Copulae and Dependence Structure

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Motivation

The importance of risk management is rapidly increasing in financial sector. Since the failure of large institutions can destabilize the whole financial system, the authorities ask for strengthened risk coverage. (e.g. Basel III)

The capital requirement from financial institutions is based on the amount of risk carried in their portfolios and therefore the appropriate measure for the risk is required.

Examples: Variance, Value at Risk, Expected Shortfall

Value at Risk

The VaR of a portfolio at level α is defined as the lower α-quantile of the distriution of the portfolio return.

$$
\begin{aligned}
& P[r \leq VaR(\alpha)] = F_r(Var(\alpha)) = \alpha \\
\Longleftrightarrow\ & VaR(\alpha) = F_r^{-1}(\alpha)
\end{aligned}
$$

Outline

1. Motivation ✓
2. Copulae and Dependence Structure
3. Empirical Study
   - 3.1 Data
   - 3.2 Procedure
   - 3.3 Result
4. Conclusion

The Limitations of Multivariate Normal Distribution

Assumptions:

1. Symmetric distribution of returns
2. The tails of the distribution are not too heavy
3. Linear dependence

These assumptions are very ristrictive and often violated.

Copulae and Dependence Structure

Sklar's Theorem states that any multivariate joint distribution can be written in terms of univariate marginal distribution functions and a copula which describes the dependence structure between variables.

The theory on modeling and estimation of univariate distributions is already well established.

Estimating dependence structure between variables with copulae will lead us to the multivariate joint distribution.

Copulae

A d-dimensional copula is a function $C : [0,1]^d \to [0,1]$ with the following properties:

1. $C(u_1, ..., u_d)$ is increasing in each component $u_i \in [0,1], i = 1, ..., d$.
2. $C(1, ..., 1, u_i, 1, ..., 1) = u_i$ for all $u_i \in [0,1], i = 1, ..., d$.
3. For all $(u_1, ..., u_d), (u_1', ..., u_d') \in [0,1]^d$ with $u_i < u_i'$ we have

$$\sum_{i_1=1}^{2} \cdots \sum_{i_d=1}^{2} (-1)^{i_1+\cdots+i_d} C(v_{j1}, ..., v_{jd}) \geq 0,$$

where $v_{j1} = u_j$ and $v_{j2} = u_j'$, for all $j = 1, ..., d$.

Copulae

For an arbitrary continuous multivariate distribution we can determine its copula from the transformation

$$C(u_1, \ldots, u_d) = F\{F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d)\}, \; u_1, \ldots, u_d \in [0, 1],$$

where F_i^{-1} are inverse marginal distribution functions.

Copulae Families

1. Simplest Copulae
   1.1 Product (independence) copula
   1.2 Frechet-Hoeffding bounds
2. Elliptical Family
   2.1 Gaussian copula
   2.2 Student's t-copula
3. Archimedean Family
   3.1 Gumbel copula
   3.2 Clayton copula
   3.3 Frank copula (elliptical Archimedian copula)

Dependence measure

1. Perfect Dependence
2. Concordance
3. Linear Correlation(Pearson's correlation)
4. Kendall's τ and Spearman's ρ
5. Tail Dependence

Kendall's τ and Spearman's ρ

Kendall's τ and Spearman's ρ provides the best alternatives to the linear correlation coefficient as a measure of dependence for nonelliptical distributions, for which the linear correlation coefficient is inappropriate and often misleading.

Since Kendall's τ is defined as a double integral of copula function, in many cases it is not straightforward to evaluate. But for an Archimedean copula Kendall's τ can be expressed as an one-dimensional integral of the generator and its derivative.

	Gumble copula	Clayton copula	Frank copula
Kendall's τ	$1 - \frac{1}{\theta}$	$\frac{\theta}{\theta+2}$	$1 - \frac{4(1-D_1(\theta))}{\theta}$

Tail dependence

The concept of tail dependence relates to the amount of dependence in the upper-right-quadrant tail or lower-left-quadrant tail of a bivariate distribution. It is a concept that is relevant for the study of dependence between extreme values.

	Gumble copula	Clayton copula	Frank copula
λ_U	$2-2^{\frac{1}{\theta}}$	-	-
λ_L	-	$2^{-\frac{1}{\theta}}$	-

Data

The dataset consists of log returns of 10 stocks, 5 of which are selected from the energy sector, while the remaining 5 are selected from the financial sector. The time span of the dataset is from 2008 to 2009 and portfolio weights are calculated based on the size of each company.

Data

Code	Energy Sector	Code	Financial Sector
CHK	Chesapeake Energy	SPGI	S&P Global, Inc.
CVX	Chevron Corp.	ICE	Intercontinental Exchange
XOM	Exxon Mobil Corp.	NDAQ	NASDAQ OMX Group
HES	Hess Corporation	PFG	Principal Financial Group
MUR	Murphy Oil	PRU	Prudential Financial

Table 1: The list of companies included in the dataset

Procedure

1. Estimate the conditional variance with a GARCH model
2. Calculate the standardized residual of the GARCH model
3. Estimate the parameters of copula with standardized residual
4. Calculate the dependency measures and VaR

Result

Gaussian Copula (Correlation Coefficient)

	CHK	CVX	XOM	HES	MUR	SPGI	ICE	NDAQ	PFG	PRU
CHK	1,00									
CVX	0,66	1,00								
XOM	0,59	0,86	1,00							
HES	0,72	0,73	0,68	1,00						
MUR	0,74	0,76	0,70	0,80	1,00					
SPGI	0,35	0,47	0,47	0,33	0,38	1,00				
ICE	0,37	0,47	0,49	0,40	0,40	0,45	1,00			
NDAQ	0,37	0,46	0,42	0,39	0,39	0,49	0,60	1,00		
PFG	0,36	0,51	0,48	0,37	0,41	0,57	0,55	0,63	1,00	
PRU	0,32	0,48	0,46	0,34	0,39	0,53	0,56	0,65	0,82	1,00

Figure 1: Pairwise Dependence structure of Gaussian copula

Result

Student's t Copula (Correlation Coefficient)

	CHK	CVX	XOM	HES	MUR	SPGI	ICE	NDAQ	PFG	PRU
CHK	1,00									
CVX	0,67	1,00								
XOM	0,61	0,87	1,00							
HES	0,73	0,74	0,69	1,00						
MUR	0,75	0,77	0,73	0,80	1,00					
SPGI	0,36	0,49	0,49	0,35	0,42	1,00				
ICE	0,38	0,48	0,51	0,42	0,42	0,46	1,00			
NDAQ	0,37	0,47	0,44	0,41	0,40	0,53	0,63	1,00		
PFG	0,37	0,51	0,50	0,39	0,43	0,61	0,57	0,65	1,00	
PRU	0,32	0,49	0,48	0,36	0,41	0,55	0,60	0,66	0,85	1,00

Figure 2: Pairwise Dependence structure of Student's t copula

Result

Frank Copula (Kendall's Tau)

	CHK	CVX	XOM	HES	MUR	SPGI	ICE	NDAQ	PFG	PRU
CHK	1,00									
CVX	0,47	1,00								
XOM	0,43	0,69	1,00							
HES	0,55	0,56	0,52	1,00						
MUR	0,56	0,58	0,54	0,61	1,00					
SPGI	0,25	0,35	0,36	0,25	0,29	1,00				
ICE	0,27	0,35	0,37	0,30	0,29	0,35	1,00			
NDAQ	0,25	0,32	0,31	0,28	0,26	0,39	0,47	1,00		
PFG	0,27	0,38	0,38	0,30	0,32	0,48	0,44	0,48	1,00	
PRU	0,22	0,34	0,34	0,26	0,29	0,42	0,45	0,48	0,68	1,00

Figure 3: Pairwise Dependence structure of Frank copula

Result

Clayton Copula (Kendall's Tau)

	CHK	CVX	XOM	HES	MUR	SPGI	ICE	NDAQ	PFG	PRU
CHK	1,00									
CVX	0,34	1,00								
XOM	0,30	0,52	1,00							
HES	0,41	0,37	0,34	1,00						
MUR	0,41	0,41	0,37	0,45	1,00					
SPGI	0,19	0,24	0,21	0,17	0,20	1,00				
ICE	0,17	0,22	0,22	0,19	0,20	0,22	1,00			
NDAQ	0,18	0,23	0,21	0,19	0,19	0,29	0,33	1,00		
PFG	0,15	0,23	0,20	0,15	0,16	0,27	0,22	0,31	1,00	
PRU	0,17	0,27	0,24	0,18	0,20	0,26	0,29	0,35	0,48	1,00

Figure 4: Pairwise Dependence structure of Clayton copula

Result

Gumbel Copula (Kendall's Tau)

	CHK	CVX	XOM	HES	MUR	SPGI	ICE	NDAQ	PFG	PRU
CHK	1,00									
CVX	0,44	1,00								
XOM	0,40	0,67	1,00							
HES	0,49	0,51	0,47	1,00						
MUR	0,52	0,55	0,50	0,56	1,00					
SPGI	0,20	0,29	0,31	0,20	0,24	1,00				
ICE	0,24	0,30	0,32	0,25	0,26	0,26	1,00			
NDAQ	0,22	0,29	0,27	0,24	0,24	0,30	0,39	1,00		
PFG	0,23	0,32	0,30	0,24	0,27	0,38	0,36	0,42	1,00	
PRU	0,19	0,30	0,29	0,20	0,25	0,33	0,38	0,43	0,62	1,00

Figure 5: Pairwise Dependence structure of Gumbel copula

Result

Lower Tail Dependence

	CHK	CVX	XOM	HES	MUR	SPGI	ICE	NDAQ	PFG	PRU
CHK	1,00									
CVX	0,51	1,00								
XOM	0,44	0,72	1,00							
HES	0,60	0,56	0,51	1,00						
MUR	0,60	0,60	0,55	0,65	1,00					
SPGI	0,24	0,33	0,27	0,18	0,25	1,00				
ICE	0,19	0,29	0,30	0,23	0,24	0,30	1,00			
NDAQ	0,21	0,32	0,26	0,22	0,22	0,42	0,49	1,00		
PFG	0,13	0,31	0,26	0,13	0,15	0,40	0,30	0,46	1,00	
PRU	0,19	0,39	0,33	0,21	0,25	0,38	0,43	0,52	0,69	1,00

Figure 6: Lower Tail Dependence from Clayton copula

Result

Upper Tail Dependence

	CHK	CVX	XOM	HES	MUR	SPGI	ICE	NDAQ	PFG	PRU
CHK	1,00									
CVX	0,53	1,00								
XOM	0,48	0,74	1,00							
HES	0,58	0,60	0,55	1,00						
MUR	0,61	0,63	0,58	0,65	1,00					
SPGI	0,25	0,37	0,38	0,26	0,31	1,00				
ICE	0,30	0,38	0,39	0,32	0,33	0,33	1,00			
NDAQ	0,28	0,37	0,34	0,31	0,31	0,37	0,48	1,00		
PFG	0,29	0,40	0,38	0,31	0,34	0,46	0,44	0,51	1,00	
PRU	0,25	0,37	0,37	0,26	0,32	0,41	0,46	0,51	0,70	1,00

Figure 7: Upper Tail Dependence from Gumbel copula

Result

	1% VaR	5% VaR
Gaussian copula	-1,760	-1,302
Student's t copula	-2,558	-1,548

Table 2: Estimated VaR based on copulae

Remarks

1. Copula can be a useful tool for detecting dependence structures, which is of important in finance, especially in risk management.
2. Dependence measures based on copula allow one to measure non linear dependence, which is not possible with usual correlaion.
3. For elliptical family or bivariate case estimating dependence structure is easy, while in multivariate Archimedean case measuring dependence is not straightforward.

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Sklar Theorem

Let F be a multivariate distribution function with margins $F_1, ..., F_d$, then there exists the copula C such that
$F(x_1, ..., x_d) = C\{F_1(x_1), ..., F_d(x_d)\}, x_1, ..., x_d \in \bar{\mathbb{R}}$.
If F_i are continuous for $i = 1, ..., d$ then C is unique. Otherwise C is uniquely determined on $F_1(\bar{\mathbb{R}}) \times \cdots \times F_d(\bar{\mathbb{R}})$.

Conversely, if C is a copula and $F_1, ..., F_d$ are univariate distribution functions, then function F defined above is a multivariate distribution function with margins $F_1, ..., F_d$.

Copulae

Density of Gaussian copula, for all $u_1, ..., ud \in [0,1]$

$$|\Sigma|^{-\frac{1}{2}} \times exp\left\{-\frac{[\Phi^{-1}(u_1), ..., \Phi^{-1}(u_d)](\Sigma^{-1} - I)[\Phi^{-1}(u_1), ..., \Phi^{-1}(u_k)]}{2}\right\}^T$$

Frank copula, $0 < \theta \leq \infty$

$$C_\theta(u_1, ..., u_d) = -\frac{1}{\theta} log\left\{1 + \frac{\prod_{i=1}^{d}(e^{-\theta u_i} - 1)}{e^{-\theta} - 1}\right\}$$

Copulae

Gumbel copula, $1 \leq \theta \leq \infty$

$$C_\theta(u_1, ..., u_d) = exp\left[-\left\{\sum_{i=1}^{d}(log\ u_i)^\theta\right\}^{\theta^{-1}}\right]$$

Clayton copula, $0 < \theta$

$$C_\theta(u_1, ..., u_d) = \left\{\left(\sum_{i=1}^{d} u_i^{-\theta}\right) - d + 1\right\}^{-\frac{1}{\theta}}$$

Kendall's τ

Let F be a continuous bivariate cumulative distribution function with the copula C. Moreover, let $(X_1, X_2) \sim F$ be independent random pairs. Then Kendall's τ_2 is given by

$$\tau_2 = 4 \iint_{[0,1]^2} C(u_1, u_2) dC(u_1, u_2) - 1$$

Spearman's ρ

Let F be a continuous bivariate distribution function with the copula C and the univariate margins F_1 and F_2 respectively. Assume that $(X_1, X_2) \sim F$. Then the Spearman's ρ is given by

$$\rho_2 = 12 \iint_{[0,1]^2} u_1 u_2 dC(u_1, u_2) - 3$$

Debye function

Debye function is defined by

$$D_k(x) = \frac{k}{x^k} \int_0^x \frac{t^k}{e^t - 1} dt,$$

for any positive integer k.