_0 to $\frac{1}{4} Q_0$

d)

$$\begin{aligned} dE = I(t)^2 \ r \ dt &= \left(\frac{dQ}{dt} \right)^2 \ r \ dt \\ &= \left(-\frac{Q_0}{Cr} e^{\frac{-t}{Cr}} \right)^2 \ r \ dt \\ &= \frac{Q_0^2}{C^2 r} e^{\frac{-2t}{Cr}} \ dt \end{aligned}$$

e)

$$\begin{aligned} E &= \frac{Q_0^2}{C^2 r} \int_0^\infty e^{\frac{-2t}{Cr}} dt \\ &= \frac{Q_0^2}{C^2 r} \frac{C}{2r} \left(1 - e^{-\infty} \right) \\ &= \frac{1}{2} \frac{Q_0^2}{C} \end{aligned}$$

f)

The energy stored in a capacitor is $\frac{1}{2} \frac{Q_0^2}{C}$ when it is fully charged to Q_0. It is the same as the total energy dissipated through the resistor calculated in **e)**.

Problem 3.3

a)

$$\begin{aligned} V_r + V_C - \varepsilon &= 0 \\ rI_r + \frac{Q}{C} - \varepsilon &= 0 \\ I_r &= \frac{\varepsilon}{r} - \frac{Q}{Cr} \end{aligned}$$

b)

$$\begin{aligned} -V_C + V_R &= 0 \\ -\frac{Q}{C} + RI_R &= 0 \\ I_R &= \frac{Q}{CR} \end{aligned}$$

c)

$$I_r - I_R = \frac{dQ}{dt}$$

d)

$$\begin{aligned} &\frac{dQ}{dt} = \frac{\varepsilon}{r} - Q\left(\frac{1}{CR} + \frac{1}{Cr}\right) \\ &\frac{dQ}{dt} + \left(\frac{1}{CR} + \frac{1}{Cr}\right)Q = \frac{\varepsilon}{r} \\ &a = \frac{1}{C}\left(\frac{1}{R} + \frac{1}{r}\right) = \frac{1}{C}\left(\frac{R+r}{Rr}\right),\ b = \frac{\varepsilon}{r} \end{aligned}$$

e)

$$Q(t) = C\varepsilon\frac{R}{R+r} + Ae^{-\frac{1}{C}\left(\frac{R+r}{Rr}\right)t}$$

f)

$$\begin{aligned} Q(0) &= C\varepsilon\frac{R}{R+r} + A \\ A &= -C\varepsilon\frac{R}{R+r} \end{aligned}$$

Homework 4

Problem 4.1

a)

$$
\begin{aligned}
F &= \int I d\vec{l} \times \vec{B} \\
&= \int_{-\infty}^{\infty} I \begin{bmatrix} dx \\ 0 \\ 0 \end{bmatrix} \times B_0 \begin{bmatrix} e^{-\left(\frac{y}{b}\right)^2} \\ e^{-\left(\frac{x}{a}\right)^2} \\ 0 \end{bmatrix} \\
&= IB_0 \int_{-\infty}^{\infty} e^{-\left(\frac{x}{a}\right)^2} \hat{k}\ dx \\
&= IB_0 a\sqrt{\pi}\hat{k}
\end{aligned}
$$

b)

$$
\begin{aligned}
F &= \int_{-\infty}^{\infty} I \begin{bmatrix} 0 \\ dy \\ 0 \end{bmatrix} \times B_0 \begin{bmatrix} e^{-\left(\frac{y}{b}\right)^2} \\ e^{-\left(\frac{x}{a}\right)^2} \\ 0 \end{bmatrix} \\
&= IB_0 \int_{-\infty}^{\infty} -e^{-\left(\frac{y}{b}\right)^2} \hat{k}\ dy \\
&= -IB_0 b\sqrt{\pi}\hat{k}
\end{aligned}
$$

c)

$$
\begin{aligned}
F &= \int_{-\infty}^{\infty} I \begin{bmatrix} \cos(\arctan m) dx \\ \sin(\arctan m) dy \\ 0 \end{bmatrix} \times B_0 \begin{bmatrix} e^{-\left(\frac{y}{b}\right)^2} \\ e^{-\left(\frac{x}{a}\right)^2} \\ 0 \end{bmatrix} \\
&= IB_0 \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{1+m^2}} e^{-\left(\frac{x}{a}\right)^2} \right) \hat{k}\ dx - \int_{-\infty}^{\infty} \left(\frac{m}{\sqrt{1+m^2}} e^{-\left(\frac{y}{b}\right)^2} \right) \hat{k}\ dy \\
&= IB_0 \sqrt{\pi} \left(\frac{a}{\sqrt{1+m^2}} - \frac{mb}{\sqrt{1+m^2}} \right) \hat{k}
\end{aligned}
$$

d)

When m = 0

$$IB_0\sqrt{\pi}\left(\frac{a}{\sqrt{1+0}} - \frac{0m}{\sqrt{1+0}}\right)\hat{k} = IB_0 a\sqrt{\pi}\hat{k}$$

When m = ∞, $\lim_{m\to\infty} \frac{1}{\sqrt{1+m^2}} = 0$, $\lim_{m\to\infty} \frac{m}{\sqrt{1+m^2}} = 1$

$$IB_0\sqrt{\pi}\left(\frac{a}{\sqrt{1+m^2}} - \frac{mb}{\sqrt{1+m^2}}\right)\hat{k} = -IB_0 b\sqrt{\pi}\hat{k}$$

Problem 4.2

a)

$$R = \rho\frac{d}{A}$$

$$\vec{j} = \frac{I}{A} = -ne\vec{v_d}$$

$$\begin{aligned}\vec{v_d} &= -\frac{I}{Ane} = -\frac{V}{RAne} = -\frac{V}{\rho\frac{d}{A}Ane} = -\frac{V}{\rho dne}\\ &= -\frac{3}{-1.6\times10^{-2}\times0.01\times2\times10^{15}\times10^{6}\times1.6\times10^{-19}}\\ &= 58.59\ m/s\end{aligned}$$

$$\vec{E} = -\vec{v_d}\times\vec{B} = -58.59\times0.1T = -5.859N/C$$

$$V = -\int\vec{E}\cdot d\vec{l} = -\int_0^{0.002} -5.859dl = 1.1718\times10^{-2}V$$

b)

$$V_H = -\int -\vec{v_d}\times\vec{B} = -\int_{y1}^{y2}\frac{V}{\rho dne}B = -(y2-y1)\frac{VB}{\rho dne}$$

Given that $y_2 - y1 > 0$, $V_H > 0$ if $e < 0$ and $V_H < 0$ if $e > 0$

Homework 5

Problem 5.1

a)

$$
\begin{aligned}
d\vec{B} &= \frac{\mu_0}{4\pi}\frac{Id\vec{l}\times\hat{r}}{r^2} = \frac{\mu_0}{4\pi}\frac{Id\vec{l}\times\vec{r}}{r^3} \\
&= \frac{\mu_0}{4\pi}\frac{I}{\sqrt{R^2+y^2}^3}\begin{bmatrix}0\\-dy\\0\end{bmatrix}\times\begin{bmatrix}-R\\-y\\0\end{bmatrix} \\
&= \frac{\mu_0}{4\pi}\frac{IR}{(R^2+y^2)^{\frac{3}{2}}}dy\ \hat{k}
\end{aligned}
$$

$$
\vec{B} = \frac{\mu_0}{4\pi}\int_0^\infty \frac{IR}{(R^2+y^2)^{\frac{3}{2}}}dy\ \hat{k} = \frac{I\mu_0}{4\pi R}\hat{k}
$$

b)

$$
\begin{aligned}
d\vec{B} &= \frac{\mu_0}{4\pi}\frac{I}{\sqrt{R^2}^3}\begin{bmatrix}\cos(\theta+\frac{\pi}{2})\\\sin(\theta+\frac{\pi}{2})\\0\end{bmatrix}Rd\theta\times R\begin{bmatrix}-\cos\theta\\-\sin\theta\\0\end{bmatrix} \\
&= \frac{\mu_0}{4\pi}\frac{I}{R}d\theta\ \hat{k}
\end{aligned}
$$

$$
\vec{B} = \frac{\mu_0}{4\pi}\int_\pi^{\frac{3}{2}\pi}\frac{I}{R}d\theta\ \hat{k} = \frac{\mu_0 I}{8R}\ \hat{k}
$$

c)

$$
\begin{aligned}
d\vec{B} &= \frac{\mu_0}{4\pi}\frac{I}{\sqrt{R^2+x^2}^3}\begin{bmatrix}dx\\0\\0\end{bmatrix}\times\begin{bmatrix}-x\\R\\0\end{bmatrix} \\
&= \frac{\mu_0}{4\pi}\frac{IR}{(R^2+x^2)^{\frac{3}{2}}}dx\ \hat{k}
\end{aligned}
$$

$$
\vec{B} = \frac{\mu_0}{4\pi}\int_0^\infty \frac{IR}{(R^2+x^2)^{\frac{3}{2}}}dx\ \hat{k} = \frac{I\mu_0}{4\pi R}\hat{k}
$$

d)

$$
\vec{B}_{\text{total}} = 2\frac{I\mu_0}{4\pi R}\hat{k} + \frac{\mu_0 I}{8R}\ \hat{k} = 8.927\times 10^{-6}T\ \hat{k}
$$

e)

They are already indicated by unit vectors

Problem 5.2

a)

$$\varepsilon = \frac{d\Phi}{dt} = B\frac{dA}{dt} = BLv$$

b)

$$I = \frac{\varepsilon}{R} = \frac{BLv}{R}$$

c)

$$F = Id\vec{l} \times \vec{B} = -ILB = -\frac{(BL)^2 v}{R}$$

d)

$$F = ma = m\frac{dv}{dt} = -\frac{(BL)^2 v}{R}$$

$$\frac{1}{v}\frac{dv}{dt} = -\frac{(BL)^2}{Rm}$$

e)

$$\int \frac{1}{v} dv = \int -\frac{(BL)^2}{Rm} dt$$

$$\ln |v| = -\frac{(BL)^2}{Rm} t + C$$

$$v = Ce^{-\frac{(BL)^2}{Rm} t}$$

$$v(0) = C = v_0 \Rightarrow C = v_0$$

$$v = v_0 e^{-\frac{(BL)^2}{Rm} t}$$

f)

$$d = \int_0^\infty v_0 e^{-\frac{(BL)^2}{Rm} t} dt = -v_0 \lim_{a\to\infty} \frac{Rm}{(BL)^2} \left[e^{-\frac{(BL)^2}{Rm} t} \right]_0^a = v_0 \frac{Rm}{(BL)^2}$$

g)

$$P(t) = \frac{(BLv)^2}{R} = \frac{(BL)^2}{R} v_0^2 \left(e^{-2\frac{(BL)^2}{Rm} t} \right)$$

h)

$$\begin{aligned}
W &= \int_0^\infty \frac{(BL)^2}{R} v_0^2 \left(e^{-2\frac{(BL)^2}{Rm}t} \right) dt \\
&= \frac{(BL)^2}{R} v_0^2 \lim_{a\to\infty} \left(-\frac{Rm}{2(BL)^2} \right) \left[e^{-2\frac{(BL)^2}{Rm}t} \right]_0^a \\
&= \frac{1}{2} m v_0^2
\end{aligned}$$

Homework 6

Problem 6.1

a)

$$v = \omega l$$

b)

$$d\varepsilon = Bvdl = B\omega l dl$$

c)

$$\varepsilon = \int_0^L B\omega l dl = B\omega \left[\frac{1}{2}l^2\right]_0^L = \frac{1}{2}B\omega L^2$$

d)

$$I = \frac{8\varepsilon}{R} = \frac{4B\omega L^2}{R}$$

e)

$$\begin{aligned}\vec{\tau} &= 8\int \vec{l} \times (Id\vec{l} \times \vec{B}) = \frac{32B\omega L^2}{R}\int \vec{l} \times (d\vec{l} \times \vec{B}) \\ &= \frac{32B\omega L^2}{R}\int_0^L ldlB\hat{k} \\ &= \frac{16B^2\omega L^4}{R}\hat{k}\end{aligned}$$

f)

$$P_{\text{mech}} = \vec{\tau} \cdot \vec{\omega} = \frac{16B^2\omega^2 L^4}{R}$$

g)

$$P_{\text{elec}} = I^2 R = \left(\frac{4B\omega L^2}{R}\right)^2 R = \frac{16B^2\omega^2 L^4}{R} = P_{\text{mech}}$$

Homework 7

Problem 7.1

a)

$$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{LC}} = 4594Hz$$

b)

$$V_R = V_{RMS}$$

c)

$$V_L = V_{RMS}\frac{\omega_0 L}{R}$$

$$Q = \frac{V_L}{V_R} = \frac{\omega_0 L}{R} = 3.4641$$

d)

$$V_R = V_{RMS}\frac{R}{\sqrt{R^2 + (1.15\omega_0 L - \frac{1}{\omega_0 C})^2}} = 0.8874V_{RMS}$$

Percent Decrease = 11.26%

e)

The resonance peak will be higher with a smaller bandwidth if the resistance if lower. Suppose the resistance has to increase by a factor of m.

$$\frac{V_R}{V_{RMS}} = \frac{mR}{\sqrt{(mR)^2 + (1.02\omega_0 L - \frac{1}{\omega_0 C})^2}} = 0.95$$

$$m = \sqrt{\frac{0.95^2(1.02\omega_0 L - \frac{1}{\omega_0 C})^2}{(1-0.95^2)R^2}} = 0.2108$$

Therefore, R needs to be decreased by $1 - 0.2108 = 78.92\%$. The new value is $R = 0.2108 \times 50\Omega = 10.54\Omega$

f)

$$Q = 16.4331$$

Problem 7.2

a)

$$\mathbf{I_1} = \frac{V_0}{\frac{1}{j\omega C}} = V_0 j\omega C$$

$$|\mathbf{I_1}| = V_0 \omega C$$

$$\phi = \arctan \frac{\omega C}{0} = \frac{\pi}{2}$$

$$I_1(t) = V_0 \omega C \sin\left(\omega t + \frac{\pi}{2}\right)$$

b)

$$\mathbf{I_2} = \frac{V_0}{R + j\omega L} = V_0 \frac{R - j\omega L}{R^2 + \omega^2 L^2}$$

$$I_2 = |\mathbf{I_2}| = \frac{V_0}{R^2 + \omega^2 L^2}\sqrt{R^2 + \omega^2 L^2} = \frac{V_0}{\sqrt{R^2 + \omega^2 L^2}}$$

$$\phi = \arctan \frac{-\omega L}{R} = -\arctan \frac{\omega L}{R}$$

$$\cos\phi = \frac{1}{\sqrt{1 + \left(\frac{\omega L}{R}\right)^2}} = \frac{R}{\sqrt{R^2 + \omega^2 L^2}}$$

$$\sin\phi = \frac{\frac{-\omega L}{R}}{\sqrt{1 + \left(\frac{\omega L}{R}\right)^2}} = \frac{-\omega L}{\sqrt{R^2 + \omega^2 L^2}}$$

c)

$$\begin{aligned}
\mathbf{I_0} = \mathbf{I_1} + \mathbf{I_2} &= V_0 j\omega C + V_0 \frac{R - j\omega L}{R^2 + \omega^2 L^2} \\
&= V_0 \frac{j\omega C(R^2 + \omega^2 L^2) + R - j\omega L}{R^2 + \omega^2 L^2} \\
&= V_0 \frac{R + j\omega(CR^2 + \omega^2 CL^2 - L)}{R^2 + \omega^2 L^2}
\end{aligned}$$

$$|\mathbf{I_0}| = \frac{V_0}{R^2 + \omega^2 L^2}\sqrt{R^2 + \omega^2(CR^2 + \omega^2 CL^2 - L)^2}$$

$$\phi = \arctan\left(\frac{\omega(CR^2 + \omega^2 CL^2 - L)}{R}\right)$$

d)

$$\cos\phi = \frac{1}{\sqrt{1 + \left(\frac{\omega(CR^2+\omega^2CL^2-L)}{R}\right)^2}} = \frac{R}{\sqrt{R^2 + \omega^2(CR^2 + \omega^2CL^2 - L)^2}}$$

Homework 8

Problem 8.1

a)

$$f_B = B\cos\phi\sin(kx-\omega t) + B\sin\phi\cos(kx-\omega t)$$

b)

$$f_A + f_B = (A + B\cos\phi)\sin(kx-\omega t) + B\sin\phi\cos(kx-\omega t)$$

c)

$$D_1^2 + D_2^2 = A^2 + B^2\cos^2\phi + 2AB\cos\phi + B^2\sin^2\phi = A^2 + 2AB\cos\phi + B^2$$

$$\tan\phi = \frac{D_2}{D_1} = \frac{B\sin\phi}{A+B\cos\phi}$$

$$f_A + f_B = \sqrt{A^2+2AB\cos\phi+B^2}\sin\left(kx-\omega t+\arctan\left(\frac{B\sin\phi}{A+B\cos\phi}\right)\right)$$

d)

$$\begin{aligned}\text{Amplitude} &= \sqrt{A^2+2AB\cos\phi+B^2}\\ \text{Speed} &= \frac{\omega}{k}\\ \text{Phase } \psi &= \arctan\frac{B\sin\phi}{A+B\cos\phi}\end{aligned}$$

The wave form is still sinusoidal

$$\begin{aligned}\phi = 0 \Rightarrow f_A + f_B &= \sqrt{A^2+2AB+B^2}\sin\left(kx-\omega t+\arctan 0\right)\\ &= (A+B)\sin(kx-\omega t)\\ &= A\sin(kx-\omega t) + B\sin(kx-\omega t)\end{aligned}$$

e)

$$\begin{aligned}f_- + f_+ &= (A\sin kx\cos\omega t - \cos kx\sin\omega t) + (A\sin kx\cos\omega t + \cos kx\sin\omega t)\\ &= 2A\sin kx\cos\omega t\end{aligned}$$

f)

$$\text{Amplitude} = 2A$$

It is sinusoidal and it is not a traveling wave.

Problem 8.2

a)

$$|\vec{E}| = \sqrt{E_0^2 \sin^2(kz - \omega t) + E_0^2 \cos^2(kz - \omega t)} = E_0$$

b)

$$|\vec{B}| = \sqrt{B_0^2 \cos^2(kz - \omega t) + B_0^2 \sin^2(kz - \omega t)} = E_0$$

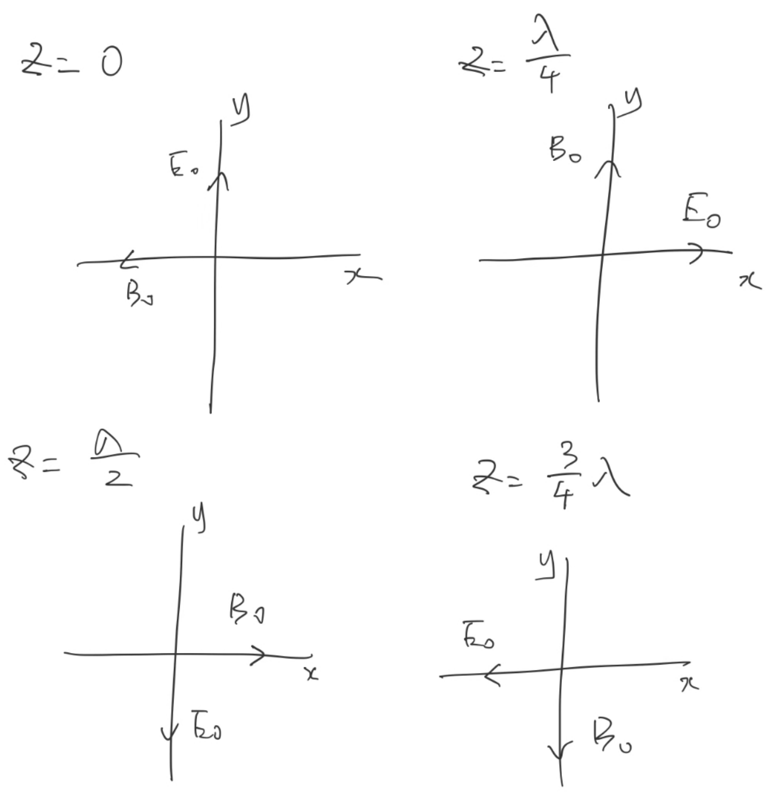
c)

$$u = \frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2 = \epsilon_0 E^2$$

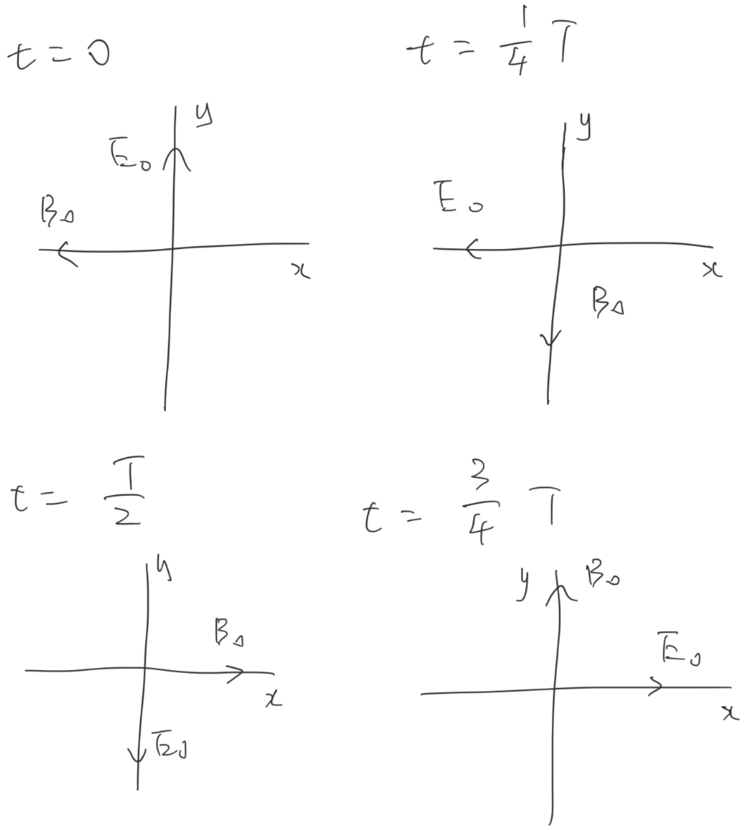
d)

$$\begin{aligned}\vec{E} \cdot \vec{B} &= \begin{bmatrix} E_0 \sin(kz - \omega t) \\ E_0 \cos(kz - \omega t) \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -B_0 \cos(kz - \omega t) \\ B_0 \sin(kz - \omega t) \\ 0 \end{bmatrix} \\ &= 0\end{aligned}$$

e)



f)



h)

$$
\begin{aligned}
\vec{E} \times \vec{B} &= \begin{bmatrix} E_0 \sin(kz - \omega t) \\ E_0 \cos(kz - \omega t) \\ 0 \end{bmatrix} \times \begin{bmatrix} -B_0 \cos(kz - \omega t) \\ B_0 \sin(kz - \omega t) \\ 0 \end{bmatrix} \\
&= E_0 B_0 \left(\sin^2(kz - \omega t) + \cos^2(kz - \omega t) \right) \hat{k} \\
&= E_0 B_0 \hat{k}
\end{aligned}
$$

i)

$$
\begin{aligned}
f_1 &= \begin{bmatrix} E_0 \sin(kz - \omega t) \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ B_0 \sin(kz - \omega t) \\ 0 \end{bmatrix} \\
&= E_0 B_0 \sin^2(kz - \omega t) \hat{k} \\
f_2 &= \begin{bmatrix} 0 \\ E_0 \cos(kz - \omega t) \\ 0 \end{bmatrix} \times \begin{bmatrix} -B_0 \cos(kz - \omega t) \\ 0 \\ 0 \end{bmatrix} \\
&= E_0 B_0 \cos^2(kz - \omega t) \hat{k} \\
f_1 + f_2 &= E_0 B_0 \left(\sin^2(kz - \omega t) + \cos^2(kz - \omega t) \right) \hat{k} \\
&= E_0 B_0 \hat{k}
\end{aligned}
$$

j)

Wave 1 oscillates in the x direction. Wave 2 oscillates in the y direction.

Problem 8.3

a)

Since $I_f = I_0 \sin^2\theta$, increasing θ by eating more doughnuts will increase the amount of light I_f detected.

b)

Assuming the incident beam is randomly polarized,

$$2 \times 10^{-4} I_0 = 0.1\% \frac{I_0}{2} \sin^2\theta$$

$$\sin\theta = \sqrt{\frac{2 \times 10^{-4} I_0}{0.05\% I_0}}$$

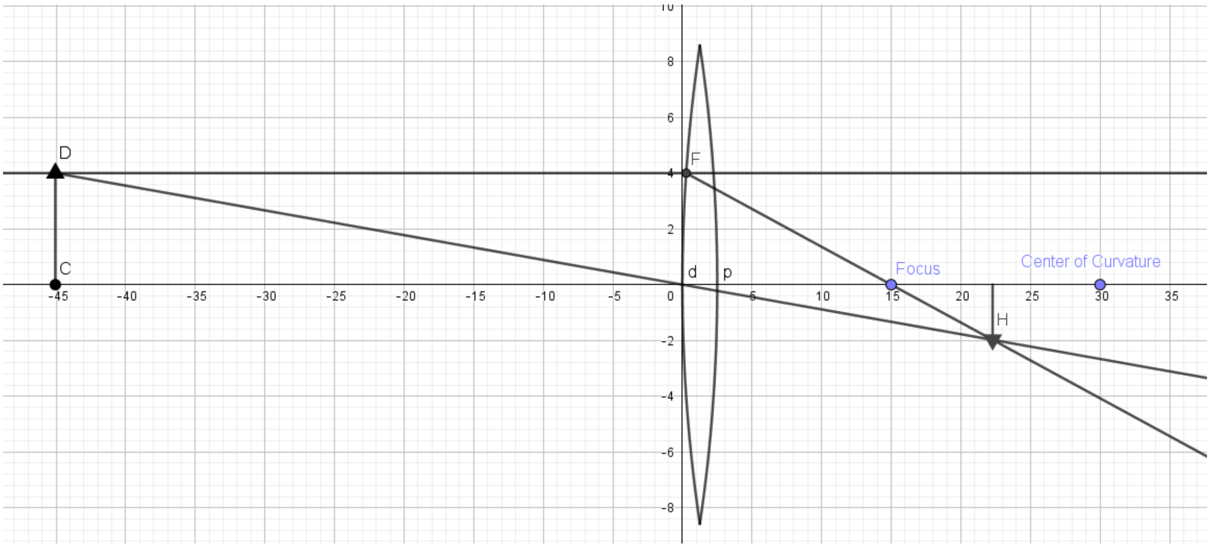
$$\theta = \arcsin\sqrt{\frac{2 \times 10^{-4}}{0.5 \times 10^{-3}}} = 39.2315°$$

$$\text{Number of doughnuts} = \frac{39.2315}{0.15} \approx 261.543 \approx 262$$

Homework 9

Problem 9.1

a)

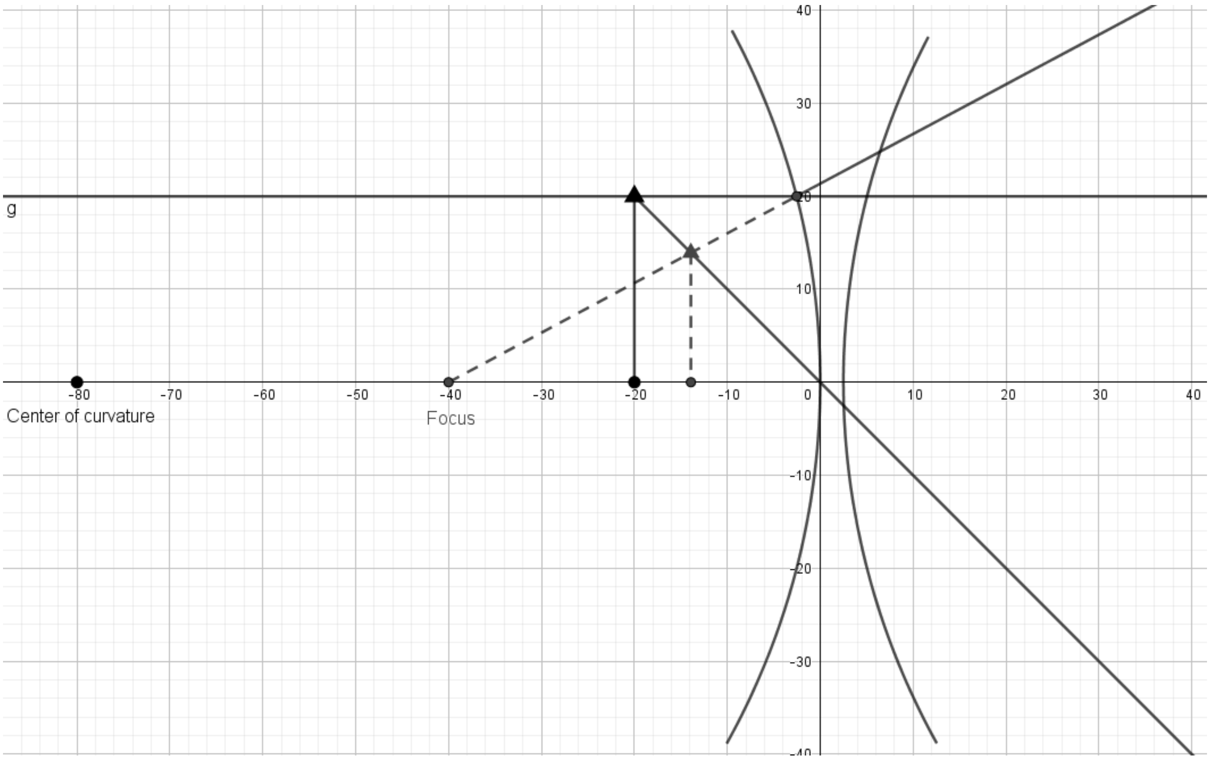


$$d_i = \left(\frac{1}{15} - \frac{1}{45}\right)^{-1} = \frac{45}{2}$$

$$M = -\frac{45/2}{45} = -\frac{1}{2}$$

The image is real and inverted.

b)

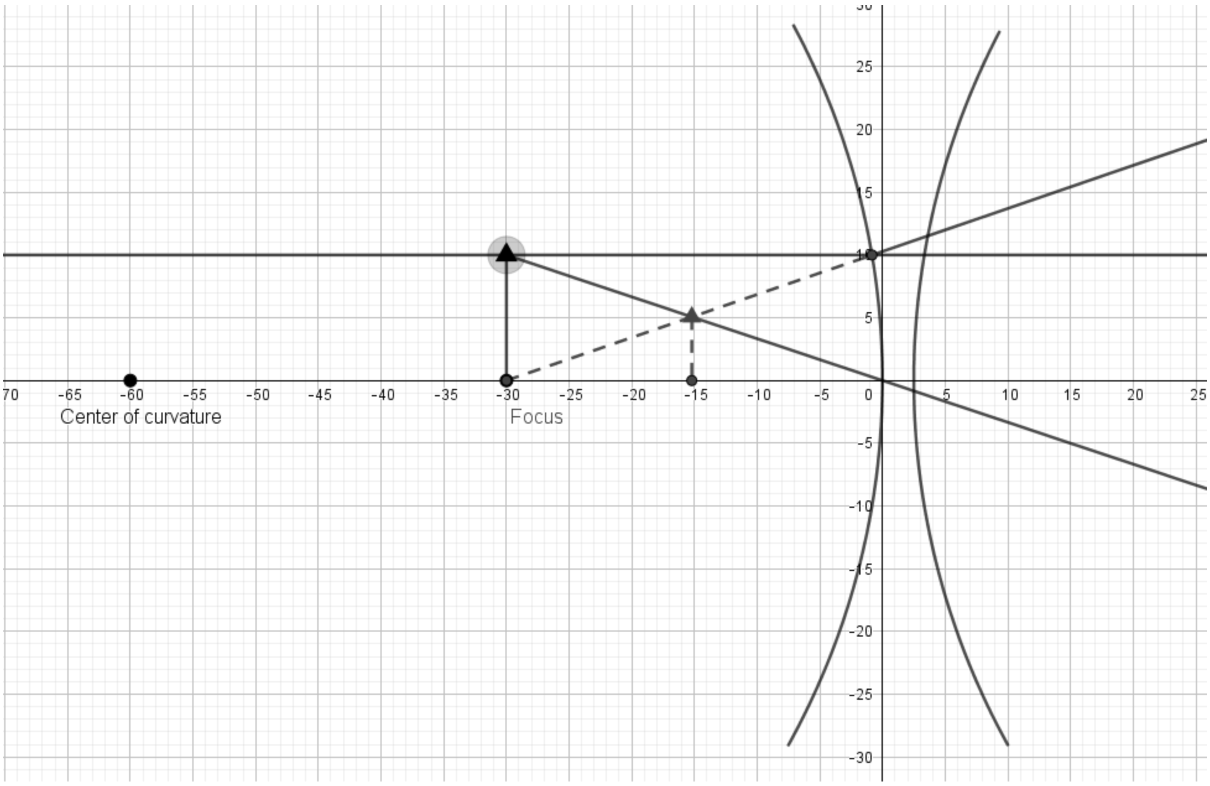


$$d_i = \left(\frac{1}{-40} - \frac{1}{20}\right)^{-1} = -\frac{40}{3}$$

$$M = -\frac{-40/3}{20} = \frac{2}{3}$$

The image is virtual and upright.

c)

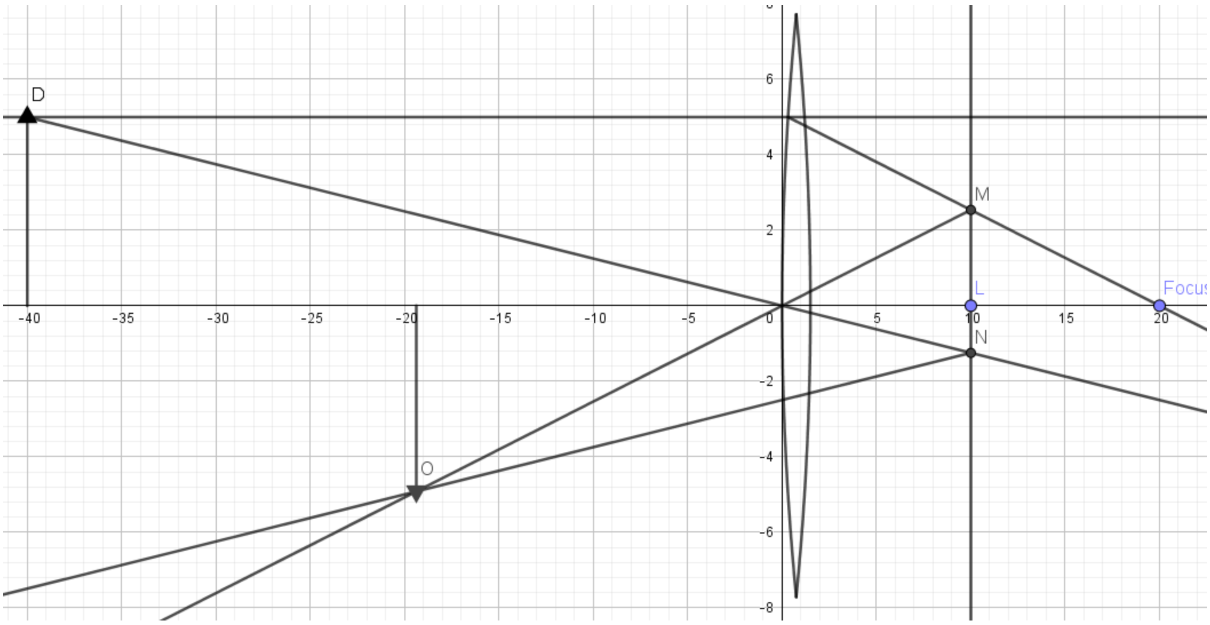


$$d_i = \left(\frac{1}{-30} - \frac{1}{30}\right)^{-1} = -15$$

$$M = -\frac{-15}{30} = \frac{1}{2}$$

The image is virtual and upright.

d)



After the converging lens,

$$d_i = \left(\frac{1}{20} - \frac{1}{40}\right)^{-1} = 40$$

$$M = -\frac{40}{40} = -1$$

For the mirror, $d_o = -40 + 10 = -30$, since the object is at the opposite side of the incoming rays. Therefore,

$$d_i = d_o = -30$$

Because d_i is negative, the object is at the same side as the incoming rays. The final position of the object is at the left of converging lens. We need to add 10cm to get the final position of the image: 20cm left to the converging lens.

Since the mirror won't change the magnification, the final magnification is -1, giving an inverted real image of the same size.

Problem 9.2

a)

$$d_{i_0} = \left(\frac{1}{5} - \frac{1}{6}\right)^{-1} = 30\text{cm}$$

$$M_0 = -\frac{d_i}{d_o} = -\frac{30}{6} = -5$$

b)

$$\begin{aligned}10 &= \left(\frac{1}{d_i} + \frac{1}{d_o}\right)^{-1} \\ &= \frac{d_i d_o}{d_o + d_i} \\ 10(d_o + d_i) &= d_i d_o \\ 10d_o &= d_i d_o - 10d_i \\ d_i &= \frac{10d_o}{d_o - 10}\end{aligned}$$

$$M = -\frac{\frac{10d_o}{d_o-10}}{d_o} M_0 = -\frac{10}{d_o - 10} M_0 = \frac{50}{d_o - 10}$$

Using $d_o = -30$,

$$d_i = \frac{10 \times -30}{-30 - 10} = 7.5$$

$$M = \frac{50}{-30 - 10} = -1.25$$