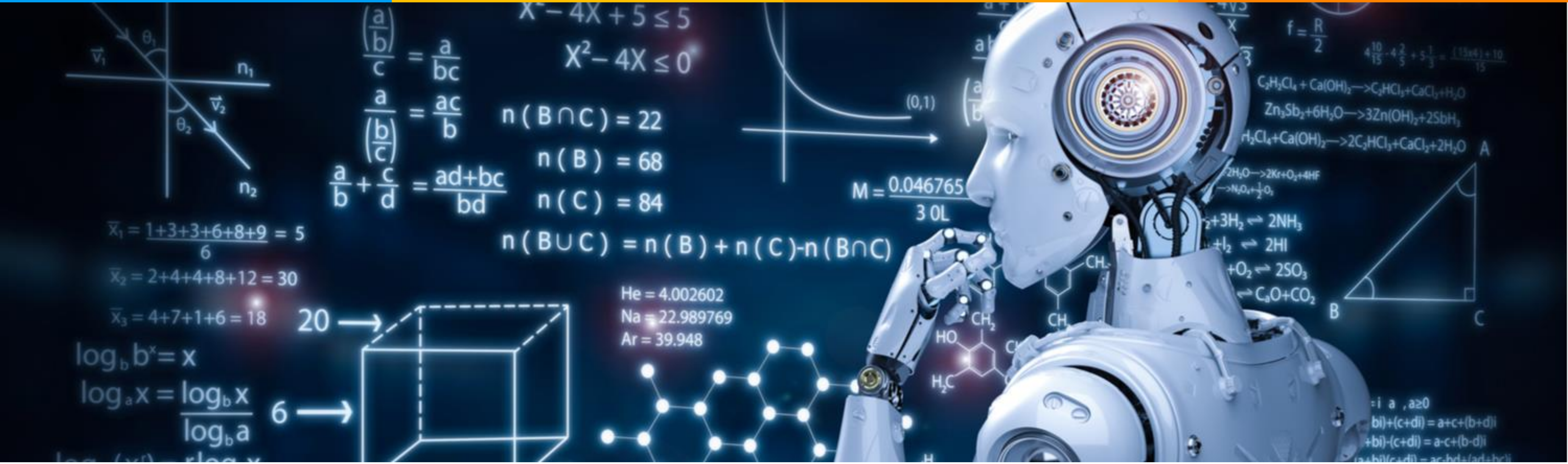
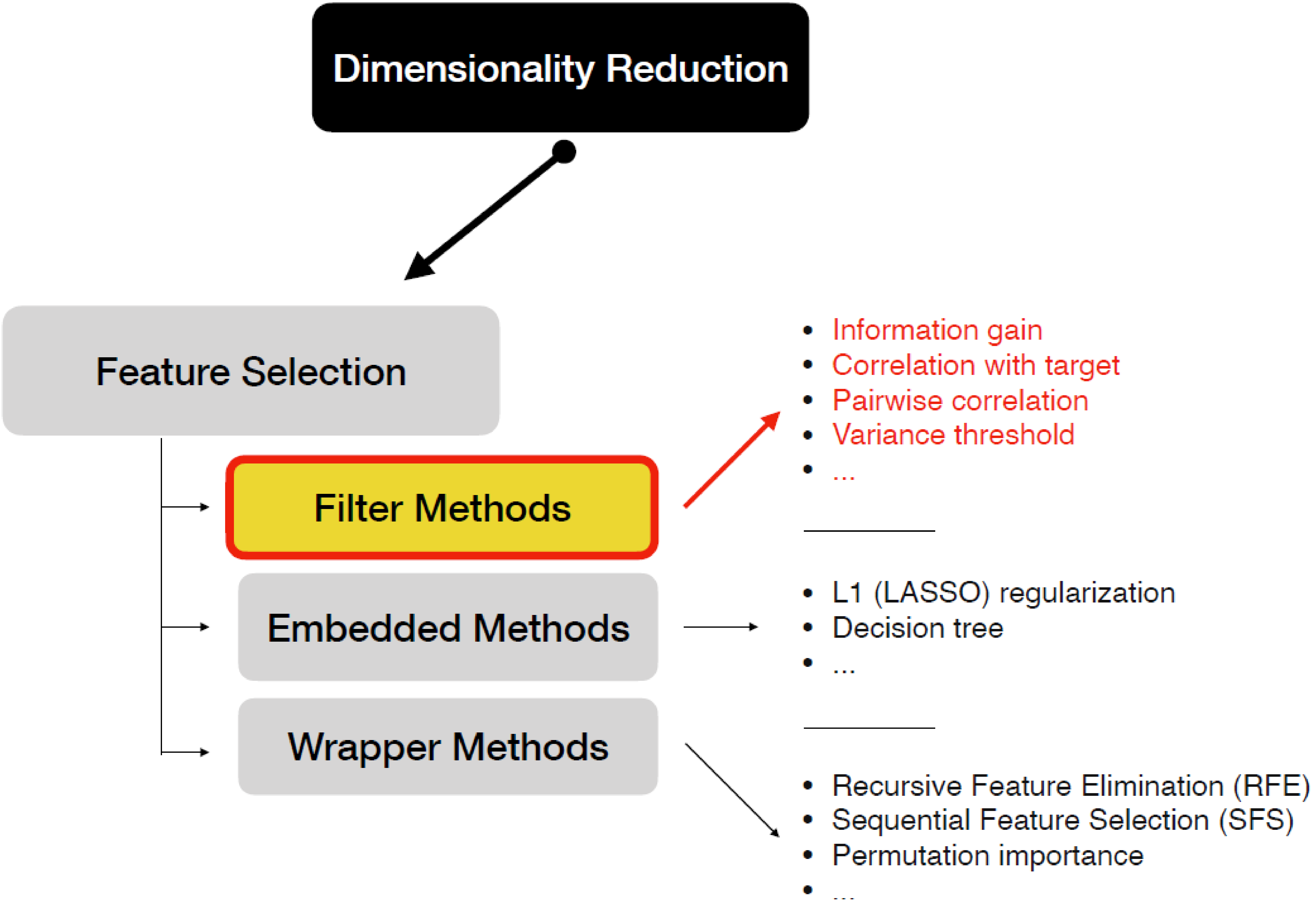
Duration: 60 min | Srikanth Shenoy | AML 5102 | M.E. AI & ML

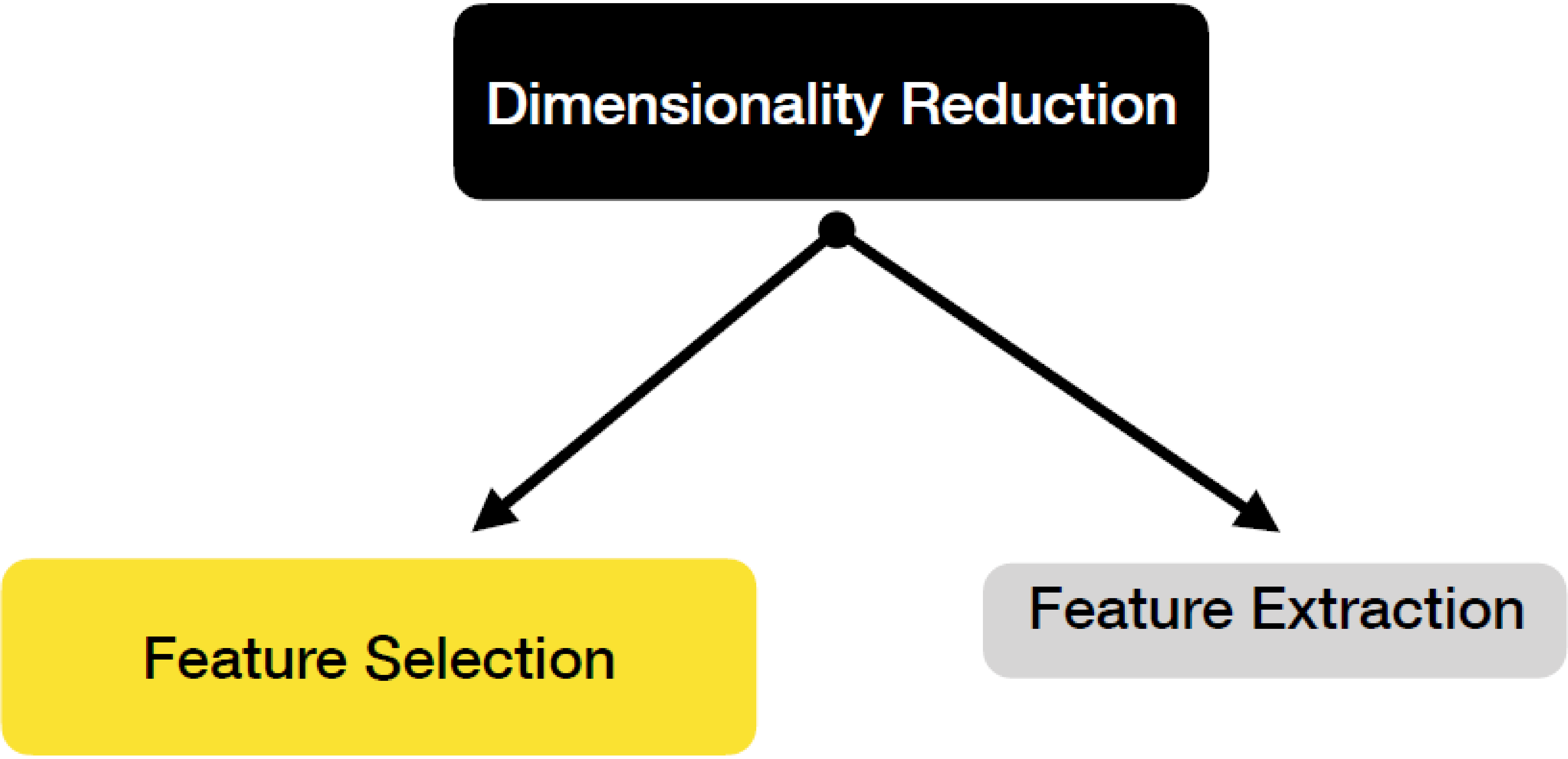


Lecture 30 & 31: Perceptron & SVM

Recap


Dimensionality Reduction
Feature Selection
Filter Methods
Embedded Methods
Wrapper Methods
• Information gain
• Correlation with target
• Pairwise correlation
• Variance threshold
• ...
• L1 (LASSO) regularization
• Decision tree
• ...
• Recursive Feature Elimination (RFE)
• Sequential Feature Selection (SFS)
• Permutation importance
• ...

Feature Extraction

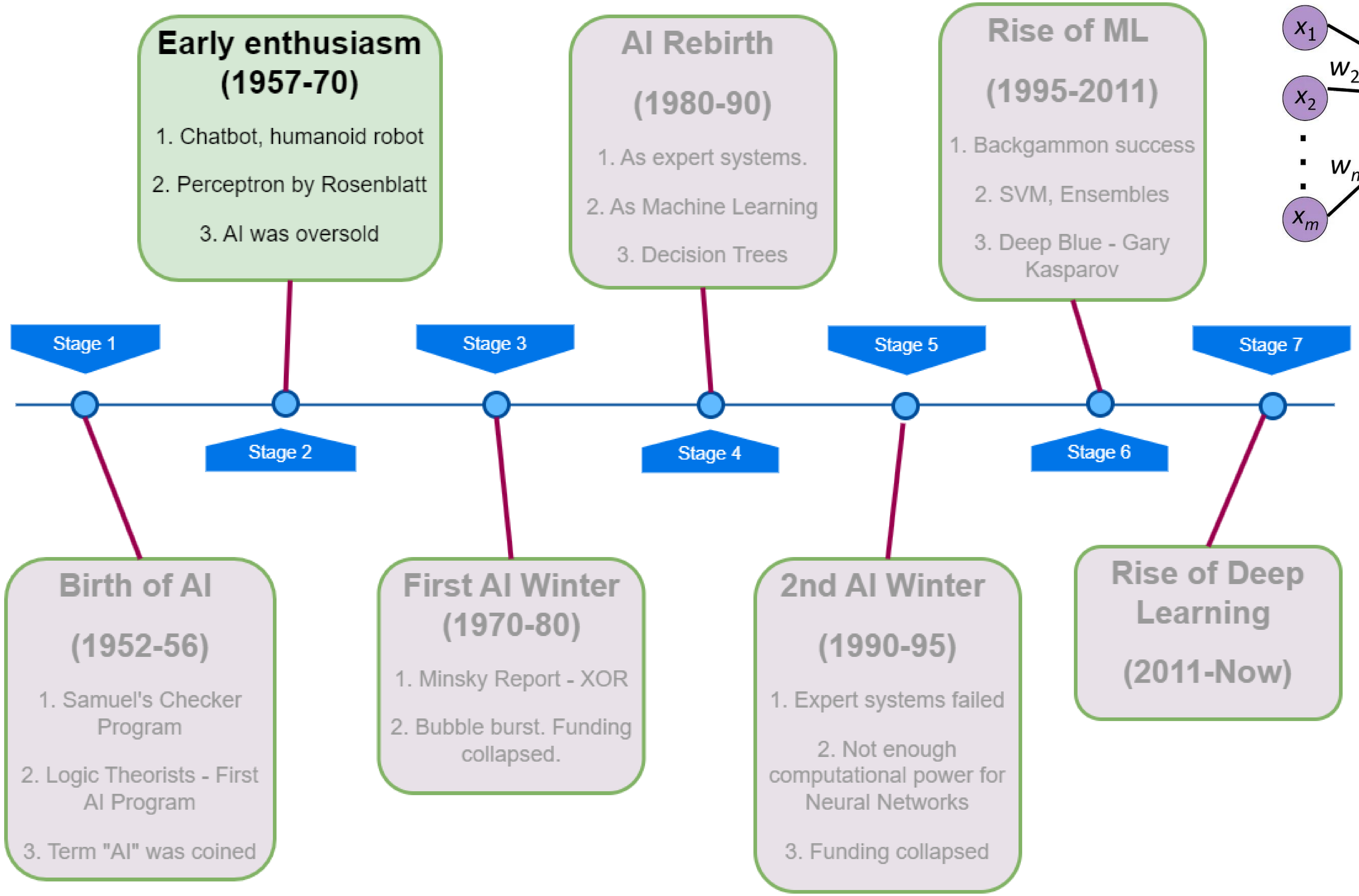


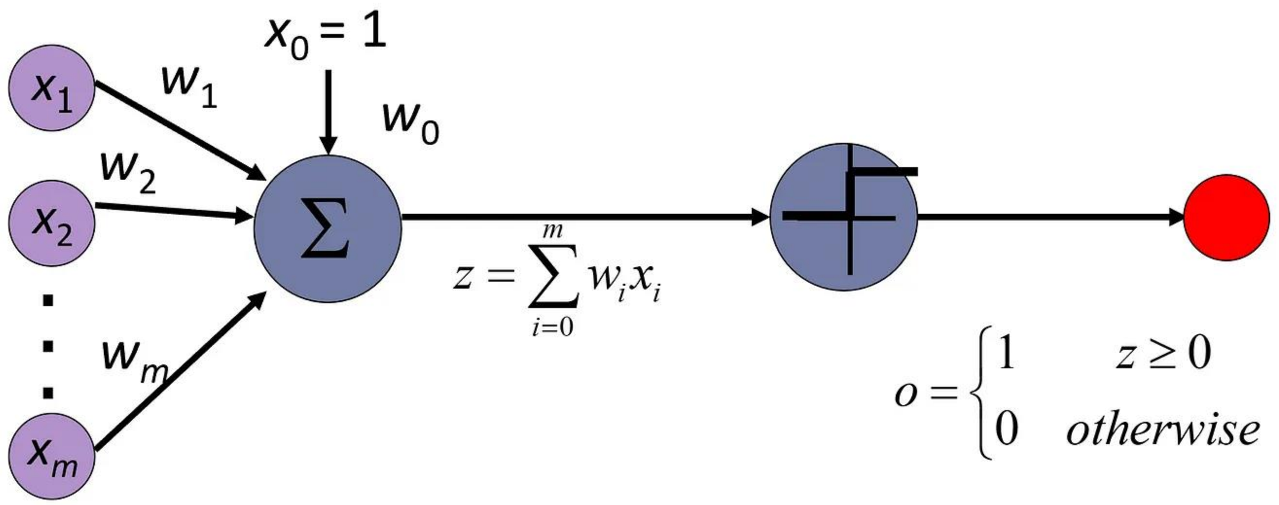
- Feature Extraction is combination of features
- Automatic elimination of unwanted features
- PCA, Kernel PCA, t-SNE, UMAP, Autoencoder


$X^2 - 4X \le 0$
$n(B \cap C) = 22$
$n(B) = 68$
$n(C) = 84$
$n(B \cup C) = n(B) + n(C) - n(B \cap C)$
$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$
$\bar{x}_2 = 2+4+4+8+12 = 30$
$\bar{x}_3 = 4+7+1+6 = 18$
$\log_b b^x = x$
$\log_a x = \frac{\log_b x}{\log_b a}$
He = 4.002602
Na = 22.989769
Ar = 39.948
(0,1)
$f = \frac{R}{2}$
n_1
n_2
20
6

Perceptron

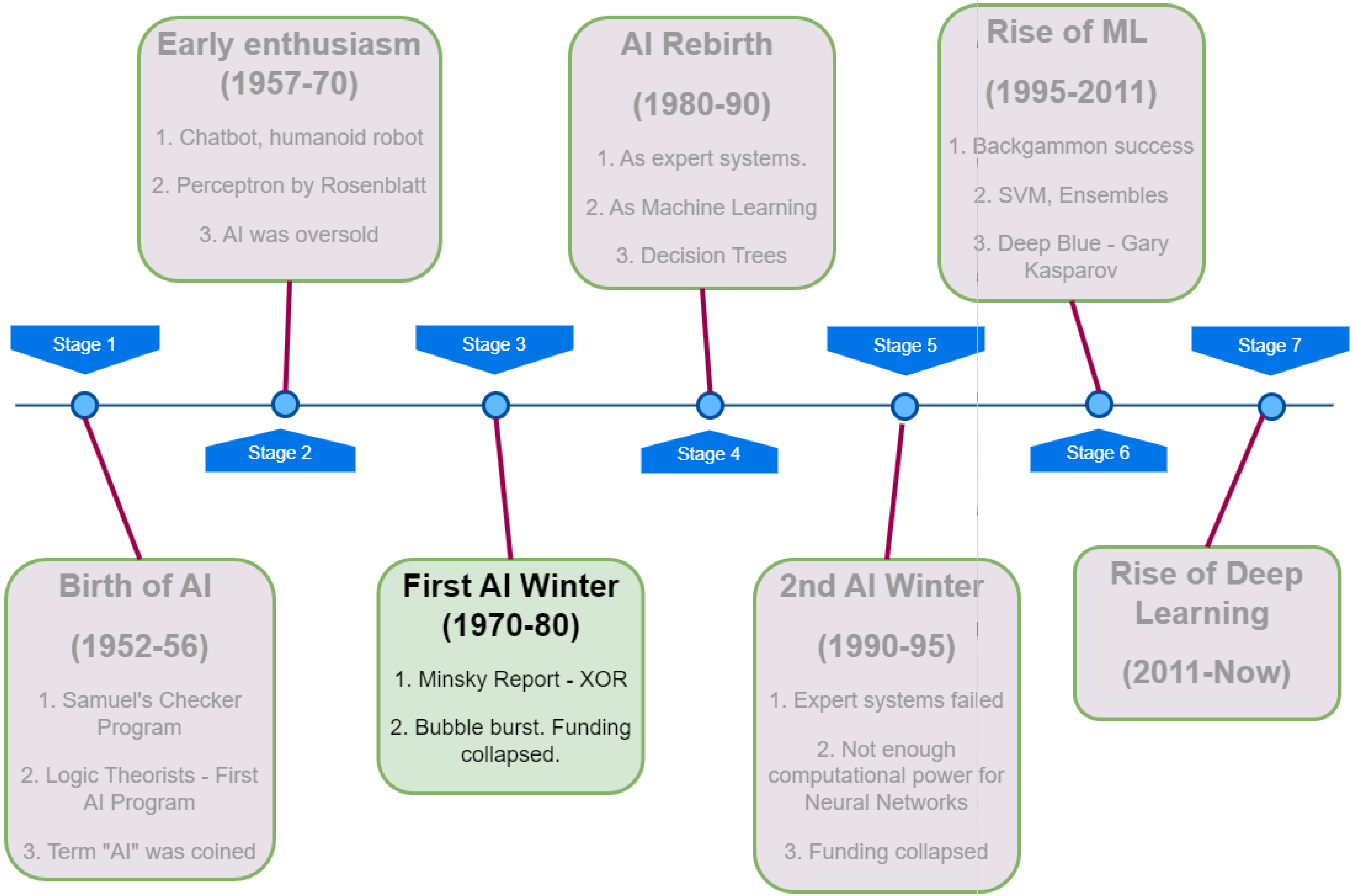
Stage 2 – Early enthusiasm



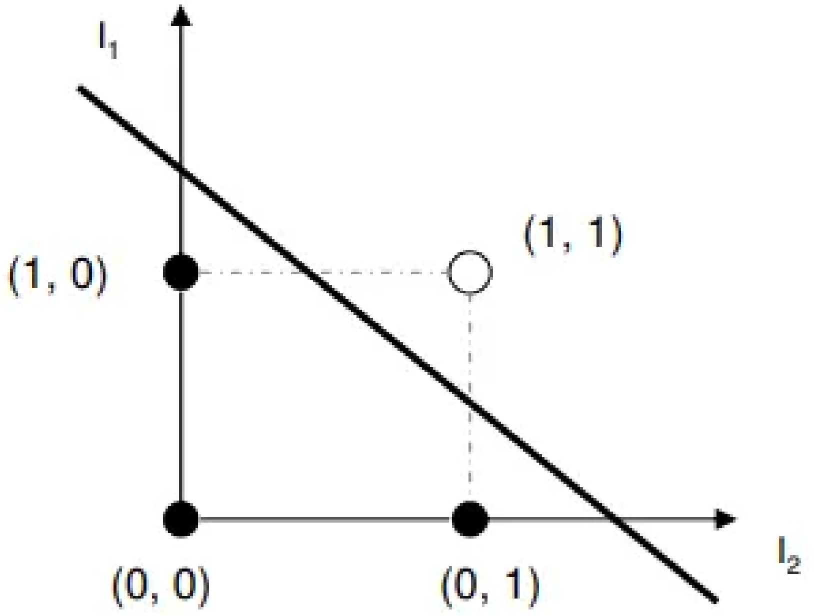


Perceptron is the idea behind neural networks

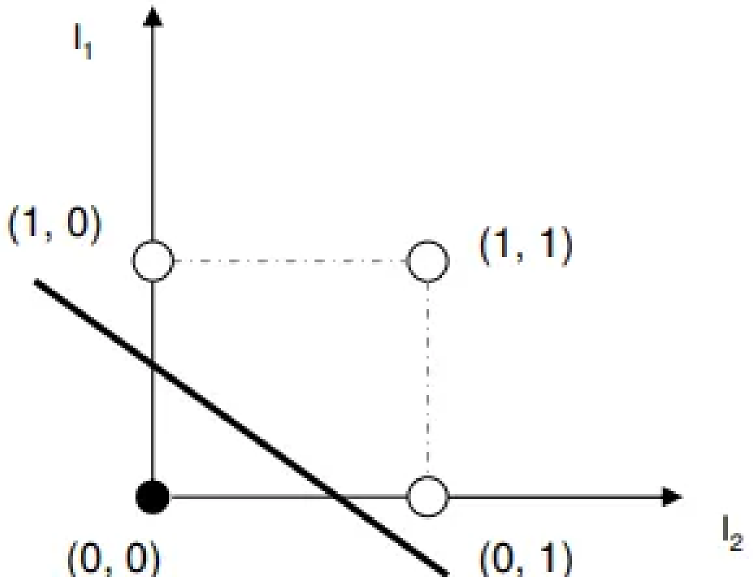
Stage 3 – First AI winter



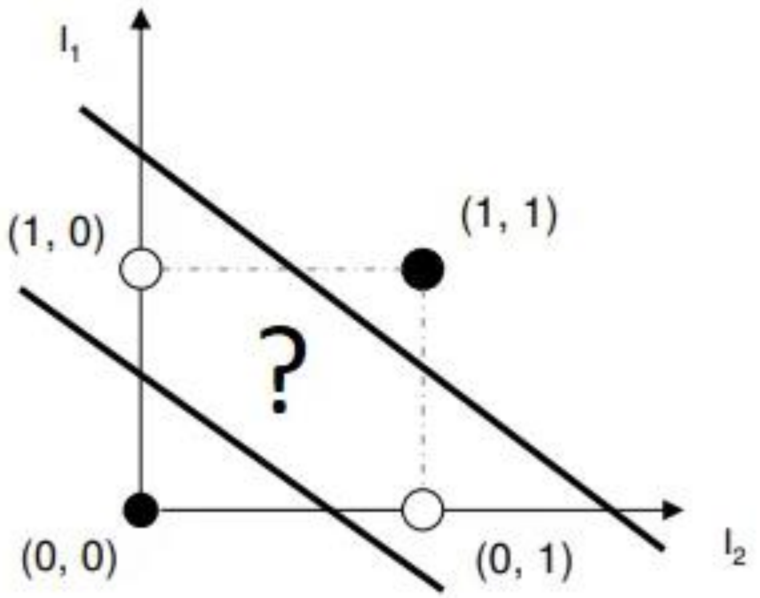
AND		
I_1	I_2	out
0	0	0
0	1	0
1	0	0
1	1	1



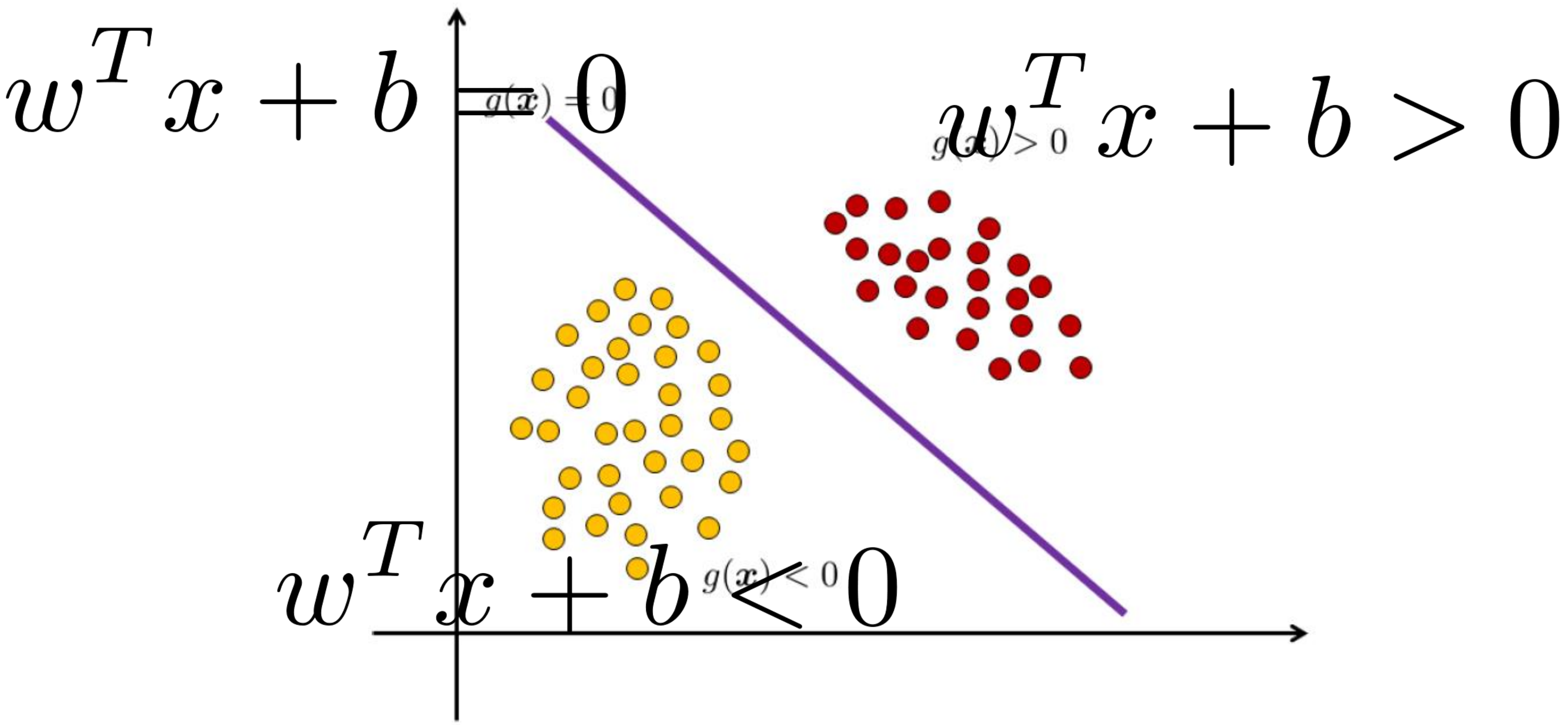
OR		
I_1	I_2	out
0	0	0
0	1	1
1	0	1
1	1	1

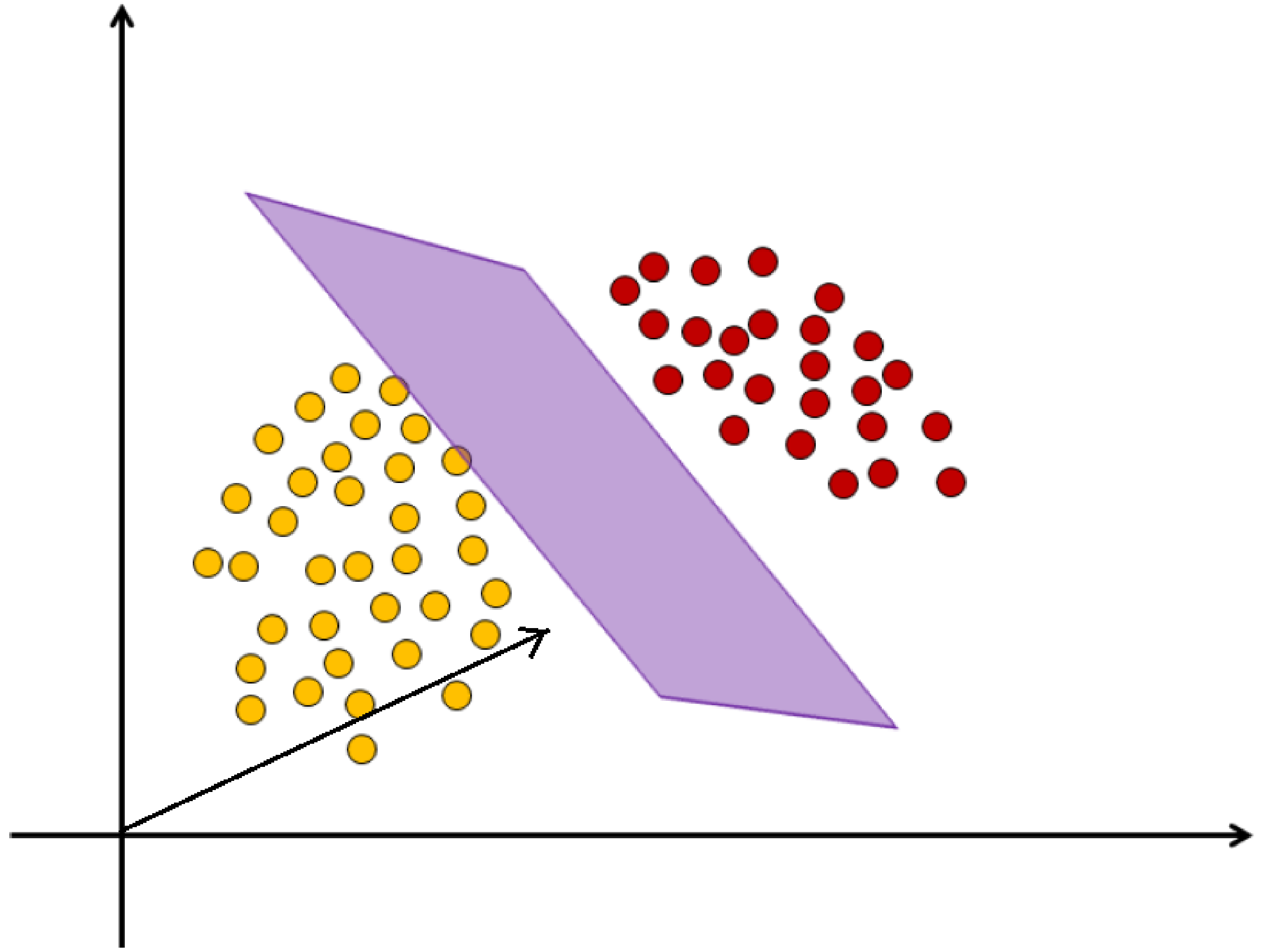


XOR		
I_1	I_2	out
0	0	0
0	1	1
1	0	1
1	1	0



Decision boundary in binary classification

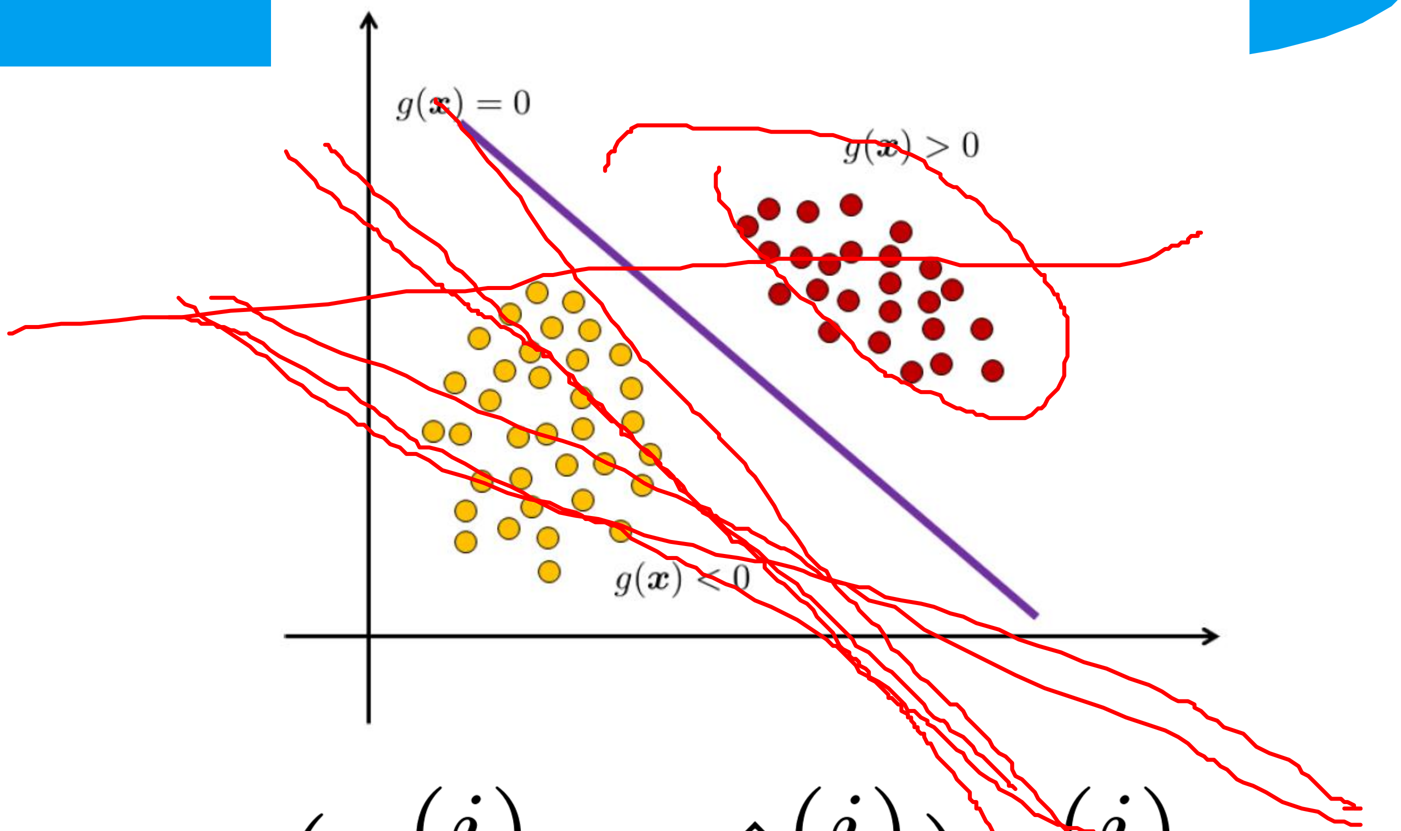




- y is +1, -1
- Calculate wTx+b for a given x
- Product of y and wTx+b
- Sign() function

y	y-hat	Prediction
1	1	Correct
-1	-1	Correct
1	-1	Incorrect
-1	1	Incorrect

Perceptron learning algorithm



- Select random w and b
- while num_iter < K
  - for each record in dataset
    - yhat = wTx + b
    - If y * yhat < 0
      - adjust w and b

$$w = w + \alpha(y^{(i)} - \hat{y}^{(i)})x^{(i)}$$

$$b = b + \alpha(y^{(i)} - \hat{y}^{(i)})$$

$$\hat{y}^{(i)}_{updated} = y^{(i)} \cdot (w^T_{updated}x^{(i)} + b_{updated})$$

Simplify

$$\hat{y}^{(i)}_{updated} = (positive \;\; number)$$

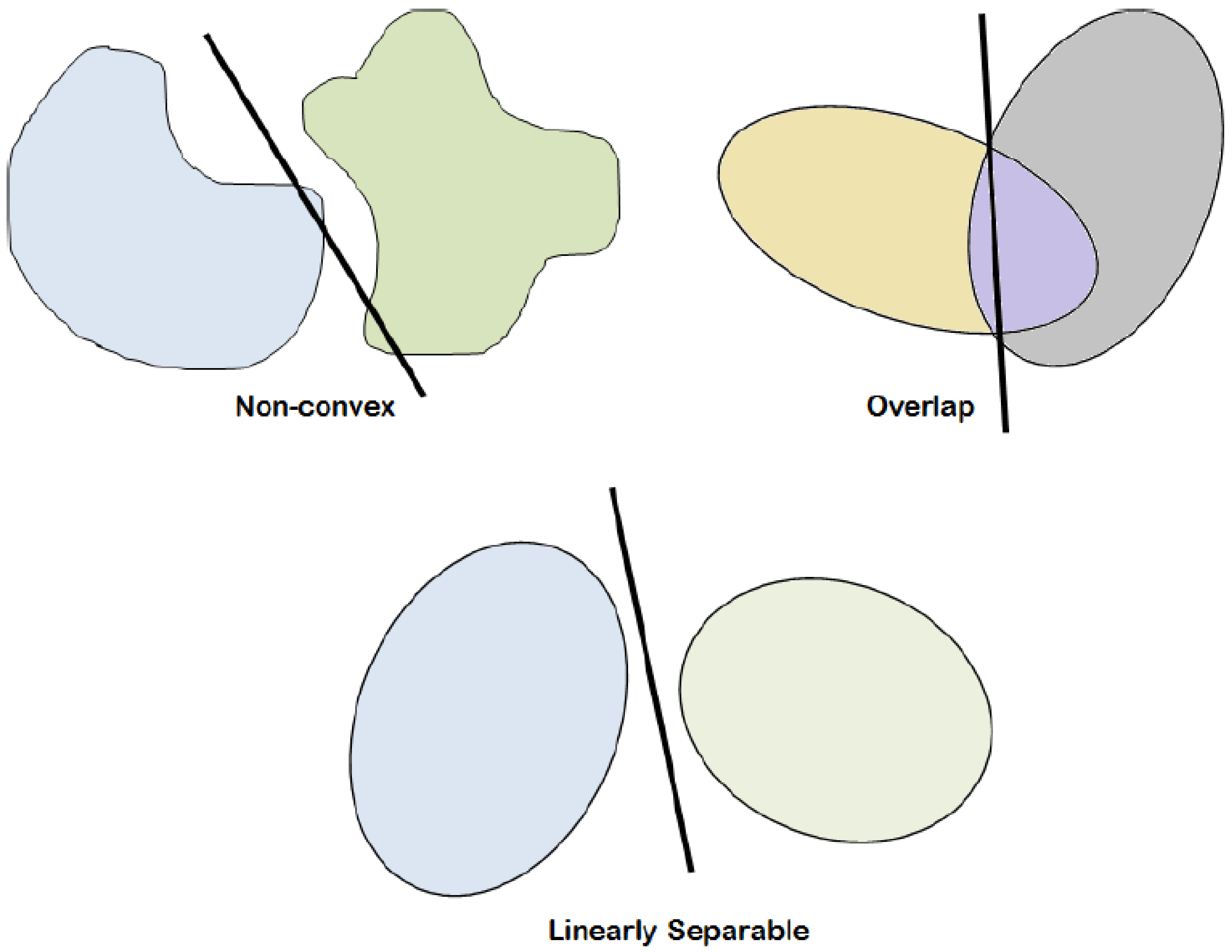
Problems with Perceptron learning algorithm

- Adjustments to w & b not systematic

$$w = w + \alpha(y^{(i)} - \hat{y}^{(i)})x^{(i)}$$

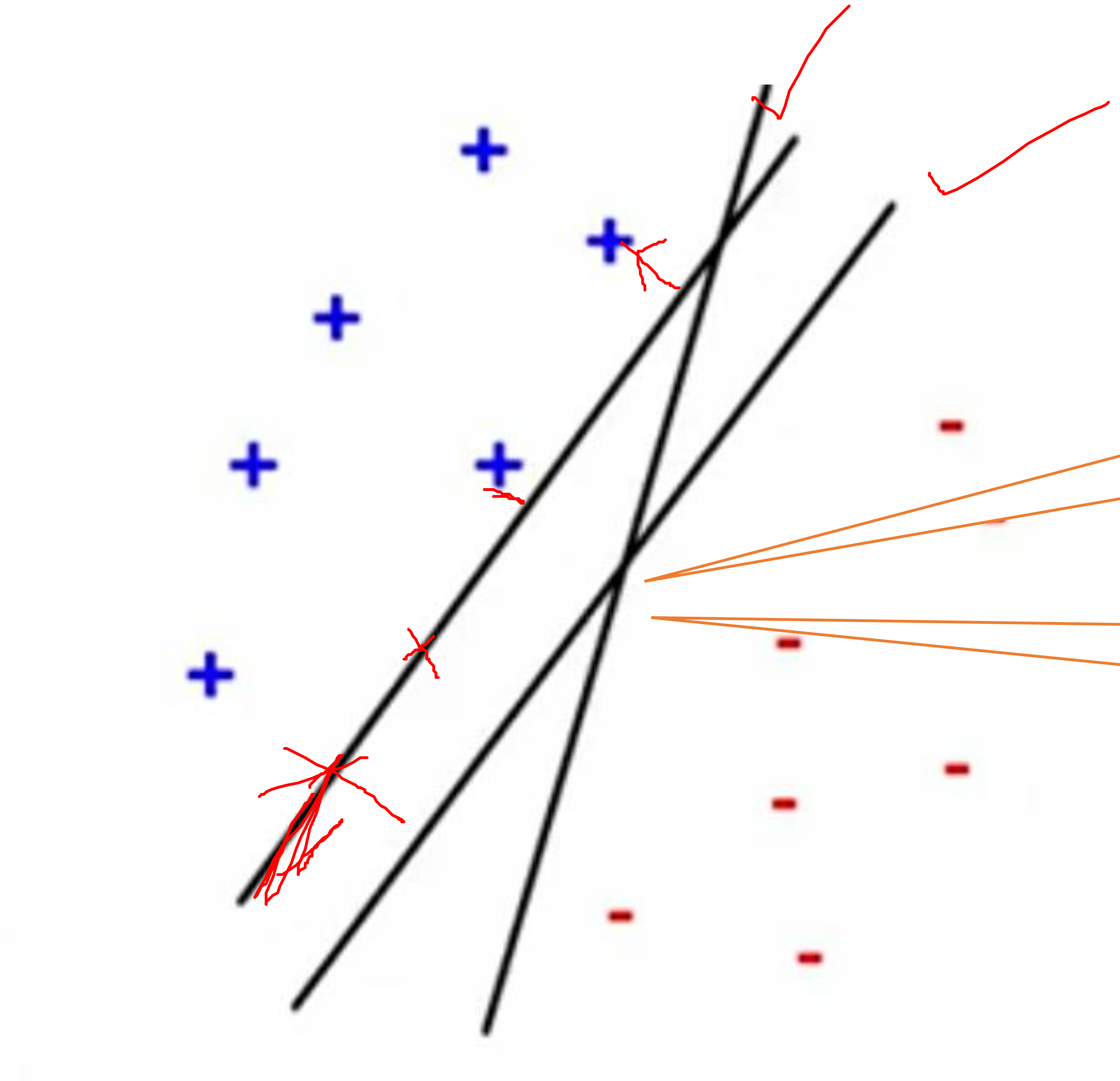
$$b = b + \alpha(y^{(i)} - \hat{y}^{(i)})$$

- Does not converge for non linear decision boundary



Problems with perceptron learning algorithm

- May get any of these boundaries

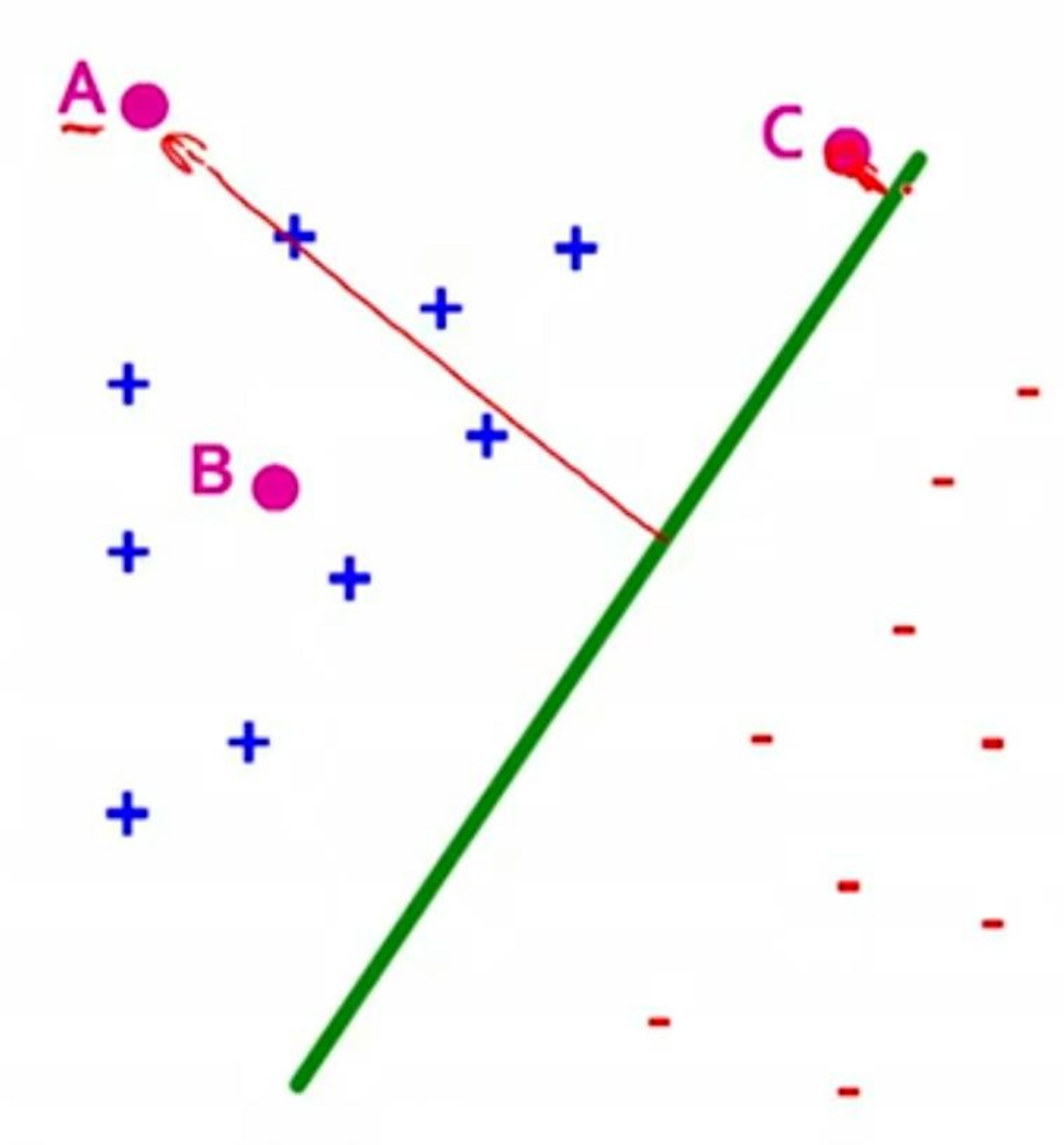


How to maximize margin?

Ans: SVM

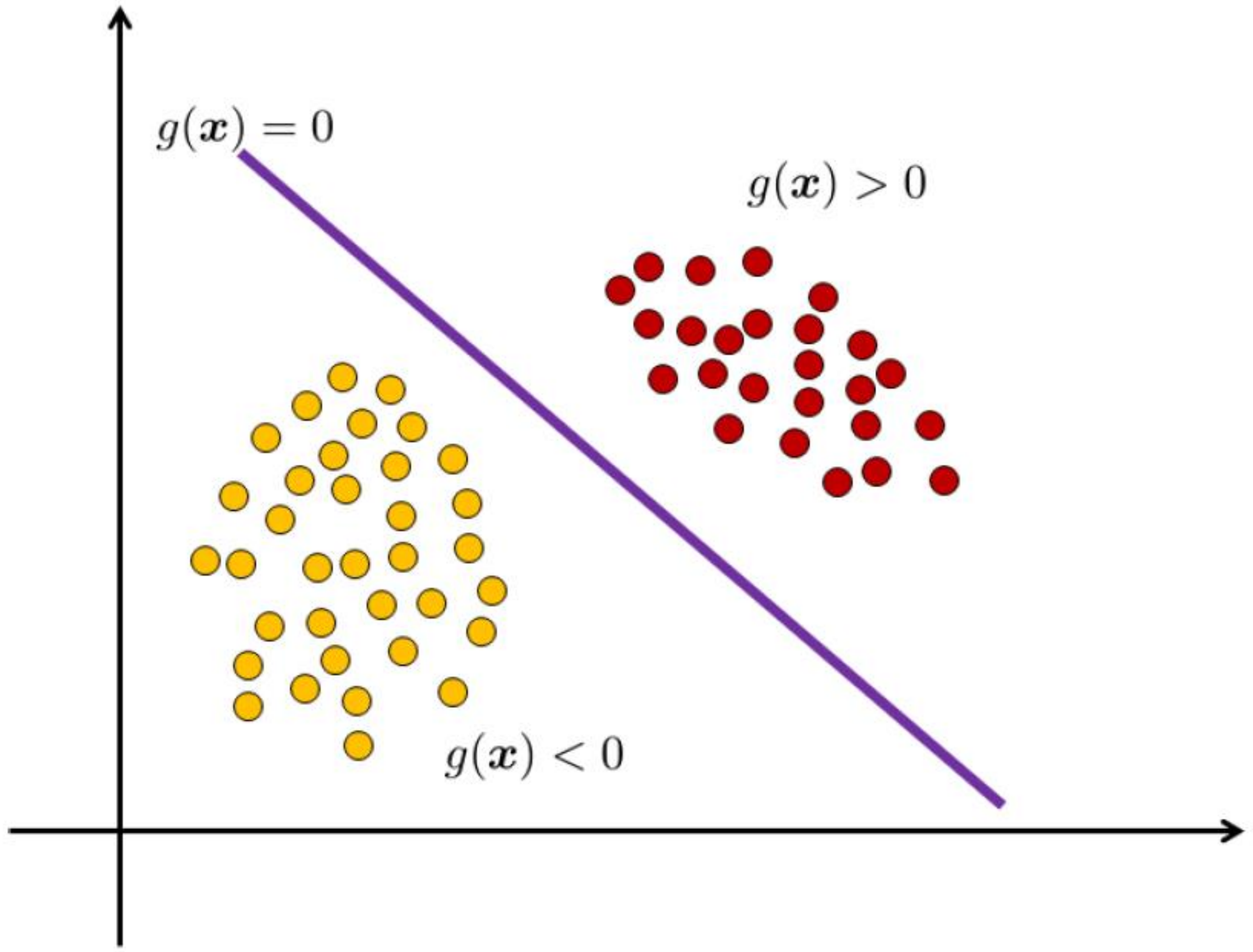
SVM intuition: Distance and confidence level

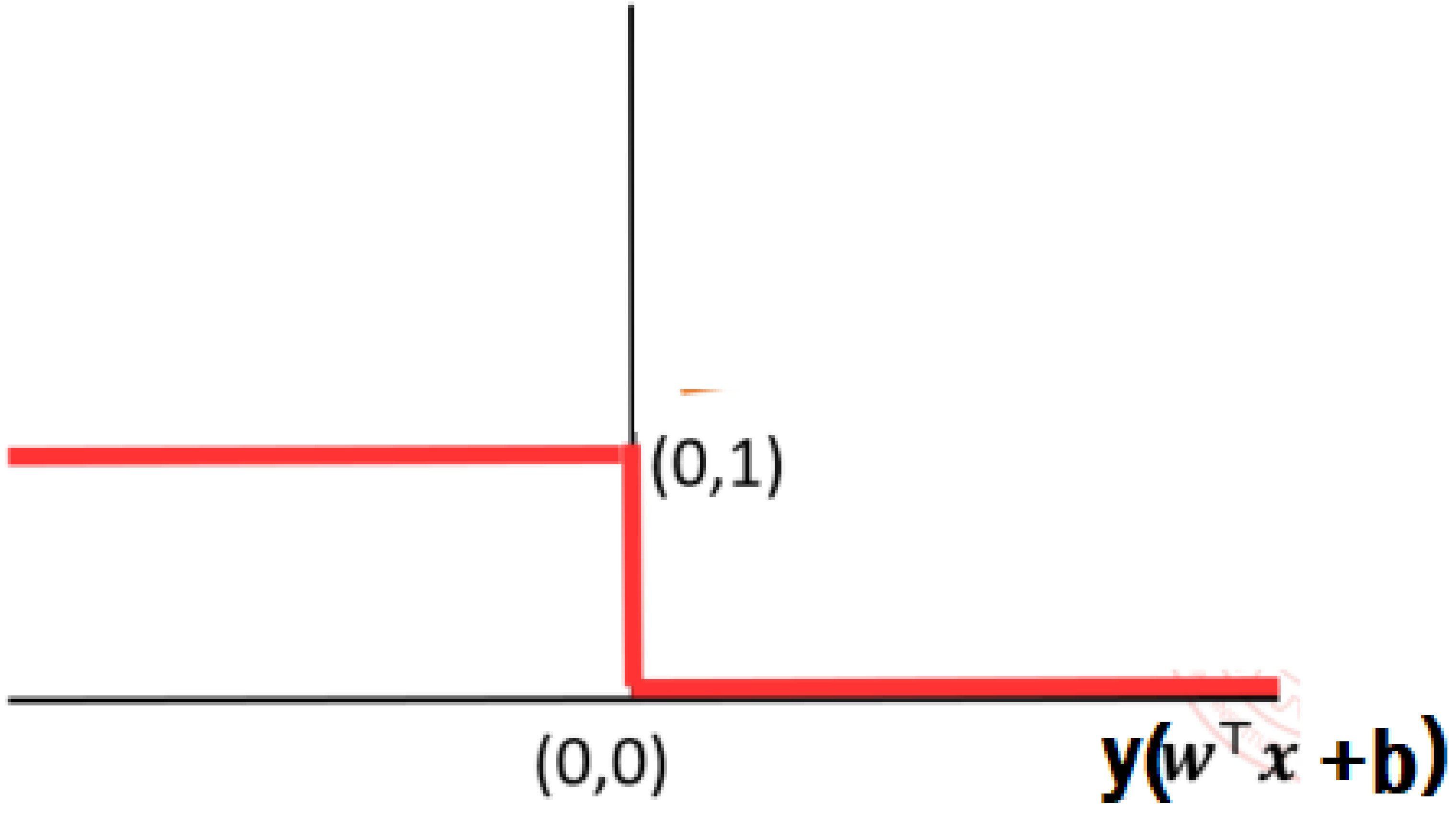
- Distance from the hyperplane is a measure of confidence of prediction



Perceptron & SVM loss functions

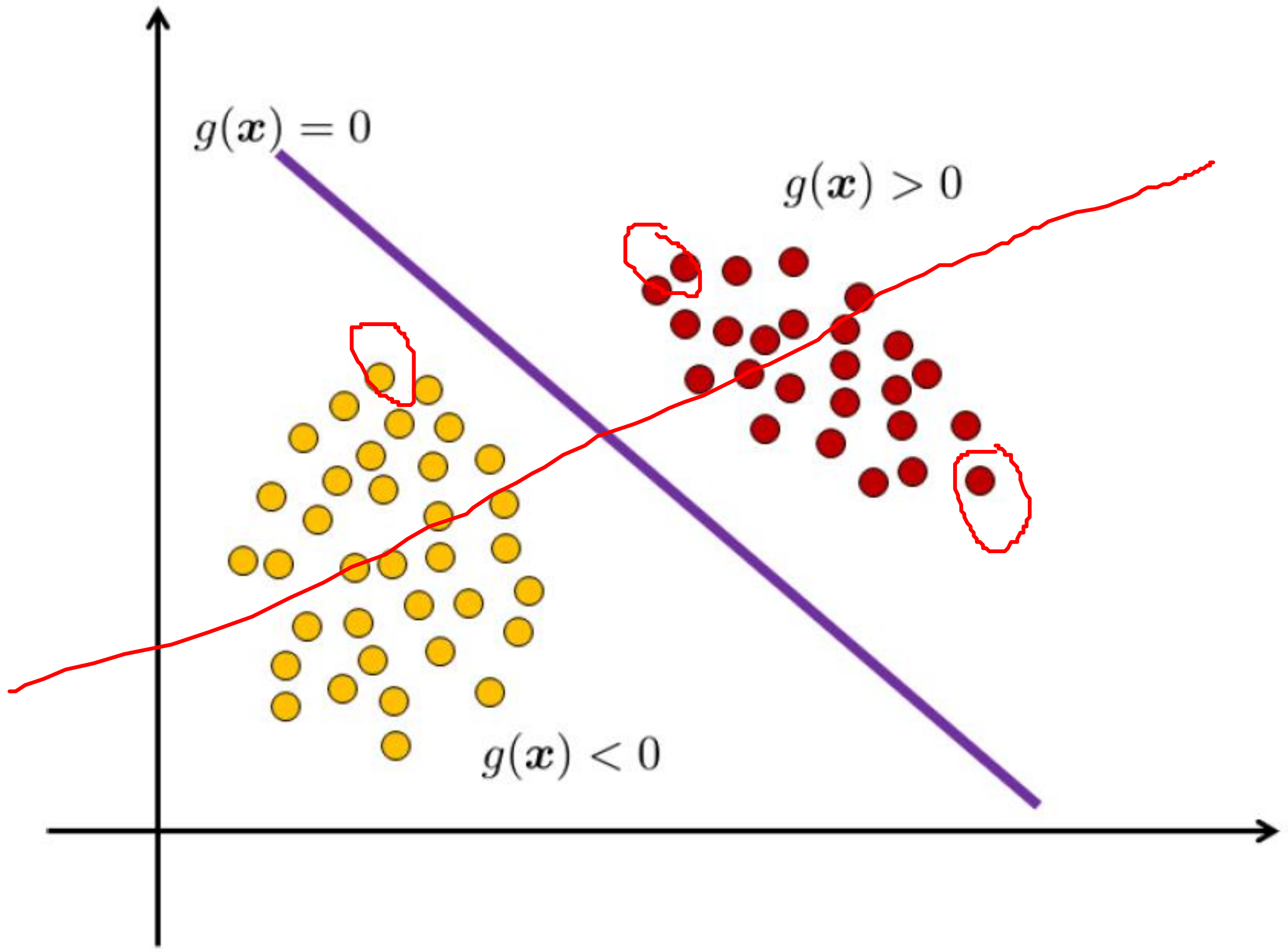
A rudimentary loss function in perceptron

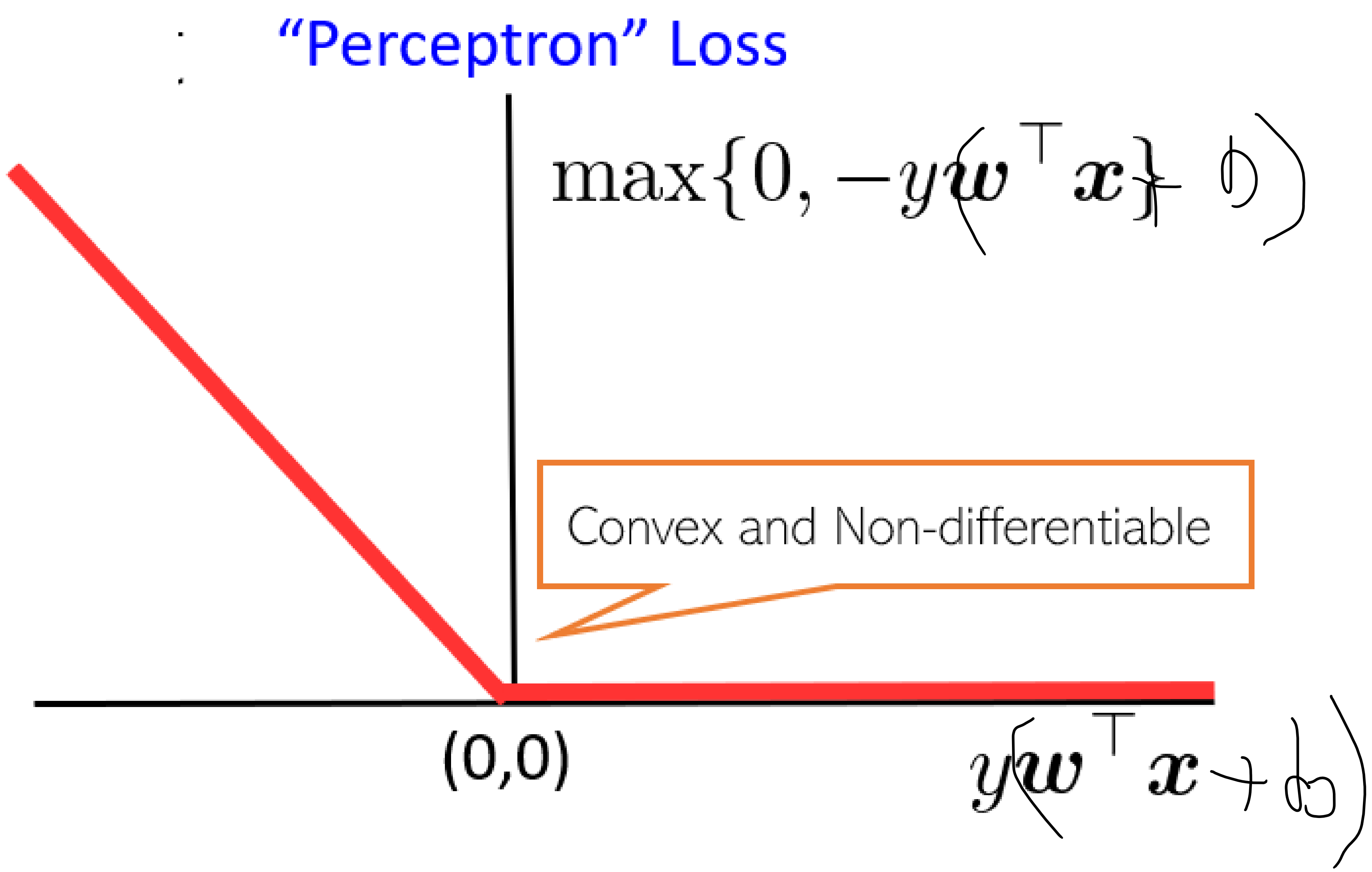




- Problems
  - Non convex, non differentiable
  - Loss does not take distance into account

A better loss function

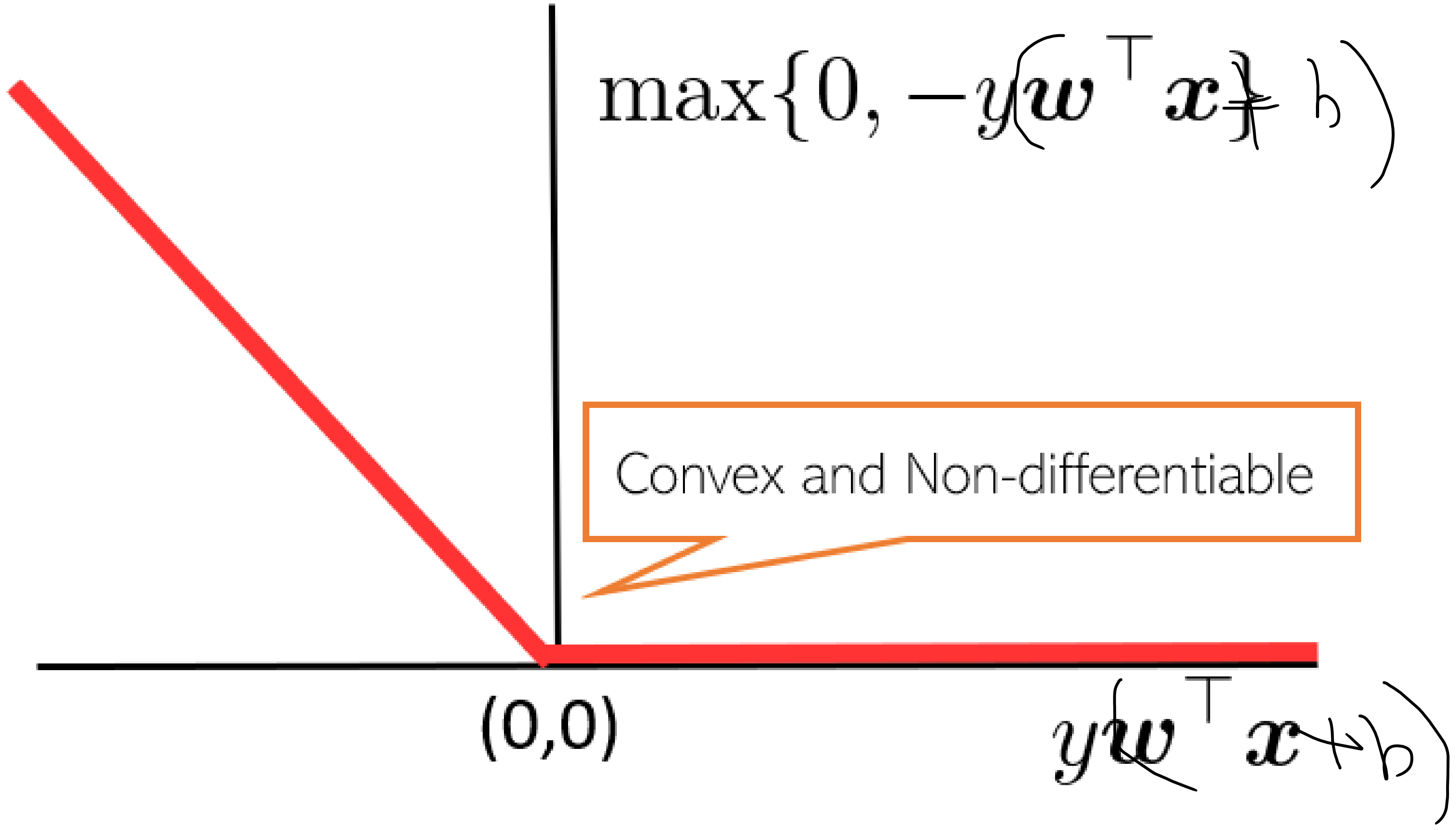




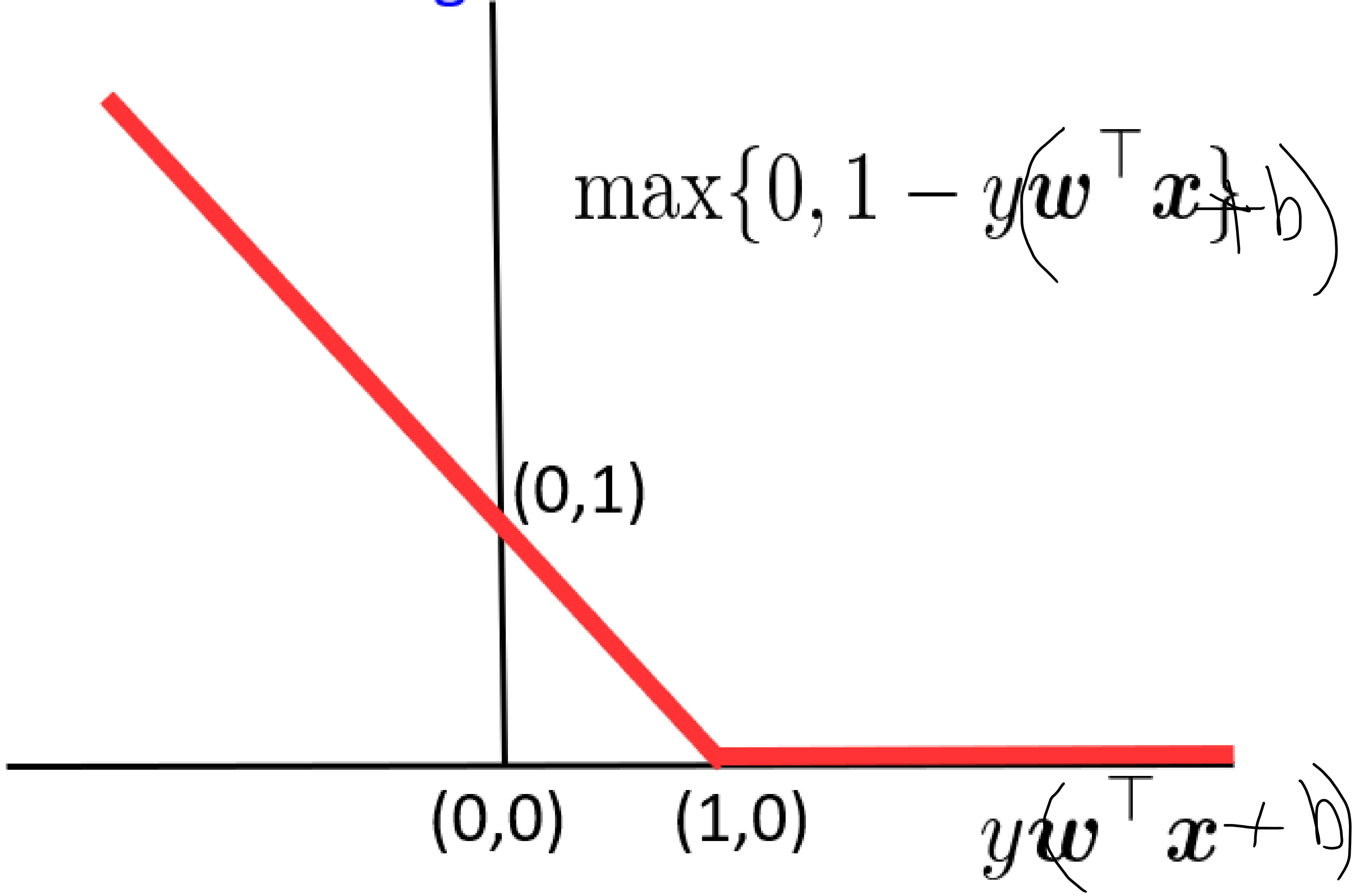
- But How to calculate distance?

An even better loss function

"Perceptron" Loss



Hinge Loss

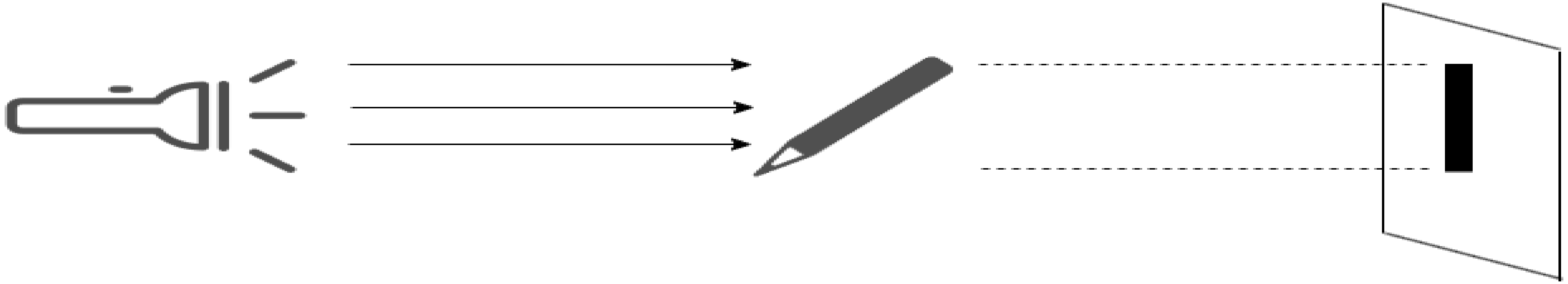


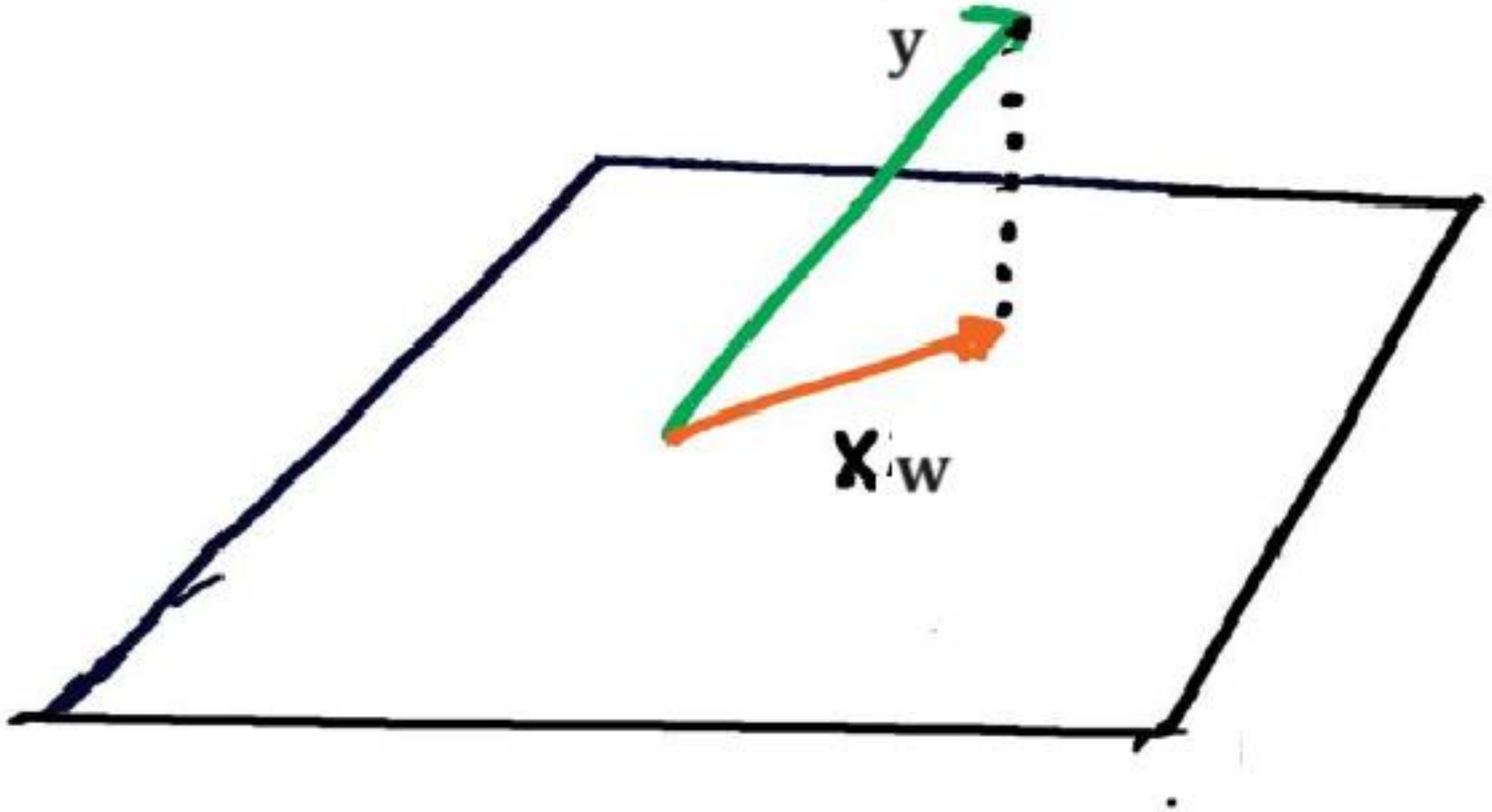
Linear Algebra of separating hyperplane

Length of the projection

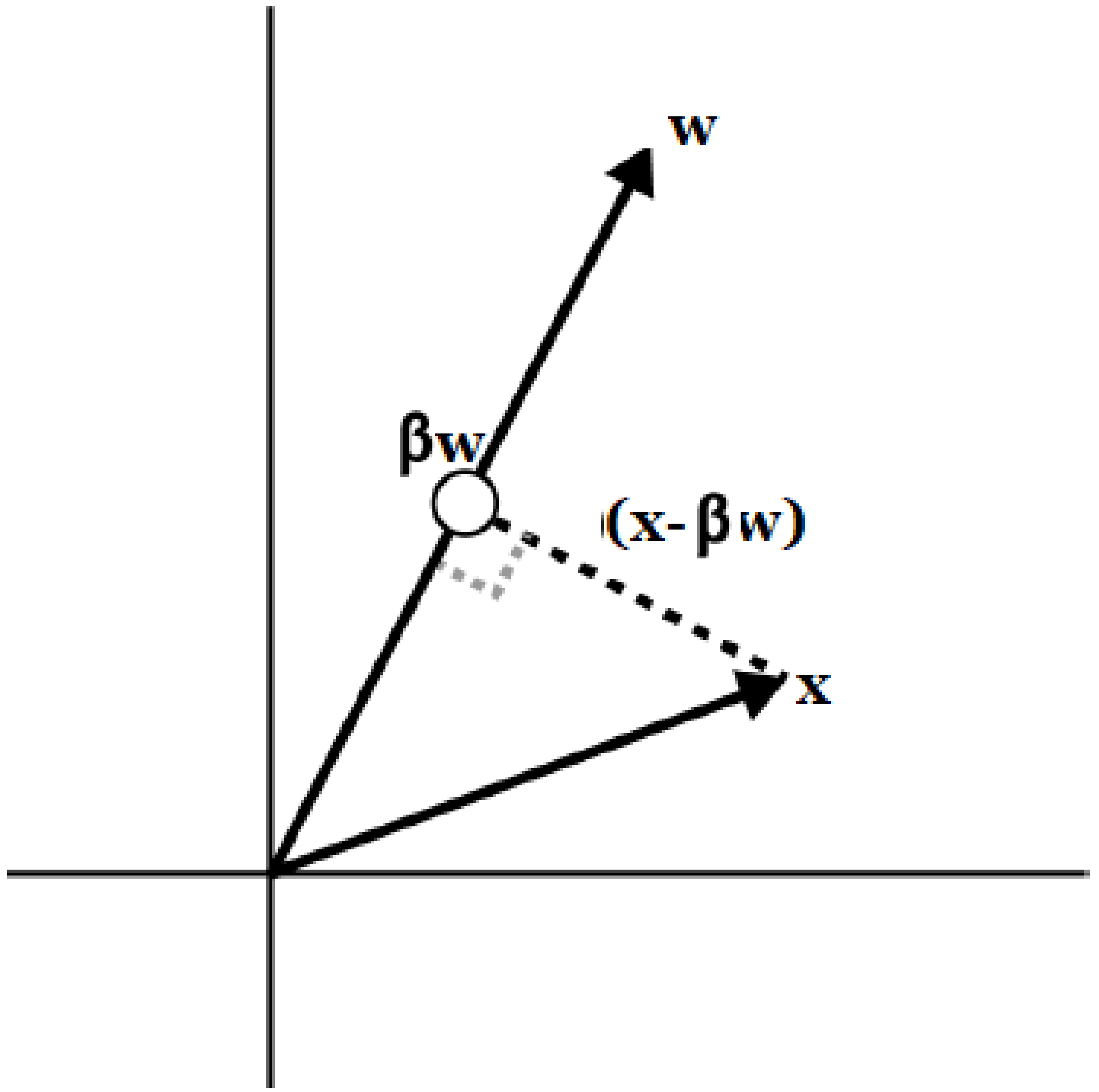
- Dot product
- Orthogonality
- Dot product of orthogonal vectors = 0

Dot Product, Shortest distance





Orthogonal component w.r.t. another vector

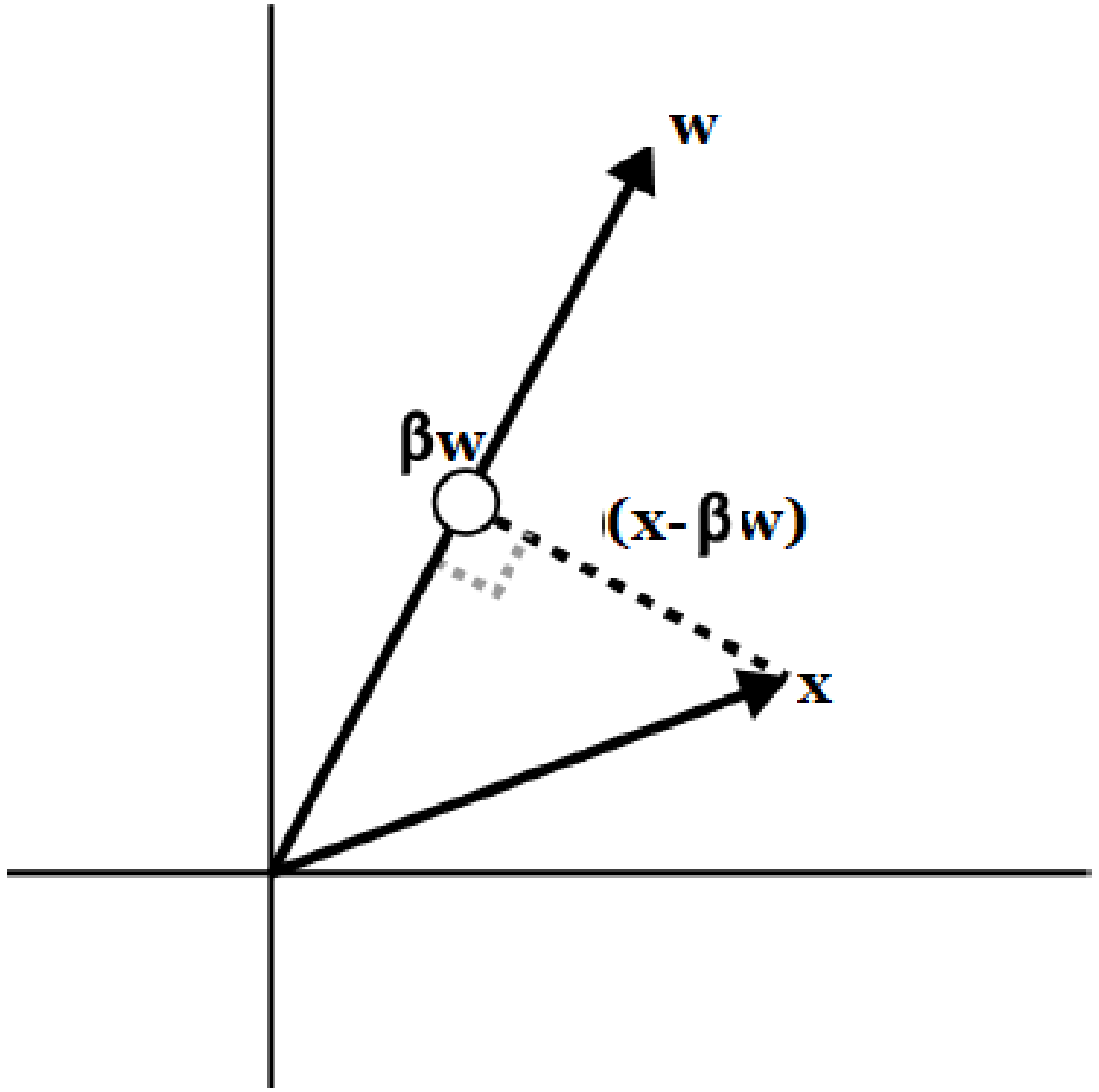


- Projection of x onto w βw
- Difference of projection vector βw and x is $x - \beta w$
- Projection vector βw is such as to minimize distance $x - \beta w$
- Then w and $x - \beta w$ are orthogonal

$$w^T(x - \beta w) = 0 \implies w^T x = \beta w^T w \implies \beta = \frac{w^T x}{w^T w}$$

$$\implies \beta w = \frac{w^T x}{\|w\|^2} w$$

Orthogonal component w.r.t. another vector



$$\beta = \frac{w^T x}{w^T w} \implies \beta w = \frac{w^T x}{\|w\|^2} w$$

$$\beta w = \frac{w^T x}{\|w\|} \frac{w}{\|w\|}$$

Unit vector in the direction of w

Length of projection $= \dfrac{w^T x}{\|w\|}$

w and separating hyperplane

- Demo $w^T x = 0$

- Demo $w^T x = k; \quad w^T x = -k$

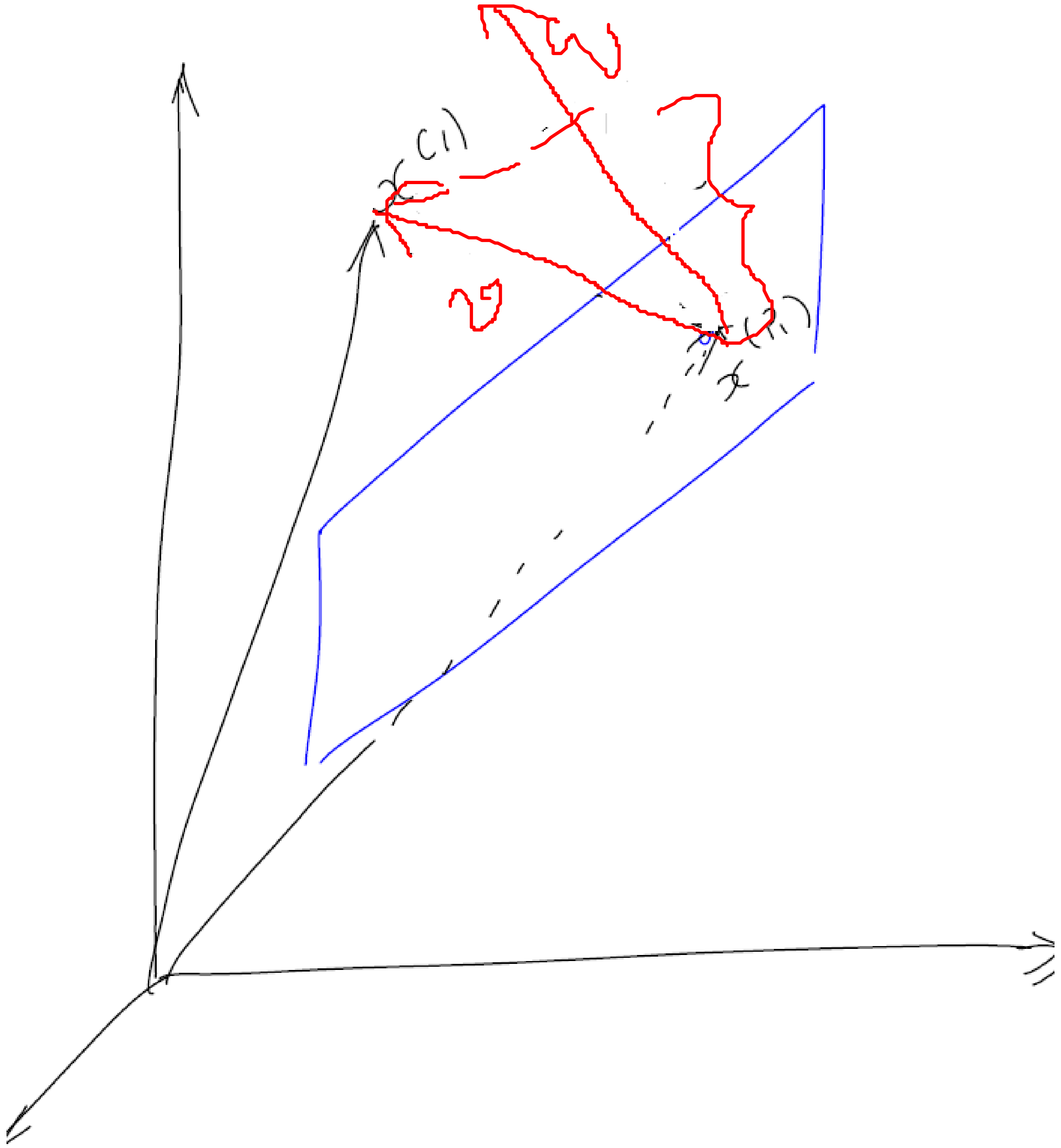
  $w^T x - k = 0; \quad w^T x + k = 0$

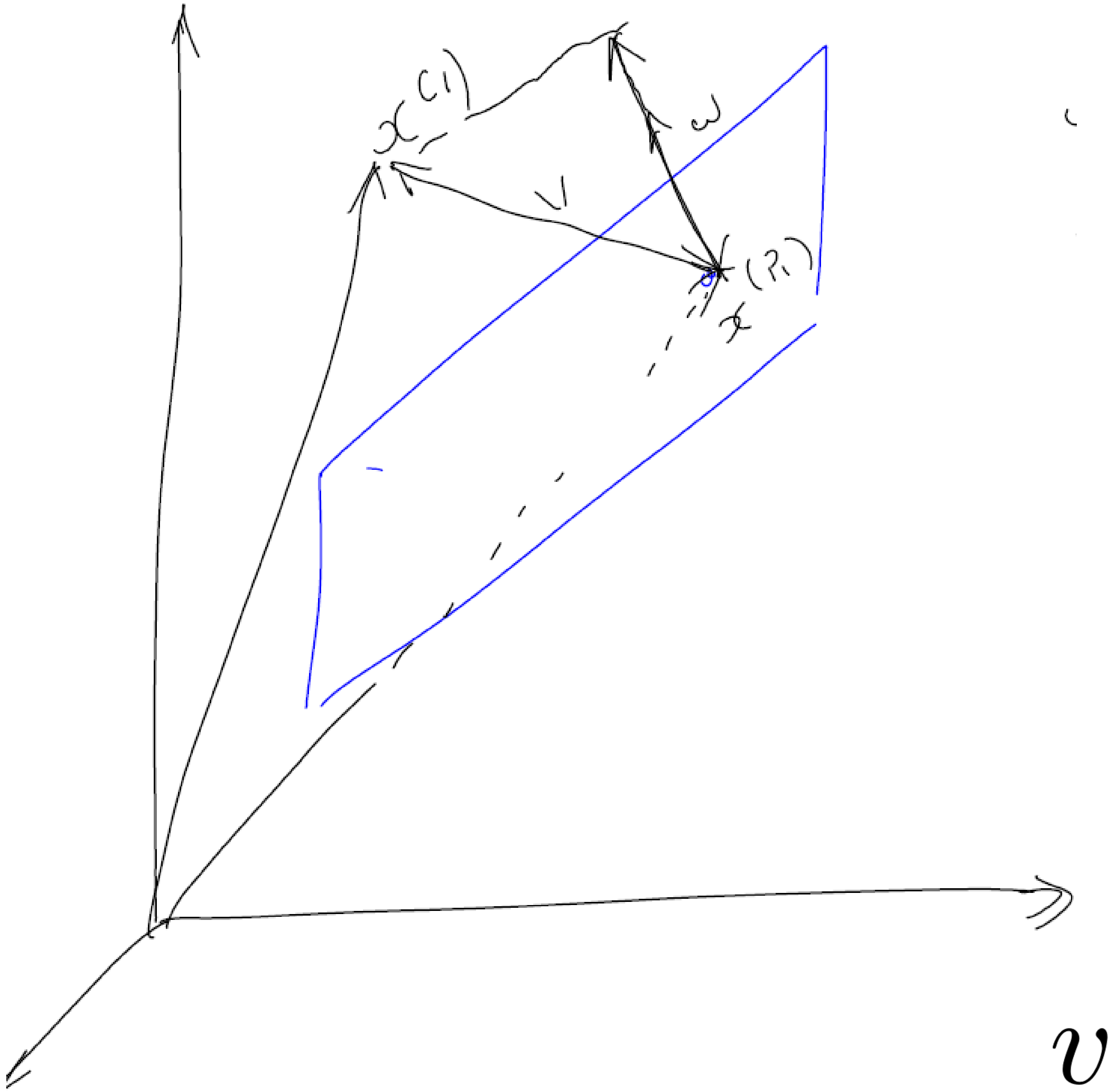
- Generic equation of hyperplane $w^T x + b = 0$

- w is orthogonal to hyperplane

- NOTE: w does not include the intercept

Distance of a point from hyperplane

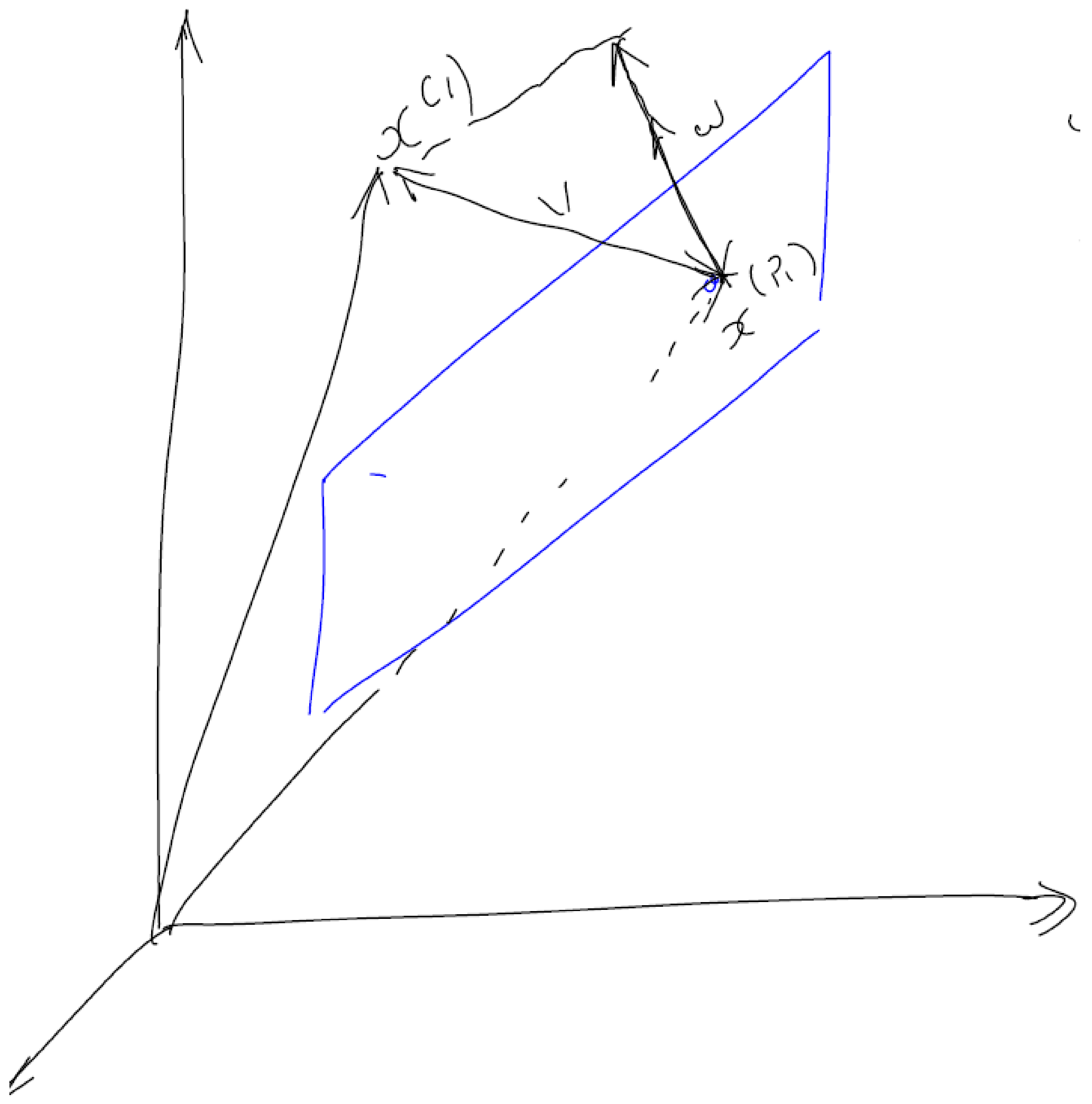




$$v = x^{(1)} - x^{(p)}$$

Dist = projection of v on w = $\dfrac{v^T w}{\|w\|}$

Distance of a point from hyperplane



$$\frac{v^T w}{\|w\|} = \frac{w^T x^{(1)} - w^T x^{(p)}}{\|w\|}$$

$$w^T x^{(p)} + b = 0$$

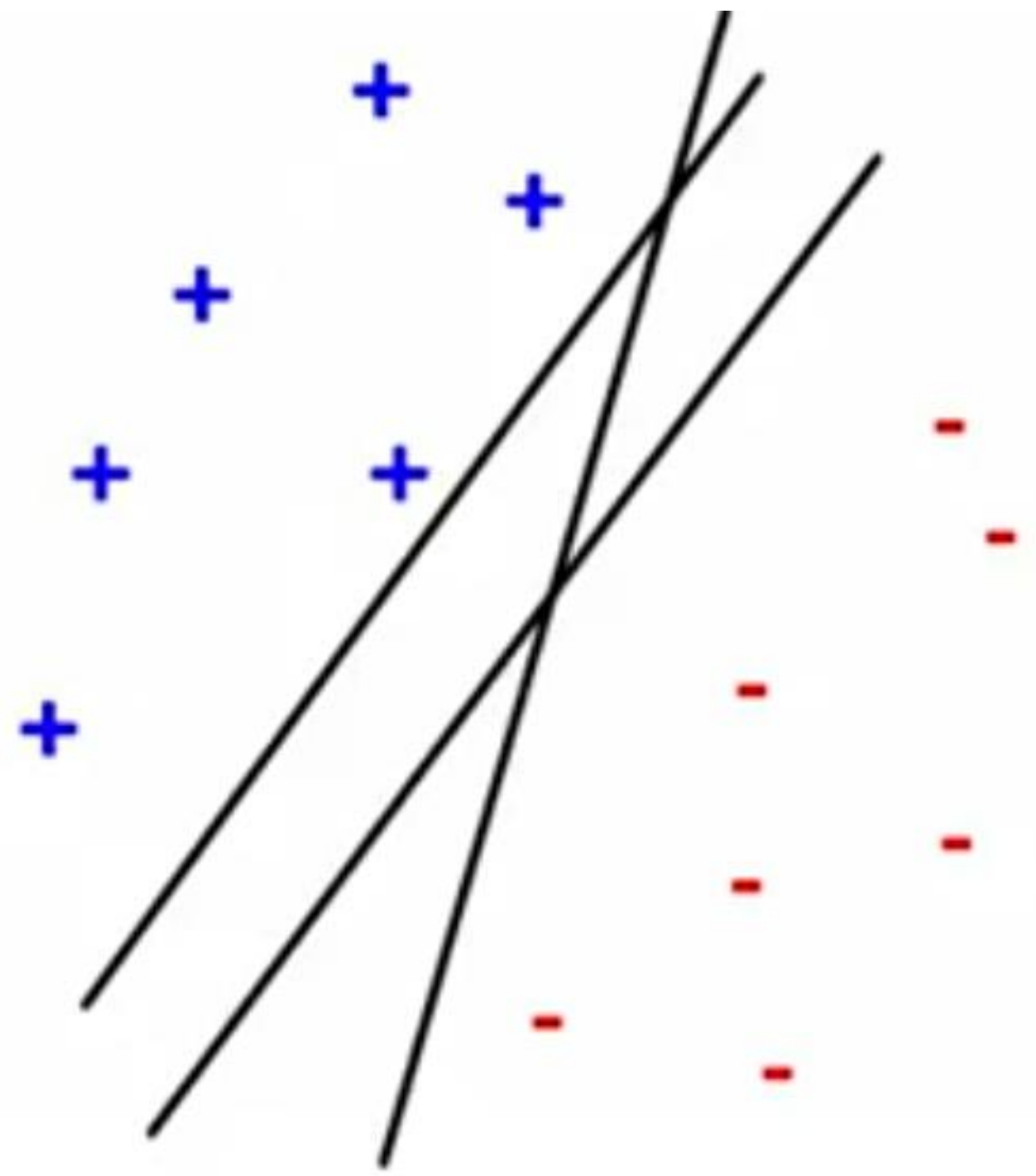
Dist = $$\frac{w^T x^{(1)} + b}{\|w\|}$$

Support Vector Machine

First attempt at SVM objective function

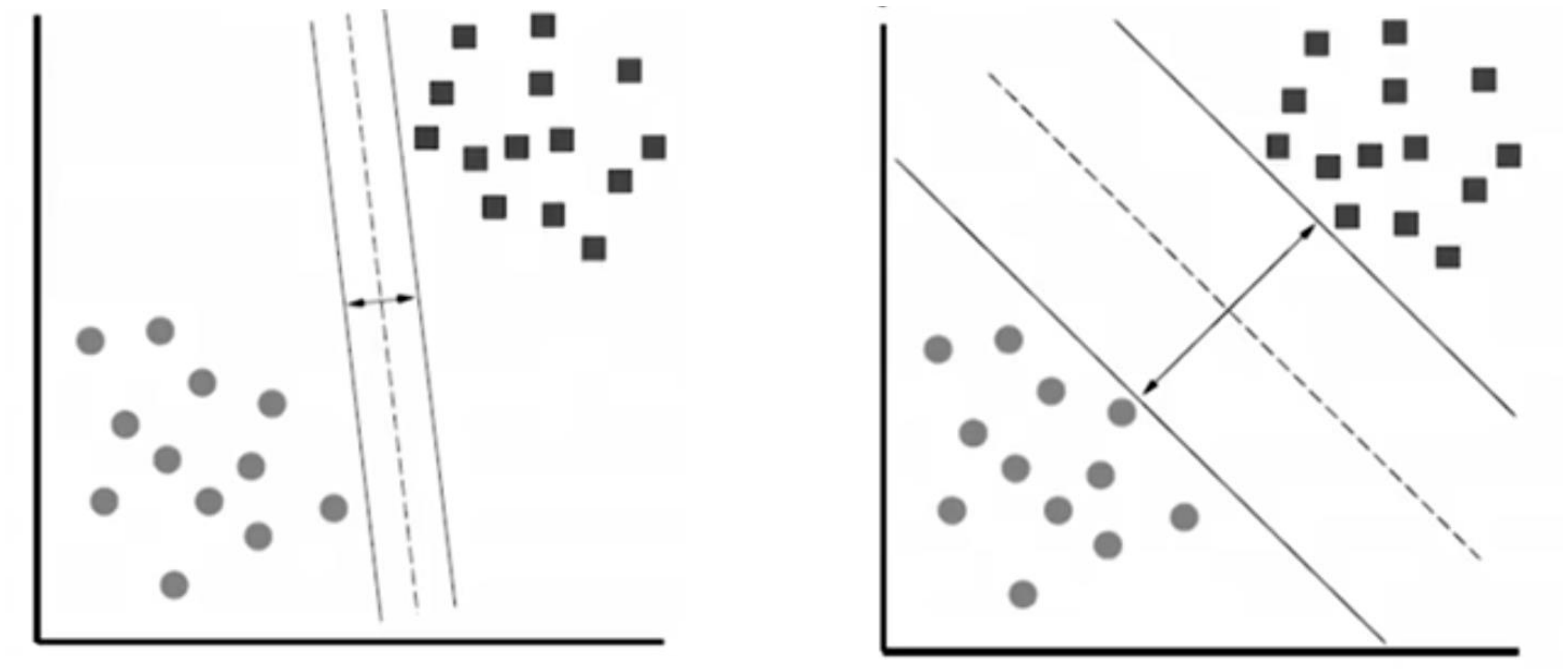
- What is the best linear separator?
- How about maximizing mean distance?

$$\mathcal{J} = \arg\max_{w} \frac{1}{m} \sum_{i=1}^{m} \frac{w^T x^{(i)} + b}{\|w\|} \quad y^{(i)}$$



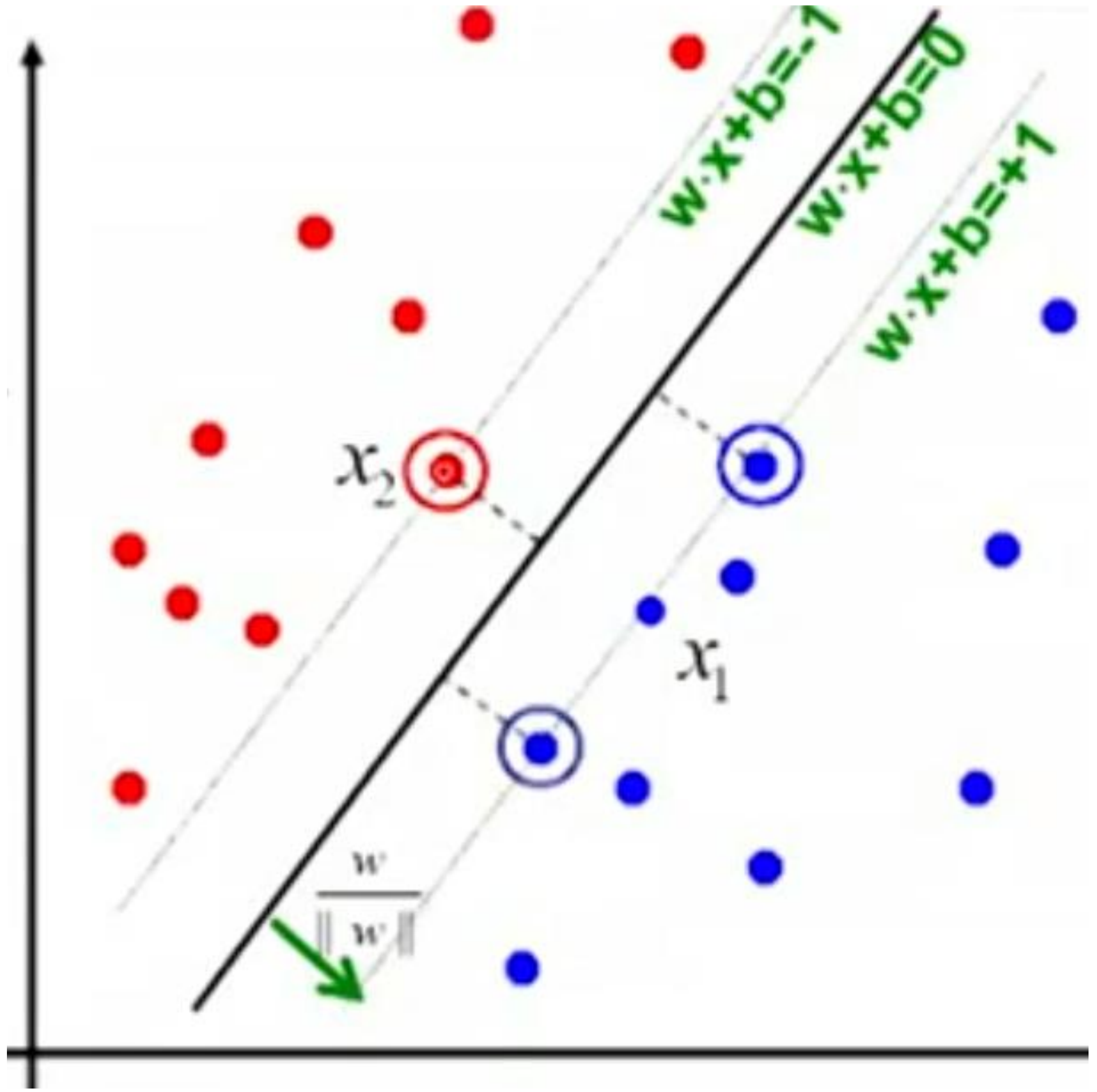
- Two problems
  - Closest points don’t add much
  - Increasing w increases avg

SVM Intuition – Margin



- Margin γ - distance of closest example from hyperplane

SVM Intuition – Support Vectors

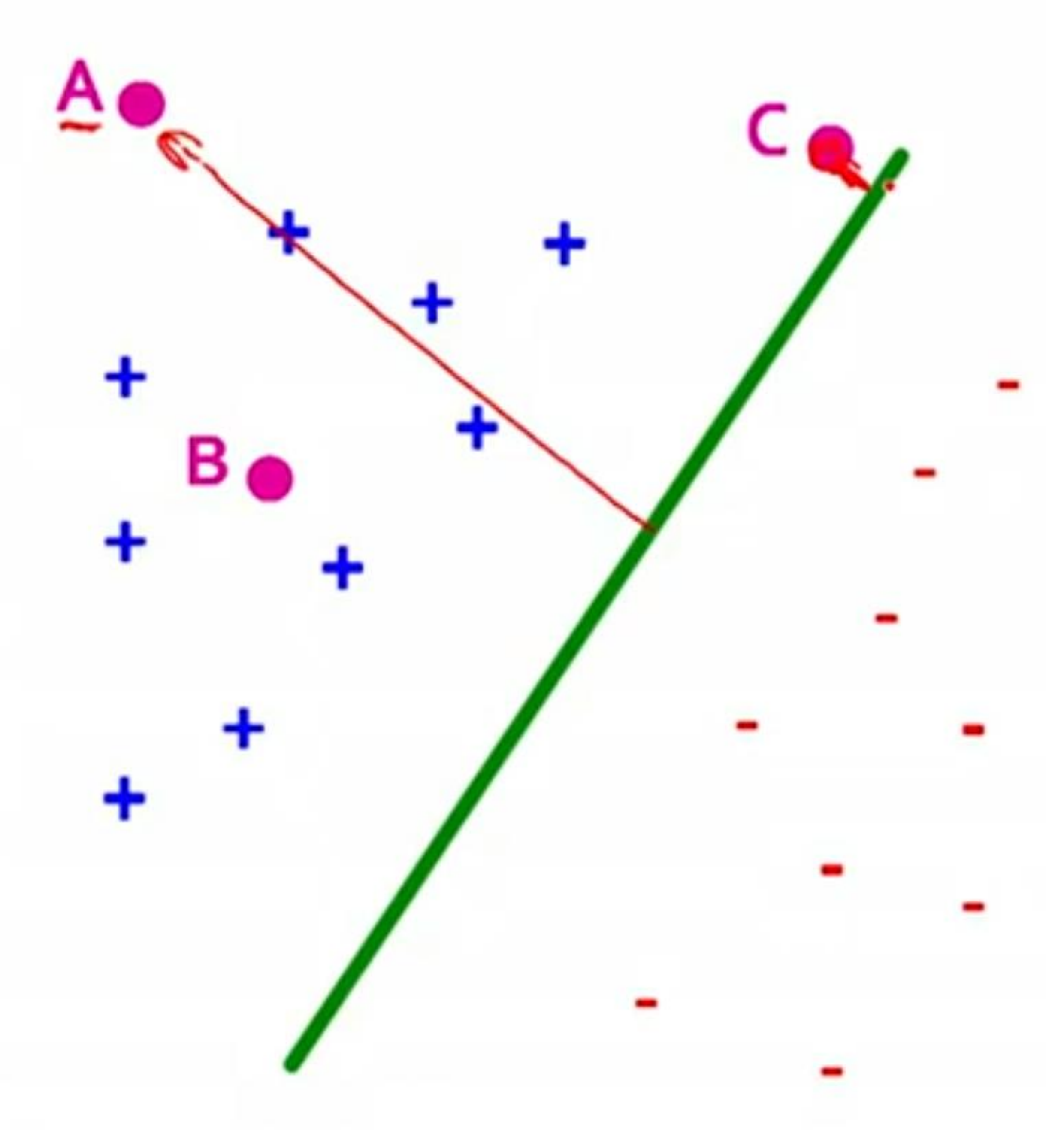


- Focus on the two outer lines instead of hyperplane
- Three data points (vectors) support the two lines
- Hence the name Support Vector Machine
- In general d dimensional data requires d+1 support vectors at minimum)

Formulate objective function using margin

- Maximum Margin
  - Pick w such that the closest data point has largest distance among all possible hyperplanes for different w

$$\gamma = \min dist$$



- For ith data point
  - dist $= \left(\dfrac{w^T x^{(i)} + b}{\|w\|}\right) y^{(i)}$

$$\mathcal{J} = \arg\max_{w,b}[min\ dist] = \arg\max_{w} \gamma$$

$$s.t. \forall i, y^{(i)}(w^T x^{(i)} + b) \geq \gamma$$

How to calculate margin?

Maximizing Margin

- Equations for supporting hyperplanes

$$w^T x^{(1)} + b = 1$$

$$w^T x^{(2)} + b = -1$$

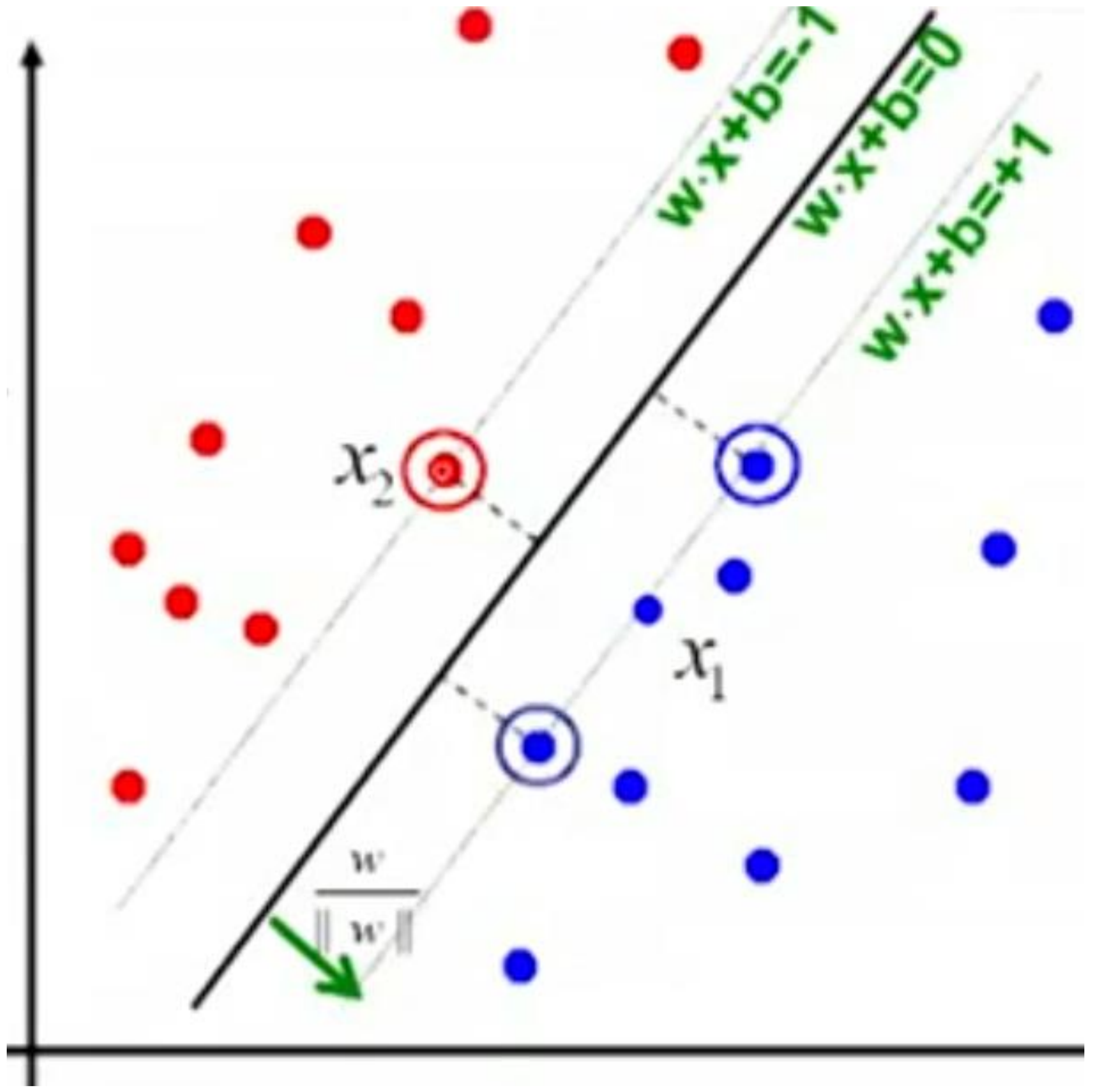
Let us assume wx+b= +/-1 for closest data point

- Relation between x1 and x2

$$x^{(1)} - x^{(2)} = 2\gamma \frac{w}{\|w\|}$$

- Solving, we get

$$\gamma = \frac{1}{\|w\|}$$



Final Objective function for SVM

- Objective function $\mathcal{J} = \arg\max_{w,b} \gamma$

$$s.t. \forall i, y^{(i)}(w^T x^{(i)} + b) \geq \gamma$$

$$\gamma = \frac{1}{\|w\|}$$

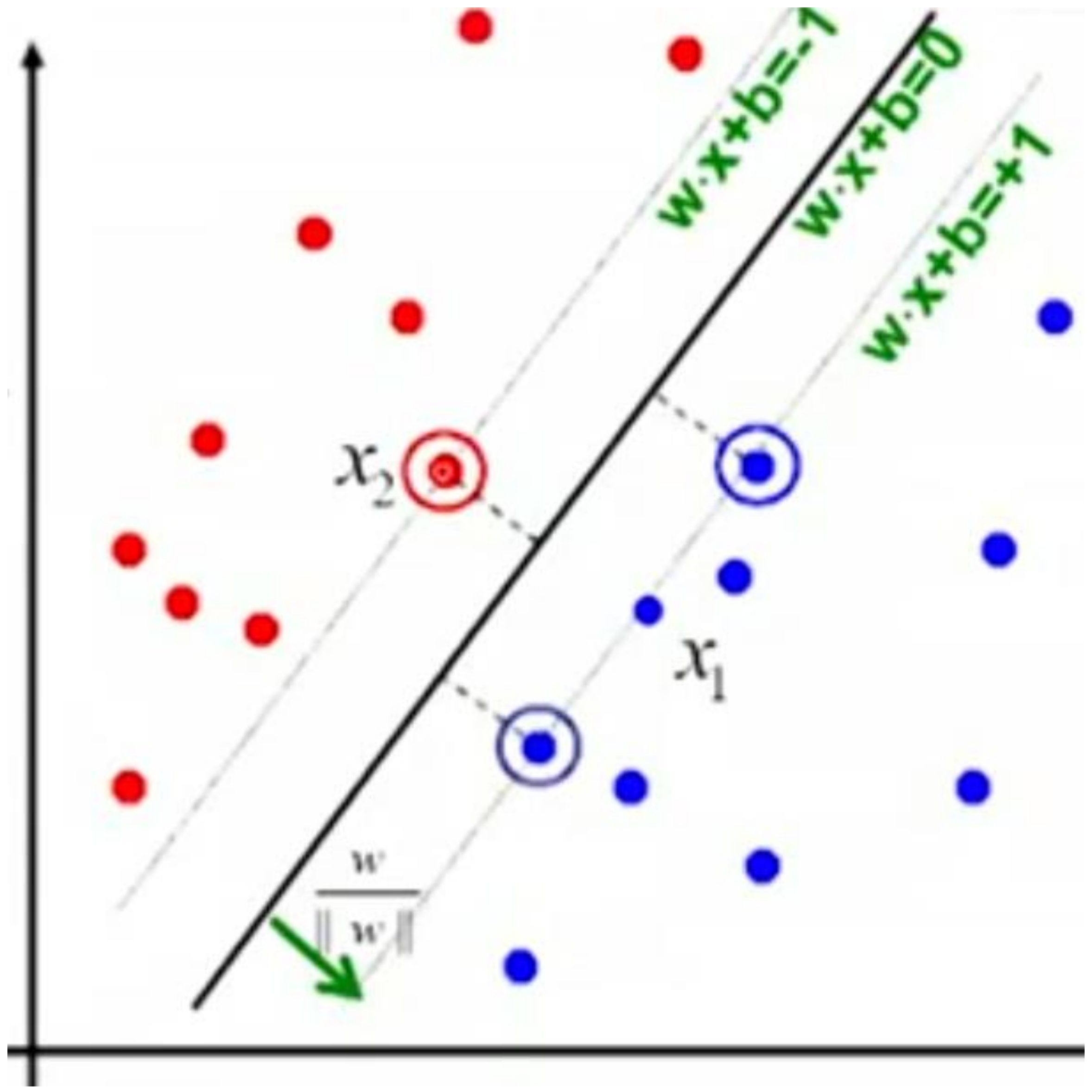
SVM with hard margin

$$\boxed{\begin{aligned} \mathcal{J} \approx \max_{w,b} \frac{1}{\|w\|} \approx \min \|w\| \approx \min \frac{\|w\|^2}{2} \\ s.t. \forall i, \quad y^{(i)}(w^T x^{(i)} + b) \geq 1 \end{aligned}}$$

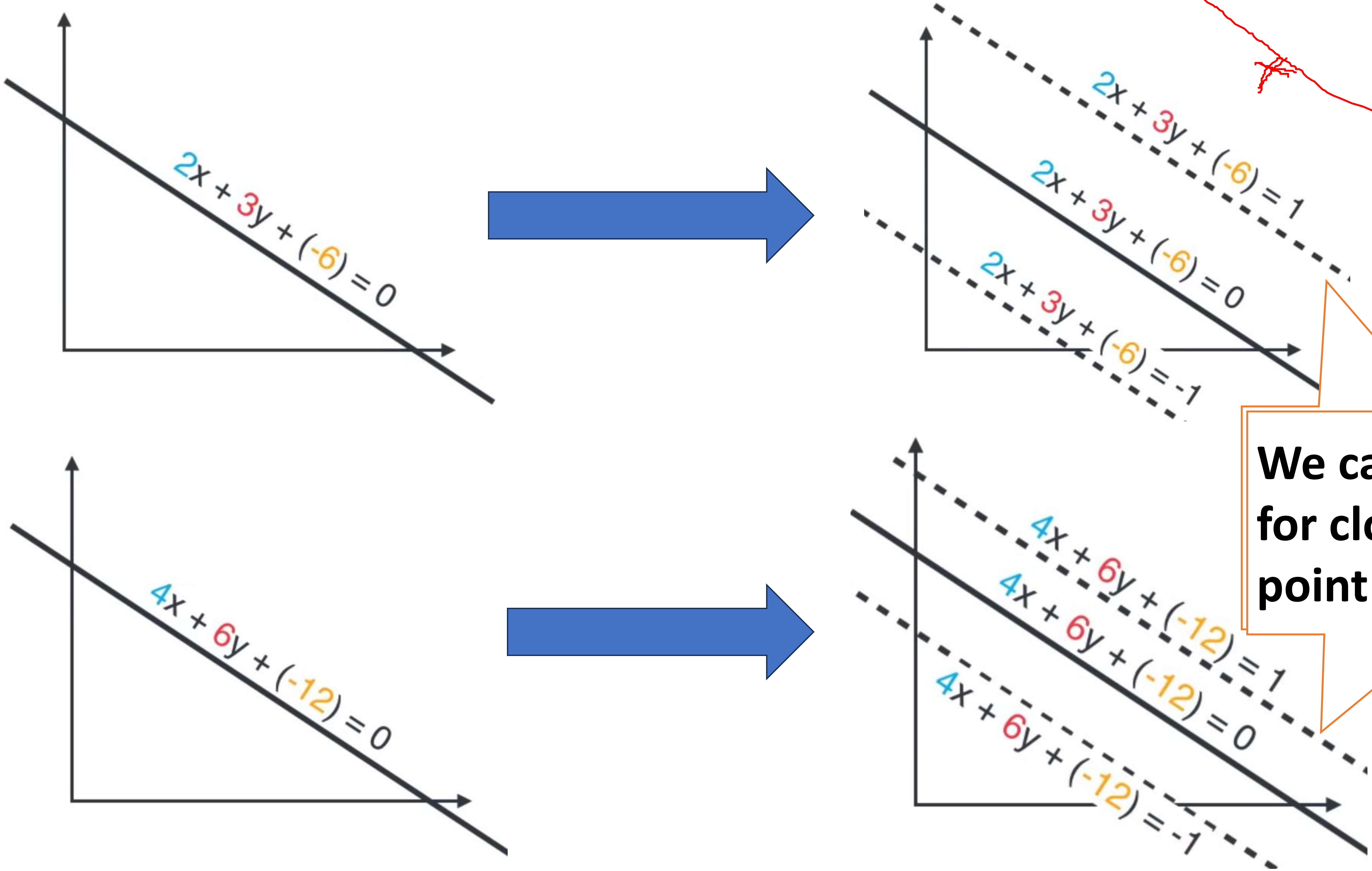
Multiple Margin Condition/Constraints

Equality holds for only support vectors

min w = 0 vector without constraints



Why wx+b = +/- 1 for supporting hyperplanes?

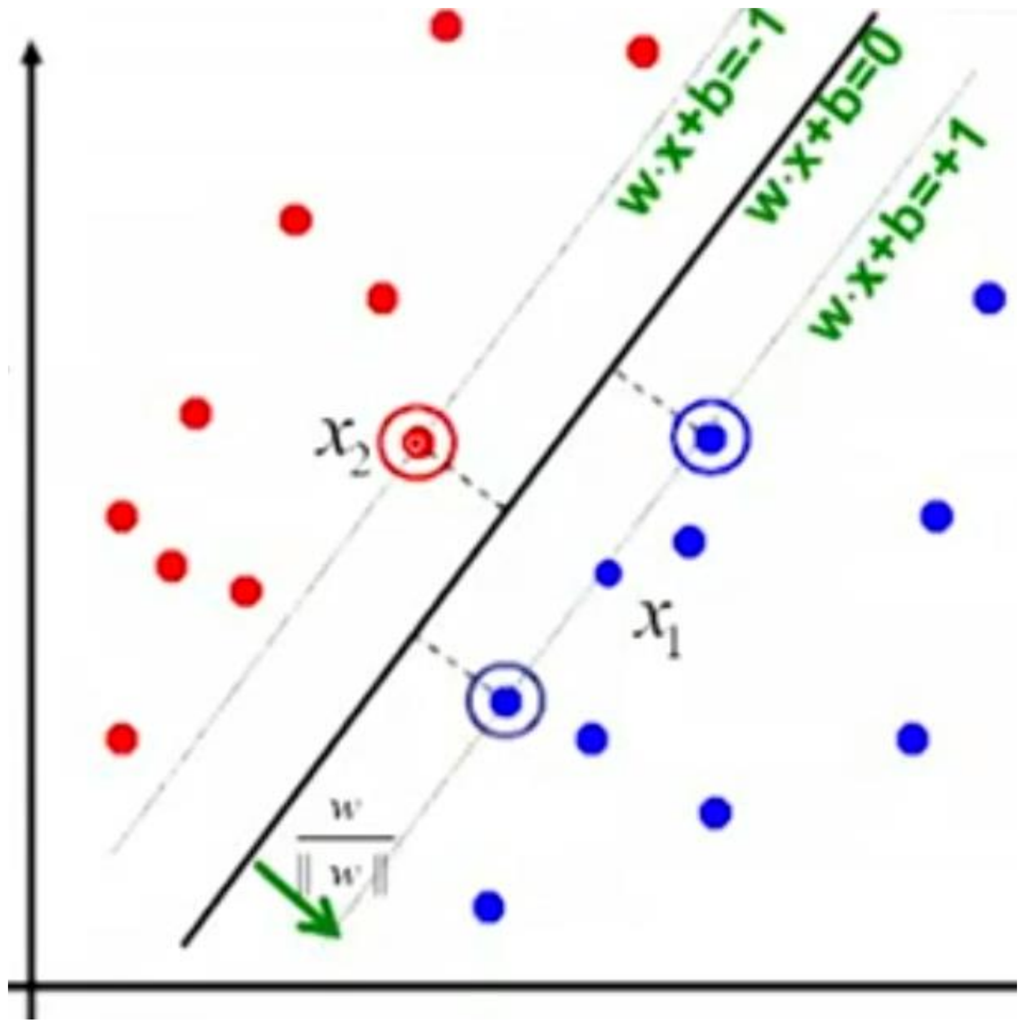


We can fit wx+b=1 for closest data point by adjusting w

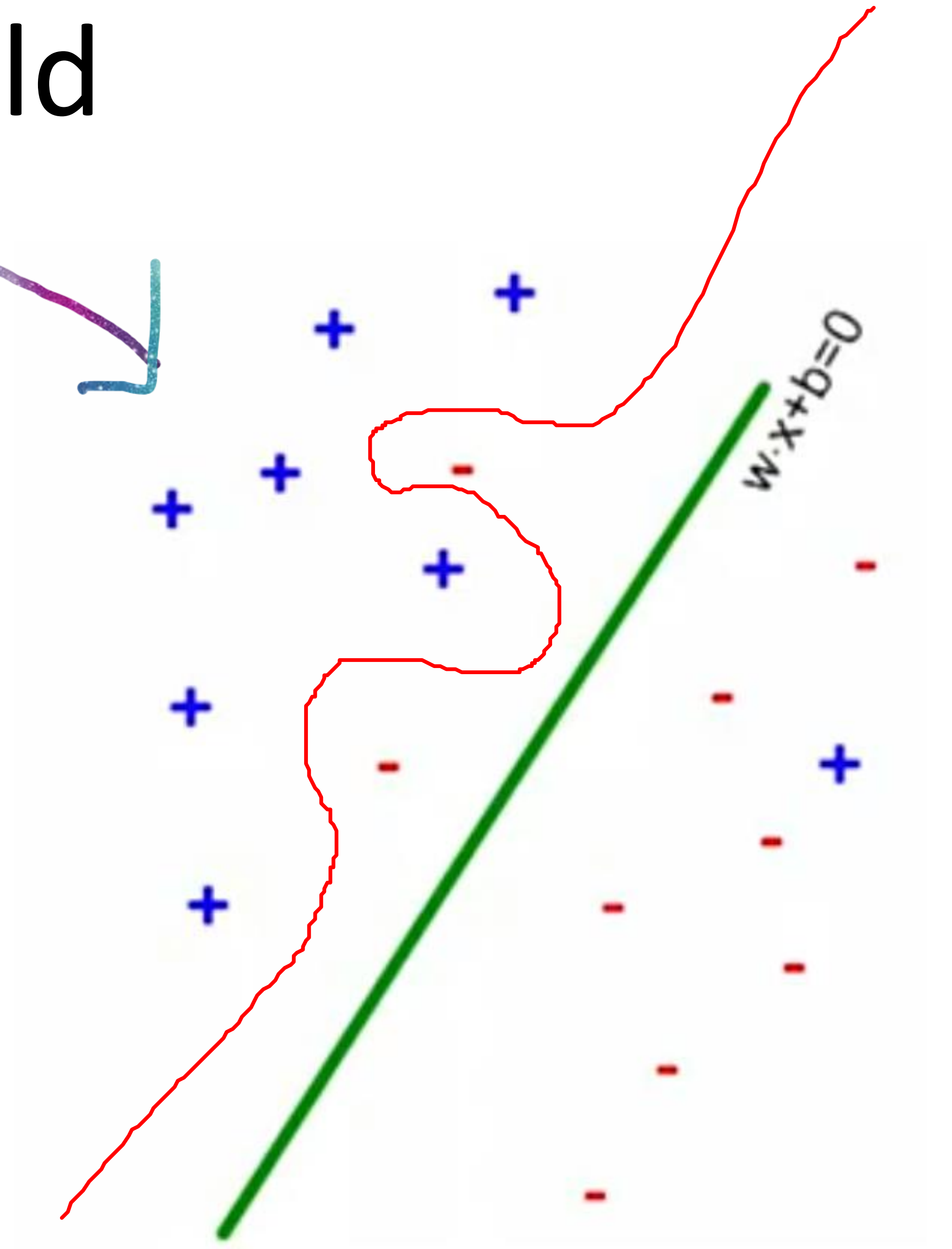
Soft Margin SVM

Recap: Objective function for Hard margin SVM

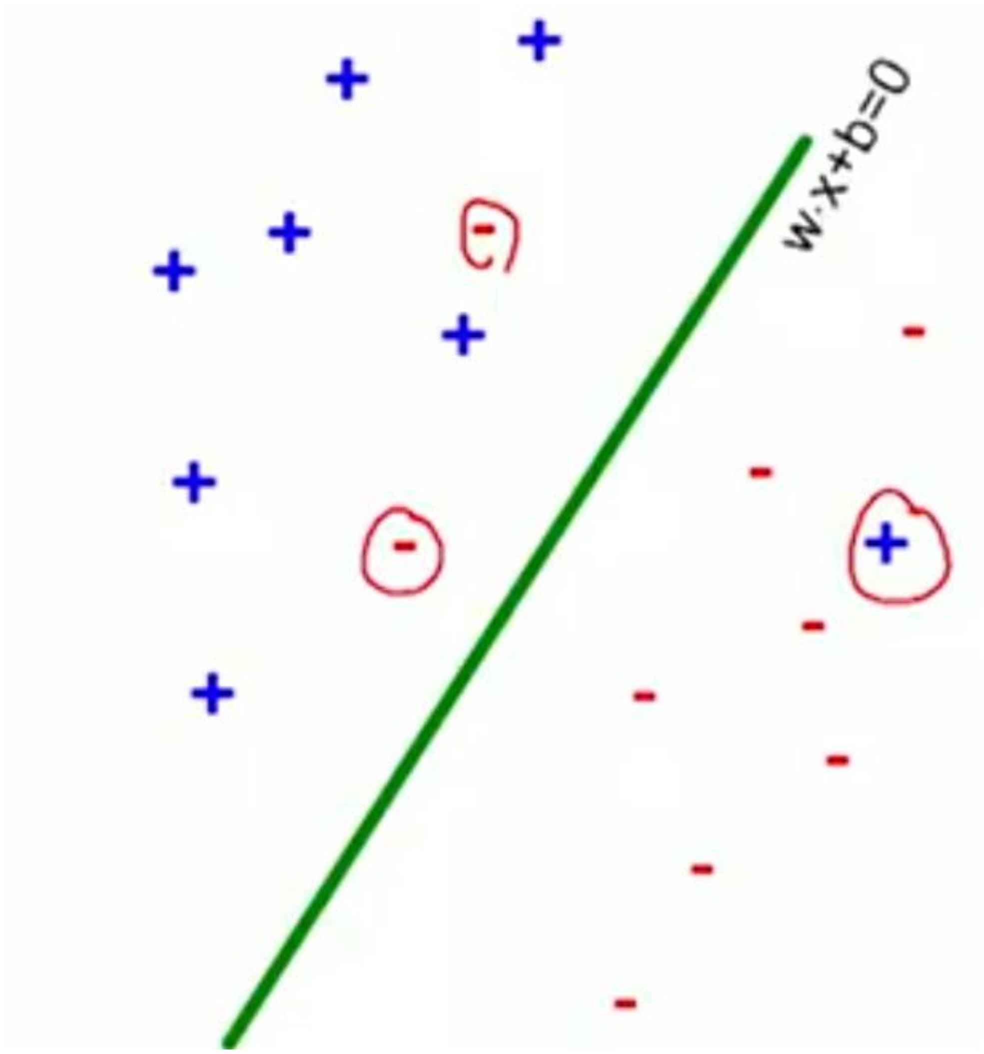
$$\mathcal{J} = \min \frac{\|w\|^2}{2} \quad s.t. \forall i, \quad y^{(i)}(w^T x^{(i)} + b) \geq 1$$



- Welcome to the real world
  - Datasets are noisy
  - Not linearly separable



Introduce penalty for mistakes

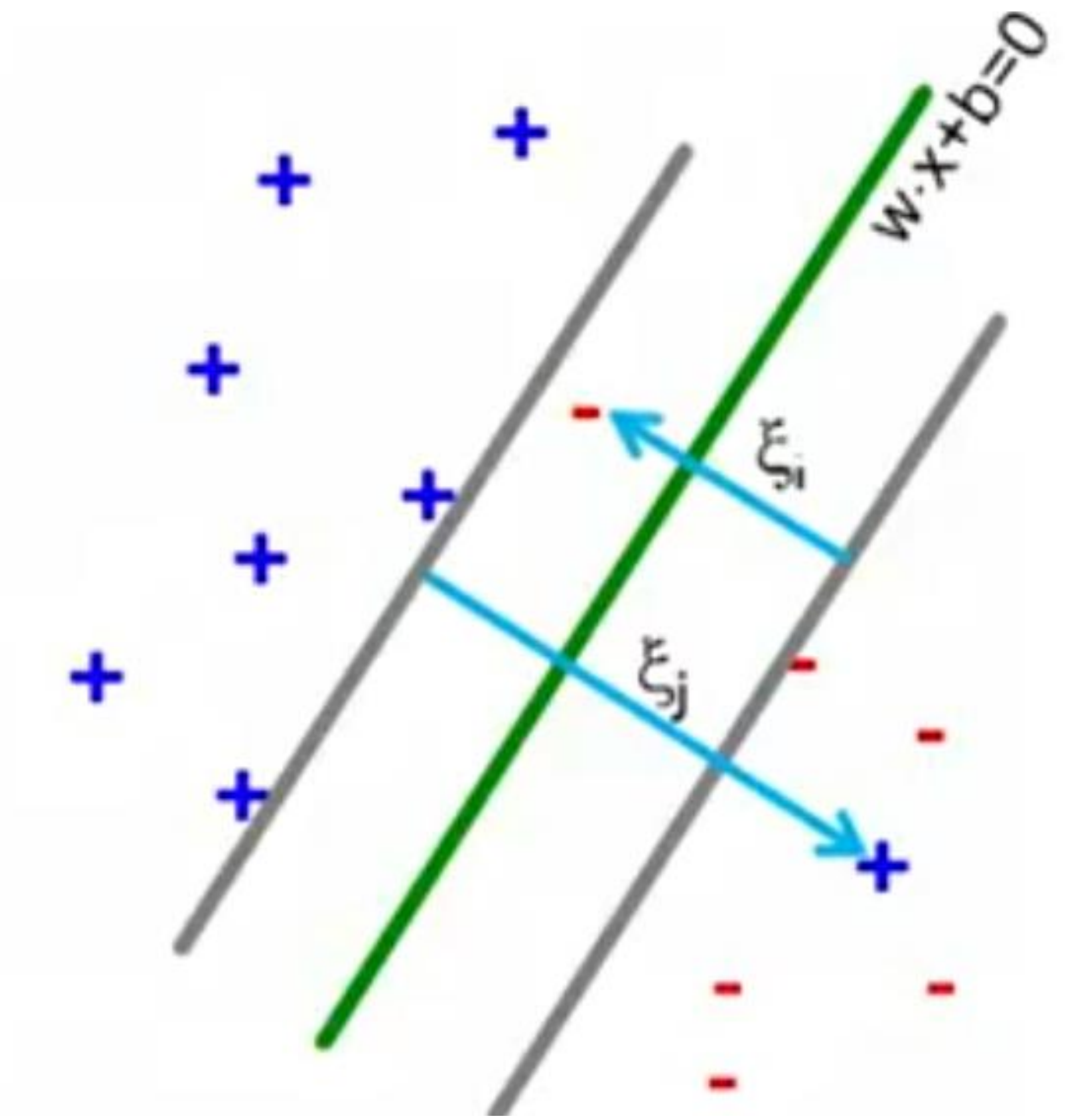


$$\mathcal{J}(w) = \min_{w,b} \frac{\|w\|^2}{2} + C \times number\ of\ mistakes$$

$$s.t. \forall i, \quad y^{(i)}(w^T x^{(i)} + b) \geq 1$$

- Find w such that number of mistakes is small
- C is determined by cross validation
- Penalizing mistakes – Not all mistakes are equally bad

Quantifying penalty for mistakes



- By introducing notion of slack variable ξ_i

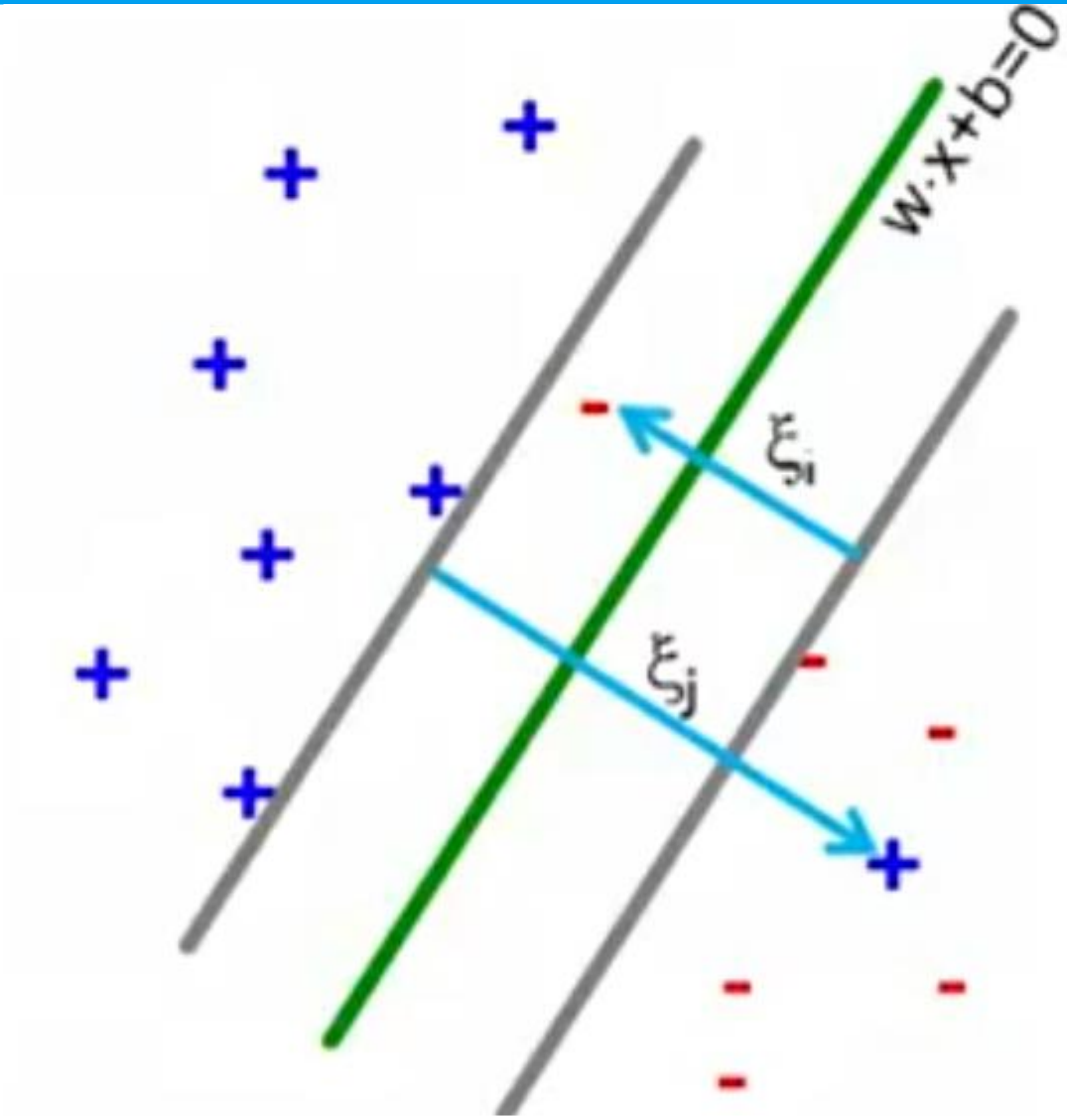
$$\mathcal{J}(w) = \min_{w,b} \frac{\|w\|^2}{2} + C\sum_{i=1}^{m} \xi_i$$

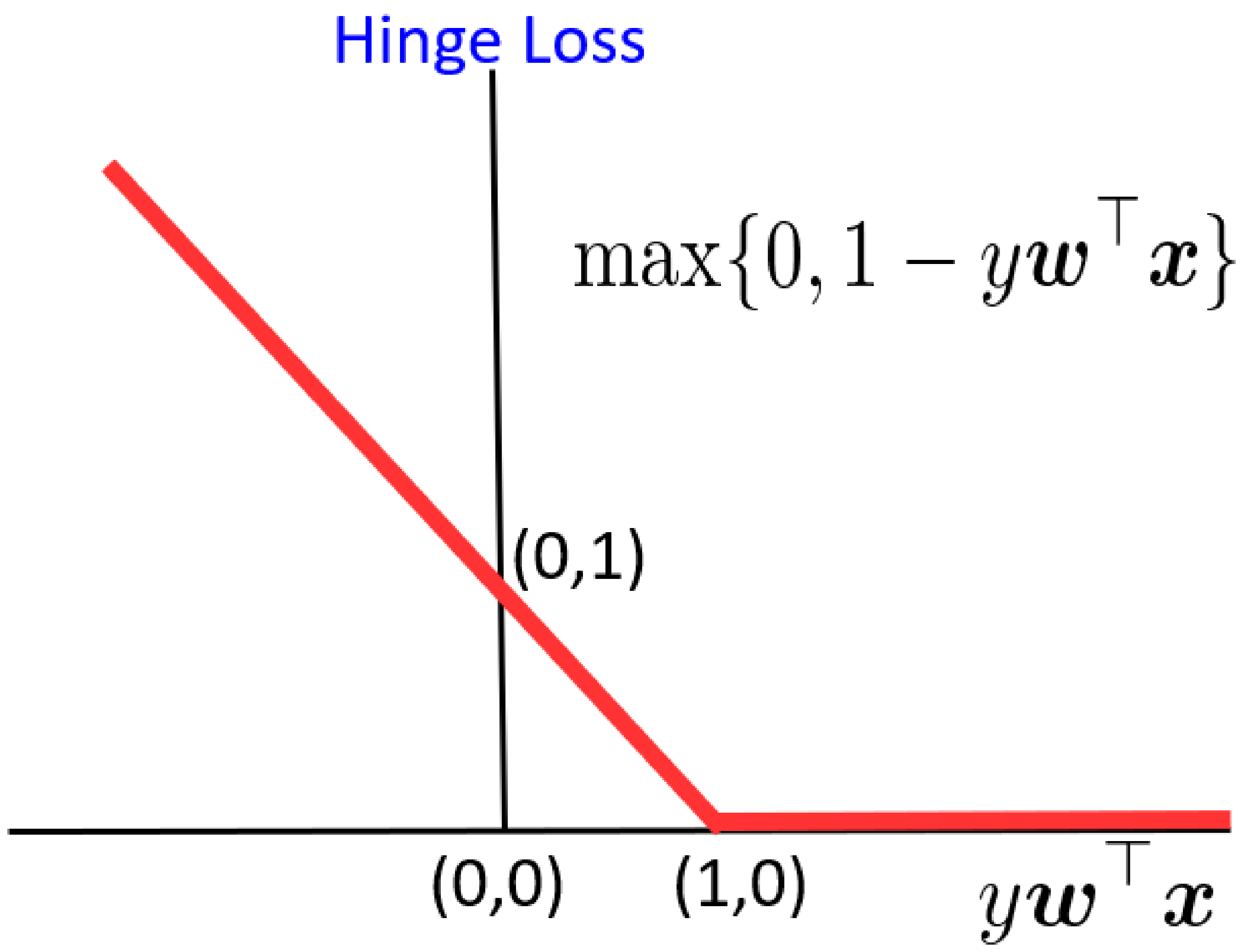
$$s.t. \forall i, \quad y^{(i)}(w^T x^{(i)} + b) \geq 1 - \xi_i$$

- If point is on the wrong side of the margin, penalty is non zero

- C = 0, no regularization
- C = large, high amount of regularization

Generic equation for SVM



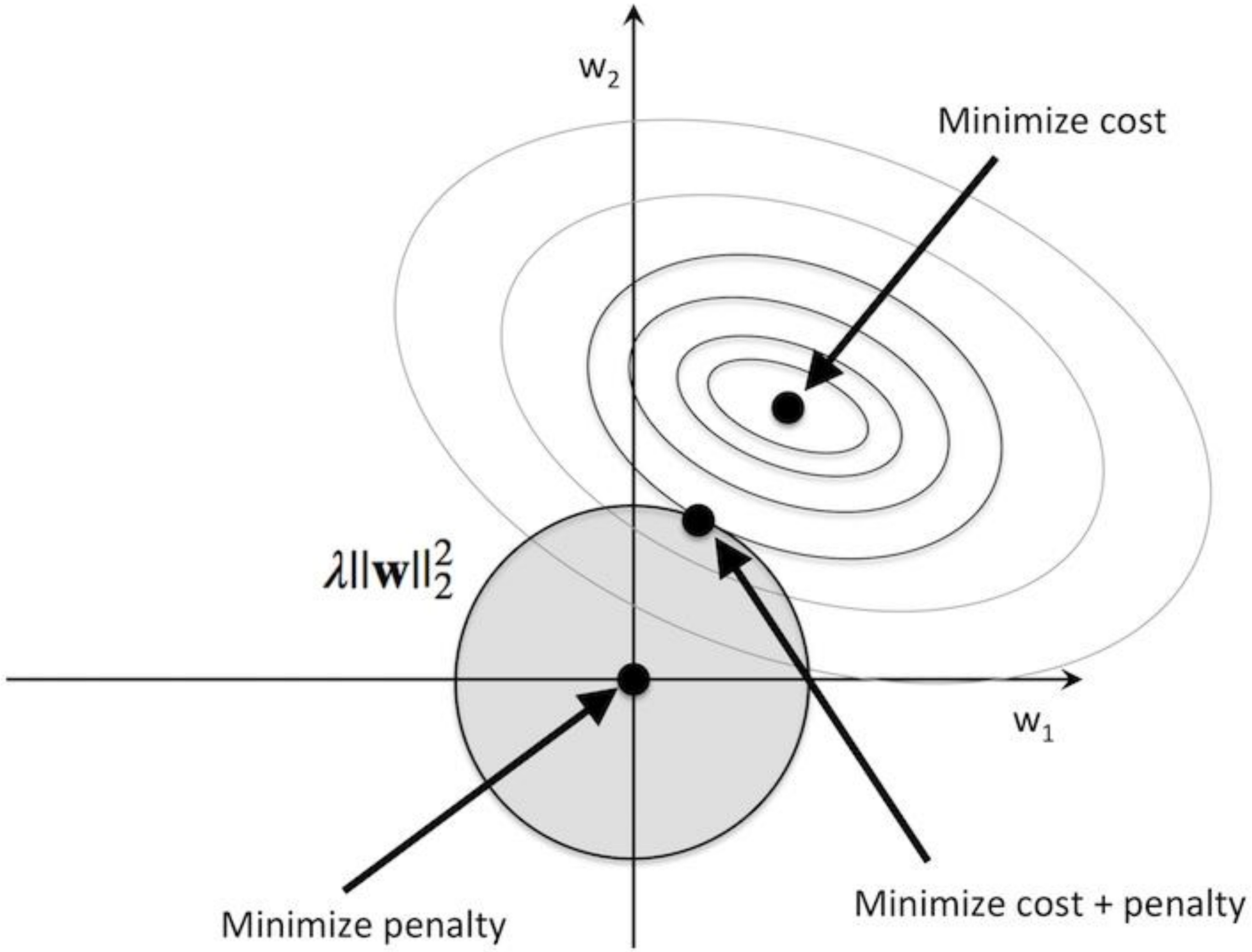


$$\mathcal{J}(w,b) = \min_{w,b} \frac{\|w\|^2}{2} + C\sum_{i=1}^{m} max(0, 1 - y^{(i)}(w^T x^{(i)} + b))$$

- Can perform gradient descent wrt w and b

SVM – Dual form

L2 Regularization in Linear Regression



$$\mathcal{J}(w) = \frac{1}{m}(Xw - y)^T(Xw - y)$$

$$\mathcal{J}(\mathbf{w}) = \frac{1}{m}\sum_{i=1}^{m}(\mathbf{w}^T x^{(i)} - y^{(i)})^2$$

Lagrange Multiplier

$$\nabla_w \mathcal{J} = \lambda \nabla_w \|w\|^2$$

Objective function in Lagrangian notation

$$\mathcal{L}(w, \lambda) = \nabla_w \mathcal{J} + \lambda \nabla_w \|w\|^2 = 0$$

J is a function of w

L is a function of w and lambda

SVM objective function in Lagrangian form

$$\mathcal{J}(w) = \min_{w,b} \frac{\|w\|^2}{2}$$

$$s.t. \forall i, \quad y^{(i)}(w^T x^{(i)} + b) \geq 1$$

Original Problem in Primal form

m separate constraints

$$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ ..\alpha_m \end{bmatrix}$$

Original problem in Lagrangian notation

No more separate constraints

$$\mathcal{L}(w, b, \alpha) = \min \left[\frac{w^T w}{2} - \sum_{i=1}^{m} \alpha^{(i)} y^{(i)} (w^T x^{(i)} + b) - 1 \right]$$

SVM objective function in dual form

Problem in primal form (Lagrangian notation)

$$\mathcal{L}(w,b,\alpha) = \min_{w,b}\left[\frac{w^T w}{2} - \sum_{i=1}^{m}\alpha^{(i)}y^{(i)}(w^T x^{(i)} + b) - 1\right]$$

Optimization in column dimension

Equivalent to

$$\alpha = \begin{bmatrix}\alpha_1\\ \alpha_2\\ ..\alpha_m\end{bmatrix}$$

Problem in dual form

$$\max_{\alpha_i \geq 0}\left[\min_{w,b}\mathcal{L}(w,b,\alpha)\right]$$

Optimization in row dimension

Solving SVM objective function in dual form

Problem in dual form

$$\boxed{\max_{\alpha_i \geq 0} \left[\min_{w,b} \mathcal{L}(w, b, \alpha) \right]}$$

$$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ ..\alpha_m \end{bmatrix}$$

$$\frac{\partial \mathcal{L}}{\partial w} = 0 \implies w = \sum_{i=1}^{m} \alpha^{(i)} y^{(i)} x^{(i)}$$

$$y = \begin{bmatrix} y_1 \\ y_2 \\ ..y_m \end{bmatrix}$$

$$\frac{\partial \mathcal{L}}{\partial b} = 0 \implies b = \alpha^T y$$

Substitute for w and b in this

$$\mathcal{L}(w, b, \alpha) = \min_{w,b} \left[\frac{w^T w}{2} - \sum_{i=1}^{m} \alpha^{(i)} y^{(i)} (w^T x^{(i)} + b) - 1 \right]$$

Solving SVM objective function in dual form

Substitute w and b

$$\max_{\alpha_i \geq 0} \left[\min_{w,b} \mathcal{L}(w, b, \alpha) \right]$$

$$\max_{\alpha^{(i)} \geq 0} \left[\sum_{i=1}^{m} \alpha^{(i)} - \frac{1}{2} \sum_{i,j} \alpha^{(i)} \alpha^{(j)} y^{(i)} y^{(j)} {x^{(i)}}^T x^{(j)} \right]$$

$$\boxed{\min_{\alpha^{(i)} \geq 0} \left[\frac{1}{2} \sum_{i,j} \alpha^{(i)} \alpha^{(j)} y^{(i)} y^{(j)} {x^{(i)}}^T x^{(j)} - \sum_{i=1}^{m} \alpha^{(i)} \right]}$$

Why solve in dual form?

$$\min_{\alpha^{(i)} \geq 0} \left[\frac{1}{2} \sum_{i,j} \alpha^{(i)} \alpha^{(j)} y^{(i)} y^{(j)} {x^{(i)}}^T x^{(j)} - \sum_{i=1}^{m} \alpha^{(i)} \right]$$

- Why dual?
- Solve in single variable vector alpha
- Most alpha are zero
- For wide data sets p >> m
- mp >> m^2
- Kernel friendly
- kernels can be solved only in dual form

$$\mathcal{J}(w) = \min_{w,b} \frac{\|w\|^2}{2}$$

$$s.t. \forall i, \quad y^{(i)}(w^T x^{(i)} + b) \geq 1$$

Further Reading

- SVM Kernels
- SVM polynomial, RBF kernels
  - Statquest by Josh Stammer (youtube)


QUESTIONS