Exploring Modern Mathematics Through the Art of Tiling

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Science Gallery Bangalore

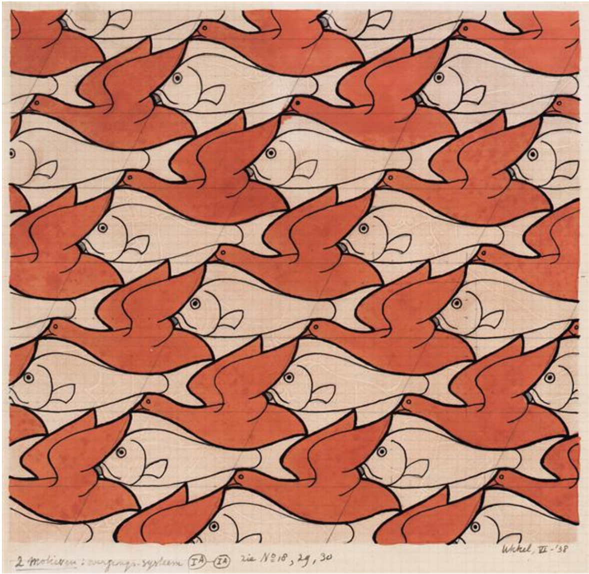


Figure: Bird Fish, 1938, MC Escher

Figure: Corridors, 2024, Photo: Aysha Mahira

Figure: Kani pashmina shawl, Varuna Anand

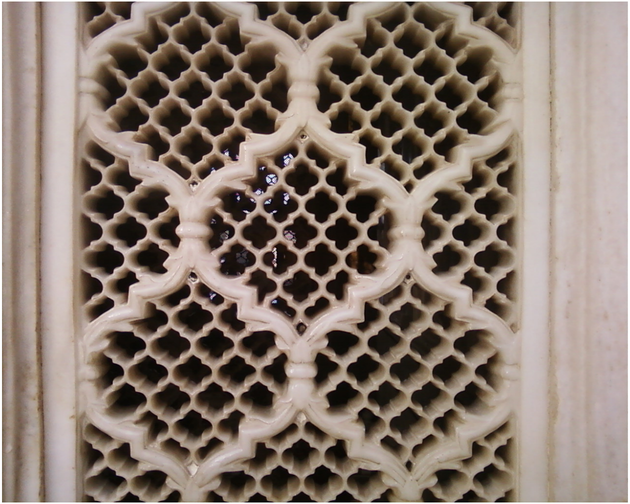


Figure: Jali at Bibi Ka Maqbara in Aurangabad. By Niranjan R. Upasani - Own work, CC BY-SA 3.0

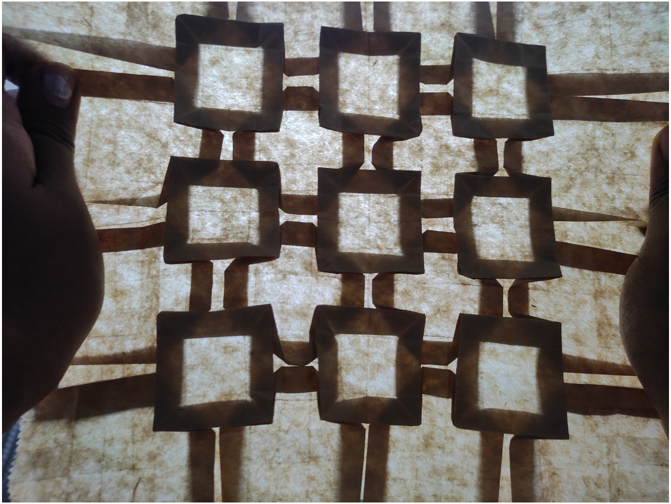


Figure: 4 - 5 square origami tessellation, pattern from Eric Gjerde, folded by dishajk

Figure: AWestern honey bee on a honeycomb, Matthew T Rader, https://matthewtrader.com, CC BY-SA 4.0

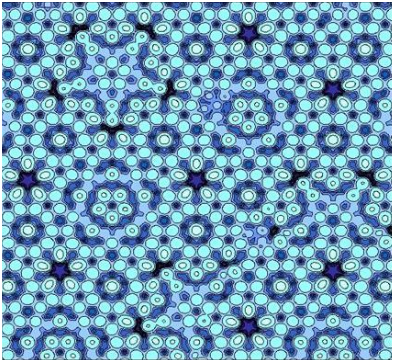


Figure: Potential energy surface $(8.6 \times 8.6nm^2)$ for silver on i-Al-Pd-Mn Quantumcrystal, Physical Review B 75, no. 6 (2007): 064205

Notice that its ordered but not periodic!

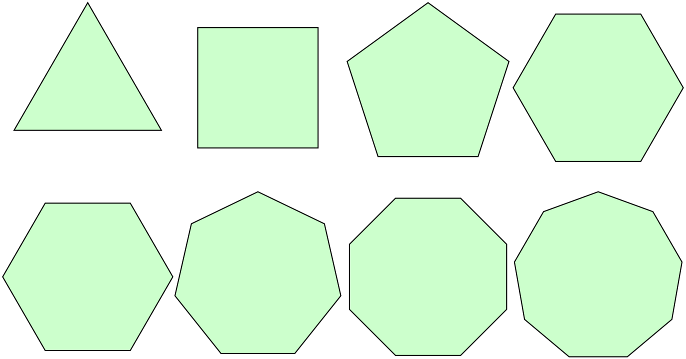
Tiling

What is Tiling?

Tiling is the covering of a surface using geometric shapes while ensuring there are no overlaps and no gaps.

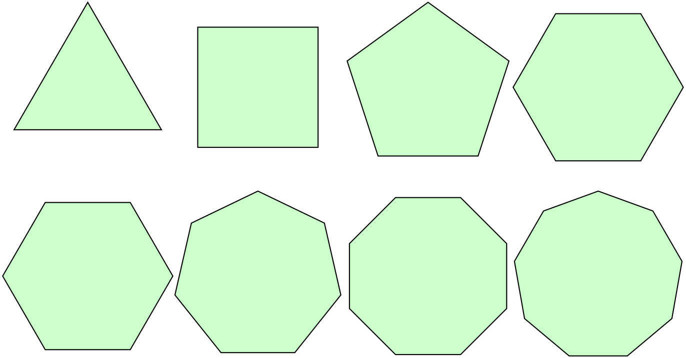
Tiling with Convex Regular Polygons

A convex regular polygon has n straight edges and n equal internal angles. It is convex because any line segment joining two vertices passes entirely through the polygon's interior.



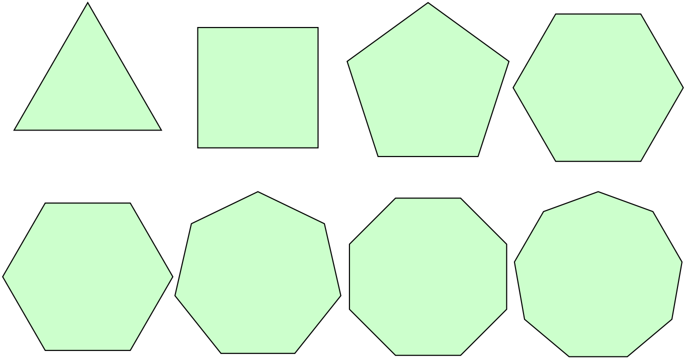
Tiling with Convex Regular Polygons

Can you tile with convex regular polygons?



Tiling with Convex Regular Polygons

How many edge to edge tilings are possible using only one type of tile?



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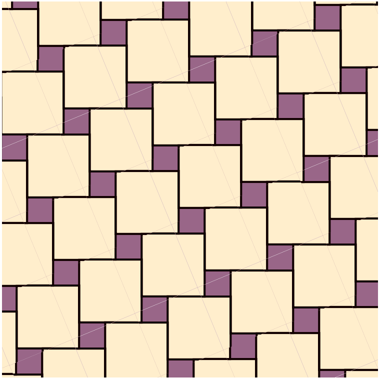


Figure: Non edge to edge tiling, in this case a Pythagorean tiling. By Arthur Baelde

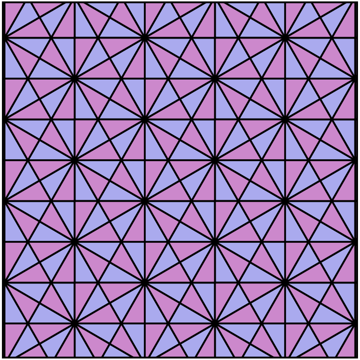
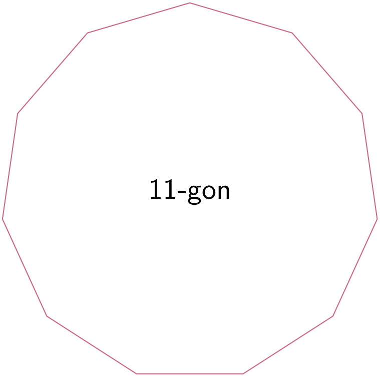


Figure: Edge to edge tiling

Tiling with Convex Regular Polygons

Internal angle of an n-sided polygon



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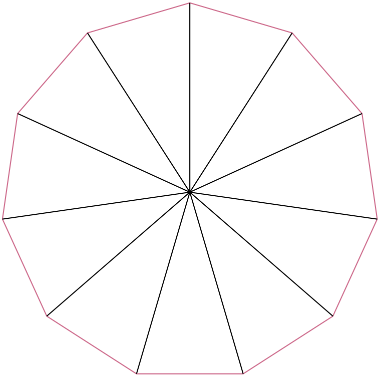

for $n = 11$, we have 11 triangles.

Tiling with Convex Regular Polygons

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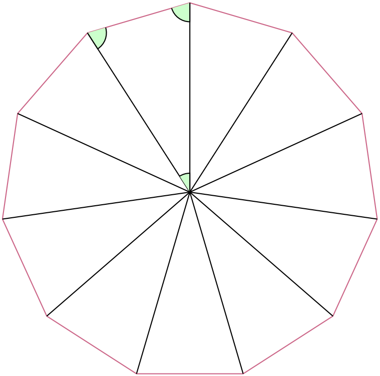

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Sum of angles of one triangle $= \pi$
$\pi = 180^\circ$

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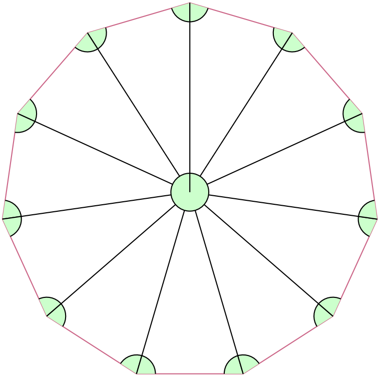

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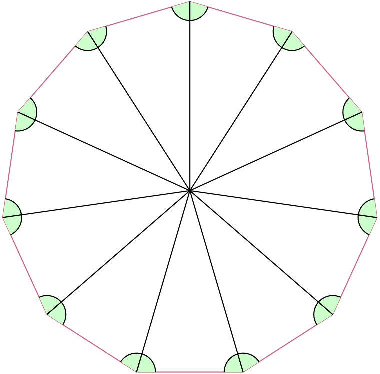
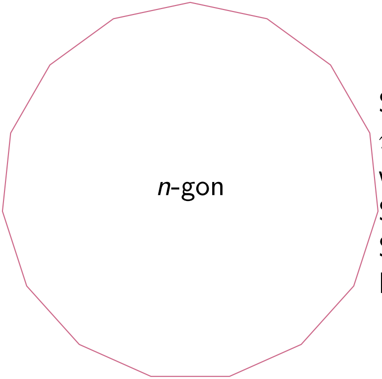
Internal angle of an *n*-sided polygon



for $n = 11$, we have 11 triangles.
Sum of angles of one triangle $= \pi$
$\pi = 180^\circ$
Sum of angles of 11 triangles $= 11\pi$
Sum of internal angles $= 11\pi - 2\pi$
Each internal angle $= (11\pi - 2\pi)/11$

Tiling with Convex Regular Polygons

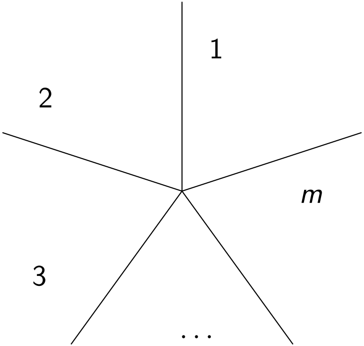
Internal angle of an n-sided polygon



Sum of angles of one triangle $= \pi$
$\pi = 180^\circ$
when we have n triangles,
Sum of angles of n triangles $= n\pi$
Sum of internal angles $= n\pi - 2\pi$
Internal angle $= (n-2)\pi/n = A_n$

Tiling with Convex Regular Polygons

How many edge to edge tilings are possible using only one type of tile?



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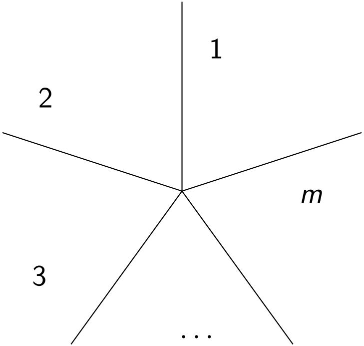

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$$\frac{1}{m} + \frac{1}{n} = \frac{1}{2}$$

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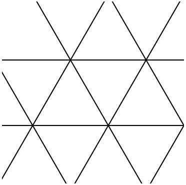

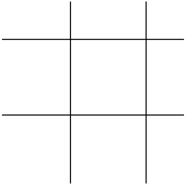

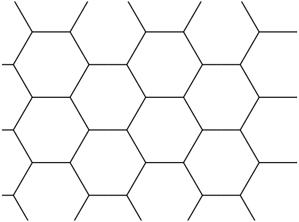

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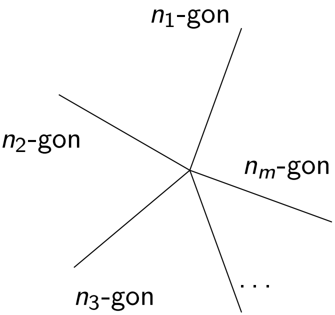

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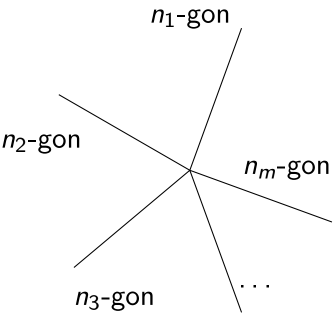

Internal angle $= (n-2)\pi/n = A_n$

$$A_{n_1} + A_{n_2} + A_{n_3} + \ldots + A_{n_m} = 2\pi$$

Tiling with Convex Regular Polygons

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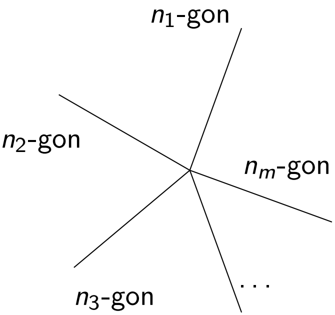

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$$A_{n_1} + A_{n_2} + A_{n_3} + \ldots + A_{n_m} = 2\pi$$

$$\frac{m}{2} - \left(\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} + \ldots + \frac{1}{n_m}\right) = 1$$

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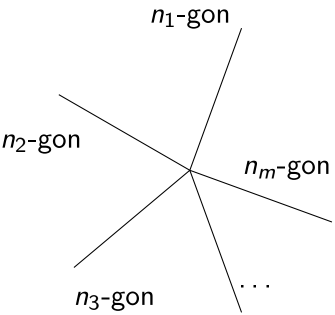

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Tiling with Convex Regular Polygons

How many edge to edge tilings are possible using more than one type of tile?
Consider $m = 3$

$$\frac{m}{2} - \left(\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} + \ldots + \frac{1}{n_m}\right) = 1$$

$$\frac{3}{2} - \left(\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3}\right) = 1$$

Tiling with Convex Regular Polygons

How many edge to edge tilings are possible using more than one type of tile?
Consider $m = 3$

n_1	n_2	n_3
3	12	12
4	6	12
4	8	8
3	7	42
3	8	24
3	9	18
3	10	15
4	5	20
5	5	10

$$\frac{m}{2} - \left(\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} + \ldots + \frac{1}{n_m}\right) = 1$$

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Tiling with Convex Regular Polygons

How many edge to edge tilings are possible using more than one type of tile?

Archimedean tilings

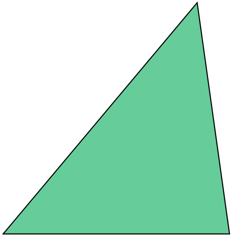
m	n_1	n_2	n_3	n_4	n_5
	3	12	12		
3	4	6	12		
	4	8	8		
	3	6	3	6	
4	3	4	6	4	
	3	3	4	3	4
5	3	3	3	4	4
	3	3	3	3	6

Tiling with Convex non-Regular Polygons

- Let's look at tilings using polygons of sides 3 (triangles), 4 (quadrilaterals), 5 (pentagons), and 6 (hexagons)

Tiling with Convex non-Regular Polygons

1. Tiling with non equilateral triangles



Tiling with Convex non-Regular Polygons

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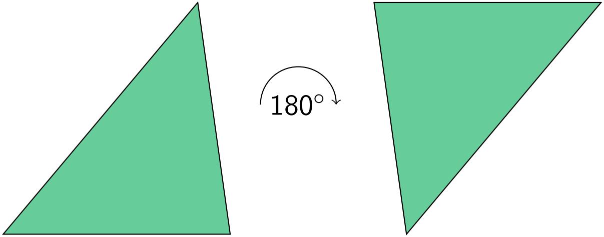
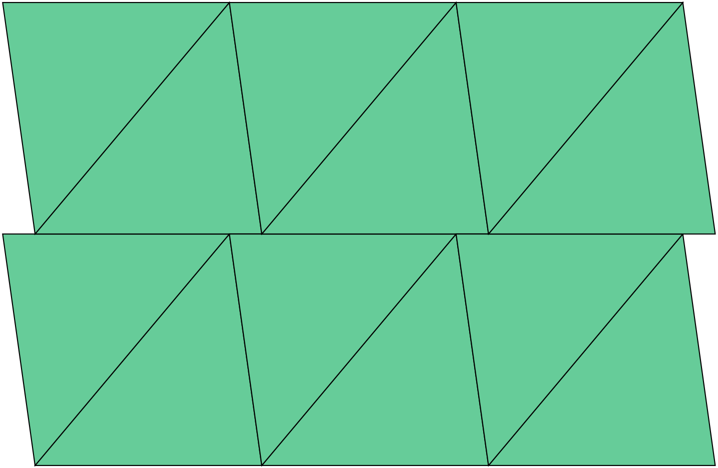


Parallelogram

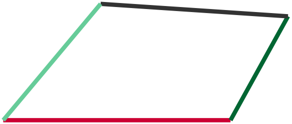
Tiling with Convex non-Regular Polygons

1. Tiling with non equilateral triangles



Tiling with Convex non-Regular Polygons

2. Tiling with quadrilaterals (4 sided figures)



Tiling with Convex non-Regular Polygons

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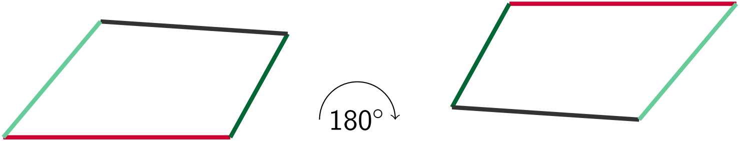

Tiling with Convex non-Regular Polygons

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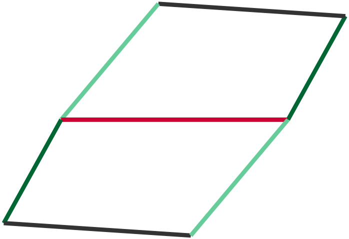

Hexagon with equal and parallel opposite edges

Tiling with Convex non-Regular Polygons

3. Tiling with pentagons

Tiling with Convex non-Regular Polygons

4. Tiling with hexagons (1918, Reinhardt)

Tiling with Convex non-Regular Polygons

Conway Criterion

Tiling with Convex non-Regular Polygons

Domino tiling

Break 1

Periodic vs. non-priodic

- In peiodic tiling, one can define a region as a 'unit cell' and the unit cell can then be used to tile the plane by translation alone - i.e. without rotating or reflecting it.

Periodic vs. non-priodic

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- Some shapes can be tiled in both ways - e.g. isoceles triangles - Central tessellations

Periodic vs. non-priodic

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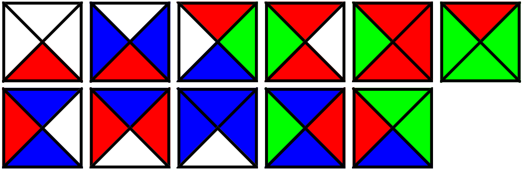
- Some shapes can tile only periodically - e.g. regular hexagon
- Some shapes can be tiled in both ways - e.g. isoceles triangles - Central tessellations
- Are there shapes that can tile only non-periodically?

Wang's Conjecture

Wang's conjecture states that if a set of tiles can tile the plane, then they can always be arranged to do so periodically (Wang 1961)

Such tiling will have a decision procedure relating this to decision questions in symbolic logic



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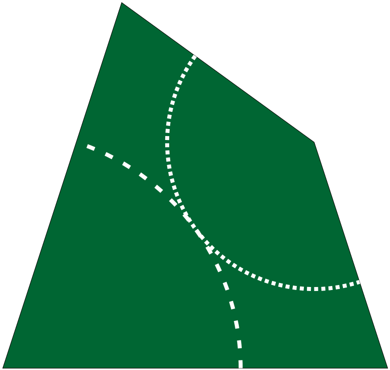
Such tiling will have a decision procedure relating this to decision questions in symbolic logic

Wang's conjecture was falsified in 1964 by R Berger by constructing a set of 20,426 tiles that tiles only periodically. He later reduced this to 104.

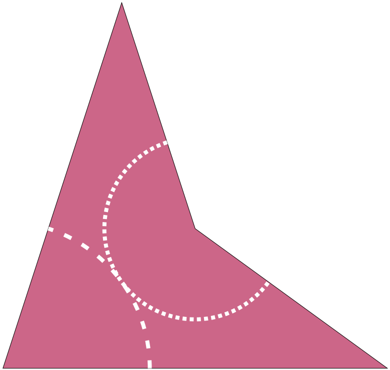
D Knuth brought this number down to 92.

Ultimately Penrose proposed a set 6 non-square tiles using which one can tile non-periodically.

Penrose Tiles

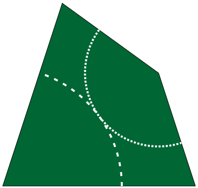


Kite



Dart

Penrose Tiles



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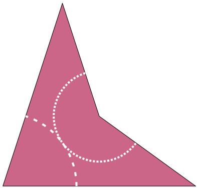
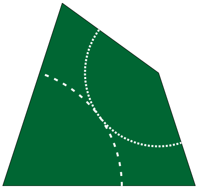


Dart

Task 1: Study the angles.

Penrose Tiles



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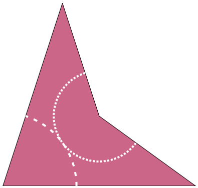
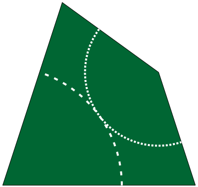


Dart

Task 2: Tiling. To force aperiodicity, ensure that abutting edges join same type of arcs (either dashed or dotted).

Penrose Tiles



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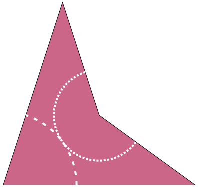


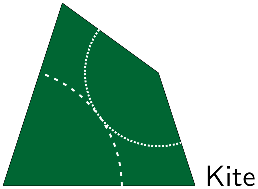
Dart

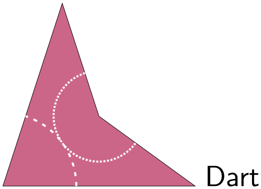
Task 3: What's the ratio of kites and darts?

Penrose Tiles

Task 4: Symmetry?

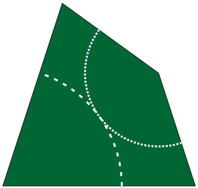
Penrose Tiles

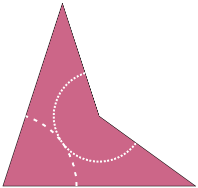




Task 5: Prove that the number of Penrose tilings is uncountable.

Penrose Tiles





Homework: Colorability, Ammann bars

Break 2

The ein - stein problem

Is there a single tile that will tile the plane only aperiodically?

The ein - stein problem

Is there a single tile that will tile the plane only aperiodically?
In March 2023, David Smith (amateur mathematician and retired print technician), Craig S. Kaplan (computer scientist and mathematician), Joseph Samuel Myers (software developer), and Chaim Goodman-Strauss (mathematician) showed that the hat tile can be used to tile aperiodically

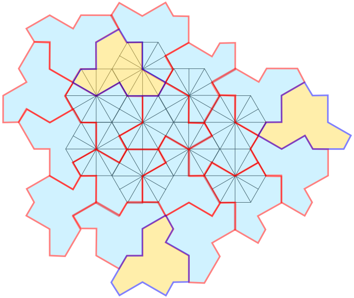


Figure: An aperiodic tiling using the hat tile by Smith, Myers, Kaplan and Goodman-Strauss, redrawn by Gringer. Note that the hat tile's mirror image is also considered

The ein - stein problem

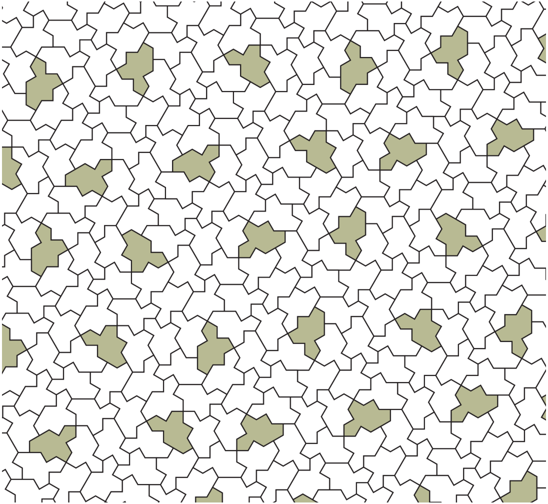


Figure: Aperiodic tiling with "Tile(1,1)" using only rotations and translations. David Smith, Joseph Samuel Myers, Craig S. Kaplan, and Chaim Goodman-Strauss, 2023

Reference

1. https://cs.uwaterloo.ca/~csk/spectre/
2. Smith, David, et al. "An aperiodic monotile." arXiv preprint arXiv:2303.10798 (2023).
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Ad break and Questions

1. icts.res.in/outreach
2. Maths Circle India
3. MathSpark
4. PrISM
5. Public Lectures at ICTS
6. Einstein Lectures
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