Mathematical Thinking
QUIZ 1
Sept 2023 term
Total marks: 20

1 Section I (7 Marks)

1. In the context of real numbers and the standard order relation, which of the following statement(s) is (are) true?
   (a) The least upper bound is the largest element in the set.
   (b) A set may have multiple least upper bounds.
   (c) The least upper bound may not be in the set itself.
   (d) The completeness axiom ensures that the real numbers have no gaps or holes.

   Answer: (c),(d)

2. If the negation of a statement is denoted as $\neg$(The statement), then which of the following options is/are true?
   (a) $\neg$(The set S is a bounded above set and *sup* S is an upper bound of S)= The set S is not a bounded above set and *sup* S is not an upper bound of S.
   (b) $\neg(p \mid ab) = p \nmid a$ and $p \nmid b$.
   (c) $\neg$(Every convergent sequence is bounded)= A bounded sequence is convergent.
   (d) $\neg$(x is an arbitrary element of the set $S \cup T$)= x is not element of S and x is not element of T

   Answer: (b),(d)

3. Which of the following options is/are true?
   (a) If a sequence $\{s_n\}$ in $\mathbb{R}$ is convergent and $s_n \geq a$ for all but finitely many n, then $\lim\limits_{n\to\infty} s_n \leq a$
   (b) Let $\{s_n\}$ be a sequence in $\mathbb{R}$. If $\lim\limits_{n\to\infty} |s_n| = 0$, then $\lim\limits_{n\to\infty} s_n = 0$
   (c) Let $\{s_n\}$ be a sequence in $\mathbb{R}$ such that $s_1 > 1$ and $s_{n+1} = 4 - \dfrac{4}{s_n}$. $\{s_n\}$ is an increasing sequence.
   (d) Let $\{s_n\}$ be a sequence in $\mathbb{R}$ such that $s_1 < 1$ and $s_{n+1} = 4 - \dfrac{4}{s_n}$. $\{s_n\}$ is an decreasing sequence.

   Answer: (b),(d)

4. Which of the following options is/are true?

   (a) Set $\{n \in \mathbb{N} \mid n \geq 5\}$ and set $\{n \in \mathbb{N} \mid n \geq 500\}$ have the same cardinality.

   (b) $\displaystyle\sum_{k=1}^{n}(3n-1) = \frac{n(6n+1)}{2}$

   (c) If S is the set $\{1,2,3,4\}$, then cardinality of the set $(S \times S) \setminus (S \times \{3\})$ is 12.

   (d) $S = -\dfrac{1^2}{2^2} + \dfrac{1^2 \cdot 3^2}{2^2 \cdot 4^2} - \dfrac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} + \ldots = \displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{1^2 \cdot 3^2 \cdot 5^2 \ldots (2n-1)^2}{2^2 \cdot 4^2 \cdot 6^2 \ldots n^2}$

   Answer: (a),(c)

5. Which of the following options is/are true?

   (a) Set of rational numbers $\mathbb{Q}$ is a field.

   (b) Set of rational numbers $\mathbb{Q}$ is a complete ordered field.

   (c) For a real number $\alpha > 0$, let set αS denoted as the set $\alpha S = \{\alpha x \mid x \in S\}$, then $sup(\alpha S) = \alpha(supS)$.

   (d) For a real number $\alpha < 0$, let set αS denoted as the set $\alpha S = \{\alpha x \mid x \in S\}$, then $sup(\alpha S) = -\alpha(supS)$

   Answer: (a),(c)

2 Section II (13 Marks)

1. Consider three sets A, B and C. Prove that $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

2. The Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$ is defined recursively by $a_1 = 1, a_2 = 1$, and $a_{n+2} = a_{n+1} + a_n$, for all $n \geq 1$. Prove that $\gcd(a_k, a_{k+1}) = 1$ for all $k \in \mathbb{N}$.

3. Consider a sequence $\{a_n\}$ such that $a_1 = 4$ and $a_{n+1} = 4^{a_n}$, for all $n > 1$. Find the remainder when a_{100} is divided by 7.

4. There exists a rational number x such that $x^2 = 3$. Prove or disprove this statement.

5. Prove or disprove the following statement: For any non-empty subset A of $\mathbb{Q}$ that is bounded from above, there exists a least upper bound for A in the set of rational numbers $\mathbb{Q}$.