1 Primal Simplex Method.

The matrix A is given by

$$\begin{bmatrix} 1.0 & 4.0 & 0.0 \\ 3.0 & -1.0 & 1.0 \end{bmatrix}.$$

The initial sets of basic and nonbasic indices are

$$\mathcal{B} = \{4, 5\} \quad \textit{and} \quad \mathcal{N} = \{1, 2, 3\}.$$

Corresponding to these sets, we have the submatrices of A:

$$B = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix} \quad N = \begin{bmatrix} 1.0 & 4.0 & 0.0 \\ 3.0 & -1.0 & 1.0 \end{bmatrix},$$

and the initial values of the basic variables are given by

$$x^*_{\mathcal{B}} = b = \begin{bmatrix} 1.0 \\ 3.0 \end{bmatrix},$$

and the initial nonbasic dual variables are simply

$$z^*_{\mathcal{N}} = -c_{\mathcal{N}} = \begin{bmatrix} -4.0 \\ -1.0 \\ -3.0 \end{bmatrix}.$$

1.1 1st Iteration.

Step 1. Since $z^*_{\mathcal{N}}$ has some negative components, the current solution is not optimal.
Step 2. Since z^*_1 = -4.0 and this is the most negative of the two nonbasic dual variables, we see that the entering index is

$$j = 1.$$

Step 3.

$$\Delta X_{\mathcal{B}} = B^{-1} N e_j = \begin{bmatrix} 1.0 & 4.0 & 0.0 \\ 3.0 & -1.0 & 1.0 \end{bmatrix} \begin{bmatrix} 1.0 \\ 0.0 \\ 0.0 \end{bmatrix} = \begin{bmatrix} 1.0 \\ 3.0 \end{bmatrix}.$$

Step 4.

$$t = \left(max \left\{ \frac{1.0}{1.0}, \frac{3.0}{3.0} \right\} \right)^{-1} = 1.0.$$

Step 5. Since the ratio that achieved the maximum in Step 4 was the 1st ratio and this ratio corresponds to basis index 4, we see that

$$i = 4.$$

Step 6.

$$\Delta z_{\mathcal{N}} = -\left(B^{-1}N\right)^{T} e_i = -\begin{bmatrix} 1.0 & 3.0 \\ 4.0 & -1.0 \\ 0.0 & 1.0 \end{bmatrix} \begin{bmatrix} 1.0 \\ 0.0 \end{bmatrix} = \begin{bmatrix} -1.0 \\ -4.0 \\ -0.0 \end{bmatrix}.$$

Step 7.

$$s = \frac{z_j^*}{\Delta z_j} = \frac{-4.0}{-1.0} = 4.0.$$

Step 8.

$$x_1^* = 1.0, x_{\mathcal{B}}^* = \begin{bmatrix} 1.0 \\ 3.0 \end{bmatrix} - 1.0 \begin{bmatrix} 1.0 \\ 3.0 \end{bmatrix} = \begin{bmatrix} 0.0 \\ 0.0 \end{bmatrix},$$

$$z_4^* = 4.0, z_{\mathcal{N}}^* = \begin{bmatrix} -4.0 \\ -1.0 \\ -3.0 \end{bmatrix} - 4.0 \begin{bmatrix} -1.0 \\ -4.0 \\ -0.0 \end{bmatrix} = \begin{bmatrix} 0.0 \\ 15.0 \\ -3.0 \end{bmatrix}.$$

Step 9. The new sets of basic and nonbasic indices are

$$\mathcal{B} = \{1, 5\} \quad and \quad \mathcal{N} = \{4, 2, 3\}.$$

Corresponding to these sets, we have the new basic and nonbasic submatrices of A,

$$B = \begin{bmatrix} 1.0 & 0.0 \\ 3.0 & 1.0 \end{bmatrix} \quad N = \begin{bmatrix} 1.0 & 4.0 & 0.0 \\ 0.0 & -1.0 & 1.0 \end{bmatrix},$$

and the new basic primal variables and nonbasic dual variables:

$$x_{\mathcal{B}}^* = \begin{bmatrix} x_1^* \\ x_5^* \end{bmatrix} = \begin{bmatrix} 1.0 \\ 0.0 \end{bmatrix}, \quad z_{\mathcal{N}}^* = \begin{bmatrix} z_4^* \\ z_2^* \\ z_3^* \end{bmatrix} = \begin{bmatrix} 4.0 \\ 15.0 \\ -3.0 \end{bmatrix}.$$

1.2 2nd Iteration.

Step 1. Since $z^*_{\mathcal{N}}$ has some negative components, the current solution is not optimal.
Step 2. Since z^*_3 = -3.0 and this is the most negative of the two nonbasic dual variables, we see that the entering index is

$$j = 3.$$

Step 3.

$$\Delta X_{\mathcal{B}} = B^{-1}Ne_j = \begin{bmatrix} 1.0 & 4.0 & 0.0 \\ -3.0 & -13.0 & 1.0 \end{bmatrix} \begin{bmatrix} 0.0 \\ 0.0 \\ 1.0 \end{bmatrix} = \begin{bmatrix} 0.0 \\ 1.0 \end{bmatrix}.$$

Step 4.

$$t = \left(max \left\{ \frac{0.0}{1.0}, \frac{1.0}{0.0} \right\} \right)^{-1} = 0.0.$$

Step 5. Since the ratio that achieved the maximum in Step 4 was the 2nd ratio and this ratio corresponds to basis index 5, we see that

$$i = 5.$$

Step 6.

$$\Delta z_{\mathcal{N}} = -\left(B^{-1}N\right)^T e_i = -\begin{bmatrix} 1.0 & -3.0 \\ 4.0 & -13.0 \\ 0.0 & 1.0 \end{bmatrix} \begin{bmatrix} 0.0 \\ 1.0 \end{bmatrix} = \begin{bmatrix} 3.0 \\ 13.0 \\ -1.0 \end{bmatrix}.$$

Step 7.

$$s = \frac{z^*_j}{\Delta z_j} = \frac{-3.0}{-1.0} = 3.0.$$

Step 8.

$$x^*_3 = 0.0, x^*_{\mathcal{B}} = \begin{bmatrix} 1.0 \\ 0.0 \end{bmatrix} - 0.0 \begin{bmatrix} 0.0 \\ 1.0 \end{bmatrix} = \begin{bmatrix} 1.0 \\ 0.0 \end{bmatrix},$$

$$z^*_5 = 3.0, z^*_{\mathcal{N}} = \begin{bmatrix} 4.0 \\ 15.0 \\ -3.0 \end{bmatrix} - 3.0 \begin{bmatrix} 3.0 \\ 13.0 \\ -1.0 \end{bmatrix} = \begin{bmatrix} -5.0 \\ -24.0 \\ 0.0 \end{bmatrix}.$$

Step 9. The new sets of basic and nonbasic indices are

$$\mathcal{B} = \{1, 3\} \quad and \quad \mathcal{N} = \{4, 2, 5\}.$$

Corresponding to these sets, we have the new basic and nonbasic submatrices of A,

$$B = \begin{bmatrix} 1.0 & 0.0 \\ 3.0 & 1.0 \end{bmatrix} \quad N = \begin{bmatrix} 1.0 & 4.0 & 0.0 \\ 0.0 & -1.0 & 1.0 \end{bmatrix},$$

and the new basic primal variables and nonbasic dual variables:

$$x^*_{\mathcal{B}} = \begin{bmatrix} x^*_1 \\ x^*_3 \end{bmatrix} = \begin{bmatrix} 1.0 \\ 0.0 \end{bmatrix}, \quad z^*_{\mathcal{N}} = \begin{bmatrix} z^*_4 \\ z^*_2 \\ z^*_5 \end{bmatrix} = \begin{bmatrix} -5.0 \\ -24.0 \\ 3.0 \end{bmatrix}.$$

1.3 3rd Iteration.

Step 1. Since $z^*_{\mathcal{N}}$ has some negative components, the current solution is not optimal.
Step 2. Since z^*_2 = -24.0 and this is the most negative of the two nonbasic dual variables, we see that the entering index is

$$j = 2.$$

Step 3.

$$\Delta X_{\mathcal{B}} = B^{-1}Ne_j = \begin{bmatrix} 1.0 & 4.0 & 0.0 \\ -3.0 & -13.0 & 1.0 \end{bmatrix} \begin{bmatrix} 0.0 \\ 1.0 \\ 0.0 \end{bmatrix} = \begin{bmatrix} 4.0 \\ -13.0 \end{bmatrix}.$$

Step 4.

$$t = \left(max \left\{ \frac{4.0}{1.0}, \frac{-13.0}{0.0} \right\} \right)^{-1} = 0.25.$$

Step 5. Since the ratio that achieved the maximum in Step 4 was the 1st ratio and this ratio corresponds to basis index 1, we see that

$$i = 1.$$

Step 6.

$$\Delta z_{\mathcal{N}} = -\left(B^{-1}N\right)^T e_i = - \begin{bmatrix} 1.0 & -3.0 \\ 4.0 & -13.0 \\ 0.0 & 1.0 \end{bmatrix} \begin{bmatrix} 1.0 \\ 0.0 \end{bmatrix} = \begin{bmatrix} -1.0 \\ -4.0 \\ -0.0 \end{bmatrix}.$$

Step 7.

$$s = \frac{z_j^*}{\Delta z_j} = \frac{-24.0}{-4.0} = 6.0.$$

Step 8.

$$x_2^* = 0.25, x_{\mathcal{B}}^* = \begin{bmatrix} 1.0 \\ 0.0 \end{bmatrix} - 0.25 \begin{bmatrix} 4.0 \\ -13.0 \end{bmatrix} = \begin{bmatrix} 0.0 \\ 3.25 \end{bmatrix},$$

$$z_1^* = 6.0, z_{\mathcal{N}}^* = \begin{bmatrix} -5.0 \\ -24.0 \\ 3.0 \end{bmatrix} - 6.0 \begin{bmatrix} -1.0 \\ -4.0 \\ -0.0 \end{bmatrix} = \begin{bmatrix} 1.0 \\ 0.0 \\ 3.0 \end{bmatrix}.$$

Step 9. The new sets of basic and nonbasic indices are

$$\mathcal{B} = \{2, 3\} \quad \textit{and} \quad \mathcal{N} = \{4, 1, 5\}.$$

Corresponding to these sets, we have the new basic and nonbasic submatrices of A,

$$B = \begin{bmatrix} 4.0 & 0.0 \\ -1.0 & 1.0 \end{bmatrix} \quad N = \begin{bmatrix} 1.0 & 1.0 & 0.0 \\ 0.0 & 3.0 & 1.0 \end{bmatrix},$$

and the new basic primal variables and nonbasic dual variables:

$$x_{\mathcal{B}}^* = \begin{bmatrix} x_2^* \\ x_3^* \end{bmatrix} = \begin{bmatrix} 0.25 \\ 3.25 \end{bmatrix}, \quad z_{\mathcal{N}}^* = \begin{bmatrix} z_4^* \\ z_1^* \\ z_5^* \end{bmatrix} = \begin{bmatrix} 1.0 \\ 6.0 \\ 3.0 \end{bmatrix}.$$

1.4 4^{th} Iteration.

Step 1. Since $z_{\mathcal{N}}^*$ has all nonnegative components, the current solution is optimal. The optimal objective function value is

$$\zeta^* = 4.0x_1^* + 1.0x_2^* + 3.0x_3^* = [10.]$$