

Solutions to  
An Introduction to Quantum Field Theory  
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## Chapter 1

Invitation: Pair Production in  $e^+e^-$  Annihilation

**No Problems**

## Chapter 2

### The Klein-Gordon Field

#### 2.1 Classical electromagnetism with no sources

The Euler-Lagrange equations of motion, with the components  $A_\mu(x)$  as the dynamical variables, are given by:

$$\partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu A_\nu)} \right) - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0,$$

where the Lagrangian density is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu},$$

and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

is the electromagnetic field tensor. For the Euler-Lagrange equations of motion,  $\mu$  and  $\nu$  are *free* indices that simply label the components  $A_\mu$  and its derivatives  $\partial_\mu A_\nu$ . In contrast, the indices for  $F_{\mu\nu}$  are *dummy* indices that are used to sum over all the components. Their label doesn't matter as long as it is distinct from the *free* indices. It is important to note that  $F_{\mu\nu}$  is antisymmetric. More precisely,

$$F_{\nu\mu} = \partial_\nu A_\mu - \partial_\mu A_\nu = -(\partial_\mu A_\nu - \partial_\nu A_\mu) = -F_{\mu\nu}.$$

Consequently, the diagonal components of  $F_{\mu\nu}$  vanish. Specifically,

$$F_{\mu\mu} = -F_{\mu\mu} \implies 2F_{\mu\mu} = 0 \implies F_{\mu\mu} = 0.$$

(a) For the second term in the Euler-Lagrange equations of motion, we have:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial A_\nu} &= \frac{\partial}{\partial A_\nu} \left( -\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \right) \rightarrow \text{apply product rule} \\
&= -\frac{1}{4} \left( F^{\alpha\beta} \frac{\partial F_{\alpha\beta}}{\partial A_\nu} + F_{\alpha\beta} \frac{\partial F^{\alpha\beta}}{\partial A_\nu} \right) \rightarrow \text{lower indices in 2nd term} \\
&= -\frac{1}{4} \left( F^{\alpha\beta} \frac{\partial F_{\alpha\beta}}{\partial A_\nu} + g^{\alpha\sigma} g^{\beta\rho} F_{\alpha\beta} \frac{\partial F_{\sigma\rho}}{\partial A_\nu} \right) \rightarrow \text{raise indices in 2nd term} \\
&= -\frac{1}{4} \left( F^{\alpha\beta} \frac{\partial F_{\alpha\beta}}{\partial A_\nu} + F^{\sigma\rho} \frac{\partial F_{\sigma\rho}}{\partial A_\nu} \right) \rightarrow \text{relabel } \sigma \text{ and } \rho \text{ to } \alpha \text{ and } \beta \\
&= -\frac{1}{4} \left( F^{\alpha\beta} \frac{\partial F_{\alpha\beta}}{\partial A_\nu} + F^{\alpha\beta} \frac{\partial F_{\alpha\beta}}{\partial A_\nu} \right) \rightarrow \text{terms are equal} \\
&= -\frac{1}{2} F^{\alpha\beta} \frac{\partial F_{\alpha\beta}}{\partial A_\nu} \rightarrow \text{second term is 0} \\
&= 0.
\end{aligned}$$

This is because

$$\frac{\partial F_{\alpha\beta}}{\partial A_\nu} = \frac{\partial}{\partial A_\nu} (\partial_\alpha A_\beta - \partial_\beta A_\alpha) = 0,$$

which follows from the fact that  $F_{\mu\nu}$  depends only on the derivative of  $A_\mu$  (i.e.,  $\partial_\mu A_\nu$ ) and not  $A_\mu$  itself. For the first term in the Euler-Lagrange equations of motion, we have:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} &= \frac{\partial}{\partial(\partial_\mu A_\nu)} \left( -\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \right) \rightarrow \text{apply product rule} \\
&= -\frac{1}{4} \left( F^{\alpha\beta} \frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)} + F_{\alpha\beta} \frac{\partial F^{\alpha\beta}}{\partial(\partial_\mu A_\nu)} \right) \rightarrow \text{lower indices in 2nd term} \\
&= -\frac{1}{4} \left( F^{\alpha\beta} \frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)} + g^{\alpha\sigma} g^{\beta\rho} F_{\alpha\beta} \frac{\partial F_{\sigma\rho}}{\partial(\partial_\mu A_\nu)} \right) \rightarrow \text{raise indices in 2nd term} \\
&= -\frac{1}{4} \left( F^{\alpha\beta} \frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)} + F^{\sigma\rho} \frac{\partial F_{\sigma\rho}}{\partial(\partial_\mu A_\nu)} \right) \rightarrow \text{relabel } \sigma \text{ and } \rho \text{ to } \alpha \text{ and } \beta \\
&= -\frac{1}{4} \left( F^{\alpha\beta} \frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)} + F^{\alpha\beta} \frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)} \right) \rightarrow \text{terms are equal} \\
&= -\frac{1}{2} F^{\alpha\beta} \frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)} \rightarrow \text{insert } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \\
&= -\frac{1}{2} F^{\alpha\beta} \frac{\partial}{\partial(\partial_\mu A_\nu)} (\partial_\alpha A_\beta - \partial_\beta A_\alpha) \rightarrow \mu, \nu = \alpha, \beta \text{ or } \beta, \alpha \text{ survive}
\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{2}F^{\alpha\beta}\left(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}\right) \rightarrow \text{distribute } F^{\alpha\beta} \\
&= -\frac{1}{2}\left(F^{\alpha\beta}\delta_{\mu\alpha}\delta_{\nu\beta} - F^{\alpha\beta}\delta_{\mu\beta}\delta_{\nu\alpha}\right) \rightarrow \text{satisfy } \delta_{ij} \text{ functions} \\
&= -\frac{1}{2}\left(F^{\mu\nu} - F^{\nu\mu}\right) \rightarrow F_{\mu\nu} \text{ is antisymmetric} \\
&= -\frac{1}{2}\left(F^{\mu\nu} + F^{\mu\nu}\right) \\
&= -F^{\mu\nu}.
\end{aligned}$$

Inserting these results into the Euler-Lagrange equations of motion, we obtain:

$$\partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu A_\nu)} \right) - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0 \implies \partial_\mu (-F^{\mu\nu}) - 0 = 0 \implies \partial_\mu F^{\mu\nu} = 0.$$

Two of Maxwell's equations can be written in standard form by making use of this result. First, consider the identity  $E^i = -F^{0i} = F^{i0}$ , where  $\nu = 0$ :

$$\begin{aligned}
0 &= \partial_\mu F^{\mu 0} \rightarrow \text{separate components} \\
&= \partial_0 F^{00} + \partial_i F^{i0} \rightarrow F^{00} = 0 \text{ and substitute } F^{i0} = E^i \\
&= \partial_i E^i \rightarrow \text{divergence of a vector field in tensor notation} \\
&= \nabla \cdot \mathbf{E} \rightarrow \text{Gauss's law.}
\end{aligned}$$

Second, consider the identity  $\epsilon^{ijk} B^k = -F^{ij} = F^{ji}$ , where  $\nu = i$ :

$$\begin{aligned}
0 &= \partial_\mu F^{\mu i} \rightarrow \text{separate components} \\
&= \partial_0 F^{0i} + \partial_j F^{ji} \rightarrow \text{substitute } E^i \text{ and } B^k \\
&= -\partial_0 E^i + \epsilon^{ijk} \partial_j B^k \rightarrow \text{2nd term is curl of a vector field in tensor notation} \\
&= \frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B} \rightarrow \text{Ampere's law.}
\end{aligned}$$

The two remaining Maxwell equations can be written in standard form by making use of these identities with the definition of  $F_{\mu\nu}$ . First, consider the case with  $\mu = i$  and  $\nu = j$ , which gives:

$$\begin{aligned}
F^{ij} &\implies \partial^i A^j - \partial^j A^i = -\epsilon^{ijk} B^k \rightarrow \text{multiply by } \epsilon^{ijm} \text{ and sum over } i, j \\
-\epsilon^{ijm} (\partial_i A^j - \partial_j A^i) &= -\epsilon^{ijm} \epsilon^{ijk} B^k \rightarrow \epsilon^{ijm} \epsilon^{ijk} = 2\delta^{mk} \\
\epsilon^{ijm} \partial_i A^j - \epsilon^{ijm} \partial_j A^i &= 2\delta^{mk} B^k \rightarrow \text{relabel } i, j \text{ to } j, i \text{ in LHS 2nd term} \\
\epsilon^{ijm} \partial_i A^j - \epsilon^{jim} \partial_i A^j &= 2\delta^{mk} B^k \rightarrow \text{swap } i, j \text{ in } \epsilon^{jim}
\end{aligned}$$

$$\epsilon^{ijm}\partial_i A^j - (-\epsilon^{ijm}\partial_i A^j) = 2\delta^{mk} B^k$$

$$2\epsilon^{ijm}\partial_i A^j = 2\delta^{mk} B^k$$

$$\epsilon^{ijm}\partial_i A^j = B^m \rightarrow \text{curl of a vector field in tensor notation}$$

$$\nabla \times \mathbf{A} = \mathbf{B} \rightarrow \text{take the divergence and recall } \nabla \cdot (\nabla \times \mathbf{A}) = 0$$

$$0 = \nabla \cdot \mathbf{B} \rightarrow \text{Gauss's law for magnetism.}$$

Finally, consider the case with  $\mu = 0$  and  $\nu = i$ , which gives:

$$F^{0i} \implies \partial^0 A^i - \partial^i A^0 = -E^i \rightarrow \text{lower an index in LHS 2nd term}$$

$$\partial^0 A^i + \partial_i A^0 = -E^i$$

$$\frac{\partial A^i}{\partial t} + \frac{\partial A^0}{\partial x^i} = -E^i \rightarrow \text{second term on LHS is the scalar potential}$$

$$-\frac{\partial \mathbf{A}}{\partial t} - \nabla \Phi = \mathbf{E} \rightarrow \text{take the curl}$$

$$-\frac{\partial}{\partial t} (\nabla \times \mathbf{A}) - \nabla \times (\nabla \Phi) = \nabla \times \mathbf{E} \rightarrow \nabla \times \mathbf{A} = \mathbf{B} \text{ from above}$$

$$-\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{E} \rightarrow \text{Faraday's law.}$$

**(b)** From Noether's theorem applied to spacetime transformations, where the components  $A_\mu(x)$  are the dynamical variables, we have the general form of the energy-momentum tensor expressed as:

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\lambda)} \partial_\nu A_\lambda - \mathcal{L} \delta^\mu{}_\nu.$$

Using information from part (a), this becomes:

$$T^\mu{}_\nu = -F^{\mu\lambda} \partial_\nu A_\lambda + \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \delta^\mu{}_\nu \rightarrow \text{apply } g^{\sigma\nu}$$

$$g^{\sigma\nu} T^\mu{}_\nu = -F^{\mu\lambda} g^{\sigma\nu} \partial_\nu A_\lambda + \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} g^{\sigma\nu} \delta^\mu{}_\nu$$

$$T^{\mu\sigma} = -F^{\mu\lambda} \partial^\sigma A_\lambda + \frac{1}{4} g^{\sigma\mu} F_{\alpha\beta} F^{\alpha\beta} \rightarrow \text{relabel } \sigma \text{ to } \nu \text{ and use } g^{\mu\nu} = g^{\nu\mu}$$

$$T^{\mu\nu} = -F^{\mu\lambda} \partial^\nu A_\lambda + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}.$$

We're told that this expression is not a symmetric tensor, but that we can remedy this by adding the term  $\partial_\lambda K^{\lambda\mu\nu}$  to  $T^{\mu\nu}$ , where  $K^{\lambda\mu\nu}$  is antisymmetric in the first two indices. The

resulting expression is:

$$\begin{aligned}
\hat{T}^{\mu\nu} &= T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu} \\
&= -F^{\mu\lambda} \partial^\nu A_\lambda + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} + \partial_\lambda K^{\lambda\mu\nu} \\
&= -F^{\mu\lambda} \partial^\nu A_\lambda + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} + \partial_\lambda F^{\mu\lambda} A^\nu,
\end{aligned}$$

where we used the provided relationship,  $K^{\lambda\mu\nu} = F^{\mu\lambda} A^\nu$ , in the last term. This expression can be further simplified as follows:

$$\begin{aligned}
\hat{T}^{\mu\nu} &= -F^{\mu\lambda} \partial^\nu A_\lambda + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} + \partial_\lambda F^{\mu\lambda} A^\nu \rightarrow \text{apply product rule} \\
&= -F^{\mu\lambda} \partial^\nu A_\lambda + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} + F^{\mu\lambda} \partial_\lambda A^\nu + A^\nu \partial_\lambda F^{\mu\lambda} \rightarrow \partial_\lambda F^{\mu\lambda} = 0 \\
&= -F^{\mu\lambda} \partial^\nu A_\lambda + F^{\mu\lambda} \partial_\lambda A^\nu + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \\
&= -F^{\mu\lambda} (\partial^\nu A_\lambda - \partial_\lambda A^\nu) + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \\
&= -F^{\mu\lambda} F^\nu{}_\lambda + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}.
\end{aligned}$$

The symmetry of this expression under the exchange  $\mu \leftrightarrow \nu$  can be demonstrated as follows:

$$\begin{aligned}
\hat{T}^{\nu\mu} &= -F^{\nu\lambda} F^\mu{}_\lambda + \frac{1}{4} g^{\nu\mu} F_{\alpha\beta} F^{\alpha\beta} \rightarrow \text{raise and lower indices and use } g^{\mu\nu} = g^{\nu\mu} \\
&= -g^{\lambda\sigma} g_{\lambda\rho} F^\nu{}_\sigma F^{\mu\rho} + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \rightarrow g^{\lambda\sigma} g_{\lambda\rho} = \delta^\sigma{}_\rho \\
&= -F^\nu{}_\sigma F^{\mu\sigma} + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \rightarrow \text{relabel } \sigma \text{ to } \lambda \\
&= -F^\nu{}_\lambda F^{\mu\lambda} + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \\
&= \hat{T}^{\mu\nu}.
\end{aligned}$$

The electromagnetic energy density,  $\varepsilon$ , is obtained from  $\hat{T}^{00}$ :

$$\begin{aligned}
\varepsilon &= \hat{T}^{00} \\
&= -F^{0\lambda} F^0{}_\lambda + \frac{1}{4} g^{00} F_{\alpha\beta} F^{\alpha\beta} \rightarrow \text{separate components in 1st term} \\
&= -F^{00} F^0{}_0 - F^{0i} F^0{}_i + \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \rightarrow F^{00} = F^0{}_0 = 0 \text{ and } E^i = -F^{0i} = F^0{}_i \\
&= E^i E^i + \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \rightarrow \text{separate components in 2nd term}
\end{aligned}$$

$$\begin{aligned}
&= E^i E^i + \frac{1}{4} \left( F_{00} F^{00} + F_{0i} F^{0i} + F_{i0} F^{i0} + F_{ij} F^{ij} \right) \rightarrow F_{00} = F^{00} = 0 \\
&= E^i E^i + \frac{1}{4} \left( E^i (-E^i) + (-E^i) E^i + (-\epsilon^{ijk} B^k) (-\epsilon^{ijl} B^l) \right) \\
&= E^i E^i + \frac{1}{4} \left( -2E^i E^i + \epsilon^{ijk} \epsilon^{ijl} B^k B^l \right) \rightarrow \epsilon^{ijk} \epsilon^{ijl} = 2\delta^{kl} \\
&= E^i E^i + \frac{1}{4} \left( -2E^i E^i + 2B^k B^k \right) \\
&= \frac{1}{2} E^i E^i + \frac{1}{2} B^k B^k \\
&= \frac{1}{2} (\mathbf{E}^2 + \mathbf{B}^2).
\end{aligned}$$

The momentum density,  $\mathbf{S}$ , is obtained from  $\hat{T}^{0i}$ :

$$\begin{aligned}
\mathbf{S} &= \hat{T}^{0i} \\
&= -F^{0\lambda} F^i_{\lambda} + \frac{1}{4} g^{0i} F_{\alpha\beta} F^{\alpha\beta} \rightarrow g^{0i} = 0 \\
&= -F^{00} F^i_0 - F^{0j} F^i_j \rightarrow F^{00} = 0 \\
&= (E^j) (\epsilon^{ijk} B^k) \\
&= \epsilon^{ijk} E^j B^k \\
&= \mathbf{E} \times \mathbf{B}.
\end{aligned}$$

## 2.2 The complex scalar field

The action for a field theory of a complex-valued scalar field obeying the Klein-Gordon equation is given by:

$$S = \int d^4x \left( \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \right).$$

The authors suggest it is easiest to consider the fields  $\phi(x)$  and  $\phi^*(x)$  as the basic dynamical variables rather than decomposing  $\phi(x)$  into its real and imaginary parts,  $\phi(x) = \phi_1(x) + i\phi_2(x)$ , where  $\phi_1(x)$  and  $\phi_2(x)$  are real Klein-Gordon fields. Therefore, we need an appropriate expression for the Fourier mode expansion of  $\phi(x)$  to describe complex-valued fields. The position-space representation of a general scalar field  $\phi(x)$  is given by its inverse Fourier transform:

$$\phi(x) = \int \frac{d^4p}{(2\pi)^4} \tilde{\phi}(p) e^{-ip \cdot x},$$

where  $\tilde{\phi}(p)$  is the Fourier transform of the field in momentum space. Inserting this expression into the Klein-Gordon equation, we obtain:

$$\begin{aligned} 0 &= \left( \partial_\mu \partial^\mu + m^2 \right) \phi(x) \\ &= \int \frac{d^4 p}{(2\pi)^4} \left( m^2 - p^2 \right) \tilde{\phi}(p) e^{-ip \cdot x}. \end{aligned}$$

This expression implies the *on-mass-shell* condition for  $\tilde{\phi}(p)$ . More precisely, in order to satisfy the Klein-Gordon equation,  $\tilde{\phi}(p)$  must vanish everywhere except where the four-momentum satisfies  $p^2 = m^2$ . Consequently, the spectrum takes the general form  $\tilde{\phi}(p) = 2\pi \delta(p^2 - m^2) f(p)$ , for some general function  $f(p)$ . The factor of  $2\pi$  ensures consistency with normalization conventions. Given that  $p^2 = (p^0)^2 - \mathbf{p}^2$ , the argument of the Dirac delta function can be rewritten as  $\delta((p^0)^2 - (E_{\mathbf{p}})^2)$ , where  $E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$ . As a result, the Dirac delta function has two roots at  $p^0 = \pm E_{\mathbf{p}}$ , so we can apply the identity:

$$\delta(f(x)) = \sum_i \frac{1}{|f'(x_i)|} \delta(x - x_i)$$

to obtain

$$\tilde{\phi}(p) = \frac{2\pi}{2E_{\mathbf{p}}} \left( \delta(p^0 - E_{\mathbf{p}}) + \delta(p^0 + E_{\mathbf{p}}) \right) f(p).$$

Inserting this expression for  $\tilde{\phi}(p)$  into the above expression for  $\phi(x)$ , and evaluating the  $p^0$  integral, we obtain:

$$\begin{aligned} \phi(x) &= \int \frac{d^4 p}{(2\pi)^4} \frac{2\pi}{2E_{\mathbf{p}}} \left( \delta(p^0 - E_{\mathbf{p}}) + \delta(p^0 + E_{\mathbf{p}}) \right) f(p) e^{-ip \cdot x} \\ &= \int \frac{d^3 p}{(2\pi)^3} \int \frac{dp^0}{2\pi} \frac{2\pi}{2E_{\mathbf{p}}} \left( \delta(p^0 - E_{\mathbf{p}}) f(p) e^{-ip \cdot x} + \delta(p^0 + E_{\mathbf{p}}) f(p) e^{-ip \cdot x} \right) \\ &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \left( f(E_{\mathbf{p}}, \mathbf{p}) e^{-i(E_{\mathbf{p}} t - \mathbf{p} \cdot \mathbf{x})} + f(-E_{\mathbf{p}}, \mathbf{p}) e^{-i(-E_{\mathbf{p}} t - \mathbf{p} \cdot \mathbf{x})} \right) \\ &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \left( f(E_{\mathbf{p}}, \mathbf{p}) e^{-i(E_{\mathbf{p}} t - \mathbf{p} \cdot \mathbf{x})} + f(-E_{\mathbf{p}}, -\mathbf{p}) e^{-i(-E_{\mathbf{p}} t - (-\mathbf{p}) \cdot \mathbf{x})} \right) \\ &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \left( f(E_{\mathbf{p}}, \mathbf{p}) e^{-i(E_{\mathbf{p}} t - \mathbf{p} \cdot \mathbf{x})} + f(-E_{\mathbf{p}}, -\mathbf{p}) e^{i(E_{\mathbf{p}} t - \mathbf{p} \cdot \mathbf{x})} \right) \\ &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left( a_{\mathbf{p}} e^{-ip \cdot x} + b_{\mathbf{p}}^\dagger e^{ip \cdot x} \right), \end{aligned}$$



where

$$a_{\mathbf{p}} = \frac{1}{\sqrt{2E_{\mathbf{p}}}} f(E_{\mathbf{p}}, \mathbf{p})$$

and

$$b_{\mathbf{p}}^\dagger = \frac{1}{\sqrt{2E_{\mathbf{p}}}} f(-E_{\mathbf{p}}, -\mathbf{p}).$$

In the classical theory,  $a_{\mathbf{p}}$  and  $b_{\mathbf{p}}$  are functions that determine the amplitude of positive-frequency and negative-frequency modes, respectively, where  $\dagger$  is just the complex conjugate. In the quantum theory,  $a_{\mathbf{p}}$  and  $b_{\mathbf{p}}$  are promoted to operators that correspond to particles and antiparticles, where  $\dagger$  is the Hermitian adjoint. If  $\phi(x)$  is real-valued (or Hermitian), then  $\phi^*(x) = \phi(x)$ , which implies  $b_{\mathbf{p}} = a_{\mathbf{p}}$  and  $b_{\mathbf{p}}^\dagger = a_{\mathbf{p}}^\dagger$ . Then, the above expression simplifies to the mode expansion for the real Klein-Gordon field.

(a) The conjugate momentum to  $\phi(x)$ , denoted  $\pi(x)$ , is given by the following:

$$\begin{aligned} \pi(x) &= \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \\ &= \frac{\partial}{\partial \dot{\phi}} \left( \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \right) \rightarrow \text{separate components and lower indices} \\ &= \frac{\partial}{\partial (\partial_0 \phi)} \left( \partial_0 \phi^* \partial_0 \phi - \partial_i \phi^* \partial_i \phi - m^2 \phi^* \phi \right) \rightarrow \text{2nd and 3rd terms give 0} \\ &= \frac{\partial (\partial_0 \phi^*)}{\partial (\partial_0 \phi)} \partial_0 \phi + \partial_0 \phi^* \frac{\partial (\partial_0 \phi)}{\partial (\partial_0 \phi)} \rightarrow \text{1st term is zero} \\ &= \partial_0 \phi^*. \end{aligned}$$

The conjugate momentum to  $\phi^*(x)$ , denoted  $\pi^*(x)$ , is determined in a similar manner:

$$\begin{aligned} \pi^*(x) &= \frac{\partial \mathcal{L}}{\partial \dot{\phi}^*} \\ &= \frac{\partial}{\partial \dot{\phi}^*} \left( \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \right) \rightarrow \text{separate components and lower indices} \\ &= \frac{\partial}{\partial (\partial_0 \phi^*)} \left( \partial_0 \phi^* \partial_0 \phi - \partial_i \phi^* \partial_i \phi - m^2 \phi^* \phi \right) \rightarrow \text{2nd and 3rd terms give 0} \\ &= \frac{\partial (\partial_0 \phi^*)}{\partial (\partial_0 \phi^*)} \partial_0 \phi + \partial_0 \phi^* \frac{\partial (\partial_0 \phi)}{\partial (\partial_0 \phi^*)} \rightarrow \text{2nd term is zero} \\ &= \partial_0 \phi. \end{aligned}$$

In order to determine the canonical commutation relations, we make use of the following

commutation relations for the creation and annihilation operators:

$$\begin{aligned}
[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] &= (2\pi)^3 \delta(\mathbf{p} - \mathbf{q}), \quad [b_{\mathbf{p}}, b_{\mathbf{q}}^\dagger] = (2\pi)^3 \delta(\mathbf{p} - \mathbf{q}), \\
[a_{\mathbf{p}}, a_{\mathbf{q}}] &= 0, \quad [b_{\mathbf{p}}, b_{\mathbf{q}}] = 0, \quad [a_{\mathbf{p}}^\dagger, a_{\mathbf{q}}^\dagger] = 0, \quad [b_{\mathbf{p}}^\dagger, b_{\mathbf{q}}^\dagger] = 0, \\
[a_{\mathbf{p}}, b_{\mathbf{q}}] &= 0, \quad [a_{\mathbf{p}}, b_{\mathbf{q}}^\dagger] = 0, \quad [a_{\mathbf{p}}^\dagger, b_{\mathbf{q}}] = 0, \quad [a_{\mathbf{p}}^\dagger, b_{\mathbf{q}}^\dagger] = 0.
\end{aligned}$$

The canonical commutation relations at  $t = 0$  are then:

$$\begin{aligned}
[\phi(\mathbf{x}), \phi(\mathbf{y})] &= \int \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \frac{1}{\sqrt{2E_{\mathbf{k}}}} \left[ a_{\mathbf{p}} e^{i\mathbf{p} \cdot \mathbf{x}} + b_{\mathbf{p}}^\dagger e^{-i\mathbf{p} \cdot \mathbf{x}}, a_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{y}} + b_{\mathbf{k}}^\dagger e^{-i\mathbf{k} \cdot \mathbf{y}} \right] \\
&= \int \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \frac{1}{\sqrt{2E_{\mathbf{k}}}} \\
&\quad \times \left( [a_{\mathbf{p}}, a_{\mathbf{k}}] e^{i\mathbf{p} \cdot \mathbf{x}} e^{i\mathbf{k} \cdot \mathbf{y}} + [a_{\mathbf{p}}, b_{\mathbf{k}}^\dagger] e^{i\mathbf{p} \cdot \mathbf{x}} e^{-i\mathbf{k} \cdot \mathbf{y}} \right. \\
&\quad \left. + [b_{\mathbf{p}}^\dagger, a_{\mathbf{k}}] e^{-i\mathbf{p} \cdot \mathbf{x}} e^{i\mathbf{k} \cdot \mathbf{y}} + [b_{\mathbf{p}}^\dagger, b_{\mathbf{k}}^\dagger] e^{-i\mathbf{p} \cdot \mathbf{x}} e^{-i\mathbf{k} \cdot \mathbf{y}} \right) \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
[\pi(\mathbf{x}), \pi(\mathbf{y})] &= \int \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 k}{(2\pi)^3} \frac{(iE_{\mathbf{p}})}{\sqrt{2E_{\mathbf{p}}}} \frac{(iE_{\mathbf{k}})}{\sqrt{2E_{\mathbf{k}}}} \left[ a_{\mathbf{p}}^\dagger e^{-i\mathbf{p} \cdot \mathbf{x}} - b_{\mathbf{p}} e^{i\mathbf{p} \cdot \mathbf{x}}, a_{\mathbf{k}}^\dagger e^{-i\mathbf{k} \cdot \mathbf{y}} - b_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{y}} \right] \\
&= \int \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 k}{(2\pi)^3} \frac{(iE_{\mathbf{p}})}{\sqrt{2E_{\mathbf{p}}}} \frac{(iE_{\mathbf{k}})}{\sqrt{2E_{\mathbf{k}}}} \\
&\quad \times \left( [a_{\mathbf{p}}^\dagger, a_{\mathbf{k}}^\dagger] e^{-i\mathbf{p} \cdot \mathbf{x}} e^{-i\mathbf{k} \cdot \mathbf{y}} - [a_{\mathbf{p}}^\dagger, b_{\mathbf{k}}] e^{-i\mathbf{p} \cdot \mathbf{x}} e^{i\mathbf{k} \cdot \mathbf{y}} \right. \\
&\quad \left. - [b_{\mathbf{p}}, a_{\mathbf{k}}^\dagger] e^{i\mathbf{p} \cdot \mathbf{x}} e^{-i\mathbf{k} \cdot \mathbf{y}} + [b_{\mathbf{p}}, b_{\mathbf{k}}] e^{i\mathbf{p} \cdot \mathbf{x}} e^{i\mathbf{k} \cdot \mathbf{y}} \right) \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
[\phi(\mathbf{x}), \pi(\mathbf{y})] &= \int \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \frac{(iE_{\mathbf{k}})}{\sqrt{2E_{\mathbf{k}}}} \left[ a_{\mathbf{p}} e^{i\mathbf{p} \cdot \mathbf{x}} + b_{\mathbf{p}}^\dagger e^{-i\mathbf{p} \cdot \mathbf{x}}, a_{\mathbf{k}}^\dagger e^{-i\mathbf{k} \cdot \mathbf{y}} - b_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{y}} \right] \\
&= \int \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \frac{(iE_{\mathbf{k}})}{\sqrt{2E_{\mathbf{k}}}} \\
&\quad \times \left( [a_{\mathbf{p}}, a_{\mathbf{k}}^\dagger] e^{i\mathbf{p} \cdot \mathbf{x}} e^{-i\mathbf{k} \cdot \mathbf{y}} - [a_{\mathbf{p}}, b_{\mathbf{k}}] e^{i\mathbf{p} \cdot \mathbf{x}} e^{i\mathbf{k} \cdot \mathbf{y}} \right. \\
&\quad \left. + [b_{\mathbf{p}}^\dagger, a_{\mathbf{k}}^\dagger] e^{-i\mathbf{p} \cdot \mathbf{x}} e^{-i\mathbf{k} \cdot \mathbf{y}} - [b_{\mathbf{p}}^\dagger, b_{\mathbf{k}}] e^{-i\mathbf{p} \cdot \mathbf{x}} e^{i\mathbf{k} \cdot \mathbf{y}} \right) \\
&= \int \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \frac{(iE_{\mathbf{k}})}{\sqrt{2E_{\mathbf{k}}}}
\end{aligned}$$

$$\begin{aligned}
& \times \left( (2\pi)^3 \delta(\mathbf{p} - \mathbf{k}) e^{i\mathbf{p} \cdot \mathbf{x}} e^{-i\mathbf{k} \cdot \mathbf{y}} - 0 \right. \\
& \left. + 0 + (2\pi)^3 \delta(\mathbf{p} - \mathbf{k}) e^{-i\mathbf{p} \cdot \mathbf{x}} e^{i\mathbf{k} \cdot \mathbf{y}} \right) \\
& = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \frac{(iE_{\mathbf{p}})}{\sqrt{2E_{\mathbf{p}}}} \left( e^{i\mathbf{p} \cdot \mathbf{x}} e^{-i\mathbf{p} \cdot \mathbf{y}} + e^{-i\mathbf{p} \cdot \mathbf{x}} e^{i\mathbf{p} \cdot \mathbf{y}} \right) \\
& = \frac{i}{2} \int \frac{d^3 p}{(2\pi)^3} \left( e^{i\mathbf{p} \cdot (\mathbf{x} - \mathbf{y})} + e^{-i\mathbf{p} \cdot (\mathbf{x} - \mathbf{y})} \right) \rightarrow \mathbf{p} \text{ to } -\mathbf{p} \text{ in 1st term} \\
& = \frac{i}{2} \int \frac{d^3 p}{(2\pi)^3} \left( e^{-i\mathbf{p} \cdot (\mathbf{x} - \mathbf{y})} + e^{-i\mathbf{p} \cdot (\mathbf{x} - \mathbf{y})} \right) \\
& = i \int \frac{d^3 p}{(2\pi)^3} \left( e^{-i\mathbf{p} \cdot (\mathbf{x} - \mathbf{y})} \right) \\
& = i \delta(\mathbf{x} - \mathbf{y}).
\end{aligned}$$

$$\begin{aligned}
[\phi^*(\mathbf{x}), \phi^*(\mathbf{y})] &= \int \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \frac{1}{\sqrt{2E_{\mathbf{k}}}} \left[ a_{\mathbf{p}}^\dagger e^{-i\mathbf{p} \cdot \mathbf{x}} + b_{\mathbf{p}} e^{i\mathbf{p} \cdot \mathbf{x}}, a_{\mathbf{k}}^\dagger e^{-i\mathbf{k} \cdot \mathbf{y}} + b_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{y}} \right] \\
&= \int \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \frac{1}{\sqrt{2E_{\mathbf{k}}}} \\
&\quad \times \left( [a_{\mathbf{p}}^\dagger, a_{\mathbf{k}}^\dagger] e^{-i\mathbf{p} \cdot \mathbf{x}} e^{-i\mathbf{k} \cdot \mathbf{y}} + [a_{\mathbf{p}}^\dagger, b_{\mathbf{k}}] e^{-i\mathbf{p} \cdot \mathbf{x}} e^{i\mathbf{k} \cdot \mathbf{y}} \right. \\
&\quad \left. + [b_{\mathbf{p}}, a_{\mathbf{k}}^\dagger] e^{i\mathbf{p} \cdot \mathbf{x}} e^{-i\mathbf{k} \cdot \mathbf{y}} + [b_{\mathbf{p}}, b_{\mathbf{k}}] e^{i\mathbf{p} \cdot \mathbf{x}} e^{i\mathbf{k} \cdot \mathbf{y}} \right) \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
[\pi^*(\mathbf{x}), \pi^*(\mathbf{y})] &= \int \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 k}{(2\pi)^3} \frac{(iE_{\mathbf{p}})}{\sqrt{2E_{\mathbf{p}}}} \frac{(iE_{\mathbf{k}})}{\sqrt{2E_{\mathbf{k}}}} \left[ b_{\mathbf{p}}^\dagger e^{-i\mathbf{p} \cdot \mathbf{x}} - a_{\mathbf{p}} e^{i\mathbf{p} \cdot \mathbf{x}}, b_{\mathbf{k}}^\dagger e^{-i\mathbf{k} \cdot \mathbf{y}} - a_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{y}} \right] \\
&= \int \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 k}{(2\pi)^3} \frac{(iE_{\mathbf{p}})}{\sqrt{2E_{\mathbf{p}}}} \frac{(iE_{\mathbf{k}})}{\sqrt{2E_{\mathbf{k}}}} \\
&\quad \times \left( [b_{\mathbf{p}}^\dagger, b_{\mathbf{k}}^\dagger] e^{-i\mathbf{p} \cdot \mathbf{x}} e^{-i\mathbf{k} \cdot \mathbf{y}} - [b_{\mathbf{p}}^\dagger, a_{\mathbf{k}}] e^{-i\mathbf{p} \cdot \mathbf{x}} e^{i\mathbf{k} \cdot \mathbf{y}} \right. \\
&\quad \left. - [a_{\mathbf{p}}, b_{\mathbf{k}}^\dagger] e^{i\mathbf{p} \cdot \mathbf{x}} e^{-i\mathbf{k} \cdot \mathbf{y}} + [a_{\mathbf{p}}, a_{\mathbf{k}}] e^{i\mathbf{p} \cdot \mathbf{x}} e^{i\mathbf{k} \cdot \mathbf{y}} \right) \\
&= 0.
\end{aligned}$$

$$[\phi^*(\mathbf{x}), \pi^*(\mathbf{y})] = \int \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \frac{(iE_{\mathbf{k}})}{\sqrt{2E_{\mathbf{k}}}} \left[ a_{\mathbf{p}}^\dagger e^{-i\mathbf{p} \cdot \mathbf{x}} + b_{\mathbf{p}} e^{i\mathbf{p} \cdot \mathbf{x}}, b_{\mathbf{k}}^\dagger e^{-i\mathbf{k} \cdot \mathbf{y}} - a_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{y}} \right]$$

$$\begin{aligned}
&= \int \int \frac{d^3p}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \frac{(iE_{\mathbf{k}})}{\sqrt{2E_{\mathbf{k}}}} \\
&\quad \times \left( [a_{\mathbf{p}}^\dagger, b_{\mathbf{k}}^\dagger] e^{-i\mathbf{p}\cdot\mathbf{x}} e^{-i\mathbf{k}\cdot\mathbf{y}} - [a_{\mathbf{p}}^\dagger, a_{\mathbf{k}}] e^{-i\mathbf{p}\cdot\mathbf{x}} e^{i\mathbf{k}\cdot\mathbf{y}} \right. \\
&\quad \left. + [b_{\mathbf{p}}, b_{\mathbf{k}}^\dagger] e^{i\mathbf{p}\cdot\mathbf{x}} e^{-i\mathbf{k}\cdot\mathbf{y}} - [b_{\mathbf{p}}, a_{\mathbf{k}}] e^{i\mathbf{p}\cdot\mathbf{x}} e^{i\mathbf{k}\cdot\mathbf{y}} \right) \\
&= \int \int \frac{d^3p}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \frac{(iE_{\mathbf{k}})}{\sqrt{2E_{\mathbf{k}}}} \\
&\quad \times \left( 0 + (2\pi)^3 \delta(\mathbf{p} - \mathbf{k}) e^{-i\mathbf{p}\cdot\mathbf{x}} e^{i\mathbf{k}\cdot\mathbf{y}} \right. \\
&\quad \left. + (2\pi)^3 \delta(\mathbf{p} - \mathbf{k}) e^{i\mathbf{p}\cdot\mathbf{x}} e^{-i\mathbf{k}\cdot\mathbf{y}} - 0 \right) \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \frac{(iE_{\mathbf{p}})}{\sqrt{2E_{\mathbf{p}}}} \left( e^{-i\mathbf{p}\cdot\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{y}} + e^{i\mathbf{p}\cdot\mathbf{x}} e^{-i\mathbf{p}\cdot\mathbf{y}} \right) \\
&= \frac{i}{2} \int \frac{d^3p}{(2\pi)^3} \left( e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} + e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \right) \rightarrow \mathbf{p} \text{ to } -\mathbf{p} \text{ in 2nd term} \\
&= \frac{i}{2} \int \frac{d^3p}{(2\pi)^3} \left( e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} + e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \right) \\
&= i \int \frac{d^3p}{(2\pi)^3} \left( e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \right) \\
&= i\delta(\mathbf{x} - \mathbf{y}).
\end{aligned}$$

$$\begin{aligned}
[\phi(\mathbf{x}), \pi^*(\mathbf{y})] &= \int \int \frac{d^3p}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \frac{(iE_{\mathbf{k}})}{\sqrt{2E_{\mathbf{k}}}} \left[ a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} + b_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}, b_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{y}} - a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{y}} \right] \\
&= \int \int \frac{d^3p}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \frac{(iE_{\mathbf{k}})}{\sqrt{2E_{\mathbf{k}}}} \\
&\quad \times \left( [a_{\mathbf{p}}, b_{\mathbf{k}}^\dagger] e^{i\mathbf{p}\cdot\mathbf{x}} e^{-i\mathbf{k}\cdot\mathbf{y}} - [a_{\mathbf{p}}, a_{\mathbf{k}}] e^{i\mathbf{p}\cdot\mathbf{x}} e^{i\mathbf{k}\cdot\mathbf{y}} \right. \\
&\quad \left. + [b_{\mathbf{p}}^\dagger, b_{\mathbf{k}}^\dagger] e^{-i\mathbf{p}\cdot\mathbf{x}} e^{-i\mathbf{k}\cdot\mathbf{y}} - [b_{\mathbf{p}}^\dagger, a_{\mathbf{k}}] e^{-i\mathbf{p}\cdot\mathbf{x}} e^{i\mathbf{k}\cdot\mathbf{y}} \right) \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
[\phi^*(\mathbf{x}), \pi(\mathbf{y})] &= \int \int \frac{d^3p}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \frac{(iE_{\mathbf{k}})}{\sqrt{2E_{\mathbf{k}}}} \left[ a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}} + b_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}}, a_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{y}} - b_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{y}} \right] \\
&= \int \int \frac{d^3p}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \frac{(iE_{\mathbf{k}})}{\sqrt{2E_{\mathbf{k}}}}
\end{aligned}$$

$$\begin{aligned}
& \times \left( [a_{\mathbf{p}}^\dagger, a_{\mathbf{k}}^\dagger] e^{-i\mathbf{p}\cdot\mathbf{x}} e^{-i\mathbf{k}\cdot\mathbf{y}} - [a_{\mathbf{p}}^\dagger, b_{\mathbf{k}}] e^{-i\mathbf{p}\cdot\mathbf{x}} e^{i\mathbf{k}\cdot\mathbf{y}} \right. \\
& \left. + [b_{\mathbf{p}}, a_{\mathbf{k}}^\dagger] e^{i\mathbf{p}\cdot\mathbf{x}} e^{-i\mathbf{k}\cdot\mathbf{y}} - [b_{\mathbf{p}}, b_{\mathbf{k}}] e^{i\mathbf{p}\cdot\mathbf{x}} e^{i\mathbf{k}\cdot\mathbf{y}} \right) \\
& = 0.
\end{aligned}$$

$$\begin{aligned}
[\phi(\mathbf{x}), \phi^*(\mathbf{y})] &= \int \int \frac{d^3p}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \frac{1}{\sqrt{2E_{\mathbf{k}}}} \left[ a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} + b_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}, a_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{y}} + b_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{y}} \right] \\
&= \int \int \frac{d^3p}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \frac{1}{\sqrt{2E_{\mathbf{k}}}} \\
&\quad \times \left( [a_{\mathbf{p}}, a_{\mathbf{k}}^\dagger] e^{i\mathbf{p}\cdot\mathbf{x}} e^{-i\mathbf{k}\cdot\mathbf{y}} + [a_{\mathbf{p}}, b_{\mathbf{k}}] e^{i\mathbf{p}\cdot\mathbf{x}} e^{i\mathbf{k}\cdot\mathbf{y}} \right. \\
&\quad \left. + [b_{\mathbf{p}}^\dagger, a_{\mathbf{k}}^\dagger] e^{-i\mathbf{p}\cdot\mathbf{x}} e^{-i\mathbf{k}\cdot\mathbf{y}} + [b_{\mathbf{p}}^\dagger, b_{\mathbf{k}}] e^{-i\mathbf{p}\cdot\mathbf{x}} e^{i\mathbf{k}\cdot\mathbf{y}} \right) \\
&= \int \int \frac{d^3p}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \frac{1}{\sqrt{2E_{\mathbf{k}}}} \\
&\quad \times \left( (2\pi)^3 \delta(\mathbf{p} - \mathbf{k}) e^{i\mathbf{p}\cdot\mathbf{x}} e^{-i\mathbf{k}\cdot\mathbf{y}} + 0 \right. \\
&\quad \left. + 0 - (2\pi)^3 \delta(\mathbf{p} - \mathbf{k}) e^{-i\mathbf{p}\cdot\mathbf{x}} e^{i\mathbf{k}\cdot\mathbf{y}} \right) \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left( e^{i\mathbf{p}\cdot\mathbf{x}} e^{-i\mathbf{p}\cdot\mathbf{y}} - e^{-i\mathbf{p}\cdot\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{y}} \right) \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \left( e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} - e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \right) \rightarrow \mathbf{p} \text{ to } -\mathbf{p} \text{ in 1st term} \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \left( e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} - e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \right) \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
[\pi(\mathbf{x}), \pi^*(\mathbf{y})] &= \int \int \frac{d^3p}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \frac{(iE_{\mathbf{p}})}{\sqrt{2E_{\mathbf{p}}}} \frac{(iE_{\mathbf{k}})}{\sqrt{2E_{\mathbf{k}}}} \left[ a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}} - b_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}}, b_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{y}} - a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{y}} \right] \\
&= \int \int \frac{d^3p}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \frac{(iE_{\mathbf{p}})}{\sqrt{2E_{\mathbf{p}}}} \frac{(iE_{\mathbf{k}})}{\sqrt{2E_{\mathbf{k}}}} \\
&\quad \times \left( [a_{\mathbf{p}}^\dagger, b_{\mathbf{k}}^\dagger] e^{-i\mathbf{p}\cdot\mathbf{x}} e^{-i\mathbf{k}\cdot\mathbf{y}} - [a_{\mathbf{p}}^\dagger, a_{\mathbf{k}}] e^{-i\mathbf{p}\cdot\mathbf{x}} e^{i\mathbf{k}\cdot\mathbf{y}} \right. \\
&\quad \left. - [b_{\mathbf{p}}, b_{\mathbf{k}}^\dagger] e^{i\mathbf{p}\cdot\mathbf{x}} e^{-i\mathbf{k}\cdot\mathbf{y}} + [b_{\mathbf{p}}, a_{\mathbf{k}}] e^{i\mathbf{p}\cdot\mathbf{x}} e^{i\mathbf{k}\cdot\mathbf{y}} \right)
\end{aligned}$$

$$\begin{aligned}
&= \int \int \frac{d^3p}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \frac{(iE_{\mathbf{p}})}{\sqrt{2E_{\mathbf{p}}}} \frac{(iE_{\mathbf{k}})}{\sqrt{2E_{\mathbf{k}}}} \\
&\quad \times \left( 0 + (2\pi)^3 \delta(\mathbf{p} - \mathbf{k}) e^{-i\mathbf{p} \cdot \mathbf{x}} e^{i\mathbf{k} \cdot \mathbf{y}} \right. \\
&\quad \left. - (2\pi)^3 \delta(\mathbf{p} - \mathbf{k}) e^{i\mathbf{p} \cdot \mathbf{x}} e^{-i\mathbf{k} \cdot \mathbf{y}} + 0 \right) \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{(iE_{\mathbf{p}})}{\sqrt{2E_{\mathbf{p}}}} \frac{(iE_{\mathbf{p}})}{\sqrt{2E_{\mathbf{p}}}} \left( e^{-i\mathbf{p} \cdot \mathbf{x}} e^{i\mathbf{p} \cdot \mathbf{y}} - e^{i\mathbf{p} \cdot \mathbf{x}} e^{-i\mathbf{p} \cdot \mathbf{y}} \right) \\
&= -\frac{1}{2} \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \left( e^{-i\mathbf{p} \cdot (\mathbf{x} - \mathbf{y})} - e^{i\mathbf{p} \cdot (\mathbf{x} - \mathbf{y})} \right) \rightarrow \mathbf{p} \text{ to } -\mathbf{p} \text{ in 2nd term} \\
&= -\frac{1}{2} \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \left( e^{-i\mathbf{p} \cdot (\mathbf{x} - \mathbf{y})} - e^{-i\mathbf{p} \cdot (\mathbf{x} - \mathbf{y})} \right) \\
&= 0.
\end{aligned}$$

Note that it is sufficient to impose the canonical commutation relations at  $t = 0$ , as they are automatically satisfied at a later time,  $t$ . This can be shown by making  $\phi$  and  $\pi$  time-dependent in the Heisenberg picture. The key idea here is outlined as follows:

$$[A(\mathbf{x}, t), B(\mathbf{y}, t)] = [e^{iHt} A(\mathbf{x}, 0) e^{-iHt}, e^{iHt} B(\mathbf{y}, 0) e^{-iHt}] = e^{iHt} [A(\mathbf{x}, 0), B(\mathbf{y}, 0)] e^{-iHt}.$$

The Hamiltonian is:

$$\begin{aligned}
H &= \int d^3x \left( \sum_i \pi_i(\mathbf{x}) \dot{\phi}_i(\mathbf{x}) - \mathcal{L} \right) \\
H &= \int d^3x \left( \pi(\mathbf{x}) \dot{\phi}(\mathbf{x}) + \pi^*(\mathbf{x}) \dot{\phi}^*(\mathbf{x}) - \partial_\mu \phi^* \partial^\mu \phi + m^2 \phi^* \phi \right) \\
&= \int d^3x \left( \pi(\mathbf{x}) \pi^*(\mathbf{x}) + \pi^*(\mathbf{x}) \pi(\mathbf{x}) - \partial_\mu \phi^* \partial^\mu \phi + m^2 \phi^* \phi \right) \\
&= \int d^3x \left( \pi(\mathbf{x}) \pi^*(\mathbf{x}) + \pi^*(\mathbf{x}) \pi(\mathbf{x}) - \partial_0 \phi^* \partial^0 \phi + \partial_i \phi^* \partial^i \phi + m^2 \phi^* \phi \right) \\
&= \int d^3x \left( \pi(\mathbf{x}) \pi^*(\mathbf{x}) + \pi^*(\mathbf{x}) \pi(\mathbf{x}) - \pi(\mathbf{x}) \pi^*(\mathbf{x}) + \partial_i \phi^* \partial^i \phi + m^2 \phi^* \phi \right) \\
&= \int d^3x \left( \pi^*(\mathbf{x}) \pi(\mathbf{x}) + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi \right).
\end{aligned}$$

The Heisenberg equation of motion for  $\phi(x)$  is:

$$\begin{aligned}
i \frac{\partial}{\partial t} \mathcal{O} &= [\mathcal{O}, H] \\
i \frac{\partial}{\partial t} \phi(x) &= \int d^3x \left[ \phi(x), \left( \pi^*(y) \pi(y) + \nabla \phi^*(y) \cdot \nabla \phi(y) + m^2 \phi^*(y) \phi(y) \right) \right]
\end{aligned}$$

$$\begin{aligned}
&= \int d^3x \left( [\phi(x), \pi^*(y)\pi(y)] + [\phi(x), \partial_i \phi^*(y) \partial^i \phi(y)] + m^2 [\phi(x), \phi^*(y)\phi(y)] \right) \\
&= \int d^3x \left( [\phi(x), \pi^*(y)]\pi(y) + \pi^*(y)[\phi(x), \pi(y)] + [\phi(x), \partial_i \phi^*(y)] \partial^i \phi(y) \right. \\
&\quad \left. + \partial_i \phi^*(y) [\phi(x), \partial^i \phi(y)] + m^2 [\phi(x), \phi^*(y)]\phi(y) + m^2 \phi^*(y) [\phi(x), \phi(y)] \right) \\
&= \int d^3x \left( 0 + \pi^*(y)(i\delta(\mathbf{x} - \mathbf{y})) + 0 + 0 + 0 + 0 \right) \\
&= i\pi^*(x),
\end{aligned}$$

which is just the conjugate momentum expression that was previously calculated. The Heisenberg equation of motion for  $\pi(x)$  is:

$$\begin{aligned}
i\frac{\partial}{\partial t}\mathcal{O} &= [\mathcal{O}, H] \\
i\frac{\partial}{\partial t}\pi^*(x) &= \left[ \pi^*(x), \int d^3y \left( \pi^*(y)\pi(y) + \nabla\phi^*(y) \cdot \nabla\phi(y) + m^2\phi^*(y)\phi(y) \right) \right] \\
&= \int d^3y [\pi^*(x), \pi^*(y)\pi(y)] \\
&\quad + \left[ \pi^*(x), \int d^3y \nabla\phi^*(y) \cdot \nabla\phi(y) \right] \\
&\quad + m^2 \int d^3y [\pi^*(x), \phi^*(y)\phi(y)] \\
&= \int d^3y \left( [\pi^*(x), \pi^*(y)]\pi(y) + \pi^*(y)[\pi^*(x), \pi(y)] \right) \\
&\quad + \left[ \pi^*(x), \int d^3y \left( \nabla \cdot (\phi^*(y)\nabla\phi(y)) - \phi^*(y)\nabla^2\phi(y) \right) \right] \\
&\quad + m^2 \int d^3y \left( [\pi^*(x), \phi^*(y)]\phi(y) + \phi^*(y)[\pi^*(x), \phi(y)] \right) \\
&= 0 + 0 \\
&\quad + \left[ \pi^*(x), \int dS \left( (\phi^*(y)\nabla\phi(y)) \cdot \hat{\mathbf{n}} \right) - \int d^3y \phi^*(y)\nabla^2\phi(y) \right] \\
&\quad + m^2 \int d^3y \left( -i\delta(\mathbf{x} - \mathbf{y})\phi(y) \right) + 0 \\
&= 0 - \int d^3y [\pi^*(x), \phi^*(y)\nabla^2\phi(y)] - im^2\phi(x) \\
&= - \int d^3y \left( \phi^*(y) [\pi^*(x), \nabla^2\phi(y)] + [\pi^*(x), \phi^*(y)] \nabla^2\phi(y) \right) - im^2\phi(x)
\end{aligned}$$

$$\begin{aligned}
&= - \int d^3y \left( \phi^*(y) (\nabla^2 [\phi^*(x), \phi(y)]) - i\delta(\mathbf{x} - \mathbf{y}) \nabla^2 \phi(y) \right) - im^2 \phi(x) \\
&= \int d^3y \left( i\delta(\mathbf{x} - \mathbf{y}) \nabla^2 \phi(y) \right) - im^2 \phi(x) \\
&= i(\nabla^2 \phi(x) - m^2 \phi(x)) \\
&= i(\nabla^2 - m^2) \phi(x),
\end{aligned}$$

where the vector identity  $\nabla \cdot (A \nabla B) = A \nabla^2 B + \nabla A \cdot \nabla B$  and Gauss' theorem were used to handle the gradient terms (the surface term is zero). We can show these equations, when combined, are indeed the Klein-Gordon equation as follows:

$$\begin{aligned}
i \frac{\partial}{\partial t} \pi^*(x) &= i(\nabla^2 - m^2) \phi(x) \\
\frac{\partial}{\partial t} \left( \frac{\partial}{\partial t} \phi(x) \right) &= (\nabla^2 - m^2) \phi(x) \\
\frac{\partial^2}{\partial t^2} \phi(x) &= (\nabla^2 - m^2) \phi(x) \\
\partial_0 \partial^0 \phi(x) &= \partial_i \partial^i \phi(x) - m^2 \phi(x) \\
\partial_0 \partial^0 \phi(x) - \partial_i \partial^i \phi(x) + m^2 \phi(x) &= 0 \\
\partial_\mu \partial^\mu \phi(x) + m^2 \phi(x) &= 0 \\
\left( \partial_\mu \partial^\mu + m^2 \right) \phi(x) &= 0 \rightarrow \text{Klein-Gordon equation.}
\end{aligned}$$

The Klein-Gordon equation for  $\phi^*(x)$  can be found in the same manner.

(b) The Hamiltonian,  $H$ , can be rewritten in terms of creation and annihilation operators as follows:

$$\begin{aligned}
H &= \int d^3x \left( \pi^*(\mathbf{x}) \pi(\mathbf{x}) + \nabla \phi^*(\mathbf{x}) \cdot \nabla \phi(\mathbf{x}) + m^2 \phi^*(\mathbf{x}) \phi(\mathbf{x}) \right) \\
&= \int d^3x \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3k}{(2\pi)^3} \\
&\quad \times \left( \frac{iE_{\mathbf{p}}}{\sqrt{2E_{\mathbf{p}}}} \frac{iE_{\mathbf{k}}}{\sqrt{2E_{\mathbf{k}}}} (b_{\mathbf{p}}^\dagger e^{ip \cdot x} - a_{\mathbf{p}} e^{-ip \cdot x}) (a_{\mathbf{k}}^\dagger e^{ik \cdot x} - b_{\mathbf{k}} e^{-ik \cdot x}) \right. \\
&\quad + \frac{(-i\mathbf{p})}{\sqrt{2E_{\mathbf{p}}}} \cdot \frac{(i\mathbf{k})}{\sqrt{2E_{\mathbf{k}}}} (a_{\mathbf{p}}^\dagger e^{ip \cdot x} - b_{\mathbf{p}} e^{-ip \cdot x}) (a_{\mathbf{k}} e^{-ik \cdot x} - b_{\mathbf{k}}^\dagger e^{ik \cdot x}) \\
&\quad \left. + m^2 \frac{1}{\sqrt{2E_{\mathbf{p}}}} \frac{1}{\sqrt{2E_{\mathbf{k}}}} (a_{\mathbf{p}}^\dagger e^{ip \cdot x} + b_{\mathbf{p}} e^{-ip \cdot x}) (a_{\mathbf{k}} e^{-ik \cdot x} + b_{\mathbf{k}}^\dagger e^{ik \cdot x}) \right)
\end{aligned}$$



$$\begin{aligned}
&= \int d^3x \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \frac{1}{\sqrt{2E_{\mathbf{k}}}} \\
&\quad \times \left( -E_{\mathbf{p}}E_{\mathbf{k}}(b_{\mathbf{p}}^{\dagger}a_{\mathbf{k}}^{\dagger}e^{i(p+k)\cdot x} - b_{\mathbf{p}}^{\dagger}b_{\mathbf{k}}e^{i(p-k)\cdot x} - a_{\mathbf{p}}a_{\mathbf{k}}^{\dagger}e^{-i(p-k)\cdot x} + a_{\mathbf{p}}b_{\mathbf{k}}e^{-i(p+k)\cdot x}) \right. \\
&\quad + \mathbf{p} \cdot \mathbf{k} (a_{\mathbf{p}}^{\dagger}a_{\mathbf{k}}e^{i(p-k)\cdot x} - a_{\mathbf{p}}^{\dagger}b_{\mathbf{k}}^{\dagger}e^{i(p+k)\cdot x} - b_{\mathbf{p}}a_{\mathbf{k}}e^{-i(p+k)\cdot x} + b_{\mathbf{p}}b_{\mathbf{k}}^{\dagger}e^{-i(p-k)\cdot x}) \\
&\quad \left. + m^2(a_{\mathbf{p}}^{\dagger}a_{\mathbf{k}}e^{i(p-k)\cdot x} + a_{\mathbf{p}}^{\dagger}b_{\mathbf{k}}^{\dagger}e^{i(p+k)\cdot x} + b_{\mathbf{p}}a_{\mathbf{k}}e^{-i(p+k)\cdot x} + b_{\mathbf{p}}b_{\mathbf{k}}^{\dagger}e^{-i(p-k)\cdot x}) \right) \\
&= \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \frac{1}{\sqrt{2E_{\mathbf{k}}}} (2\pi)^3 \\
&\quad \times \left( -E_{\mathbf{p}}E_{\mathbf{k}}(b_{\mathbf{p}}^{\dagger}a_{\mathbf{k}}^{\dagger}\delta(\mathbf{p}+\mathbf{k}) - b_{\mathbf{p}}^{\dagger}b_{\mathbf{k}}\delta(\mathbf{p}-\mathbf{k}) - a_{\mathbf{p}}a_{\mathbf{k}}^{\dagger}\delta(\mathbf{p}-\mathbf{k}) + a_{\mathbf{p}}b_{\mathbf{k}}\delta(\mathbf{p}+\mathbf{k})) \right. \\
&\quad + \mathbf{p} \cdot \mathbf{k} (a_{\mathbf{p}}^{\dagger}a_{\mathbf{k}}\delta(\mathbf{p}-\mathbf{k}) - a_{\mathbf{p}}^{\dagger}b_{\mathbf{k}}^{\dagger}\delta(\mathbf{p}+\mathbf{k}) - b_{\mathbf{p}}a_{\mathbf{k}}\delta(\mathbf{p}+\mathbf{k}) + b_{\mathbf{p}}b_{\mathbf{k}}^{\dagger}\delta(\mathbf{p}-\mathbf{k})) \\
&\quad \left. + m^2(a_{\mathbf{p}}^{\dagger}a_{\mathbf{k}}\delta(\mathbf{p}-\mathbf{k}) + a_{\mathbf{p}}^{\dagger}b_{\mathbf{k}}^{\dagger}\delta(\mathbf{p}+\mathbf{k}) + b_{\mathbf{p}}a_{\mathbf{k}}\delta(\mathbf{p}+\mathbf{k}) + b_{\mathbf{p}}b_{\mathbf{k}}^{\dagger}\delta(\mathbf{p}-\mathbf{k})) \right) \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \\
&\quad \times \left( -E_{\mathbf{p}}^2b_{\mathbf{p}}^{\dagger}a_{-\mathbf{p}}^{\dagger} + E_{\mathbf{p}}^2b_{\mathbf{p}}^{\dagger}b_{\mathbf{p}} + E_{\mathbf{p}}^2a_{\mathbf{p}}a_{\mathbf{p}}^{\dagger} - E_{\mathbf{p}}^2a_{\mathbf{p}}b_{-\mathbf{p}} \right. \\
&\quad + \mathbf{p}^2a_{\mathbf{p}}^{\dagger}a_{\mathbf{p}} + \mathbf{p}^2a_{\mathbf{p}}^{\dagger}b_{-\mathbf{p}}^{\dagger} + \mathbf{p}^2b_{\mathbf{p}}a_{-\mathbf{p}} + \mathbf{p}^2b_{\mathbf{p}}b_{\mathbf{p}}^{\dagger} \\
&\quad \left. + m^2a_{\mathbf{p}}^{\dagger}a_{\mathbf{p}} + m^2a_{\mathbf{p}}^{\dagger}b_{-\mathbf{p}}^{\dagger} + m^2b_{\mathbf{p}}a_{-\mathbf{p}} + m^2b_{\mathbf{p}}b_{\mathbf{p}}^{\dagger} \right) \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \\
&\quad \times \left( (-E_{\mathbf{p}}^2 + \mathbf{p}^2 + m^2)b_{\mathbf{p}}^{\dagger}a_{-\mathbf{p}}^{\dagger} + (-E_{\mathbf{p}}^2 + \mathbf{p}^2 + m^2)a_{\mathbf{p}}b_{-\mathbf{p}} \right. \\
&\quad \left. + E_{\mathbf{p}}^2a_{\mathbf{p}}a_{\mathbf{p}}^{\dagger} + \mathbf{p}^2a_{\mathbf{p}}^{\dagger}a_{\mathbf{p}} + m^2a_{\mathbf{p}}^{\dagger}a_{\mathbf{p}} + E_{\mathbf{p}}^2b_{\mathbf{p}}^{\dagger}b_{\mathbf{p}} + \mathbf{p}^2b_{\mathbf{p}}b_{\mathbf{p}}^{\dagger} + m^2b_{\mathbf{p}}b_{\mathbf{p}}^{\dagger} \right) \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \left( E_{\mathbf{p}}^2(a_{\mathbf{p}}^{\dagger}a_{\mathbf{p}} + (2\pi)^3\delta(0)) + \mathbf{p}^2a_{\mathbf{p}}^{\dagger}a_{\mathbf{p}} + m^2a_{\mathbf{p}}^{\dagger}a_{\mathbf{p}} \right. \\
&\quad \left. + E_{\mathbf{p}}^2b_{\mathbf{p}}^{\dagger}b_{\mathbf{p}} + \mathbf{p}^2(b_{\mathbf{p}}^{\dagger}b_{\mathbf{p}} + (2\pi)^3\delta(0)) + m^2(b_{\mathbf{p}}^{\dagger}b_{\mathbf{p}} + (2\pi)^3\delta(0)) \right) \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \left( (E_{\mathbf{p}}^2 + \mathbf{p}^2 + m^2)a_{\mathbf{p}}^{\dagger}a_{\mathbf{p}} + E_{\mathbf{p}}^2(2\pi)^3\delta(0) \right)
\end{aligned}$$

$$\begin{aligned}
& + (E_{\mathbf{p}}^2 + \mathbf{p}^2 + m^2) b_{\mathbf{p}}^\dagger b_{\mathbf{p}} + \mathbf{p}^2 (2\pi)^3 \delta(0) + m^2 (2\pi)^3 \delta(0) \Big) \\
& = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \left( 2E_{\mathbf{p}}^2 a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + 2E_{\mathbf{p}}^2 b_{\mathbf{p}}^\dagger b_{\mathbf{p}} + 2E_{\mathbf{p}}^2 (2\pi)^3 \delta(0) \right) \\
& = \int \frac{d^3 p}{(2\pi)^3} E_{\mathbf{p}} \left( a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + b_{\mathbf{p}}^\dagger b_{\mathbf{p}} + (2\pi)^3 \delta(0) \right) \\
& = \int \frac{d^3 p}{(2\pi)^3} E_{\mathbf{p}} \left( a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + b_{\mathbf{p}}^\dagger b_{\mathbf{p}} \right),
\end{aligned}$$

where the  $\delta(0)$  term was dropped in the last step. To show that this theory contains two sets of particles with mass  $m$ , we can interpret the Hamiltonian in terms of its action on the one-particle states  $|\mathbf{p}\rangle_a = a_{\mathbf{p}}^\dagger |0\rangle$  and  $|\mathbf{p}\rangle_b = b_{\mathbf{p}}^\dagger |0\rangle$ , where the state  $|0\rangle$  is the ground state, as follows:

$$\begin{aligned}
H |\mathbf{p}\rangle_a &= \int \frac{d^3 q}{(2\pi)^3} E_{\mathbf{q}} \left( a_{\mathbf{q}}^\dagger a_{\mathbf{q}} + b_{\mathbf{q}}^\dagger b_{\mathbf{q}} \right) a_{\mathbf{p}}^\dagger |0\rangle \\
&= \int \frac{d^3 q}{(2\pi)^3} E_{\mathbf{q}} \left( a_{\mathbf{q}}^\dagger a_{\mathbf{q}} a_{\mathbf{p}}^\dagger |0\rangle + b_{\mathbf{q}}^\dagger b_{\mathbf{q}} a_{\mathbf{p}}^\dagger |0\rangle \right) \\
&= \int \frac{d^3 q}{(2\pi)^3} E_{\mathbf{q}} \left( (2\pi)^3 \delta(\mathbf{q} - \mathbf{p}) a_{\mathbf{p}}^\dagger |0\rangle + b_{\mathbf{q}}^\dagger a_{\mathbf{p}}^\dagger b_{\mathbf{q}} |0\rangle \right) \\
&= E_{\mathbf{p}} a_{\mathbf{p}}^\dagger |0\rangle \\
&= E_{\mathbf{p}} |\mathbf{p}\rangle_a.
\end{aligned}$$

Similarly,

$$H |\mathbf{p}\rangle_b = E_{\mathbf{p}} |\mathbf{p}\rangle_b.$$

Therefore, the Hamiltonian describes two sets of particles: one created by  $a_{\mathbf{p}}^\dagger$  and the other by  $b_{\mathbf{p}}^\dagger$ , each with momentum  $\mathbf{p}$  and energy  $E_{\mathbf{p}}$ . Since  $E_{\mathbf{p}}^2 = p^2 + m^2$ , both particles have the same mass  $m$ .

(c) The conserved charge,  $Q$ , can be rewritten in terms of creation and annihilation operators as follows:

$$\begin{aligned}
Q &= \int d^3 x \frac{i}{2} \left( \phi^*(\mathbf{x}) \pi^*(\mathbf{x}) - \pi(\mathbf{x}) \phi(\mathbf{x}) \right) \\
&= \int d^3 x \int \frac{d^3 p}{(2\pi)^3} \int \frac{d^3 k}{(2\pi)^3} \frac{i}{2} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \frac{1}{\sqrt{2E_{\mathbf{k}}}} \\
&\quad \times \left( (iE_{\mathbf{k}}) (a_{\mathbf{p}}^\dagger e^{ip \cdot x} + b_{\mathbf{p}} e^{-ip \cdot x}) (b_{\mathbf{k}}^\dagger e^{ik \cdot x} - a_{\mathbf{k}} e^{-ik \cdot x}) \right)
\end{aligned}$$

$$\begin{aligned}
& - (iE_{\mathbf{p}})(a_{\mathbf{p}}^{\dagger}e^{ip \cdot x} - b_{\mathbf{p}}e^{-ip \cdot x})(a_{\mathbf{k}}e^{-ik \cdot x} + b_{\mathbf{k}}^{\dagger}e^{ik \cdot x}) \\
& = \int d^3x \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3k}{(2\pi)^3} \frac{i}{2} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \frac{1}{\sqrt{2E_{\mathbf{k}}}} \\
& \quad \times \left( (iE_{\mathbf{k}})(a_{\mathbf{p}}^{\dagger}b_{\mathbf{k}}^{\dagger}e^{ip \cdot x}e^{ik \cdot x} - a_{\mathbf{p}}^{\dagger}a_{\mathbf{k}}e^{ip \cdot x}e^{-ik \cdot x} + b_{\mathbf{p}}b_{\mathbf{k}}^{\dagger}e^{-ip \cdot x}e^{ik \cdot x} - b_{\mathbf{p}}a_{\mathbf{k}}e^{-ip \cdot x}e^{-ik \cdot x}) \right. \\
& \quad \left. - (iE_{\mathbf{p}})(a_{\mathbf{p}}^{\dagger}a_{\mathbf{k}}e^{ip \cdot x}e^{-ik \cdot x} + a_{\mathbf{p}}^{\dagger}b_{\mathbf{k}}^{\dagger}e^{ip \cdot x}e^{ik \cdot x} - b_{\mathbf{p}}a_{\mathbf{k}}e^{-ip \cdot x}e^{-ik \cdot x} - b_{\mathbf{p}}b_{\mathbf{k}}^{\dagger}e^{-ip \cdot x}e^{ik \cdot x}) \right) \\
& = \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3k}{(2\pi)^3} \frac{i}{2} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \frac{1}{\sqrt{2E_{\mathbf{k}}}} \\
& \quad \times \left( (iE_{\mathbf{k}})(a_{\mathbf{p}}^{\dagger}b_{\mathbf{k}}^{\dagger}\delta(\mathbf{p} + \mathbf{k}) - a_{\mathbf{p}}^{\dagger}a_{\mathbf{k}}\delta(\mathbf{p} - \mathbf{k}) + b_{\mathbf{p}}b_{\mathbf{k}}^{\dagger}\delta(\mathbf{p} - \mathbf{k}) - b_{\mathbf{p}}a_{\mathbf{k}}\delta(\mathbf{p} + \mathbf{k})) \right. \\
& \quad \left. - (iE_{\mathbf{p}})(a_{\mathbf{p}}^{\dagger}a_{\mathbf{k}}\delta(\mathbf{p} - \mathbf{k}) + a_{\mathbf{p}}^{\dagger}b_{\mathbf{k}}^{\dagger}\delta(\mathbf{p} + \mathbf{k}) - b_{\mathbf{p}}a_{\mathbf{k}}\delta(\mathbf{p} + \mathbf{k}) - b_{\mathbf{p}}b_{\mathbf{k}}^{\dagger}\delta(\mathbf{p} - \mathbf{k})) \right) \\
& = \int \frac{d^3p}{(2\pi)^3} \frac{i}{2} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \\
& \quad \times \left( (iE_{\mathbf{p}})(a_{\mathbf{p}}^{\dagger}b_{-\mathbf{p}}^{\dagger} - a_{\mathbf{p}}^{\dagger}a_{\mathbf{p}} + b_{\mathbf{p}}b_{\mathbf{p}}^{\dagger} - b_{\mathbf{p}}a_{-\mathbf{p}}) \right. \\
& \quad \left. - (iE_{\mathbf{p}})(a_{\mathbf{p}}^{\dagger}a_{\mathbf{p}} + a_{\mathbf{p}}^{\dagger}b_{-\mathbf{p}}^{\dagger} - b_{\mathbf{p}}a_{-\mathbf{p}} - b_{\mathbf{p}}b_{\mathbf{p}}^{\dagger}) \right) \\
& = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} (a_{\mathbf{p}}^{\dagger}a_{\mathbf{p}} - b_{\mathbf{p}}b_{\mathbf{p}}^{\dagger}) \\
& = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} (a_{\mathbf{p}}^{\dagger}a_{\mathbf{p}} - b_{\mathbf{p}}^{\dagger}b_{\mathbf{p}} - (2\pi)^3\delta(0)) \\
& = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} (a_{\mathbf{p}}^{\dagger}a_{\mathbf{p}} - b_{\mathbf{p}}^{\dagger}b_{\mathbf{p}}),
\end{aligned}$$

where the  $\delta(0)$  term was dropped in the last step. To evaluate the charge carried by each particle type, we can interpret the operator  $Q$  in terms of its action on the one-particle states  $|\mathbf{p}\rangle_a = a_{\mathbf{p}}^{\dagger}|0\rangle$  and  $|\mathbf{p}\rangle_b = b_{\mathbf{p}}^{\dagger}|0\rangle$ , where the state  $|0\rangle$  is the ground state, as follows:

$$\begin{aligned}
Q|\mathbf{p}\rangle_a &= \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} (a_{\mathbf{q}}^{\dagger}a_{\mathbf{q}} - b_{\mathbf{q}}^{\dagger}b_{\mathbf{q}})a_{\mathbf{p}}^{\dagger}|0\rangle \\
&= \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} (a_{\mathbf{q}}^{\dagger}a_{\mathbf{q}}a_{\mathbf{p}}^{\dagger}|0\rangle - b_{\mathbf{q}}^{\dagger}b_{\mathbf{q}}a_{\mathbf{p}}^{\dagger}|0\rangle) \\
&= \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} ((2\pi)^3\delta(\mathbf{q} - \mathbf{p})a_{\mathbf{p}}^{\dagger}|0\rangle - b_{\mathbf{q}}^{\dagger}a_{\mathbf{p}}^{\dagger}b_{\mathbf{q}}|0\rangle)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} a_{\mathbf{p}}^\dagger |0\rangle \\
&= \frac{1}{2} |\mathbf{p}\rangle_a.
\end{aligned}$$

Similarly,

$$H |\mathbf{p}\rangle_b = -\frac{1}{2} |\mathbf{p}\rangle_b.$$

Therefore, the particles created by  $a_{\mathbf{p}}^\dagger$  carry a charge of  $+1/2$ , while the particles created by  $b_{\mathbf{p}}^\dagger$  carry a charge of  $-1/2$ .

(d) For the case of two complex Klein-Gordon fields with the same mass, labeled as  $\phi_a(x)$ , where  $a = 1, 2$ , the action can be written as:

$$S = \int d^4x \left( \partial_\mu \phi_a^* \partial^\mu \phi_a - m^2 \phi_a^* \phi_a \right).$$

To identify the conserved charges, we first need to determine the continuous transformations that leave the Lagrangian invariant. These transformations are of the form  $\phi_a \rightarrow U_{ab} \phi_b$ , where  $U_{ab}$  are components of a linear operator. Since we have two fields, transformations satisfying these conditions are unitary transformations described by the unitary group  $U(2)$ , which is the set of all  $2 \times 2$  unitary matrices. This set forms a non-Abelian group with the group operation being matrix multiplication. It is important to note that  $U_{ab}$  are the elements of these matrices. To analyze the symmetry structure of this theory, we make use of the generators associated with the  $U(1)$  and  $SU(2)$  subgroups of  $U(2)$ . The  $U(1)$  subgroup has one generator, denoted  $T_0$ , given by the identity matrix multiplied by  $i$ , i.e.,  $T_0 = i\mathbb{I}$ . The Lagrangian is invariant under the transformation:

$$\phi_a \rightarrow (e^{-\alpha T_0})_{ab} \phi_b,$$

where  $\alpha$  is an infinitesimal parameter. Furthermore, this transformation gives the following infinitesimal change:

$$(e^{-\alpha T_0})_{ab} \phi_b = (\mathbb{I} - i\alpha \mathbb{I})_{ab} \phi_b + \mathcal{O}(\alpha^2) \approx \phi_a - i\alpha \phi_a.$$

For  $U(1)$ , we have the following Noether current:

$$\begin{aligned}
j^\mu(x) &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \Delta \phi_a + \Delta \phi_a^* \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a^*)} \\
&= (\partial_\mu \phi_a^*) (-i\alpha \phi_a) + (i\alpha \phi_a^*) (\partial_\mu \phi_a) \\
&= i \left( \phi_a^* (\partial_\mu \phi_a) - (\partial_\mu \phi_a^*) \phi_a \right),
\end{aligned}$$

where we have arbitrarily chosen  $\alpha$ . Accordingly, the single charge from this symmetry, which

is the generalization of part (c), is given by:

$$\begin{aligned}
Q &= \int j^0 d^3x \\
&= i \int d^3x (\phi_a^* \dot{\phi}_a - \dot{\phi}_a^* \phi_a) \rightarrow \text{use conjugate momenta notation} \\
&= i \int d^3x (\phi_a^* \pi_a^* - \pi_a \phi_a).
\end{aligned}$$

The U(2) subgroup has three generators, which are the Pauli matrices,  $\sigma_k$ , multiplied by  $i/2$ , i.e.,  $T_k = \frac{i}{2}\sigma_k$  for  $k = 1, 2, 3$ . The Lagrangian is invariant under the transformation:

$$\phi_a \rightarrow (e^{-\beta_k T_k})_{ab} \phi_b,$$

where  $\beta_k$  is an infinitesimal parameter. Similarly, this transformation gives the following infinitesimal change:

$$(e^{-\beta_k T_k})_{ab} \phi_b = (\mathbb{I} - \frac{i}{2} \beta_k \sigma_k)_{ab} \phi_b + \mathcal{O}(\beta_k^2) \approx \phi_a - \frac{i}{2} (\beta_k \sigma_k)_{ab} \phi_b.$$

For SU(2), we have the following Noether currents:

$$\begin{aligned}
j^{\mu,k}(x) &= \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \Delta^k \phi_a + \Delta^k \phi_a^* \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a^*)} \\
&= \left( \partial_\mu \phi_a^* \right) \left( -\frac{i}{2} (\beta_k \sigma_k)_{ab} \phi_b \right) + \left( \frac{i}{2} (\beta_k \sigma_k)_{ab}^* \phi_b^* \right) \left( \partial_\mu \phi_a \right) \\
&= \frac{i}{2} \left( (\sigma_k)_{ab}^* \phi_b^* (\partial_\mu \phi_a) - (\partial_\mu \phi_a^*) (\sigma_k)_{ab} \phi_b \right) \rightarrow (\sigma_k)_{ab}^* = (\sigma_k)_{ba} \text{ and relabel indices} \\
&= \frac{i}{2} \left( (\sigma_k)_{ab} \phi_a^* (\partial_\mu \phi_b) - (\partial_\mu \phi_a^*) (\sigma_k)_{ab} \phi_b \right) \\
&= \frac{i}{2} \left( \phi_a^* (\sigma_k)_{ab} (\partial_\mu \phi_b) - (\partial_\mu \phi_a^*) (\sigma_k)_{ab} \phi_b \right)
\end{aligned}$$

where we have arbitrarily chosen  $\beta_k$ . Accordingly, the three charges from this symmetry are given by:

$$\begin{aligned}
Q^k &= \int j^{0,k} d^3x \\
&= \frac{i}{2} \int d^3x \left( \phi_a^* (\sigma_k)_{ab} \dot{\phi}_b - \dot{\phi}_a^* (\sigma_k)_{ab} \phi_b \right) \rightarrow \text{use conjugate momenta notation} \\
&= \frac{i}{2} \int d^3x \left( \phi_a^* (\sigma_k)_{ab} \pi_b^* - \pi_a (\sigma_k)_{ab} \phi_b \right).
\end{aligned}$$

These charges having the commutation relations of angular momentum is shown as follows:

$$\begin{aligned}
[Q^i, Q^j] &= -\frac{1}{4} \int d^3x \int d^3y \left[ \left( \phi_a^*(\sigma_i)_{ab} \pi_b^* - \pi_a(\sigma_i)_{ab} \phi_b \right), \left( \phi_c^*(\sigma_j)_{cd} \pi_d^* - \pi_c(\sigma_j)_{cd} \phi_d \right) \right] \\
&= -\frac{1}{4} \int d^3x \int d^3y \\
&\quad \times \left( \left[ \phi_a^*(\sigma_i)_{ab} \pi_b^*, \phi_c^*(\sigma_j)_{cd} \pi_d^* \right] - \left[ \phi_a^*(\sigma_i)_{ab} \pi_b^*, \pi_c(\sigma_j)_{cd} \phi_d \right] \right. \\
&\quad \left. - \left[ \pi_a(\sigma_i)_{ab} \phi_b, \phi_c^*(\sigma_j)_{cd} \pi_d^* \right] + \left[ \pi_a(\sigma_i)_{ab} \phi_b, \pi_c(\sigma_j)_{cd} \phi_d \right] \right) \\
&= -\frac{1}{4} \int d^3x \int d^3y (\sigma_i)_{ab} (\sigma_j)_{cd} \\
&\quad \times \left( \left[ \phi_a^* \pi_b^*, \phi_c^* \pi_d^* \right] - \left[ \phi_a^* \pi_b^*, \pi_c \phi_d \right] - \left[ \pi_a \phi_b, \phi_c^* \pi_d^* \right] + \left[ \pi_a \phi_b, \pi_c \phi_d \right] \right) \\
&= -\frac{1}{4} \int d^3x \int d^3y (\sigma_i)_{ab} (\sigma_j)_{cd} \\
&\quad \times \left( \phi_a^* \left[ \pi_b^*, \phi_c^* \right] \pi_d^* + \left[ \phi_a^*, \phi_c^* \right] \pi_b^* \pi_d^* + \left[ \phi_c^*, \phi_a^* \right] \pi_b^* \pi_d^* + \phi_c^* \left[ \phi_a^*, \pi_d^* \right] \pi_b^* \right. \\
&\quad - \phi_a^* \left[ \pi_b^*, \pi_c \right] \phi_d - \left[ \phi_a^*, \pi_c \right] \pi_b^* \phi_d - \pi_c \phi_a^* \left[ \pi_b^*, \phi_d \right] - \pi_c \left[ \phi_a^*, \phi_d \right] \pi_b^* \\
&\quad - \pi_a \left[ \phi_b, \phi_c^* \right] \pi_d^* - \left[ \pi_a, \phi_c^* \right] \phi_b \pi_d^* - \phi_c^* \pi_a \left[ \phi_b, \pi_d^* \right] - \phi_c^* \left[ \pi_a, \pi_d^* \right] \phi_b \\
&\quad \left. + \pi_a \left[ \phi_b, \pi_c \right] \phi_d + \left[ \pi_a, \pi_c \right] \phi_b \phi_d + \pi_c \pi_a \left[ \phi_b, \phi_d \right] + \pi_c \left[ \pi_a, \phi_d \right] \phi_b \right) \\
&= -\frac{1}{4} \int d^3x \int d^3y (\sigma_i)_{ab} (\sigma_j)_{cd} \\
&\quad \times \left( \phi_a^* (-i\delta(\mathbf{x} - \mathbf{y})) \pi_d^* + 0 + 0 + \phi_c^* (i\delta(\mathbf{x} - \mathbf{y})) \pi_b^* \right. \\
&\quad - 0 - 0 - 0 - 0 \\
&\quad - 0 - 0 - 0 - 0 \\
&\quad \left. + \pi_a (i\delta(\mathbf{x} - \mathbf{y})) \phi_d + 0 + 0 + \pi_c (-i\delta(\mathbf{x} - \mathbf{y})) \phi_b \right) \\
&= -\frac{1}{4} \int d^3x (\sigma_i)_{ab} (\sigma_j)_{cd} \left( -i\phi_a^* \pi_d^* + i\phi_c^* \pi_b^* + i\pi_a \phi_d - i\pi_c \phi_b \right) \\
&= \frac{i}{4} \int d^3x (\sigma_i)_{ab} (\sigma_j)_{cd} \left( \phi_a^* \pi_d^* - \phi_c^* \pi_b^* - \pi_a \phi_d + \pi_c \phi_b \right) \rightarrow \text{relabel indices}
\end{aligned}$$

$$\begin{aligned}
&= \frac{i}{4} \int d^3x \left( (\sigma_i)_{ab}(\sigma_j)_{cd} - (\sigma_i)_{cd}(\sigma_j)_{ab} \right) \left( \phi_a^* \pi_d^* - \pi_a \phi_d \right) \\
&= \frac{i}{4} \int d^3x \left( 2i\epsilon_{ijk}(\sigma_k)_{ad}\delta_{bc} \right) \left( \phi_a^* \pi_d^* - \pi_a \phi_d \right) \\
&= i\epsilon_{ijk} \int d^3x \frac{i}{2} \left( \phi_a^*(\sigma_k)_{ad}\pi_d^* - \pi_a(\sigma_k)_{ad}\phi_d \right) \\
&= i\epsilon_{ijk} Q^k.
\end{aligned}$$

These results can be naturally generalized to the case of  $n$  identical complex scalar fields. In this context, the relevant symmetry group becomes the unitary group  $U(n)$ . This group has  $n^2$  generators: one corresponding to  $U(1)$  and  $n^2 - 1$  corresponding to  $SU(n)$ . The associated Noether currents,  $j^{\mu,k}$ , and conserved charges,  $Q^k$ , have the same form as above. Additionally, the Pauli matrices are replaced by the generators  $T_k$  of  $SU(n)$ , and the Levi-Civita symbol is replaced by the structure constants  $f^{ijk}$ , which define the commutation relations between the generators. To address the footnote, you can treat the  $n$  complex scalar fields by their real and imaginary components as  $2n$  real fields. Then, instead of transformations described by the unitary group  $U(n)$ , we consider transformations described by the special orthogonal group  $SO(N)$ , which has  $N(N - 1)/2$  generators, or  $n(2n - 1)$  generators with  $N = 2n$ .

## 2.3 Spacelike propagation amplitude

Evaluate the function

$$\langle 0 | \phi(x)\phi(y) | 0 \rangle = D(x - y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} e^{-ip \cdot (x-y)},$$

for  $(x - y)$  spacelike so that  $(x - y)^2 = -r^2$ , explicitly in terms of Bessel functions. For the case where  $(x - y)$  is spacelike, we have  $x^0 - y^0 = 0$  and  $\mathbf{x} - \mathbf{y} = \mathbf{r}$ . Using this, the propagation amplitude becomes:

$$\begin{aligned}
D(x - y) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} e^{-ip \cdot (x-y)} \rightarrow \text{apply spacelike condition} \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} e^{i\mathbf{p} \cdot \mathbf{r}} \rightarrow \text{switch to spherical coordinates and rewrite } E_{\mathbf{p}} \\
&= \frac{1}{(2\pi)^3} \int_0^\infty p^2 dp \int_0^\pi \sin(\theta) d\theta \int_0^{2\pi} d\phi \frac{e^{ipr \cos(\theta)}}{2\sqrt{p^2 + m^2}} \rightarrow \text{do } \phi \text{ integral} \\
&= \frac{(2\pi)}{2(2\pi)^3} \int_0^\infty p^2 dp \int_0^\pi \sin(\theta) d\theta \frac{e^{ipr \cos(\theta)}}{\sqrt{p^2 + m^2}} \rightarrow \text{u substitution for } \theta \text{ integral} \\
&= \frac{1}{2(2\pi)^2} \int_0^\infty p^2 dp \int_{-1}^1 du \frac{e^{ipru}}{\sqrt{p^2 + m^2}} \rightarrow \text{do } u \text{ integral}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2(2\pi)^2} \int_0^\infty p^2 dp \frac{1}{\sqrt{p^2 + m^2}} \frac{2 \sin(pr)}{pr} \rightarrow \text{simplify} \\
&= \frac{1}{r(2\pi)^2} \int_0^\infty dp \frac{p \sin(pr)}{\sqrt{p^2 + m^2}} \rightarrow \text{use Gradstein and Ryzhik equation 3.754.2} \\
&= \frac{1}{r(2\pi)^2} \left( -\frac{\partial}{\partial r} K_0(rm) \right) \rightarrow \frac{\partial}{\partial r} K_0(rm) = -m K_1(rm) \\
&= \frac{1}{(2\pi)^2} \frac{m}{r} K_1(rm) \rightarrow K_\nu \text{ are the modified Bessel functions.}
\end{aligned}$$



## Chapter 3

### The Dirac Field

- 3.1 Lorentz group
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- 3.3 Spinor products
- 3.4 Majorana fermions
- 3.5 Supersymmetry
- 3.6 Fierz transformations
- 3.7 Discrete symmetries  $P$ ,  $C$ , and  $T$
- 3.8 Bound states

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**Final Project**

**Radiation of Gluon Jets**

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- 20.2 Decay modes of the  $W$  and  $Z$  bosons
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