Mathematical Thinking - Week 2
Activity Questions

September 28, 2023

Contents

1 A Trip to Cantorsville

1. What are other ways in which the manager of the Hilbert Hotel in Cantorsville could have accommodated the people coming from the infinitely many Hilbert Hotels?
2. What are other ways in which the manager of the Hilbert Hotel in Cantorsville could have accomodated the people coming from the infinitely many Hilbert Hotels if it is alright to leave some of the rooms empty?
   Hint: There are infinitely many prime numbers.

2 Cantor's Diagonalization Argument

1. Each real number $r \in [0, 1)$ can be denoted as $r_i = 0.d_{i1}d_{i2}d_{i3}d_{i4}\ldots$ for $i \in \mathbb{N}$, e.g. $1/2$ can be written as $0.5000\ldots$ where $d_{i1} = 5$, $d_{i2} = 0$, $d_{i3} = 0$, and so on, for some $i \in \mathbb{N}$. Can you construct a bijection $f : \mathbb{N} \to [0, 1)$? If not, use Cantor's Diagonalization Argument to show that such a function would be surjective.

3 Towards the Real Numbers

1. Show that there does not exist any rational number x for which $x^2 = 3$.
2. Verify each of the ten field axioms for rationals using the algebra of integers.
3. Show that addition preserves order for rational numbers. If $a \geq b$, then $a + c \geq b + c$ for any $a, b, c \in \mathbb{Q}$.

4 Ordered Field

1. Provide examples that illustrate each of the three order axioms for rational numbers.
2. If $a > 0$ and $b < 0$. Show that $ab < 0$.

5 Completeness Axiom

1. In the lecture we find that $\mathbb{Q}$ has *holes*. Give more examples of sets of rational numbers that does not have the least upper bound.

6 The Least Upper Bound Property

1. Assume that Amri runs the first half of the marathon in one hour and that his average speed each hour is half of his average speed in the previous hour. Work out how much of the marathon he would run in 6 hours? Will he ever completeness the marathon?
2. Using induction show that $2^n > n$ for every natural number n. Conclude that $2^{-n} < 1/n$ for every natural number n.

7 Mathematical Logic and Statements

1. Show that for $a, b \in \mathbb{R}$, if $|a - b| < \epsilon$ for all $\epsilon > 0$, then $a = b$.