Power Logarithmic Inequality

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Abstract

In this article we will see a new identity $A^{C-1} > C \cdot \ln A$, which can be used to proof few useful identities and also can be used in different approximation.

To reach our goal we will consider functions of the form

$$f(x) = \frac{A^x}{B} - 1 \tag{1}$$

Here A and B are real positive numbers.
Later we will put some more restrictions on them according to our needs.

Restricting the Function.

Our method of finding out the inequality, heavily depends upon the **area defination of definite integral**.
To make our work easy, we will only consider the A and B such that the **eq-1** $f(1) = 0$. This gives us

$$f(x) = \frac{A^x}{A} - 1 \tag{2}$$

The Inequality and It's application

Theorem 1 (Power-Log Inequality)**.** *If A and C are both real numbers with $C > 1$ and $A \geq e$, then*

$$A^{C-1} > C \cdot \ln A$$

Proof. In the **Fig-1**, we have drawn the graph of $f(x)$ and as you can see it goes through the point $(1, 0)$.
From this,

$$\text{Area under the Curve } = \Delta = \int_1^C \Big(\frac{A^x}{A} - 1\Big)dx \tag{3}$$

Which on simplifying gives us,

$$\Delta = \frac{1}{\ln A}\left(\frac{A^C}{A} - C \cdot \ln A\right) + \left(1 - \frac{1}{\ln A}\right) \tag{4}$$

Notice, **if the curve is always above x-axis**, then we can be always sure that $\Delta \geq 0$. Also we can see that we can have many conditions on how Δ can be zero,but we are only interested in the particular case for which both the **terms individually greater than zero**,i.e.,

$$1 - \frac{1}{\ln A} \geq 0 \Rightarrow A \geq e \tag{5}$$

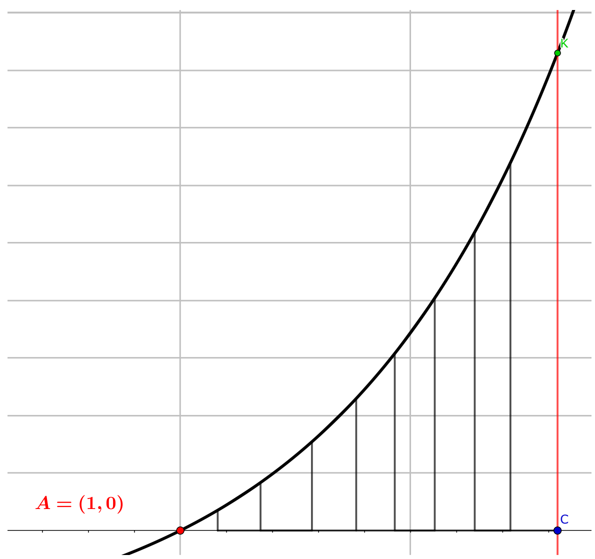

Figure 1: Graph of f(x) and C is any real number with C¿1.

From the other one,

$$\frac{1}{\ln A}\left(\frac{A^C}{A} - C \cdot \ln A\right) > 0 \Rightarrow A^{C-1} > C \cdot \ln A \tag{6}$$

And hence this gives our result.

□

Note: It should be noted that $C > 1$. As if C is less than 1 then the area is negative and the sign of the inequality flips and if $C = 1$, then the **area under the curve is zero**.

Remark. *As $A^{C-1} > C \cdot \ln A$, we can also write*

$$A^{C-1} > C \tag{7}$$

As $ln(A) \geq 1$ for all $A \geq e$

This new inequality is indeed powerful. It can give us a general proof for a more general series of power inequalities.

Theorem 2 (A-B Inequality)**.** *If A and B are real numbers and also $e \leq A < B$, then*

$$A^B > B^A$$

Proof. In **equation-7**, if we use $C = \frac{B}{A}(B > A)$,

$$A^{\frac{B}{A}-1} > \frac{B}{A} \Rightarrow A^{\frac{B}{A}} > B$$

which gives us our result.

□

Remark. *If you use $B = \pi$ in the Theorem-2 or $C = \frac{B}{e}$ in Theorem-1, then you will find another famous inequality*

$$e^B > B^e \tag{8}$$

This last one also gives us another famous inequality

$$e^\pi > \pi^e$$

Theorem 3 (Pseudo AM-GM inequality)**.** *If A and B are two real numbers such that $e \leq A, B$ then,*

$$\frac{A+B}{2} > \ln(AB)^{\frac{1}{2}}$$

Proof. Suppose, A and B are two real numbers and $A, B \geq e$. Hence they both satisfy the condition of $Theorem - 1$, i.e.,

$$A > 2 \cdot \ln A \quad \text{ and } \quad B > 2 \cdot \ln B$$

Here we have taken $C = 2$.
Adding this two equations,

$$\frac{A+B}{2} > \ln A + \ln B$$

From this,

$$\frac{A+B}{2} > \ln AB > \frac{\ln AB}{2}$$

And hence gives us our result.

□

As you have seen, this **Power Logarithmic Inequality** can be used in many ways, giving us beautiful results.

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