Maths Circle Challenge

April 4, 2025

Dividing a Field

Problem 1. A farmer's field is irregularly shaped, as shown in the figure. The terrain is flat and the boundary lines are all in the North-South or East-West direction. The farmer has two sons and wants to divide the land into two pieces of land with equal area. He has before him a map of the land and an old school geometry box. The box has a compass and a ruler, whose markings are faded and unreadable. Can you help the farmer divide the land with the instruments he has?

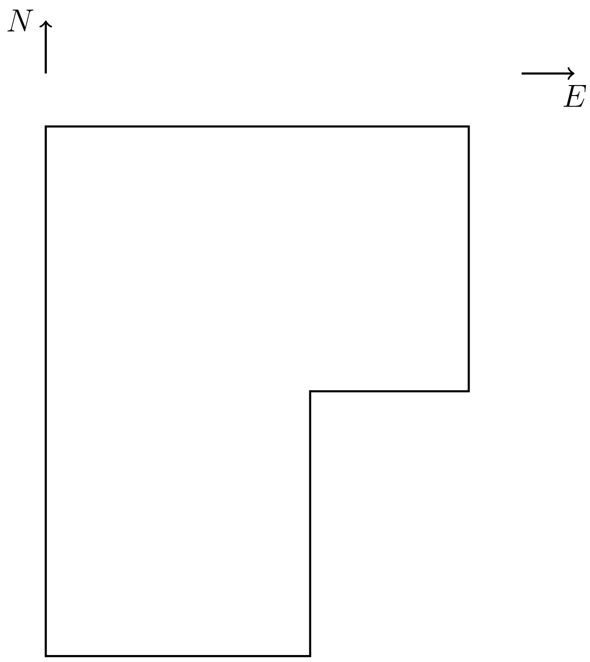

Figure 1: Figure shows an irregular piece of land owned by the Farmer. The problem is to divide the land into two pieces of equal area using only a straight edge and compass.

Solution: A line through the centre of the rectangle divides it into two equal areas.

Step 1: Find the O_a centre of the extended rectangle $ABCD$.

Step 2: Find the centre of the missing rectangle portion, call it O_b.

Step 3: Connect O_a and O_b.

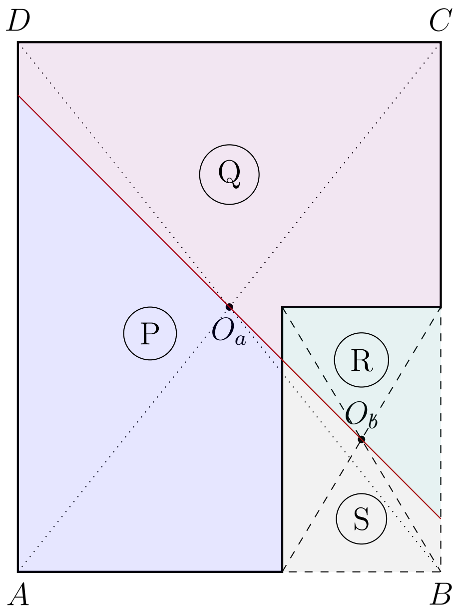

Figure 2: The line through O_a and O_b divides the extended rectangle into four regions - P, Q, R, and S.

$$\begin{aligned}&P + S = Q + R,\\ &S = R,\\ &\text{and therefore } P = Q\end{aligned}$$

Problem 2. Prove that for any integer n, $n^3 + 11n$ is a multiple of 6.

Solution:

Case 1a: $n \equiv 0 \pmod 2, n$ is even

then $n^3 + 11n \equiv 0 \pmod 2$

Case 1b: $n \equiv 1 \pmod 2$, n is odd

then $n^3 \equiv 1 \pmod 2$ and $11n \equiv 1 \pmod 2$

so $n^3 + 11n \equiv 0 \pmod 2$

$n^3 + 11n$ is divisible by 2 for all n.

Case 2a: $n \equiv 0 \pmod 3$

then $n^3 + 11n \equiv 0 \pmod 3$

Case 2b: $n \equiv 1 \pmod 3$, $n = 3x + 1$

$n^3 + 11n = 12 + 42x + 27x^2 + 27x^3 = 3 \times (4 + 14x + 9x^2 + 9x^3) \equiv 0 \pmod 3$

Case 2c: $n \equiv 2 \pmod 3$, $n = 3x + 2$

$n^3 + 11n = 30 + 69x + 54x^2 + 27x^3 = 10 + 23x + 18x^2 + 9x^3 \equiv 0 \pmod 3$

Method 2:

Let $n^3 + 11n = n(n^2 + 11) = p$

when n is odd, $(n^2 + 11)$ is even, making p even.

when n is even, p is even.

when n is a multiple of 3, p is also a multiple of 3.

$n^2 + 11 = (n + 1)(n - 1) + 12.$

When n is not a multiple of 3, either $n + 1$ or $n - 1$ is a multiple of 3.

This makes $n^2 + 11$ a multiple of 3 when n is not.

Method 3: Proof by induction

$$\begin{aligned}
&\text{Let } P(n) \text{ be the statement that } n^3 + 11n \equiv 0 \pmod 6\\
&\text{For } n = 1,\ n^3 + 11n = 12 \equiv 0 \pmod 6\\
&P(1) \text{ is correct.}\\
&\text{Assume } P(k) \text{ is correct.}\\
&k^3 + 11k \equiv 0 \pmod 6.\\
&(k+1)^3 + 11(k+1) = k^3 + 11k + 12 + 3k(k+1)\\
&3k(k+1) \equiv 0 \pmod 2 \text{ for any } k\\
&3k(k+1) \equiv 0 \pmod 3 \text{ for any } k\\
&(k+1)^3 + 11(k+1) \equiv 0 \pmod 6 \text{ or } P(k+1) \text{ is correct.}
\end{aligned}$$

The same proof can be used for $n < 0,\ n \in \mathbb{Z}$. Let $n = -m$ where $m \in \mathbb{Z}^+$. $(-m)^3 + 11(-m) = -(m^3 + 11m)$. Since $m^3 + 11m \equiv 0 \pmod 6$ for $m \in \mathbb{Z}^+$, $-(m^3 + 11m) \equiv 0 \pmod 6$.

Method 4:

$$\begin{aligned}
&n^3 + 11n = n(n^2 + 11) = n((n+1)(n-1) + 12) = (n+1)n(n-1) + 12n\\
&(n+1)n(n-1) \equiv 0 \pmod 2\ \forall\, n \in \mathbb{Z}\\
&\text{and } (n+1)n(n-1) \equiv 0 \pmod 3\ \forall\, n \in \mathbb{Z}\\
&\implies (n+1)n(n-1) \equiv 0 \pmod 6\ \forall\, n \in \mathbb{Z}\\
&12n \equiv 0 \pmod 6\ \forall\, n \in \mathbb{Z}\\
&\implies (n+1)n(n-1) + 12n = n^3 + 11n \equiv 0 \pmod 6\ \forall\, n \in \mathbb{Z}
\end{aligned}$$

Problem 3. A couple (let's call them Sharmila and Prakash) invite five couples over for dinner. When they meet, there are introductions and some of the people shake hands. Of course, one does not shakes one's own hand and that of his/her spouse. Sharmila notices that of the other people (excluding herself) in the gathering, no two people shake the same number of hands. How many hands does Prakash shake?

Solution: There are 11 people, each engaging in a handshake with some or none of the others. Each person shakes a number of hands from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Let P_k be the person who shook k hands.

P_{10} shook everyone's hands except their spouse's and P_0. This accounts for 13 people (P_{10}, thier spouse, the 10 people they shake hands with, and P_0). P_0 and P_{10} are a couple. P_1 shakes hands only with P_{10}. P_9 has to shake 8 more hands other than that of P_0, P_1, P_{10}, their spouse, or self. The only way to ensure distinct handshakes is for P_9 to be the spouse of P_1. By extending this logic, one can see that P_{10-n} and P_n are a couple for $n \in \{0, 1, 2, 3, 4\}$ and P_5 is unpaired.

This means that P_5 and another person both shook the same number of hands.

Since no two people shake the same number of hands (excluding Sharmila), but we now have two people shaking 5 hands. Thus, P_5 must be Prakash who shakes the same number of hands as his spouse Sharmila.

Prakash shook hands with 5 people.

Problem 4. Let us make the following definition. We call any finite sequence of English letters "a word" (whether or not it can be found in a dictionary). For example, we can form six words using the letters A, B, and C each exactly once: ABC, ACB, BAC, BCA, CAB, and CBA. In the following calculate the number of different words that can be obtained by rearranging the letters of the word.

(a) MESOPOTAMIA

Solution: The formula for the number of distinct permutations of a multiset is:

$$\frac{n!}{k_1!k_2!...k_m!}$$

$$\frac{11!}{2!2!2!} = 4989600$$

(b) SCRAMBLE

Solution:

$$8! = 40320$$

(c) JUXTAPOSITION

Solution:

$$\frac{13!}{2!2!2!} = 778377600$$

(d) VIOLIN

Solution:

$$\frac{6!}{2!} = 360$$

(e) MISSISSIPPI

Solution:

$$\frac{11!}{4!4!2!} = 34650$$

Problem 5. Mr. and Mrs. Sharma have four children - three boys and a girl - who each likes one of the colours - blue, green, red, yellow - and one of the letters - P, Q, R, S. The oldest child likes the letter Q. The youngest child likes green. Aditya likes the letter S. Bhanumati, the girl, has an older brother who likes R. The one who likes blue isn't the oldest. The one who likes red likes the letter P. Chetan likes yellow. Based on the above facts, Deepak is the

A. youngest child

B. third child

C. second child

D. oldest child

Solution:

	three boys and a girl
	blue, green, red, yellow
	P, Q, R, S
1 Q 2 3 4	The oldest child likes the letter Q
1 Q 2 3 4 green	The youngest child likes green.
2, 3, or 4 Boy S Aditya	Aditya likes the letter S
2 or 3 Boy R 3 or 4 Girl Bhanumati	Bhanumati, the girl, has an older brother who likes R.

1	Boy	yellow	Q	Chethan
2	Boy	blue	R	Deepak
3	Girl	red	P	Bhanumati
4	Boy	green	S	Aditya

Problem 6. When we throw a die, the numbers one to six are all equally likely. When we throw a pair of dice, the "outcome" of the throw is usually defined as the sum of the two numbers appearing on top of the dice. Let us change the rules and define the "outcome" as the product of the two numbers.

How many distinct outcomes are there? What is the chance (probability) that the outcome is

(a) a prime number

Solution:

There are 18 distinct outcomes = $\{1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, 24, 25, 30, 36\}$

The outcomes with prime numbers are $\{(1, 2), (2, 1), (1, 3), (3, 1), (1, 5), (5, 1)\}$.
P(prime)= $6/36 = 1/6$

(b) a perfect square

Solution: Favourable outcomes are $\{(1, 1), (2, 2), (3, 3), (1, 4), (4, 1), (4, 4), (5, 5), (6, 6)\}$.
P= $8/36 = 2/9$.

(c) a triangular number

Solution: Triangular numbers are of the form $n(n + 1)/2$. Favourable outcomes are $\{(1, 1), (1, 3), (3, 1), (3, 2), (2, 3), (2, 5), (5, 2), (5, 3), (3, 5), (6, 6)\}$.
P $= 10/36 = 5/18$.

(d) an even number

Solution: P(even) = 1 - P(odd). Odd numbers happen when outcomes are of the form (odd, odd). There 9 (odd,odd) outcomes.
P $1 - 9/36 = 3/4$

(e) an odd number

Solution: From previous answer P $= 1/4$

(f) both triangular and square

Solution: The only triangular number that is also a square is 1. $\{(1,1),(6,6)\}$ is the favourable outcome.
P $= 2/36 = 1/18$.

Problem 7. 10 cables were laid across the Zambezi river in a remote part of Africa. After laying the cables, the engineer realised that he had forgotten to label them, so he didn't know which of the 10 ends on one bank corresponded to the 10 ends on the other. He had at his disposal a multimeter to test continuity, some copper wire to connect the ends on each bank and a boat to cross the river with. As you may know, the Zambezi river has herds of hippos and these animals can be dangerous to crossing boats. It would be wise to minimise the number of river crossings.

How would the engineer figure out the corresponding cables with a minimum number of crossings?