

## Constraining the Interiors of Asteroids Through Close Encounters

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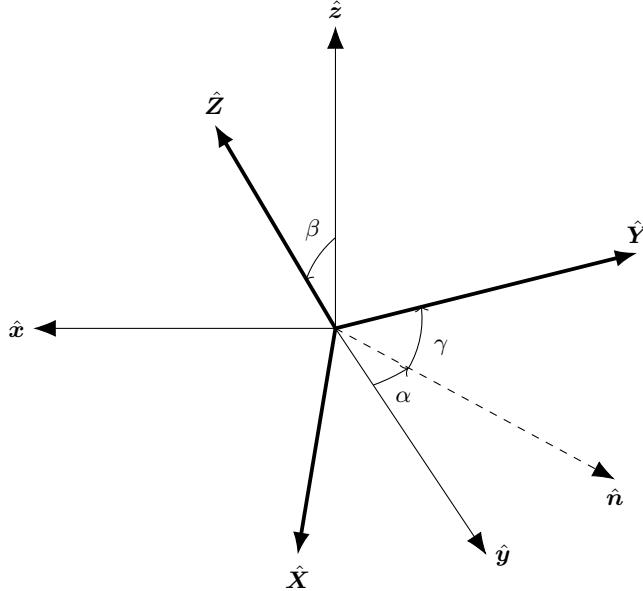
### ABSTRACT

We investigate the degree to which asteroid interior density distributions can be extracted from rotational velocity data gathered during a close encounter. We derive the equations of motion for an asteroid's orientation and angular velocity during a close encounter to arbitrary precision and use them to generate synthetic rotational velocity data for a representative asteroid on a close Earth flyby. We re-extract the properties of the asteroid from this synthetic data and confirm the fit's accuracy. We investigate the dependence of the best fit uncertainty on asteroid moment of inertia, observational precision and cadence, the flyby's orbital parameters, the initial spin pole direction, the quantity of near-pericenter data, and the oblateness of the central body. Finally, we discuss the degeneracy between the parameters that can be extracted from flyby data and the actual density distribution and propose four models to construct a representative density distribution.

*Keywords:* Keywords, em-dash separated

### 1. INTRODUCTION

- Asteroids simulated in binaries ?
- Previous work on the math of asteroid rotations ??
- CNEOS: 4 asteroids  $D > 100m$  are coming to within 1 LD of Earth in the next 10 years, 29 with  $D > 10m$ .
- Moskowitz survey of Duende: ? with companion observational article ?. Demonstrates that moment of inertia can be extracted from a tumbling asteroid via lightcurve analysis in the zero-torque regime (Fourier transform of lightcurve to pick up on two frequencies and assign precession and nutation frequencies to them.) Assumes constant geometry.
- Study of asteroid binary to extract density ?
- Scheeres theoretical paper ? detailing the different forces and torques felt by an asteroid.
- Paper stating in general that asteroids can be tidally disrupted on hyperbolic flyby. ?, plus similar paper ? for tidal disruption
- Another theoretical paper ? inspired by Paul
- Apophis will definitely pass by Earth and will definitely significantly change its rotation state, but will not undergo tidal disruption. ?



**Figure 1.**  $z - y - z$  Euler angles used in this work to express the orientation of the asteroid. Orientation is expressed as a rotation from the body-fixed axes to the inertial axes.

## 2. ASTEROID MODEL

In an effort to make a general asteroid model, we will consider the entire effect of the tidal torque applied to an asteroid from a planet, rather than solving for the first order perturbation. To better understand the effect, we will not model perturbing effects from other bodies such as the Sun or other planets. However, nearby bodies, such as moons or rings, are considered in a later section. We will also model the asteroid as a rigid body, whose density distribution is fixed throughout the flyby.

Most asteroids are observed to have attained their minimum energy rotation state, so we will also assume that the asteroid's initial state aligns its rotational velocity parallel to the principal axis with maximal moment of inertia.

### 2.1. Coordinates

We will make use of two frames of reference to model this system. One is the “inertial frame,” with axes denoted by  $\hat{X}$ ,  $\hat{Y}$ ,  $\hat{Z}$ . As the name suggests, these axes are inertial, with  $\hat{X}$  pointing to the asteroid pericenter and  $\hat{Z}$  pointing parallel to the orbit angular momentum. The origin of this frame is set to the center of mass of the central body. We will assume that the mass distribution of the central body is known in this inertial frame.

Our second frame is the “body-fixed” frame, denoted by  $\hat{x}$ ,  $\hat{y}$ ,  $\hat{z}$ . This frame is fixed with respect to the body’s principle axes and rotates with the body, with its origin at the body’s center of mass. We will solve for the asteroid’s mass distribution with reference to the body-fixed frame. For definiteness, we define  $\hat{z}$  to be the principal axis with maximal MOI, and  $\hat{x}$  to have minimal MOI.

We define a rotation matrix  $M$  such that  $M\mathbf{r} = \mathbf{R}$ , where  $\mathbf{r}$  is in the body-fixed frame and  $\mathbf{R}$  is in the inertial frame. It will be useful for us to represent this matrix in two different ways; one is with a normalized quaternion  $\tilde{\mathbf{q}}$ , and the other is with  $z - y - z$  Euler angles named  $\alpha$ ,  $\beta$ , and  $\gamma$ . In particular, we define

$$M = R_z(\alpha)R_y(\beta)R_z(\gamma) \quad (1)$$

where  $R_i(\theta)$  is a rotation around the  $i$ th unit vector by  $\theta$  (figure 1).

We use the quaternion  $\tilde{\mathbf{q}}$  as our dynamical variable because Euler angles suffer from gimbal lock. For the sake of this paper, we write quaternions as  $\tilde{\mathbf{q}} = r + i\mathbf{i} + j\mathbf{j} + k\mathbf{k}$ , with the complex conjugate written as  $\tilde{\mathbf{q}}^* = r - i\mathbf{i} - j\mathbf{j} - k\mathbf{k}$ . The inverse is then  $\tilde{\mathbf{q}}^{-1} = \tilde{\mathbf{q}}^*/(\tilde{\mathbf{q}}\tilde{\mathbf{q}}^*)$ , and we define real and imaginary parts as  $\Re\tilde{\mathbf{q}} = r$ ;  $\Im\tilde{\mathbf{q}} = i\hat{x} + j\hat{y} + k\hat{z}$ . See a description of quaternions in Ref. ?

Normalized quaternions express rotations of an angle  $\theta$  by  $\tilde{\mathbf{q}} = \cos \frac{\theta}{2} + \sin \frac{\theta}{2}\tilde{\mathbf{n}}$  where  $\tilde{\mathbf{n}} = 0 + n_x\mathbf{i} + n_y\mathbf{j} + n_z\mathbf{k}$ . This rotation may be applied to any vector  $\mathbf{r}$  by

$$\mathbf{R} = \Im[\tilde{\mathbf{q}}\tilde{\mathbf{r}}\tilde{\mathbf{q}}^{-1}]. \quad (2)$$

Due to the real part of  $\tilde{\mathbf{q}}$  which does not vanish for  $\theta \rightarrow 0$ , quaternions do not suffer from gimbal lock. Using the above notation, we can give the full equation of quaternion multiplication:

$$(q_0 + \tilde{\mathbf{q}})(p_0 + \tilde{\mathbf{p}}) = q_0 p_0 - \mathbf{p} \cdot \mathbf{q} + p_0 \mathbf{q} + q_0 \mathbf{p} + \mathbf{q} \times \mathbf{p}. \quad (3)$$

Note that quaternion multiplication is not commutative.

We convert from Euler angles to quaternions by writing equation 1 in terms of quaternions:

$$\tilde{\mathbf{q}} = \begin{pmatrix} \cos \frac{\alpha}{2} \\ 0 \\ 0 \\ \sin \frac{\alpha}{2} \end{pmatrix} \begin{pmatrix} \cos \frac{\beta}{2} \\ 0 \\ \sin \frac{\beta}{2} \\ 0 \end{pmatrix} \begin{pmatrix} \cos \frac{\gamma}{2} \\ 0 \\ 0 \\ \sin \frac{\gamma}{2} \end{pmatrix} = \begin{pmatrix} \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \cos \frac{\gamma}{2} - \frac{\alpha}{2} \cos \frac{\beta}{2} \sin \sin \frac{\gamma}{2} \\ \frac{\alpha}{2} \sin \frac{\beta}{2} \cos \sin \frac{\gamma}{2} - \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \cos \frac{\gamma}{2} \\ \frac{\alpha}{2} \sin \frac{\beta}{2} \cos \cos \frac{\gamma}{2} + \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \sin \frac{\gamma}{2} \\ \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \sin \frac{\gamma}{2} + \frac{\alpha}{2} \cos \frac{\beta}{2} \sin \cos \frac{\gamma}{2} \end{pmatrix}. \quad (4)$$

This equation can be reversed for  $\alpha, \beta, \gamma$  as follows:

$$\tan \alpha = \frac{jk - ir}{jr + ik} \quad \cos \beta = 1 - 2(i^2 + j^2) \quad \tan \gamma = \frac{jk + ir}{jr - ik}. \quad (5)$$

## 2.2. Tidal Torque

Derivations for the tidal torque experienced by a rigid body in the gravitational field of a larger mass have been computed by several previous studies ????, sometimes in terms of the moment of inertia of the rigid body (or higher order moments of inertia), and to varying degrees of precision. We present a novel derivation of the tidal torque experienced by a rigid body in a gravitational field of another in terms of (complex) spherical harmonic coefficients of the bodies' mass distributions, which are closely related to their moments of inertia. This formulation of the problem will be more useful for this study. The calculation is performed to all orders of distance between the two bodies.

We start by expressing the gravitational potential energy field of the central body in its most general form:

$$V(\mathbf{R}') = -G \int_{\mathcal{P}} d^3 R \rho_M(\mathbf{R}) \frac{1}{|\mathbf{R} - \mathbf{R}'|}. \quad (6)$$

This integral is computed in the inertial frame and has units of energy per unit mass. We represent the central body density as  $\rho_M$  and its volume as  $\mathcal{P}$ . To reduce equation 6, we define the unnormalized spherical harmonics  $Y_{\ell m}(\theta, \phi) = P_{\ell m}(\cos \theta) e^{im\phi}$ , where  $P_{\ell m}$  are the associated Legendre Polynomials without the Condon-Shortley phase. The regular and irregular spherical harmonics are then defined as

$$S_{\ell m}(\mathbf{r}) = (-1)^m (\ell - m)! \frac{Y_{\ell m}(\hat{\mathbf{r}})}{r^{\ell+1}} \quad R_{\ell m}(\mathbf{r}) = (-1)^m \frac{r^\ell}{(\ell + m)!} Y_{\ell m}(\hat{\mathbf{r}}). \quad (7)$$

Assume that we can construct a sphere, centered at the center of mass of the central body, so that  $\rho_M = 0$  outside the sphere and  $\rho_m = 0$  inside the sphere. Then  $R' > R$  in equation 6, and Ref. ? gives the identity

$$\frac{1}{|\mathbf{R} - \mathbf{R}'|} = \sum_{\ell, m} R_{\ell m}(\mathbf{R}) S_{\ell m}^*(\mathbf{R}'), \quad (8)$$

where the sum is shorthand for  $\sum_{\ell, m} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell}$ .

Suppose the vector  $\mathbf{D}$  points from the center of mass of the central body to the center of mass of the asteroid, again in the inertial frame. Then we may write  $\mathbf{R}' = \mathbf{D} + \mathbf{U}$ , and Ref. ? gives

$$S_{\ell m}(\mathbf{R}') = \sum_{\ell', m'} (-1)^{\ell'} R_{\ell' m'}^*(\mathbf{U}) S_{\ell + \ell', m + m'}(\mathbf{D}). \quad (9)$$

Define the complex, unitless constants

$$J_{\ell m} = \int_{\mathcal{P}} d^3 R \rho_M(\mathbf{R}) \frac{R_{\ell m}(\mathbf{R})}{\mu_M a_M^\ell} \quad K_{\ell m} = \int_{\mathcal{A}} d^3 r \rho_m(\mathbf{r}) \frac{R_{\ell m}(\mathbf{r})}{\mu_m a_m^\ell} \quad (10)$$

where the integrals are taken in the inertial and the body-fixed frames respectively. Here,  $\rho_m$  denotes the density of the asteroid and  $\mathcal{A}$  denotes its volume. The symmetry relations of  $R_{\ell m}$  guarantee  $J_{\ell m} = (-1)^m J_{\ell, -m}^*$  and similarly

for  $K_{\ell m}$ . We have chosen  $\mu_M$ ,  $\mu_m$  to be the masses of the central body and asteroid respectively, and  $a_M$  and  $a_m$  to be scale lengths of both bodies (defined more precisely later). Then we can express the potential as

$$V(\mathbf{U}) = -G \left[ \sum_{\ell, m} \mu_M a_M^\ell J_{\ell m} \sum_{\ell', m'} (-1)^{\ell'} R_{\ell' m'}(\mathbf{U}) S_{\ell+\ell', m+m'}^*(\mathbf{D}) \right]. \quad (11)$$

To make use of  $K_{\ell m}$  in equation 10, we need to rotate  $\mathbf{U}$  from the inertial frame into  $\mathbf{u}$  in the body fixed frame via  $\mathbf{U} = M\mathbf{u}$ . The rotation of a spherical harmonic satisfies

$$Y_{\ell m}(M\mathbf{u}) = \sum_{m'=-\ell}^{\ell} (-1)^{m+m'} \sqrt{\frac{(\ell-m')!(\ell+m)!}{(\ell+m')!(\ell-m)!}} \mathcal{D}_{mm'}^\ell(M)^* Y_{\ell m'}(\mathbf{u}). \quad (12)$$

where  $\mathcal{D}$  are the Wigner- $D$  matrices. The corollary of this formula for the regular solid spherical harmonic is

$$R_{\ell m}(M\mathbf{u}) = \sum_{m'=-\ell}^{\ell} \sqrt{\frac{(\ell-m')!(\ell+m')!}{(\ell-m)!(\ell+m)!}} \mathcal{D}_{mm'}^\ell(M)^* R_{\ell m'}(\mathbf{u}) \quad (13)$$

where  $\mathcal{D}_{mm'}^\ell$  are the Wigner- $D$  matrix elements. In practice, these are computed from the Euler angles corresponding to the rotation  $M$ , which we specified in section 2.1 are  $z-y-z$  Euler angles. With this choice of Euler angles, the Wigner- $D$  matrix elements are real with matrix elements given by

$$d_{mm'}^\ell(\beta) = \sqrt{(\ell+m)!(\ell-m)!(\ell+m')!(\ell-m')!} \quad (14)$$

$$\sum_{s=s_{\min}}^{s_{\max}} \left[ \frac{(-1)^{m-m'+s} (\cos \frac{\beta}{2})^{2\ell+m'-m-2s} (\sin \frac{\beta}{2})^{m-m'+2s}}{(\ell+m'-s)!(\ell-m-s)!(m-m'+s)!s!} \right] \quad (15)$$

where  $s_{\min} = \max(0, m' - m)$  and  $s_{\max} = \min(\ell + m', \ell - m)$ , and  $\mathcal{D}_{mm'}^\ell(\alpha, \beta, \gamma) = d_{mm'}^j e^{-im\alpha} e^{-im'\gamma}$ .

Combining equations 13 and 11,

$$V(\mathbf{u}) = -G \left[ \sum_{\ell, m} \mu_M a_M^\ell J_{\ell m} \sum_{\ell', m'} (-1)^{\ell'} S_{\ell+\ell', m+m'}^*(\mathbf{D}) \sum_{m''=-\ell'}^{\ell'} \sqrt{\frac{(\ell'-m'')!(\ell'+m'')!}{(\ell'-m')!(\ell'+m')!}} \mathcal{D}_{m'm''}^{\ell'}(M)^* R_{\ell'm''}(\mathbf{u}) \right]. \quad (16)$$

The tidal torque applied to an asteroid at a point  $\mathbf{u}$  is given by  $d\boldsymbol{\tau}(\mathbf{u}) = -\rho_m(\mathbf{u}) d^3 u (\mathbf{u} \times (\nabla V_{\mathbf{u}}(\mathbf{u})))$  in the body-fixed frame, which leaves

$$\boldsymbol{\tau} = G \int_A d^3 u \rho_m(\mathbf{u}) \mathbf{u} \times \nabla \left[ \sum_{\ell, m} \mu_M a_M^\ell J_{\ell m} \sum_{\ell', m'} (-1)^{\ell'} S_{\ell+\ell', m+m'}^*(\mathbf{D}) \right. \quad (17)$$

$$\left. \sum_{m''=-\ell'}^{\ell'} \sqrt{\frac{(\ell'-m'')!(\ell'+m'')!}{(\ell'-m')!(\ell'+m')!}} \mathcal{D}_{m'm''}^{\ell'}(M)^* R_{\ell'm''}(\mathbf{u}) \right]. \quad (18)$$

Using the identity

$$\mathbf{u} \times \nabla R_{\ell m}(\mathbf{u}) = \frac{1}{2} [(i\hat{x} - \hat{y})(\ell - m + 1) R_{\ell, m-1}(\mathbf{u})] \quad (19)$$

$$+ (i\hat{x} + \hat{y})(\ell + m + 1) R_{\ell, m+1}(\mathbf{u}) + 2im\hat{z} R_{\ell m}(\mathbf{u}), \quad (20)$$

which was derived from Ref. ?'s analysis of the angular momentum operators applied to spherical harmonics, we write equation 18 as

$$\boldsymbol{\tau} = G \frac{\mu_m \mu_M}{2} \left[ \sum_{\ell, m} a_M^\ell J_{\ell m} \sum_{\ell', m'} a_m^{\ell'} S_{\ell+\ell', m+m'}^*(\mathbf{D}) (-1)^{\ell'} \sum_{m''=-\ell'}^{\ell'} \sqrt{\frac{(\ell'-m'')!(\ell'+m'')!}{(\ell'-m')!(\ell'+m')!}} \mathcal{D}_{m'm''}^{\ell'}(M)^* \right. \quad (21)$$

$$\left. \left[ (i\hat{x} - \hat{y})(\ell' - m'' + 1) K_{\ell', m''-1} + (i\hat{x} + \hat{y})(\ell' + m'' + 1) K_{\ell', m''+1} + 2im''\hat{z} K_{\ell'm''} \right] \right]. \quad (22)$$

This equation contains complex values, but it was numerically confirmed that  $\boldsymbol{\tau}$  is a real-valued vector.

### 2.3. Interpretation of Tidal Torque

The rest of this paper is devoted to interpreting the properties of equation 22 through a fit to data generated with this equation of motion, but several properties are already evident through the functional form.

Firstly, each term is weighted by a factor of  $a_M^\ell a_m^{\ell'} / D^{\ell+\ell'+1}$ . Since  $D \gg a_M \gg a_m$  in general, the leading order torques come from low  $\ell$  and  $\ell'$ . However,  $\ell' = 0$  contributes nothing because the inner bracketed term is zero, and  $K_{\ell'=1} = 0$  if we choose the origin of the body-fixed axes to be the center of mass of the asteroid. The leading order of equation 22 is therefore proportional to  $1/D^3$  with  $\ell = 0$  and  $\ell' = 2$ . We therefore call  $K_{2m}$  and  $\gamma_0$  the “low-order” parameters because they are found in the leading order term of equation 22. The next order is  $K_{3m}$ , and other orders after that are terms with  $J_{2m}$  and  $K_{2m}$  as well as  $J_{00}$  and  $K_{4m}$ .

It is also useful to note when equation 22 is zero, since in these cases we expect to be able to determine little about the asteroid. Most important is when  $\omega \parallel \hat{\mathbf{Z}}$ ; i.e., when  $\beta = 0$ . In this case, the Wigner- $D$  matrix elements are  $D_{mm'}^\ell \propto \delta_{mm'}$  and  $D_{mm}^\ell = D_{-m,-m}^{\ell*}$ . Note that the inner bracketed term of equation 22 transforms under  $m \rightarrow -m$  as  $[ \cdot ] \rightarrow -(-1)^m [ \cdot ]^*$ . Since  $S_{\ell m}$  and  $D_{\ell m}$  transform to their conjugates and  $J_{\ell m} \rightarrow (-1)^m J_{\ell m}^*$  while none of the other terms transform, the entire equation for tidal torque transforms to the opposite of its conjugate. Since  $\tau$  is invariant under this transformation, which is equivalent to a rearrangement of terms, we must have  $\tau = -\tau^*$ . Since  $\tau \in \mathbb{R}^3$ , we have  $\tau = 0$ .

Restricting to the first order of equation 22,  $\tau$  has two other important zeros. Writing the position vector  $\mathbf{D}$  in spherical coordinates  $D, \theta, \phi$  and setting  $\theta = \pi/2$  for simplicity, we note that  $\tau = 0$  when (1)  $\alpha = \phi$  and  $\beta = \pi/2$ , and when (2)  $\alpha = \phi + \pi/2$  and  $\gamma$  is an integer multiple of  $\pi/2$ . At  $\phi = 0$ , where  $D$  is smallest and  $\tau$  is generally greatest, these positions correspond to (1)  $\omega \parallel \hat{\mathbf{X}}$  and (2)  $\omega$  is in the  $\hat{\mathbf{Y}} - \hat{\mathbf{Z}}$  with a condition on  $\gamma$ . When the latter condition on  $\gamma$  is not met,  $\tau \propto \hat{\mathbf{Z}} K_{22} \sin(2\gamma)$ . Since  $\gamma$  changes as the asteroid rotates and depends on the initial condition  $\gamma_0$ , whether the  $\gamma$  condition is met at perigee is more or less random.

### 2.4. Moment of Inertia

The inertia tensor of the asteroid is expressed as a matrix  $I$ . Written in terms of spherical harmonics in the body-fixed frame, we have

$$I = \int_A d^3r \rho_m(\mathbf{r}) \begin{pmatrix} y^2 + z^2 & -xy & -xz \\ -xy & x^2 + z^2 & -xz \\ -xz & -yz & x^2 + y^2 \end{pmatrix} \quad (23)$$

$$= \frac{2}{3} \mu_m a_m^2 \begin{pmatrix} K_{20} - 3K_{2,-2} - 3K_{22} + 1 & 3i(K_{22} - K_{2,-2}) & \frac{3}{2}(K_{21} - K_{2,-1}) \\ 3i(K_{22} - K_{2,-2}) & K_{20} + 3K_{2,-2} + 3K_{22} + 1 - \frac{3}{2}i(K_{21} + K_{2,-1}) & \\ \frac{3}{2}(K_{21} - K_{2,-1}) & -\frac{3}{2}i(K_{21} + K_{2,-1}) & -2K_{20} + 1 \end{pmatrix} \quad (24)$$

where we have defined, for neatness,

$$a_m^2 = \frac{1}{\mu_m} \int_A d^3r \rho_m(\mathbf{r}) r^2. \quad (25)$$

With this definition of  $a_m$ , and an arbitrary definition of  $a_M$  which may be taken to be similar to equation 25 or a more convenient unit such as the central body radius, the parameters  $J_{\ell m}$  and  $K_{\ell m}$  of equation 10 are completely defined. Note that  $a_m$  is roughly to the radius of the asteroid under certain assumptions; if the asteroid is a sphere of uniform density and radius  $r$ , then  $a_m = r\sqrt{3/5}$ .

Henceforth, we define the body-fixed axis to be coincident with the principal axes of the asteroids, so that the off-diagonal components of  $I$  are zero.

### 2.5. Equation of Motion

As a simplifying assumption, we will not model the effect of the central body’s non-sphericity on the asteroid orbit. This effect has been studied in much detail and is small. We therefore separate the position dynamical variables from the angular variables and derive the standard equations of motion for asteroid position in the inertial frame:

$$\dot{\mathbf{V}} = -\frac{G\mu_M}{r^3} \mathbf{D} \quad \dot{\mathbf{D}} = \mathbf{V}. \quad (26)$$

The position variables will no longer be discussed.

Let  $\omega$  be the rotational velocity of the asteroid in the body-fixed frame. We have two equations of motion for our system; one governs the evolution of  $\omega$  and the other governs the evolution of  $q$ . Starting with the latter, consider a fixed point  $r$  in the body-fixed frame on a rotating asteroid. The corresponding point  $R = \Im[\tilde{q}r\tilde{q}^{-1}]$  in the inertial frame satisfies  $\dot{R} = \Im[\tilde{q}(\omega \times r)\tilde{q}^{-1}]$ . Furthermore, we can demand that  $\omega \perp r$  so that  $\omega \times r = \Im[\tilde{\omega}r]$ . We obtain another equation for  $R$  by differentiating equation 2 with  $\dot{r} = 0$ . Setting these equal,

$$\dot{\tilde{q}}\tilde{r}\tilde{q}^{-1} + \tilde{q}\tilde{r}\dot{\tilde{q}}^{-1} = \tilde{q}\tilde{\omega}\tilde{r}\tilde{q}^{-1}. \quad (27)$$

Substituting  $r \rightarrow \tilde{q}^{-1}R\tilde{q}$  and acknowledging that  $\Re[\dot{\tilde{q}}\tilde{q}^{-1}] = 0$ ,

$$\dot{\tilde{q}}\tilde{q}^{-1}R + R\tilde{q}\dot{\tilde{q}}^{-1} = 2\dot{\tilde{q}}\tilde{q}^{-1}\tilde{R} = \tilde{q}\tilde{\omega}\tilde{q}^{-1}R. \quad (28)$$

Since  $R$  is arbitrary, this simplifies to

$$2\dot{\tilde{q}}\tilde{q}^{-1} = \tilde{q}\tilde{\omega}\tilde{q}^{-1} \Rightarrow \dot{\tilde{q}} = \frac{1}{2}\tilde{q}\tilde{\omega}. \quad (29)$$

The second equation of motion are Euler's equations of motion given our expression for tidal torque. Namely,

$$I_1\dot{\omega}_1 - \omega_2\omega_3(I_2 - I_3) = \tau_1 \quad (30)$$

$$I_2\dot{\omega}_2 - \omega_3\omega_1(I_3 - I_1) = \tau_2 \quad (31)$$

$$I_3\dot{\omega}_3 - \omega_1\omega_2(I_1 - I_2) = \tau_3 \quad (32)$$

The components of  $\tau$  are given by equation 22, which was also in the body-fixed frame.

## 2.6. Initial Values

We assume that the asteroid begins the simulation in its minimum energy configuration; that is, with  $\omega \parallel \hat{z}$ . The initial conditions are therefore that  $\omega = \omega\hat{z}$ , and the quaternion is fixed to rotate  $\hat{z}$  to  $\Omega$  in the inertial frame, which fixes Euler angles  $\beta$  and  $\alpha$ . The third Euler angle  $\gamma$  represents the initial roll of the asteroid, and therefore cannot be determined from physical data. However, it is an initial condition of the simulation. Once the Euler angles are known, the quaternion is computed via equation 4.

## 2.7. Fit Parameters

By requiring that the  $\hat{x}, \hat{y}, \hat{z}$  axes coincide with principal axes, we force the diagonal components of  $I$  to be zero. Thus, we may require  $K_{21} = K_{2,-1} = 0$  and  $\Re K_{22} = \Im K_{2,-2} = 0$  with no loss of generality. By requiring that the body-fixed frame be located at the center of mass of the asteroid, we also require  $K_{1m} = 0$ . Finally,  $K_{00} = 1$  follows automatically from the definition. Thus,  $K_{20}$ ,  $\Re K_{22}$ , and  $K_{\ell>2,m}$  are the only physical parameters of the asteroid density distribution, beyond  $a_m$ .

As discussed before, the leading order of equation 22 is  $\ell' = 2$ . Therefore, the equation is proportional to  $\mu_m a_m^2$ . Factoring out this product results in an equation of motion independent of  $\mu_m$ , meaning that the asteroid dynamics are unaffected by the mass of the asteroid. This is expected, and it means that the density distribution of the asteroid can only be determined up to a scale factor.

Since the coefficient of  $J_{lm}K_{l'm'}$  is proportional to  $a_M^\ell a_m^{\ell'} / D^{\ell+\ell'+1}$ , to first order in the asteroid density distribution, the system is also independent of  $a_m$ . When  $K_{\ell>2}$  terms are included,  $a_m$  scales the size of the higher order terms.

We require that the moment of inertia matrix be positive-definite, which bounds our asteroid parameters. Specifically, we have  $K_{20} \leq \frac{1}{2}$ , and  $|K_{22}| \leq (K_{20} + 1)/6$ . We can also choose without loss of generality to let the maximum moment of inertia align with  $\hat{z}$ , so that  $1 - 2K_{20} \geq K_{20} + 1 + 6|K_{22}|$ . Finally, we wish to require that the density of the asteroid be everywhere positive. One consequence of this requirement is that the sum of any two distinct moments of inertia cannot be greater than the third, by the integral definition of moment of inertia in equation 24. Three inequalities result from this, but only one is not covered by the above restrictions. Namely,  $I_{xx} + I_{yy} \geq I_{zz} \Rightarrow K_{20} \geq -1/4$ . The intersection of these requirements forces  $K_{20}$  and  $K_{22}$  to lie in the triangle defined by

$$-\frac{1}{4} \leq K_{20} \leq 0 \quad |K_{22}| \leq -\frac{K_{20}}{2}. \quad (33)$$

We have one additional fit parameter, which is the only initial condition not fixed by the problem setup or observations. Namely, the initial value of the first Euler angle  $\alpha_0$ . Due to the fact that the principal axes  $\hat{\mathbf{x}}$  and  $\hat{\mathbf{y}}$  are orthogonal, increments of  $\gamma_0$  by  $\pi/2$  correspond to flipping the body-fixed axes so that no physical change in the asteroid model occurs. We must therefore limit the range of  $\gamma_0$  to width  $\pi/2$ . Arbitrarily, we choose  $\gamma_0 \in [-\pi/4, \pi/4]$

### 3. EXPERIMENT DESIGN

#### 3.1. *Simulation Design*

Given the observed orbit of an asteroid and observational data of its resolved spin over time, we built a simulation that would integrate equations 29 and 32 to generate synthetic data. Inputs to the simulation were (1) the  $\ell$  and  $\ell'$  values at which to truncate the potential energy sum in equation 22, (2) the cadence of observations, (3) the  $J_{\ell m}$  parameters and mass and radius of the central body, (4) trial asteroid parameters  $K_{\ell m}$ ,  $\gamma_0$  and  $a_m$ , (5) the initial rotational velocity of the asteroid in the inertial frame, and (6) the asteroid's orbital parameters.

The orbital parameters we use to describe the orbit are the hyperbolic excess velocity  $v_\infty$  and the pericenter altitude  $r_p$ . The orbit orientation is fixed to lie in the  $\hat{\mathbf{X}} - \hat{\mathbf{Y}}$  plane, with pericenter lying on  $\hat{\mathbf{X}}$ . The position of the satellite as a function of time is therefore generated by numerically integrating the Newtonian equation of motion (equation 26) from initial values  $\mathbf{r}(t=0) = r_p \hat{\mathbf{X}}$  and  $\mathbf{v}(t=0) = \hat{\mathbf{Y}} \sqrt{v_\infty^2 + 2G\mu_M/r_p}$

Once the inputs were specified, the moments of inertia were pre-computed from  $K_{\ell m}$ , as was the orbital path of the satellite as a function of time. Equations 29 and 32 were then integrated by the Runge-Kutta fourth order method with a time step of *describe* seconds, with a custom implementation of the spherical harmonics and Wigner-D matrices to improve runtime. At fixed time intervals of one cadence, the rotational velocity in the body-fixed frame  $\omega_{xyz}$  was converted via equation 2 to the inertial frame to produce the synthetic data set.

We begin our simulation at  $D = 10r_p$ . Since the leading order of the equations of motion is  $\ell' = 2, \ell = 0$ , this corresponds roughly to a torque of  $10^{-3}$  times the maximum torque at perigee. Unless otherwise indicated, the simulation is terminated at  $D = 10r_p$  as well.

#### 3.2. *Fitting Method*

To analyze the degree to which asteroid density distributions can be determined from flyby rotational velocity data, we use the forward model defined in section 3.1 to generate synthetic data, randomize the data according to one of the observational uncertainty model defined in section 3.3, then use the same forward model to fit to the data and recover parameters.

After randomizing the data, we isolate a set of parameters which are likely to provide good fits. To do this, we minimize the  $\chi^2$  value of the data given the model using the Nelder-Mead algorithm, implemented in `scipy`?. Our starting points for the minimization are spaced uniformly randomly across the allowed parameter space, which is the set of parameters with the triangle described by equation 33. The other parameter bounds are  $\gamma_0 \in [-\pi/4, \pi/4]$ ,  $|\Re K_{l>2}| \leq 1$ , and  $|\Im K_{l>2}| \leq 1$ .

If the minimization is conducted over too many parameter dimensions, convergence can be slow or can fail. However, the parameters  $K_{\ell m}$  are naturally separated into groups by order of  $\ell$ , since the torque applied by successive increment to  $\ell$  is suppressed by  $a_m/D$ . Thus, we fit the  $\ell = 2$  and  $\gamma_0$  parameters first, then lock them in place and fit the  $\ell = 3$  parameters, etc. The result is a set of points in parameter space which minimize the  $\chi^2$ .

Even with this ordering of parameters, the system's strong dependence on asteroid shape makes convergence rare. The data is therefore truncated at distance  $D = fr_p$ , where  $f$  is a manually-set fraction. Once minimization is conducted for the truncated data, the resulting parameters are used as the initial point for a minimization over the entire data set. Experimentation revealed that  $f = 2$  was appropriate, where the cut is performed after the perigee for the  $\ell = 2$  parameters and before the perigee for the  $\ell = 3$  parameters.

We are interested in finding not just the optimal parameters, but the probability distributions of each fit parameter. To do this, we use an Affine Invariant Markov Chain Monte Carlo Ensemble Sampler (MCMC), implemented in the Python package `emcee`, to conduct the fit to data ?. This fit method requires the parameter space to be populated with "walkers," indicating the a guess at the true parameters of the system. We center 32 walkers on the minimizing parameters computed by the minimization of  $\chi^2$ , and spread out the walkers such that the number density of walkers is proportional to the likelihood ( $\ln \mathcal{L} = -\chi^2/2$ ) at each point. This is done explicitly by computing the Hessian  $\Sigma$  of the  $\chi^2$  value at each minimizing point  $\theta_0$  and expressing the likelihood locally as a multi-dimensional Gaussian

$$\mathcal{L}(\boldsymbol{\theta}) \propto \exp\left(-\frac{1}{2}(\boldsymbol{\theta} - \boldsymbol{\theta}_0)^T \Sigma(\boldsymbol{\theta} - \boldsymbol{\theta}_0)\right). \quad (34)$$

We may change bases into a frame in which the parameters are uncorrelated by multiplying by  $V^T$ , where  $V$  is the matrix of eigenvectors of  $\Sigma$ . Populating the walkers in this uncorrelated space according to Gaussian standard deviations given by the eigenvalues of  $\Sigma$ , we can convert these walkers back to  $\boldsymbol{\theta}$ -space by multiplying by  $V$ . In practice, we only compute the components of  $\Sigma$  for parameters that are found in the same minimization step; for example, we do not compute the entries of  $\Sigma$  corresponding to  $K_{20}$  and  $K_{30}$ , setting it equal to zero instead.

Occasionally, the likelihood is non-Gaussian enough for  $\Sigma$  to have negative or otherwise nonsensical eigenvalues (this occurs for strongly varying likelihood or due to numerical errors in the calculation of the Hessian). In these cases, we choose eigenvalues of between *limit* and *limit*.

With the initial walkers distributed, we compute the posterior probability distribution of  $\boldsymbol{\theta}$  given flat priors in the range outlined above for each parameter. We run the MCMC until results converge (defined as the condition that the autocorrelation time is less than one hundredth of the current iteration number and that the fractional difference between autocorrelation times is under 1%), or for 100,000 iterations, whichever is smaller. The MCMC is observed only to run for all 100,000 iterations if the data is insufficiently precise to constrain some parameters.

### 3.3. Uncertainty Models

We model uncertainties in experimental data according to several methods, comparing them to determine the dependence of our conclusions on the sensitivity model used. In every case, the sensitivity model is used both to add randomness to the data set after it is generated, and to weight the  $\chi^2$  statistic.

#### 3.3.1. Nominal Method

The most detailed uncertainty model we use, and the one we use most often, is called the “nominal method.” It rotates each angular velocity vector  $\omega$  by some angle  $\Theta$  drawn from a normal distribution centered on  $\Theta = 0$  with standard deviation  $\sigma_\theta$ . The rotation is done around an arbitrary axis perpendicular to  $\omega$ . We also allow the length of  $\omega$  to change by a fraction  $\rho$  drawn from a log-normal distribution

$$P(\rho) = \frac{1}{\rho\sqrt{2\pi\sigma_\rho^2}} \exp\left(-\frac{\ln^2 \rho}{2\sigma_\rho^2}\right) \quad (35)$$

so that  $\ln\rho$  is normally distributed with mean zero (no change in the length of  $\omega$ ). Both uncertainties  $\sigma_\theta$  and  $\sigma_\rho$  are general parameters that can be tweaked to model uncertainty in spin axis direction and period. In our case, we often set  $\sigma_\rho = 0$  for simplicity. *Explain why with evidence.*

Practically, the randomization of the data is done by creating a rotation matrix  $M$  that rotates  $\hat{\mathbf{Z}}$  to the true rotational velocity vector  $\boldsymbol{\omega}$ . This same rotation matrix is then used to rotate the random vector  $\boldsymbol{\Omega} = \rho\omega(\sin\Theta\cos\Phi\hat{\mathbf{X}} + \sin\Theta\sin\Phi\hat{\mathbf{Y}} + \cos\Theta\hat{\mathbf{Z}})$  to the randomized velocity vector. Here,  $\Phi$  is a uniformly random variable in the range  $[0, \pi]$ . Specifically, we write this matrix as

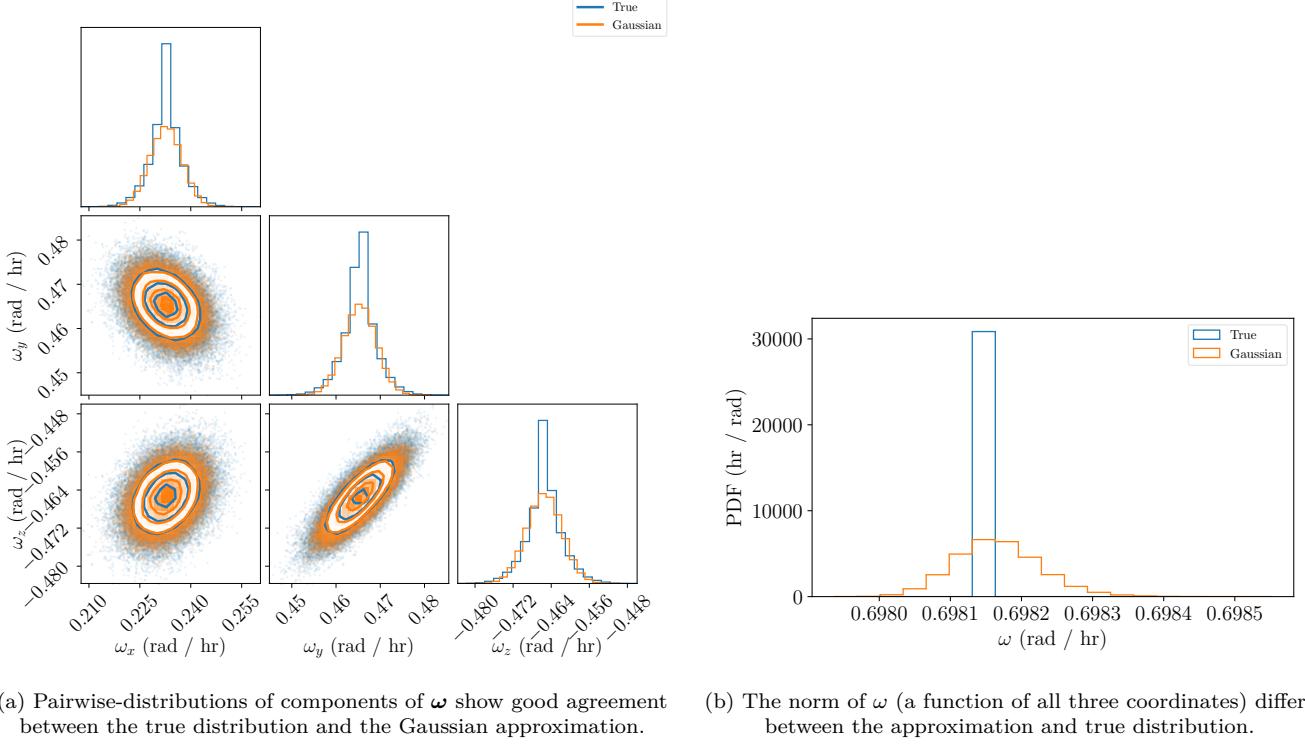
$$M = \rho \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \quad (36)$$

where  $\theta$  and  $\phi$  are the spherical coordinates of the original velocity vector  $\boldsymbol{\omega}$ .

Equation 36 is sufficient to randomize synthetic data, but an analytical expression for the coordinates of  $\boldsymbol{\omega}$  is required to define the  $\chi^2$  value used for fitting. We approximate the distribution of each spin coordinate as normal, with covariance given by the covariance of the true distribution. Based on equation 36, the covariance matrix is

$$\text{Cov}(\boldsymbol{\omega}) = \frac{1}{2} e^{-\sigma_\theta^2 + 2\sigma_\rho^2} \left[ \left( e^{-\sigma_\theta^2} - 2e^{-\sigma_\rho^2} + \cosh(\sigma_\theta^2) \right) \begin{pmatrix} \omega_x^2 & \omega_x\omega_y & \omega_x\omega_z \\ \omega_x\omega_y & \omega_y^2 & \omega_y\omega_z \\ \omega_x\omega_z & \omega_y\omega_z & \omega_z^2 \end{pmatrix} + \sinh(\sigma_\theta^2) \mathbb{1} \right]. \quad (37)$$

The difference between this Gaussian approximation and the true distribution are shown in figure 2 (a), which shows the distribution of the components of  $10^5$  angular velocity vectors and their correlations, randomly generated for a true velocity vector given by the reference value (appendix A), both using the true uncertainty method and the Gaussian approximation. The true distribution is more likely to yield results close to the true value than the Gaussian



(a) Pairwise-distributions of components of  $\omega$  show good agreement between the true distribution and the Gaussian approximation.

(b) The norm of  $\omega$  (a function of all three coordinates) differs between the approximation and true distribution.

**Figure 2.** The true distribution (blue) and the Gaussian approximation (orange) with the same covariance (orange) for the components of angular velocity  $\omega$  under the nominal uncertainty method.

approximation. Agreement also worsens in the case where one of the components of  $\omega$  nears zero (not shown). This may have the result of our likelihood not perfectly representing our system, and the fit probabilities will be slightly inaccurate.

A larger source of disagreement between the Gaussian approximated likelihood and the true distribution is the correlation between all three components of  $\omega$ . If two components of  $\omega$ , the third component is known up to its sign except when  $\sigma_\rho \neq 0$ . However, the covariance matrix of  $\omega$  does not capture this, nor do the distributions displayed in figure 2 (a). Therefore, when  $\sigma_\rho \simeq 0$ , the Gaussian likelihood function may poorly represent the model. To demonstrate this, we show a histogram of the norm of  $\omega$  for the Gaussian approximated distribution and the true distribution in figure 2 (b), which demonstrates that statistics involving all three components of  $\omega$  can differ highly between the approximation and the true distribution.

Aware of the failings of the Gaussian likelihood assumption, we then define the  $\chi^2$  statistic to be

$$\chi^2 = \sum_{i=0} (\omega_i^* - \omega_i)^T \text{Cov}^{-1}(\omega_i^*) (\omega_i^* - \omega_i) \quad (38)$$

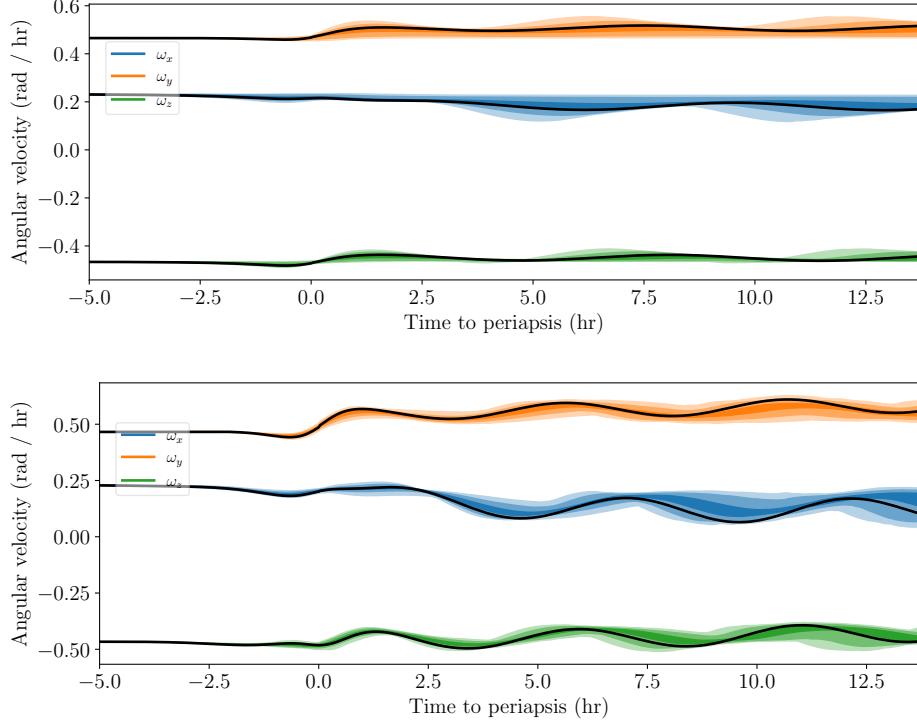
where  $\omega_i$  is the  $i$ th expected spin and  $\omega_i^*$  is the data. We also refer to the reduced chi squared,  $\chi_r^2 = \chi^2/(3N)$ , where  $N$  is the number of  $\omega_i$  in the data set.

*All this text is explanatory material for what I've been doing, but in the paper only the true likelihood will be there probably.*

Given the above explanation of the nominal uncertainty method, we define the likelihood of our data to be

$$\ln \mathcal{L} = \sum_{i=0} -\frac{\cos^{-1}(\omega_i^* \cdot \omega_i)^2}{2\sigma_\theta^2} - \frac{\ln(\omega_i^*/\omega_i)^2}{2\sigma_\rho^2} - \ln \frac{\omega_i^*}{\omega_i}. \quad (39)$$

Sometimes we compute the value  $\chi_r^2 = -2\ln \mathcal{L}/N$  where  $N$  is the number of spin vectors in the data set. For uncorrelated uncertainty *Check the assumptions on which the chi squared distribution is met* we expect  $\chi^2 \approx 1$  for large  $N$ . However, due to the highly correlated nature of our data, this limit occurs in our data set.



**Figure 3.** Angular velocity data simulated for the symmetric (above) and asymmetric (below) reference asteroid densities and the reference orbit (black lines). Also plotted is the deviation of the data for posterior-PDF-distributed perturbations to the asteroid shape (bands) using the nominal uncertainty method. Bands contain 68.3%, 95.5%, and 99.7% of the 1000 simulations run.

In figure 3 we present example data for this asteroid using the two reference asteroid and orbit parameters described in appendix A. Additionally, one thousand simulations are also run with asteroid parameters  $\gamma_0$ ,  $K_{20}$ , and  $K_{22}$  changed slightly, distributed according to the posterior distribution obtained by a fit to synthetic data altered according to the nominal uncertainty method (section 3.3) and a fit method described in section 3.2. An observational uncertainty of  $\sigma_\theta = 1.0$  was used. However, due to the large amount of data, even this large observational uncertainty does not lead to visible deviations from the true data. So the posterior distribution is widened by a factor  $N$ , where  $N$  is the number of data points in the data set. Bands containing 68.3%, 95.5%, and 99.7% of these additional simulations are plotted in figure 3. This choice of  $N$  is made merely for illustrative purposes; the scale of the bands in figure 3 have little meaning in an absolute sense, but are meaningful when comparing two bands or two points in one band.

### 3.3.2. Position dependent method

In practice, precision on the asteroid spin axis will increase with decreasing distance *Citation needed*. The nominal sensitivity model does not account for this; however, we can approximate this effect by choosing  $\sigma_\theta$  from the nominal method to be position-dependent. Specifically, we choose

$$\sigma_\theta(D) = \text{unimplemented}. \quad (40)$$

We do not expect this change to differ qualitatively from the nominal method because resolution is most important near pericenter, where  $D$  is stationary to first order.

### 3.4. System Setup

We study several environments for our asteroid, including (1) a flyby of a single, spherical planet, (2) a close flyby of a ringed planet, (3) a flyby of a planet with a moon, and (4) a close flyby of a planet with a moon. The qualitative and quantitative differences between these flyby types are discussed below.

### 3.4.1. One Spherical Planet

The simplest setup for an asteroid flyby is a close flyby of a single, spherical body. We call this central body a “planet” because it is likely that a body massive enough to observably torque an asteroid would be a planet. For such a body, the only relevant physical parameter is its mass,  $\mu_M$ , since  $J_{00} = 1$  and the other  $J_{\ell m}$  parameters are zero.

### 3.4.2. Ringed Planet

Equation 22 demonstrates that parameters controlling non-sphericity in the central body ( $J_{2m}$ ,  $J_{3m}$ , etc.) are damped by a factor of  $(a_M/D)^\ell$ , where  $a_M$  can be thought to be the radius of the central body. Furthermore, the  $J_{1m}$  terms are all zero by choosing the center of mass of the central body to be the origin of the coordinate system. Thus, we are interested in the extreme values of  $J_{\ell \geq 2}$  and whether they have a significant effect on the observability of asteroid density distributions. An example of such an extreme value is the presence of a planetary ring.

For a ring of inner radius  $a$ , outer radius  $b$ , and mass  $\mu_r$  oriented in the  $\hat{\mathbf{X}} - \hat{\mathbf{Y}}$  plane, the only changed  $J_{\ell m}$ s have  $m = 0$  by rotational symmetry. Furthermore,  $Y_{\ell, m} = 0$  on the equator for odd  $m$ , and  $(-1)^{\ell/2} \ell! / (2^\ell (\ell/2)!^2)$  otherwise. Thus, we have

$$J_{\ell m} = \begin{cases} \frac{2\pi\sigma}{\mu_M + \pi(b^2 - a^2)\sigma} \frac{(-1)^{\ell/2}}{a_M^\ell} \frac{b^{\ell+2} - a^{\ell+2}}{2^\ell (\ell/2)!^2 (\ell+2)} & \text{for } m = 0, \ell \text{ even, } \ell \neq 0 \\ 1 & \ell = 1 \\ 0 & \text{else} \end{cases} \quad (41)$$

where  $\mu_M$  is the mass of the planet without the ring and  $\sigma_s$  is the surface density of the ring.

The oblateness  $\epsilon$ , also called  $J_2$  of a body is defined as  $\epsilon = (I_z - I_x)/(\mu_M a_M^2)$ , where  $I_x = I_y$  (that is,  $J_{22} = 0$ ) and  $a_M$  is the true radius of the body, not the value defined in equation 25. Expressed in terms of  $J_{\ell m}$ , this is  $\epsilon = -2J_{20}$ . We can compare typical oblateness to the  $J_{20}$  value from equation 41 to see which effect is more dramatic.

Using data from the NASA Saturn Fact Sheet, we arrive at  $J_{20} \sim 10^{-8}$  from rings, but  $10^{-4}$  from oblateness. So oblateness is in fact the stronger effect.

*Don't actually use this section in the paper. I'm keeping it here as a math repository. Change the section to an oblate planet.*

### 3.5. Planet with a Moon: Distant Flyby

Consider a system with a moon of mass  $\mu_o$  at constant displacement  $\mathbf{R}$  from the central body. We assume that both the planet and the moon are spherical. As long as the distance from the barycenter of the planet-moon system to the asteroid  $D$  always satisfies  $D > r$  we are free to use the formalism developed above, absorbing the moon into  $J_{\ell m}$ . Making use of the identity ?

$$R_{\ell m}(\mathbf{R} + \mathbf{R}') = \sum_{\ell'=0}^{\ell} \sum_{m'=-\ell'}^{\ell} R_{\ell' m'}(\mathbf{R}') R_{\ell-\ell', m-m'}(\mathbf{R}) \quad (42)$$

*Is this true?* for  $R > R'$ , we have

$$J_{\ell m} = \frac{1}{(\mu_M + \mu_o)a_M^\ell} \left[ \delta_{\ell 0} \delta_{m 0} \mu_M + \int_{\mathcal{P}} d^3 r' \rho_o(\mathbf{r}') R_{\ell m}(\mathbf{R} + \mathbf{r}') \right] \quad (43)$$

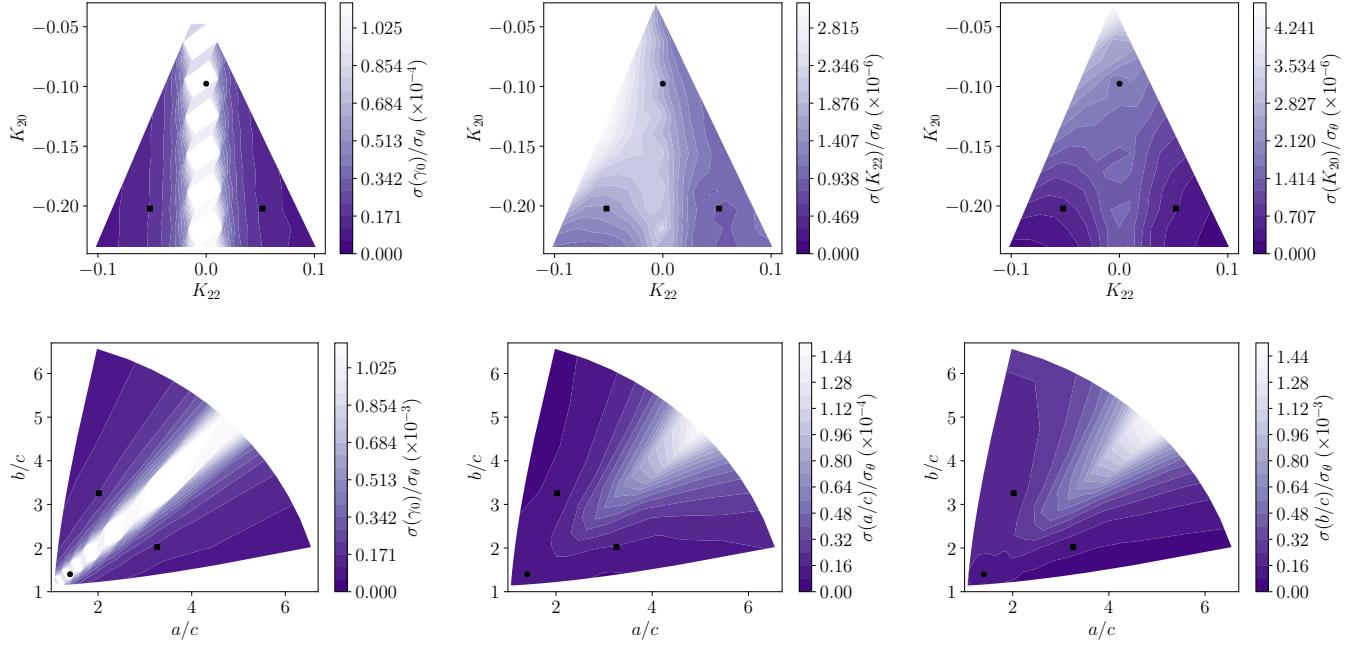
$$J_{\ell m} = \frac{1}{(\mu_M + \mu_o)} \left[ \mu_M \delta_{\ell 0} \delta_{m 0} + \mu_o \frac{R_{\ell m}(\mathbf{R})}{a_M^\ell} \right] \quad (44)$$

where the  $\delta$ s are Kronecker deltas. It is apparent that if  $a_M^\ell$  remains the radius of the planet,  $|J_{\ell m}| \rightarrow \infty$  as  $\ell \rightarrow \infty$ . However, since each  $J_{\ell m}$  is damped by  $a_M^\ell/D^\ell$  and since  $R < D$  by construction, the series in equation 22 remains roughly geometric and does not blow up. For the sake of this calculation, we neglect motion of the moon during the flyby.

For our tests, we use parameters similar to our Moon: we set  $\mu_o = 0.01\mu_M$  and  $R = 60a_M$ .

### 3.6. Planet with a Moon: Close Flyby

If, contrary to the previous section, we want to fly by both the moon and the planet of a binary system, we must consider the tidal torque contributions from both bodies separately. The formalism above cannot be extended to write



**Figure 4.** Ratios of fit uncertainties (the standard deviations of parameter posterior distributions) to experimental uncertainty ( $\sigma_\theta$ ) for fit parameters  $\gamma_0$ ,  $K_{22}$ , and  $K_{20}$  (top row) and  $\gamma_0$ ,  $a/c$ , and  $b/c$  (bottom row). Also shown are the reference asteroid shapes used elsewhere in this paper (appendix A); the symmetric case is marked with a circle and the asymmetric with a square.

one set of  $J_{\ell m}$  for both the moon and planet. To do this, we generate two separate paths of  $\mathbf{D}(t)$ , one pointing from the central body to the asteroid and one pointing from the moon to the asteroid. Both are defined in the inertial reference frames. Then the torque from both can be computed by equation 22 and added to get the net torque on the asteroid. Again, we neglect motion of the moon during the flyby.

#### 4. RESULTS

To interpret the capacity of this tidal torque technique to constrain the density distribution of asteroids, we generate synthetic data in multiple simulated settings and fit to this data using an MCMC, reporting the standard deviation of the posterior parameter distributions as a measure of the effectiveness of the fit. Fits with narrow posterior distributions are considered more effective than those with wide posterior distributions, and the correlations between parameters are also of interest when computing uncertainties on physical models for the asteroid shape.

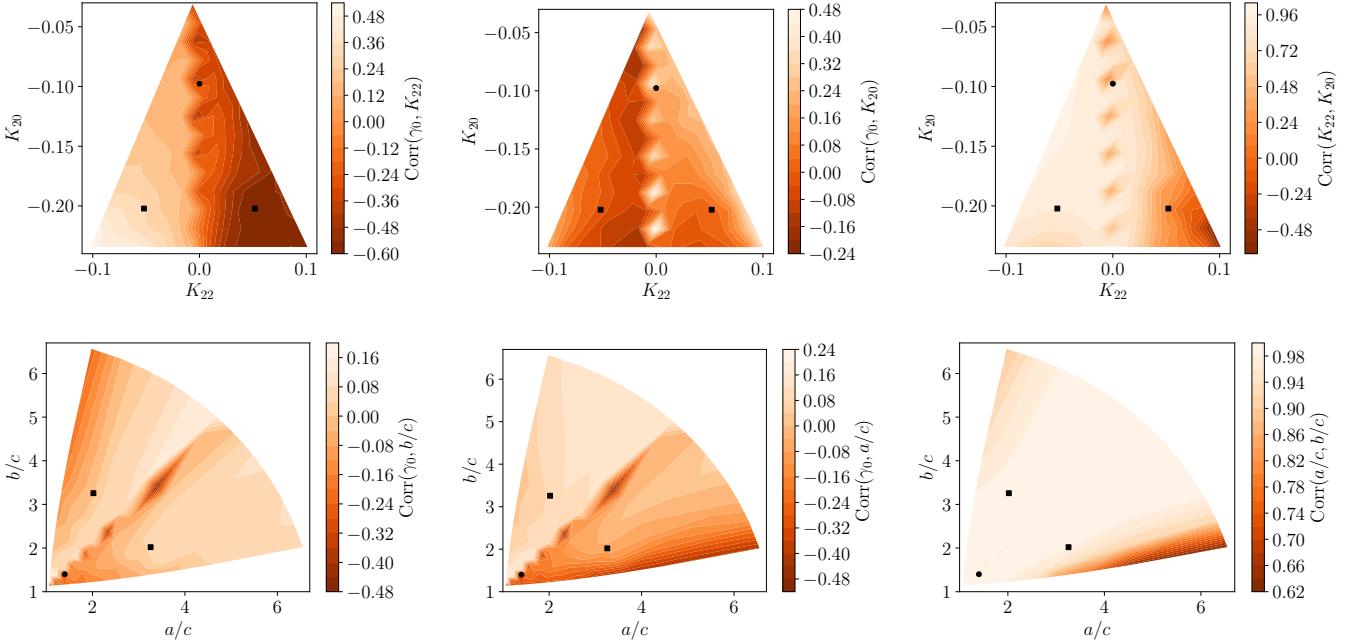
The distributions are nearly Gaussian in most cases (see appendix ??) and all fits are run until convergence.

##### 4.1. Dependence on asteroid parameters

Of first importance is the dependence of parameter uncertainties on the true parameter values. To measure this, we run 120 fits, all for the two  $K_{\ell=2}$  parameters and  $\gamma_0$  with the standard orbit (appendix ??) and experimental uncertainty  $\sigma_\theta = 0.01$  with the nominal uncertainty model (section 3.3). Each fit is run for  $\gamma_0 = \pi/8$  and  $K_{20}, K_{22}$  distributed in a triangular lattice with 17 layers, spanning the parameter space. The 33 cases with extremal values of  $K_{22}$  are not fitted as being too close to the edge of the parameter space.

The parameters  $K_{22}$  and  $K_{20}$  are then converted to the more physical parameters  $a/c$  and  $b/c$  under the assumption of an elliptical asteroid of uniform density with minor axes  $a, b, c$  (section 5.1.1). Uncertainties on  $\gamma_0$  and the two shape parameters are given in figure 4, and correlations between pairs of parameters ( $\text{Corr}(X, Y) = \text{Var}(X, Y)/(\sigma_X \sigma_Y)$ ) are given in figure 5.

These figures demonstrate several key properties. Figure 4 demonstrates that fit uncertainties for all three parameters are generally for asteroids with an axis of rotational symmetry — i.e.  $a = b$  or  $K_{22} = 0$ . This is likely due to the fact that the local axes, which are required to define the parameters, are set such that they align with the asteroid principal axes, and therefore are degenerate in the case of a symmetrical asteroid. For similar reasons, uncertainty in



**Figure 5.** Correlations between fit uncertainties and experimental uncertainty ( $\sigma_\theta$ ) for fit parameters  $\gamma_0$ ,  $K_{22}$ , and  $K_{20}$  (top row) and  $\gamma_0$ ,  $a/c$ , and  $b/c$  (bottom row). Also shown are the reference asteroid shapes used elsewhere in this paper (appendix A); the symmetric case is marked with a circle and the asymmetric with a square.

Benchmark configurations	$\sigma(\gamma_0) / \sigma_\theta$	$\sigma(K_{22} [a/c]) / \sigma_\theta$	$\sigma(K_{20} [b/c]) / \sigma_\theta$	$\text{Corr}(\gamma_0, K_{22})$	$\text{Corr}(\gamma_0, K_{20})$	$\text{Corr}(K_{22}, K_{20})$
Symmetric: $a = b$	36	$1.9 \times 10^{-6}$	$1.7 \times 10^{-5}$	-0.40	0.36	0.19
	36	$9.4 \times 10^{-5}$	$1.9 \times 10^{-4}$	-0.42	-0.29	0.96
Asymmetric: $a < b$	$1.6 \times 10^{-5}$	$1.5 \times 10^{-6}$	$5.4 \times 10^{-6}$	0.32	-0.0053	0.88
	$1.6 \times 10^{-5}$	$9.4 \times 10^{-5}$	$1.8 \times 10^{-4}$	0.082	-0.019	0.99
Asymmetric: $a > b$	$1.5 \times 10^{-5}$	$9.1 \times 10^{-7}$	$4.8 \times 10^{-6}$	-0.58	0.017	0.15
	$1.5 \times 10^{-5}$	$1.5 \times 10^{-4}$	$1.0 \times 10^{-4}$	-0.049	0.061	0.98

**Table 1.** Values for the standard deviation and correlation of  $\gamma_0$ ,  $K_{22}$ , and  $K_{20}$  (top entry of each row) or  $\gamma_0$ ,  $a/c$ , and  $b/c$  respectively (bottom entry of each row) for the symmetric and asymmetric reference points. The large  $\sigma(\gamma_0)$  for the symmetric benchmark reflects the fact that  $\gamma_0$  is not defined for a symmetric body, and the posterior probability distribution is poorly constrained.

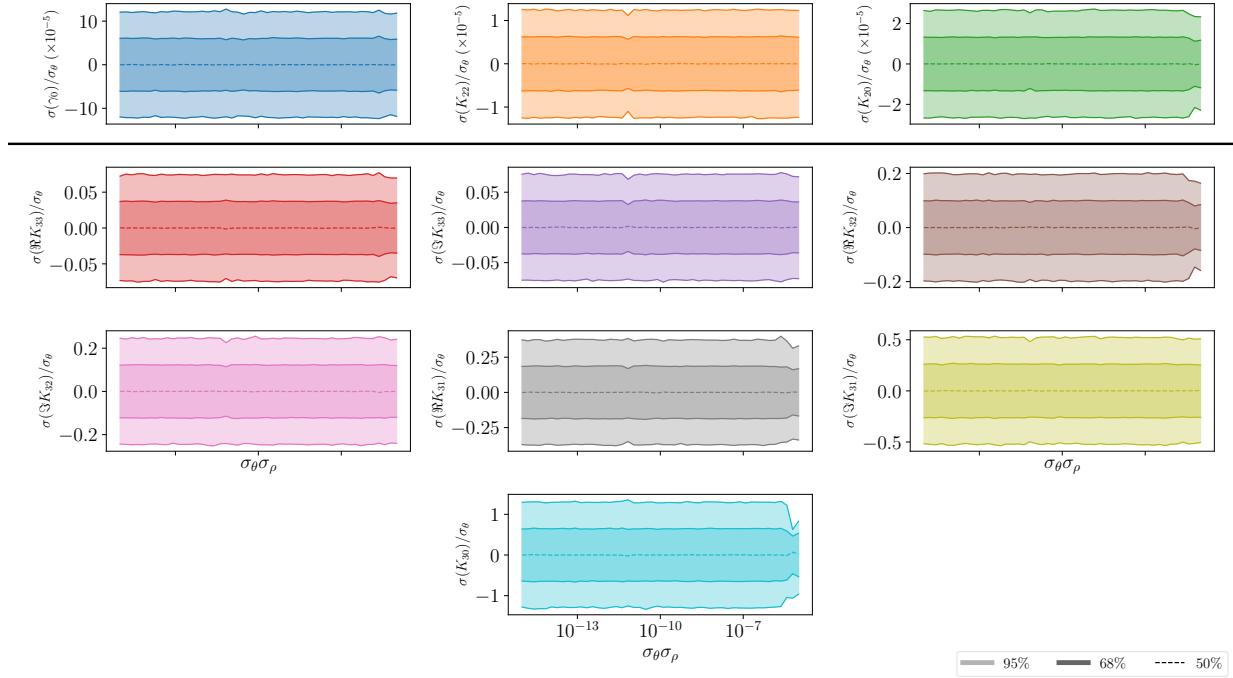
$\gamma_0$  especially is large for  $a \approx b \approx c$ , which is the spherical case in which no tidal torque is generated. Beyond these two trends, uncertainty in  $a/c$  and  $b/c$  appear to scale linearly with these ratios.

Figure 5 demonstrates that the parameters are generally correlated. Specifically,  $\gamma_0$  and  $|K_{22}|$  are correlated for large  $|K_{22}|$  which corresponds to highly asymmetric asteroids. Likewise, for the correlation of  $a/c$  and  $b/c$  with  $\gamma_0$ .  $\text{Corr}(\gamma_0, K_{20})$  is less, however.  $K_{20}$  and  $K_{22}$  are strongly correlated in general as are  $a/c$  and  $b/c$ .

Also shown in figures 4 and 5 are the reference asteroid configurations which we study when fixing the asteroid shape and varying other parameters. We note that these reference configurations capture the above trends in variance and correlation, so that they can be viewed as representative of all but the most extreme asteroid configurations. The measured values of variance and covariance are displayed in table 1.

#### 4.2. Dependence on observational precision

Depending on the size of the asteroid, its orbital elements, and the availability of observation, uncertainty on the rotational pole of the asteroid is expected to vary between flybys. A detailed model of uncertainty in the rotational pole of the asteroid (a model that might simulate a full lightcurve, for example) is beyond the scope of this paper.



**Figure 6.** Ratio of 1- and 2- $\sigma$  uncertainties of the marginal PDFs to the observational uncertainty  $\sigma_\theta$  as a function of the product  $\sigma_\theta\sigma_\rho$ . A horizontal line divides the precise  $\ell = 2$  parameters from the others. Except at large  $\sigma_\theta$ , the ratio is observed to be constant. The asymmetric asteroid benchmark is used. Values of the ratio for other asteroid models can be computed from figure 4.

However, in this section we demonstrate how the uncertainty in the fit parameters is governed by the observational uncertainty in rotational pole  $\sigma_\theta$  and in the period  $\sigma_\rho$  as a rule-of-thumb to guide future investigation.

We represent 1- $\sigma$  uncertainty in the fit parameters as the difference between the 84th or 15th percentile of the marginal posterior PDF for each fit parameter and its mean. For a multivariate Gaussian posterior PDF (which must also have Gaussian marginal PDFs), this 1- $\sigma$  uncertainty is exactly the standard deviation of the marginal PDF. We perform a similar exercise for the 2- $\sigma$  uncertainty.

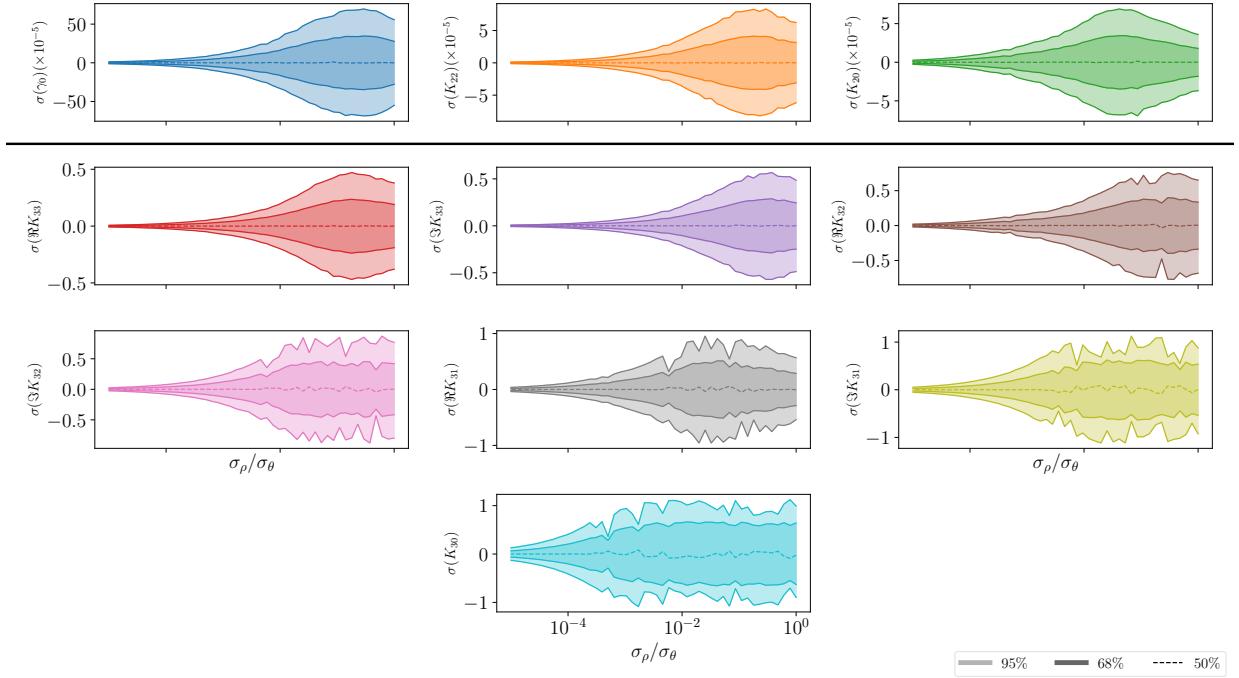
We expect that high observational uncertainty corresponds to high uncertainty in the best-fit parameters. To show this trend, in figure 6 we plot the best-fit uncertainty as a function of the product of the two uncertainty types, with the ratio fixed at  $\sigma_\rho/\sigma_\theta = 10^{-5}$  (the benchmark value) so that we are only sensitive to the total uncertainty. To check deviation from a linear trend, we plot the ratio of fit uncertainty to  $\sigma_\theta$ . The figure shows no deviation from a linear trend except for high observational uncertainty, at which point the posterior distributions begin to fill the prior distributions and are no longer Gaussian.

To test the relative importance of  $\sigma_\rho$  and  $\sigma_\theta$ , we also show best-fit uncertainty as a function of the ratio  $\sigma_\rho/\sigma_\theta$  with the product fixed at  $\sigma_\rho\sigma_\theta = 10^{-7}$  (the benchmark value) in figure 7. If  $\sigma_\rho$  and  $\sigma_\theta$  were interchangeable in terms of their effect on precision of the posterior distributions, (e.g., if the experimental uncertainty were a function of  $\sigma_\rho\sigma_\theta$ ) we might expect no dependence on the ratio of the uncertainties. However, we see relatively low uncertainty for low  $\sigma_\rho/\sigma_\theta$  in figure 7, with roughly stable (or slightly decreasing) fit uncertainty for  $\sigma_\rho \approx \sigma_\theta$ . Therefore, for fixed product  $\sigma_\rho\sigma_\theta$ , low  $\sigma_\rho$  compared to  $\sigma_\theta$  will yield more precise fits.

Due to the constancy of the ratio of fit to observational uncertainty for fixed  $\sigma_\rho\sigma_\theta$ , the ratio can be extracted for other asteroid configurations from figure 4, and we are not much concerned with the value of  $\sigma_\theta$  used throughout this paper because we are aware that results can be easily rescaled.

#### 4.3. Dependence on cadence

To analyze the degree to which our fit accuracy and precision depends on the rapidity of observation, we simulate a flyby of the reference asteroid with data taken at fixed intervals (“cadences”) of 2 minutes to one hour, perform a fit to each, and compare the precision and accuracy of the fit results as a function of cadence.



**Figure 7.** 1- and 2- $\sigma$  uncertainties of the marginal PDFs as a function of the ratio  $\sigma_\rho/\sigma_\theta$ . A horizontal line divides the precise  $\ell = 2$  parameters from the others. Period uncertainty  $\sigma_\rho$  is shown to be more vital than pole uncertainty  $\sigma_\theta$ .

Figure 8 shows the 68th and 95th percentiles of the posterior PDF for each fit parameter as a function of evenly-spaced cadences from 2 to 60 minutes. The percentiles are divided by  $\sigma_\theta$  (which is constant in this figure) since they are expected to be linearly proportional to  $\sigma_\theta$  in general. As expected, the precision of the fit result decreases with increased cadence (less data). However, the decrease is not uniform. All parameters show a slow increase in posterior PDF width until  $\Delta t \approx 40$  min, at when a sharp increase PDF width occurs, often followed by a slight decrease before the uncertainty rises again. *Say more about this once the orbital information comes through.*

Figure 9 shows the difference between best fit parameter results and the true values as a function of cadence. No divergence is seen between then true and best-fit results for high cadence, demonstrating that even a small amount of data is sufficient to extract accurate parameters for the asteroid density distribution, albeit with the precision penalties shown in figure 8

#### 4.4. Dependence on orbital parameters

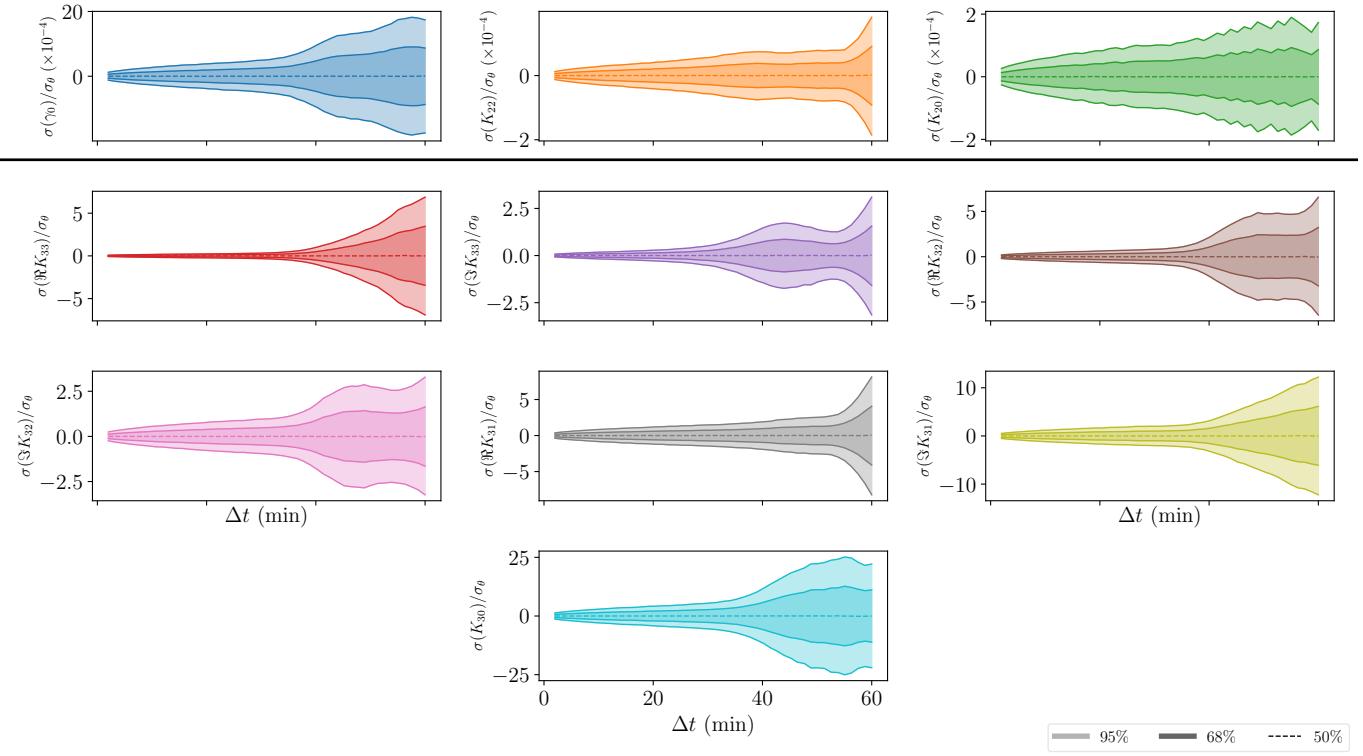
The dependency of the best-fit parameter uncertainty on the orbit of the asteroid is analyzed in this section. We parameterize the orbit by  $v_\infty$ , the excess velocity, and  $r_p$ , the distance of the perigee. We do not change the plane of the orbit because this is equivalent to rotating  $J_{\ell m}$ , and since we use the benchmark model for this section (with a spherical planet) orbital plane changes have no effect.

Figure 10 shows the best-fit uncertainty divided by (constant)  $\sigma_\theta$  as a function of  $v_\infty$ . The true  $K_{\ell m}$  parameters are extracted in all cases, generally with larger uncertainty for large  $v_\infty$ . This trend is in agreement with intuition, since flybys with larger  $v_\infty$  occur over a faster time range. Uncertainty in the high- and low-order parameters increases at a similar rate with  $v_\infty$ .

Figure 10 also demonstrates a strong dependence on the exact value of  $v_\infty$  through small-scale oscillations in precision as a function of  $v_\infty$ . This is because uncertainty is strongly dependent on the amplitude of oscillations after the encounter *how?*, and these oscillations are highly sensitive to initial conditions such as  $v_\infty$ .

#### 4.5. Dependence on spin pole

In addition to the physical parameters of the asteroid, the initial orientation of the asteroid spin pole strongly effects the precision of the best-fit parameters attainable from the data. For example, it was pointed out that  $\omega \propto \hat{\mathbf{Z}}$  gives  $\tau = 0$  in section 2.3. Since  $\dot{\omega} = 0$  until torque is applied to start tumbling, we expect  $\omega$  to be constant throughout



**Figure 8.** The 68th and 95th percentile envelopes for the posterior PDFs associated with ten fit parameters as a function of the cadence of the data set, plotted as a function of cadence.

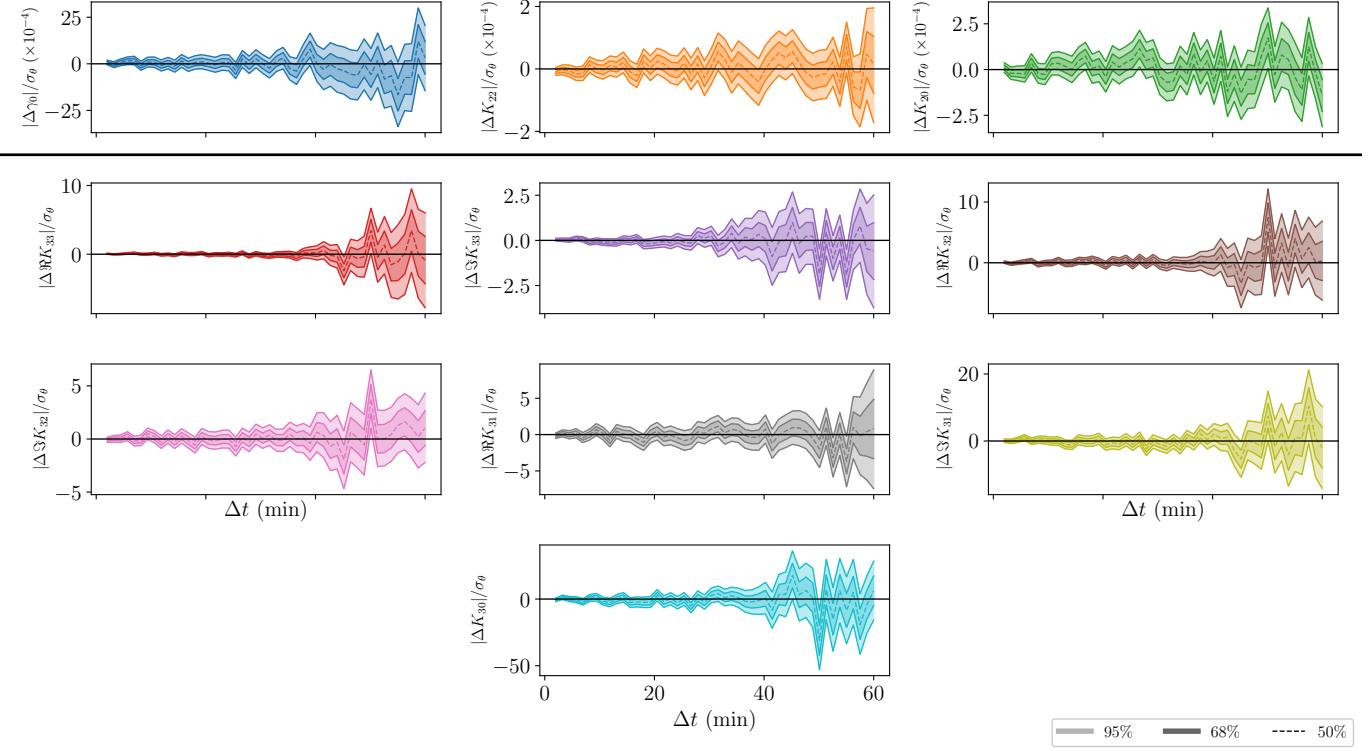
the data set in this case, and the fit parameters  $K_{\ell m}$  will be invisible. We also expect to see decreasing fit precision near this limit.

Figure 11 displays the  $1\sigma$  uncertainties on the first ten fit parameters of the asymmetric reference asteroid flyby as a function of the initial direction of the spin pole. In all cases, the asteroid's initial period was set to the reference 9 hours. The data is presented mapped onto a sphere which indicates the direction of the spin pole, in the Mollweide projection. To highlight common features between the ten parameters, the average standard deviation is presented in figure 12. The average is weighted such that the uncertainty map for each parameter contributes an equal amount (the weight of each is set to one-tenth of its mean). Also shown in 12 is the map of the average uncertainty with a polar projection, to show the high uncertainty near  $\omega \propto \hat{Z}$ .

To generate figures 11 and 12, the sphere was populated with 64 sample sites spaced on a Fibonacci sphere  $(\theta_i, \phi_i) = (1 - (2i + 1)/64, 2\pi i / \varphi)$ , where  $\varphi \approx 1.618$  is the golden ratio. This yielded points that were roughly evenly distributed to avoid discontinuities in the figures. The locations of these 64 points are plotted in figure 12 for reference.

Figure 12 confirms our expectation that the fit parameters would be uncertain near  $\omega \propto \hat{Z}$ , and it displays other points of lack of precision: those of  $\omega \propto \pm \hat{Y}$ . Not all of the parameters shown in figure 11 demonstrate this bare point: for example,  $K_{20}$  and  $K_{30}$  are not as affected by  $\omega \propto \hat{Y}$ . The fact that these regions of imprecision line up with  $\pm \hat{Y}$  is not guaranteed, and it is a coincidence that the reference asteroid happens to. In general, these region will be placed at other points on the equator at a location determined by the asteroid's  $K_{\ell m}$  and  $\gamma_0$ .

This dependence on asteroid parameters is because, as section 2.3 points out, certain directions of  $\omega$  give can be shown via equation 22 to give  $\tau = 0$  regardless of  $K_{\ell m}$ . Other locations give  $\tau \parallel \hat{Z} \perp \omega$ , which may cause imprecision both because it restricting  $\tau$  to only one coordinate lowers its magnitude, and because  $d/dt|\omega|$  is minimized when  $\tau \perp \omega$  and the asteroid is not tumbling. Of these low-torque locations, those that do not depend on  $\gamma$  all satisfy either  $\omega \parallel \hat{Z}$  (zero torque) or  $\omega_\phi = \phi + n\pi/2$ , where  $n$  is an integer,  $\omega_\phi$  is the azimuthal coordinate of  $\omega$  in the global frame, and  $\phi$  is the azimuthal location of the asteroid. As the asteroid encounters the central body,  $\phi$  sweeps over a large range so that the low-torque loci associated with several values of  $n$  affect the dynamics of the asteroid, and the



**Figure 9.** The difference between the fit parameters and the true parameters, with 68th and 95th percentile envelopes shown, for ten fit parameters as a function of cadence. No significant divergence is seen for high cadence. *Is this figure necessary?*

degree to which the asteroid's dynamics are affected depends on  $K_{ellm}$ . Thus, exactly what initial spin pole direction maximizes uncertainty in the fit parameters depends on  $K_{\ell m}$ , but it is expected to lie in the orbital plane.

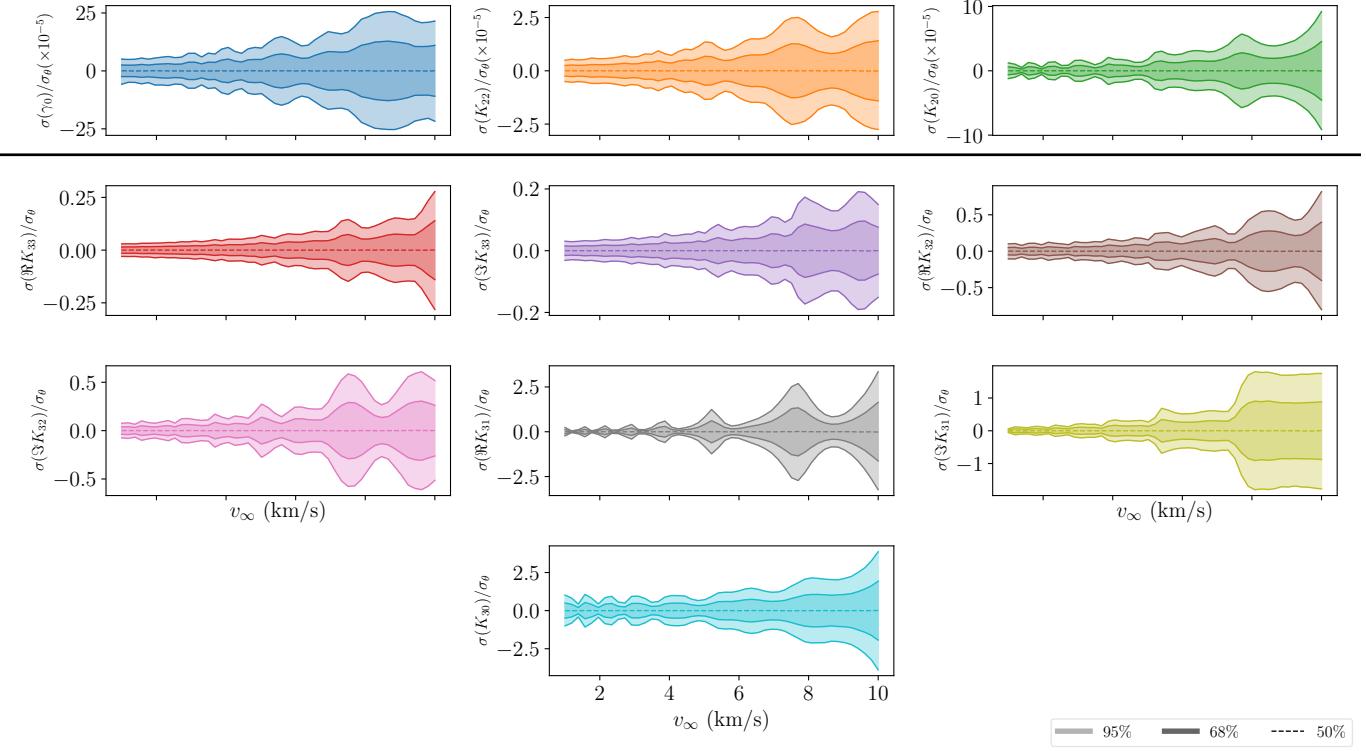
Since  $K_{20}$  is not controlled by torque at this point, and some show lack of precision at  $\omega \propto -\hat{Y} + \frac{1}{2}\hat{Z}$  as well. Better precision, however, was generally attained when aligning parallel or antiparallel to  $\hat{\mathbf{X}}$ , the perigee vector. Initial poles with negative latitude also often relatively precise estimates. However, the average best-fit is not very volatile; it rarely grows to twice its mean over the sphere. Exactly which regions of the sphere yield better precision is likely controlled by particulars of the reference asteroid rather than fundamental properties of asteroid flybys, since no cases except when  $\omega \propto \hat{Z}$  does  $\tau = 0$ .

The increased uncertainty in the non-polar regions stems from the way the  $K_{\ell m}$  are correlated. (If two  $K_{\ell m}$  are highly correlated, we “double count” uncertainty because uncertainty in one parameter will yield uncertainty in the other). A metric of average uncertainty which is immune to correlation is the trace of the covariance matrix, since the trace of a covariance matrix transformed by an orthogonal matrix is the trace of the original covariance matrix. Note that this trace is also the sum of eigenvectors of the covariance matrix, which is the sum of the variances of uncorrelated parameters. This covariance matrix trace is plotted in figure ??.

#### 4.6. Dependence on blocking out perigee

Much of the data available concerning an asteroid's rotational velocity is captured long before or after the flyby perigee, where tidal torque is low. The angular velocity evolution here is dominated by torque-free precession dictated by the moment of inertia components, so that zero-torque data can still be used to fix  $K_{20}$  and  $K_{22}$ . However,  $K_{3m}$  are not extractable from precession data alone. We are therefore curious how much of the sensitivity of the system to  $K_{3m}$  components is due only data taken during the encounter perigee.

To test this, we mask the perigee of the counter by removing data within time  $T$  of the perigee, where  $T$  ranges from 0 to 3 hours. To prevent lack of precision on  $K_{\ell m}$  induced by lower amounts of data for high  $T$ , we always cut 3 hr– $T$  from the beginning and end of the data set, so that all data sets produced for all  $T$  have the same length of data before and after the perigee. We then fit the same asteroid model to the cut data for all  $T$  and plot the standard



**Figure 10.** Best-fit uncertainties as a function of  $v_\infty$ , the excess velocity of the asteroid orbit. The three higher order parameters (top) are plotted with the seven lower order parameters. A general trend of increasing uncertainty for higher  $v_\infty$  is visible.

deviations of the parameter posterior PDFs for (1)  $\gamma_0$  and  $K_{2m}$  as a control, and (2)  $K_{3m}$ , to test whether they can still be fitted for without data concerning the perigee. We use the asymmetric benchmark asteroid configuration.

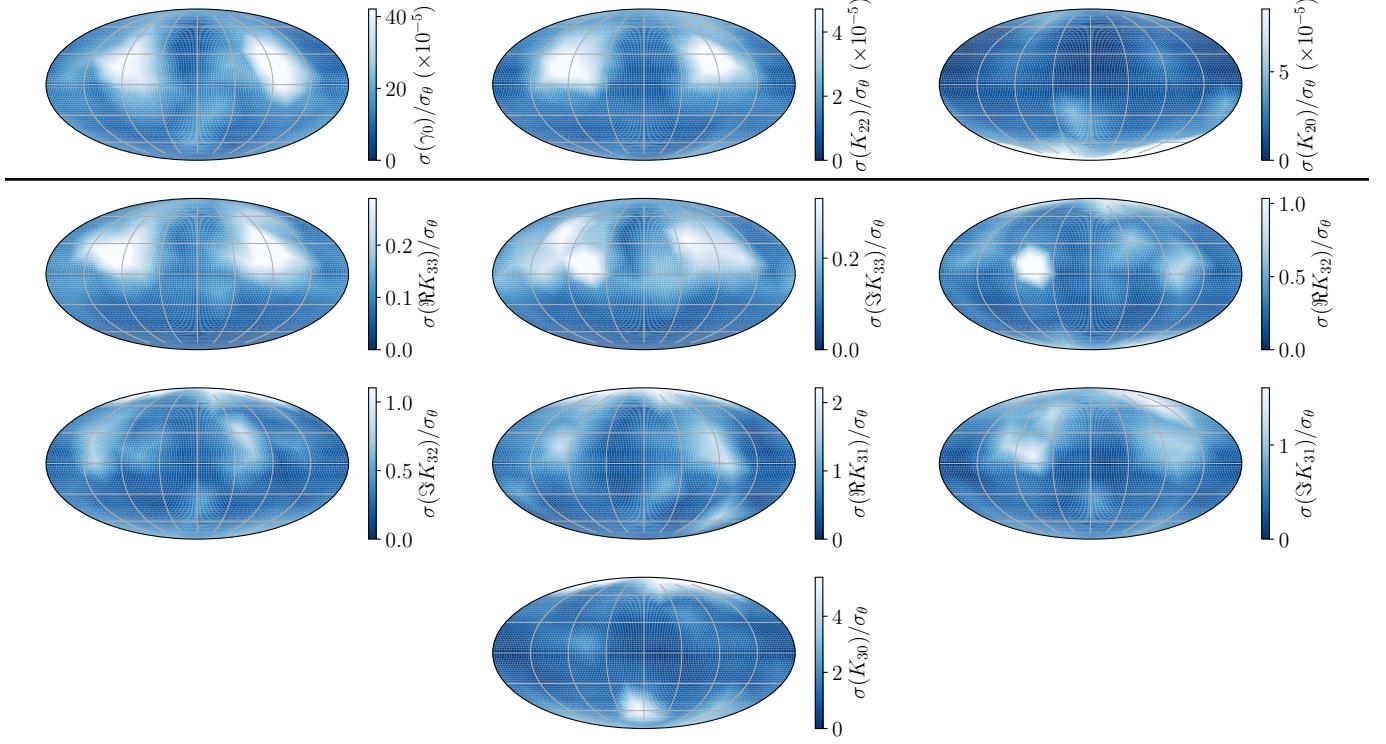
Figure 13 shows the best fit  $1\sigma$  and  $2\sigma$  uncertainty in the first order  $K_{2m}$  and  $\gamma_0$  and higher order  $K_{3m}$  parameters as a function of the amount of perigee data  $T$  that has been cut from the synthetic data set. Since torque is highest at perigee, we expect that region of the data to contain the most information about  $K_{\ell m}$ , and therefore that uncertainty should increase monotonically with  $T$ , which is observed. We also see that this increase halts after about 2 hours of data has been cut for the first order parameters, likely because the majority of the tidal torque occurs within 2 hours of perigee. The exact time of this threshold is dependent on the asteroid orbital parameters.

The higher order parameters show different behavior, increasing in  $\sigma$  at smaller  $T$  than the first-order parameters, and continuing to increase even for  $T > 2$  hr. Qualitatively, this is expected since the torque is sensitive to the first-order parameters to order  $D^{-3}$  and to the second-order parameters to order  $D^{-4}$ , so that the second-order parameters are expected to be more dominated by data near the center of the orbit than the first-order parameters. It is clear from figure 13, however, that not all the higher order parameters increase in uncertainty at the same rate as each other, nor do they increase in uncertainty uniformly with  $T$ ; once  $T \sim 1$  hr is cut out, further cuts to  $T$  do not damage precision in  $K_{3m}$  as much until  $T \gtrsim 2$  hr is reached.

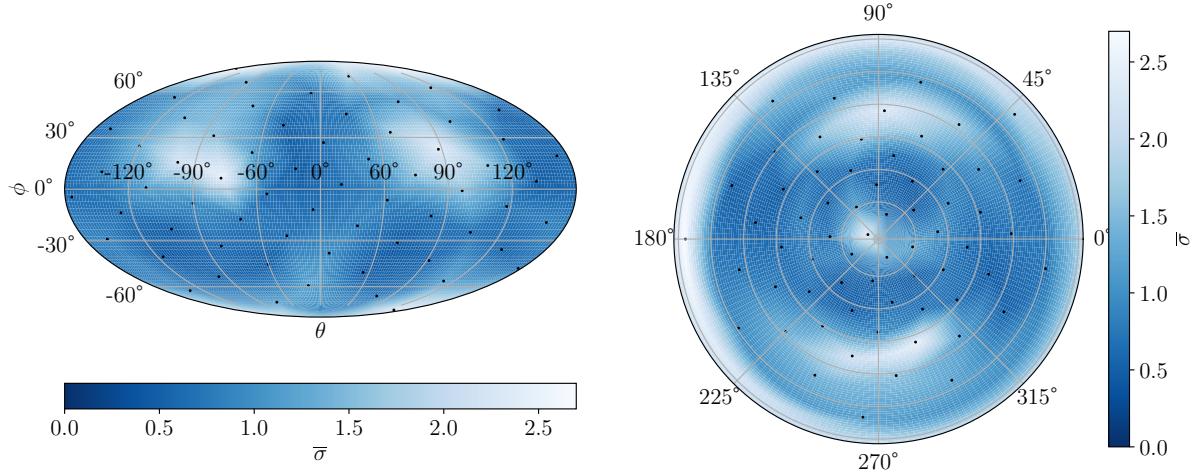
#### 4.7. Oblate planet

Until this point, we have assumed an asteroid flyby of Earth because observations are most plentiful for near-Earth asteroids. However, it is worth considering how observations taken around another planet — by a large Earth-orbiting telescope or an interplanetary mission, for instance — could affect the efficacy of this model.

In particular, we consider how high planetary oblateness and the presence of moons affects the torque dynamics of asteroids. Other central body parameters worth of mention are the central body mass and radius. However, the dependence of tidal torque on these properties is clear from equation 22 without requiring a simulation. If the central planet mass  $\mu_M$  is increased, tidal torque increases uniformly and we expect greater precision in our measurements. If the planetary radius is increased, we also expect an increase in the distance  $D$  to the planet. Therefore, we consider



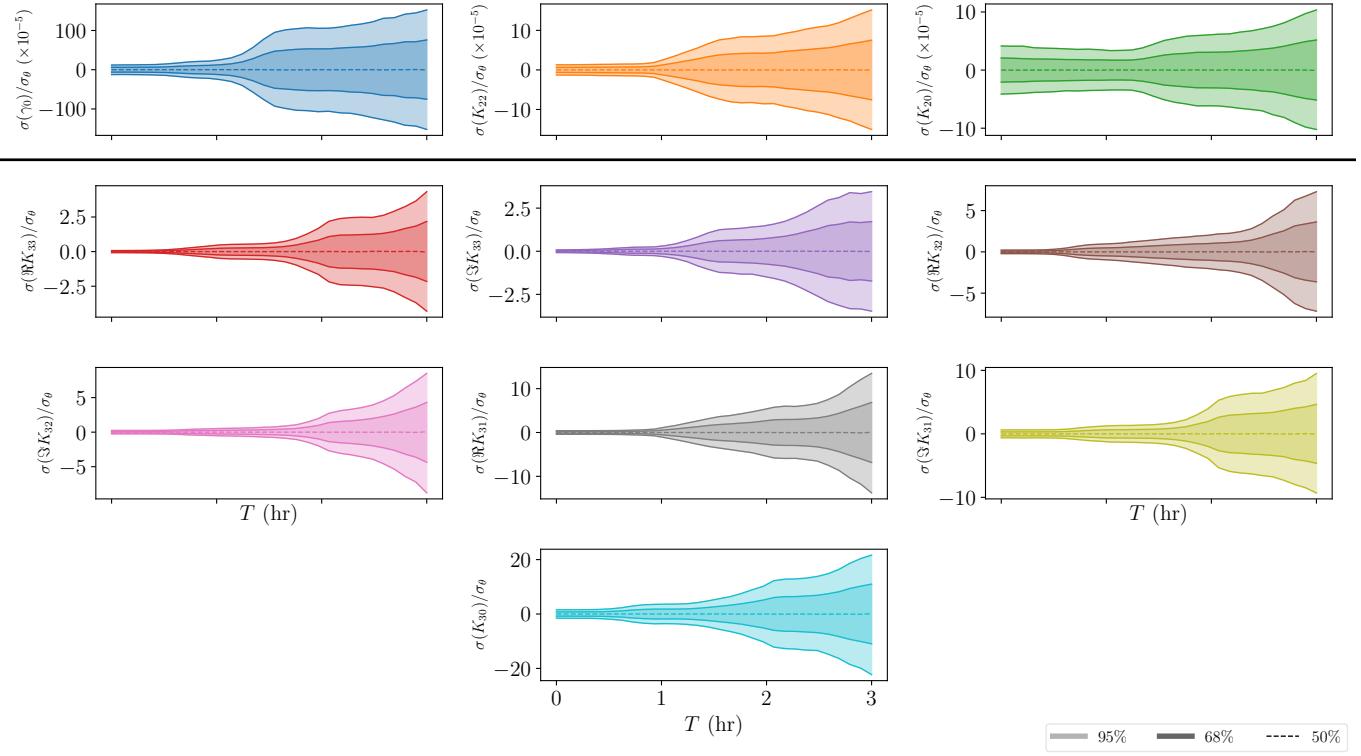
**Figure 11.**  $1 - \sigma$  uncertainties in the first 10 asteroid fit parameters as a function of the initial direction of spin. All maps are made in the Mollweide projection and shown in the inertial  $\hat{X} - \hat{Y} - \hat{Z}$  frame.



**Figure 12.** The weighted average of the uncertainties shown in figure 11, in Mollweide (left) and polar (right) projections. See text for a description of how the average was computed.

changes in  $(a_M/D)$ . Each  $J_{\ell m}$  term is multiplied by a factor of  $(a_M/D)^{\ell}$ , so an increase in planetary radius without a change in mass only affects the subdominant terms of 22.

Depending on the composition and rotation rate of the planet which the asteroid is encountering, the oblateness (equivalently, the planet's  $J_{20}$  or  $J_2$  value) may vary. The  $J_{1m}$  values are fixed at zero by the constraint that the center of mass of the central body must be at the origin. Equation 22 shows that the presence of a large  $J_{20}$  value introduces



**Figure 13.** Best fit uncertainty in the first (top row) and second (bottom three rows) order parameters as a function of  $T$ , the amount of perigee observational data cut from the data set. All data within  $T$  hours of perigee is cut, and the data set is correspondingly extended before and after the perigee so that its total size is unchanged.

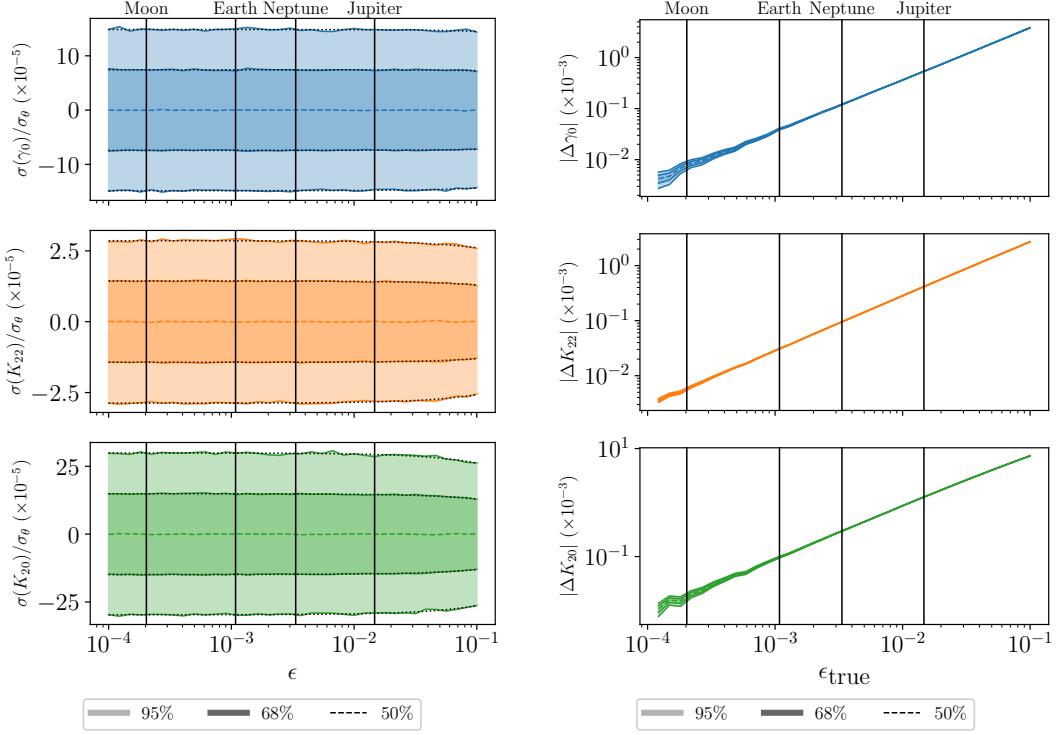
a second factor through which the  $K_{2m}$  parameters can affect the tidal torque, which is damped by an extra factor of  $D^{-2}$  compared to the  $J_{00}$  term. In this section, we ask what can be gained by a large oblateness.

Another factor work consideration is the presence of a planetary ring, which will affect  $J_2$ . However, using the properties of Saturn’s rings as an benchmark, we compute that the oblateness of the planet far outstrips the increase in  $J_2$  caused by the ring, so we ignore this scenario henceforth. *Cite an appendix?*

For our purposes, we define oblateness to be  $\epsilon = (I_z - I_x)/(\mu_M a_M^2)$  for central body moments of inertia  $I_{x,y,z}$  where  $I_x = I_y$  (that is,  $J_{22} = 0$ ) and  $a_M$  is the true radius of the body, not the value defined in equation 25. (Some sources such as ? take this definition of  $\epsilon$  as their definition of  $J_2$ . Expressed in terms of  $J_{\ell m}$ , this is  $\epsilon = -2J_{20}$ . Given this conversion between  $\epsilon$  and  $J_{20}$ , we generate simulated data sets and fit  $K_{2m}$  parameters to them for logarithmically spaced values of  $\epsilon$  between  $10^{-4}$  and  $10^{-1}$ . The 68th and 95th percentiles deviations, computed as in figure 4, are shown in figure 14.

Figure 14 shows a slight dependence of best-fit uncertainty on oblateness, as expected due to the weakness of the  $J_{20}$  term in equation 22 compared to the dominant  $J_{00}$  term. However, the fit parameter uncertainty does slightly decrease as oblateness increases on further study. The trend of the decrease is linear, as shown by the close agreement between the overlaid best fit linear trend between the 1- and 2- $\sigma$  best fit uncertainties to the measured values. We compute an average slope of the 1- $\sigma$  trend divided by the uncertainty for zero eccentricity as  $\Delta\sigma/\Delta\epsilon/\sigma_{\epsilon=0} = -0.25$  for  $\gamma_0$ ,  $-0.94$  for  $K_{22}$ , and  $-1.2$  for  $K_{20}$ . Other asteroid parameters  $K_{3m}$  will likely be affected by oblateness to similar degrees, however fitting these extra parameters is computationally more expensive.

The second figure of figure 14 shows divergence of best-fit parameters from the true values when an incorrect oblateness is used. Specifically, the figure shows the difference between parameter values extracted via a model with  $\epsilon = 0$  and the true parameter values used to generate data with oblateness  $\epsilon \neq 0$ . The uncertainty on fit parameters is roughly constant over variations in  $\epsilon_{\text{true}}$ , but very quickly (even for  $\epsilon_{\text{true}} = 10^{-4}$ ) the fit values become inconsistent with the true values. The observational uncertainty used here was the benchmark value, but even if observational uncertainty were increased by a factor of 10 (resulting in a factor of 10 increase in best-fit uncertainty — see section



**Figure 14.** (Left) Ratio of 1- and 2- $\sigma$  uncertainties of the marginal PDFs to the observational uncertainty  $\sigma_\theta$  as a function of the oblateness  $\epsilon$  of the central planet. Linear best fit lines (black, dotted) are plotted over the 1- and 2- $\sigma$  uncertainties. (Right) The deviation of fit parameters from the true parameters when a zero oblateness model is fitted to data with oblateness  $\epsilon$ . Also shown in both figures are the oblatenesses of reference Solar System bodies ?.

4.2) the figure shows that the best fit results would still exclude the true results by  $> 2\sigma$  if the oblateness of Earth were approximated as zero. For 100-fold increase in uncertainty, mismodeling Earth's oblateness would not lead to by  $2\sigma$  divergence, but mismodeling Jupiter would.

Figure 14 also includes reference Solar System  $J_2$  values listed in ? for comparison. However truly comparing these reference oblatenesses is complicated by the fact that the  $J_{20}$  term of equation 22 is modified by a factor of  $(a_M/D)^2$ . If the perigee of the orbit is fixed at the benchmark value of 5 planetary radii, we expect this term to remain roughly comparable between the different reference planets. However, if  $a_M/D$  increases for larger planet mass, which seems likely given that a larger planet presents a larger target, we expect the large-mass reference planets to show more dependence on oblateness than figure 14 suggests. In general, larger mass planets are gas giants and therefore have higher oblatenesses than rocky planets. Therefore, we expect more dependence on oblateness for gas giants.

## 5. INTERPRETATION OF THE FIT PARAMETERS

The previous sections have demonstrated that the parameters  $K_{\ell m}$  can be extracted from a fit to spin data taken of an asteroid encounter given  $a_m$ . However, the most interesting physical property of the asteroid is the interior density distribution  $\rho(\mathbf{r})$ . Unfortunately, the density distribution is strongly degenerate given fixed parameters  $K_{\ell m}$  and  $a_m$ . In this section, we outline four methods to extract density distributions under assumptions that reduce the degeneracy.

The first simplifying step we take is to extract  $\eta(\mathbf{r}) = \rho(\mathbf{r})/\mu_m$  instead of  $\rho(\mathbf{r})$  from  $K_{\ell m}$  and  $a_m$ ; we do this because tidal torque data is not affected by the mass of the asteroid and the average density of the asteroid will therefore never be constrained. We further simplify matters by defining  $H_{\ell m} = a_m^\ell K_{\ell m}$ , so that the task is to find  $\eta(\mathbf{r})$  such that

$$a_m^2 = \int_{\mathcal{A}} d^3 r \eta(\mathbf{r}) r^2 \quad H_{\ell m} = \int_{\mathcal{A}} d^3 r \eta(\mathbf{r}) R_{\ell m}(\mathbf{r}) \quad (45)$$

is satisfied with  $a_m^2$  and  $H_{\ell m}$  known. We made the shift from  $K_{\ell m}$  to  $H_{\ell m}$  because the constraints are now linear in  $\eta(\mathbf{r})$ ; thus, if  $\eta_1(\mathbf{r})$  and  $\eta_2(\mathbf{r})$  both satisfy equation 45, then  $(\eta_1 + \eta_2)/2$  does as well.

Equation 45 amounts to  $N$  constraints, when  $N$  values of  $H_{\ell m}$  are known. This is because  $H_{00} = 1$  by definition.

To resolve further degeneracy, we either assume (1) that the density  $\eta$  is constant and solve for the shape (section 5.1), or (2) that the shape is fixed and solve for  $\eta(\mathbf{r})$ . In the latter case, a further simplification is needed. We propose three models: one in which  $\nabla^2 \eta = 0$  (section 5.2), one which averages over an ensemble of extracted distributions (section 5.3), and one which defines a likelihood on density distributions and find that which maximizes the likelihood (section 5.4).

### 5.1. Constant Density Model

Suppose our asteroid has constant density  $\eta_0$  and known parameters  $H_{\ell m}$  and  $a_m^2$ , and that the asteroid is “star shaped” in that every point on its surface maps to a unique set of spherical coordinates  $\theta, \phi$ , and  $r(\theta, \phi)$ . We use the helpful identity

$$H_{\ell m} = \eta_0 \int_{\mathcal{A}} d^3 r R_{\ell m}(\mathbf{r}) = \eta_0 \int_{\mathcal{A}} d^3 r \nabla \cdot \mathbf{v}(\mathbf{r}) = \eta_0 \oint_{\partial \mathcal{A}} d^2 \mathbf{r} \cdot \mathbf{v}(\mathbf{r}) \quad (46)$$

where  $\mathbf{v}(\mathbf{r}) = \hat{\mathbf{r}} R_{\ell m}(\mathbf{r}) r / (3 + \ell)$ . We know the area element satisfies  $d^2 \mathbf{r} = \partial \mathbf{r} / \partial \theta \times \partial \mathbf{r} / \partial \phi$  in our coordinates, and when dotted with  $\mathbf{v} \propto \hat{\mathbf{r}}$ , this gives

$$H_{\ell m} = \frac{\eta_0}{3 + \ell} \int_0^\pi d\theta \sin \theta \int_0^{2\pi} d\phi r(\theta, \phi)^3 R_{\ell m}(r(\theta, \phi), \theta, \phi). \quad (47)$$

The integrand of each  $H_{\ell m}$  therefore is proportional to  $r(\theta, \phi)^{(3+\ell)}$  times constants and a spherical harmonic. We can write  $r(\theta, \phi) = \sum_{\ell m} Y_{\ell m}^* C_{\ell m}$  without loss of generality. To keep  $r(\theta, \phi) \in \mathbb{R}$ , we require  $C_{\ell m}^* = (-1)^m (\ell - m)! / (\ell + m) C_{l,-m}$  so that the  $\ell, m$  term of the sum for  $r$  is the conjugate of the  $\ell, -m$  term.

With this definition of  $r(\theta, \phi)$ , equation 47 becomes a polynomial of degree  $3 + \ell$  in terms of  $C_{\ell' m'}$  and integrals of products of spherical harmonics. These integrals can be precomputed, since they are taken over the whole sphere  $\theta \in (0, \pi)$  and  $\phi \in (0, 2\pi)$ . Equation 47 can then be numerically solved to get  $C_{\ell m}$ . To remove the degeneracy caused by the factor of  $\eta_0$ , we evaluate  $H_{\ell m} / H_{00}$  instead of  $H_{\ell m}$ . We also force the solution to reproduce the correct value of  $a_m^2 / H_{00}$ , calculated as

$$a_m^2 = \eta_0 \int_{\mathcal{A}} d^3 r r^2 = \frac{\eta_0}{5} \oint_{\mathcal{A}} d^2 \mathbf{r} \cdot \mathbf{r} (r^2) = \frac{\eta_0}{5} \int_0^\pi d\theta \sin \theta \int_0^{2\pi} d\phi r(\theta, \phi)^5. \quad (48)$$

#### 5.1.1. Special Case: Constant Density Ellipsoid

Suppose that only  $K_{\ell=2}$  are known. Then the above method is not necessary to compute the uniform density shape; we can assume the asteroid is an ellipsoid and solve for the axis lengths, which are

$$a = \sqrt{\frac{5}{3}} a_m \sqrt{1 - 2K_{20} + 12K_{22}} \quad b = \sqrt{\frac{5}{3}} a_m \sqrt{1 - 2K_{20} - 12K_{22}} \quad c = \sqrt{\frac{5}{3}} a_m \sqrt{1 + 4K_{20}}. \quad (49)$$

These arise out of computing the moments of inertia for a constant density ellipsoid and using equation 24 to relate the result to  $K_{\ell m}$  and  $a_m$ . The values  $a$ ,  $b$ , and  $c$ , represent semi-axes aligned with  $\hat{\mathbf{x}}$ ,  $\hat{\mathbf{y}}$ , and  $\hat{\mathbf{z}}$  respectively.

Again, the density of this asteroid cannot be extracted because, as is a general principle of gravitational interactions, the mass of the asteroid does not affect its dynamics. If  $a_m$  is unknown, the ratios  $a/c$  and  $b/c$  can be computed instead. Since  $I_{zz}$  is the greatest of the moments of inertia, we expect  $a/c > 1$  and  $b/c > 1$ .

### 5.2. Fixed Surface $\nabla^2 \rho = 0$ model

Now suppose the shape of the asteroid is known, perhaps by a simplifying guess or perhaps by direct observation during a flyby. Then a density distribution  $\eta(\mathbf{r})$  can be solved for which is consistent with observed values of  $H_{\ell m}$  and  $a_m^2$ . Supposing  $\nabla^2 \eta = 0$ , we write  $\eta(\mathbf{r}) = \sum_{\ell m} C_{\ell m} R_{\ell m}(\mathbf{r})^*$ . Thus,  $H_{\ell m} = \sum_{\ell' m'} C_{\ell' m'} B_{\ell \ell' m m'}$  and  $a_m^2 = \sum_{\ell' m'} C_{\ell' m'} v_{\ell' m'}$  where

$$B_{\ell \ell' m m'} = \int_{\mathcal{A}} d^3 r R_{\ell m}(\mathbf{r}) R_{\ell' m'}(\mathbf{r})^* \quad v_{\ell' m'} = \int_{\mathcal{A}} d^3 r R_{\ell' m'}(\mathbf{r})^* r^2 \quad (50)$$

can be precomputed due to the shape of  $\mathcal{A}$  being known. If we write  $H_{\ell m}$  as a vector  $\mathbf{H}$  and similarly for  $\mathbf{C}$ , we can write  $B_{\ell \ell' m m'}$  as a matrix  $B$ , obtaining  $\mathbf{H} = \mathbf{BC}$ . Combining this with the equation  $a_m^2 = \mathbf{v} \cdot \mathbf{C}$ , we get an

overdetermined matrix equation  $\mathbf{d} = \mathbf{AC}$  where  $A$  is  $B$  with  $\mathbf{v}$  added as an additional row, and  $\mathbf{d}$  is  $\mathbf{H}$  with  $a_m^2$  added. It follows that the density distribution coefficients that minimize the least squares distance between  $\mathbf{d}$  and  $\mathbf{AC}$  are

$$\mathbf{C} = \mathbf{A}^+ \mathbf{d} \quad (51)$$

where  $\mathbf{A}^+$  is the Moore-Penrose pseudoinverse ?, defined by  $\mathbf{A}^+ = \mathbf{A}^*(\mathbf{AA}^*)^{-1}$  where  $\mathbf{A}^*$  is the adjoint of  $\mathbf{A}$ .

Physically, there is no reason to expect  $\nabla^2 \rho = 0$ . However, the density distribution derived from the  $C_{\ell m}$ s can be viewed as a projection from the true density distribution into the closest density distribution with  $\nabla^2 \rho = 0$  everywhere, and therefore an approximation of the true distribution. This formulation for the density distribution of the asteroid is also conceptually useful because when  $\mathcal{A}$  is spherical,  $B_{\ell\ell'mm'} \propto \delta_{\ell\ell'} \delta_{mm'}$ . Thus, the fit coefficients  $K_{\ell m}$  are proportional to  $C_{\ell m}$ , with proportionality constants governed by the values of  $B_{\ell\ell'mm'}$  and  $a_m^\ell$ . For a nearly spherical asteroid, small off-diagonal terms will be added to  $B$ , but we still expect  $K_{\ell m}$  to roughly correspond to  $C_{\ell m}$ .

### 5.3. Fixed Surface Ensemble Average Model

Now we approach the problem of solving for the density distribution of an asteroid from  $H_{\ell m}$  and  $a_m$  with more general assumptions. Suppose the shape of the asteroid is known, where “known” in this case means that an indicator function  $\mathbf{1}(\mathbf{r})$  exists such that  $\mathbf{1}(\mathbf{r}) = 1$  inside the asteroid and 0 elsewhere. Then we suggest a numerical algorithm to compute a representative distribution of the asteroid density distribution as follows.

1. Fill the asteroids with  $n$  uniformly random “seed points” by distributing them inside a bounding box containing the asteroid and rejecting those with positions  $\mathbf{r}_i$  such that  $\mathbf{1}(\mathbf{r}_i) = 0$ .
2. Construct  $n$  regions  $\mathcal{V}_i$  around each point such that the  $i$ th region is the set of all points inside the asteroid closest to the  $i$ th point. Suppose  $\mathcal{V}_i$  has uniform density  $\rho_i$ .
3. Equation 45 then becomes two linear equations  $a_m^2 = \mathbf{v} \cdot \boldsymbol{\rho}$  and  $\mathbf{H} = \mathbf{A}\boldsymbol{\rho}$  where  $\rho_i$  are the components of  $\boldsymbol{\rho}$  and  $\mathbf{H}$  are  $H_{\ell m}$  arranged as a vector. The components of  $\mathbf{v}$  and the matrix  $A$  are given by

$$v_i = \int_{\mathcal{V}_i} d^3 r r^2 \quad A_{(\ell m),j} = \int_{\mathcal{V}_i} d^3 r R_{\ell m}. \quad (52)$$

Precompute  $\mathbf{v}$  and  $A$ .

4. We can write equation 45 now as a matrix equation  $\mathbf{d} = M\boldsymbol{\rho}$ , where  $M = A$  with  $\mathbf{v}$  added as an additional row, and  $\mathbf{d} = \mathbf{K}$  with  $a_m^2$  added in the corresponding entry. Thus,  $\boldsymbol{\rho} = M^{-1}\mathbf{d}$  can be directly computed. Note that  $M$  is  $N + 1 \times N + 1$ , with the extra constraint enforcing that  $\int_{\mathcal{A}} d^3 r \eta(\mathbf{r}) = 1$ .
5. Go back to step 1 and compute new density maps with new draws of seed points. After sufficiently many computations, return the average of all the density distributions.

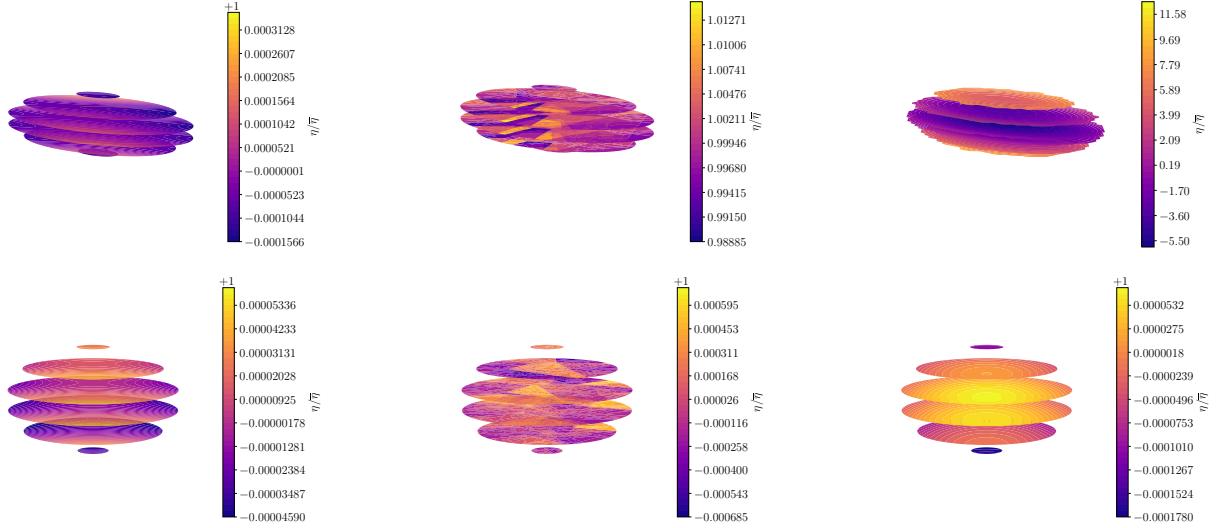
### 5.4. Fixed Surface Maximum Likelihood Finite Element Model

We present a final method to extract the density distribution of an asteroid with fixed surface and parameters  $H_{\ell m}$  and  $a_m^2$ . Suppose the asteroid is divided into  $N$  finite, constant density units with volume  $\Delta V$  (a square grid, for instance). We further write the density of the  $i$ th unit as  $\eta_i = C_i / \Delta V$ , where  $C_i$  is a free real parameter. Approximating each unit as small, we may write

$$a_m^2 = \sum_i^N C_i r_i^2 \quad H_{\ell m} = \sum_i^N C_i R_{\ell m}(\mathbf{r}_i) \quad (53)$$

where  $\mathbf{r}_i$  are the centers of the units. Equation 53 is a matrix equation, which can be written in the form  $\mathbf{d} = \mathbf{AC}$  where  $a_m$  and  $H_{\ell m}$  have been arranged as a vector  $\mathbf{d}$  and  $\mathbf{C}$  contains the  $N$  values of  $C_i$ . The matrix  $A$  is non-square and cannot be inverted to find  $\mathbf{C}$  because equation 53 has many solutions. However, by imposing a probability distribution on  $\mathbf{C}$ , we find one configuration of  $\mathbf{C}$  that maximizes likelihood.

Suppose that the likelihood for  $\mathbf{C}$  is multivariate independent Gaussian distributed with each  $C_i$  having the same standard deviation. The log-likelihood is therefore  $\ln \mathcal{L} \propto -(x - \mu)^2$ . We expect that  $C_i$  sum to one by the definition of  $C_i$ , so we set  $\mu = 1/N$ . Thus, the value of  $\mathbf{C}$  we choose is the value which minimizes  $|\mathbf{C} - \mathbf{1}/N|^2$ , where  $\mathbf{1}$  is an  $N$ -dimensional vector of only ones.



**Figure 15.** Density distributions extracted for the harmonic (*left*), ensemble average (*center*), and likelihood (*right*) models, with  $K_{\ell m}$  values equal to the asymmetric (*top*) and symmetric (*bottom*) benchmark values, and ellipsoidal shapes equal to the shape of the associated constant-density ellipsoid. The same length scale is used in all six images.

This problem has already been solved in the context of the Moore-Penrose pseudoinverse. First, we define  $\mathbf{k} = \mathbf{K} - \mathbf{A}\mathbf{1}/N$ , so that if  $\mathbf{k} = \mathbf{A}\mathbf{c}$ , then  $\mathbf{C} = \mathbf{c} + \mathbf{1}/N$  solves equation 53. We therefore attempt to minimize  $|\mathbf{c}|^2$  to maximize the likelihood of density distributions.

The Moore-Penrose pseudoinverse of  $A$  is defined as  $A^+ = A^*(AA^*)^{-1}$ , where  $A^*$  is the adjoint of  $A$ ? The solution that minimizes  $|\mathbf{c}|^2$  is  $\mathbf{c} = A^+\mathbf{k}$ .

### 5.5. Comparison of density distribution models

In this section, we compare the above methods of extracting asteroid density distributions given sample values for  $K_{\ell m}$ , to demonstrate their differences in behavior. To display the density distributions (and to derive them for the likelihood and ensemble average models), we subdivide the asteroid into a lattice of uniform-density cubes, each with a side length of ten meters. In the case of the ensemble average model, we average over 48 ensembles.

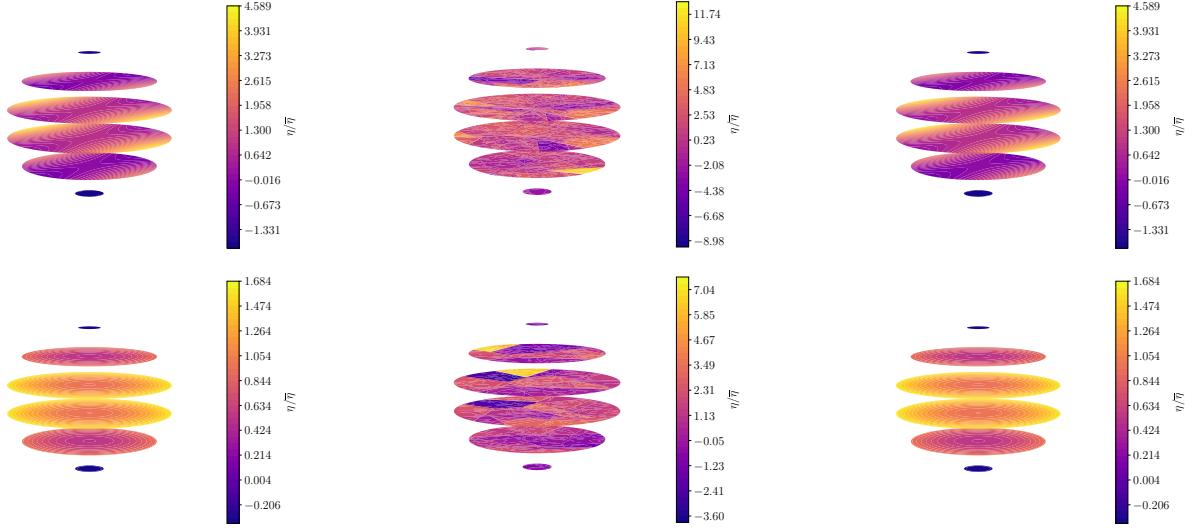
The sample  $K_{\ell m}$  values we investigate are the symmetric and asymmetric benchmark values listed in appendix A. We execute the models for both benchmarks, and for the three models which require a fixed surface, we use both a spherical surface (radius equal to  $\sqrt{5/3}a_m$ ) and an elliptical (axes set by section 5.1.1) model.

Figure 15 shows the density distribution produced by the three methods that fix the surface of the asteroid (the harmonic, ensemble average, and likelihood methods) for the symmetric and asymmetric benchmarks. Since the fixed shape is that of the constant-density ellipsoid associated with the same  $K_{\ell m}$ s, we expect nearly constant density distributions. The harmonic and likelihood models produce the most uniform density distributions, with the likelihood model producing slightly less uniform distributions for the asymmetric case. The non-uniformity of the harmonic model appears to occur by extreme edges of the asteroid possessing different densities than the majority of the asteroid, whereas the likelihood model appears to establish a more radial gradient. The ensemble average shows a stochastic randomness of the density distribution throughout the asteroid.

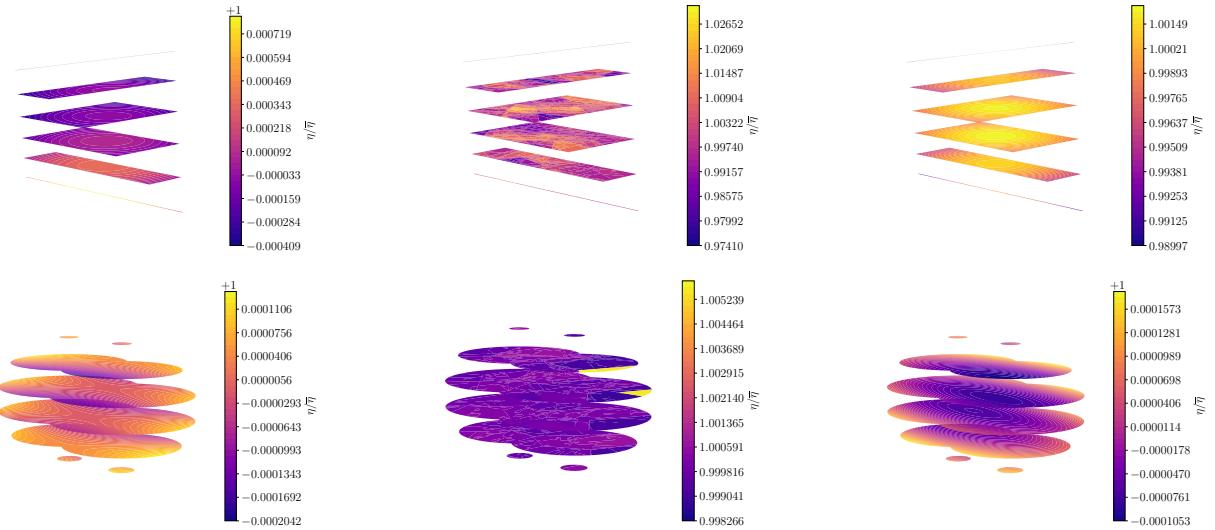
In contrast, figure 16 shows density distributions extracted for spherical shapes with the asymmetric and symmetric benchmark  $K_{\ell m}$  parameters. As expected, distinctly non-uniform densities are produced; indeed, negative densities are produced. For the ensemble average model, solutions with negative density regions could be discarded. But negative density values for the other two models are unavoidable without changes to the model, such as a different likelihood distribution for the density of each chunk.

Figure 16 shows that the harmonic and likelihood models produce very similar results *Is this because of non-uniformity in density or because of the spherical surface?*. The ensemble average value again produces a more noisy distribution.

In addition to these two shape constraints placed on benchmark asteroid parameters, we consider two asteroid shapes of note: a tetrahedron and a “dumb-bell.” These shapes are interesting because tetrahedral shape models for



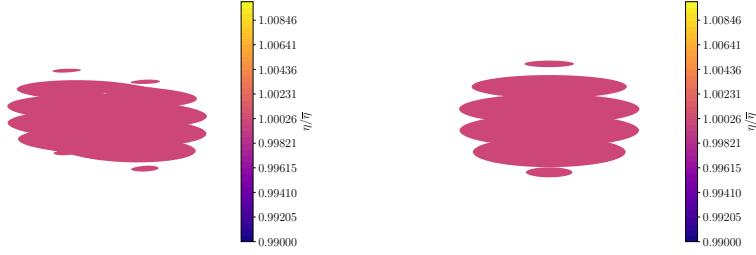
**Figure 16.** Density distributions extracted for the harmonic (*left*), ensemble average (*center*), and likelihood (*right*) models, with  $K_{\ell m}$  values equal to the asymmetric (*top*) and symmetric (*bottom*) benchmark values, and spherical shapes with radius  $r = \sqrt{5/3}a_m$ . The same length scale is used in all six images.



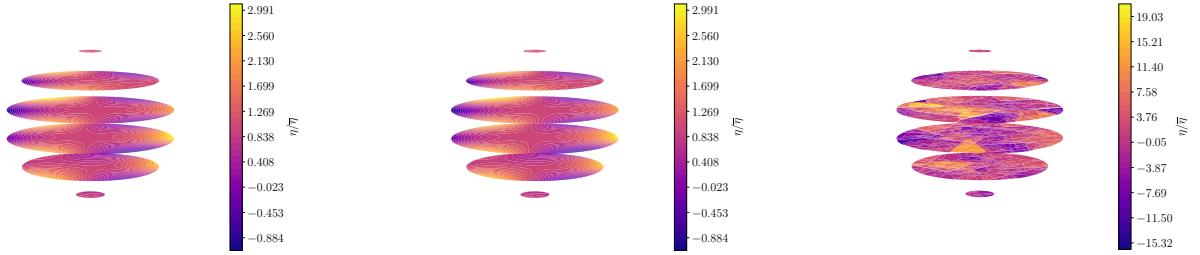
**Figure 17.** Density distributions extracted for the harmonic (*left*), ensemble average (*center*), and likelihood (*right*) models, with a tetrahedral asteroid shape (*top*) and a “dumb-bell” asteroid shape (*bottom*).  $K_{\ell m}$  values were extracted assuming uniform density for both asteroids.

asteroids have hitherto been explored *Citation needed* and a dumb-bell shape can be taken as a proxy for a contact binary asteroid. The tetrahedron is composed out of four equilateral triangular sides and the dumb-bell consists of two spheres with radius  $r$ , separated by distance  $r/2$ . These shapes are fixed and values of  $K_{\ell m}$  are calculated assuming constant density. Then the three shape models are run on the resulting  $K_{\ell m}$  to reproduce the density distribution of the asteroid (figure 17). Once again, the harmonic and the likelihood models produce nearly uniform density distributions, with the ensemble method also producing a nearly uniform distribution. Similar to before, the likelihood model tends to show radial density distribution and the harmonic model tends to produce axial differences in density. All three models seem well suited for the both shapes.

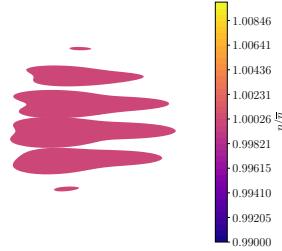
Finally, we discuss the model which fixes constant density and solves for the asteroid surface. This model was run for the two benchmark  $K_{\ell m}$  cases considered. A cutout diagram of the resulting density distribution is shown in figure



**Figure 18.** Density distributions extracted for the constant-density model, from the asymmetric (*left*) and symmetric (*right*) benchmark  $K_{\ell m}$  parameters.



**Figure 19.** Density distributions extracted for the three fixed-surface models, for a spherical asteroid with  $I_x = I_y = I_z$  but nonzero  $K_{3m}$ .



**Figure 20.** Asteroid shape extracted for an asteroid with  $I_x = I_y = I_z$  but nonzero  $K_{3m}$  as a demonstration of the effects of the higher order  $K_{3m}$  moments.

18. For the symmetric case, we see an ellipsoid of revolution closely resembling the true asteroid ellipsoid. For the asymmetric case, we see a two-lobed asymmetric formation with equal  $K_{\ell m}$  to the asymmetric ellipsoid because a triaxial ellipsoid cannot be represented with spherical harmonics of degree  $\ell \leq 2$ . The model therefore takes the closest approximating route.

The figure demonstrates the limitations on the uniform density model: only a small number of shapes are accessible given fixed parameters  $K_{\ell m}$  which may in fact be drawn from a diverse set of shapes from ellipsoids to tetrahedra to dumb-bells. However, a smooth shape distribution approximating the true distribution can nevertheless be fitted.

### 5.6. High order parameters

In the above demonstrations of the four density distribution models, we test examples where only the  $K_{2m}$  parameters are known. In this section, we show density distributions extracted for example values of known  $K_{3m}$ .

Suppose  $K_{2m} = 0$ . For the models that fix the surface of the asteroid, we assume a spherical shape, which is the uniform-density shape associated with  $K_{2m} = 0$ . We introduce  $K_{3m}$  parameters uniformly randomly chosen in  $(-0.01, 0.01)$ ; larger values of  $K_{3m}$  tended to lead to highly negative density distributions. The three fixed-shape density distributions are shown in figure 19 and the constant-density distribution is shown in figure 20.

Beyond demonstrating the effectiveness of the density distribution algorithms when  $K_{3m}$  parameters are introduced, figure 19 demonstrates again the similarity between the likelihood and the harmonic models, and shows that both

$\ell$	$\Re K_{\ell 3}$	$\Im K_{\ell 3}$	$\Re K_{\ell 2}$	$\Im K_{\ell 2}$	$\Re K_{\ell 1}$	$\Im K_{\ell 1}$	$K_{\ell 0}$
0							—
1					$x$	$y$	$z$
2			—	$x, y$	$x, z$	$y, z$	—
3	$x$	$y$	$z$	$x, y, z$	$x$	$y$	$z$

**Table 2.** For each  $K_{\ell m}$  value from  $\ell = 0$  to 3, the density distribution mirror symmetries which set  $K_{\ell m} = 0$  are listed. Mirror symmetries around the axis  $x$  are marked with  $x$  and so on. A dash indicates that no mirror symmetries set  $K_{\ell m} = 0$ .

still prefer smoothly varying distributions. The ensemble distribution is much less smoothly varying. More drawings of Voronoi cells would likely smooth the distribution further. All three density distributions produce negative (and therefore unphysical) density distributions.

Figure 20 shows that the constant density model is able to find a smooth surface to match an asteroid with nonzero  $K_{3m}$  despite the negative densities in the fixed-surface model. Specifically, the surface model appears to have removed parts of the sphere that corresponded to low density in the fixed-surface model and extruded the asteroid shape in regions of high density. This fact demonstrates the sensitivity of the fixed-surface density distributions of figure 19 to the surface model; a relatively small change to the shape of the distribution can change the density distribution from unphysical (negative) to physical (positive — indeed constant). It also suggests that a reasonable way to adjust a surface model to avoid unphysical density distributions is to remove regions with low density and add to regions with high density.

*Should I explore the location / size of a rock? How should I talk about the scale size of the high order moments? By my random draw method? By calculating them for an ellipsoid, tetrahedron, and dumbbell?*

### 5.7. Structure of density moments

In this section, we qualitatively describe the properties of the density moments for the sake of interpreting them.

First, we mention the symmetry property  $K_{\ell m} = (-)^m K_{\ell, -m}^*$  which causes each degree  $\ell$  to contain  $2\ell + 1$  degrees of freedom (Further degrees of freedom are removed by fixing the origin of the coordinate system, its orientation, and the asteroid's mass). Also,  $\Im K_{\ell 0} = 0$ . Each  $K_{\ell m}$  contains a factor of  $r^\ell/a_m^\ell$  in its integrand, and since we generally expect  $r \sim a_m$ , naively we expect all  $K_{\ell m}$  to be of the same order of magnitude. However, the fact that  $\rho > 0$  everywhere induces other complications. Specifically, *The part listed in the paper for  $K_{20}$* . For the allowed values of  $K_{3m}$  are more difficult to constrain, but we note that  $|K_{\ell m}| \gtrsim 0.01$  often lead to negative density distributions. Furthermore, of the four shapes studied in section 5.5 (a sphere, ellipsoid, tetrahedron, and dumbbell), only the tetrahedron has nonzero  $K_{3m}$  and even for the tetrahedron,  $K_{3m} = 0$  except  $\Re K_{32} = -0.036$ .

Symmetries of the asteroid density model may set some  $K_{\ell m} = 0$ . Specifically, the parity properties of  $R_{\ell m}$  under axis flips along  $x$ ,  $y$ , and  $z$  guarantee that asteroid density distributions that are flip-symmetric along  $z$  have  $K_{\ell m} = 0$  when  $\ell + m$  is odd. Flip-symmetry along  $x$  guarantees  $\Re K_{\ell m} = 0$  for  $m$  odd and  $\Im K_{\ell m} = 0$  for  $m$  even. Flip-symmetry along  $y$  guarantees  $\Im K_{\ell m} = 0$  always. These results are summarized in table 2.

The first three rows of table 2 are as intuitively expected.  $K_{00}$  (mass) is not zeroed by any mirror symmetries. This fact is even more clear from the definition of  $K_{\ell m}$ , which normalizes by mass such that  $K_{00} = 1$ . But even without this normalization,  $K_{00}$  would not be zeroed by symmetries. Also,  $K_{1m}$  each correspond to one symmetry axis because they are the coordinates of the center of mass. Finally,  $K_{21}$  and  $\Im K_{22}$  are each zeroed by two symmetry axes because they express coordinate system orientation. The other two degree-two moments  $K_{20}$  and  $\Re K_{22}$  are not zeroed by mirror symmetries because they are moment of inertia ratios and must be positive.

The last row of table 2 shows that all  $K_{3m}$  moments are zeroed by at least one symmetry. Therefore, a highly non-mirror symmetric distribution is necessary to produce large  $K_{3m}$  values. This is why the constant density sphere, ellipsoid, and dumbbell shapes all have  $K_{3m} = 0$ .

## 6. DENSITY DISTRIBUTIONS, BUT REDONE

In this section, we describe a pipeline to convert the density moments  $K_{\ell m}$  to a density distribution  $\rho(\mathbf{r})$  which correctly reproduces the density moments via equation 10. We will also produce a function  $\sigma_\rho(\mathbf{r})$  which represents the uncertainty in density propagated from uncertainty in the  $K_{\ell m}$  moments. Unfortunately,  $\rho(\mathbf{r})$  represents infinitely many degrees of freedom, whereas generally only a few  $K_{\ell m}$  are known. Additional assumptions must therefore

be made to produce one  $\rho(\mathbf{r})$  solution. One assumption we will make is that the surface of the asteroid has been accurately determined, from light curve data for instance. A method of computing a density distribution when no asteroid surface is known is presented in appendix [Appendix](#). A few example sets of solutions are described below, but first we investigate the problem in a general formulation, agnostic of what assumptions beyond a known surface are chosen to eliminate the degeneracy.

### 6.1. General formulation

We assume that an indicator function  $\mathbb{1}(\mathbf{r})$  has been determined such that  $\mathbb{1}(\mathbf{r}) = 1$  inside the asteroid and 0 outside, where  $\mathbb{1}(\mathbf{r})$  is defined in some frame whose orientation with respect to Earth is known. The initial orientation of the asteroid's spin pole with respect to Earth is also known, which fixes the initial orientation of the local frame (the asteroid's frame) with respect to Earth up to one parameter  $\gamma_0$ , which is attained by a fit as seen in previous sections.

We define a new coordinate system, the “hybrid frame,” which coincides exactly with the initial orientation of the asteroid assuming that the fit result for  $\gamma_0$  is perfectly accurate. The indicator function  $\mathbb{1}$  is then assumed to be known exactly in this frame, and the  $K_{\ell m}$  components are known in the local frame which aligns with the hybrid frame up to uncertainty in  $\gamma_0$ . We will solve for  $\rho(\mathbf{r})$  in the hybrid frame given fit results  $K_{\ell m}$  and the known  $a_m$ .

The density moments  $K_{\ell m}$  defined in equation 10 are not linear in  $\rho(\mathbf{r})$ , but we may fix  $a_m$  at the value assumed during the fit, removing the non-linearity induced by division by  $a_m$ . Furthermore, since the system is independent of the total asteroid mass, we set  $\mu_m$  equal to an arbitrary constant which makes equation 10 linear in  $\rho(\mathbf{r})$ . Similarly, the form of equation 25 guarantees that  $a_m^2$  is linear in  $\rho(\mathbf{r})$ . Equations 25 and the  $\ell, m = 0, 0$  component of equation 10 can be used to enforce the choices of  $\mu_m$  and  $a_m^2$ .

Suppose that we restrict the number of degrees of freedom of  $\rho(\mathbf{r})$  from infinity to  $m$  by explicitly defining some function  $\rho(\mathbf{r}, \mathbf{D})$  for an  $m$ -dimensional vector  $\mathbf{D}$  which controls the free parameters of  $\rho$ . We will leave the explicit definition of  $\rho$  to the model descriptions below, but for now we assume that  $\rho$  is linear in  $\mathbf{D}$ ; i.e.,

$$\rho(\mathbf{r}, \mathbf{D}) = \mathbf{B}(\mathbf{r}) \cdot \mathbf{D} \quad (54)$$

for a  $m$ -dimensional vector  $\mathbf{B}(\mathbf{r})$ . Thus, equations ?? and ?? represent a linear function that gives  $K_{\ell m}$  in terms of  $\mathbf{D}$ . We further assume that the model describes a way to reverse this equation, to write

$$\mathbf{D} = A\mathbf{K}. \quad (55)$$

Here, we have arranged  $a_m^2$  and the components of  $K_{\ell m}$  into a vector  $\mathbf{K}$ , which we say has  $n$  dimensions. The order of this arrangement is irrelevant, as long as it is kept consistent.

To calculate the uncertainty on  $\mathbf{D}$  and therefore  $\rho(\mathbf{r})$ , we first need the covariance matrix  $\Sigma_K$  for  $\mathbf{K}$ . First, assume that the hybrid frame is offset from the local frame by some small angle  $\Delta\gamma$ , which results from uncertainty in  $\gamma_0$ . Then

$$K_{\ell m}^{\text{hybrid}} = e^{-im\Delta\gamma} K_{\ell m}^{\text{local}}. \quad (56)$$

Since  $K_{\ell m}^{\text{local}}$  was obtained by an MCMC fit, a large set of posterior-distributed samples are available for  $\Delta\gamma$  and  $K_{\ell m}^{\text{local}}$ , and the covariance matrix  $\Sigma_K$  can be computed statistically by applying equation 56 to the samples. Then, propagation of uncertainty guarantees that the covariance matrix of  $\mathbf{D}$  is  $\Sigma_D = A\Sigma_K A^T$  and the density distribution and uncertainty on density distribution are equal to

$$\rho(\mathbf{r}) = \rho(\mathbf{r}, A\mathbf{K}) \quad \sigma_\rho^2(\mathbf{r}) = \mathbf{B}(\mathbf{r})^T A \Sigma_K A^T \mathbf{B}(\mathbf{r}). \quad (57)$$

The job of a model is therefore to restrict the space of valid density distributions by defining the  $m \times n$ -dimensional constant matrix  $A$  and the  $m$ -dimensional vector  $\mathbf{B}(\mathbf{r})$  such that equations 55 and 54 are true. Then the density distribution and its uncertainty are given by equation 57.

### 6.2. Likelihood model

In order to restrict the space of density distributions that correspond to the known  $H_{\ell m}$  so that only one distribution remains, we define a function  $\mathcal{L}$  which takes large values for “nice” density distributions according to some metric of niceness. We then take the density distribution which maximizes  $\mathcal{L}$ . In this way,  $\mathcal{L}$  can be thought of a likelihood for the density distribution, but it should not be confused with the likelihood used to extract the fit parameters  $K_{\ell m}$ .

To employ this method, we divide the asteroid into a square grid of  $m \gg n$  elements, each of which is assumed to have constant density  $\rho_0$  plus some deviation  $D_i$ . This defines the model function  $\mathbf{B}(\mathbf{r})$ , which is zeroed in all components except for the  $i$ th, where  $i$  is the index of the grid element that contains  $\mathbf{r}$ .

The task is to define a likelihood  $\mathcal{L}(\mathbf{D})$  such that the minimum satisfies equation 55. This happens to be an uncorrelated Gaussian likelihood

$$\mathcal{L}(\mathbf{D}) = \prod_{i=1}^m \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{D_i^2}{2\sigma^2}\right) \quad (58)$$

with free parameters  $\mu$  and  $\sigma$ . These parameters do not affect the location of the maximum, so we do not define them. Given this likelihood, the log likelihood is proportional to  $-\|\mathbf{D}\|^2$ . Minimizing the norm of  $\mathbf{D}$  is therefore equivalent to finding the maximum likelihood.

To minimize this norm, we use the fact that equations 10 and 25 correspond exactly to a matrix equation  $\mathbf{K} = M\mathbf{D}$ , where the  $i$ th entry of every row of the  $n \times m$  matrix  $M$  is the integral presented in 10 or 25 evaluated over the  $i$ th finite element. Since  $m > n$ , we define the Moore-Penrose inverse of  $M$  as

$$M^+ = M^*(MM^*)^{-1} \quad (59)$$

where  $M^*$  is the adjoint of  $M$ . The value of  $\mathbf{D}$  that minimizes  $\|\mathbf{D}\|^2$  and also satisfies equation 55 is then  $A = M^+$ , which fully defines the model.

Note that this model is fast to compute; assuming fast matrix multiplication, the most time-consuming step is the matrix inversion of equation 59. However,  $MM^*$  is an  $n$ -dimensional square matrix, which is very small compared to the number of finite elements  $m$ .

### 6.3. Harmonic model

We now explore a model that seeks to narrow down the space of allowed density distributions a different way than the likelihood model; we define the “nice” density distributions directly. They are the density distributions with zero Laplacian:  $\nabla^2\rho = 0$  inside the asteroid. The expansion of such a function in spherical coordinates is given by

$$\rho(\mathbf{r}) = \sum_{\ell,m} \frac{D_{\ell m}}{a_m^\ell} R_{\ell m}^*(\mathbf{r}) \quad (60)$$

where the terms which lead to  $\rho \rightarrow \infty$  at the origin have been removed and  $D_{\ell m}$  are free parameters. The complex conjugate was added for convenience. By setting a maximum on  $\ell$ , we restrict the number of degrees of freedom to  $(\ell_{\max} + 1)^2$ . Choosing the same maximum  $\ell$  as the maximum  $\ell$  for the  $H_{\ell m}$  moments, we have  $m = n - 1$  coefficients  $D_{\ell m}$  which can be stacked into an  $m$ -dimensional vector  $\mathbf{D}$ .

Inserting equation 60 into equations ?? and ??, we get

$$K_{\ell',m'} = \sum_{\ell m} \frac{1}{\mu_m a_m^{\ell'} a_m^\ell} D_{\ell m} \int_A d^3 r R_{\ell m}^*(\mathbf{r}) R_{\ell' m'}(\mathbf{r}) \quad (61)$$

This is an over-determined matrix equation  $\mathbf{K} = M\mathbf{D}$ , where  $M$  is an  $n \times m$  matrix. The Moore-Penrose inverse can therefore be used again to find an inverse  $A = M^+$  which yields approximately correct  $\mathbf{D}$  (approximate in that the norm of the error vector between  $\mathbf{K}$  and  $M\mathbf{D}$  is minimized). However, the form of the inverse changes due to the matrix now being taller than it is wide. The Moore-Penrose is now

$$M^+ = (M^* M)^{-1} M^* \quad (62)$$

and the model concludes with  $A = M^+$ .

Note that, in the special case where the asteroid is a sphere of radius  $R$ , the matrix defined by equation 61 is diagonal, with entries

$$M_{\ell m; \ell' m'} = \frac{4\pi R^3}{\mu_m} \frac{R^{2\ell}}{a_m^{2\ell}} \frac{\delta_{\ell\ell'} \delta_{mm'}}{(4\ell^2 + 8\ell + 3)(\ell - m)!(\ell + m)!}. \quad (63)$$

A nonspherical perturbation will introduce small off-diagonal entries to  $M$ . This diagonal represents an interpretation of the physical meaning of  $K_{\ell m}$ ; they are directly related through this form of  $M$  and the matrix equation  $\mathbf{K} = M\mathbf{D}$  to the coefficients of the spherical harmonic expansion of the asteroid in the spherical case.

	N			
	1	2	3	4
1	2	5	8	11
d	5	8	11	14
3	7	10	13	16

**Table 3.** Total degrees of freedom  $D$  as a function of  $N$ , the number of lumps modeled, and  $d$ , the number of independent moment of inertia axes considered for each lump. The configurations with  $D \leq$  the five known parameters not including  $K_{3m}$  are colored black, and those including seven  $K_{3m}$  parameters are colored gray.

#### 6.4. Lump model

The above two simple models produce smooth density distributions that generally extend non-uniformly over large regions. In this section, we define a more complicated model which identifies discrete regions differing from the overall density of the asteroid to best fit the  $K_{\ell m}$  fit results.

Suppose the asteroid contains  $N$  ‘lumps’ of constant mass  $\mu_i$  displaced by distance  $\mathbf{x}_i$  from the asteroid center of mass, and immersed in a constant density overall asteroid shape which is known in the form of  $\mathbf{1}(\mathbf{r})$ . For simplicity, we assume that all  $N$  regions are ellipsoids with  $d$  independent axis lengths. (For example,  $d = 3$  corresponds to an asymmetric ellipsoid,  $d = 2$  corresponds to a symmetric ellipsoid, and  $d = 1$  corresponds to a sphere.) Recall that  $\mathbf{1}(\mathbf{r})$  is known in the hybrid frame where the origin is the center of mass of the asteroid. The displacement of the asteroid shape from the center of mass, which is the opposite of the net displacement of the discrete regions, is therefore observable from light curve analysis. The model therefore has  $4N - 3$  positional degrees of freedom and rotational degrees of freedom laid out in table 3. These degrees of freedom should be compared to the known density moments, of which there is a square number  $(L + 1)^2$ . However, three of these are the center of mass of the asteroid which is guaranteed, and one is the unconstrained total mass. So the total number of known parameters is  $(L + 1)^2 - 4$ .

The net  $K_{\ell m}$  components of such an ensemble obey

$$K_{\ell m} = K_{\ell m}^{\prime 0} + \sum_{i=1}^N K_{\ell m}^{\prime i} \quad a_m^2 = a_0'^2 + \sum_{i=1}^N a_i'^2 \quad (64)$$

where  $K_{\ell m}^i$  and  $a_i^2$  obey the same definition as  $K_{\ell m}$  in equation 10 and  $a_m$  in equation 25 respectively. Crucially, note that the normalizing factors  $1/(\mu_m a_m^\ell)$  and  $1/\mu_m$  in equations 10 and 25 do not change to their counterparts for each lump; they remain the known asteroid-wide values. The zero-indexed parameters indicate the density moments of the asteroid medium surrounding the lumps.

The prime in equation 64 denotes that the moments are calculated in the hybrid frame. We can relate the primed moments to the moments calculated relative to each body’s center of mass via the translation rules for solid spherical harmonics:

$$K_{\ell m}^i = \sum_{\ell' m'} (-1)^{\ell-\ell'} R_{\ell-\ell', m-m'}(\mathbf{x}_i) K_{\ell' m'}^i \quad a_i'^2 = a_i^2 + x_i^2 K_{00} \quad (65)$$

The dummy indices  $\ell', m'$  should only be summed over values in which  $\ell - \ell' \geq 0$  and  $|m - m'| \leq \ell - \ell'$ . Here,  $K_{00}^i$  is the added mass  $\mu_i$  of lump  $i$ , while  $K_{1m} = 0$  and  $K_{2m}$  incorporate its orientation and moment of inertia ratios. The volume is constrained by  $a_i^2$ . These values map directly onto an ellipsoid representation via equations 49, so that if  $K_{\ell m}^i$ ,  $a_i^2$ ,  $\mu_i$ , and  $x_i$  are known, then the density distribution is known. Note that if  $\mathbf{x}_i$  is known, then equation 65 are linear in  $K_{\ell m}^i$ . Therefore, if we define  $\mathbf{D}$  to contain  $K_{2m}^i$  and  $a_i^2$  in some order, then equations 65 and 64 define a matrix equation  $\mathbf{K} = \mathbf{C} + \mathbf{M}\mathbf{D}$  which can be solved by setting  $A$  equal to the Moore-Penrose inverse of  $M$ . Henceforth, we assume that the dimensions of  $\mathbf{D}$  are less than or equal to the number of known parameters (the shaded regions of table 3), so that  $A$  is given by equation 62). Note that the  $\mathbf{C}$  term of this equation came from the expansion of  $K_{\ell m}^{\prime 0}$  and  $a_0'^2$ , which is written in terms of the already-known displacement of the asteroid surface from its center of mass.

This model’s strategy will be to choose  $\mathbf{x}_i$  and  $\mu_i$  via a nonlinear process, then fit for the values  $K_{\ell m}^i$  and  $a_i^2$  using the linear format described in the introduction to this section. The constraints on  $\mathbf{x}_i$  and  $|\mu_i|$  are primarily

$$\sum_{i=1}^N \mathbf{x}_i \mu_i + \mu_0 \mathbf{x}_0 = 0 \quad (66)$$

where  $\mathbf{x}_0$  is the known displacement of the centroid of the asteroid model from the center of mass. There are also additional constraints, such as that  $\mathbf{x}_i$  should lie inside the asteroid which can be enforced manually. The overall mass of the asteroid is  $\mu_m = \mu_0 + \sum_{i=1}^N \mu_i$ , and has been set, so we may eliminate  $\mu_0$  from equation 66. The constraint is now a function of  $\mu_i$  and  $\mathbf{x}_i$  for  $i \geq 1$ . The matrix equation defining  $M$  is overdetermined, but we would like it to have a solution nevertheless. We therefore solve for  $\mathbf{x}_i$  and  $\mu_i$  to yield an  $M$  with solutions. This is done by minimizing

$$|(M(\mathbf{x}_i, \mu_i)M^+(\mathbf{x}_i, \mu_i) - \mathbf{1})(\mathbf{K} - \mathbf{C})|^2. \quad (67)$$

If  $\mathbf{x}_i$  and  $\mu_i$  are chosen to minimize this function, with the constraint of 66 obeyed, then  $A = M^+$  and  $\mathbf{B}(\mathbf{r})$  following from the resulting values of  $K_{\ell m}^i$  and  $a_i^2$  defines the model. If the resulting density distribution is somehow excluded (it predicts negative density distributions, the lumps extend outside the asteroid, etc.), then another minimum of equation 67 can be used, or another combination of  $N$  and  $d$  listed in table 3.

## 7. CONCLUSION

*Facilities:* MIT Supercloud

*Software:* emcee, scipy.optimize.minimize, numdifftools.Hessian

## APPENDIX

### A. REFERENCE FLYBY CONFIGURATIONS

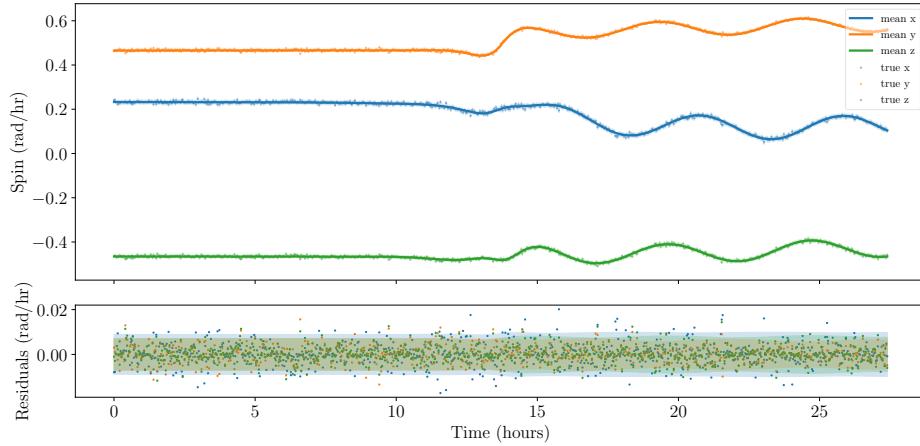
Except when otherwise mentioned, we use the following fit parameters when running the simulation. Many of the parameter choices are made to maximize the quality of observations (a close orbit, large asteroid, etc.) This is so that if it is desired to test the dependence of fit results on perigee distance, for instance, the other parameters will not be fixed at values that are so unfavorable to observations that good fits are already excluded. Otherwise, the fit results for all perigee distances would always be poor and the dependence on perigee would be difficult to measure.

1. An orbit around Earth with  $6 \text{ km s}^{-1}$  excess velocity and perigee at 5 Earth radii. *Cite Apophis close approach*
2. An initial roll of  $\gamma_0 = \pi/8$ .
3. A cadence of 2 minutes and observational uncertainty of  $\sigma_\theta = 0.01$  and  $\sigma_\rho/\sigma_\theta = 10^{-5}$ .
4. A rotational period of 9 hours, with the rotational velocity vector distributed between the  $\hat{\mathbf{X}}$ ,  $\hat{\mathbf{Y}}$ , and  $\hat{\mathbf{Z}}$  axes in a  $1 : 2 : -2$  ratio.
5. An asteroid with radius  $a_m = 1 \text{ km}$  and  $K_{3m} = 0$ . For  $K_{22}$  and  $K_{20}$ , we use two standard values. One with  $(K_{22}, K_{20}) = (0, -0.097)$  and one with  $(0.052, -0.202)$ . Including the third point obtained by reflection  $K_{22} \rightarrow -K_{22}$ , these are the three points that minimize the mean distance between an arbitrary point in the allowed parameter space (equation 33) and these reference values. The first point is called the symmetric case because the corresponding uniform-density-ellipsoid model is rotationally symmetric around  $\hat{\mathbf{z}}$ . The second case and its reflection are called the asymmetric cases. Values of  $(0.052, -0.202)$  have  $a < b$  in the ellipsoid model, and the reflected value has  $a > b$ . If not specified, we use the  $a < b$  case. Specifically, the asymmetric case has  $a = 1839 \text{ m}$ ,  $b = 1140 \text{ m}$ , and  $c = 565 \text{ m}$ , while the symmetric case has  $a = b = 1411 \text{ m}$  and  $c = 1008 \text{ m}$ .

### B. EXAMPLE FIT RESULTS

In the interest of demonstrating the result of a fit to synthetic data, we present the results of a fit to synthetic data generated with the nominal sensitivity model and  $\sigma_\theta = 9.5 \times 10^{-3}$ . The symmetric reference asteroid is used. The reduced  $\chi^2$  value of this fit (defined as  $\chi^2/N$ , where  $N$  is the number of degrees of freedom) was 0.83, and fit results were in good agreement with the true parameters. Nearly 12,000 MCMC iterations were required before convergence.

Figure 21 shows the data with the best-fit overlaid and the residuals. Uncertainties are plotted on the residuals corresponding to the square root of the diagonal entries of the covariance matrix — i.e.,  $1-\sigma$  uncertainties. However,



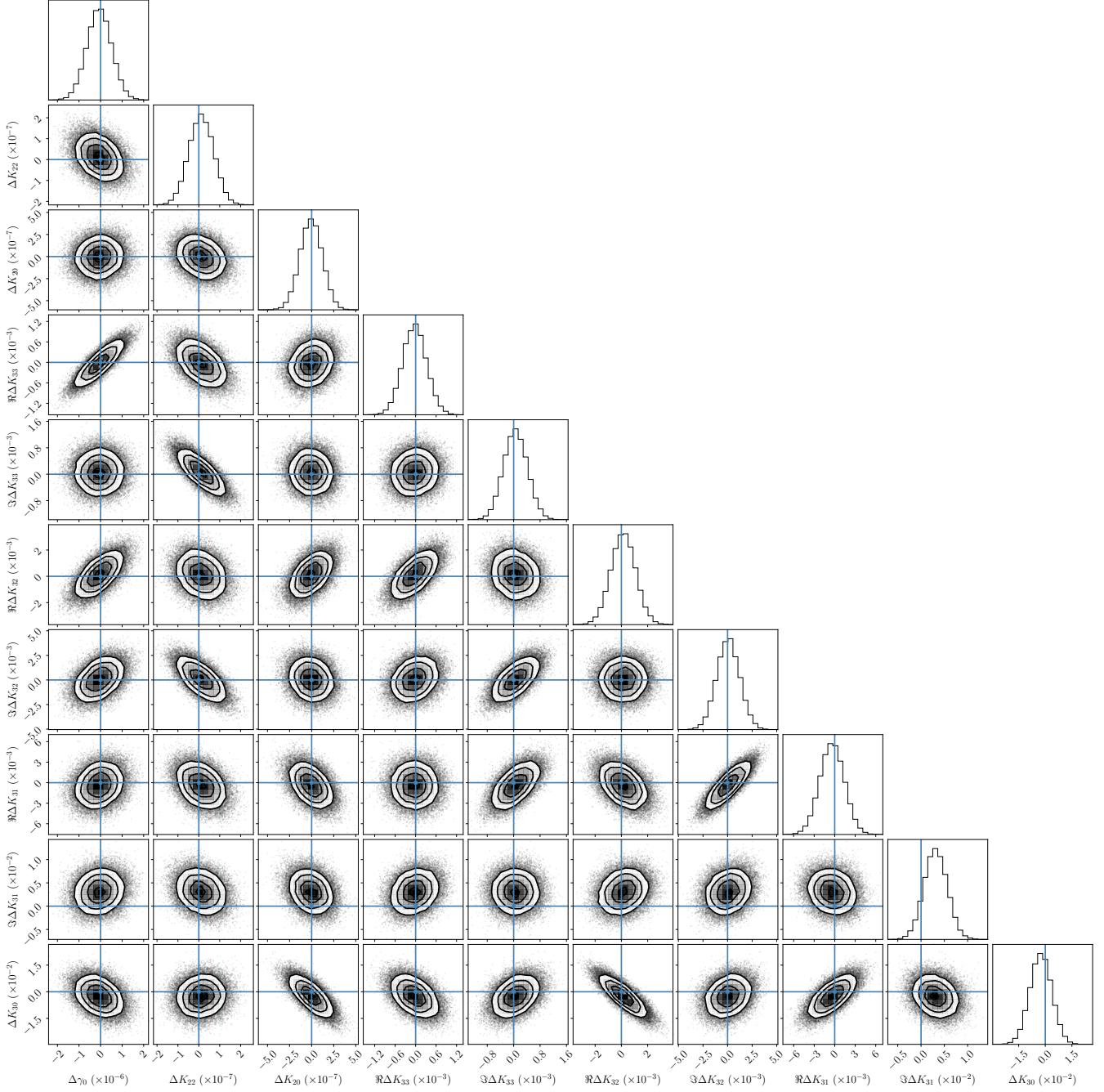
**Figure 21.** Data, best fit results, and residuals for a fit to synthetic data simulated for an asymmetric reference asteroid with  $\sigma_\theta = 9.5 \times 10^{-3}$ . The standard deviation of the data is plotted as an uncertainty band in the residuals plot, but these do not capture covariance between data points, so that the fact that many residuals lie outside the band does not disqualify the fit, which has a reduced  $\chi^2$  value of 0.68.

the data often exceeds these uncertainty bands. This is because uncertainties are correlated in truth, and points that fall outside of the band are not necessary in conflict with the best fit.

Figure 22 shows a corner plot of the ten fitted parameters' posterior distributions, both marginalized to be a function of one variable and of two. The true parameters are shown as lines.

Figure 21 demonstrates that a good fit to data was achieved, despite the small error bars and strong dependence of the model on initial conditions. Figure 22 shows that the posterior distributions are largely Gaussian-distributed, with several of the parameters strongly correlated. The true values lie within  $2\sigma$  contours in most cases, as expected. Parameters are computed to very high accuracy —  $O(10^{-6})$  for  $\gamma_0$  and  $O(10^{-7})$  for  $K_{2m}$  (the first tier of parameters as ordered by the sensitivity of the asteroid system to their values), and  $O(10^{-4}) - O(10^{-2})$  for  $K_{3m}$ . These ranges should be compared to the width of the prior distributions for these parameters, which is  $O(1)$  for all. The success of this fit demonstrates that these higher order parameters can be fitted to data as long as good enough precision on the data is achieved.

## REFERENCES



**Figure 22.** A corner plot for the fit results ten parameters to synthetic data simulated for an asymmetric reference asteroid with  $\sigma_\theta = 9.5 \times 10^{-3}$ . Marginal posterior PDFs for each parameter are shown (histograms) along with two-dimensional PDFs (contours). Individual points in the contours are samples from the MCMC fit, and the contours enclose 1-, 2-, and 3- $\sigma$  of them. True values (which are zero here due to the parameters being displayed relative to the true values) are also shown (blue lines).