Solitons in a Bose-Einstein Condensate

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Introduction

The atoms in a Bose-Einstein condensate are all described by the same wavefunction, since they are in the same quantum state, which includes each atom's interaction with the other atoms in the condensate. Introducing an pseudopotential term $g|\psi|^2$ which characterises the interactions between the particles into the dimensionless Schrödinger equation yields the Gross-Pitaevskii equation []

$$i\frac{\partial\psi}{\partial t} = -\frac{\partial^2\psi}{\partial x^2} + (V + g|\psi|^2)\psi, \tag{1}$$

where ψ is the atomic wavefunction, t and x are the rescaled time and length variables, V is the external potential (if present) and g is the interaction parameter. It is usual to define ζ as a parameter to characterise width, with units of inverse length. The normalised wavefunction is then

$$\psi(x) = \sqrt{\frac{\zeta}{2}}\,\mathrm{sech}\,(\zeta x)e^{ivx+\phi}, \tag{2}$$

where v is the velocity of the soliton and ϕ is a phase factor. Substituting the $v = 0$ case into the time-independent Schrodinger equation and setting E to be ζ^2, we find $g = 4\zeta$.

Propagating a Soliton

The Gross-Pitaevskii equation was solved numerically using the split-step Fourier method [].

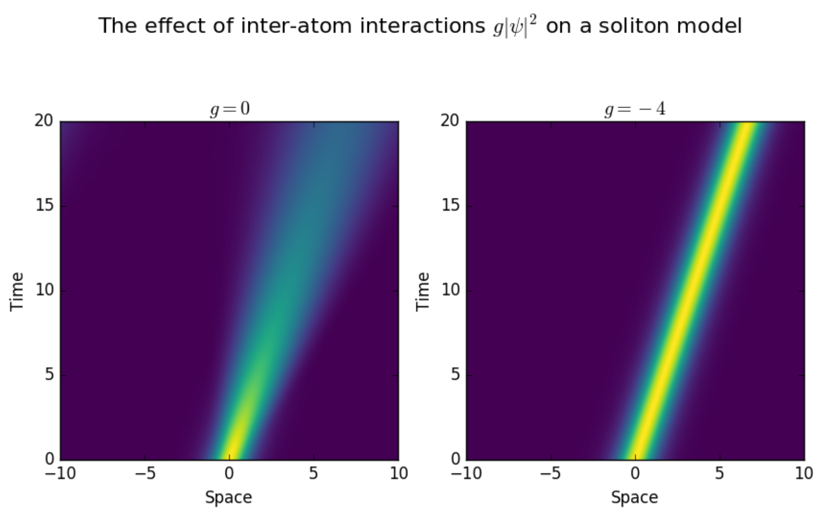

Figure 1: $n = 50$, $p = 0.6$

Richardson Growth Model

This model is constructed by attaching independent Poisson Processes $T_n^{(x,y)}$ of rate 1 for each of its y neighbours, where n denotes the n^{th} arrival, to x. If at time $T_n^{(x,y)}$ we have $x \in \xi_t$ and $y \notin \xi_t$ we add y to ξ_t.

Take $d = 1$, $A = \{0\}$ and let $t(n) = \inf\{t : n \in \xi_t^{\{0\}}\}$, that is, $t(n)$ is the time taken for $n \in \mathbb{Z}$ to become occupied. Suppose $n > 0$, then the only way to occupy n is to first occupy $1, 2, \cdots, n-1$ successively. The interarrival times are exponentially distributed with parameter 1, and the memoryless property of the exponential distribution gives us that each successive occupation is independent. Hence by an application of the strong law of large numbers we get that $t(n)/n \to 1$ almost surely. So we can see that ξ_t grows somewhat linearly when $d = 1$.
Now for $d \geq 1$ define a thickened version of our set of occupied sites, $\xi_t^{\{0\}}$, as follows

$$\bar{\xi}_t^{\{0\}} := \left\{x + y : x \in \xi_t^{\{0\}}, y \in \left[-\frac{1}{2}, \frac{1}{2}\right]^d\right\}$$

This gives rise to the following result for the limiting shape.

Theorem. *Let $\epsilon > 0$ be given, then $\exists$ a convex set A s.t.*

$$P\Big[(1-\epsilon)tA \subset \bar{\xi}_t^{\{0\}} \subset (1+\epsilon)tA\Big] \to 1 \textit{ as } t \to \infty$$

If $\xi_{t(x)+u}^{(x,t(x))}$ is the set of sites that can be reached after time u from x at time $t(x)$, we define

$$t(x,y) = \inf\{u : y \in \xi_{t(x)+u}^{(x,t(x))}\}$$

We have that $t(x,y)$ is independent of $t(x)$ by properties of the Poisson process and $t(x,y)$ has the same distribution has $t(y-x)$ by translational symmetry of the lattice.

Taking $n \in \mathbb{Z}$ one can use the subadditive ergodic theorem to obtain that $t(0,nx)/n \to$ some limit $\mu(x)$ as $n \to \infty$, so in particular we have that $\xi_t^{\{0\}}$ grows linearly in each direction. By setting $t(x) = \inf\{t : x \in \bar{\xi}_t^{\{0\}}\}$ we extend $t(x)$ to all $x \in \mathbb{R}^d$. It is possible to show that

$$\frac{t(nx)}{n} \to \mu(x) \text{ a.s. } \forall x \in \mathbb{R}^d$$

It turns out that μ, given by $\inf_{m\geq 1} \frac{E(t(mx))}{m}$, defines a norm on $\mathbb{R}^d$. The convex set A in the theorem is given by the unit ball in that norm. In particular for $\mathbb{Z}^2$ we see that $\bar{\xi}_t^{\{0\}}/t$ is, in the sense of the norm μ, roughly circular. However it is very difficult to compute μ from this expression, so it does not really tell us much about how our process is growing.

To get a better feel for the norm and the limiting shape of the growth model we will pass to a discrete version of the process on $\mathbb{Z}^2$.

Flat edges

We identify an embedded discrete contact process given by $\zeta_n(x) = \eta_n(x, n-x)$, where $x \in \mathbb{Z}$. It is known for the contact process that $\exists\ p_0 < 1$ such that $p > p_0 \Rightarrow P(\zeta_n \not\equiv 0 \text{ for all } n) > 0$, so the inclusion $\{x \in \mathbb{Z}^2 : \eta_n(x) = 1\} \supset \{(y, n-y) : \zeta_n(y) = 1\}$ gives us that for $P(\zeta_n \not\equiv 0 \text{ for all } n) > 0$

$$B_p := \{x : \mu_p(x) \leq 1\} \cap \{x : x_1 + x_2 = 1\} \neq \phi$$

Symmetry arguments yield that $(1/2, 1/2) \in B_p$, and if we define $p_{cr} = \inf\{p : P(\zeta_n \not\equiv 0 \text{ for all } n) > 0\}$, then the result reads that for $p > p_{cr}, (1/2, 1/2) \in B_p$ and $\mu_p(1/2, 1/2) = 1$.

It can be shown that for $p > p_{cr}$, we have $\partial B_p \cap \{x : x_1 + x_2 = 1\}$ is an interval of length at least $2\sqrt{2}\,[p - p_{cr}]$. This means that in the limit of our rescaled set of occupied points, the boundary has edges that look flat.

In fact simulations of the discrete Richardson growth model, you see that for p close to 1, the limiting shape is close to a diamond, but as p varies from 1 to 0, the limiting shape becomes more and more circular.

Discrete simulations in Python

Colouring only the boundary, we can see the shape of the rescaled set of occupied points.

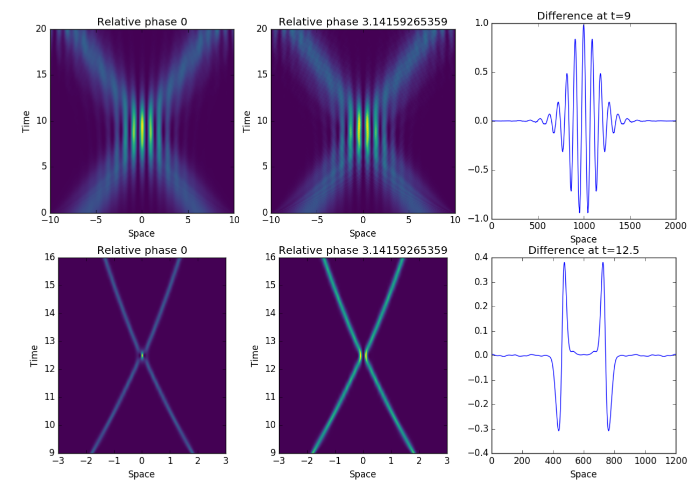

Figure 2: $n = 50$, $p = 0.6$

The two-type Richardson growth model

For two different types of particles, type I and type II, on $\mathbb{Z}^d$ we define the two-type Richardson growth model in a similar manner. Vacant sites are occupied by a particle of type I or type II at a rate given by the number of neighbours of a given type multiplied by parameters λ_1, λ_2 respectively. Once a particle occupies a site, it occupies it for all future times. Starting from a single occupied site for each type, it is not clear whether the sets of occupied sites for each types can both grow to be infinite. For example it can happen that one set completely surrounds the other and prevents further growth.

It is conjectured that for $d \geq 2$, starting from single sites, the probability that the sets of occupied sites can both grow to infinite size is positive if and only if $\lambda_1 = \lambda_2$.

It is known that for all but at most countably many values of $\lambda := \lambda_2/\lambda_1$, the probability that both sets of occupied sites can grow to be infinite is 0.

Voter Model

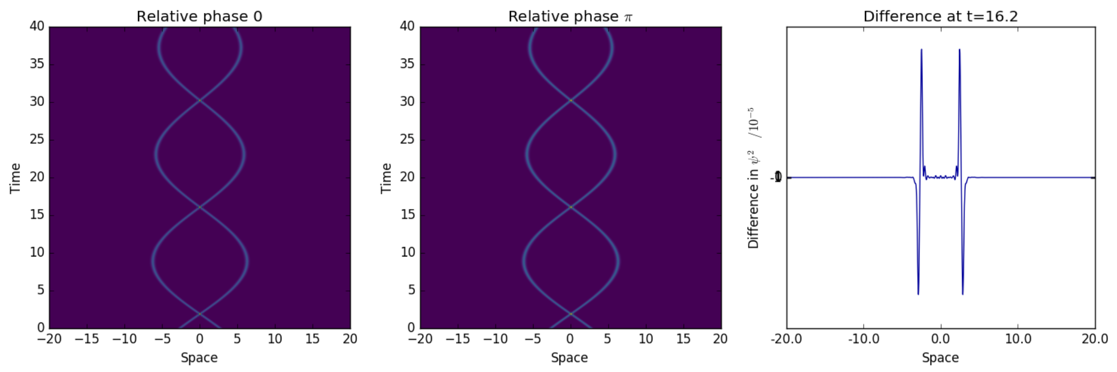

Figure 3: $n = 50$, $p = 0.6$

Threshold voter model

The voter model we defined above is in fact a particular case of the linear voter model. There are nonlinear models such as the threshold voter model. If we intersect $\mathbb{Z}^d$ with a compact, convex and symmetric set $X \subset \mathbb{R}^d$ we obtain a neighbourhood $\mathcal{N} = X \cap \mathbb{Z}^d$ of 0. For a positive integer T, the threshold model corresponding to $\mathcal{N}$ and T has rate function

$$c(x, \eta) = \begin{cases} 1 & \text{if } \#\{y \in x + \mathcal{N} : \eta(y) \neq \eta(x)\} \geq T \\ 0 & \text{otherwise.} \end{cases}$$

Final Remarks

Each of the models presented here can be studied in greater generality, for various index sets I. There are many results for processes taking $I = V$ for some vertex set V of a connected undirected graph $G = (V, E)$, whose vertices have bounded degrees.

References

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