Solitons in a Bose-Einstein Condensate

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Introduction

Interacting Particle Systems is an area of Probability, that spans several fields of study. Typically one starts with a countable number of particles, which would evolve as independent Markov chains on some countable state space I, and imposes a local interaction between these particles. This means that the evolution of an individual particle on I no longer has the Markov property. Understanding how these systems develops over time is now much more involved.

While the motivation for Interacting Particle Systems initially came from problems in statistical mechanics and from trying to understand phase transition, the area is now widely applicable to other areas such as population evolution, modelling infections, modelling computer viruses, and behavioural systems.

We consider continuous time Markov processes, denoted η_t, in a state space $S = \{0,1\}^I$ where I is the countable collection of sites. The interpretation of the values $0, 1$ changes for various models, for example they could denote whether a particle occupies a particular site. The dynamics of these processes are specified by transition rates between states.

Processes and Models

There are lots of different types of processes and models, such as the Richardson growth model, the contact process, the voter model and the exclusion process. The interactions on these models can lead to some surprising behaviour. In particular we will concentrate on the behaviours of the Richardson growth model and the voter model when the set of sites $I = \mathbb{Z}^d$.

The effect of inter-atom interactions $g|\psi|^2$ on a soliton model

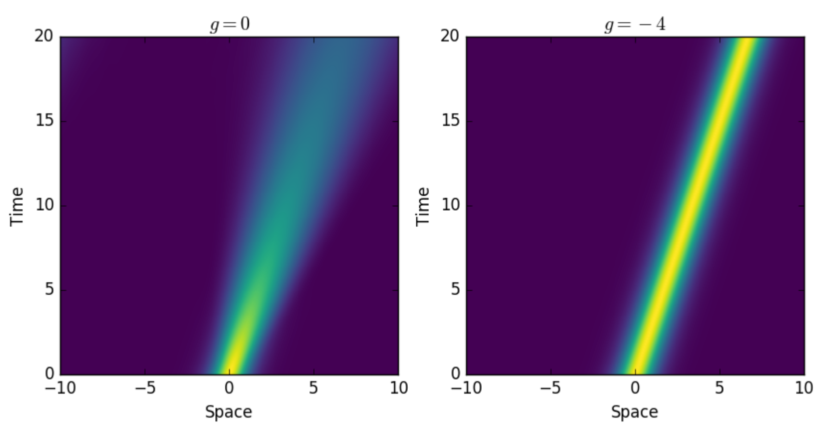

Figure 1: $n = 50$, $p = 0.6$

Contact Process

We consider an undirected $G = (V, E)$ whose vertices have bounded degrees. For the contact process on a graph with parameter λ on G we take $I = V$. Each vertex represents an individual, which we call infected if $\eta_t(x) =$ and healthy if $\eta_t(x) = 0$. Infected individuals recover independently at rate 1 and infect neighbouring healthy vertices at a rate λ. This gives the following rate function

$$c(x,\eta) = \begin{cases} 1 & \text{if } \eta(x) = 1 \\ \lambda \sum_{y:|x-y|=1} \eta(y) & \text{if } \eta(x) = 0 \end{cases}$$

The contact process can be used to model things like infections in populations, and the spread of computer viruses. If the graph is finite the infection will die off almost surely, but we can ask questions about the survival time of the process under various conditions. We can also ask questions about the long time behaviour of the process on infinite graphs such as lattices.

Richardson Growth Model

This model is constructed by attaching independent Poisson Processes $T_n^{(x,y)}$ of rate 1 for each of its y neighbours, where n denotes the n^{th} arrival, to x. If at time $T_n^{(x,y)}$ we have $x \in \xi_t$ and $y \notin \xi_t$ we add y to ξ_t.

Take $d = 1$, $A = \{0\}$ and let $t(n) = \inf\{t : n \in \xi_t^{\{0\}}\}$, that is, $t(n)$ is the time taken for $n \in \mathbb{Z}$ to become occupied. Suppose $n > 0$, then the only way to occupy n is to first occupy $1, 2, \cdots, n-1$ successively. The interarrival times are exponentially distributed with parameter 1, and the memoryless property of the exponential distribution gives us that each successive occupation is independent. Hence by an application of the strong law of large numbers we get that $t(n)/n \to 1$ almost surely. So we can see that ξ_t grows somewhat linearly when $d = 1$.
Now for $d \geq 1$ define a thickened version of our set of occupied sites, $\xi_t^{\{0\}}$, as follows

$$\bar{\xi}_t^{\{0\}} := \left\{ x + y : x \in \xi_t^{\{0\}}, y \in \left[-\frac{1}{2}, \frac{1}{2}\right]^d \right\}$$

This gives rise to the following result for the limiting shape.

Theorem. *Let $\epsilon > 0$ be given, then $\exists$ a convex set A s.t.*

$$P\Big[(1-\epsilon)tA \subset \bar{\xi}_t^{\{0\}} \subset (1+\epsilon)tA\Big] \to 1 \text{ as } t \to \infty$$

If $\xi_{t(x)+u}^{(x,t(x))}$ is the set of sites that can be reached after time u from x at time $t(x)$, we define

$$t(x,y) = \inf\{u : y \in \xi_{t(x)+u}^{(x,t(x))}\}$$

We have that $t(x,y)$ is independent of $t(x)$ by properties of the Poisson process and $t(x,y)$ has the same distribution has $t(y-x)$ by translational symmetry of the lattice.

Taking $n \in \mathbb{Z}$ one can use the subadditive ergodic theorem to obtain that $t(0,nx)/n \to$ some limit $\mu(x)$ as $n \to \infty$, so in particular we have that $\xi_t^{\{0\}}$ grows linearly in each direction. By setting $t(x) = \inf\{t : x \in \bar{\xi}_t^{\{0\}}\}$ we extend $t(x)$ to all $x \in \mathbb{R}^d$. It is possible to show that

$$\frac{t(nx)}{n} \to \mu(x) \text{ a.s. } \forall x \in \mathbb{R}^d$$

It turns out that μ, given by $\inf_{m \geq 1} \frac{E(t(mx))}{m}$, defines a norm on $\mathbb{R}^d$. The convex set A in the theorem is given by the unit ball in that norm. In particular for $\mathbb{Z}^2$ we see that $\bar{\xi}_t^{\{0\}}/t$ is, in the sense of the norm μ, roughly circular. However it is very difficult to compute μ from this expression, so it does not really tell us much about how our process is growing.

To get a better feel for the norm and the limiting shape of the growth model we will pass to a discrete version of the process on $\mathbb{Z}^2$.

Flat edges

We identify an embedded discrete contact process given by $\zeta_n(x) = \eta_n(x, n-x)$, where $x \in \mathbb{Z}$. It is known for the contact process that $\exists\ p_0 < 1$ such that $p > p_0 \Rightarrow P(\zeta_n \not\equiv 0 \text{ for all } n) > 0$, so the inclusion $\{x \in \mathbb{Z}^2 : \eta_n(x) = 1\} \supset \{(y, n-y) : \zeta_n(y) = 1\}$ gives us that for $P(\zeta_n \not\equiv 0 \text{ for all } n) > 0$

$$B_p := \{x : \mu_p(x) \leq 1\} \cap \{x : x_1 + x_2 = 1\} \neq \phi$$

Symmetry arguments yield that $(1/2, 1/2) \in B_p$, and if we define $p_{cr} = \inf\{p : P(\zeta_n \not\equiv 0 \text{ for all } n) > 0\}$, then the result reads that for $p > p_{cr}$, $(1/2, 1/2) \in B_p$ and $\mu_p(1/2, 1/2) = 1$.

It can be shown that for $p > p_{cr}$, we have $\partial B_p \cap \{x : x_1 + x_2 = 1\}$ is an interval of length at least $2\sqrt{2}\,[p - p_{cr}]$. This means that in the limit of our rescaled set of occupied points, the boundary has edges that look flat.

In fact simulations of the discrete Richardson growth model, you see that for p close to 1, the limiting shape is close to a diamond, but as p varies from 1 to 0, the limiting shape becomes more and more circular.

Discrete simulations in Python

Colouring only the boundary, we can see the shape of the rescaled set of occupied points.

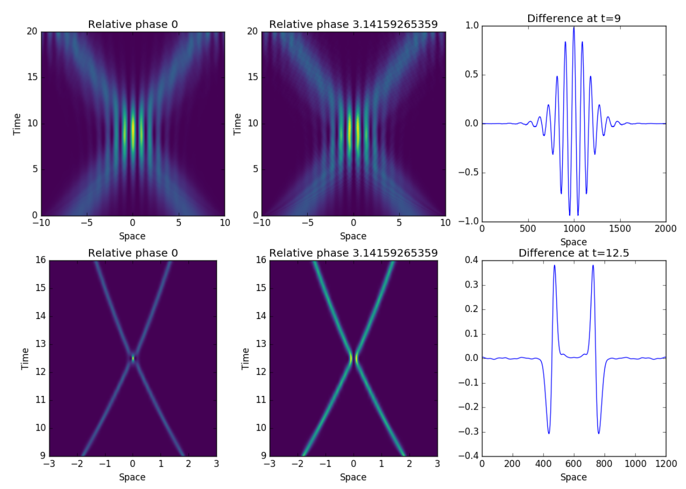

Figure 2: $n = 50$, $p = 0.6$

The two-type Richardson growth model

For two different types of particles, type I and type II, on $\mathbb{Z}^d$ we define the two-type Richardson growth model in a similar manner. Vacant sites are occupied by a particle of type I or type II at a rate given by the number of neighbours of a given type multiplied by parameters λ_1, λ_2 respectively. Once a particle occupies a site, it occupies it for all future times. Starting from a single occupied site for each type, it is not clear whether the sets of occupied sites for each types can both grow to be infinite. For example it can happen that one set completely surrounds the other and prevents further growth.

It is conjectured that for $d \geq 2$, starting from single sites, the probability that the sets of occupied sites can both grow to infinite size is positive if and only if $\lambda_1 = \lambda_2$.

It is known that for all but at most countably many values of $\lambda := \lambda_2/\lambda_1$, the probability that both sets of occupied sites can grow to be infinite is 0.

Voter Model

To each $x \in \mathbb{Z}^d$ we attach independent Poisson processes T_n^x of rate 1, and let Y_n^x be independent i.i.d. sequences, with $P(Y_n^x = y) = 1/2d$ for all y with $|y| = 1$. The individual x changes their opinion for the nth time at time T_n^x to that of $x + Y_n^x$. The process can be represented graphically on $\mathbb{Z}^d \times [0, \infty)$ by drawing arrows from $(x + Y_n^x, T_n^x)$ to (x, T_n^x), and we also write a δ at (x, T_n^x) for each x and n.

Weak Harmonic Confining Potential

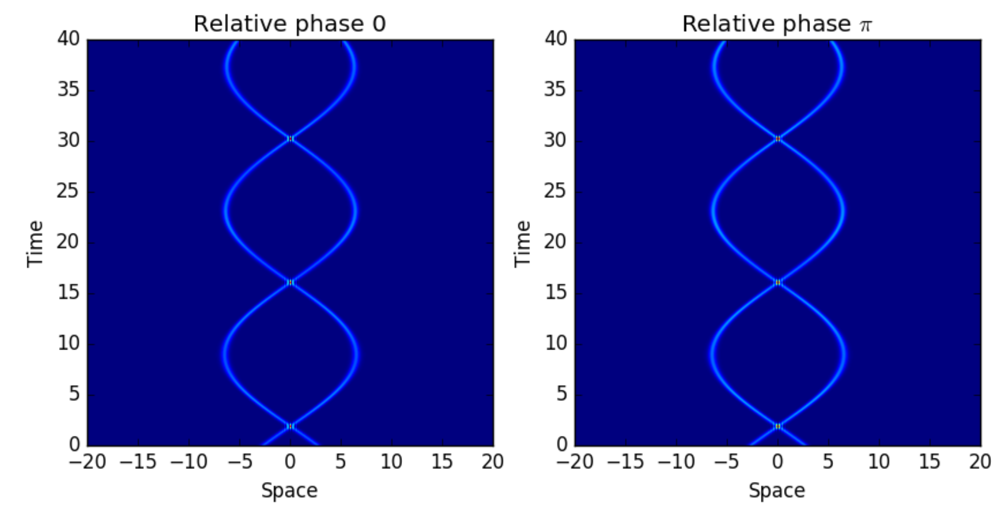

Figure 3: $n = 50$, $p = 0.6$

Limiting distributions

It is easy to see that $\eta \equiv 0$ and $\eta \equiv 1$ are invariant for the process, as if all individuals are of the same opinion, nobody can change. However it is not clear that the process has other invariant extremal measures. The trick here is to relate our dual process to a simple random walk, which is recurrent in dimension $d = 1, 2$ and transient for $d \geq 3$.

From this we can obtain that the voter model on $\mathbb{Z}^d$ reaches a consensus for $d \leq 2$, but for $d \geq 3$ we get a family of limiting distributions. In particular we get a different limiting distribution for every $p \in [0, 1]$ if we start from an initial configuration which puts $\eta_0(x) = 1$ with probability p, i.e. the product measure with density p.

Threshold voter model

The voter model we defined above is in fact a particular case of the linear voter model. There are nonlinear models such as the threshold voter model. If we intersect $\mathbb{Z}^d$ with a compact, convex and symmetric set $X \subset \mathbb{R}^d$ we obtain a neighbourhood $\mathcal{N} = X \cap \mathbb{Z}^d$ of 0. For a positive integer T, the threshold model corresponding to $\mathcal{N}$ and T has rate function

$$c(x,\eta) = \begin{cases} 1 & \text{if } \#\{y \in x + \mathcal{N} : \eta(y) \neq \eta(x)\} \geq T \\ 0 & \text{otherwise.} \end{cases}$$

This essentially gives that individuals have an area of effect on the opinions of neighbouring individuals. While this seems like a minor extension its behaviour can be strange, even in $d = 1$. For example it is possible for the system to get trapped in a non-trivial extremal measure in the sense that, starting from any initial configuration, each $x \in \mathbb{Z}$ will only change its opinion finitely many times. In the case we say that the process fixates. If we consider $\mathcal{N} = \{-1, 0, 1\}$ and $T = 2$, the system can become trapped in

$$\cdots\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ \cdots$$

More generally if $\eta_t(x) = \eta_t(x+1)$ then these two individuals have the same opinion for all future times. In contrast this situation does not arise in linear voter models. In particular

$$\text{the process fixates when } T > \frac{\#\mathcal{N} - 1}{2}$$

In one dimension, if any interval of length T is constant at any time, then individuals in the interval will never change their opinion, so it is easy to see why it fixates in this case. In fact for dimension one, when $\mathcal{N} = \{-T, \cdots, T\}$, the limiting distributions are given by consensus.

Final Remarks

Each of the models presented here can be studied in greater generality, for various index sets I. There are many results for processes taking $I = V$ for some vertex set V of a connected undirected graph $G = (V, E)$, whose vertices have bounded degrees.

The behaviours of many models, even simple ones such as the Richardson growth model, can have some very non-trivial properties. While there are lots of known results for these models, small changes in the local dynamics can lead to completely different global behaviours, which are often difficult to understand.

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