SPD Derivation with Black-Scholes Put Option Price

Weongi Woo

Ladislaus von Bortkiewicz Chair of Statistics
Humboldt–Universität zu Berlin
http://lvb.wiwi.hu-berlin.de

State-Price Density

State-Price Density (SPD) is the density function of a Risk Neutral (RN) euqivalent martingale measure for option pricing. It carries important information on the behavior and expectations of the market and it often serves as a base for option pricing and hedging.
Breeden and Litzenberger suggested that one can replicate Arrow-Debreu Price using the Butterfly Spread Strategy on European call options.
The SPD $f(\cdot)$ may be expressed as

$$f(K) = e^{r(T-t)} \frac{\partial^2 C_t(K,T)}{\partial K^2}$$

In the same manner, one can also replicate the Arrow-Debreu Price with a portfolio of European put options. This can be done with Butterfly Spread Streategy with European put options.

This leads the SPD to be expressed as

$$f(K) = e^{r(T-t)} \frac{\partial^2 P_t(K,T)}{\partial K^2}$$

Black-Scholes Put Price Model

$$C_t = S_t e^{-\delta\tau}\Phi(d_1) - K e^{-r\tau}\Phi(d_2)$$

$$d_1 = \frac{\log(\frac{S_t}{K}) + (r - \delta + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$$

$$d_2 = d_1 - \sigma\sqrt{\tau}$$

By the Put-Call Parity : $C_t - P_t = S_t e^{-\delta\tau} - K e^{-r\tau}$

$$\Longrightarrow \; P_t = K e^{-r\tau}\Phi(-d_2) - S_t e^{-\delta\tau}\Phi(-d_1)$$

Note: $\Phi(\cdot)$ and $\varphi(\cdot)$ are standard normal distribution function and density function, respectively.

The second partial derivative of P w.r.t. K

In order to estimate the SPD, we need to calculate the second partial derivative of European Put Option pricing function, i.e.

$$\frac{\partial^2 P_t(K,T)}{\partial K^2} = \frac{\partial^2 Ke^{-r\tau}\Phi(-d_2) - S_t e^{-\delta\tau}\Phi(-d_1)}{\partial K^2}$$

For concise expression, we first introduce a Lemma and calculate the elements of partial derivatives.

Lemma

Lemma : $K e^{-r\tau} \varphi(-d_2) = S_t e^{-\delta\tau} \varphi(-d_1)$

Proof:

$$\Longleftrightarrow \frac{S_t}{K} e^{(r-\delta)\tau} = \frac{\varphi(-d_2)}{\varphi(-d_1)} = \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(-d_2)^2}}{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(-d_1)^2}} = e^{\frac{d_1^2 - d_2^2}{2}}$$

$$\Longleftrightarrow \log\left(\frac{S_t}{K}\right) + (r-\delta)\tau = \frac{(d_1+d_2)(d_1-d_2)}{2}$$

$$= \frac{(2d_1 - \sigma\sqrt{\tau})(\sigma\sqrt{\tau})}{2} = \log\left(\frac{S_t}{K}\right) + (r-\delta)\tau \qquad \square$$

The Elements of Partial Derivative

The first order partial derivative of $-d_1$ and $-d_2$ w.r.t. K.

$$
\begin{aligned}
d_1 &= \frac{\log(\frac{S_t}{K}) + (r - \delta + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \\
d_2 &= d_1 - \sigma\sqrt{\tau}
\end{aligned}
\qquad \Longrightarrow \qquad
\begin{aligned}
\frac{\partial(-d_1)}{\partial K} &= \frac{1}{K\sigma\sqrt{\tau}} \\
\frac{\partial(-d_2)}{\partial K} &= \frac{1}{K\sigma\sqrt{\tau}}
\end{aligned}
$$

The First Partial Derivative

As first step, we calculate the first order partial derivative of P_t w.r.t. K

$$\frac{\partial P_t(K,T)}{\partial K} = \frac{\partial}{\partial K}\left(Ke^{-r\tau}\Phi(-d_2) - S_t e^{-\delta\tau}\Phi(-d_1)\right)$$

$$= e^{-r\tau}\Phi(-d_2) + Ke^{-r\tau}\varphi(-d_2)\frac{1}{K\sigma\sqrt{\tau}} - S_t e^{-\delta\tau}\varphi(-d_1)\frac{1}{K\sigma\sqrt{\tau}}$$

The First Partial Derivative

As first step, we calculate the first order partial derivative of P_t w.r.t. K

$$\frac{\partial P_t(K,T)}{\partial K} = \frac{\partial}{\partial K}\left(Ke^{-r\tau}\Phi(-d_2) - S_t e^{-\delta\tau}\Phi(-d_1)\right)$$

$$= \underbrace{e^{-r\tau}\Phi(-d_2) + Ke^{-r\tau}\underbrace{\varphi(-d_2)\frac{1}{K\sigma\sqrt{\tau}}}_{\textit{chain rule}}}_{\textit{product rule}} - S_t e^{-\delta\tau}\underbrace{\varphi(-d_1)\frac{1}{K\sigma\sqrt{\tau}}}_{\textit{chain rule}}$$

The First Partial Derivative

As first step, we calculate the first order partial derivative of P_t w.r.t. K

$$\frac{\partial P_t(K,T)}{\partial K} = \frac{\partial}{\partial K}\left(Ke^{-r\tau}\Phi(-d_2) - S_t e^{-\delta\tau}\Phi(-d_1)\right)$$
$$= e^{-r\tau}\Phi(-d_2) + \left[Ke^{-r\tau}\varphi(-d_2) - S_t e^{-\delta\tau}\varphi(-d_1)\right]\frac{1}{K\sigma\sqrt{\tau}}$$

The First Partial Derivative

As first step, we calculate the first order partial derivative of P_t w.r.t. K

$$
\begin{aligned}
\frac{\partial P_t(K,T)}{\partial K} &= \frac{\partial}{\partial K}\left(Ke^{-r\tau}\Phi(-d_2) - S_t e^{-\delta\tau}\Phi(-d_1)\right) \\
&= e^{-r\tau}\Phi(-d_2) + \underbrace{\left[Ke^{-r\tau}\varphi(-d_2) - S_t e^{-\delta\tau}\varphi(-d_1)\right]}_{\stackrel{Lemma}{=}0}\frac{1}{K\sigma\sqrt{\tau}} \\
&= e^{-r\tau}\Phi(-d_2)
\end{aligned}
$$

The Second Partial Derivative

Now we calculate the second order partial derivative of P_t w.r.t. K

$$\frac{\partial^2 P_t(K,T)}{\partial K^2} = \frac{\partial}{\partial K}\left(e^{-r\tau}\Phi(-d_2)\right)$$

$$= e^{-r\tau}\varphi(-d_2)\frac{1}{K\sigma\sqrt{\tau}} = \frac{e^{-r\tau}}{K\sigma\sqrt{\tau}}\varphi(-d_2)$$

State Price Density Function

The resulting State Price Density Function is

$$f(K) = e^{r\tau}\frac{\partial^2 P_t(K,T)}{\partial K^2} = e^{r\tau}\frac{e^{-r\tau}}{K\sigma\sqrt{\tau}}\varphi(-d_2) = \frac{1}{K\sigma\sqrt{\tau}}\varphi(-d_2)$$

$$= \frac{1}{K\sigma\sqrt{2\pi\tau}}e^{-\frac{(-d_2)^2}{2}}$$

This is identical to the SPD derived with Black-Scholes Call Price.

Greeks: Delta & Gamma

Greeks are the quantities representing the sensitivity of the price of derivatives to a change in underlying parameters on which the value of an instrument is dependent.

Delta measures the sensitivity of option price to changes in the underlying stock price. Delta hedging is achived by try to make the value of the portfolio for small time intervals as insensitive as possible to small changes in the price of the underlying stock.

Gamma measures the sensitivity of Delta to changes in the underlying stock price. Gamma hedging will make the portfolio value even more insensitive to changes in the stock price.

One can also calculate the Delta and Gamma based on Black-Scholes option pricing model.

We use the previous lemma and the following fact.

$$
\begin{aligned}
d_1 &= \frac{\log(\frac{S_t}{K}) + (r - \delta + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \\
d_2 &= d_1 - \sigma\sqrt{\tau}
\end{aligned}
\qquad \Longrightarrow \quad \frac{\partial d_1}{\partial S_t} = \frac{\partial d_2}{\partial S_t} = \frac{1}{S_t \sigma \sqrt{\tau}}
$$

Delta

$$\begin{aligned}
\Delta_{BS} &= \frac{\partial C_{BS}}{\partial S_t} = \frac{\partial}{\partial S_t}\left(S_t e^{-\delta\tau}\Phi(d_1) - Ke^{-r\tau}\Phi(d_2)\right) \\
&= e^{-\delta\tau}\Phi(d_1) + S_t e^{-\delta\tau}\varphi(d_1)\frac{1}{S_t\sigma\sqrt{\tau}} - Ke^{-r\tau}\varphi(d_2)\frac{1}{S_t\sigma\sqrt{\tau}} \\
&= e^{-\delta\tau}\Phi(d_1) + \underbrace{\left[S_t e^{-\delta\tau}\varphi(d_1) - Ke^{-r\tau}\varphi(d_2)\right]}_{\overset{Lemma}{=}0}\frac{1}{S_t\sigma\sqrt{\tau}} \\
&= e^{-\delta\tau}\Phi(d_1)
\end{aligned}$$

Gamma

$$
\Gamma_{BS} = \frac{\partial^2 C_{BS}}{\partial S_t^2} = \frac{\partial \Delta_{BS}}{\partial S_t} = \frac{\partial}{\partial S_t}\left(e^{-\delta\tau}\Phi(d_1)\right)
$$

$$
= e^{-\delta\tau}\varphi(d_1)\frac{1}{S_t\sigma\sqrt{\tau}} = \frac{e^{-\delta\tau}}{S_t\sigma\sqrt{\tau}}\varphi(d_1)
$$

Application

We have derived the SPD based on Breeden and Litzenberger method as well as Delta and Gamma of options. As an application we estimate the SPD and Greeks of Dax data on 15.01.1997 for $\tau \in \{0.125, 0.25, 0.375\}$, with local polynomial estimation (LPE).

XFGSPDoneday

Application

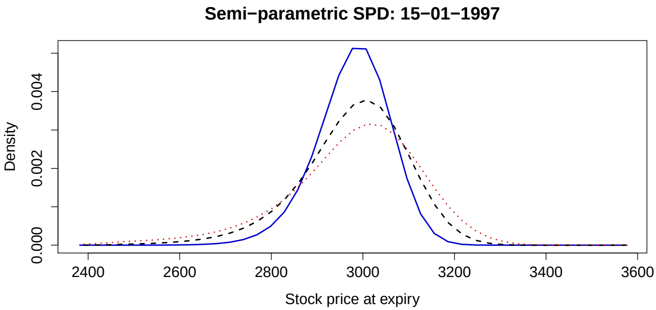
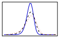


Figure 1: SPD of 15.01.1997 Dax for $\tau \in \{0.125, 0.25, 0.375\}$



Application

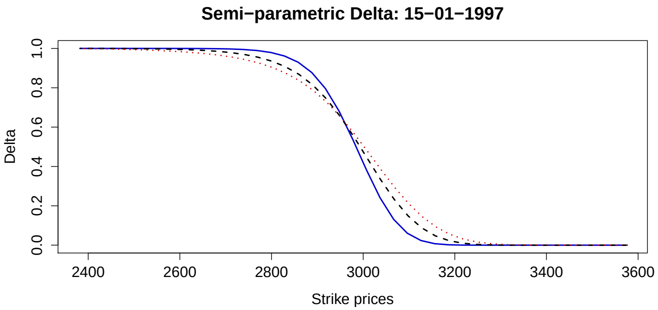


Figure 2: Delta of 15.01.1997 Dax for $\tau \in \{0.125, 0.25, 0.375\}$

Application

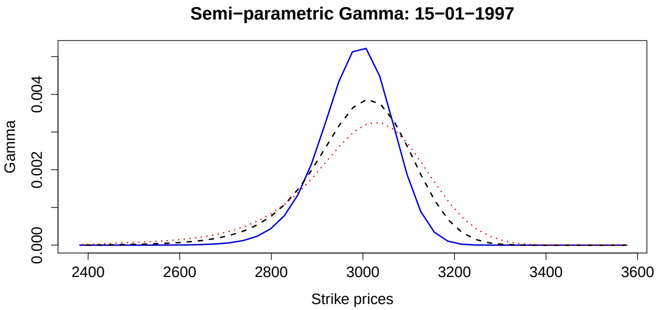


Figure 3: Gamma of 15.01.1997 Dax for $\tau \in \{0.125, 0.25, 0.375\}$

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Butterfly Spread with European Put Option

	$S_T \leq K - \Delta K$	S_T	$S_T \leq K + \Delta K$
A long put at $K - \Delta K$	$(K - \Delta K) - S_T$	0	0
A long put at $K + \Delta K$	$(K + \Delta K) - S_T$	$(K + \Delta K) - K$	0
Two short put at K	$-2(K - S_T)$	0	0
Total	0	ΔK	0

For $\frac{1}{\Delta K}$ shares of this portfolio, the payoff will be 1 at $S_T = K$ and 0 otherwise.