Solutions to
An Introduction to Quantum Field Theory
by M.E. Peskin and D.V. Schroeder

James A. Stewart

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2.1 Classical electromagnetism with no sources

The Euler-Lagrange equations of motion, with the components $A_\mu(x)$ as the dynamical variables, are given by:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right) - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0,$$

where the Lagrangian density is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu},$$

and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

is the electromagnetic field tensor. For the Euler-Lagrange equations of motion, μ and ν are *free* indices that simply label the components A_μ and its derivatives $\partial_\mu A_\nu$. In contrast, the indices for $F_{\mu\nu}$ are *dummy* indices that are used to sum over all the components. Their label doesn't matter as long as it is distinct from the *free* indices. It is important to note that $F_{\mu\nu}$ is antisymmetric. More precisely,

$$F_{\nu\mu} = \partial_\nu A_\mu - \partial_\mu A_\nu = -\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) = -F_{\mu\nu}.$$

Consequently, the diagonal components of $F_{\mu\nu}$ vanish. Specifically,

$$F_{\mu\mu} = -F_{\mu\mu} \implies 2F_{\mu\mu} = 0 \implies F_{\mu\mu} = 0.$$

(a) For the second term in the Euler-Lagrange equations of motion, we have:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial A_\nu} = \frac{\partial}{\partial A_\nu}\bigg(-\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\bigg) &\rightarrow \text{ apply product rule} \\
&= -\frac{1}{4}\bigg(F^{\alpha\beta}\frac{\partial F_{\alpha\beta}}{\partial A_\nu} + F_{\alpha\beta}\frac{\partial F^{\alpha\beta}}{\partial A_\nu}\bigg) \rightarrow \text{ lower indices in 2nd term} \\
&= -\frac{1}{4}\bigg(F^{\alpha\beta}\frac{\partial F_{\alpha\beta}}{\partial A_\nu} + g^{\alpha\sigma}g^{\beta\rho}F_{\alpha\beta}\frac{\partial F_{\sigma\rho}}{\partial A_\nu}\bigg) \rightarrow \text{ raise indices in 2nd term} \\
&= -\frac{1}{4}\bigg(F^{\alpha\beta}\frac{\partial F_{\alpha\beta}}{\partial A_\nu} + F^{\sigma\rho}\frac{\partial F_{\sigma\rho}}{\partial A_\nu}\bigg) \rightarrow \text{ relabel } \sigma \text{ and } \rho \text{ to } \alpha \text{ and } \beta \\
&= -\frac{1}{4}\bigg(F^{\alpha\beta}\frac{\partial F_{\alpha\beta}}{\partial A_\nu} + F^{\alpha\beta}\frac{\partial F_{\alpha\beta}}{\partial A_\nu}\bigg) \rightarrow \text{ terms are equal} \\
&= -\frac{1}{2}F^{\alpha\beta}\frac{\partial F_{\alpha\beta}}{\partial A_\nu} \rightarrow \text{ second term is 0} \\
&= 0.
\end{aligned}
$$

This is because

$$
\frac{\partial F_{\alpha\beta}}{\partial A_\nu} = \frac{\partial}{\partial A_\nu}(\partial_\alpha A_\beta - \partial_\beta A_\alpha) = 0,
$$

which follows from the fact that $F_{\mu\nu}$ depends only on the derivative of A_μ (i.e., $\partial_\mu A_\nu$) and not A_μ itself. For the first term in the Euler-Lagrange equations of motion, we have:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = \frac{\partial}{\partial(\partial_\mu A_\nu)}\bigg(-\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\bigg) &\rightarrow \text{ apply product rule} \\
&= -\frac{1}{4}\bigg(F^{\alpha\beta}\frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)} + F_{\alpha\beta}\frac{\partial F^{\alpha\beta}}{\partial(\partial_\mu A_\nu)}\bigg) \rightarrow \text{ lower indices in 2nd term} \\
&= -\frac{1}{4}\bigg(F^{\alpha\beta}\frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)} + g^{\alpha\sigma}g^{\beta\rho}F_{\alpha\beta}\frac{\partial F_{\sigma\rho}}{\partial(\partial_\mu A_\nu)}\bigg) \rightarrow \text{ raise indices in 2nd term} \\
&= -\frac{1}{4}\bigg(F^{\alpha\beta}\frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)} + F^{\sigma\rho}\frac{\partial F_{\sigma\rho}}{\partial(\partial_\mu A_\nu)}\bigg) \rightarrow \text{ relabel } \sigma \text{ and } \rho \text{ to } \alpha \text{ and } \beta \\
&= -\frac{1}{4}\bigg(F^{\alpha\beta}\frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)} + F^{\alpha\beta}\frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)}\bigg) \rightarrow \text{ terms are equal} \\
&= -\frac{1}{2}F^{\alpha\beta}\frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)} \rightarrow \text{ insert } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \\
&= -\frac{1}{2}F^{\alpha\beta}\frac{\partial}{\partial(\partial_\mu A_\nu)}\Big(\partial_\alpha A_\beta - \partial_\beta A_\alpha\Big) \rightarrow \mu,\nu = \alpha,\beta \text{ or } \beta,\alpha \text{ survive}
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{2}F^{\alpha\beta}\Big(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}\Big) \;\rightarrow\; \text{distribute } F^{\alpha\beta} \\
&= -\frac{1}{2}\Big(F^{\alpha\beta}\delta_{\mu\alpha}\delta_{\nu\beta} - F^{\alpha\beta}\delta_{\mu\beta}\delta_{\nu\alpha}\Big) \;\rightarrow\; \text{satisfy } \delta_{ij} \text{ functions} \\
&= -\frac{1}{2}\Big(F^{\mu\nu} - F^{\nu\mu}\Big) \;\rightarrow\; F_{\mu\nu} \text{ is antisymmetric} \\
&= -\frac{1}{2}\Big(F^{\mu\nu} + F^{\mu\nu}\Big) \\
&= -F^{\mu\nu}.
\end{aligned}
$$

Inserting these results into the Euler-Lagrange equations of motion, we obtain:

$$
\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) - \frac{\partial\mathcal{L}}{\partial A_\nu} = 0 \implies \partial_\mu\big(-F^{\mu\nu}\big) - 0 = 0 \implies \partial_\mu F^{\mu\nu} = 0.
$$

Two of Maxwell's equations can be written in standard form by making use of this result. First, consider the identity $E^i = -F^{0i} = F^{i0}$, where $\nu = 0$:

$$
\begin{aligned}
0 &= \partial_\mu F^{\mu 0} \;\rightarrow\; \text{separate components} \\
&= \partial_0 F^{00} + \partial_i F^{i0} \;\rightarrow\; F^{00} = 0 \text{ and substitute } F^{i0} = E^i \\
&= \partial_i E^i \;\rightarrow\; \text{divergence of a vector field in tensor notation} \\
&= \nabla\cdot\mathbf{E} \;\rightarrow\; \text{Gauss's law.}
\end{aligned}
$$

Second, consider the identity $\epsilon^{ijk}B^k = -F^{ij} = F^{ji}$, where $\nu = i$:

$$
\begin{aligned}
0 &= \partial_\mu F^{\mu i} \;\rightarrow\; \text{separate components} \\
&= \partial_0 F^{0i} + \partial_j F^{ji} \;\rightarrow\; \text{substitute } E^i \text{ and } B^k \\
&= -\partial_0 E^i + \epsilon^{ijk}\partial_j B^k \;\rightarrow\; \text{2nd term is curl of a vector field in tensor notation} \\
&= \frac{\partial\mathbf{E}}{\partial t} - \nabla\times\mathbf{B} \;\rightarrow\; \text{Ampere's law.}
\end{aligned}
$$

The two remaining Maxwell equations can be written in standard form by making use of these identities with the definition of $F_{\mu\nu}$. First, consider the case with $\mu = i$ and $\nu = j$, which gives:

$$
\begin{aligned}
F^{ij} \implies \partial^i A^j - \partial^j A^i &= -\epsilon^{ijk}B^k \;\rightarrow\; \text{multiply by } \epsilon^{ijm} \text{ and sum over } i, j \\
-\epsilon^{ijm}\big(\partial_i A^j - \partial_j A^i\big) &= -\epsilon^{ijm}\epsilon^{ijk}B^k \;\rightarrow\; \epsilon^{ijm}\epsilon^{ijk} = 2\delta^{mk} \\
\epsilon^{ijm}\partial_i A^j - \epsilon^{ijm}\partial_j A^i &= 2\delta^{mk}B^k \;\rightarrow\; \text{relabel } i, j \text{ to } j, i \text{ in LHS 2nd term} \\
\epsilon^{ijm}\partial_i A^j - \epsilon^{jim}\partial_i A^j &= 2\delta^{mk}B^k \;\rightarrow\; \text{swap } i, j \text{ in } \epsilon^{jim}
\end{aligned}
$$

$$
\begin{aligned}
\epsilon^{ijm}\partial_i A^j - \big(-\epsilon^{ijm}\partial_i A^j\big) &= 2\delta^{mk}B^k \\
2\epsilon^{ijm}\partial_i A^j &= 2\delta^{mk}B^k \\
\epsilon^{ijm}\partial_i A^j &= B^m \;\rightarrow\; \text{curl of a vector field in tensor notation} \\
\nabla\times\mathbf{A} &= \mathbf{B} \;\rightarrow\; \text{take the divergence and recall } \nabla\cdot\big(\nabla\times\mathbf{A}\big)=0 \\
0 &= \nabla\cdot\mathbf{B} \rightarrow \text{Gauss's law for magnetism.}
\end{aligned}
$$

Finally, consider the case with $\mu = 0$ and $\nu = i$, which gives:

$$
\begin{aligned}
F^{0i} \implies \partial^0 A^i - \partial^i A^0 &= -E^i \;\rightarrow\; \text{lower an index in LHS 2nd term} \\
\partial^0 A^i + \partial_i A^0 &= -E^i \\
\frac{\partial A^i}{\partial t} + \frac{\partial A^0}{\partial x^i} &= -E^i \;\rightarrow\; \text{second term on LHS is the scalar potential} \\
-\frac{\partial\mathbf{A}}{\partial t} - \nabla\Phi &= \mathbf{E} \;\rightarrow\; \text{take the curl} \\
-\frac{\partial}{\partial t}\left(\nabla\times\mathbf{A}\right) - \nabla\times\left(\nabla\Phi\right) &= \nabla\times\mathbf{E} \;\rightarrow\; \nabla\times\mathbf{A} = \mathbf{B} \text{ from above} \\
-\frac{\partial\mathbf{B}}{\partial t} &= \nabla\times\mathbf{E} \;\rightarrow\; \text{Faraday's law.}
\end{aligned}
$$

(b) From Noether's theorem applied to spacetime transformations, where the components $A_\mu(x)$ are the dynamical variables, we have the general form of the energy-momentum tensor expressed as:

$$
T^\mu{}_\nu = \frac{\partial\mathcal{L}}{\partial\big(\partial_\mu A_\lambda\big)}\partial_\nu A_\lambda - \mathcal{L}\delta^\mu{}_\nu .
$$

Using information from part (a), this becomes:

$$
\begin{aligned}
T^\mu{}_\nu &= -F^{\mu\lambda}\partial_\nu A_\lambda + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\delta^\mu{}_\nu \;\rightarrow\; \text{apply } g^{\sigma\nu} \\
g^{\sigma\nu}T^\mu{}_\nu &= -F^{\mu\lambda}g^{\sigma\nu}\partial_\nu A_\lambda + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}g^{\sigma\nu}\delta^\mu{}_\nu \\
T^{\mu\sigma} &= -F^{\mu\lambda}\partial^\sigma A_\lambda + \frac{1}{4}g^{\sigma\mu}F_{\alpha\beta}F^{\alpha\beta} \;\rightarrow\; \text{relabel } \sigma \text{ to } \nu \text{ and use } g^{\mu\nu} = g^{\nu\mu} \\
T^{\mu\nu} &= -F^{\mu\lambda}\partial^\nu A_\lambda + \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}.
\end{aligned}
$$

We're told that this expression is not a symmetric tensor, but that we can remedy this by adding the term $\partial_\lambda K^{\lambda\mu\nu}$ to $T^{\mu\nu}$, where $K^{\lambda\mu\nu}$ is antisymmetric in the first two indices. The resulting expression is:

$$
\begin{aligned}
\widehat{T}^{\mu\nu} &= T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu} \\
&= -F^{\mu\lambda}\partial^\nu A_\lambda + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} + \partial_\lambda K^{\lambda\mu\nu} \\
&= -F^{\mu\lambda}\partial^\nu A_\lambda + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} + \partial_\lambda F^{\mu\lambda} A^\nu,
\end{aligned}
$$

where we used the provided relationship, $K^{\lambda\mu\nu} = F^{\mu\lambda}A^\nu$, in the last term. This expression can be further simplified as follows:

$$
\begin{aligned}
\widehat{T}^{\mu\nu} &= -F^{\mu\lambda}\partial^\nu A_\lambda + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} + \partial_\lambda F^{\mu\lambda} A^\nu \;\rightarrow\; \text{apply product rule} \\
&= -F^{\mu\lambda}\partial^\nu A_\lambda + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} + F^{\mu\lambda}\partial_\lambda A^\nu + A^\nu \partial_\lambda F^{\mu\lambda} \;\rightarrow\; \partial_\lambda F^{\mu\lambda} = 0 \\
&= -F^{\mu\lambda}\partial^\nu A_\lambda + F^{\mu\lambda}\partial_\lambda A^\nu + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \\
&= -F^{\mu\lambda}\big(\partial^\nu A_\lambda - \partial_\lambda A^\nu\big) + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \\
&= -F^{\mu\lambda}F^{\nu}{}_{\lambda} + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}.
\end{aligned}
$$

The symmetry of this expression under the exchange $\mu \leftrightarrow \nu$ can be demonstrated as follows:

$$
\begin{aligned}
\widehat{T}^{\nu\mu} &= -F^{\nu\lambda}F^{\mu}{}_{\lambda} + \frac{1}{4} g^{\nu\mu} F_{\alpha\beta} F^{\alpha\beta} \;\rightarrow\; \text{raise and lower indices and use } g^{\mu\nu} = g^{\nu\mu} \\
&= -g^{\lambda\sigma} g_{\lambda\rho} F^{\nu}{}_{\sigma} F^{\mu\rho} + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \;\rightarrow\; g^{\lambda\sigma} g_{\lambda\rho} = \delta^{\sigma}{}_{\rho} \\
&= -F^{\nu}{}_{\sigma} F^{\mu\sigma} + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \;\rightarrow\; \text{relabel } \sigma \text{ to } \lambda \\
&= -F^{\nu}{}_{\lambda} F^{\mu\lambda} + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \\
&= \widehat{T}^{\mu\nu}.
\end{aligned}
$$

The electromagnetic energy density, ε, is obtained from $\widehat{T}^{00}$:

$$
\begin{aligned}
\varepsilon &= \widehat{T}^{00} \\
&= -F^{0\lambda}F^{0}{}_{\lambda} + \frac{1}{4} g^{00} F_{\alpha\beta} F^{\alpha\beta} \;\rightarrow\; \text{separate components in 1st term} \\
&= -F^{00}F^{0}{}_{0} - F^{0i}F^{0}{}_{i} + \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \;\rightarrow\; F^{00} = F^{0}{}_{0} = 0 \text{ and } E^i = -F^{0i} = F^{0}{}_{i} \\
&= E^i E^i + \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \;\rightarrow\; \text{separate components in 2nd term}
\end{aligned}
$$

$$
\begin{aligned}
&= E^i E^i + \frac{1}{4}\Big(F_{00}\,F^{00} + F_{0i}\,F^{0i} + F_{i0}\,F^{i0} + F_{ij}\,F^{ij}\Big) \;\rightarrow\; F_{00} = F^{00} = 0 \\
&= E^i E^i + \frac{1}{4}\Big(E^i\big(-E^i\big) + \big(-E^i\big)E^i + \big(-\epsilon^{ijk}B^k\big)\big(-\epsilon^{ijl}B^l\big)\Big) \\
&= E^i E^i + \frac{1}{4}\Big(-2E^i E^i + \epsilon^{ijk}\epsilon^{ijl}B^k B^l\Big) \;\rightarrow\; \epsilon^{ijk}\epsilon^{ijl} = 2\delta^{kl} \\
&= E^i E^i + \frac{1}{4}\Big(-2E^i E^i + 2B^k B^k\Big) \\
&= \frac{1}{2}E^i E^i + \frac{1}{2}B^k B^k \\
&= \frac{1}{2}\big(\mathbf{E}^2 + \mathbf{B}^2\big).
\end{aligned}
$$

The momentum density, $\mathbf{S}$, is obtained from $\widehat{T}^{0i}$:

$$
\begin{aligned}
\mathbf{S} &= \widehat{T}^{0i} \\
&= -F^{0\lambda}F^{i}_{\ \lambda} + \frac{1}{4}g^{0i}F_{\alpha\beta}\,F^{\alpha\beta} \;\rightarrow\; g^{0i} = 0 \\
&= -F^{00}F^{i}_{\ 0} - F^{0j}F^{i}_{\ j} \;\rightarrow\; F^{00} = 0 \\
&= \big(E^j\big)\big(\epsilon^{ijk}B^k\big) \\
&= \epsilon^{ijk}E^j B^k \\
&= \mathbf{E}\times\mathbf{B}.
\end{aligned}
$$

2.2 The complex scalar field

The action for a field theory of a complex-valued scalar field obeying the Klein-Gordon equation is given by:

$$
S = \int d^4x\Big(\partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi\Big).
$$

The authors suggest it is easiest to consider the fields $\phi(x)$ and $\phi^*(x)$ as the basic dynamical variables rather than decomposing $\phi(x)$ into its real and imaginary parts, $\phi(x) = \phi_1(x) + i\phi_2(x)$, where $\phi_1(x)$ and $\phi_2(x)$ are real Klein-Gordon fields. Therefore, we need an appropriate expression for the Fourier mode expansion of $\phi(x)$ to describe complex-valued fields. The position-space representation of a general scalar field $\phi(x)$ is given by its inverse Fourier transform:

$$
\phi(x) = \int \frac{d^4p}{(2\pi)^4}\tilde{\phi}(p)e^{-ip\cdot x},
$$

where $\tilde{\phi}(p)$ is the Fourier transform of the field in momentum space. Inserting this expression into the Klein-Gordon equation, we obtain:

$$\begin{aligned}
0 &= \Big(\partial_\mu \partial^\mu + m^2\Big)\phi(x) \\
&= \int \frac{d^4p}{(2\pi)^4}\Big(m^2 - p^2\Big)\tilde{\phi}(p)e^{-ip\cdot x}.
\end{aligned}$$

This expression implies the *on-mass-shell* condition for $\tilde{\phi}(p)$. More precisely, in order to satisfy the Klein-Gordon equation, $\tilde{\phi}(p)$ must vanish everywhere except where the four-momentum satisfies $p^2 = m^2$. Consequently, the spectrum takes the general form $\tilde{\phi}(p) = 2\pi\delta(p^2 - m^2)f(p)$, for some general function $f(p)$. The factor of 2π ensures consistency with normalization conventions. Given that $p^2 = (p^0)^2 - \mathbf{p}^2$, the argument of the Dirac delta function can be rewritten as $\delta((p^0)^2 - (E_{\mathbf{p}})^2)$, where $E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$. As a result, the Dirac delta function has two roots at $p^0 = \pm E_{\mathbf{p}}$, so we can apply the identity:

$$\delta(f(x)) = \sum_i \frac{1}{|f'(x_i)|}\delta(x - x_i)$$

to obtain

$$\tilde{\phi}(p) = \frac{2\pi}{2E_{\mathbf{p}}}\Big(\delta(p^0 - E_{\mathbf{p}}) + \delta(p^0 + E_{\mathbf{p}})\Big)f(p).$$

Inserting this expression for $\tilde{\phi}(p)$ into the above expression for $\phi(x)$, and evaluating the p^0 integral, we obtain:

$$\begin{aligned}
\phi(x) &= \int \frac{d^4p}{(2\pi)^4}\frac{2\pi}{2E_{\mathbf{p}}}\Big(\delta(p^0 - E_{\mathbf{p}}) + \delta(p^0 + E_{\mathbf{p}})\Big)f(p)e^{-ip\cdot x} \\
&= \int \frac{d^3p}{(2\pi)^3}\int \frac{dp^0}{2\pi}\frac{2\pi}{2E_{\mathbf{p}}}\Big(\delta(p^0 - E_{\mathbf{p}})f(p)e^{-ip\cdot x} + \delta(p^0 + E_{\mathbf{p}})f(p)e^{-ip\cdot x}\Big) \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\Big(f(E_{\mathbf{p}}, \mathbf{p})e^{-i(E_{\mathbf{p}}t - \mathbf{p}\cdot\mathbf{x})} + f(-E_{\mathbf{p}}, \mathbf{p})e^{-i(-E_{\mathbf{p}}t - \mathbf{p}\cdot\mathbf{x})}\Big) \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\Big(f(E_{\mathbf{p}}, \mathbf{p})e^{-i(E_{\mathbf{p}}t - \mathbf{p}\cdot\mathbf{x})} + f(-E_{\mathbf{p}}, -\mathbf{p})e^{-i(-E_{\mathbf{p}}t - (-\mathbf{p})\cdot\mathbf{x})}\Big) \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\Big(f(E_{\mathbf{p}}, \mathbf{p})e^{-i(E_{\mathbf{p}}t - \mathbf{p}\cdot\mathbf{x})} + f(-E_{\mathbf{p}}, -\mathbf{p})e^{i(E_{\mathbf{p}}t - \mathbf{p}\cdot\mathbf{x})}\Big) \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\Big(a_{\mathbf{p}}e^{-ip\cdot x} + b^\dagger_{\mathbf{p}}e^{ip\cdot x}\Big),
\end{aligned}$$

where

$$a_{\mathbf{p}} = \frac{1}{\sqrt{2E_{\mathbf{p}}}} f(E_{\mathbf{p}}, \mathbf{p})$$

and

$$b_{\mathbf{p}}^\dagger = \frac{1}{\sqrt{2E_{\mathbf{p}}}} f(-E_{\mathbf{p}}, -\mathbf{p}).$$

In the classical theory, $a_{\mathbf{p}}$ and $b_{\mathbf{p}}$ are functions that determine the amplitude of positive-frequency and negative-frequency modes, respectively, where $\dagger$ is just the complex conjugate. In the quantum theory, $a_{\mathbf{p}}$ and $b_{\mathbf{p}}$ are promoted to operators that correspond to particles and antiparticles, where $\dagger$ is the Hermitian adjoint. If $\phi(x)$ is real-valued (or Hermitian), then $\phi^*(x) = \phi(x)$, which implies $b_{\mathbf{p}} = a_{\mathbf{p}}$ and $b_{\mathbf{p}}^\dagger = a_{\mathbf{p}}^\dagger$. Then, the above expression simplifies to the mode expansion for the real Klein-Gordon field.

(a) The conjugate momentum to $\phi(x)$, denoted $\pi(x)$, is given by the following:

$$\begin{aligned}
\pi(x) &= \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \\
&= \frac{\partial}{\partial \dot{\phi}} \Big(\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \Big) \ \rightarrow \ \text{separate components and lower indices} \\
&= \frac{\partial}{\partial(\partial_0 \phi)} \Big(\partial_0 \phi^* \partial_0 \phi - \partial_i \phi^* \partial_i \phi - m^2 \phi^* \phi \Big) \ \rightarrow \ \text{2nd and 3rd terms give 0} \\
&= \frac{\partial(\partial_0 \phi^*)}{\partial(\partial_0 \phi)} \partial_0 \phi + \partial_0 \phi^* \frac{\partial(\partial_0 \phi)}{\partial(\partial_0 \phi)} \ \rightarrow \ \text{1st term is zero} \\
&= \partial_0 \phi^*.
\end{aligned}$$

The conjugate momentum to $\phi^*(x)$, denoted $\pi^*(x)$, is determined in a similar manner:

$$\begin{aligned}
\pi^*(x) &= \frac{\partial \mathcal{L}}{\partial \dot{\phi}^*} \\
&= \frac{\partial}{\partial \dot{\phi}^*} \Big(\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \Big) \ \rightarrow \ \text{separate components and lower indices} \\
&= \frac{\partial}{\partial(\partial_0 \phi^*)} \Big(\partial_0 \phi^* \partial_0 \phi - \partial_i \phi^* \partial_i \phi - m^2 \phi^* \phi \Big) \ \rightarrow \ \text{2nd and 3rd terms give 0} \\
&= \frac{\partial(\partial_0 \phi^*)}{\partial(\partial_0 \phi^*)} \partial_0 \phi + \partial_0 \phi^* \frac{\partial(\partial_0 \phi)}{\partial(\partial_0 \phi^*)} \ \rightarrow \ \text{2nd term is zero} \\
&= \partial_0 \phi.
\end{aligned}$$

In order to determine the canonical commutation relations, we make use of the following commutation relations for the creation and annihilation operators:

$$\begin{aligned}
&\left[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger\right] = (2\pi)^3\delta(\mathbf{p}-\mathbf{q}),\ \left[b_{\mathbf{p}}, b_{\mathbf{q}}^\dagger\right] = (2\pi)^3\delta(\mathbf{p}-\mathbf{q}),\\
&\left[a_{\mathbf{p}}, a_{\mathbf{q}}\right] = 0,\ \left[b_{\mathbf{p}}, b_{\mathbf{q}}\right] = 0,\ \left[a_{\mathbf{p}}^\dagger, a_{\mathbf{q}}^\dagger\right] = 0,\ \left[b_{\mathbf{p}}^\dagger, b_{\mathbf{q}}^\dagger\right] = 0,\\
&\left[a_{\mathbf{p}}, b_{\mathbf{q}}\right] = 0,\ \left[a_{\mathbf{p}}, b_{\mathbf{q}}^\dagger\right] = 0,\ \left[a_{\mathbf{p}}^\dagger, b_{\mathbf{q}}\right] = 0,\ \left[a_{\mathbf{p}}^\dagger, b_{\mathbf{q}}^\dagger\right] = 0.
\end{aligned}$$

The canonical commutation relations at $t=0$ are then:

$$\begin{aligned}
\left[\phi(\mathbf{x}), \phi(\mathbf{y})\right] &= \int\int \frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\frac{1}{\sqrt{2E_{\mathbf{k}}}}\left[a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} + b_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}, a_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{y}} + b_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{y}}\right]\\
&= \int\int \frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\frac{1}{\sqrt{2E_{\mathbf{k}}}}\\
&\quad\times\Big(\left[a_{\mathbf{p}}, a_{\mathbf{k}}\right]e^{i\mathbf{p}\cdot\mathbf{x}}e^{i\mathbf{k}\cdot\mathbf{y}} + \left[a_{\mathbf{p}}, b_{\mathbf{k}}^\dagger\right]e^{i\mathbf{p}\cdot\mathbf{x}}e^{-i\mathbf{k}\cdot\mathbf{y}}\\
&\quad+ \left[b_{\mathbf{p}}^\dagger, a_{\mathbf{k}}\right]e^{-i\mathbf{p}\cdot\mathbf{x}}e^{i\mathbf{k}\cdot\mathbf{y}} + \left[b_{\mathbf{p}}^\dagger, b_{\mathbf{k}}^\dagger\right]e^{-i\mathbf{p}\cdot\mathbf{x}}e^{-i\mathbf{k}\cdot\mathbf{y}}\Big)\\
&= 0.
\end{aligned}$$

$$\begin{aligned}
\left[\pi(\mathbf{x}), \pi(\mathbf{y})\right] &= \int\int \frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\frac{(iE_{\mathbf{p}})}{\sqrt{2E_{\mathbf{p}}}}\frac{(iE_{\mathbf{k}})}{\sqrt{2E_{\mathbf{k}}}}\left[a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}} - b_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}}, a_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{y}} - b_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{y}}\right]\\
&= \int\int \frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\frac{(iE_{\mathbf{p}})}{\sqrt{2E_{\mathbf{p}}}}\frac{(iE_{\mathbf{k}})}{\sqrt{2E_{\mathbf{k}}}}\\
&\quad\times\Big(\left[a_{\mathbf{p}}^\dagger, a_{\mathbf{k}}^\dagger\right]e^{-i\mathbf{p}\cdot\mathbf{x}}e^{-i\mathbf{k}\cdot\mathbf{y}} - \left[a_{\mathbf{p}}^\dagger, b_{\mathbf{k}}\right]e^{-i\mathbf{p}\cdot\mathbf{x}}e^{i\mathbf{k}\cdot\mathbf{y}}\\
&\quad- \left[b_{\mathbf{p}}, a_{\mathbf{k}}^\dagger\right]e^{i\mathbf{p}\cdot\mathbf{x}}e^{-i\mathbf{k}\cdot\mathbf{y}} + \left[b_{\mathbf{p}}, b_{\mathbf{k}}\right]e^{i\mathbf{p}\cdot\mathbf{x}}e^{i\mathbf{k}\cdot\mathbf{y}}\Big)\\
&= 0.
\end{aligned}$$

$$\begin{aligned}
\left[\phi(\mathbf{x}), \pi(\mathbf{y})\right] &= \int\int \frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\frac{(iE_{\mathbf{k}})}{\sqrt{2E_{\mathbf{k}}}}\left[a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} + b_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}, a_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{y}} - b_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{y}}\right]\\
&= \int\int \frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\frac{(iE_{\mathbf{k}})}{\sqrt{2E_{\mathbf{k}}}}\\
&\quad\times\Big(\left[a_{\mathbf{p}}, a_{\mathbf{k}}^\dagger\right]e^{i\mathbf{p}\cdot\mathbf{x}}e^{-i\mathbf{k}\cdot\mathbf{y}} - \left[a_{\mathbf{p}}, b_{\mathbf{k}}\right]e^{i\mathbf{p}\cdot\mathbf{x}}e^{i\mathbf{k}\cdot\mathbf{y}}\\
&\quad+ \left[b_{\mathbf{p}}^\dagger, a_{\mathbf{k}}^\dagger\right]e^{-i\mathbf{p}\cdot\mathbf{x}}e^{-i\mathbf{k}\cdot\mathbf{y}} - \left[b_{\mathbf{p}}^\dagger, b_{\mathbf{k}}\right]e^{-i\mathbf{p}\cdot\mathbf{x}}e^{i\mathbf{k}\cdot\mathbf{y}}\Big)\\
&= \int\int \frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\frac{(iE_{\mathbf{k}})}{\sqrt{2E_{\mathbf{k}}}}
\end{aligned}$$

$$
\begin{aligned}
&\times \Big((2\pi)^3 \delta(\mathbf{p}-\mathbf{k}) e^{i\mathbf{p}\cdot\mathbf{x}} e^{-i\mathbf{k}\cdot\mathbf{y}} - 0 \\
&+ 0 + (2\pi)^3 \delta(\mathbf{p}-\mathbf{k}) e^{-i\mathbf{p}\cdot\mathbf{x}} e^{i\mathbf{k}\cdot\mathbf{y}} \Big) \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_\mathbf{p}}} \frac{(iE_\mathbf{p})}{\sqrt{2E_\mathbf{p}}} \Big(e^{i\mathbf{p}\cdot\mathbf{x}} e^{-i\mathbf{p}\cdot\mathbf{y}} + e^{-i\mathbf{p}\cdot\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{y}} \Big) \\
&= \frac{i}{2} \int \frac{d^3p}{(2\pi)^3} \Big(e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} + e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \Big) \;\rightarrow\; \mathbf{p} \text{ to } -\mathbf{p} \text{ in 1st term} \\
&= \frac{i}{2} \int \frac{d^3p}{(2\pi)^3} \Big(e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} + e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \Big) \\
&= i \int \frac{d^3p}{(2\pi)^3} \Big(e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \Big) \\
&= i\delta(\mathbf{x}-\mathbf{y}).
\end{aligned}
$$

$$
\begin{aligned}
\big[\phi^*(\mathbf{x}), \phi^*(\mathbf{y})\big] &= \int\int \frac{d^3p}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_\mathbf{p}}} \frac{1}{\sqrt{2E_\mathbf{k}}} \Big[a^\dagger_\mathbf{p} e^{-i\mathbf{p}\cdot\mathbf{x}} + b_\mathbf{p} e^{i\mathbf{p}\cdot\mathbf{x}}, a^\dagger_\mathbf{k} e^{-i\mathbf{k}\cdot\mathbf{y}} + b_\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{y}} \Big] \\
&= \int\int \frac{d^3p}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_\mathbf{p}}} \frac{1}{\sqrt{2E_\mathbf{k}}} \\
&\quad \times \Big(\big[a^\dagger_\mathbf{p}, a^\dagger_\mathbf{k}\big] e^{-i\mathbf{p}\cdot\mathbf{x}} e^{-i\mathbf{k}\cdot\mathbf{y}} + \big[a^\dagger_\mathbf{p}, b_\mathbf{k}\big] e^{-i\mathbf{p}\cdot\mathbf{x}} e^{i\mathbf{k}\cdot\mathbf{y}} \\
&\quad + \big[b_\mathbf{p}, a^\dagger_\mathbf{k}\big] e^{i\mathbf{p}\cdot\mathbf{x}} e^{-i\mathbf{k}\cdot\mathbf{y}} + \big[b_\mathbf{p}, b_\mathbf{k}\big] e^{i\mathbf{p}\cdot\mathbf{x}} e^{i\mathbf{k}\cdot\mathbf{y}} \Big) \\
&= 0.
\end{aligned}
$$

$$
\begin{aligned}
\big[\pi^*(\mathbf{x}), \pi^*(\mathbf{y})\big] &= \int\int \frac{d^3p}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \frac{(iE_\mathbf{p})}{\sqrt{2E_\mathbf{p}}} \frac{(iE_\mathbf{k})}{\sqrt{2E_\mathbf{k}}} \Big[b^\dagger_\mathbf{p} e^{-i\mathbf{p}\cdot\mathbf{x}} - a_\mathbf{p} e^{i\mathbf{p}\cdot\mathbf{x}}, b^\dagger_\mathbf{k} e^{-i\mathbf{k}\cdot\mathbf{y}} - a_\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{y}} \Big] \\
&= \int\int \frac{d^3p}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \frac{(iE_\mathbf{p})}{\sqrt{2E_\mathbf{p}}} \frac{(iE_\mathbf{k})}{\sqrt{2E_\mathbf{k}}} \\
&\quad \times \Big(\big[b^\dagger_\mathbf{p}, b^\dagger_\mathbf{k}\big] e^{-i\mathbf{p}\cdot\mathbf{x}} e^{-i\mathbf{k}\cdot\mathbf{y}} - \big[b^\dagger_\mathbf{p}, a_\mathbf{k}\big] e^{-i\mathbf{p}\cdot\mathbf{x}} e^{i\mathbf{k}\cdot\mathbf{y}} \\
&\quad - \big[a_\mathbf{p}, b^\dagger_\mathbf{k}\big] e^{i\mathbf{p}\cdot\mathbf{x}} e^{-i\mathbf{k}\cdot\mathbf{y}} + \big[a_\mathbf{p}, a_\mathbf{k}\big] e^{i\mathbf{p}\cdot\mathbf{x}} e^{i\mathbf{k}\cdot\mathbf{y}} \Big) \\
&= 0.
\end{aligned}
$$

$$
\big[\phi^*(\mathbf{x}), \pi^*(\mathbf{y})\big] = \int\int \frac{d^3p}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_\mathbf{p}}} \frac{(iE_\mathbf{k})}{\sqrt{2E_\mathbf{k}}} \Big[a^\dagger_\mathbf{p} e^{-i\mathbf{p}\cdot\mathbf{x}} + b_\mathbf{p} e^{i\mathbf{p}\cdot\mathbf{x}}, b^\dagger_\mathbf{k} e^{-i\mathbf{k}\cdot\mathbf{y}} - a_\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{y}} \Big]
$$

$$
\begin{aligned}
&= \int\int \frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\frac{(iE_{\mathbf{k}})}{\sqrt{2E_{\mathbf{k}}}} \\
&\quad \times \Big(\big[a^\dagger_{\mathbf{p}}, b^\dagger_{\mathbf{k}}\big]e^{-i\mathbf{p}\cdot\mathbf{x}}e^{-i\mathbf{k}\cdot\mathbf{y}} - \big[a^\dagger_{\mathbf{p}}, a_{\mathbf{k}}\big]e^{-i\mathbf{p}\cdot\mathbf{x}}e^{i\mathbf{k}\cdot\mathbf{y}} \\
&\quad + \big[b_{\mathbf{p}}, b^\dagger_{\mathbf{k}}\big]e^{i\mathbf{p}\cdot\mathbf{x}}e^{-i\mathbf{k}\cdot\mathbf{y}} - \big[b_{\mathbf{p}}, a_{\mathbf{k}}\big]e^{i\mathbf{p}\cdot\mathbf{x}}e^{i\mathbf{k}\cdot\mathbf{y}}\Big) \\
&= \int\int \frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\frac{(iE_{\mathbf{k}})}{\sqrt{2E_{\mathbf{k}}}} \\
&\quad \times \Big(0 + (2\pi)^3\delta(\mathbf{p}-\mathbf{k})e^{-i\mathbf{p}\cdot\mathbf{x}}e^{i\mathbf{k}\cdot\mathbf{y}} \\
&\quad + (2\pi)^3\delta(\mathbf{p}-\mathbf{k})e^{i\mathbf{p}\cdot\mathbf{x}}e^{-i\mathbf{k}\cdot\mathbf{y}} - 0\Big) \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\frac{(iE_{\mathbf{p}})}{\sqrt{2E_{\mathbf{p}}}}\Big(e^{-i\mathbf{p}\cdot\mathbf{x}}e^{i\mathbf{p}\cdot\mathbf{y}} + e^{i\mathbf{p}\cdot\mathbf{x}}e^{-i\mathbf{p}\cdot\mathbf{y}}\Big) \\
&= \frac{i}{2}\int \frac{d^3p}{(2\pi)^3}\Big(e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} + e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\Big) \;\rightarrow\; \mathbf{p} \text{ to } -\mathbf{p} \text{ in 2nd term} \\
&= \frac{i}{2}\int \frac{d^3p}{(2\pi)^3}\Big(e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} + e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\Big) \\
&= i\int \frac{d^3p}{(2\pi)^3}\Big(e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\Big) \\
&= i\delta(\mathbf{x}-\mathbf{y}).
\end{aligned}
$$

$$
\begin{aligned}
\big[\phi(\mathbf{x}), \pi^*(\mathbf{y})\big] &= \int\int \frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\frac{(iE_{\mathbf{k}})}{\sqrt{2E_{\mathbf{k}}}}\Big[a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} + b^\dagger_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}}, b^\dagger_{\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{y}} - a_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{y}}\Big] \\
&= \int\int \frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\frac{(iE_{\mathbf{k}})}{\sqrt{2E_{\mathbf{k}}}} \\
&\quad \times \Big(\big[a_{\mathbf{p}}, b^\dagger_{\mathbf{k}}\big]e^{i\mathbf{p}\cdot\mathbf{x}}e^{-i\mathbf{k}\cdot\mathbf{y}} - \big[a_{\mathbf{p}}, a_{\mathbf{k}}\big]e^{i\mathbf{p}\cdot\mathbf{x}}e^{i\mathbf{k}\cdot\mathbf{y}} \\
&\quad + \big[b^\dagger_{\mathbf{p}}, b^\dagger_{\mathbf{k}}\big]e^{-i\mathbf{p}\cdot\mathbf{x}}e^{-i\mathbf{k}\cdot\mathbf{y}} - \big[b^\dagger_{\mathbf{p}}, a_{\mathbf{k}}\big]e^{-i\mathbf{p}\cdot\mathbf{x}}e^{i\mathbf{k}\cdot\mathbf{y}}\Big) \\
&= 0.
\end{aligned}
$$

$$
\begin{aligned}
\big[\phi^*(\mathbf{x}), \pi(\mathbf{y})\big] &= \int\int \frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\frac{(iE_{\mathbf{k}})}{\sqrt{2E_{\mathbf{k}}}}\Big[a^\dagger_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}} + b_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}}, a^\dagger_{\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{y}} - b_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{y}}\Big] \\
&= \int\int \frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\frac{(iE_{\mathbf{k}})}{\sqrt{2E_{\mathbf{k}}}}
\end{aligned}
$$

$$
\begin{aligned}
&\times \Big(\big[a^\dagger_{\mathbf{p}}, a^\dagger_{\mathbf{k}}\big] e^{-i\mathbf{p}\cdot\mathbf{x}} e^{-i\mathbf{k}\cdot\mathbf{y}} - \big[a^\dagger_{\mathbf{p}}, b_{\mathbf{k}}\big] e^{-i\mathbf{p}\cdot\mathbf{x}} e^{i\mathbf{k}\cdot\mathbf{y}} \\
&+ \big[b_{\mathbf{p}}, a^\dagger_{\mathbf{k}}\big] e^{i\mathbf{p}\cdot\mathbf{x}} e^{-i\mathbf{k}\cdot\mathbf{y}} - \big[b_{\mathbf{p}}, b_{\mathbf{k}}\big] e^{i\mathbf{p}\cdot\mathbf{x}} e^{i\mathbf{k}\cdot\mathbf{y}} \Big) \\
&= 0.
\end{aligned}
$$

$$
\begin{aligned}
\big[\phi(\mathbf{x}), \phi^*(\mathbf{y})\big] &= \int\int \frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\frac{1}{\sqrt{2E_{\mathbf{k}}}}\Big[a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} + b^\dagger_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}}, a^\dagger_{\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{y}} + b_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{y}}\Big] \\
&= \int\int \frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\frac{1}{\sqrt{2E_{\mathbf{k}}}} \\
&\quad \times \Big(\big[a_{\mathbf{p}}, a^\dagger_{\mathbf{k}}\big] e^{i\mathbf{p}\cdot\mathbf{x}} e^{-i\mathbf{k}\cdot\mathbf{y}} + \big[a_{\mathbf{p}}, b_{\mathbf{k}}\big] e^{i\mathbf{p}\cdot\mathbf{x}} e^{i\mathbf{k}\cdot\mathbf{y}} \\
&\quad + \big[b^\dagger_{\mathbf{p}}, a^\dagger_{\mathbf{k}}\big] e^{-i\mathbf{p}\cdot\mathbf{x}} e^{-i\mathbf{k}\cdot\mathbf{y}} + \big[b^\dagger_{\mathbf{p}}, b_{\mathbf{k}}\big] e^{-i\mathbf{p}\cdot\mathbf{x}} e^{i\mathbf{k}\cdot\mathbf{y}} \Big) \\
&= \int\int \frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\frac{1}{\sqrt{2E_{\mathbf{k}}}} \\
&\quad \times \Big((2\pi)^3\delta(\mathbf{p}-\mathbf{k}) e^{i\mathbf{p}\cdot\mathbf{x}} e^{-i\mathbf{k}\cdot\mathbf{y}} + 0 \\
&\quad + 0 - (2\pi)^3\delta(\mathbf{p}-\mathbf{k}) e^{-i\mathbf{p}\cdot\mathbf{x}} e^{i\mathbf{k}\cdot\mathbf{y}} \Big) \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\Big(e^{i\mathbf{p}\cdot\mathbf{x}} e^{-i\mathbf{p}\cdot\mathbf{y}} - e^{-i\mathbf{p}\cdot\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{y}} \Big) \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\Big(e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} - e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \Big) \ \rightarrow\ \mathbf{p} \text{ to } -\mathbf{p} \text{ in 1st term} \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\Big(e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} - e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \Big) \\
&= 0.
\end{aligned}
$$

$$
\begin{aligned}
\big[\pi(\mathbf{x}), \pi^*(\mathbf{y})\big] &= \int\int \frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\frac{(iE_{\mathbf{p}})}{\sqrt{2E_{\mathbf{p}}}}\frac{(iE_{\mathbf{k}})}{\sqrt{2E_{\mathbf{k}}}}\Big[a^\dagger_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}} - b_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}}, b^\dagger_{\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{y}} - a_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{y}}\Big] \\
&= \int\int \frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\frac{(iE_{\mathbf{p}})}{\sqrt{2E_{\mathbf{p}}}}\frac{(iE_{\mathbf{k}})}{\sqrt{2E_{\mathbf{k}}}} \\
&\quad \times \Big(\big[a^\dagger_{\mathbf{p}}, b^\dagger_{\mathbf{k}}\big] e^{-i\mathbf{p}\cdot\mathbf{x}} e^{-i\mathbf{k}\cdot\mathbf{y}} - \big[a^\dagger_{\mathbf{p}}, a_{\mathbf{k}}\big] e^{-i\mathbf{p}\cdot\mathbf{x}} e^{i\mathbf{k}\cdot\mathbf{y}} \\
&\quad - \big[b_{\mathbf{p}}, b^\dagger_{\mathbf{k}}\big] e^{i\mathbf{p}\cdot\mathbf{x}} e^{-i\mathbf{k}\cdot\mathbf{y}} + \big[b_{\mathbf{p}}, a_{\mathbf{k}}\big] e^{i\mathbf{p}\cdot\mathbf{x}} e^{i\mathbf{k}\cdot\mathbf{y}} \Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \int\int \frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\frac{(iE_{\mathbf{p}})}{\sqrt{2E_{\mathbf{p}}}}\frac{(iE_{\mathbf{k}})}{\sqrt{2E_{\mathbf{k}}}} \\
&\quad \times \Big(0 + (2\pi)^3\delta(\mathbf{p}-\mathbf{k})e^{-i\mathbf{p}\cdot\mathbf{x}}e^{i\mathbf{k}\cdot\mathbf{y}} \\
&\quad - (2\pi)^3\delta(\mathbf{p}-\mathbf{k})e^{i\mathbf{p}\cdot\mathbf{x}}e^{-i\mathbf{k}\cdot\mathbf{y}} + 0\Big) \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{(iE_{\mathbf{p}})}{\sqrt{2E_{\mathbf{p}}}}\frac{(iE_{\mathbf{p}})}{\sqrt{2E_{\mathbf{p}}}}\Big(e^{-i\mathbf{p}\cdot\mathbf{x}}e^{i\mathbf{p}\cdot\mathbf{y}} - e^{i\mathbf{p}\cdot\mathbf{x}}e^{-i\mathbf{p}\cdot\mathbf{y}}\Big) \\
&= -\frac{1}{2}\int \frac{d^3p}{(2\pi)^3}E_{\mathbf{p}}\Big(e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} - e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\Big) \ \rightarrow\ \mathbf{p} \text{ to } -\mathbf{p} \text{ in 2nd term} \\
&= -\frac{1}{2}\int \frac{d^3p}{(2\pi)^3}E_{\mathbf{p}}\Big(e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} - e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\Big) \\
&= 0.
\end{aligned}
$$

Note that it is sufficient to impose the canonical commutation relations at $t = 0$, as they are automatically satisfied at a later time, t. This can be shown by making ϕ and π time-dependent in the Heisenberg picture. The key idea here is outlined as follows:

$$\big[A(\mathbf{x},t), B(\mathbf{y},t)\big] = \big[e^{iHt}A(\mathbf{x},0)e^{-iHt}, e^{iHt}B(\mathbf{y},0)e^{-iHt}\big] = e^{iHt}\big[A(\mathbf{x},0), B(\mathbf{y},0)\big]e^{-iHt}.$$

The Hamiltonian is:

$$
\begin{aligned}
H &= \int d^3x\Big(\sum_i \pi_i(\mathbf{x})\dot{\phi}_i(\mathbf{x}) - \mathcal{L}\Big) \\
H &= \int d^3x\Big(\pi(\mathbf{x})\dot{\phi}(\mathbf{x}) + \pi^*(\mathbf{x})\dot{\phi}^*(\mathbf{x}) - \partial_\mu\phi^*\partial^\mu\phi + m^2\phi^*\phi\Big) \\
&= \int d^3x\Big(\pi(\mathbf{x})\pi^*(\mathbf{x}) + \pi^*(\mathbf{x})\pi(\mathbf{x}) - \partial_\mu\phi^*\partial^\mu\phi + m^2\phi^*\phi\Big) \\
&= \int d^3x\Big(\pi(\mathbf{x})\pi^*(\mathbf{x}) + \pi^*(\mathbf{x})\pi(\mathbf{x}) - \partial_0\phi^*\partial^0\phi + \partial_i\phi^*\partial^i\phi + m^2\phi^*\phi\Big) \\
&= \int d^3x\Big(\pi(\mathbf{x})\pi^*(\mathbf{x}) + \pi^*(\mathbf{x})\pi(\mathbf{x}) - \pi(\mathbf{x})\pi^*(\mathbf{x}) + \partial_i\phi^*\partial^i\phi + m^2\phi^*\phi\Big) \\
&= \int d^3x\Big(\pi^*(\mathbf{x})\pi(\mathbf{x}) + \nabla\phi^*\cdot\nabla\phi + m^2\phi^*\phi\Big).
\end{aligned}
$$

The Heisenberg equation of motion for $\phi(x)$ is:

$$
\begin{aligned}
i\frac{\partial}{\partial t}\mathcal{O} &= \big[\mathcal{O}, H\big] \\
i\frac{\partial}{\partial t}\phi(x) &= \int d^3x\Big[\phi(x), \Big(\pi^*(y)\pi(y) + \nabla\phi^*(y)\cdot\nabla\phi(y) + m^2\phi^*(y)\phi(y)\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \int d^3x\Big(\big[\phi(x),\pi^*(y)\pi(y)\big] + \big[\phi(x),\partial_i\phi^*(y)\partial^i\phi(y)\big] + m^2\big[\phi(x),\phi^*(y)\phi(y)\big]\Big) \\
&= \int d^3x\Big(\big[\phi(x),\pi^*(y)\big]\pi(y) + \pi^*(y)\big[\phi(x),\pi(y)\big] + \big[\phi(x),\partial_i\phi^*(y)\big]\partial^i\phi(y) \\
&\quad + \partial_i\phi^*(y)\big[\phi(x),\partial^i\phi(y)\big] + m^2\big[\phi(x),\phi^*(y)\big]\phi(y) + m^2\phi^*(y)\big[\phi(x),\phi(y)\big]\Big) \\
&= \int d^3x\Big(0 + \pi^*(y)(i\delta(\mathbf{x}-\mathbf{y})) + 0 + 0 + 0 + 0\Big) \\
&= i\pi^*(x),
\end{aligned}
$$

which is just the conjugate momentum expression that was previously calculated. The Heisenberg equation of motion for $\pi^*(x)$ is:

$$
\begin{aligned}
i\frac{\partial}{\partial t}\mathcal{O} &= \big[\mathcal{O},H\big] \\
i\frac{\partial}{\partial t}\pi^*(x) &= \Big[\pi^*(x),\int d^3y\Big(\pi^*(y)\pi(y) + \nabla\phi^*(y)\cdot\nabla\phi(y) + m^2\phi^*(y)\phi(y)\Big)\Big] \\
&= \int d^3y\big[\pi^*(x),\pi^*(y)\pi(y)\big] \\
&\quad + \Big[\pi^*(x),\int d^3y\nabla\phi^*(y)\cdot\nabla\phi(y)\Big] \\
&\quad + m^2\int d^3y\Big[\pi^*(x),\phi^*(y)\phi(y)\Big] \\
&= \int d^3y\Big(\big[\pi^*(x),\pi^*(y)\big]\pi(y) + \pi^*(y)\big[\pi^*(x),\pi(y)\big]\Big) \\
&\quad + \Big[\pi^*(x),\int d^3y\Big(\nabla\cdot\big(\phi^*(y)\nabla\phi(y)\big) - \phi^*(y)\nabla^2\phi(y)\Big)\Big] \\
&\quad + m^2\int d^3y\Big(\big[\pi^*(x),\phi^*(y)\big]\phi(y) + \phi^*(y)\big[\pi^*(x),\phi(y)\big]\Big) \\
&= 0 + 0 \\
&\quad + \Big[\pi^*(x),\int dS\Big(\big(\phi^*(y)\nabla\phi(y)\big)\cdot\hat{\mathbf{n}}\Big) - \int d^3y\phi^*(y)\nabla^2\phi(y)\Big] \\
&\quad + m^2\int d^3y\Big(-i\delta(\mathbf{x}-\mathbf{y})\phi(y)\Big) + 0 \\
&= 0 - \int d^3y\Big[\pi^*(x),\phi^*(y)\nabla^2\phi(y)\Big] - im^2\phi(x) \\
&= -\int d^3y\Big(\phi^*(y)\Big[\pi^*(x),\nabla^2\phi(y)\Big] + \Big[\pi^*(x),\phi^*(y)\Big]\nabla^2\phi(y)\Big) - im^2\phi(x)
\end{aligned}
$$

$$
\begin{aligned}
&= -\int d^3y\Big(\phi^*(y)\big(\nabla^2\big[\phi^*(x),\phi(y)\big]\big) - i\delta(\mathbf{x}-\mathbf{y})\nabla^2\phi(y)\Big) - im^2\phi(x) \\
&= \int d^3y\Big(i\delta(\mathbf{x}-\mathbf{y})\nabla^2\phi(y)\Big) - im^2\phi(x) \\
&= i\big(\nabla^2\phi(x) - m^2\phi(x)\big) \\
&= i\big(\nabla^2 - m^2\big)\phi(x),
\end{aligned}
$$

where the vector identity $\nabla\cdot(A\nabla B) = A\nabla^2 B + \nabla A\cdot\nabla B$ and Gauss' theorem were used to handle the gradient terms (the surface term is zero). We can show these equations, when combined, are indeed the Klein-Gordon equation as follows:

$$
\begin{aligned}
i\frac{\partial}{\partial t}\pi^*(x) &= i\big(\nabla^2 - m^2\big)\phi(x) \\
\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}\phi(x)\right) &= \big(\nabla^2 - m^2\big)\phi(x) \\
\frac{\partial^2}{\partial t^2}\phi(x) &= \big(\nabla^2 - m^2\big)\phi(x) \\
\partial_0\partial^0\phi(x) &= \partial_i\partial^i\phi(x) - m^2\phi(x) \\
\partial_0\partial^0\phi(x) - \partial_i\partial^i\phi(x) + m^2\phi(x) &= 0 \\
\partial_\mu\partial^\mu\phi(x) + m^2\phi(x) &= 0 \\
\Big(\partial_\mu\partial^\mu + m^2\Big)\phi(x) &= 0 \;\rightarrow\; \text{Klein-Gordon equation.}
\end{aligned}
$$

The Klein-Gordon equation for $\phi^*(x)$ and $\pi(x)$ can be found in the same manner.

(b) The Hamiltonian, H, can be rewritten in terms of creation and annihilation operators as follows:

$$
\begin{aligned}
H = &\int d^3x\Big(\pi^*(\mathbf{x})\pi(\mathbf{x}) + \nabla\phi^*(\mathbf{x})\cdot\nabla\phi(\mathbf{x}) + m^2\phi^*(\mathbf{x})\phi(\mathbf{x})\Big) \\
= &\int d^3x\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3k}{(2\pi)^3} \\
&\times\Bigg(\frac{iE_{\mathbf{p}}}{\sqrt{2E_{\mathbf{p}}}}\frac{iE_{\mathbf{k}}}{\sqrt{2E_{\mathbf{k}}}}\big(b^\dagger_{\mathbf{p}}e^{ip\cdot x} - a_{\mathbf{p}}e^{-ip\cdot x}\big)\big(a^\dagger_{\mathbf{k}}e^{ik\cdot x} - b_{\mathbf{k}}e^{-ik\cdot x}\big) \\
&+ \frac{(-i\mathbf{p})}{\sqrt{2E_{\mathbf{p}}}}\cdot\frac{(i\mathbf{k})}{\sqrt{2E_{\mathbf{k}}}}\big(a^\dagger_{\mathbf{p}}e^{ip\cdot x} - b_{\mathbf{p}}e^{-ip\cdot x}\big)\big(a_{\mathbf{k}}e^{-ik\cdot x} - b^\dagger_{\mathbf{k}}e^{ik\cdot x}\big) \\
&+ m^2\frac{1}{\sqrt{2E_{\mathbf{p}}}}\frac{1}{\sqrt{2E_{\mathbf{k}}}}\big(a^\dagger_{\mathbf{p}}e^{ip\cdot x} + b_{\mathbf{p}}e^{-ip\cdot x}\big)\big(a_{\mathbf{k}}e^{-ik\cdot x} + b^\dagger_{\mathbf{k}}e^{ik\cdot x}\big)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&= \int d^3x \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \frac{1}{\sqrt{2E_{\mathbf{k}}}} \\
&\quad \times \Bigg(- E_{\mathbf{p}} E_{\mathbf{k}} \big(b^\dagger_{\mathbf{p}} a^\dagger_{\mathbf{k}} e^{i(p+k)\cdot x} - b^\dagger_{\mathbf{p}} b_{\mathbf{k}} e^{i(p-k)\cdot x} - a_{\mathbf{p}} a^\dagger_{\mathbf{k}} e^{-i(p-k)\cdot x} + a_{\mathbf{p}} b_{\mathbf{k}} e^{-i(p+k)\cdot x} \big) \\
&\quad + \mathbf{p}\cdot\mathbf{k} \, \big(a^\dagger_{\mathbf{p}} a_{\mathbf{k}} e^{i(p-k)\cdot x} - a^\dagger_{\mathbf{p}} b^\dagger_{\mathbf{k}} e^{i(p+k)\cdot x} - b_{\mathbf{p}} a_{\mathbf{k}} e^{-i(p+k)\cdot x} + b_{\mathbf{p}} b^\dagger_{\mathbf{k}} e^{-i(p-k)\cdot x} \big) \\
&\quad + m^2 \big(a^\dagger_{\mathbf{p}} a_{\mathbf{k}} e^{i(p-k)\cdot x} + a^\dagger_{\mathbf{p}} b^\dagger_{\mathbf{k}} e^{i(p+k)\cdot x} + b_{\mathbf{p}} a_{\mathbf{k}} e^{-i(p+k)\cdot x} + b_{\mathbf{p}} b^\dagger_{\mathbf{k}} e^{-i(p-k)\cdot x} \big) \Bigg) \\
&= \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \frac{1}{\sqrt{2E_{\mathbf{k}}}} (2\pi)^3 \\
&\quad \times \Bigg(- E_{\mathbf{p}} E_{\mathbf{k}} \big(b^\dagger_{\mathbf{p}} a^\dagger_{\mathbf{k}} \delta(\mathbf{p}+\mathbf{k}) - b^\dagger_{\mathbf{p}} b_{\mathbf{k}} \delta(\mathbf{p}-\mathbf{k}) - a_{\mathbf{p}} a^\dagger_{\mathbf{k}} \delta(\mathbf{p}-\mathbf{k}) + a_{\mathbf{p}} b_{\mathbf{k}} \delta(\mathbf{p}+\mathbf{k}) \big) \\
&\quad + \mathbf{p}\cdot\mathbf{k} \, \big(a^\dagger_{\mathbf{p}} a_{\mathbf{k}} \delta(\mathbf{p}-\mathbf{k}) - a^\dagger_{\mathbf{p}} b^\dagger_{\mathbf{k}} \delta(\mathbf{p}+\mathbf{k}) - b_{\mathbf{p}} a_{\mathbf{k}} \delta(\mathbf{p}+\mathbf{k}) + b_{\mathbf{p}} b^\dagger_{\mathbf{k}} \delta(\mathbf{p}-\mathbf{k}) \big) \\
&\quad + m^2 \big(a^\dagger_{\mathbf{p}} a_{\mathbf{k}} \delta(\mathbf{p}-\mathbf{k}) + a^\dagger_{\mathbf{p}} b^\dagger_{\mathbf{k}} \delta(\mathbf{p}+\mathbf{k}) + b_{\mathbf{p}} a_{\mathbf{k}} \delta(\mathbf{p}+\mathbf{k}) + b_{\mathbf{p}} b^\dagger_{\mathbf{k}} \delta(\mathbf{p}-\mathbf{k}) \big) \Bigg) \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \\
&\quad \times \Bigg(- E^2_{\mathbf{p}} b^\dagger_{\mathbf{p}} a^\dagger_{-\mathbf{p}} + E^2_{\mathbf{p}} b^\dagger_{\mathbf{p}} b_{\mathbf{p}} + E^2_{\mathbf{p}} a_{\mathbf{p}} a^\dagger_{\mathbf{p}} - E^2_{\mathbf{p}} a_{\mathbf{p}} b_{-\mathbf{p}} \\
&\quad + \mathbf{p}^2 a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + \mathbf{p}^2 a^\dagger_{\mathbf{p}} b^\dagger_{-\mathbf{p}} + \mathbf{p}^2 b_{\mathbf{p}} a_{-\mathbf{p}} + \mathbf{p}^2 b_{\mathbf{p}} b^\dagger_{\mathbf{p}} \\
&\quad + m^2 a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + m^2 a^\dagger_{\mathbf{p}} b^\dagger_{-\mathbf{p}} + m^2 b_{\mathbf{p}} a_{-\mathbf{p}} + m^2 b_{\mathbf{p}} b^\dagger_{\mathbf{p}} \Bigg) \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \\
&\quad \times \Bigg(\big(- E^2_{\mathbf{p}} + \mathbf{p}^2 + m^2 \big) b^\dagger_{\mathbf{p}} a^\dagger_{-\mathbf{p}} + \big(- E^2_{\mathbf{p}} + \mathbf{p}^2 + m^2 \big) a_{\mathbf{p}} b_{-\mathbf{p}} \\
&\quad + E^2_{\mathbf{p}} a_{\mathbf{p}} a^\dagger_{\mathbf{p}} + \mathbf{p}^2 a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + m^2 a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + E^2_{\mathbf{p}} b^\dagger_{\mathbf{p}} b_{\mathbf{p}} + \mathbf{p}^2 b_{\mathbf{p}} b^\dagger_{\mathbf{p}} + m^2 b_{\mathbf{p}} b^\dagger_{\mathbf{p}} \Bigg) \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \Bigg(E^2_{\mathbf{p}} \big(a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + (2\pi)^3 \delta(0) \big) + \mathbf{p}^2 a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + m^2 a^\dagger_{\mathbf{p}} a_{\mathbf{p}} \\
&\quad + E^2_{\mathbf{p}} b^\dagger_{\mathbf{p}} b_{\mathbf{p}} + \mathbf{p}^2 \big(b^\dagger_{\mathbf{p}} b_{\mathbf{p}} + (2\pi)^3 \delta(0) \big) + m^2 \big(b^\dagger_{\mathbf{p}} b_{\mathbf{p}} + (2\pi)^3 \delta(0) \big) \Bigg) \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \Bigg(\big(E^2_{\mathbf{p}} + \mathbf{p}^2 + m^2 \big) a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + E^2_{\mathbf{p}} (2\pi)^3 \delta(0) \big)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \left(E_{\mathbf{p}}^2 + \mathbf{p}^2 + m^2\right) b_{\mathbf{p}}^\dagger b_{\mathbf{p}} + \mathbf{p}^2 (2\pi)^3 \delta(0) + m^2 (2\pi)^3 \delta(0) \Big) \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \left(2E_{\mathbf{p}}^2 a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + 2E_{\mathbf{p}}^2 b_{\mathbf{p}}^\dagger b_{\mathbf{p}} + 2E_{\mathbf{p}}^2 (2\pi)^3 \delta(0) \right) \\
&= \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \left(a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + b_{\mathbf{p}}^\dagger b_{\mathbf{p}} + (2\pi)^3 \delta(0) \right) \\
&= \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \left(a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + b_{\mathbf{p}}^\dagger b_{\mathbf{p}} \right),
\end{aligned}
$$

where the $\delta(0)$ term was dropped in the last step. To show that this theory contains two sets of particles with mass m, we can interpret the Hamiltonian in terms of its action on the one-particle states $|\mathbf{p}\rangle_a = a_{\mathbf{p}}^\dagger |0\rangle$ and $|\mathbf{p}\rangle_b = b_{\mathbf{p}}^\dagger |0\rangle$, where the state $|0\rangle$ is the ground state, as follows:

$$
\begin{aligned}
H |\mathbf{p}\rangle_a &= \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} \left(a_{\mathbf{q}}^\dagger a_{\mathbf{q}} + b_{\mathbf{q}}^\dagger b_{\mathbf{q}} \right) a_{\mathbf{p}}^\dagger |0\rangle \\
&= \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} \left(a_{\mathbf{q}}^\dagger a_{\mathbf{q}} a_{\mathbf{p}}^\dagger |0\rangle + b_{\mathbf{q}}^\dagger b_{\mathbf{q}} a_{\mathbf{p}}^\dagger |0\rangle \right) \\
&= \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} \left((2\pi)^3 \delta(\mathbf{q} - \mathbf{p}) a_{\mathbf{q}}^\dagger |0\rangle + b_{\mathbf{q}}^\dagger a_{\mathbf{p}}^\dagger b_{\mathbf{q}} |0\rangle \right) \\
&= E_{\mathbf{p}} a_{\mathbf{p}}^\dagger |0\rangle \\
&= E_{\mathbf{p}} |\mathbf{p}\rangle_a .
\end{aligned}
$$

Similarly,

$$
H |\mathbf{p}\rangle_b = E_{\mathbf{p}} |\mathbf{p}\rangle_b .
$$

Therefore, the Hamiltonian describes two sets of particles: one created by $a_{\mathbf{p}}^\dagger$ and the other by $b_{\mathbf{p}}^\dagger$, each with momentum $\mathbf{p}$ and energy $E_{\mathbf{p}}$. Since $E_{\mathbf{p}}^2 = p^2 + m^2$, both particles have the same mass m.

(c) The conserved charge, Q, can be rewritten in terms of creation and annihilation operators as follows:

$$
\begin{aligned}
Q &= \int d^3x \frac{i}{2} \left(\phi^*(\mathbf{x}) \pi^*(\mathbf{x}) - \pi(\mathbf{x}) \phi(\mathbf{x}) \right) \\
&= \int d^3x \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3k}{(2\pi)^3} \frac{i}{2} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \frac{1}{\sqrt{2E_{\mathbf{k}}}} \\
&\quad \times \Big((iE_{\mathbf{k}}) \left(a_{\mathbf{p}}^\dagger e^{ip\cdot x} + b_{\mathbf{p}} e^{-ip\cdot x} \right) \left(b_{\mathbf{k}}^\dagger e^{ik\cdot x} - a_{\mathbf{k}} e^{-ik\cdot x} \right)
\end{aligned}
$$

$$- (iE_{\mathbf{p}})\big(a_{\mathbf{p}}^\dagger e^{ip\cdot x} - b_{\mathbf{p}} e^{-ip\cdot x}\big)\big(a_{\mathbf{k}} e^{-ik\cdot x} + b_{\mathbf{k}}^\dagger e^{ik\cdot x}\big)\Big)$$

$$= \int d^3x \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3k}{(2\pi)^3} \frac{i}{2} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \frac{1}{\sqrt{2E_{\mathbf{k}}}}$$

$$\times \Big((iE_{\mathbf{k}})\big(a_{\mathbf{p}}^\dagger b_{\mathbf{k}}^\dagger e^{ip\cdot x} e^{ik\cdot x} - a_{\mathbf{p}}^\dagger a_{\mathbf{k}} e^{ip\cdot x} e^{-ik\cdot x} + b_{\mathbf{p}} b_{\mathbf{k}}^\dagger e^{-ip\cdot x} e^{ik\cdot x} - b_{\mathbf{p}} a_{\mathbf{k}} e^{-ip\cdot x} e^{-ik\cdot x}\big)$$

$$- (iE_{\mathbf{p}})\big(a_{\mathbf{p}}^\dagger a_{\mathbf{k}} e^{ip\cdot x} e^{-ik\cdot x} + a_{\mathbf{p}}^\dagger b_{\mathbf{k}}^\dagger e^{ip\cdot x} e^{ik\cdot x} - b_{\mathbf{p}} a_{\mathbf{k}} e^{-ip\cdot x} e^{-ik\cdot x} - b_{\mathbf{p}} b_{\mathbf{k}}^\dagger e^{-ip\cdot x} e^{ik\cdot x}\big)\Big)$$

$$= \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3k}{(2\pi)^3} \frac{i}{2} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \frac{1}{\sqrt{2E_{\mathbf{k}}}}$$

$$\times \Big((iE_{\mathbf{k}})\big(a_{\mathbf{p}}^\dagger b_{\mathbf{k}}^\dagger \delta(\mathbf{p}+\mathbf{k}) - a_{\mathbf{p}}^\dagger a_{\mathbf{k}} \delta(\mathbf{p}-\mathbf{k}) + b_{\mathbf{p}} b_{\mathbf{k}}^\dagger \delta(\mathbf{p}-\mathbf{k}) - b_{\mathbf{p}} a_{\mathbf{k}} \delta(\mathbf{p}+\mathbf{k})\big)$$

$$- (iE_{\mathbf{p}})\big(a_{\mathbf{p}}^\dagger a_{\mathbf{k}} \delta(\mathbf{p}-\mathbf{k}) + a_{\mathbf{p}}^\dagger b_{\mathbf{k}}^\dagger \delta(\mathbf{p}+\mathbf{k}) - b_{\mathbf{p}} a_{\mathbf{k}} \delta(\mathbf{p}+\mathbf{k}) - b_{\mathbf{p}} b_{\mathbf{k}}^\dagger \delta(\mathbf{p}-\mathbf{k})\big)\Big)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{i}{2} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \frac{1}{\sqrt{2E_{\mathbf{p}}}}$$

$$\times \Big((iE_{\mathbf{p}})\big(a_{\mathbf{p}}^\dagger b_{-\mathbf{p}}^\dagger - a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + b_{\mathbf{p}} b_{\mathbf{p}}^\dagger - b_{\mathbf{p}} a_{-\mathbf{p}}\big)$$

$$- (iE_{\mathbf{p}})\big(a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + a_{\mathbf{p}}^\dagger b_{-\mathbf{p}}^\dagger - b_{\mathbf{p}} a_{-\mathbf{p}} - b_{\mathbf{p}} b_{\mathbf{p}}^\dagger\big)\Big)$$

$$= \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \Big(a_{\mathbf{p}}^\dagger a_{\mathbf{p}} - b_{\mathbf{p}} b_{\mathbf{p}}^\dagger \Big)$$

$$= \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \Big(a_{\mathbf{p}}^\dagger a_{\mathbf{p}} - b_{\mathbf{p}}^\dagger b_{\mathbf{p}} - (2\pi)^3 \delta(0) \Big)$$

$$= \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \Big(a_{\mathbf{p}}^\dagger a_{\mathbf{p}} - b_{\mathbf{p}}^\dagger b_{\mathbf{p}} \Big),$$

where the $\delta(0)$ term was dropped in the last step. To evaluate the charge carried by each particle type, we can interpret the operator Q in terms of its action on the one-particle states $|\mathbf{p}\rangle_a = a_{\mathbf{p}}^\dagger |0\rangle$ and $|\mathbf{p}\rangle_b = b_{\mathbf{p}}^\dagger |0\rangle$, where the state $|0\rangle$ is the ground state, as follows:

$$\begin{aligned}
Q\,|\mathbf{p}\rangle_a &= \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} \Big(a_{\mathbf{q}}^\dagger a_{\mathbf{q}} - b_{\mathbf{q}}^\dagger b_{\mathbf{q}} \Big) a_{\mathbf{p}}^\dagger |0\rangle \\
&= \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} \Big(a_{\mathbf{q}}^\dagger a_{\mathbf{q}} a_{\mathbf{p}}^\dagger |0\rangle - b_{\mathbf{q}}^\dagger b_{\mathbf{q}} a_{\mathbf{p}}^\dagger |0\rangle \Big) \\
&= \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} \Big((2\pi)^3 \delta(\mathbf{q}-\mathbf{p}) a_{\mathbf{q}}^\dagger |0\rangle - b_{\mathbf{q}}^\dagger a_{\mathbf{p}}^\dagger b_{\mathbf{q}} |0\rangle \Big)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2} a^\dagger_{\mathbf{p}} \left|0\right\rangle \\
&= \frac{1}{2} \left|\mathbf{p}\right\rangle_a .
\end{aligned}
$$

Similarly,

$$
Q \left|\mathbf{p}\right\rangle_b \; = \; -\frac{1}{2} \left|\mathbf{p}\right\rangle_b .
$$

Therefore, the particles created by $a^\dagger_{\mathbf{p}}$ carry a charge of $+1/2$, while the particles created by $b^\dagger_{\mathbf{p}}$ carry a charge of $-1/2$.

(d) For the case of two complex Klein-Gordon fields with the same mass, labeled as $\phi_a(x)$, where $a = 1, 2$, the action can be written as:

$$
S = \int d^4x \Big(\partial_\mu \phi_a^* \partial^\mu \phi_a - m^2 \phi_a^* \phi_a \Big).
$$

To identify the conserved charges, we first need to determine the continuous transformations that leave the Lagrangian invariant. These transformations are of the form $\phi_a \to U_{ab} \phi_b$, where U_{ab} are components of a linear operator. Since we have two fields, transformations satisfying these conditions are unitary transformations described by the unitary group U(2), which is the set of all 2×2 unitary matrices. This set forms a non-Abelian group with the group operation being matrix multiplication. It is important to note that U_{ab} are the elements of these matrices. To analyze the symmetry structure of this theory, we make use of the generators associated with the U(1) and SU(2) subgroups of U(2). The U(1) subgroup has one generator, denoted T^0, given by the identity matrix multiplied by i, i.e., $T^0 = i\mathbb{I}$. The Lagrangian is invariant under the transformation:

$$
\phi_a \to (e^{-\alpha T^0})_{ab} \phi_b,
$$

where α is an infinitesimal parameter. Furthermore, this transformation gives the following infinitesimal change:

$$
(e^{-\alpha T^0})_{ab} \phi_b = (\mathbb{I} - i\alpha\mathbb{I})_{ab} \phi_b + \mathcal{O}(\alpha^2) \approx \phi_a - i\alpha \phi_a .
$$

For U(1), we have the following Noether current:

$$
\begin{aligned}
j^\mu(x) &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \Delta\phi_a + \Delta\phi_a^* \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a^*)} \\
&= \big(\partial_\mu \phi_a^*\big)\big(-i\alpha\phi_a\big) + \big(i\alpha\phi_a^*\big)\big(\partial_\mu \phi_a\big) \\
&= i\Big(\phi_a^* \big(\partial_\mu \phi_a\big) - \big(\partial_\mu \phi_a^*\big)\phi_a\Big),
\end{aligned}
$$

where we have arbitrarily chosen α. Accordingly, the single charge from this symmetry, which is the generalization of part (c), is given by:

$$
\begin{aligned}
Q &= \int j^0 d^3x \\
&= i \int d^3x \big(\phi_a^* \dot{\phi}_a - \dot{\phi}_a^* \phi_a\big) \;\; \rightarrow \;\text{use conjugate momenta notation} \\
&= i \int d^3x \big(\phi_a^* \pi_a^* - \pi_a \phi_a\big).
\end{aligned}
$$

The U(2) subgroup has three generators, which are the Pauli matrices, σ^k, multiplied by $i/2$, i.e., $T^k = \frac{i}{2}\sigma^k$ for $k = 1, 2, 3$. The Lagrangian is invariant under the transformation:

$$
\phi_a \rightarrow (e^{-\beta^k T^k})_{ab}\phi_b,
$$

where β^k is an infinitesimal parameter. Similarly, this transformation gives the following infinitesimal change:

$$
(e^{-\beta^k T^k})_{ab}\phi_b = (\mathbb{I} - \frac{i}{2}\beta^k\sigma^k)_{ab}\phi_b + \mathcal{O}((\beta^k)^2) \approx \phi_a - \frac{i}{2}(\beta^k\sigma^k)_{ab}\phi_b.
$$

For SU(2), we have the following Noether currents:

$$
\begin{aligned}
j^{\mu,k}(x) &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\Delta^k\phi_a + \Delta^k\phi_a^*\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a^*)} \\
&= \Big(\partial_\mu \phi_a^*\Big)\Big(-\frac{i}{2}(\beta^k\sigma^k)_{ab}\phi_b\Big) + \Big(\frac{i}{2}(\beta^k\sigma^k)^*_{ab}\phi_b^*\Big)\Big(\partial_\mu\phi_a\Big) \\
&= \frac{i}{2}\Big((\sigma^k)^*_{ab}\phi_b^*(\partial_\mu\phi_a) - (\partial_\mu\phi_a^*)(\sigma^k)_{ab}\phi_b\Big) \;\; \rightarrow \;\; (\sigma^k)^*_{ab} = (\sigma^k)_{ba} \text{ and relabel indices} \\
&= \frac{i}{2}\Big((\sigma^k)_{ab}\phi_a^*(\partial_\mu\phi_b) - (\partial_\mu\phi_a^*)(\sigma^k)_{ab}\phi_b\Big) \\
&= \frac{i}{2}\Big(\phi_a^*(\sigma^k)_{ab}(\partial_\mu\phi_b) - (\partial_\mu\phi_a^*)(\sigma^k)_{ab}\phi_b\Big)
\end{aligned}
$$

where we have arbitrarily chosen β^k. Accordingly, the three charges from this symmetry are given by:

$$
\begin{aligned}
Q^k &= \int j^{0,k} d^3x \\
&= \frac{i}{2}\int d^3x \Big(\phi_a^*(\sigma^k)_{ab}\dot{\phi}_b - \dot{\phi}_a^*(\sigma^k)_{ab}\phi_b\Big) \;\; \rightarrow \;\text{use conjugate momenta notation} \\
&= \frac{i}{2}\int d^3x \Big(\phi_a^*(\sigma^k)_{ab}\pi_b^* - \pi_a(\sigma^k)_{ab}\phi_b\Big).
\end{aligned}
$$

These charges having the commutation relations of angular momentum is shown as follows:

$$
\begin{aligned}
\Big[Q^i, Q^j\Big] =& -\frac{1}{4}\int d^3x \int d^3y \; \Big[\Big(\phi_a^*(\sigma^i)_{ab}\pi_b^* - \pi_a(\sigma^i)_{ab}\phi_b\Big), \Big(\phi_c^*(\sigma^j)_{cd}\pi_d^* - \pi_c(\sigma^j)_{cd}\phi_d\Big)\Big] \\
=& -\frac{1}{4}\int d^3x \int d^3y \\
&\times \bigg(\Big[\phi_a^*(\sigma^i)_{ab}\pi_b^*, \phi_c^*(\sigma^j)_{cd}\pi_d^*\Big] - \Big[\phi_a^*(\sigma^i)_{ab}\pi_b^*, \pi_c(\sigma^j)_{cd}\phi_d\Big] \\
&- \Big[\pi_a(\sigma^i)_{ab}\phi_b, \phi_c^*(\sigma^j)_{cd}\pi_d^*\Big] + \Big[\pi_a(\sigma^i)_{ab}\phi_b, \pi_c(\sigma^j)_{cd}\phi_d\Big]\bigg) \\
=& -\frac{1}{4}\int d^3x \int d^3y \; (\sigma^i)_{ab}(\sigma^j)_{cd} \\
&\times \bigg(\Big[\phi_a^*\pi_b^*, \phi_c^*\pi_d^*\Big] - \Big[\phi_a^*\pi_b^*, \pi_c\phi_d\Big] - \Big[\pi_a\phi_b, \phi_c^*\pi_d^*\Big] + \Big[\pi_a\phi_b, \pi_c\phi_d\Big]\bigg) \\
=& -\frac{1}{4}\int d^3x \int d^3y \; (\sigma^i)_{ab}(\sigma^j)_{cd} \\
&\times \bigg(\phi_a^*\Big[\pi_b^*, \phi_c^*\Big]\pi_d^* + \Big[\phi_a^*, \phi_c^*\Big]\pi_b^*\pi_d^* + \Big[\phi_c^*, \phi_a^*\Big]\pi_b^*\pi_d^* + \phi_c^*\Big[\phi_a^*, \pi_d^*\Big]\pi_b^* \\
&- \phi_a^*\Big[\pi_b^*, \pi_c\Big]\phi_d - \Big[\phi_a^*, \pi_c\Big]\pi_b^*\phi_d - \pi_c\phi_a^*\Big[\pi_b^*, \phi_d\Big] - \pi_c\Big[\phi_a^*, \phi_d\Big]\pi_b^* \\
&- \pi_a\Big[\phi_b, \phi_c^*\Big]\pi_d^* - \Big[\pi_a, \phi_c^*\Big]\phi_b\pi_d^* - \phi_c^*\pi_a\Big[\phi_b, \pi_d^*\Big] - \phi_c^*\Big[\pi_a, \pi_d^*\Big]\phi_b \\
&+ \pi_a\Big[\phi_b, \pi_c\Big]\phi_d + \Big[\pi_a, \pi_c\Big]\phi_b\phi_d + \pi_c\pi_a\Big[\phi_b, \phi_d\Big] + \pi_c\Big[\pi_a, \phi_d\Big]\phi_b\bigg) \\
=& -\frac{1}{4}\int d^3x \int d^3y \; (\sigma^i)_{ab}(\sigma^j)_{cd} \\
&\times \bigg(\phi_a^*\big(-i\delta_{bc}\delta(\mathbf{x}-\mathbf{y})\big)\pi_d^* + 0 + 0 + \phi_c^*\big(i\delta_{ad}\delta(\mathbf{x}-\mathbf{y})\big)\pi_b^* \\
&- 0 - 0 - 0 - 0 \\
&- 0 - 0 - 0 - 0 \\
&+ \pi_a\big(i\delta_{bc}\delta(\mathbf{x}-\mathbf{y})\big)\phi_d + 0 + 0 + \pi_c\big(-i\delta_{ad}\delta(\mathbf{x}-\mathbf{y})\big)\phi_b\bigg) \\
=& -\frac{1}{4}\int d^3x \; (\sigma^i)_{ab}(\sigma^j)_{cd}\bigg(-i\delta_{bc}\phi_a^*\pi_d^* + i\delta_{ad}\phi_c^*\pi_b^* + i\delta_{bc}\pi_a\phi_d - i\delta_{ad}\pi_c\phi_b\bigg) \\
=& \frac{i}{4}\int d^3x \; \bigg(\phi_a^*(\sigma^i\sigma^j)_{ad}\pi_d^* - \phi_c^*(\sigma^j\sigma^i)_{cb}\pi_b^* - \pi_a(\sigma^i\sigma^j)_{ad}\phi_d + \pi_c(\sigma^j\sigma^i)_{cd}\phi_b\bigg)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{i}{4}\int d^3x\, \left(\phi_a^*(\sigma^i\sigma^j)_{ab}\pi_b^* - \phi_a^*(\sigma^j\sigma^i)_{ab}\pi_b^* - \pi_a(\sigma^i\sigma^j)_{ab}\phi_b + \pi_a(\sigma^j\sigma^i)_{ab}\phi_b\right) \\
&= \frac{i}{4}\int d^3x\, \left(\phi_a^*\Big((\sigma^i\sigma^j)_{ab} - (\sigma^j\sigma^i)_{ab}\Big)\pi_b^* - \pi_a\Big((\sigma^i\sigma^j)_{ab} - (\sigma^j\sigma^i)_{ab}\Big)\phi_b\right) \\
&= \frac{i}{4}\int d^3x\, \left(\phi_a^*\big[\sigma^i,\sigma^j\big]_{ab}\pi_b^* - \pi_a\big[\sigma^i,\sigma^j\big]_{ab}\phi_b\right) \\
&= \frac{i}{4}\int d^3x\, \left(\phi_a^*\Big(2i\epsilon^{ijk}(\sigma^k)_{ab}\Big)\pi_b^* - \pi_a\Big(2i\epsilon^{ijk}(\sigma^k)_{ab}\Big)\phi_b\right) \\
&= i\epsilon^{ijk}\int d^3x\, \frac{i}{2}\Big(\phi_a^*(\sigma^k)_{ab}\pi_b^* - \pi_a(\sigma^k)_{ab}\phi_b\Big) \\
&= i\epsilon^{ijk}Q^k.
\end{aligned}
$$

These results can be naturally generalized to the case of n identical complex scalar fields. In this context, the relevant symmetry group becomes the unitary group $\mathrm{U}(n)$. This group has n^2 generators: one corresponding to $\mathrm{U}(1)$ and n^2-1 corresponding to $\mathrm{SU}(n)$. The associated Noether currents, $j^{\mu,k}$, and conserved charges, Q^k, have the same form as above. Additionally, the Pauli matrices are replaced by the generators T^k of $\mathrm{SU}(n)$, and the Levi-Civita symbol is replaced by the structure constants f^{ijk}, which define the commutation relations between the generators. To address the footnote, you can treat the n complex scalar fields by their real and imaginary components as $2n$ real fields. Then, instead of transformations described by the unitary group $\mathrm{U}(n)$, we consider transformations described by the special orthogonal group $\mathrm{SO}(N)$, which has $N(N-1)/2$ generators, or $n(2n-1)$ generators with $N=2n$.

2.3 Spacelike propagation amplitude

Evaluate the function

$$
\langle 0\,|\,\phi(x)\phi(y)\,|\,0\rangle \;=D(x-y)=\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-ip\cdot(x-y)},
$$

for $(x-y)$ spacelike so that $(x-y)^2=-r^2$, explicitly in terms of Bessel functions. For the case where $(x-y)$ is spacelike, we have $x^0-y^0=0$ and $\mathbf{x}-\mathbf{y}=\mathbf{r}$. Using this, the propagation amplitude becomes:

$$
\begin{aligned}
D(x-y) &= \int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-ip\cdot(x-y)} \;\rightarrow\; \text{apply spacelike condition} \\
&= \int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{i\mathbf{p}\cdot\mathbf{r}} \;\rightarrow\; \text{switch to spherical coordinates and rewrite } E_{\mathbf{p}} \\
&= \frac{1}{(2\pi)^3}\int_0^\infty p^2dp\int_0^\pi \sin(\theta)d\theta\int_0^{2\pi}d\phi\frac{e^{ipr\cos(\theta)}}{2\sqrt{p^2+m^2}} \;\rightarrow\; \text{do } \phi \text{ integral}
\end{aligned}
$$

$$= \frac{(2\pi)}{2(2\pi)^3} \int_0^\infty p^2 dp \int_0^\pi \sin(\theta) d\theta \frac{e^{ipr\cos(\theta)}}{\sqrt{p^2+m^2}} \;\rightarrow\; \text{u substitution for } \theta \text{ integral}$$

$$= \frac{1}{2(2\pi)^2} \int_0^\infty p^2 dp \int_{-1}^1 du \frac{e^{ipru}}{\sqrt{p^2+m^2}} \;\rightarrow\; \text{do } u \text{ integral}$$

$$= \frac{1}{2(2\pi)^2} \int_0^\infty p^2 dp \frac{1}{\sqrt{p^2+m^2}} \frac{2\sin(pr)}{pr} \;\rightarrow\; \text{simplify}$$

$$= \frac{1}{r(2\pi)^2} \int_0^\infty dp \frac{p\sin(pr)}{\sqrt{p^2+m^2}} \;\rightarrow\; \text{use Gradstein and Ryzhik equation 3.754.2}$$

$$= \frac{1}{r(2\pi)^2} \left(-\frac{\partial}{\partial r} K_0(rm) \right) \;\rightarrow\; \frac{\partial}{\partial r} K_0(rm) = -mK_1(rm)$$

$$= \frac{1}{(2\pi)^2} \frac{m}{r} K_1(rm) \;\rightarrow\; K_\nu \text{ are the modified Bessel functions.}$$

Chapter 3

The Dirac Field

3.1 Lorentz group

(a) The commutation relations of the generators of the group of Lorentz transformations are given by:

$$\left[J^{\mu\nu}, J^{\rho\sigma}\right] = i\Big(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}\Big),$$

where the differential operators $J^{\mu\nu}$ are antisymmetric, i.e., $J^{\nu\mu} = -J^{\mu\nu}$. First, from the definition of L^i, we can calculate the following useful relation:

$$\begin{aligned} L^k &= \frac{1}{2}\epsilon^{klm}J^{lm} \;\rightarrow\; \text{multiply by } \epsilon^{ijk} \\ \epsilon^{ijk}L^k &= \frac{1}{2}\epsilon^{ijk}\epsilon^{klm}J^{lm} \\ &= \frac{1}{2}\left(\delta^{il}\delta^{jm} - \delta^{im}\delta^{jl}\right)J^{lm} \\ &= \frac{1}{2}\left(J^{ij} - J^{ji}\right) \\ &= \frac{1}{2}\left(J^{ij} + J^{ij}\right) \\ &= J^{ij}. \end{aligned}$$

The commutation relations between two generators of a rotation are:

$$\left[L^i, L^j\right] = \left[\frac{1}{2}\epsilon^{imn}J^{mn}, \frac{1}{2}\epsilon^{jkl}J^{kl}\right]$$

$$
\begin{aligned}
&= \frac{1}{4}\epsilon^{imn}\epsilon^{jkl}\big[J^{mn}, J^{kl}\big] \\
&= \frac{i}{4}\epsilon^{imn}\epsilon^{jkl}\Big(g^{nk}J^{ml} - g^{mk}J^{nl} - g^{nl}J^{mk} + g^{ml}J^{nk}\Big) \\
&= \frac{i}{4}\epsilon^{imn}\Big(- \epsilon^{jnl}J^{ml} + \epsilon^{jml}J^{nl} + \epsilon^{jkn}J^{mk} - \epsilon^{jkm}J^{nk}\Big) \\
&= \frac{i}{4}\epsilon^{imn}\Big(\epsilon^{jln}J^{ml} + \epsilon^{jml}J^{nl} + \epsilon^{jkn}J^{mk} + \epsilon^{jmk}J^{nk}\Big) \\
&= \frac{i}{4}\Bigg(\big(\delta^{ij}\delta^{ml} - \delta^{il}\delta^{mj}\big)J^{ml} + \big(\delta^{ij}\delta^{nl} - \delta^{il}\delta^{nj}\big)J^{nl} \\
&\qquad + \big(\delta^{ij}\delta^{mk} - \delta^{ik}\delta^{mj}\big)J^{mk} + \big(\delta^{ij}\delta^{nk} - \delta^{ik}\delta^{nj}\big)J^{nk}\Bigg) \\
&= \frac{i}{4}\Bigg(\delta^{ij}J^{ll} - J^{ji} + \delta^{ij}J^{ll} - J^{ji} + \delta^{ij}J^{kk} - J^{ji} + \delta^{ij}J^{kk} - J^{ji}\Bigg) \\
&= \frac{i}{4}\Bigg(0 + J^{ij} + 0 + J^{ij} + 0 + J^{ij} + 0 + J^{ij}\Bigg) \\
&= iJ^{ij} \;\; \rightarrow \;\; \text{use above relation} \\
&= i\epsilon^{ijk}L^k.
\end{aligned}
$$

The commutation relations between two generators of a boost are:

$$
\begin{aligned}
\big[K^i, K^j\big] &= \big[J^{0i}, J^{0j}\big] \\
&= i\Big(g^{i0}J^{0j} - g^{00}J^{ij} - g^{ij}J^{00} + g^{0j}J^{i0}\Big) \\
&= i\Big(0 - J^{ij} - 0 + 0\Big) \\
&= \, - iJ^{ij} \;\; \rightarrow \;\; \text{use above relation} \\
&= \, - i\epsilon^{ijk}L^k.
\end{aligned}
$$

The commutation relations between generators of a rotation and a boost are:

$$
\begin{aligned}
\big[L^i, K^j\big] &= \left[\frac{1}{2}\epsilon^{ikl}J^{kl}, J^{0j}\right] \\
&= \frac{i}{2}\epsilon^{ikl}\Big(g^{l0}J^{kj} - g^{k0}J^{lj} - g^{lj}J^{k0} + g^{kj}J^{l0}\Big) \\
&= \frac{i}{2}\epsilon^{ikl}\Big(0 - 0 - g^{lj}J^{k0} + g^{kj}J^{l0}\Big) \\
&= \frac{i}{2}\Big(\epsilon^{ikj}J^{k0} - \epsilon^{ijl}J^{l0}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{i}{2}\Big(\epsilon^{ikj}\big(-K^k\big) - \epsilon^{ijl}\big(-K^l\big)\Big) \\
&= \frac{i}{2}\Big(\epsilon^{ijk}K^k + \epsilon^{ijl}K^l\Big) \;\rightarrow\; \text{relabel } l \text{ to } k \text{ in second term} \\
&= i\epsilon^{ijk}K^k.
\end{aligned}
$$

The commutation relations between J_+^i and J_+^j are:

$$
\begin{aligned}
\big[J_+^i, J_-^j\big] &= \bigg[\frac{1}{2}\big(L^i + iK^i\big), \frac{1}{2}\big(L^j - iK^j\big)\bigg] \\
&= \frac{1}{4}\Big(\big[L^i, L^j\big] - i\big[L^i, K^j\big] + i\big[K^i, L^j\big] + \big[K^i, K^j\big]\Big) \\
&= \frac{1}{4}\Big(i\epsilon^{ijk}L^k - i\big(i\epsilon^{ijk}K^k\big) + i\big(-i\epsilon^{jik}K^k\big) + \big(-i\epsilon^{ijk}L^k\big)\Big) \\
&= \frac{1}{4}\Big(-i\big(i\epsilon^{ijk}K^k\big) + i\big(i\epsilon^{ijk}K^k\big)\Big) \\
&= 0.
\end{aligned}
$$

The commutation relations between $J_\pm^i$ and $J_\pm^j$ are:

$$
\begin{aligned}
\big[J_\pm^i, J_\pm^j\big] &= \bigg[\frac{1}{2}\big(L^i \pm iK^i\big), \frac{1}{2}\big(L^j \pm iK^j\big)\bigg] \\
&= \frac{1}{4}\Big(\big[L^i, L^j\big] \pm i\big[L^i, K^j\big] \pm i\big[K^i, L^j\big] - \big[K^i, K^j\big]\Big) \\
&= \frac{1}{4}\Big(i\epsilon^{ijk}L^k \pm i\big(i\epsilon^{ijk}K^k\big) \pm i\big(-i\epsilon^{jik}K^k\big) - \big(-i\epsilon^{ijk}L^k\big)\Big) \\
&= \frac{1}{4}\Big(i\epsilon^{ijk}L^k \pm i\big(i\epsilon^{ijk}K^k\big) \pm i\big(i\epsilon^{ijk}K^k\big) + i\epsilon^{ijk}L^k\Big) \\
&= \frac{1}{2}\Big(i\epsilon^{ijk}L^k \pm i\big(i\epsilon^{ijk}K^k\big)\Big) \\
&= \frac{1}{2}i\epsilon^{ijk}\Big(L^k \pm iK^k\Big) \\
&= i\epsilon^{ijk}J_\pm^k.
\end{aligned}
$$

(b) An infinitesimal Lorentz transformation, under rotations $\boldsymbol{\theta}$ and boosts $\boldsymbol{\beta}$, is given by:

$$
\Phi \rightarrow \Big(1 - i\boldsymbol{\theta}\cdot\mathbf{L} - i\boldsymbol{\beta}\cdot\mathbf{K}\Big)\Phi.
$$

In order to write the transformation laws of the 2-component objects transforming according to the $(j_+, j_-) = (\frac{1}{2}, 0)$ and $(j_+, j_-) = (0, \frac{1}{2})$ representations of the Lorentz group, we first need to determine $\mathbf{L}$ and $\mathbf{K}$ in terms of $\mathbf{J}$. By adding the expressions for $\mathbf{J}_+$ and $\mathbf{J}_-$, we obtain:

$$\begin{aligned}\mathbf{J}_+ + \mathbf{J}_- &= \frac{1}{2}\big(\mathbf{L} + i\mathbf{K}\big) + \frac{1}{2}\big(\mathbf{L} - i\mathbf{K}\big)\\ &= \mathbf{L}.\end{aligned}$$

Similarly, by subtracting the expressions for $\mathbf{J}_+$ and $\mathbf{J}_-$, we obtain:

$$\begin{aligned}\mathbf{J}_+ - \mathbf{J}_- &= \frac{1}{2}\big(\mathbf{L} + i\mathbf{K}\big) - \frac{1}{2}\big(\mathbf{L} - i\mathbf{K}\big)\\ &= i\mathbf{K}.\end{aligned}$$

In the $(j_+, j_-) = (\frac{1}{2}, 0)$ representation, we have $\mathbf{J}_+$ represented by $\frac{1}{2}\boldsymbol{\sigma}$ and $\mathbf{J}_-$ represented by 0. For the generators of rotations, this gives:

$$\begin{aligned}\mathbf{L} &= \mathbf{J}_+ + \mathbf{J}_-\\ &= \frac{\boldsymbol{\sigma}}{2},\end{aligned}$$

and for the generators of boosts:

$$\begin{aligned}\mathbf{K} &= -i\big(\mathbf{J}_+ - \mathbf{J}_-\big)\\ &= -i\frac{\boldsymbol{\sigma}}{2}.\end{aligned}$$

Inserting these expressions for $\mathbf{L}$ and $\mathbf{K}$ into the above expression for an infinitesimal Lorentz transformation, we have:

$$\begin{aligned}\Phi &\to \Big(1 - i\boldsymbol{\theta}\cdot\mathbf{L} - i\boldsymbol{\beta}\cdot\mathbf{K}\Big)\Phi\\ &= \Big(1 - i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2} - \boldsymbol{\beta}\cdot\frac{\boldsymbol{\sigma}}{2}\Big)\Phi,\end{aligned}$$

which is precisely the transformation ψ_L. In the $(j_+, j_-) = (0, \frac{1}{2})$ representation, we have $\mathbf{J}_+$ represented by 0 and $\mathbf{J}_-$ represented by $\frac{1}{2}\boldsymbol{\sigma}$. For the generators of rotations, this gives:

$$\begin{aligned}\mathbf{L} &= \mathbf{J}_+ + \mathbf{J}_-\\ &= \frac{\boldsymbol{\sigma}}{2},\end{aligned}$$

and for the generators of boosts:

$$\begin{aligned}\mathbf{K} &= -i\big(\mathbf{J}_+ - \mathbf{J}_-\big)\\ &= i\frac{\boldsymbol{\sigma}}{2}.\end{aligned}$$

Inserting these expressions for $\mathbf{L}$ and $\mathbf{K}$ into the above expression for an infinitesimal Lorentz transformation, we have:

$$\begin{aligned}\Phi \to& \Big(1 - i\boldsymbol{\theta}\cdot\mathbf{L} - i\boldsymbol{\beta}\cdot\mathbf{K}\Big)\Phi \\ &= \Big(1 - i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2} + \boldsymbol{\beta}\cdot\frac{\boldsymbol{\sigma}}{2}\Big)\Phi,\end{aligned}$$

which is precisely the transformation ψ_R.

(c) Part...

3.2 The Gordon identity

Something...

3.3 Spinor products

(a) Part...
(b) Part...
(c) Part...

3.4 Majorana fermions

(a) Part...
(b) Part...
(c) Part...
(d) Part...
(e) Part...

3.5 Supersymmetry

(a) Part...
(b) Part...
(c) Part...

3.6 Fierz transformations

(a) Part...
(b) Part...
(c) Part...

3.7 Discrete symmetries P, C, and T

(a) Part...
(b) Part...
(c) Part...

3.8 Bound states

(a) Part...
(b) Part...

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4.1 Creation of Klein-Gordon particles by a classical source

(a) Part...
(b) Part...
(c) Part...
(d) Part...
(e) Part...
(f) Part...

4.2 Decay of a scalar particle

Something...

4.3 Linear sigma model

(a) Part...
(b) Part...
(c) Part...
(d) Part...

4.4 Rutherford scattering

(a) Part...
(b) Part...
(c) Part...

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