Mathematical thinking
Questions
Week 2

1. Let A and B be nonempty sets and $A \subset B$. Suppose set B has a least upper bound. Then
   (i) Prove that set A also has a least upper bound.
   (ii) Prove that the supremum of A is less than or equal to the supremum of B.

2. Let F be an ordered field and $x, y, z \in F$. Using only the definitions, prove that if $x > 0$ and $y < z$, then $xy < xz$.

3. Which of the following is/are irrational numbers?
   (a) π
   (b) $\sqrt{25}$
   (c) $\sqrt{5}$
   (d) $\sqrt{8}$

4. Let a and b be two real numbers such that $a < b$. Prove that there exist infinitely many real numbers α such that $a < \alpha < b$.

5. Which of the following options is/are true?
   (a) If p is any prime number, then $\sqrt{p}$ is an integer.
   (b) If x is an irrational number and y is a rational number, then $x + y$ is an irrational number.
   (c) If x is an irrational number and y is a rational number, then $x - y$ is an integer.
   (d) If x is an irrational number and y is a rational number, then $\dfrac{x}{y}$ is a rational number.

6. Let S be a set of real numbers. Set S is said to be bounded if there exist two real numbers α and β such that $\alpha < x < \beta, \forall x \in S$. Prove that any subset of S has the supremum in $\mathbb{R}$.

7. Give an example (with the explanation) of a set of real numbers that has an upper bound but not the supremum in the set.

8. Find the supremum of the set $\left\{1 - \dfrac{(-1)^n}{n} \mid n \in \mathbb{N}\right\}$.

9. Consider the set $S = \mathbb{Q}^+ \cup \{0\}$, where $\mathbb{Q}^+$ is the collection of all positive rational numbers. Prove or disprove that S is a field.

10. Prove that for any $\epsilon > 0$ there exist a integer m such that $\dfrac{1}{m} < \epsilon$.

11. Consider the set $S = \{5^n \mid n \in \mathbb{N}\}$. Prove that the set S has no upper bound.

12. Which of the following sets have an upper bound but not the supremum in the set.

    (a) $D = \{x \in \mathbb{Q} \mid \sin(\pi x) = 0\}$

    (b) $S = \{x \in \mathbb{Q}' \mid 0 < x < \pi\}$. $\mathbb{Q}'$ is the set of all irrational numbers.

    (c) $T = \{1/n \mid n \in \mathbb{N}\}$

    (d) $U = \{x \in \mathbb{R} \mid x^2 < 4\}$

13. Negate the statement "Either X is true, or Y is true, but not both".

14. A function f in 2 variables is said to be *good* if "For every real number x, there exists a real number y such that $f(x, y) = 0$". What does it mean to say that a function f is not good.

15. Let $S = \{\dfrac{(-1)^n}{2^n} | n \in \mathbb{N}\}$. Find the $Sup(S)$.
    Answer: 0.25

16. Which of the following statements is equivalent to the statement "not(For all real numbers satisfying $a < b$, there exists an $n \in \mathbb{N}$ such that $a + \dfrac{1}{n} < b$')"?

    (a) There exist real numbers satisfying $a < b$ where $a + \dfrac{1}{n} < b$ for all $n \in \mathbb{N}$.

    (b) For some real numbers $a < b$, there exists an $n \in \mathbb{N}$ such that $a + \dfrac{1}{n} < b$'.

    (c) For all real numbers satisfying $a < b$ where $a + \dfrac{1}{n} < b$ for all $n \in \mathbb{N}$.

    (d) For some real numbers $a < b$ where $a + \dfrac{1}{n} < b$ for some $n \in \mathbb{N}$.

    Answer: (a)