Chapter 7

Solitons

7.1 Introduction

Formally, a *soliton* is a solitary wave which asymptotically preserves its shape and velocity upon nonlinear interaction with other solitary waves, or more generally, with another (arbitrary) localized disturbance.

Here, a solitary wave is a traveling wave solution f, which has the form $f(x,t) = g(x-vt) = g(z)$, and whose asymptotic states for $z \to \pm\infty$ are equal.

However, the formal definition is commonly not used in physics, and the subtle distinction between solitons and solitary waves is dropped. Depending on the community, the term soliton refers to localised solutions of a 1D non-linear wave equation where dispersion is exactly cancelled by the nonlinearity or is cancelled by nonlinearity together with linear confinement. Examples include water waves, optical solitons in fibres, and matter-wave solitons in atomic Bose-Einstein condensates.

Here we will focus on matter-wave solitons produced using atomic Bose-Einstein condensates, first because they provide a model system to study the particle and wave like properties of solitons and second because there are exciting experiments happening here in Durham!

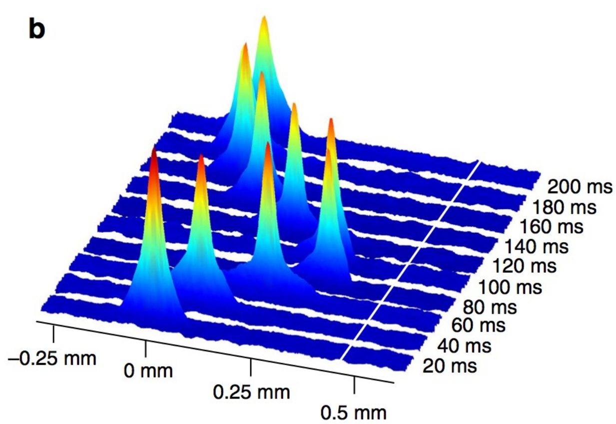

Figure 7.1: Image showing the atomic density for a matter wave soliton reflected from a laser beam ‘barrier’. From Marchant, A. L. *et al. Controlled formation and reflection of a bright solitary matter-wave.* Nat. Commun. **4**, 1865 (2013). doi: 10.1038/ncomms2893

7.2 NLSE

To model solitons we will use the non-linear Schrödinger equation (NLSE). This description is appropriate for optical solitons and matter-wave 'solitons'. Due to the analogy with the linear Schrödinger equation we will discuss matter wave solitons in more detail here. For a single atom, the de Broglie wave in the position basis satisfies a 1D wave equation of the form

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi \tag{7.1}$$

where V is an external potential. This is the linear Schrödinger equation. For N interacting atoms in a Bose-Einstein condensate (BEC) the effect of interactions is to add a term proportional to the density giving a non-linear Schrödinger equation (NLSE) of the form

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi + g|\psi|^2\psi \ , \tag{7.2}$$

where g is a constant relating to the strength of the interactions. The constant g may be either positive or negative, but only a negative sign corresponding to attractive interactions leads to localised solutions. This equation can be written in dimensionless form

$$\mathrm{i}\frac{\partial\tilde{\psi}}{\partial t} = -\frac{\partial^2\tilde{\psi}}{\partial x^2} + \tilde{V}\psi - \tilde{g}|\tilde{\psi}|^2\tilde{\psi} \ . \tag{7.3}$$

This equation also describe solitons in optical fibres, where 'effective' interaction between photons arises due to the **optical Kerr effect** in the optical fibre. Is the non-linearity due to the optical Kerr effect generally positive or negative? Why?

7.3 Numerical methods

We need to solve an equation of the form

$$\mathrm{i}\frac{\partial\tilde{\psi}}{\partial t} = \mathbf{H}\tilde{\psi} \ . \tag{7.4}$$

An approximate solution to this equation has the form

$$\tilde{\psi}(x, t+\delta t) \approx \mathrm{e}^{-\mathrm{i}\mathbf{H}\delta t}\tilde{\psi}(x,t) \ . \tag{7.5}$$

For a NLSE this is only approximate because $\mathbf{H}$ depends on $\tilde{\psi}$.

7.3.1 Split-step Fourier

In the split-step Fourier method we split the 'Hamiltonian' into a kinetic and potential term using the approximation

$$\mathrm{e}^{-\mathrm{i}(\mathbf{T}+\mathbf{V})\delta t} \approx \mathrm{e}^{-i\mathbf{V}\delta t/2}\mathrm{e}^{-i\mathbf{T}\delta t}\mathrm{e}^{-i\mathbf{V}\delta t/2} \ . \tag{7.6}$$

The Fourier 'trick' is to recognise that in Fourier space the operator $\partial/\partial x$ becomes k, so we can write

$$\tilde{\psi}(x, t+\delta t) = \mathrm{e}^{-\mathrm{i}\mathbf{V}\delta t/2}\mathcal{F}^{-1}\left[e^{-\mathrm{i}k^2\delta t}\mathcal{F}\left[\mathrm{e}^{-\mathrm{i}\mathbf{V}\delta t/2}\tilde{\psi}(x,t)\right]\right] \ .$$

This is exactly the same as the angular spectrum method used in the L2 optics course (see Optics f2f).

7.4 Milestone

A good way to start is to first implement the diffraction alone, and propagate an initial Gaussian [$\psi = \exp(-x^2)$] profile and compare to the analytical solution. Then, use a linear harmonic oscillator to exactly cancel the diffraction, and check that the profile $|\psi|^2$ of the wavefunction does not change upon propagation.

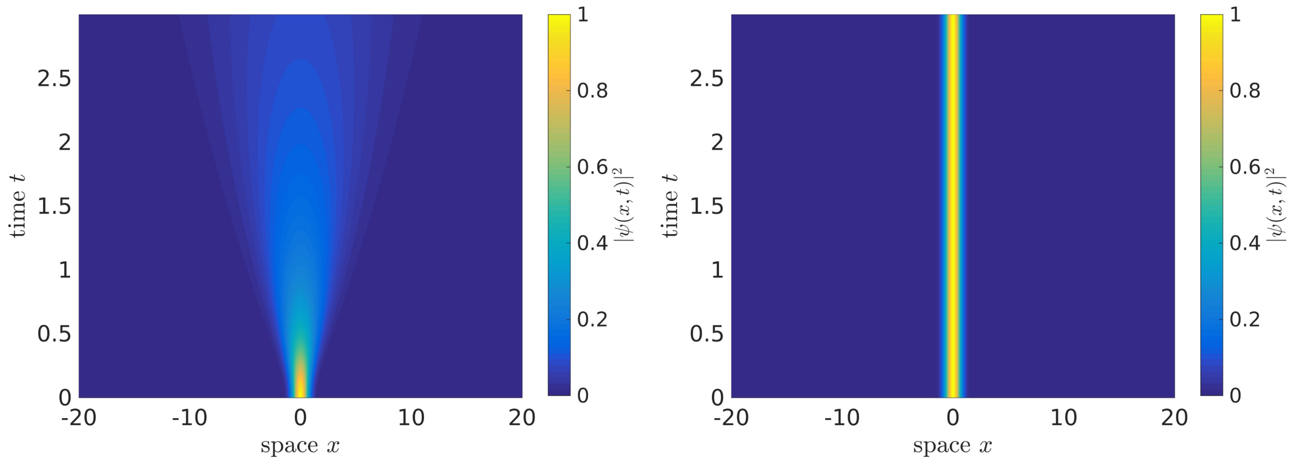

Figure 7.2: Colormap on left showing a gaussian input into the linear Schrödinger equation. Right: the same input but with the parabolic confining potential added.

Now try the NLSE. The soliton solution is in the form of a sech-function, see e.g. `https://en.wikipedia.org/wiki/Soliton_(optics)`. Plot a colormap showing the intensity (modulus squared of the amplitude).

Finally, the your Milestone program should create propagating a sech soliton

$$\psi = \frac{\sqrt{2\xi}}{cosh(x\sqrt{\xi})} e^{ivx} \tag{7.7}$$

(with family parameter $\xi = 1$ and velocity $v = L/4$, where L is the box length, for example). Show that its shape does not change as it propogates and that its norm ($\int |\psi|^2 dx$) is conserved to better than 0.1% accuracy over a time $t = \frac{5}{\xi}$.

7.5 Extending the milestone

Now that you have created a code that accurately propogates solitions, you can extend you investigation in many different directions. For example, Figure 7.3 illustrates what happens when two solitions colide.

The next steps are to test the accuracy of your numerics and then apply the optimised code to address specific physics questions. Some interesting topics include solitons in optical fibres and the dynamics of matter wave solitons in harmonic traps, reflection from a barrier (including 'quantum' reflection), collisions, etc.

Here are some more ideas that you could use as the starting point for your research.

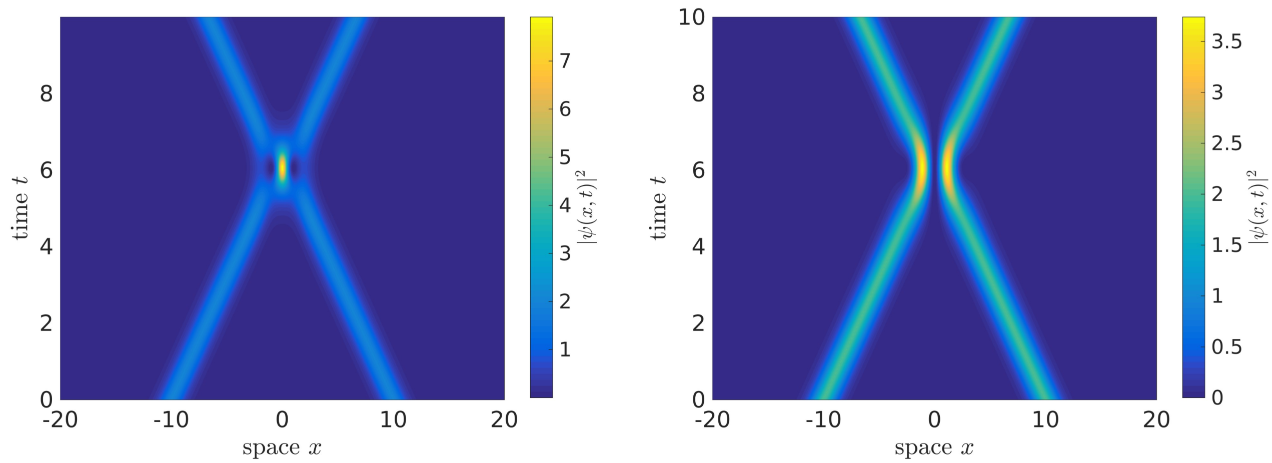

Figure 7.3: These figures illustrate the colision of two solitons. In the left hand panel, the solitons are in phase and combine to form a bright spot as they collide. In the right panel, the solitons are out of phase and appear to bounce off each other.

- Investigate the importance of the amplitude of the initial wave. Show that solitons may be 'dark' as well as bright.

- Transverse modes in an optical fibre.

- Collisions of matter wave solitons, A. D. Martin *et al.*, *Bright solitary-matter-wave collisions in a harmonic trap: Regimes of soliton like behavior* Phys. Rev. A **77**, 013620 (2008).

- Quantum reflection, S. L. Cornish *et al.*, *Quantum reflection of bright matter-wave solitons* Physica D **238**, 1299 (2009).

- Optical solitons and freak waves, B. Kibler *et al.*, *Observation of Kuznetsov-Ma soliton dynamics in optical fibre*, Scientific Reports **2**, 463 (2012).

- Modulational instability and self-organization of light, F. Maucher *et al.*, Phys. Rev. Lett. **116**, 163902 (2016), F. Maucher *et al*, Optical Data Processing and Storage, **3**, 13 (2017).

- Brownian motion and radiative loss of solitons in fibres, V Folli, C Conti, Phys. Rev. Lett. **104** 193901 (2010), F. Maucher *et al* Phys. Rev. A 85, 063803 (2012)