Limits - Sep 15 2023

Leaning objectives:

1. Explain using both words and pictures the meanings of:

   - $\lim_{x\to a} f(x) = L$
   - $\lim_{x\to a^-} f(x) = L$
   - $\lim_{x\to a^+} f(x) = L$
   - $\lim_{x\to\infty} f(x) = L$
   - $\lim_{x\to-\infty} f(x) = L$

   when L is a real number or positive or negative infinity
2. Find the limit of a function at a point given the graph of the function.
3. Evaluate limits of rational, trigonometric, exponential, and logarithmic functions.

We'll be using a lot of piecewise-defined functions when we talk about limits. These often arise naturally from models. We'll start with two examples of pricing models.

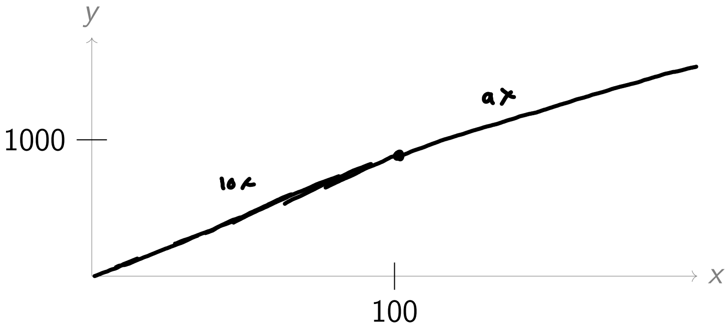
Pricing Model 1: Bulk Discount

The cost per widget, for the first 100 widgets, is \$10. After the first 100, each additional widget costs only \$9.

x = units

Let $f(x)$ be the cost for x widgets (x is real, nonnegative).

$$f(x) = \begin{cases} 10x & \text{if } x \le 100 \\ (10 \cdot 100) + 9(x - 100) & x > 100 \end{cases}$$



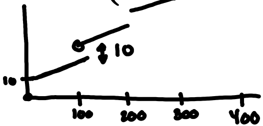
Pricing Model 2: Shipping Fees

Each widget costs \$10. There is a shipping fee of \$10 per box. At most 100 widgets can be shipped per box.

Let $f(x)$ be the cost for x widgets (x is real, nonnegative). There are various ways to write this function; one way is piecewise:

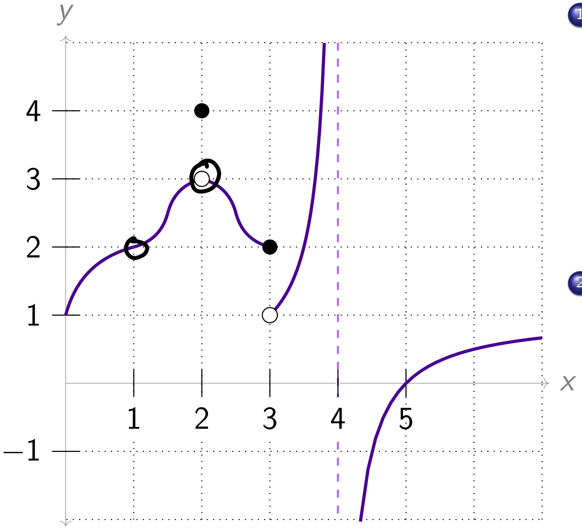
$$f(x) = \begin{cases} 0 & \text{for } x = 0 \\ 10 + 10x & \text{for } 0 < x \leq 100 \\ 20 + 10x & \text{for } 100 < x \leq 200 \\ 30 + 10x & \text{for } 200 < x \leq 300 \\ 40 + 10x & \text{for } 300 < x \leq 400 \\ & \text{etc} \end{cases}$$

Sketch $f(x)$.



Small classes will pick up on the discussion of limits to investigate continuity.

Consider the function $f(x)$ whose graph is below.



1. If x is "sufficiently close" (but not equal) to 1, then $f(x)$ is "arbitrarily close" to

   $\underline{\ 2\ }$. $x = 1$

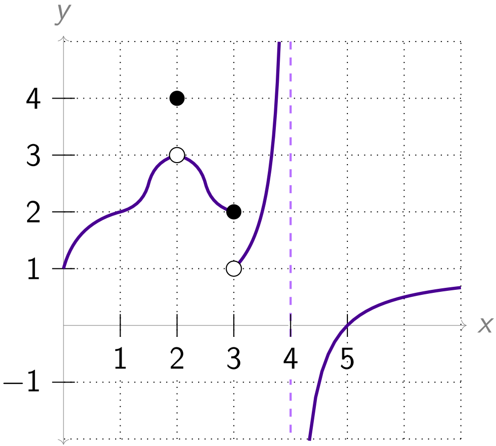
2. If x is "sufficiently close" (but not equal) to 2, then $f(x)$ is "arbitrarily close" to

   $\underline{\ 3\ }$. $\lim_{x \to 2} f(x) = 3$

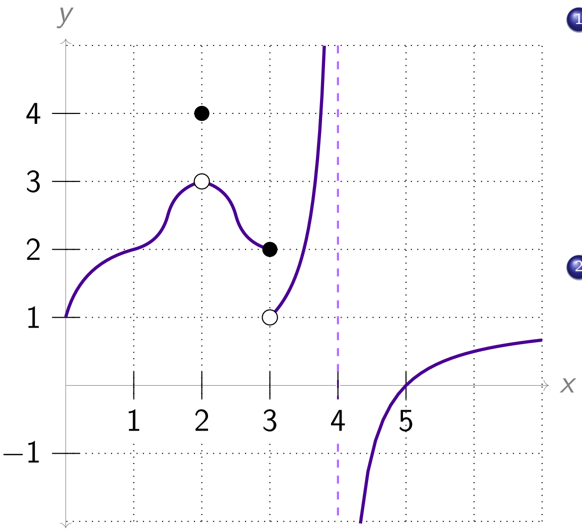
   "f(2) is expected to be 3 since 3 is approached"

Definition 1.3.3 in CLP textbook

For real numbers a and L: $\lim_{x \to a} f(x) = L$ (read "the limit of $f(x)$ as x approaches a is equal to L) means $f(x)$ is arbitrarily close to L provided x is sufficiently close to a (but not equal to a).



Consider the function $f(x)$ whose graph is below.



1. What value (if any) is $f(x)$ arbitrarily close to if x is sufficiently close to 3 and $x < 3$?

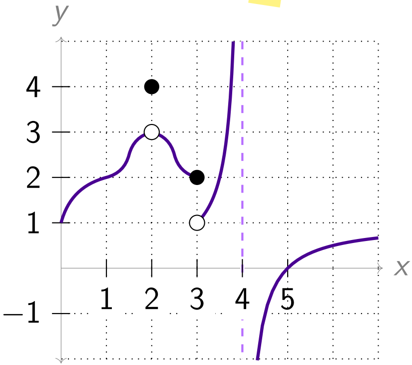
   $\lim_{x \to 3^-} f(x) = 1$

2. What value (if any) is $f(x)$ arbitrarily close to if x is sufficiently close to 3 and $x > 3$?

   $\lim_{x \to 3^+} f(x) = -1$

Definition 1.3.7 in CLP textbook

We write $\lim\limits_{x \to a^-} f(x) = K$ when the value of $f(x)$ gets arbitrarily close to K when $x < a$ and x is sufficiently close to a.
Since the x-values are always less than a, we say that x approaches a from below or from the left. This is also often called the left-hand limit since the x-values lie to the left of a on a sketch of the graph.



Definition 1.3.7 in CLP textbook

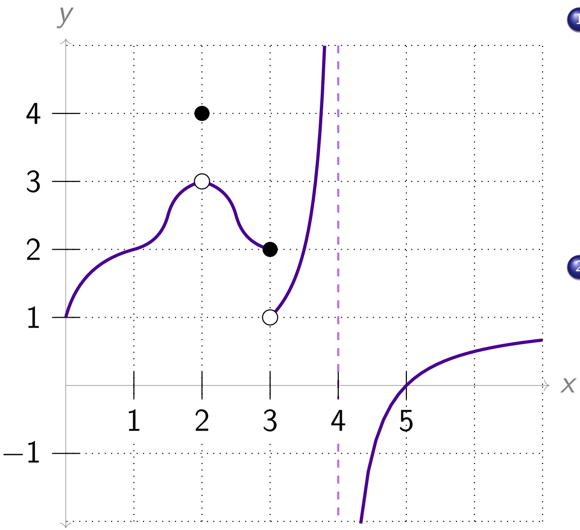
We write $\lim_{x \to a^-} f(x) = K$ when the value of $f(x)$ gets arbitrarily close to K when $x < a$ and x is sufficiently close to a.
Since the x-values are always less than a, we say that x approaches a from below or from the left. This is also often called the left-hand limit since the x-values lie to the left of a on a sketch of the graph.

(The limit from the right is defined similarly.)

Theorem 1.3.8 in CLP textbook

$\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$

Consider the function $f(x)$ whose graph is below.



1. What value (if any) is $f(x)$ arbitrarily close to if x is sufficiently close to 4 and $x < 4$?

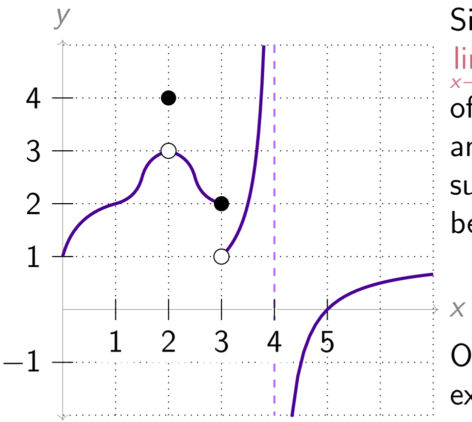
   $\lim_{x \to 4^-} f(x) = \infty$

2. What value (if any) is $f(x)$ arbitrarily close to if x is sufficiently close to 4 and $x > 4$?

   $\lim_{x \to 4^+} f(x) = \infty$

Definition 1.3.10

We write $\lim_{x\to a} f(x) = \infty$ when the value of $f(x)$ becomes arbitrarily large and positive when x is sufficiently close to a, without being exactly a.



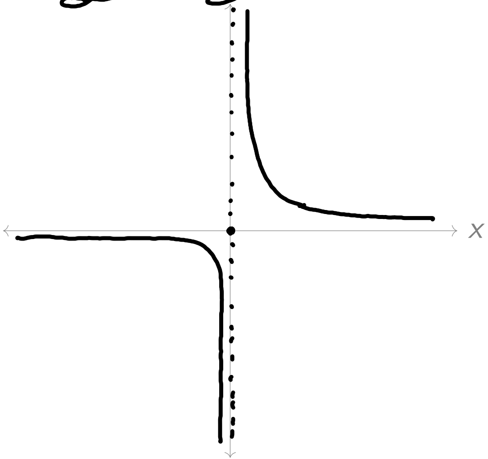
Similarly, we write $\lim_{x\to a} f(x) = -\infty$ when the value of $f(x)$ becomes arbitrarily large and negative when x is sufficiently close to a, without being exactly a.

One-sided limits also work as expected.

Find $\lim_{x \to 0^-} \frac{1}{x}$, $\lim_{x \to 0^+} \frac{1}{x}$, and $\lim_{x \to 0} \frac{1}{x}$. $=$ DNE , no one value is approached

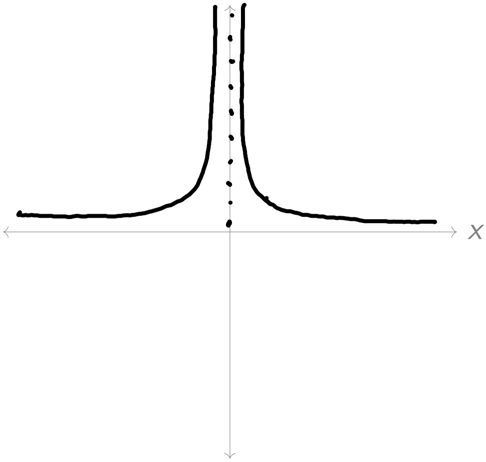
$-\infty$ ∞

limit is invalid. :(

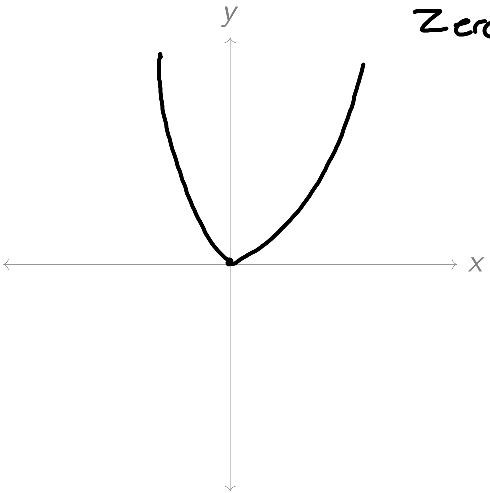


Find $\lim_{x\to 0^-} \frac{1}{x^2}$, $\lim_{x\to 0^+} \frac{1}{x^2}$, and $\lim_{x\to 0} \frac{1}{x^2}$. $= \infty$, both one-sided limits are equal to each other.

∞ ∞ ∞

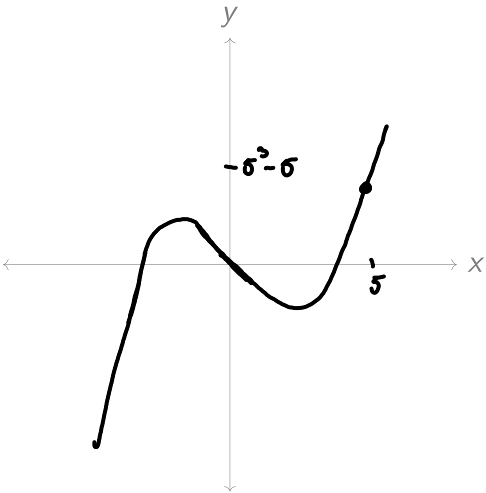


Find $\lim_{x \to 0} x^2 = 0$, one sided limits approach zero.

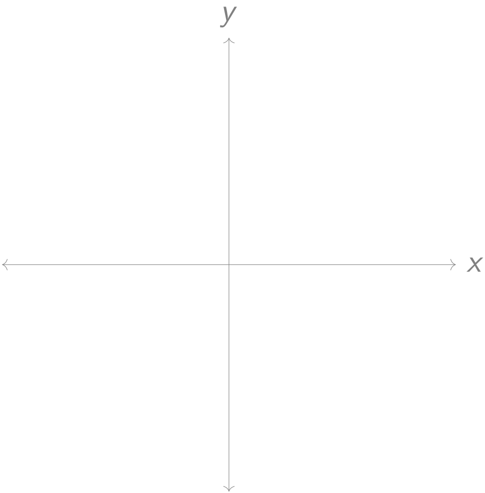


Find $\lim_{x \to 5} (x^3 - x)$. $x(x^2 - 1)$

$\lim_{x \to 5} (x^3 - x) = 5^3 - 5 = 120$



Find $\lim_{x \to 5} (x^3 - x)$.



→ continuous everywhere, no disconts.

Polynomials are "nice" functions: to evaluate a limit, you can just plug in the appropriate value of x.

This property is related to continuity, which you'll explore further in your small classes.

Find $\lim_{x \to 2} \dfrac{x-2}{x^2+x-6}$.

numerator: $2-2=0$

denom: $4+2-6=0$

Plugging in does not yield limit, since func is UD @ zero. Simplify func then get lim.

This is known as "indeterminate form"

$$\lim_{x \to 2} \frac{(x-\cancel{2})}{(x+3)(x-\cancel{2})}$$

$$\lim_{x \to 2} \frac{1}{(x+3)} = \frac{1}{5}$$

for all values of x that aren't 2

$$\frac{1}{(x+3)} = \frac{(x-2)}{x^2+x-6}$$

so since we are interested in the nums AROUND 2, we can get the lim of simplified $f(x)$.

We can use some of the "dominance" behaviour we discussed in the last large class to evaluate the limits below:

n^x over powers x^n

- $\lim\limits_{x\to\infty} \dfrac{e^x}{x} = \infty$
- $\lim\limits_{x\to\infty} \dfrac{x}{e^x} = 0$
- $\lim\limits_{x\to\infty} \dfrac{\log x}{\sqrt{x}} =$

all these examples are in indeterminate form...
- numerator reaches infinity
- denom reaches infinity

use dominance rules from previous lecture to know how big n and d are relative to each other

x^n overpowers $\log x$, since log is inverse of exponential

exp =   log = , SLOW

Asymptotes

Learning Objectives:

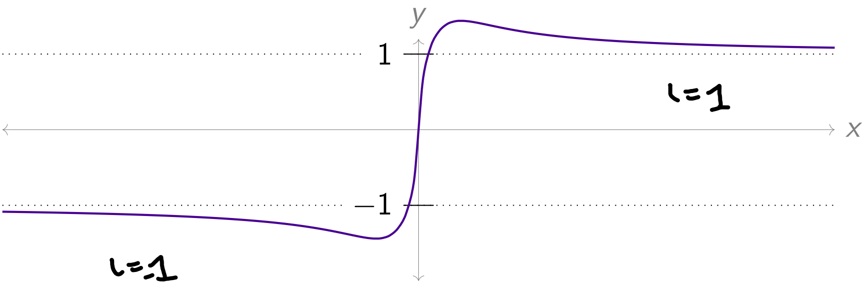
1. Explain using both informal language and the language of limits what it means for a function to have a horizontal or vertical asymptote.
2. Given a simple function, find its vertical and horizontal asymptotes by asymptotic reasoning or by using algebra.

Let L be a real number and let $f(x)$ be a function. If

$$\lim_{x\to\infty} f(x) = L \text{ or } \lim_{x\to-\infty} f(x) = L$$

L = constant

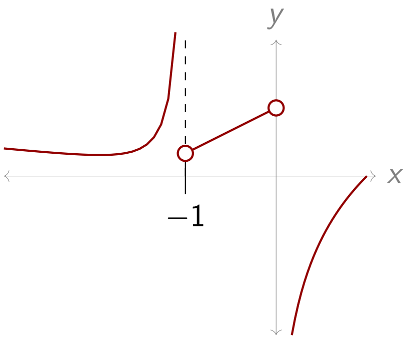
then we say $y = L$ is a horizontal asymptote of $f(x)$.



Let a be a real number and let $f(x)$ be a function.
If a function $f(x)$ has

$$\lim_{x \to a} f(x) = \infty \text{ or } \lim_{x \to a} f(x) = -\infty$$

or if a one-sided limit as x approaches a is infinite, then we way $f(x)$ has a vertical asymptote at $x = a$.



HA rules:

- if n degr e d degree no HA.
- if n° = d H- = $\frac{\text{leading coef}}{\text{leading coef}}$
- if n° < d H- = 0
- if not appl ab, take infinite lim.

$$f(x) = \frac{x-1}{x+3}$$

y int = $-\frac{1}{3}$

x int @ 1

, or devid VA @ -3

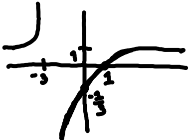
HA = 1

- Find the intercepts of $f(x)$.
- Find any horizontal asymptotes of $f(x)$.
- Find any vertical asymptotes of $f(x)$, and the associated (one-sided) limits.
- Use your results to make a sketch of $y = f(x)$.

for infinite lims:

- analyze bigger power/dominance
- divide n and d by highest degree of x in denom.

f(x)



- use number line with zeroes for numerator and denom to find out when each is pos/neg

ie: num line for x-1

$-\quad 1 \quad +$

Find the horizontal and vertical asymptotes of

$$f(x) = \frac{\sqrt{x^2+1}}{x+1}.$$

HA: infinite limit of f(x).

using dominance

@ ∞ $\sqrt{x^2+1} = \sqrt{x^2} = x$

@ ∞ $x+1 = x$

$\lim_{x\to\infty} \frac{x}{x} = 1$

$\lim_{x\to-\infty} \frac{\sqrt{x^2}}{x} = -1$

VA @ $x = -1$

($x+1$ doesn't cancel and $= 0$ @ -1)

$\lim_{x\to-1} f(x) = \infty$

alt method (divide n and d by x)

$x = \sqrt{x^2}$ for ($x>0$) $x = -\sqrt{x^2}$ for ($x<0$)

$\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x+1} \cdot \frac{1/x}{1/x} = \frac{\frac{\sqrt{x^2+1}}{x}}{1+\frac{1}{x}} = \frac{\sqrt{\frac{x^2+1}{x^2}}}{1+\cancel{\frac{1}{x}}}$ plug $= \frac{\sqrt{1}}{1} = 1$

frequently DO NOT

Can a function cross its own horizontal asymptote?

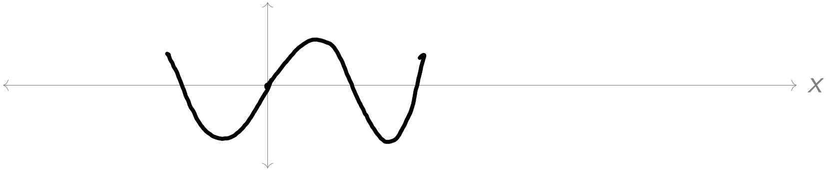
A. Yes @ inflection point / concavity switch.

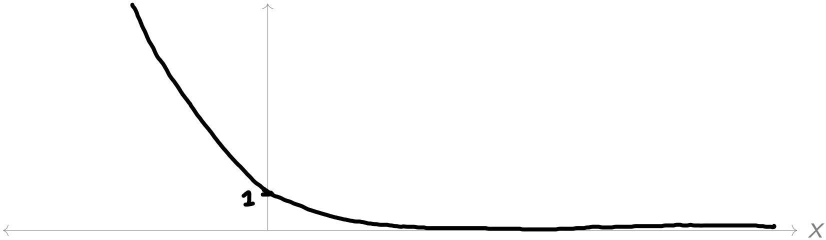
B. No

C. Not sure

Sketching $f(x) = e^{-x} \sin x$

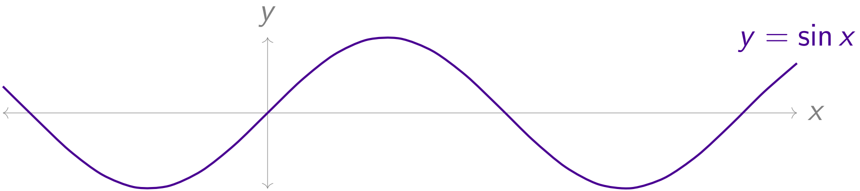
Sketch $y = \sin x$ and $y = e^{-x}$. - flipped in y ö



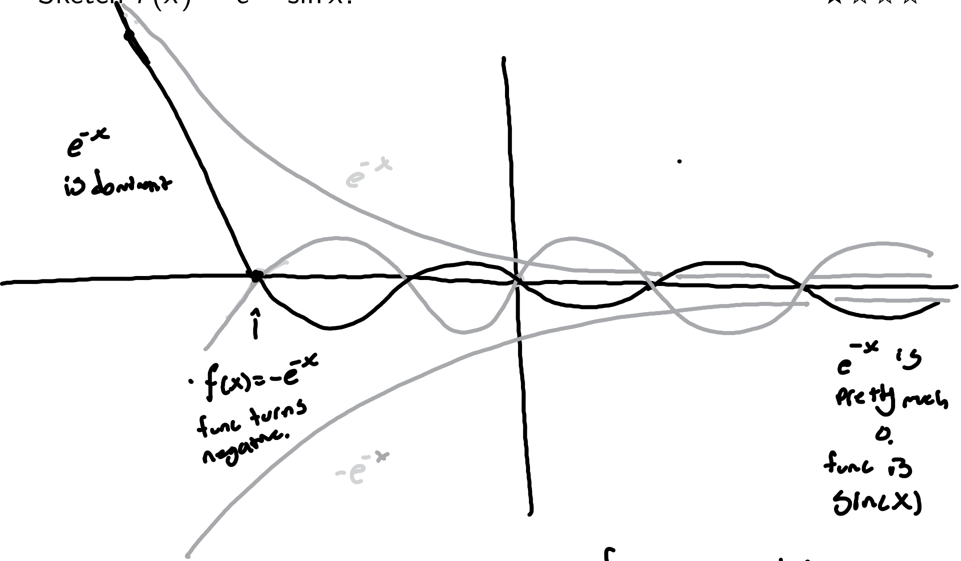


Similar to small class, where you sketched $e^x \cos x$, if you got far enough

x	$\sin x$	$e^{-x}\sin x$	
0	0	0	
$\frac{\pi}{2}$	1	$e^{-(\pi/2)}$	e^{-x}
π	0	0	
$\frac{3\pi}{2}$	-1	$-e^{-(3\pi/2)}$	$-e^{-x}$
2π	0	0	



Sketch $f(x) = e^{-x} \sin x$. ★☆☆☆



func crosses HA.

Time permitting:

Evaluate: ★☆☆☆

$$\lim_{x\to-\infty} \frac{\sqrt{x^6+x^3+2}}{2x^3-8x+1}$$

@ ∞ $\sqrt{x^6}$

@ ∞ $2x^3$

$\lim_{x\to\infty} = \frac{x^3}{2x^3} = \frac{1}{2}$

$\lim_{x\to-\infty} = \frac{x^3}{-2x^3} = -\frac{1}{2}$

neg at $-\infty$

Evaluate the limit below. You may leave your answer in calculator-ready form. ★★☆☆

$$\lim_{x\to\frac{5}{3}} \frac{3x^3-5x^2+6x-10}{3x^2-5x}$$

Can't Just Plug

$(3x^3-5x^2)+(6x-10) \rightarrow$ by grouping

$x^2(3x-5)+2(3x-5)$

$(x^2+2)(3x-5)$

$$\frac{(x^2+2)\cancel{(3x-5)}}{x\cancel{(3x-5)}}$$

$$\lim_{x\to\frac{5}{3}} \frac{(x^2+2)}{x}$$

$$\frac{\frac{25}{9}+2}{\frac{5}{3}} = \frac{\frac{43}{9}}{\frac{5}{3}} = \frac{\frac{43}{3}}{5}$$

Find all asymptotes of $f(x) = \dfrac{\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{2}}}{x^2 - 4}$. ★★☆☆

simplify to see if u can cancel

$$= \frac{\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{2}}}{(x-2)(x+2)} \cdot (\sqrt{2}\sqrt{x})^{\text{lcm}}$$

$$= \frac{\sqrt{2} - \sqrt{x}}{(x-2)(x+2)(\sqrt{2}\sqrt{x})} \cdot (\sqrt{2} + \sqrt{x})^{\text{conjugate}}$$

$$= \frac{2 - x}{(x-2)(x+2)(\sqrt{2x})(\sqrt{2}+\sqrt{x})}$$

HA = 0

$$= \frac{-\cancel{(x-2)}}{\cancel{(x-2)}(x+2)(\sqrt{2x})(\sqrt{2}+\sqrt{x})}$$

VA = @ 0, −2

$$\frac{-1}{(x+2)(\sqrt{2x})(\sqrt{2}+\sqrt{x})}$$

Evaluate the following limits.

★☆☆☆

1. $\lim_{x\to\infty} 2^x$

2. $\lim_{x\to-\infty} 2^x$

3. $\lim_{x\to\infty} 2^{-x}$

4. $\lim_{x\to-\infty} 2^{-x}$

CLP Problem 1.5.3-5; warm-up for following question

Find all asymptotes of $y = 2^{\frac{1}{x}}$. ★★☆☆

1. Evaluate $\lim_{x\to\infty} \cos x$ ★☆☆☆

2. Evaluate $\lim_{x\to\infty} [x - 3x^5 + 100x^2]$ ★☆☆☆

CLP Problem 1.5.6-7

Given the function shown below, evaluate: ★☆☆☆

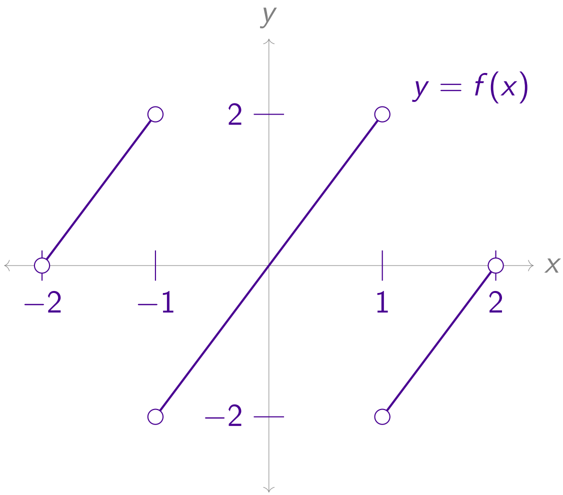
(a) $\displaystyle\lim_{x\to-1^-} f(x)$

(b) $\displaystyle\lim_{x\to-1^+} f(x)$

(c) $\displaystyle\lim_{x\to-1} f(x)$

(d) $\displaystyle\lim_{x\to-2^+} f(x)$

(e) $\displaystyle\lim_{x\to2^-} f(x)$



Find all asymptotes of $f(x) = \dfrac{x(2x+1)(x-7)}{3x^3-81}$. ★☆☆☆

Use dominance to evaluate ★☆☆☆

$$\lim_{x\to\infty} \frac{\sqrt{3x^8+7x^4}+10}{x^4-2x^2+1}$$

Use dominance to evaluate ★☆☆☆

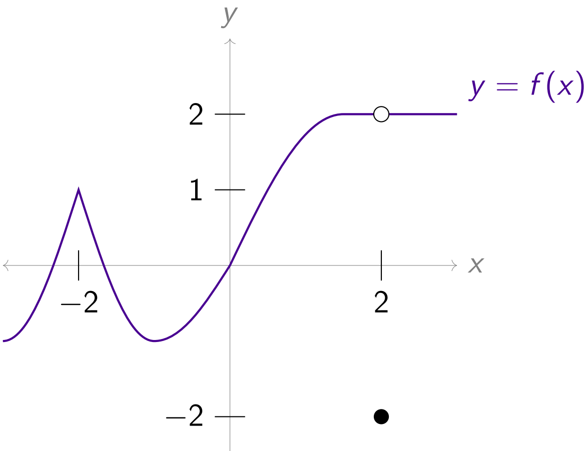
$$\lim_{x\to\infty} \frac{\sqrt{4x+2}}{3x+4}$$

Given the function shown below, evaluate the following: ★☆☆☆

(a) $\lim_{x \to -2} f(x)$

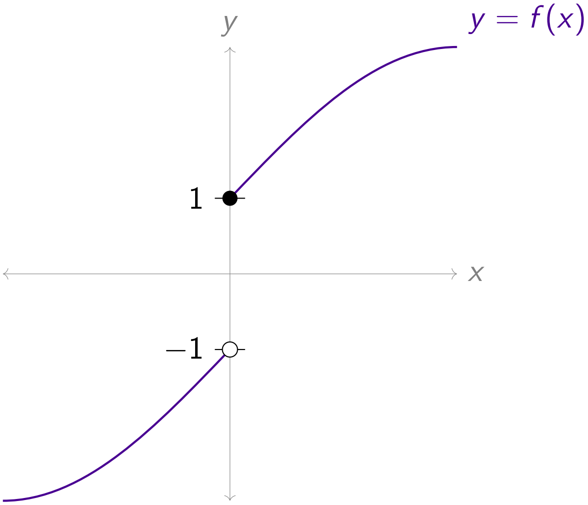
(b) $\lim_{x \to 0} f(x)$

(c) $\lim_{x \to 2} f(x)$



CLP Problem 1.3.1

Given the function shown below, evaluate $\lim_{x \to 0} f(x)$. ★☆☆☆



Evaluate $\lim_{x \to 3} f(x)$, where ★☆☆☆

$$f(x) = \begin{cases} \sin x & x \leq 2.9 \\ x^2 & x > 2.9 \end{cases}$$

Limits of Rational Functions

- $\lim\limits_{x \to \infty} \dfrac{x^2 + 2x + 1}{3x^2 - 5x + 6}$
- $\lim\limits_{x \to \infty} \dfrac{x^3 + 2x + 1}{3x^2 - 5x + 6}$
- $\lim\limits_{x \to \infty} \dfrac{x^2 + 2x + 1}{3x^3 - 5x + 6}$

Limits of Rational Functions

Let $p(x)$ and $q(x)$ be polynomials.
Case 1: Suppose p and q have the same degree.

$$\lim_{x\to\infty} \frac{p(x)}{q(x)} \qquad \lim_{x\to-\infty} \frac{p(x)}{q(x)}$$

Both limits above are of the form $\frac{a}{b}$, where a is the coefficient of the largest power of x in $p(x)$, and b is the coefficient of the largest power of x in $q(x)$.

For example:

$$\lim_{x\to-\infty} \frac{7x - 19x^3 + 8}{x + 2x^3 - 1} =$$

Limits of Rational Functions

Let $p(x)$ and $q(x)$ be polynomials.
Case 2: Suppose q has a higher degree than p.

$$\lim_{x\to\infty} \frac{p(x)}{q(x)} = 0 = \lim_{x\to-\infty} \frac{p(x)}{q(x)}$$

When x is a very large (positive or negative) number, the denominator is a very large (positive or negative) number, dominating the numerator.

For example:

$$\lim_{x\to-\infty} \frac{7x - 19x^3 + 8}{x + 2x^4 - 1} =$$

Limits of Rational Functions

Let $p(x)$ and $q(x)$ be polynomials.
Case 3: Suppose p has a higher degree than q.

$$\lim_{x\to\infty} \frac{p(x)}{q(x)} \qquad \lim_{x\to-\infty} \frac{p(x)}{q(x)}$$

When x is a very large (positive or negative) number, the numerator is a very large (positive or negative) number, dominating the denominator. So the limits above diverge to infinity or negative infinity.

For example:

$$\lim_{x\to\infty} \frac{7x - 19x^4 + 8}{x + 2x^3 - 1} = \qquad \lim_{x\to-\infty} \frac{7x - 19x^4 + 8}{x + 2x^3 - 1} =$$