

Mathematical Thinking

Activity Questions

Week 6

1 Why Study Sequences?

1. For the sequence a_n constructed using the continued fraction representation of $\sqrt{2}$, show that

$$a_{n+1} = 1 + \frac{1}{1 + a_n}$$

- 2.

$$\text{Let } a_n = \left(1 + \frac{1}{n}\right)^n.$$

Show that the k th term in the binomial expansion of a_n is less than that of a_{n+1} .

2 Binomial Coefficients

1.
 - a. $\gcd(2a, 2b) = k$
 - b. $\gcd(2a, a - b) = k$
 - c. $\gcd(a, b) = k$
 - d. $\gcd(a + b, 2b) = k$

Answer: (b), (d)

2. Every odd integer is of the form _____.
 - a. either $1 + (n - 1)^2$ or $2 + (2n - 1)^2, n \in \mathbb{N}$
 - b. either $6n - 3$ or $6n - 5, n \in \mathbb{N}$
 - c. $3n - 2, n \in \mathbb{N}$
 - d. either $4n - 1$ or $4n - 3, n \in \mathbb{N}$

Answer: (d)

3. Suppose $\gcd(a, b) = 1$. If $c \mid a$ and $d \mid b$, then prove that $\gcd(c, d) = 1$. (2 marks)
4. If $a \mid bc$ and $\gcd(a, b) = 1$, then prove that $a \mid c$. (2 marks)
5. Using the induction method prove that for any $n \in \mathbb{N}$, $2 \mid n(n+1)$ and $3 \mid n(n+1)(n+2)$.
6. Consider two integers x and y such that $6x + 12y = 3$. Prove or disprove that such integers exist or not. If it exists then write some values of x and y .

graded

1. Find the remainder when $2^{20} + 3^{30} + 4^{40} + 5^{50}$ is divided by 7. (2 marks)

Answer: 6

2. Let $a, b \in \mathbb{N}$. Which of the following options is (are) always true? (2 marks)
 - a. If $a^3 \mid b^3$, then $a \mid b$.
 - b. If $a^a \mid b^b$, then $a \mid b$.
 - c. If $a^b \mid b^a$, then $a \mid b$.
 - d. If $a^2 \mid 2b^2$, then $a \mid b$.

Answer: (a),(b),(d)

3. Let $m, n \in \mathbb{N}$ and $\gcd(m, n) = 1$. Which of the following options is (are) always true? (1 mark)
 - a. $\exists x, y \in \mathbb{Z}$ such that $mx - ny = 1$.
 - b. $\exists x, y \in \mathbb{Z}$ such that $mx + ny = mn$.
 - c. $\gcd(xm, xn) = 1 \forall x \in \mathbb{Z}$.
 - d. $\gcd(xm, yn) = 1 \forall x, y \in \mathbb{Z}$ and $\gcd(x, y) = 1$.

Answer: (a), (b)

4. Which of the following equations have solutions $a, b \in \mathbb{Z}$? (2 marks)
 - a. $12a + 20b = 42$
 - b. $152a + 102b = 3$
 - c. $23a + 11b = 120$
 - d. $21a + 91b = 50$

Answer: (c)

5. Consider a number $a = a_1 a_2 a_3 \dots a_n$, where $a_1, a_2, a_3, \dots, a_n$ are the digits for integers a . If a is divisible by 8, then prove that the number which is formed by the last 3 digits ($a_{n-2} a_{n-1} a_n$) of a is divisible by 8. (3 marks)
6. Using the division algorithm show that if $n \mid m$ and $n \mid k$, then $n^2 \mid mk$. (2 marks)
7. Prove that $\gcd(\gcd(m, n), k) = \gcd(m, \gcd(n, k))$. (4 marks)
8. Let r be the remainder obtained when m is divided by n . Let $g = \gcd(m, n)$ and $g' = \gcd(n, r)$. Prove that $g' \mid g$. (4 marks)