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APPLIED PROBABILITY AND STATISTICS | PROBLEM SET – 1

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1. A student taking a test has to select 7 out of 10 questions. How many different choices does she have if -

(a) there are no other restrictions?

= Then the answer is 10C7 = **120 choices**

(b) she has to answer exactly 2 of the last 4?

= first 5 questions have to be answered from first 6 questions = 6C5

And

Exactly last 2 questions have to be answered from last 4 questions = 4C2

= 6C5 * 4C2 = **36 choices**

(c) she has to answer exactly 2 of the first 6?

= since only 2 questions must be answered in the first 6, only 4 will be left out of which 5 cannot be chosen we have **0 choices** here.

(d) she has to answer at least 3 of the first 5?

= There are 3 cases here

I) she can answer 3 out of first 5 and 4 out of last 5 questions = 5C3 * 5C4 = 10 * 4 = 50

OR

II) 4 of first 5 and 3 out of last 5 = 5C4 * 5C3 = 5 * 10 = 50

OR

III) 5 of first 5 and 2 out of last 5 = 5C5 * 5C2 = 10

Total ways = 50 + 50 + 10 = **110 ways**

2. A bone marrow transplant can be made only between two people with all six human leukocyte antigens (HLA) being compatible. Each person's halotype consists of 2 HLA-A antigens, 2 HLA-B antigens, and 2 HLA-DR antigens. There are a total of 18, 40, and 14 antigens for types A, B, and DR, respectively. The order of the antigens is irrelevant, and the types can be repeated. How many possible antigen strings are there?

= We need to take 2 HLA-A, 2 HLA-B, 2 HLA-DR antigens respectively, and since order does not matter, and types can be repeated =

= 18C2 * 40C2 * 14C2 = 153 * 780 * 91 = **10859940** antigen strings are possible

3. If eight identical blackboards are to be divided among 4 schools, how many divisions are possible? What if each school must receive at least one blackboard?

= First 4 boards are divided among 4 schools where each school gets one board, and the remaining 4 boards can be divided using the stars and bars concept (sampling without replacement and order does not matter) - so that way we get

(r – 1) C (n – 1) = 7c3 = **35 ways.**

4. 9 computers are brought in for servicing (and machines are serviced one at a time). Of the 9 computers, 3 are PCs (Personal Computer), 4 are Macs, and 2 are Linux machines. Assume that all computers of the same type are indistinguishable (i.e., all the PCs are indistinguishable, all the Macs are indistinguishable, etc.).

(a) In how many distinguishable ways can the computers be ordered for servicing?

= 9C3 * 6C4 * 2C2 = **1260 ways**

(b) In how many distinguishable ways can the computers be ordered if the first 5 machines serviced must include all 4 Macs?

= here order matters and selection happens without replacement so

= 5P4 * 5P5 = **14400 ways**

(c) In how many distinguishable ways can the computers be ordered if 2 PCs must be in the first three and 1 PC must be in the last three computers serviced?

= same as above

= 3P2 * 6P6 * 2P1 = **8640 ways**

5. 100 units of stabilizing weights are to be placed into 5 vehicles. Because of different vehicle characteristics, vehicle 1 needs at least 10 units, vehicles 2 and 3 at least 12 each, vehicles 6 and 7 travel in a convoy and they need at least 4 combined. How many distributions of these weight units are feasible?

= we must distribute 100 units among 5 vehicles such that each of these vehicle gets some minimum number of units – 10,12,12, and last two vehicles need 4 units minimum combined.

= Here r = 100 and n = 4

= (r - (10+ 12 + 12 + 4) + (10+12+12+4) – 1) C (n – 1)

= (100 – 1) C (4 – 1) = 99 C 3 = **156849 ways**

6. Eleven soccer players are to be divided into 4 functional groups: 3 forwards, 3 midfields, 4 defenses, and 1 goalie. There are only 2 people who can play goalie. Both of these two players can play any other position. Of the remaining 9, 4 can play only forward or midfield; the other 5 can play only defense or midfield. We want to calculate the number of possible ways to divide the team into the 4 functional groups.

= There are 3 cases to consider here

= 2C1 * [CASE 1 + CASE 2 + CASE 3]

= CASE I: Of the two people who can play goalie, one goes to goalie and other goes to defense

= 5C3 [defense] * 6C3 [midfield] * 3C3 [forwards]

= 200

OR

= CASE II: Of the two people who can play goalie, one goes to goalie and other goes to midfields

= 5C4 [defense] * 3C3 [midfield] * 4C3 [forwards] = 20

OR

= CASE III: Of the two people who can play goalie, one goes to goalie and other goes to forwards

= 5C4 [defense] * 5C3 [midfield] * 3C3 [forwards] = 50

Total = 2 [200 + 20 + 50] = 540 ways

7. In how many ways can r identical server requests be distributed among n servers so that the ith server receives at least mi requests, for each i =1, 2, . . ., n? You can assume that r ≥ (m1 + m2 + · · + mn).

= [number of ways to distribute r identical objects into n distinct bins such that no bin is empty]

= if requests should be distributed in such a way that each ith server gets mi number of requests, we are left with = [r - (m1 + m2 + … + mn)] requests

= To distribute these remaining requests among n servers we can use combinations formula as below (sampling with replacement and order doesn't matter)

= **(r - (m1 + m2 + ... + mn) + (m1 + m2 + ... + mn) – 1) C (n – 1)**

= (r-1) C (n-1)

8. Suppose a particle starting from the origin can move only up or down; the binomial option pricing model addresses stock price movements using such an

idea. Show that the number of ways the particle can move from the origin to position k in n steps is n C (n + k) / 2.

→ To show that a particle reaches position 'k' in 'n' steps, in $^{n}C_{\frac{n+k}{2}}$ ways,

consider 2 cases from given diagram.

(1) case 1: To reach 2^{nd} position (k = 2) from above in 2 steps there is 1 way.

n = 2
k = 2

n + k = 4 (even)

$^{2}C_{4/2} = {}^{2}C_{2} = \boxed{1}$

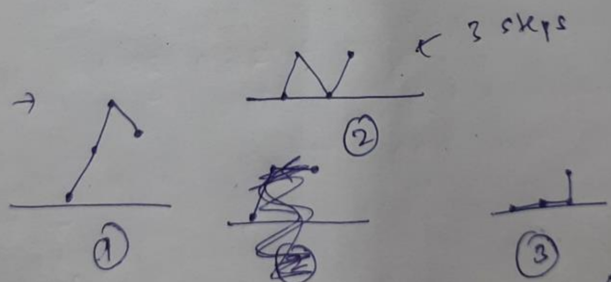
(2) case 2: To reach '1' in diagram (2 upward and 1 downward movement)

k = 1 n = 3

n + k = 4 (even)

$^{3}C_{4/2} = {}^{3}C_{2} = \boxed{3 \text{ ways}}$

3 ways →



(3 steps)

→ to reach position 2 from origin in 3 steps.

=

From above observations, the formula can be generalized to n C (n+k / 2) ways.

9. Imagine a criminal appeals court consisting of five judges; let’s call them A, B, C, D, and E. The judges meet regularly to vote (independently, of course) on the fate

of prisoners who have petitioned for a review of their convictions. The result of each of the court's deliberations is determined by a simple majority; for a petitioner to be granted or denied a new trial requires three or more votes. Based on long-term record keeping, it is known that A votes correctly 95% of the time; i.e., when A votes to either uphold or to reverse the original conviction, he is wrong only 5% of the time. Similarly, B, C, D, and E vote correctly 95%, 90%, and 80% of the time. (There are, of course, two different ways a judge can make a mistake. The judge may uphold a conviction, with new evidence later showing that the petitioner was in fact innocent. Or the judge may vote to reverse a conviction when in fact the petitioner is actually guilty, as determined by the result of a second conviction at the new trial.) Suppose we want to calculate the probability that the court, as an entity, makes an incorrect decision.

(a) Write the sample space showing any two outcomes in it clearly. Explain what the outcomes mean. Hint: consider {0, 1}.

= If 1 means a judge voted correctly and 0 means judge voted incorrectly.

= **(1, 0, 1, 1, 0) and (0, 1, 1, 0, 0).**

(b) How many outcomes n are there in the sample space?

= Since every judge can vote correctly or incorrectly – there are 2 possible outcomes.

= **2 * 2 * 2 * 2 * 2 = 32 = n** [total number of outcomes in the sample space]

(c) How many outcomes n(E) are there in the event of interest? Hint: consider selecting at least 3 slots from 5.

= here the event of interest is when court makes an incorrect decision

= 3 judges vote incorrectly + 4 judges vote incorrectly + all 5 judges vote incorrectly

= 5C3 + 5C4 + 5C5 = 10 + 5 + 1 = **16 outcomes**

= clearly, this means in total number of outcomes (32), half outcomes are correct decisions = 16 and rest 16 outcomes are incorrect decisions.

(d) Explain briefly why or why not the probability of the event of interest can be calculated as n(E)/n.

= basic probability definition is number of desired outcomes by total outcomes, but this is valid when all the outcomes in sample spaces are equally likely to occur.

= n(E) - is event of interest which can be anything like court makes an incorrect decision and n is total number of outcomes in sample space, **but n(E)/n - doesn’t give probability** because here each judge has different probabilities of voting correctly or incorrectly and they have to be considered to calculate the actual probability of interest. The sample space does not consist of equally likely events.

10. Mr. Brown needs to take 1 tablet of type A and 1 tablet of type B together on a regular basis. One tablet of type A corresponds to a 1 mg dosage, and so does 1 tablet of type B. He keeps these two types of tablets in two separately labeled bottles as they cannot be differentiated easily. One day, on a business trip, Mr. Brown brought 10 tablets of type A and 10 tablets of type B. Unfortunately, he drops the bottles and breaks them. He does not have the time to go to a pharmacy to buy a new set of tablets, but he needs to take his required dosage of both tablets A and B. The safe dosage that he needs for both tablets A and B is given by 0.9 mg ≤ safe dosage ≤ 1.1 mg. Taking either an excess or a shortage of the required intake will result in serious health issues.

(a) Suppose that after investigating the broken bottles, Mr. Brown finds 2 tablets that are still intact in the bottle for tablet A. The other 18 tablets are found to be mixed in a pile. Is it better for him to take one known tablet from the bottle and one from the pile, or take two tablets from the pile? Answer this by calculating the respective probabilities that he will not have any serious health issues for both options.

CASE 1: he takes 1 tablet from the pile and 1 tablet from the bottle

= probability = (desired mg / total mg)

= [(0.9+1+1.1)] / 18 * [½] = **8.3%**

CASE 2: He takes both tablets from the pile

= [(0.9+1+1.1) * 2] / 20 = **30%**

Based on the above probability values I think it's better if chooses to go by **case 2**

(b) Suppose that after investigating the broken bottles, Mr. Brown finds that the tablets are all mixed up. What is the probability that he will not have any serious health issues if he randomly picks 2 tablets?

= The probability of picking tablet from A and the probability of picking tablet from B

= 1/10 * 1/10 = 1/100 = **1%**

11. A total of 28% of American males smoke cigarettes, 7% smoke cigars, and 5% smoke both cigars and cigarettes. Let A and B represent the events that a randomly chosen person is a cigarette smoker and a cigar smoker, respectively. Explain in plain English what the following compound events represent and calculate their probabilities: (1) $(A \cup B)$ c (2) $B \cap A$

= (1) $P(A) + P(B) - P(A \cap B) = 28/100 + 7/100 - 5/100 =$ **0.3**

= (2) From the question we know about 5% of people smoke both cigars and cigarettes = **0.05**

12. What is more likely? Provide quantitative support.

(a) Obtaining at least one 6 in 4 rolls of a single die.

= the probability of not getting a 6 in a single roll of die = 5/6

= the probability of not getting a 6 in 4 rolls of die = (5/6) ^ 4

= the probability of getting at least one 6 in 4 rolls of dice = 1 - (5/6) ^4 = **51.77%**

(b) Obtaining at least one 12 in 24 rolls of a pair of dice.

= the probability of not getting a 12 in a single pair of die is 35/36

= the probability of not getting a 12 in 24 rolls of a pair of die is (35/36) ^24

= the probability of getting at least one 12 in 24 rolls of a pair of dice = 1 - (35/36) ^24 = **49.19%**

Therefore, case (a) is more likely as it has more probability.

13. Data was collected from the residents of a town and displayed as follows:

		Income <\$25k	\$25k – \$70k	> 70k
Age (years)	< 25	952	1,050	53
	25 – 45	456	2,055	1,570
	> 45	54	952	1,008

Total number of residents in town = 8150 residents.

(a) What fraction of people are less than 25 years old?

= ((952 + 1050 + 53)/ 8150) * 100 = **25.21 %**

(b) What is the probability that a randomly chosen person is more than 25 years old?

= ((456+2055+1570+54+952+1008)/8150) * 100 = **74.78%**

(c) What fraction of people earn less than $70,000?

= ((952 + 1050 + 456 + 2055 + 54 + 952) / 8150) * 100 = **67.71%**

(d) What is the probability that a randomly chosen person is less than 25 years old and earns more than $70,000?

= (53 / 8150) * 100 = **0.65%**

(e) What fraction of people among those who earn less than $25,000 are between 25-45 years old?

= (456 / 8150) * 100 = **5.59%**

(f) If the next random person you see happens to be more than 45 years old, what is the probability that the person earns less than $70,000?

= ((54 + 952) /2014) * 100 = **49.95%**