

ICTS-RRI Maths Circle, Saturday 13 January, 2024

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Divisibility

The division algorithm is a fundamental mathematical concept that provides a systematic way to divide one integer by another, with the result expressed as a quotient and a remainder. The division algorithm ensures that when you divide a by b , you can express a as a multiple of b plus a remainder, and the remainder is always less than the absolute value of the divisor b . We say that b divides a when the remainder is zero.

Tests of divisibility are rules or criteria that help determine whether a given number is divisible by another without performing the actual division. You may have already come across these in your school.

Let's consider a problem: What is the remainder when 2^{100} is divided by 101?

We can solve such problems without actually performing the division. Here are some warm-up problems to help you get started.

1. Check if the following numbers are divisible by 4:

- (a) 632,
- (b) 7896,
- (c) 245,
- (d) 1584

2. Check if the following numbers are divisible by 6:

- (a) 846,
- (b) 729,
- (c) 312,
- (d) 990

3. Check if the following numbers are divisible by 8:

- (a) 1248,
- (b) 572,
- (c) 896,
- (d) 633

4. Investigate the divisibility by 5 of the sum of the squares of the first 10 positive integers:
 $1^2 + 2^2 + 3^2 + \dots + 10^2$.

5. Find the smallest positive integer that is divisible by 8, consists of only the digits 1 and 0, and each digit appears at least once.
6. Investigate the divisibility by 9 of the sum of the cubes of the first 5 positive integers: $1^3 + 2^3 + 3^3 + 4^3 + 5^3$.
7. Determine the largest five-digit number that is divisible by 10, has all distinct digits, and the sum of its digits is 25.

Modular Arithmetic and Test for Divisibility by 7

Let's take a step back and try to understand how these tests for divisibility can be constructed. One way to go about this is by examining the remainders when powers of 10, 10^0 , 10^1 , 10^2 , \dots , are divided by a specific number.

Compute the remainders when the powers of 10 (10^0 , 10^1 , 10^2 , \dots) are divided by 3. Do you observe any patterns or repetitions in the remainders? Repeat this exercise for the number 7. Based on your observation, propose a divisibility test for determining whether a number is divisible by 7?

What can be said about the remainders when powers of a different number, say 8 (8^0 , 8^1 , 8^2 , \dots) are divided by 3 or 7? Do you notice any patterns?