03 October 2023 02:35 PM

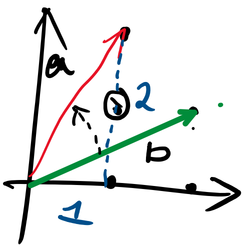
Dot product between two vectors

$$a = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \Rightarrow a \cdot b = 1*4 + 2*5 + 3*6$$
$$= 32$$

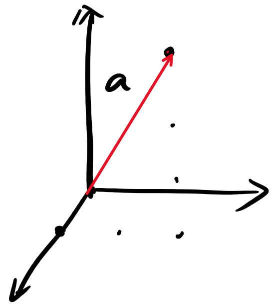
- What does the dot product tell us about the vectors?

- Geometric interpretation of a vector

$$a = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
$$b = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$



$$a = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$



- Geometric length of a vector $\|a\|_2 = \sqrt{a_1^2 + a_2^2}$

(L2 norm of a) $= \sqrt{1^2 + 2^2}$

- Angle between vectors $\left\{ -1 \leq \dfrac{a \cdot b}{\|a\| \, \|b\|} \leq 1 \right.$ (Cauchy-Schwarz inequality)

$$\boxed{\cos(\angle a, b) = \frac{a \cdot b}{\|a\| \, \|b\|}}$$

$\cos(0) = 1$
$\cos(\pi) = -1$
$\cos(\pi/2) = 0$

$$\boxed{\begin{array}{l} a \cdot b = 0 \\ \angle a, b = \pi/2 \end{array}}$$

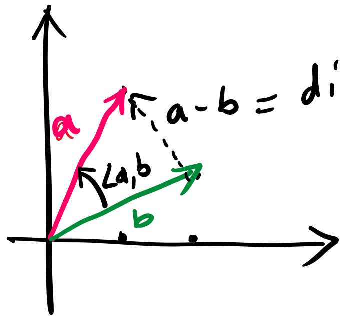
$$\boxed{\begin{array}{l} a \cdot b = -ve \\ \angle a, b > \pi/2 \end{array}}$$

$$\boxed{\begin{array}{l} a \cdot b = +ve \\ \angle a, b < \pi/2 \end{array}}$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Distance between vectors

$$\boxed{\angle a,b < \pi/2}$$



$a - b$ = difference vector $= \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

How big is the difference vector $= \|a-b\|_2$

$$= \left\| \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\| = \sqrt{(-1)^2 + 1^2}$$

$$= \sqrt{2}$$

- An important relationship : $a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

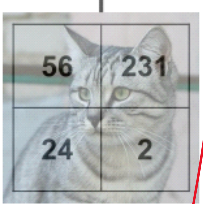
$$\|a\|^2 = a \cdot a$$

$$\Downarrow \qquad \Downarrow$$

$$\left(\sqrt{a_1^2 + a_2^2}\right)^2 \Longleftrightarrow a_1^2 + a_2^2$$

Example with an image with 4 pixels, and 3 classes (cat/dog/ship)

Stretch pixels into column



Input image

Weight matrix W

$$\begin{bmatrix} w_{11} & w_{12} & w_{13} & w_{14} \\ w_{21} & w_{22} & w_{23} & w_{24} \\ w_{31} & w_{32} & w_{33} & w_{34} \end{bmatrix}$$

Sample vector x

$$\begin{bmatrix} 56 \\ 231 \\ 24 \\ 2 \end{bmatrix}$$

Weights for cats

$$W_1 = \begin{bmatrix} w_{11} \\ w_{12} \\ w_{13} \\ w_{14} \end{bmatrix}$$

weights for dogs

$$W_2 = \begin{bmatrix} w_{21} \\ w_{22} \\ w_{23} \\ w_{24} \end{bmatrix}$$

weights for ship

$$W_3 = \begin{bmatrix} w_{31} \\ w_{32} \\ w_{33} \\ w_{34} \end{bmatrix}$$

$$\boxed{\begin{array}{l} \text{Cat score} = W_1 \cdot x \\ \text{Dog score} = W_2 \cdot x \\ \text{ship score} = W_3 \cdot x \end{array}}$$

↳ matrix-vector product Wx

W → 3×4-vector

x → 4-vector

Why did we not do this?

$$\begin{matrix} w_1 \cdot x \\ w_2 \cdot x \\ w_3 \cdot x \end{matrix} \quad \overset{?}{=} \quad \underbrace{\begin{bmatrix} w_{11} & w_{21} & w_{31} \\ w_{12} & w_{22} & w_{32} \\ w_{13} & w_{23} & w_{33} \\ w_{14} & w_{24} & w_{34} \end{bmatrix}}_{W} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{x} = \begin{bmatrix} w_1 \cdot x & w_2 \cdot x & w_3 \cdot x \end{bmatrix}$$

A matrix-vector product Wx = sequence of dot products

= dot products between the rows of matrix W (seen as vectors) and the vector x

Tensor-vector product

E.g. $T \to (4, 3, 2)$, $x \to (2,)$

Matrix-vector product

E.g. $A \to (3, 4)$, $x \to (4,)$

Linear combination of columns of a matrix A

1st row of A

2nd row of A

3rd row of A

$$\underbrace{\begin{bmatrix} 1 & 2 & -1 & -1 \\ 2 & 4 & -2 & 3 \\ -1 & 1 & -2 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{x} = \left\{ \begin{array}{l} \begin{bmatrix} a^{(1)} \cdot x \\ a^{(2)} \cdot x \\ a^{(3)} \cdot x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \\ x_1 a_1 + x_2 a_2 + x_3 a_3 + x_4 a_4 = \begin{bmatrix} \ \end{bmatrix} \end{array} \right.$$

1st column of A

2nd column of A

... A

2nd column ,

3rd column of A

4th column of A

$x_1 a_1 + x_2 a_2 + x_3 a_3 + x_4 a_4$ = linear combination of columns of A using the elements of x as multipliers

Matrix-vector product

E.g. $A = \begin{bmatrix} \boxed{1 \quad 2 \quad 3} \\ \boxed{4 \quad 5 \quad 6} \end{bmatrix}$, $x = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \times 7 + 2 \times 8 + 3 \times 9 \\ 4 \times 7 + 5 \times 8 + 6 \times 9 \end{bmatrix}$

2×3   3   2

$\| $

Ax is defined as $= \begin{bmatrix} (\text{1st row of } A) \cdot x \\ (\text{2nd row of } A) \cdot x \end{bmatrix} = \begin{bmatrix} a^{(1)} \cdot x \\ a^{(2)} \cdot x \end{bmatrix}$

Tensor-Vector product

E.g. $T = \begin{bmatrix} \boxed{\begin{bmatrix} 76 & 120 \\ 74 & 124 \\ 78 & 136 \end{bmatrix}} \\ \boxed{\begin{bmatrix} 78 & 136 \\ 80 & 124 \\ 70 & 120 \end{bmatrix}} \end{bmatrix}$ → Timestamp-0, → Timestamp-1 , $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

2×3×2   2

Timestamps, patients, features

$$\begin{bmatrix} 76 & 120 \\ 74 & 124 \\ 78 & 136 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 76 \\ 74 \\ 78 \end{bmatrix}$$

$$\begin{bmatrix} 78 & 136 \\ 80 & 124 \\ 70 & 120 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 78 \\ 80 \\ 70 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 76 & 74 & 78 \\ 78 & 80 & 70 \end{bmatrix}$$

2×3

Matrix-matrix Product

1st row of A   1st column of B

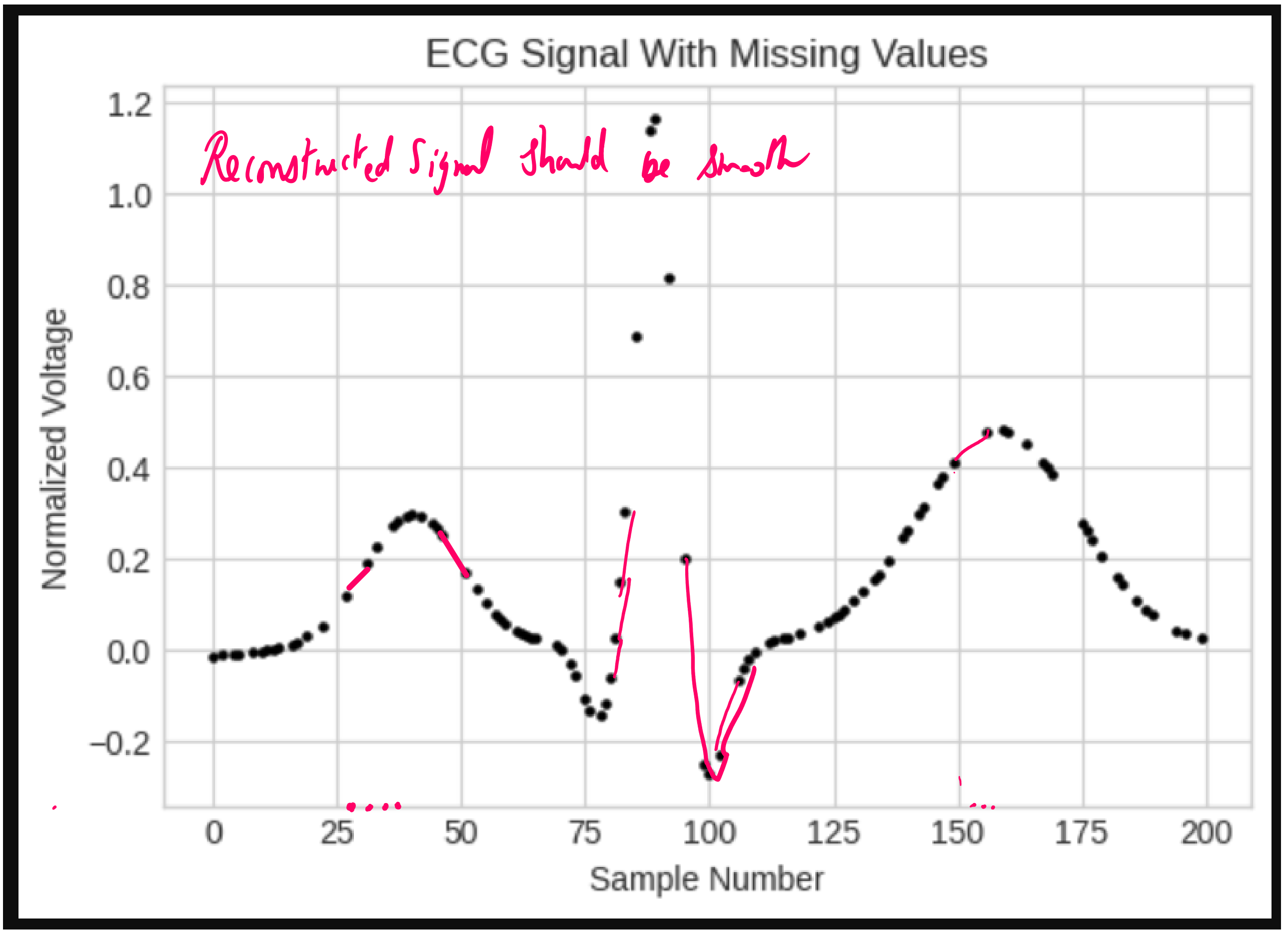
Matrix-Matrix Product

E.g. $A = \begin{bmatrix} \boxed{1 \quad 2 \quad 3} \\ \boxed{4 \quad 5 \quad 6} \end{bmatrix}$ $\cdot$ $B = \begin{bmatrix} 7 & 10 \\ 8 & 11 \\ 9 & 12 \end{bmatrix}$ $= \begin{bmatrix} a^{(1)} \cdot b_1 & a^{(1)} \cdot b_2 \\ a^{(2)} \cdot b_1 & a^{(2)} \cdot b_2 \end{bmatrix}$

2×3 $\quad$ 3×2 $\quad$ 2×2

1st row of A $\quad$ 1st column of B

2nd row of A $\quad$ 1st column of B



$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}. \quad x = \begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ x_4 \\ v_2 \\ v_3 \end{bmatrix}.$$

missing ECG values

$$\begin{bmatrix} x_1 \\ x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ x_4 \\ v_2 \\ v_3 \end{bmatrix}.$$

Transposes

Filter matrix for known ECG values

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

ECG vector with unknown values turned to zeros

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_4 \end{bmatrix}. = \begin{bmatrix} x_1 \\ x_2 \\ 0 \\ x_4 \\ 0 \\ 0 \end{bmatrix}$$

vector of known ECG values

$$\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\\0&0&1\\0&0&0\\0&0&0\end{bmatrix}}_{S_1}\underbrace{\begin{bmatrix}x_1\\x_2\\x_4\end{bmatrix}}_{\Downarrow\ x_{known}} + \underbrace{\begin{bmatrix}0&0&0\\0&0&0\\1&0&0\\0&0&0\\0&1&0\\0&0&1\end{bmatrix}}_{S_2}\underbrace{\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}}_{\Downarrow\ x_{unknown}} = \underbrace{\begin{bmatrix}x_1\\x_2\\v_1\\x_4\\v_2\\v_3\end{bmatrix}}_{\Downarrow\ x}$$

x Full ECG vector with missing values

$$\Downarrow \qquad\qquad \Downarrow$$

$$\begin{bmatrix}x_1\\x_2\\0\\x_4\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\v_1\\0\\v_2\\v_3\end{bmatrix}$$

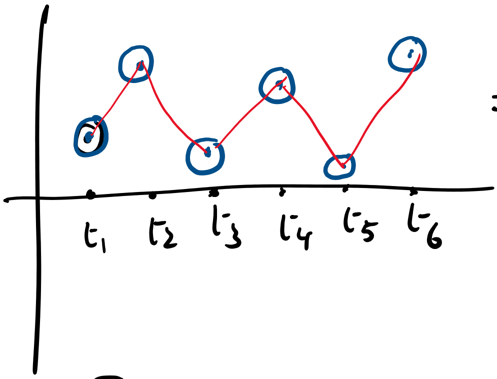
$$\Rightarrow \boxed{\underbrace{x}_{\Downarrow\ \text{ECG vector}} = \underbrace{S_1 x_{known}}_{\text{Known part of the ECG vector}} + \underbrace{S_2 x_{unknown}}_{\text{unknown part of the ECG vector}}}$$

Aside

$$\underbrace{\begin{bmatrix}1&-2&1&0&0&0\\0&1&-2&1&0&0\\0&0&1&-2&1&0\\0&0&0&1&-2&1\end{bmatrix}}_{D}\begin{bmatrix}y_1\\y_2\\y_3\\y_4\\y_5\\y_6\end{bmatrix} = \begin{bmatrix}\boxed{y_1-2y_2+y_3}\\y_2-2y_3+y_4\\y_3-2y_4+y_5\\y_4-2y_5+y_6\end{bmatrix}$$

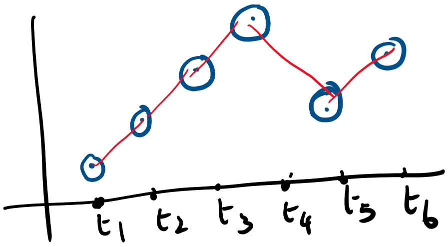
$$(y_1-y_2)-(y_2-y_3)$$

$$y = \begin{bmatrix} 15 \\ 30 \\ 10 \\ 25 \\ 5 \\ 30 \end{bmatrix} \Rightarrow Dy = \begin{bmatrix} 15 - 60 + 10 \\ 30 - 20 + 25 \\ 10 - 50 + 5 \\ 25 - 10 + 30 \end{bmatrix} = \begin{bmatrix} -35 \\ 35 \\ -35 \\ 45 \end{bmatrix}$$

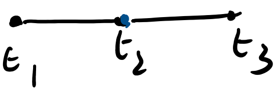


$\Rightarrow$ oscillatory signal

$$y = \begin{bmatrix} 15 \\ 20 \\ 25 \\ 30 \\ 20 \\ 25 \end{bmatrix}, \quad Dy = \begin{bmatrix} 15 - 40 + 25 \\ 20 - 50 + 30 \\ 25 - 60 + 20 \\ 30 - 40 + 25 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -15 \\ 15 \end{bmatrix}$$



Discrete version of second derivative



(Approx)

$$\frac{y_2 - y_1}{t_2 - t_1} = \frac{\text{change in temperature}}{\text{change in time}} = \frac{°C}{Hr}$$

Sensitivity of temperature w.r.t. time

$$\frac{y_3 - y_2}{t_3 - t_2} = \frac{°C}{Hr}$$

$\rightarrow$ Rms(x)

Detour

Detour

$$x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \boxed{\frac{\|x\|}{\sqrt{n}}} = \frac{\sqrt{1^2+1^2+1^2+1^2}}{\sqrt{4}} = \frac{\sqrt{4}}{\sqrt{4}} = 1$$

Rms(x)

$$x = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \quad \frac{\|x\|}{\sqrt{n}} = 1$$

Reconstruction goal

{ the reconstructed ECG signal should be as smooth as possible

$$\mathrm{RMS}\Big(D\big(\underbrace{S_1 x_{known} + S_2 x_{unknown}}_{\text{ECG vector}}\big)\Big)$$ to be as small as possible $\Rightarrow$ $$\frac{\|D(S_1 x_{known} + S_2 x_{unknown})\|_2}{\underbrace{\sqrt{n}}_{\text{constant}}}$$

Should be as small as possible

$\Rightarrow$ Find the vector $x_{unknown}$ s.t. $\boxed{\|D(S_1 x_{known} + S_2 x_{unknown})\|_2^2}$ is as small as possible.

b = 4-vector A = 4x3-matrix unknown vector $\cancel{x}$ = 3-vector

$$\Rightarrow \boxed{\underbrace{\underbrace{DS_1}_{4\times6} \underbrace{}_{6\times3}}_{4\times3} \underbrace{x_{known}}_{3\times1}} + \boxed{\underbrace{DS_2}_{4\times 6 \;\; 6\times3}}\boxed{\underbrace{x_{unknown}}_{3\times1}} \Rightarrow \|b + Ax\|_2^2$$

Not all linear systems of equations $Ax = b$ are

Not all linear systems of equations $Ax = b$ are solvable

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

D matrix

Standard least squares problem

Find a vector x s.t. $\|Ax - b\|_2^2$ is minimized

A ⇓ Known matrix; b ⇒ Known vector

In Python, we say linalg.lstsq(A,b) to get the solution for the least squares problem.

For the ECG data, recall that we wanted to minimize

$$\| \underbrace{DS_1 x_{known}}_{-b} + \underbrace{DS_2}_{A} \underbrace{x_{unknown}}_{x} \|_2^2$$

Match A and b in the expression above with $\|Ax - b\|_2^2$

Solving systems of equations

Consider the following model for opinion formation among n individuals, each of whom interact with a certain number of individuals in the group. The numerical value of the ith person's opinion is denoted as x_i. The value of x_i is influenced by the following:

- The ith person's self opinion denoted as s_i
- The opinions of the remaining individuals x_j, where $j = 1, 2, \ldots, n$ and $j \neq i$.

Assuming that the ith person gives a weightage w_{ij} to the jth person's opinion, we can compute x_i as follows:

$$x_i = \frac{s_i + \sum_{j \neq i} w_{ij} x_j}{1 + \sum_{j \neq i} w_{ij}}, \quad i = 1, \ldots, n.$$

1. From the equation above, what do you see is the weightage that a person gives to their own opinion?
2. The equation above can be written as $(A + I)x = s$, where A is an $n \times n$-matrix and I represents the identity matrix. What are the elements of the matrix A, vectors x and s?

Scenario: 4 people in a network

Self-opinion vector $= s = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix}$ Can be calculated based on individual history.

Opinion vector $= x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ that is built after interaction with others

Weights matrix $= W = \begin{bmatrix} w_{11} & w_{12} & w_{13} & w_{14} \\ w_{21} & w_{22} & w_{23} & w_{24} \\ w_{31} & w_{32} & w_{33} & w_{34} \\ w_{41} & w_{42} & w_{43} & w_{44} \end{bmatrix}$

1

weightage that the 1st individual gives to the 2nd individual's opinion

$$x_i = \frac{s_i + \sum_{j \neq i} w_{ij} x_j}{1 + \sum_{j \neq i} w_{ij}},$$

Suppose $i = 1$, the 1st individual

Opinion of 1st individual $x_1 = \dfrac{s_1 + w_{12}x_2 + w_{13}x_3 + w_{14}x_4}{\underbrace{w_{11}}_{=1} + w_{12} + w_{13} + w_{14}}$

$$\Rightarrow x_1 \underbrace{w_{11}}_{=1} + \boxed{x_1 w_{12}} + x_1 w_{13} + x_1 w_{14} = s_1 + w_{12}x_2 + w_{13}x_3 + w_{14}x_4$$

$$\Rightarrow \underbrace{x_1}_{\text{unknown}}(1 + \underbrace{w_{12}}_{\text{known}} + \underbrace{w_{13}}_{\text{known}} + \underbrace{w_{14}}_{\text{known}}) + (-\underbrace{w_{12}}_{\text{known}})\underbrace{x_2}_{\text{unknown}} + (-\underbrace{w_{13}}_{\text{known}})\overbrace{x_3}^{\text{unknown}} + (-\underbrace{w_{14}}_{\text{known}})\overbrace{x_4}^{\text{unknown}} = \underbrace{s_1}_{\text{known}}$$

$$\Rightarrow \begin{cases} w_{12}(x_1 - x_2) + w_{13}(x_1 - x_3) + w_{14}(x_1 - x_4) + 1.x_1 = s_1 \\ w_{21}(x_2 - x_1) + 1.x_2 + w_{23}(x_2 - x_3) + w_{24}(x_2 - x_4) = s_2 \\ w_{31}(x_3 - x_1) + w_{32}(x_3 - x_2) + 1.x_3 + w_{34}(x_3 - x_4) = s_3 \\ w_{41}(x_4 - x_1) + w_{42}(x_4 - x_2) + w_{43}(x_4 - x_3) + 1.x_4 = s_4 \end{cases}$$

Given the weights matrix W and the self-opinion vector s, we get 4 equations in 4 unknowns

Recap of the matrix-vector product

$$A = \begin{bmatrix} 1 & 2 & -1 & -1 \\ 2 & 4 & -2 & 3 \\ -1 & 1 & -2 & 4 \end{bmatrix}, \quad x = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$Ax = \begin{bmatrix} a^{(1)} \cdot x \\ a^{(2)} \cdot x \\ a^{(3)} \cdot x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$a^{(1)} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ -1 \end{bmatrix}$$

Linear combination of the columns of A using the elements of x as multipliers

$$a_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, a_2 = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}, a_3 = \begin{bmatrix} -1 \\ -2 \\ -2 \end{bmatrix}, a_4 = \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}$$

$$\Rightarrow x_1 a_1 + x_2 a_2 + x_3 a_3 + x_4 a_4$$

$$= -1\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + (1)\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} + (1)\begin{bmatrix} -1 \\ -2 \\ -2 \end{bmatrix} + (0)\begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Same answer

In general, if

A: $m \times n$-matrix

x: n-vector

Then

$$Ax = \begin{cases} \begin{bmatrix} a^{(1)} \cdot x \\ a^{(2)} \cdot x \\ \vdots \\ a^{(m)} \cdot x \end{bmatrix} & m \text{ dot products} \\ x_1 a_1 + x_2 a_2 + \cdots + x_n a_n & \text{linear combination} \end{cases}$$

System of equations with

Solution exists means the System is consistent. otherwise

System of equations with

Solution exists means the system is consistent. Otherwise the system is inconsistent

(1) no solution: $\begin{cases} x_1 + x_2 = 1 \\ x_1 + x_2 = 5 \end{cases}$ no solution

(2) unique solution: $\begin{cases} x_1 + x_2 = 1 \\ x_1 + 2x_2 = 1 \end{cases} \Rightarrow \boxed{\begin{matrix} x_1 = 1 \\ x_2 = 0 \end{matrix}}$ unique solution

(3) Infinitely many solutions: $\{ x_1 + x_2 = 1 \Rightarrow \boxed{\begin{matrix} x_1 = 1 \\ x_2 = 0 \end{matrix}}, \begin{bmatrix} x_1 = 0.5 \\ x_2 = 0.5 \end{bmatrix}, \begin{bmatrix} x_1 = 0.4 \\ x_2 = 0.6 \end{bmatrix}$ infinitely many solutions

Elementary row operations

1. Divide/multiply a row by a non zero constant.
2. Subtract a scalar multiple of one row from another row.
3. Exchange two rows.

$\boxed{\begin{matrix} x_1 + x_2 = 1 \\ x_1 + 2x_2 = 1 \end{matrix}} \Longrightarrow \begin{matrix} 2(x_1 + x_2) = 2 \\ x_1 + 2x_2 = 1 \end{matrix} \Rightarrow \boxed{\begin{matrix} 2x_1 + 2x_2 = 2 \\ x_1 + 2x_2 = 1 \end{matrix}}$

$\boxed{\begin{matrix} x_1 + x_2 = 1 \\ x_1 + 2x_2 = 1 \end{matrix}} \Longrightarrow \boxed{\begin{matrix} x_1 + 2x_2 = 1 \\ x_1 + x_2 = 1 \end{matrix}}$

$\boxed{\begin{matrix} x_1 + x_2 = 1 \\ x_1 + 2x_2 = 1 \end{matrix}} \xRightarrow{Eq① = Eq① + 2 * Eq②} \boxed{\begin{matrix} 3x_1 + 5x_2 = 3 \\ x_1 + 2x_2 = 1 \end{matrix}}$

Reduced row echelon form (RREF)

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 2 & 2 & 0 & 1 & 6 \\ 0 & 1 & -1 & 1 & 3 \\ -1 & -2 & 1 & 1 & -1 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

1st column, 2nd column, 4th column

$\Downarrow$

x_1, x_2, x_4 as the pivot variables

$$\underbrace{\begin{bmatrix} -1 & -2 & 1 & 1 & | & -1 \end{bmatrix}}_{\text{Original augmented matrix}} \qquad \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}}_{\text{rref}}$$

Variables and x_3 is the free variable

$$\Downarrow$$

$$\begin{cases} x_1 + x_2 = 2 \\ 2x_1 + 2x_2 + x_4 = 6 \\ x_2 - x_3 + x_4 = 3 \\ -x_1 - 2x_2 + x_3 + x_4 = -1 \end{cases} \Longrightarrow \begin{cases} x_1 + x_3 = 1 \\ x_2 - x_3 = 1 \\ x_4 = 2 \\ \boxed{\cancel{0 = 0}} \end{cases}$$

anything about x_3 here? Nothing!

x_3 can be anything

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 - x_3 \\ 1 + x_3 \\ x_3 \\ 2 \end{bmatrix} \Rightarrow \underset{\substack{\Downarrow \\ \text{unknown vector}}}{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

such that

$$\Rightarrow \left\{ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \;\middle|\; x_3 \in \mathbb{R} \right\}$$

<u>No solution case</u>

$$x_1 + x_2 = 1$$
$$x_1 + x_2 = 5$$

$$\Rightarrow \text{Augmented matrix} = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 1 & 5 \end{array}\right]$$

$$\xrightarrow{\text{Row 2} = \text{Row 2} - \text{Row 1}} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 4 \end{array}\right]$$

Infinitely many Solutions case $\quad x_1 + x_2 = 1$

Infinitely many solutions case

$x_1 + x_2 = 1$

$2x_1 + 2x_2 = 2$

$\Rightarrow$ Augmented matrix $= \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 2 & 2 \end{array}\right]$

RREF $= \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$

Example, Solving $Ax = b$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & -1 & 1 \\ 2 & 4 & -2 & 3 & 3 \\ -1 & 1 & -2 & 4 & 2 \end{array}\right]$$

A (first four columns), b (last column)

Original augmented matrix

```
(Matrix([
[1, 0,  1, 0, -2/5],
[0, 1, -1, 0,  4/5],
[0, 0,  0, 1,  1/5]]), (0, 1, 3))
```

RREF using sympy

Solution

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_3 - \frac{2}{5} \\ x_3 + \frac{4}{5} \\ x_3 \\ \frac{1}{5} \end{bmatrix} = x_3 \underbrace{\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}}_{\text{solution to } Ax=0} + \underbrace{\begin{bmatrix} -\frac{2}{5} \\ \frac{4}{5} \\ 0 \\ \frac{1}{5} \end{bmatrix}}_{\text{particular solution}}$$

Solution vector x

RHS vector b

$$A\left(x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -2/5 \\ 4/5 \\ 0 \\ 1/5 \end{bmatrix} \right) \stackrel{?}{=} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

$$A\left(x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right) + A \begin{bmatrix} -2/5 \\ 4/5 \\ 0 \\ 1/5 \end{bmatrix}$$

$$= x_3 * A \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + A \begin{bmatrix} -2/5 \\ 4/5 \\ 0 \\ 1/5 \end{bmatrix}$$

$$x_3 * \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

✓

ALA Sessional-2 Review

(i) Matrices : matrix-vector and matrix-matrix products

questions will be scenario-based

E.g. 5 stations and 8 paths

$S_1 \; S_2 \; S_3 \; S_4 \; S_5$

$\{ S_1 \; S_4 \; S_5, \quad S_2 \; S_3 \; S_4, \quad S_1 \; S_2 \; S_3 \; S_4 \}$

stations →

paths ↓ $[P]$ → 1st row of $P = p^{(1)}$

1st column of $P = P_1$

P : 8×5-matrix

P_1 : 8-vector, E.g $P_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ (Station-1)

→ station-1 is in the first path

→ station-2 is in the second path

→ station-1 is not in the third path

$p^{(1)}$: 5-vector, E.g $p^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ (Path-1)

→ Path-1 contains station-1

→ Path-1 does not contain station-2

E.g.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ \ldots \end{bmatrix} \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \times 7 + 2 * 8 + 3 * 9 \\ 4 * 7 + 5 * 8 + 6 * 9 \end{bmatrix}$$

$$\begin{bmatrix} \boxed{4 \quad 5 \quad 6} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} 8 \\ 9 \end{matrix}} \end{bmatrix} = \begin{bmatrix} 4*7+5*8+6*9 \end{bmatrix}$$

The 8x5-matrix P

$$\mathbb{1}\text{-vector} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad P\mathbb{1} \text{ is a special matrix-vector product}$$

$$P = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

p_1, $p^{(1)T}$

$$\Rightarrow P\mathbb{1} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 2 \\ 4 \\ 3 \\ 0 \\ 5 \\ 2 \end{bmatrix}$$

No. of stations in path-1

No. of stations in path-2

E.g. What about the matrix-vector product $P^T\mathbb{1}$?

5x8-matrix   8-vector

$(P^T\mathbb{1})_1$ = 1st component of the vector

vector = How many paths in which station-1 shows up

$(P\mathbb{1})_3$ = 3rd component of the vector

vector = How many stations are in the 3rd path

Recall the unit vectors: $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$Pe_1 = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 1 * \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + 0 * \cdots\cdots$$

= Paths in which station-1 shows up $= P_1 =$ 1st column of P

Recall example

$$P = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad x = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

$$Px = \begin{cases} \begin{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \\ \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \end{bmatrix} & \text{Dot-product version} \\ 7\begin{bmatrix} 1 \\ 4 \end{bmatrix} + 8\begin{bmatrix} 2 \\ 5 \end{bmatrix} + 9\begin{bmatrix} 3 \\ 6 \end{bmatrix} & \text{linear combination version} \end{cases}$$

$$e_1^T \underbrace{P e_3}_{\text{matrix-vector product}} = \; ?$$

matrix-vector product $= P_3 =$ 3rd column of matrix P

= Paths in which station-3 shows up

$$e_1^T P_3 = P_3^T e_1 = P_3 \cdot e_1 = e_1 \cdot P_3 \;-\; \text{1st component of}$$

shows up

$$\underbrace{e_1^T P_3 = P_3^T e_1}_{\text{matrix-vector product}} = \underbrace{P_3 \cdot e_1 = e_1 \cdot P_3}_{\text{Dot-product}} = \text{1st component of the vector } P_3$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$P_3 \cdot e_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0$$

In general
$$\begin{cases} P e_i = P_i = i\text{th column of matrix } P \\ e_i^T P e_j = (i,j)\text{th element of matrix } P \\ e_i^T P = p^{(i)} = i\text{th row of matrix } P \end{cases}$$

Properties of matrix-vector product: $(AB)^T = B^T A^T$, $(A^T)^T = A$

$$e_i^T P = (\underset{A}{P^T} \underset{B}{e_i})^T = \underset{B^T}{e_i^T} \underset{A^T}{(P^T)^T} = e_i^T P$$

ith column of P^T = ith row of P

Build a matrix-vector product from a description

E.g. a signal sampled at 6 time stamps

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}.$$

Samples at only the even time stamps

$$= \underset{\text{Sampling matrix}}{S}\ \underset{\text{signal}}{x} = \begin{bmatrix} x_2 \\ x_4 \\ x_6 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_4 \\ x_6 \end{bmatrix}$$

What should be the matrix S such that we weight three successive timestamps equally to generate a new value.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \rightarrow \begin{array}{l} \frac{1}{3}\cdot x_1 + \frac{1}{3}\cdot x_2 + \frac{1}{3}\cdot x_3 \\ \frac{1}{3} x_2 + \frac{1}{3} x_3 + \frac{1}{3} x_4 \\ ? \\ ? \end{array}$$

$$\underbrace{\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}}_{x}$$

Matrix-Matrix Product

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & 10 \\ 8 & 11 \\ 9 & 12 \end{bmatrix}$$

2×3 $\qquad\qquad$ 3×2

matrix-vector product $\qquad$ matrix-vector product

product $\begin{bmatrix} \begin{bmatrix} * \\ * \end{bmatrix} \end{bmatrix}$ product $\begin{bmatrix} \begin{bmatrix} * \\ * \end{bmatrix} \end{bmatrix}$

2x2

$$P = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}, \quad P^T = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

8 x 5 - matrix   5 x 8 - matrix

Which matrix-matrix product makes sense: P^TP ✓ or PP^T ✓ or P^2?

P^TP: 5x8, 8x5 ⇒ 5x5 - matrix

PP^T: 8x5, 5x8 ⇒ 8x8 - matrix

$P^2 = P \cdot P$: 8x5, 8x5 → inner dimensions do not match

$$[AB]_{i,j} = a^{(i)} \cdot b_j = a^{(i)T} b_j$$

$a^{(i)}$ ⇓ ith row of A b_j ⇓ jth column of B

$$[P^TP]_{ij} = (\text{ith row of } P^T) \cdot (\text{jth column of } P)$$

⇓

$$(\text{ith column of } P) \cdot (\text{jth column of } P)$$

$$= p_i \cdot p_j = p_i^T p_j$$

for e.g $i=1, j=2 \Rightarrow [P^TP]_{1,2} = p_1 \cdot p_2$

$$\begin{array}{ccccc} p_1 & p_2 & & & \\ \uparrow & \uparrow & & & \\ \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$p_1 \cdot p_2 = 1$ = no. of paths common

$$P = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$P_1 \cdot P_2 = 1$ = no. of paths common to station-1 and station-2

How about $P_1 - P_2$? $\quad P_1 - P_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Always do this first

Is the system consistent or inconsistent?

RREF

```
[1, 0,  1, 0, | -2/5],
[0, 1, -1, 0, |  4/5],
[0, 0,  0, 1, |  1/5]]
```

$0 = 1/5$, not possible $\Rightarrow$ System is inconsistent

$$\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ 1 & 0 & 1 & 0 & -2/5 \\ 0 & 1 & -1 & 0 & 4/5 \\ 0 & 0 & 0 & 1 & 1/5 \end{array}$$

x_1, x_2, x_4 = pivot variables

x_3 = free variable

$$x_1 + x_3 = -2/5$$
$$x_2 - x_3 = 4/5$$
$$x_4 = 1/5$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2/5 - x_3 \\ 4/5 + x_3 \\ x_3 \\ 1/5 \end{bmatrix}$$

$$\Rightarrow \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 4/5 \\ 0 \\ 1/5 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \text{ where } x_3 \in \mathbb{R}$$

This system has infinitely many solutions, but there is an easy choice of x_3 which is equal to 0.

$$\underbrace{\begin{bmatrix} 1 & 2 & -1 & -1 \\ 2 & 4 & -2 & 3 \\ -1 & 1 & -2 & 4 \end{bmatrix}}_{\text{Coefficient matrix } A} \underbrace{\left(\begin{bmatrix} -2/5 \\ 4/5 \\ 0 \\ 1/5 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right)}_{\text{Solution vector } x} = \underbrace{\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}}_{\text{Right hand side vector } b}$$

$$\begin{bmatrix} 1 & 2 & -1 & -1 \\ 2 & 4 & -2 & 3 \\ -1 & 1 & -2 & 4 \end{bmatrix} \begin{bmatrix} -2/5 \\ 4/5 \\ 0 \\ 1/5 \end{bmatrix} + x_3 \begin{bmatrix} 1 & 2 & -1 & -1 \\ 2 & 4 & -2 & 3 \\ -1 & 1 & -2 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

Matrix-vector product $= \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$

matrix-vector product $= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\underbrace{\begin{bmatrix} 1 & 2 & -1 & -1 \\ 2 & 4 & -2 & 3 \\ -1 & 1 & -2 & 4 \end{bmatrix}}_{A} \left(\underbrace{x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}}_{\text{solution to } Ax=0} + \underbrace{\begin{bmatrix} -\frac{2}{5} \\ \frac{4}{5} \\ 0 \\ \frac{1}{5} \end{bmatrix}}_{\text{particular solution}} \right)$$

$$\underbrace{\begin{bmatrix}-1 & 1 & -2 & 4\end{bmatrix}}_{A} \qquad \underbrace{\begin{bmatrix}0\end{bmatrix}}_{\text{solution to } Ax=0} \qquad \underbrace{\begin{bmatrix}\tfrac{1}{5}\end{bmatrix}}_{\text{particular solution}}$$

Solution vector

and there are infinitely many of them

$$= \underbrace{\begin{bmatrix}1 & 2 & -1 & -1\\ 2 & 4 & -2 & 3\\ -1 & 1 & -2 & 4\end{bmatrix}}_{A}\left(x_3\begin{bmatrix}-1\\1\\1\\0\end{bmatrix}\right) + \underbrace{\underbrace{\begin{bmatrix}1 & 2 & -1 & -1\\ 2 & 4 & -2 & 3\\ -1 & 1 & -2 & 4\end{bmatrix}}_{A}\begin{bmatrix}-2/5\\4/5\\0\\1/5\end{bmatrix}}$$

$$= x_3\underbrace{\underbrace{\begin{bmatrix}1 & 2 & -1 & -1\\ 2 & 4 & -2 & 3\\ -1 & 1 & -2 & 4\end{bmatrix}}_{A}\begin{bmatrix}-1\\1\\1\\0\end{bmatrix}} + \begin{bmatrix}1\\3\\2\end{bmatrix}$$

$$= x_3\begin{bmatrix}0\\0\\0\end{bmatrix} + \begin{bmatrix}1\\3\\2\end{bmatrix} = \begin{bmatrix}1\\3\\2\end{bmatrix}$$

Linear combination of vectors

$$a_1 = \begin{bmatrix}1\\2\end{bmatrix}, \quad a_2 = \begin{bmatrix}2\\1\end{bmatrix}$$

One linear combination: $0 * \begin{bmatrix}1\\2\end{bmatrix} + 0 * \begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

Multipliers

Another linear combination: $1 * \begin{bmatrix}1\\2\end{bmatrix} + 0 * \begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$

One more linear combination: $1 * \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 * \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

One last linear combination: $\underbrace{? * \begin{bmatrix} 1 \\ 2 \end{bmatrix} + ? * \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$

Matrix-vector product

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

R.H.S. vector is special, it is the zero vector

Augmented matrix $= \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 1 & 0 \end{array}\right] \xrightarrow{\text{Elementary row operations}} \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \end{array}$

RREF

$\Rightarrow \boxed{x_1 = 0 \text{ and } x_2 = 0}$

The only multipliers that can result in a zero linear combination of the vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ are zeros. In other words,

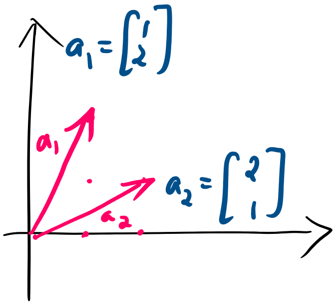
$A x = 0$ is possible only when $x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, the zero vector

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

a_1 a_2

$\Rightarrow$ The vectors a_1 and a_2 (columns of matrix A) are said to be linearly independent.

Geometric perspective of linear independence



The two vectors are not aligned

E.g. $a_1 = \begin{bmatrix} 72 \\ 76 \\ 80 \end{bmatrix}$ (HR), $a_2 = \begin{bmatrix} 144 \\ 152 \\ 160 \end{bmatrix}$ (BP)

Are a_1 and a_2 linearly independent?

$$Ax = 0 \Rightarrow \begin{bmatrix} 72 & 144 \\ 76 & 152 \\ 80 & 160 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Augmented matrix

$$\left[\begin{array}{cc|c} 72 & 144 & 0 \\ 76 & 152 & 0 \\ 80 & 160 & 0 \end{array}\right] \xrightarrow[\text{row operations}]{\text{Elementary}} \begin{array}{c} \begin{array}{cc} x_1 \checkmark & x_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \\ \text{RREF} \end{array}$$

$$x_1 + 2x_2 = 0 \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$
$$x_2 = x_2 \Rightarrow$$

Choose the free variable $x_2 = 1 \Rightarrow x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$$\Rightarrow x_1 a_1 + x_2 a_2 = -2a_1 + 1 \cdot a_2 = 0$$

$\Rightarrow \boxed{a_2 = 2a_1} \Rightarrow$ $\boxed{BP = 2 * HR \text{ is what we understand}}$

The vectors a_1 and a_2 are linearly dependent

Suppose we had HR, BP, and TEMP. We observe that
(BPM) (mm Hg) (°C)

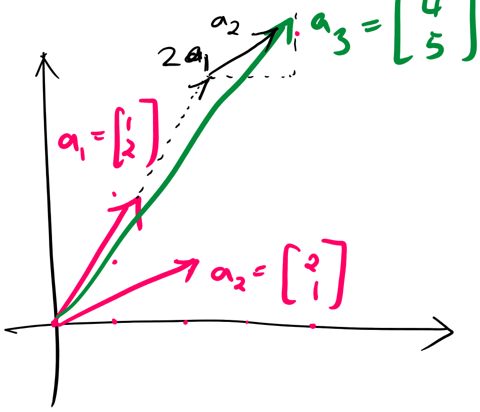
$$\boxed{BP = 1.2 * HR + 1.05 * TEMP}$$

$$\boxed{\text{Haemoglobin} = \beta_0 + \beta_1 * HR + \beta_2 * BP + \beta_3 * TEMP}$$

Linear model

E.g.

$$\underbrace{\begin{array}{ccc} \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 5 \end{bmatrix} \\ a_1 \quad a_2 \quad a_3 \end{array}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\text{0 vector}} \Rightarrow x_1 a_1 + x_2 a_2 + x_3 a_3 = 0$$



$$\begin{array}{c} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \\ x_1 \quad x_2 \quad x_3 \\ \text{pivot} \quad \text{pivot} \quad \text{free} \end{array} = \text{RREF of the augmented matrix}$$

$\Rightarrow x_1 + 2x_3 = 0 \Rightarrow x_1 = -2x_3$
$x_2 + 1 \cdot x_3 = 0 \Rightarrow x_2 = -x_3$
$x_3 = x_3 \Rightarrow x_3 = x_3$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$$

So our choice for $x_3 = 1 \Rightarrow x = \begin{bmatrix} -2 \\ \dots \end{bmatrix} \Rightarrow -2a_1 + (-1)a_2 + a_3 = 0$

Easy choice for $x_3 = 1 \Rightarrow x = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \Rightarrow -2a_1 + (-1)a_2 + a_3 = 0$

$\Rightarrow \boxed{a_3 = 2a_1 + a_2}$

<u>E.g.</u> $A = \begin{bmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{bmatrix}$, are the columns linearly dependent or independent?

Solving $Ax = 0$ (RHS vector is the zero vector)

$$RREF = \left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 - x_3 - 2x_4 = 0 \\ x_2 + 2x_3 + 3x_4 = 0 \end{array}$$

$$\underbrace{x_1 \quad x_2}_{\text{Pivots}} \quad \underbrace{x_3 \quad x_4}_{\text{free}}$$

$$\Rightarrow \begin{array}{l} x_1 = x_3 + 2x_4 \\ x_2 = -2x_3 - 3x_4 \\ x_3 = x_3 \\ x_4 = x_4 \end{array}$$

$$\Rightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

Easy choices for x_3, x_4 are $x_3 = 1$ and $x_4 = 1$

$$\Rightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ 1 \\ 1 \end{bmatrix} \Rightarrow 3a_1 - 5a_2 + 1 \cdot a_3 + 1 \cdot a_4 = 0$$

$\Rightarrow$ Columns of A are linearly dependent

<u>For an $m \times n$ - matrix</u>

(1) $m > n$ (more rows than columns)

no. of pivots = at most n

E.g. $m=5, n=2$, RREF =
$$\begin{array}{cc|c} x_1 & x_2 & \\ \end{array}$$
$$\left[\begin{array}{cc|c} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & * \end{array}\right]$$

(2) $m < n$ (more columns than rows)

no. of pivots = at most m

E.g. $m=2, n=5$, RREF =
$$\begin{array}{cc} x_1 & x_2 \end{array}$$
$$\left[\begin{array}{ccccc|c} 1 & 0 & * & * & * & * \\ 0 & 1 & * & * & * & * \end{array}\right]$$

(3) $m = n$ (same no. of rows and colunns)

no. of pivots = at most m = at most n

E.g. $m=5, n=5$, RREF =
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} \longrightarrow$$

```
[1, 0, -1, -2, 0]
[0, 1,  2,  3, 0]
[0, 0,  0,  0, 0]
```

Coefficient matrix

RREF

$$\boxed{\begin{aligned} x_1 - x_3 - 2x_4 &= 0, \\ x_2 + 2x_3 + 3x_4 &= 0, \end{aligned}} \Rightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 + 2x_4 \\ -2x_3 - 3x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

$$a_1 - 2a_2 + a_3 = 0 \Rightarrow a_3 = -a_1 + 2a_2.$$

$$2a_1 - 3a_2 + a_4 = 0 \Rightarrow a_4 = -2a_1 + 3a_2.$$

Columns corresponding to the free variables can be written as a linear combination of columns corresponding to the pivot variables

$\Downarrow$

The only columns in A that matter are a_1 and a_2

Column space of matrix A denoted as C(A)

The set of all possible vectors that can be generated using a linear combination of the columns of A

$$\begin{array}{c} A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} \\ \quad\ \ a_1 \quad a_2 \quad a_3 \quad a_4 \end{array}$$

$$C(A) = \left\{ \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 + \alpha_4 a_4 \;\middle|\; \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R} \right\}$$

$$= \left\{ \alpha_1 a_1 + \alpha_2 a_2 \;\middle|\; \alpha_1, \alpha_2 \in \mathbb{R} \right\}$$

Why do we need to define a 'column space' of a matrix?

$\boxed{\text{To solve } \underset{m\times n}{A}\,\underset{n}{x} = \underset{m}{b}}$ If this system is consistent, there is a set of values $x_1, x_2, \cdots, x_n$ s.t.

$$x_1 a_1 + x_2 a_2 + x_3 a_3 + \cdots + x_n a_n = b$$

E.g. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, solve $Ax = b$

Augmented matrix = $\begin{array}{cc} x_1 & x_2 \end{array}$ $\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = 1 \\ x_2 = 2 \end{array}$

Now we have $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, Solve $Ax = b$

Augmented matrix = $\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{array}\right]$ → Inconsistent

$\boxed{\text{Is } b \in C(A)?}$

$C(A) = \left\{ \alpha_1 a_1 + \alpha_2 a_2 \,\middle|\, \alpha_1, \alpha_2 \in \mathbb{R} \right\}$

$= \left\{ \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{bmatrix} \,\middle|\, \alpha_1, \alpha_2 \in \mathbb{R} \right\}$

Can we find a 'compromise' solution

- We want to solve $Ax = b$
- There is no s.t. $Ax = b$. Why? Because $b \notin C(A)$
- Let us find a compromise solution x s.t.

$\|Ax - b\|^2$ is as small as possible

$$= \left\| \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\|^2$$

$$= \left\| \begin{bmatrix} x_1 \\ \cdot \end{bmatrix} - \begin{bmatrix} 1 \\ \cdot \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} x_1 - 1 \\ \cdot \end{bmatrix} \right\|^2$$

$$= \left\| \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} x_1 - 1 \\ x_2 - 2 \\ -3 \end{bmatrix} \right\|$$

$$\Rightarrow x_1 = 1, \; x_2 = 2$$

The least squares solution to $Ax = b$ is obtained by minimizing $\|Ax - b\|^2$ $\xrightarrow[\text{Calculation}]{\text{Gradient}}$ $\boxed{x = A(A^TA)^{-1}A^Tb}$

E.g. $\|2x - 4\|^2$, $x = 2$

$$\left\| \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 4 \\ 6 \end{bmatrix} \right\|^2, \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

- Projection of a vector onto the direction of another vector

E.g.

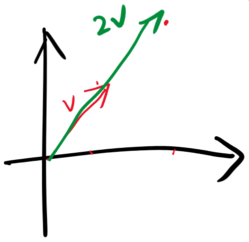
Data matrix

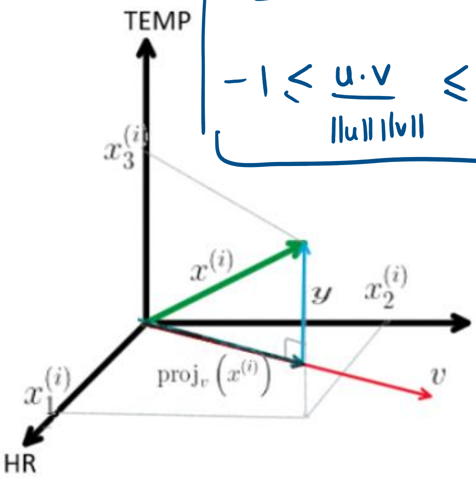
	HR	BP	Temp
Patient-1	76	126	38.0
Patient-2	74	120	38.0
Patient-3	72	118	37.5
Patient-4	78	136	37.0

$$x^{(1)} = \begin{bmatrix} 76 \\ 126 \\ 38 \end{bmatrix}$$

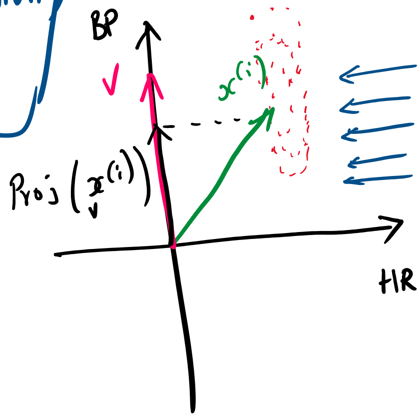
E.g.

$v = \begin{bmatrix}1\\2\end{bmatrix}, \quad 2v = 2\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}2\\4\end{bmatrix}$



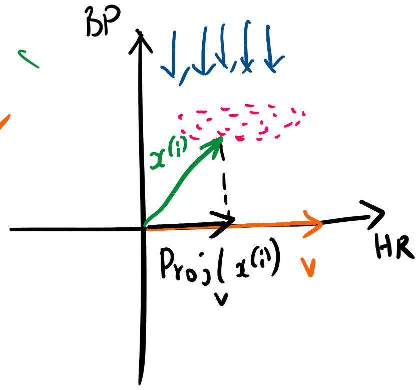


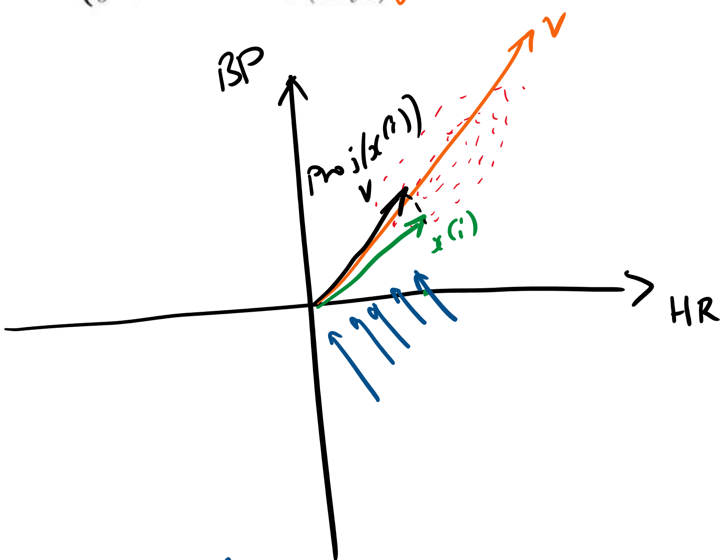
$$\boxed{\begin{gathered}\angle(u,v) = \cos^{-1}\left(\frac{u\cdot v}{\|u\|\,\|v\|}\right)\\ -1 \le \frac{u\cdot v}{\|u\|\,\|v\|} \le 1\end{gathered}}$$



We can derive an expression for the projection as follows:

$$\begin{cases}\text{proj}_v\left(x^{(i)}\right) & = cv, \text{ for some unknown constant } c \text{ (why?)} \\ y & = x^{(i)} - \text{proj}_v\left(x^{(i)}\right) \text{ (why?)} \\ y\cdot v & = 0 \text{ (why?)}\end{cases}$$





$y\cdot v = 0$ (y and v are orthogonal to each other)

$\Rightarrow \left[x^{(i)} - \text{Proj}_v(x^{(i)})\right]\cdot v = 0$

$\Rightarrow x^{(i)}\cdot v - \underbrace{\text{Proj}_v(x^{(i)})}_{cv}\cdot v = 0$

$\Rightarrow x^{(i)}\cdot v - (cv)\cdot v = 0 \Rightarrow c(v\cdot v) = x^{(i)}\cdot v$

$$\Rightarrow c = \frac{x^{(i)}\cdot v}{v\cdot v} \Rightarrow \boxed{Proj_v\, x^{(i)} = cv = \left[\frac{x^{(i)}\cdot v}{v\cdot v}\right] v}$$

$$\boxed{v\cdot v = \|v\|^2}, \quad v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

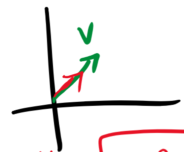
$$v\cdot v = v_1^2 + v_2^2$$

$$\|v\|^2 = \left(\sqrt{v_1^2 + v_2^2}\right)^2 = v_1^2 + v_2^2$$

$$Proj_v\,(x^{(i)}) = \left(\frac{x^{(i)}\cdot v}{\|v\|}\right)\frac{v}{\|v\|} \qquad = \|v\|^2 = v\cdot v$$

unit vector (direction)
length = 1

E.g. $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$



$$\frac{v}{\|v\|} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \quad \left\|\frac{v}{\|v\|}\right\| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = 1$$

$$Proj_v\,(x^{(i)}) = \boxed{\frac{x^{(i)}\cdot v}{\|v\|}}\ \boxed{\frac{v}{\|v\|}}$$

Magnitude of the projection (Shadow length) — Direction of projection

Suppose the vector v is already a unit vector,

$$\boxed{Proj_v\,(x^{(i)}) = (x^{(i)}\cdot v)\, v}$$

Block matrix-vector operations

$$\begin{bmatrix} (x^{(1)})^T v \\ (x^{(2)})^T v \\ (x^{(3)})^T v \\ (x^{(4)})^T v \end{bmatrix} = \begin{bmatrix} x^{(1)T} \\ x^{(2)T} \\ x^{(3)T} \\ x^{(4)T} \end{bmatrix} v = Xv$$

$\Downarrow$ vector

$$\begin{bmatrix} 1\cdot 5 \\ 2\cdot 5 \\ 3\cdot 5 \\ 4\cdot 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} 5$$

$\left\lfloor \left(x^{(4)}\right)^{\mathrm{T}} \underline{v} \right\rfloor$

$\lfloor x^{(4)T} \rfloor$ vector

$\Downarrow$

matrix

	HR	BP	Temp
Patient-1	76	126	38.0
Patient-2	74	120	38.0
Patient-3	72	118	37.5
Patient-4	78	136	37.0

Data matrix in terms of its rows

$$= X = \begin{bmatrix} x^{(1)T} \\ x^{(2)T} \\ x^{(3)T} \\ x^{(4)T} \end{bmatrix}$$

Data matrix in terms of its columns

$$= [x_1 \;\; x_2 \;\; x_3]$$

a = vector of heart rates

b = vector of BP

$\dfrac{a \cdot b}{\|a\| \|b\|}$ is guaranteed to be between -1 and 1

covariance between HR and BP

$$\boxed{\dfrac{a_{MC} \cdot b_{MC}}{\|a_{MC}\| \, \|b_{MC}\|}}$$

is also guaranteed to be between -1 and 1

Correlation coeff. between HR and BP

This tells us how HR and BP are correlated.