Modeling Saturation in Industrial Growth

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Abstract Long-time saturation in industrial growth has been modeled by a logistic equation of arbitrary degree of nonlinearity. Equipartition between nonlinearity and exponential growth in the integral solution of this logistic equation gives a nonlinear time scale for the onset of saturation. Predictions can be made about the limiting values of the annual revenue and the human resource content that an industrial organization may attain. These variables have also been modeled to set up an autonomous first-order dynamical system, whose equilibrium condition forms a stable node (an attractor state) in a related phase portrait. The theoretical model has received strong support from all relevant data pertaining to the well-known global company, *IBM*.

1 Introduction

The present global economic recession has made it imperative to devise mathematical models of high quantitative accuracy for understanding economic stagnation in industries [1]. The health of a company can be judged from the revenue that it generates and the human resource that it employs in achieving its objectives. Precise numerical measures of all these variables can be made, affording a clear understanding of industrial growth pattern and, consequently, allowing for a mathematical model to be framed for it.

Even when an industrial organization displays noticeable growth in its early stages, there is a saturation of this growth towards a terminal end after the elapse of a certain scale of time [2]. As the system size begins to grow through time, a self-regulatory mechanism drives the system towards a terminal state. Therefore, the effectiveness of any mathematical model that purports to explain saturation in industrial growth, lies in studying the global growth behavior of a company, whose

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operating space is on the largest available scale, and, as an additional advantage of operating on these scales, whose overall growth pattern becomes free of local inhomogeneities. To this end the growth trends of the annual revenue and the human resource strength of the multi-national company, *IBM*, have been analyzed here. Data about its annual revenue generation, the net annual earnings and the cumulative human resource strength, dating from the year 1914, have been published on the company website.[1] Both the capacity for revenue generation and the human resource content of *IBM*, over a period of more than ninety years of the existence of the company, show an initial phase of exponential growth, to be followed later by a slow drive towards saturation. An understanding of the general nature of this saturation, and other adversities lying ahead, can enable a company to apply corrective measures at the right juncture. This ought to be the guiding principle behind a feasible management strategy for long-term growth, especially in the case of organizations that are still in their early stages. As a result there will be a more effective formulation and implementation of innovative growth strategies, like the "Blue Ocean" strategy [3].

2 A Nonlinear Model for Growth

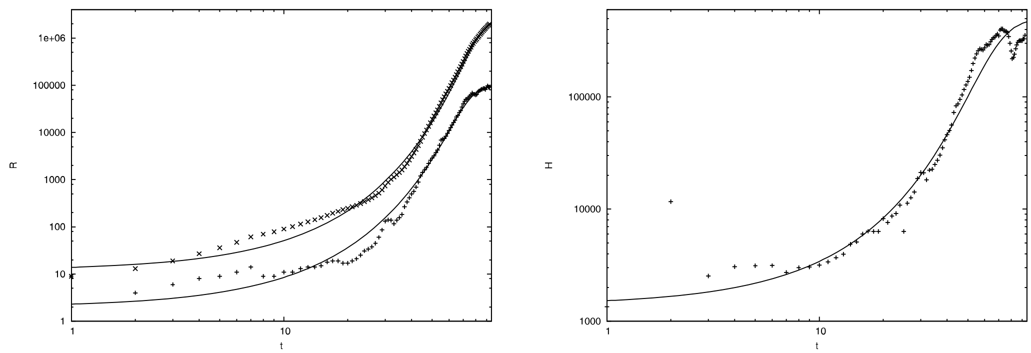
Regarding the study of growth from industrial data, a preceding work [4] has pedagogically underlined the relevance of various model differential equations of increasing complexity. Saturation in growth can be described by a logistic differential equation, as it is done to study the growth of a population [5, 6, 7]. Along the same lines, a generalization of the logistic prescription, to any arbitrary degree of nonlinearity, is being posited here, to follow industrial growth through time, t. Such a logistic equation will read as

$$\dot{\phi}(t) = \lambda \phi \left(1 - \eta \phi^{\alpha}\right) , \tag{1}$$

where ϕ can be any relevant variable to guage the health of a firm, like its annual revenue (or cumulative revenue growth) and human resource strength. The parameters α and η are, respectively, the nonlinear saturation exponent, and the "tuning" parameter for nonlinearity. A primary factor contributing to growth saturation is the space within which an organization can thrive. If this space is constrained to be of finite size (practically it has to be so), then terminal behavior becomes a certainty. This brings growth to a slow halt. Indeed, saturation in growth due to finite-size effects is understood well by now in other situations of economic interest where physical models can be applied [8]. The adverse conditions against growth can be further aggravated by the presence of rival organizations competing for the same space.

Integration of Eq. (1), which is a nonlinear differential equation, yields the general integral solution (for $\alpha \neq 0$),

[1] `http://www-03.ibm.com/ibm/history`.



(a) The lower curve gives the model fit for the annual revenue generated by *IBM*. The fit by the theoretical model agrees well on nonlinear time scales, for $\alpha = 1$, $\lambda = 0.145$ and $\eta = 10^{-5}$. The *cumulative* growth of the annual revenue generated by *IBM* is fitted by the upper curve. This fit is given by $\eta = 4 \times 10^{-7}$

(b) The growth of the human resource strength against time is fitted globally by the theoretical model for $\alpha = 1$, $\lambda = 0.09$ and $\eta = 2 \times 10^{-6}$. There has been a noticeable depletion of human resource on the same nonlinear saturation time scale for revenue growth, i.e. $75 - 80$ years

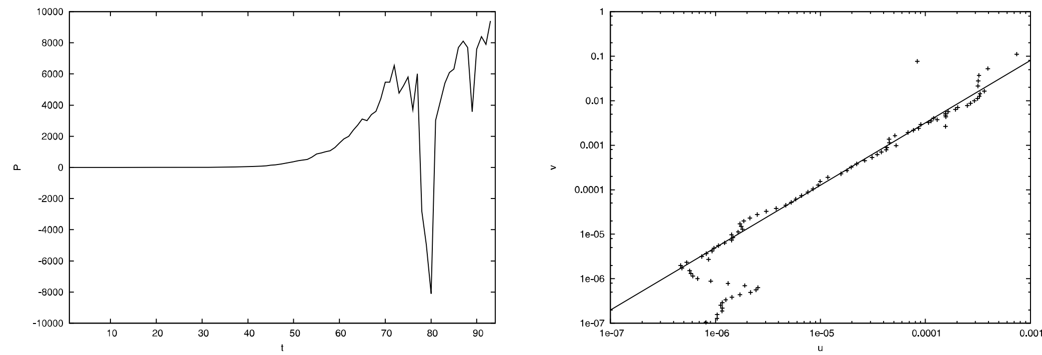
Fig. 1 Fit of the logistic equation with revenue and human resources growth

$$\phi(t) = \left[\eta + c^{-\alpha} \exp(-\alpha\lambda t)\right]^{-1/\alpha}, \tag{2}$$

in which c is an integration constant. The fit of the foregoing integral solution with the data has been shown in Figure 1, whose left panel gives a log-log plot of the annual revenue, R, that *IBM* has generated over time, t. Here the annual revenue has been measured in millions of dollars, and time has been scaled in years. The data and the theoretical model given by Eq. (2) agree well with each other, especially on mature time scales, when nonlinear saturation is conspicuous. The left panel in Figure 1 also shows the plot of the *cumulative* growth of the *IBM* revenue.

While the early stages of growth is exponential, the later stages shift into a saturation mode. A limiting value in relation to this saturated state is given by $\dot{\phi} = 0$ in Eq. (1), leading to $\phi_{\text{sat}} = \eta^{-1/\alpha}$. Using the values of α and η, by which the saturation properties of the plot in Figure 1 have been fitted, a prediction can be made that the maximum possible annual revenue that *IBM* can generate will be about 100 billion dollars. Similar claims can be made about the limiting value of the human resources of *IBM*, which is another important indicator of the prevailing state of a company. The data for the human resource of *IBM* have been plotted in the right panel in Figure 1. Going by the values of α and η needed for the model fit here, the maximum possible human resource that *IBM* can viably employ is predicted to be about $500,000$ strong.

A point of great interest here is that the growth data have been fitted well by the simplest possible case of nonlinearity, given by $\alpha = 1$. This will place the present mathematical problem in the same class of the logistic differential equation devised by Verhulst to study population dynamics [5, 6, 7]. This equation has also been applied satisfactorily to a wide range of other cases involving growth [5, 6, 7].



(a) The net annual earnings (in millions of dollars) made by *IBM* has shown steady growth, except for the early years of the 1990s decade, which was about 80 years of the existence of the company. Around this time the company suffered major losses in its net earnings, and this time scale corresponds very closely to the time scale for the onset of nonlinear saturation in revenue growth

(b) The straight-line fit validates the logistic equation model. The slope of the straight line is given as 1.4, and it closely matches the value of 1.6 that can be obtained from the parameter fitting in Figure 1. The cusp at the bottom left is due to the loss of human resource. Growth, corresponding to the positive slope in this plot, can be modeled well by the logistic equation

Fig. 2 The two plots here support modeling by the logistic equation

The time scale for the onset of nonlinearity can be fixed by requiring the two terms on the right hand side of Eq. (2) to be in rough equipartition with each other. This will yield the nonlinear time scale as $t_{\mathrm{nl}} \sim -(\alpha\lambda)^{-1}\ln|\eta c^{\alpha}|$, from which, making use of the values of α, η and c needed to calibrate the *IBM* revenue data (both the annual revenue and the cumulative revenue), one gets $t_{\mathrm{nl}} \sim 75-80$ years. An indirect confirmation about the relevance of this time scale comes from the plot in the left panel of Figure 2, which shows the growth of the net annual earnings of *IBM* (labelled as P, the profit, scaled in millions of dollars), against time, t (in years). The company suffered major reverses in its net earnings around 1991-1993 (upto 8 billion dollars in 1993), which was indeed very close to 80 years of the company, since its inception in 1914.

3 A Dynamical Systems Modeling

It is clear that there is a strong correlation among the variables by which the state of an industrial organization is monitored. The concept of the "Balanced Scorecard" is somewhat related to this principle [9]. The growth rate of any relevant variable will have a correlated functional dependence on the current state of all the other variables. If an industrial organization generates enough revenue, it becomes financially viable for it to maintain a sizeable human resource pool, while the human resource strength will translate into a greater ability to generate revenue. In this manner both the revenue and the human resource content of an organization will sustain the growth of each other. Considering a general revenue variable, R (which can be either be the annual revenue or the cumulative revenue), its coupled dynamic growth along with the human resource, H, can be stated mathematically by the relations $\dot{R} = \rho\,(R,H)$and $\dot{H} = \sigma\,(R,H)$. The foregoing coupled set of autonomous first-order differential equations forms a two-dimensional system, and the equilibrium condition of this dynamical system is obtained when $\dot{R} = \dot{H} = 0$. The corresponding coordinates for this condition in the H-R plane may be labelled (H_0,R_0). Since the terminal state implies the cessation of all growth in time, it is now possible to argue that the equilibrium state in the H-R plane actually represents a terminal state in real time growth.

One might describe the individual growth patterns of R and H by simply using an *uncoupled* logistic equation for either variable. This will go as $\dot{R}(t) = \lambda_r R\,(1-\eta_r R^{\alpha_r})$and $\dot{H}(t) = \lambda_h H\,(1-\eta_h H^{\alpha_h})$, with the subscripts r and h in the parameters α, λ and η, indicating that R and H will each, in general, have its own different set of parameter values. The integral solution in the H-R plane can be transformed in a compact power-law form as $v = \kappa u^{\beta}$ under the definitions that $v = R^{-\alpha_r} - \eta_r$, $u = H^{-\alpha_h} - \eta_h$, and $\beta = (\alpha_r/\alpha_h) \times (\lambda_r/\lambda_h)$, with κ being an integration constant. The power-law behavior implies that a log-log plot of v against u will be a straight line with a slope, β. This fact has been shown in the right panel of Figure 2. In this plot, v has been defined in terms of the cumulative revenue, and the slope of the resulting straight line is given by $\beta \simeq 1.4$, which is quite close to the value of $\beta \simeq 1.6$, found simply by taking the ratio of the respective theoretical values of λ, chosen to fit the empirical data in Figure 1. The cusp in the bottom left corner of the plot has arisen because of an irregular depletion of human resource in *IBM* in the early 1990s. However, the lower arm of the cusp has nearly the same positive slope as the straight-line fit. This shows that intermittent deviations do not affect the overall course of the evolutionary growth process [6].

By use of the logistic equation, the limiting state for industrial growth is represented by a stable node in the phase portrait of an autonomous first-order dynamical system [5]. Extending this argument, the limiting state can be perceived to be an attractor state, towards which there will be an asymptotic approach through an infinite passage of time [5].

Acknowledgements The author thanks A. Basu, J. K. Bhattacharjee, S. Bhattacharya, B. K. Chakrabarti, I. Dutta, T. Ghose, W. C. Kim, A. Kumar, A. Marjit, S. Marjit, S. Roy Chowdhury, H. Singharay, J. Spohrer and V. M. Yakovenko for useful comments and suggestions. A. Varkey helped in collecting data.

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