UE22MA251 LINEAR ALGEBRA AND ITS APPLICATIONS

<u>Unit-2-</u> **Four Fundamental Subspaces & Linear Transformations**:

Linear Independence, Basis and Dimensions, Row reduced Echelon form, Sum of subspaces, Direct Sums, The Four Fundamental Subspaces, Rank-Nullity theorem. Linear Transformations, Algebra of Linear transformations, Invertible maps, Isomorphisms. Applications.

Self-Learning Component: Examples of Vector Spaces and Subspaces.

Class No.	Portions to be covered
18-20	Linear Dependence, Independence, Span, Basis and Dimension
21-22	Echelon Form, Row Reduced Echelon Form, Pivot Variables , Free variables
23	Sum of Subspaces, Direct Sums
24	**Matlab Class Number 3 – LU Decomposition**
25-27	The Four Fundamental Subspaces-Column Space and Null Space
28-29	Row Space, Left Null Space
30	Uniqueness & Existence of Inverses, Rank-Nullity theorem
31-32	Linear Transformations, Transformations represented by Matrices
33-34	Algebra of Linear transformations, Invertible maps, Isomorphisms
35	**Matlab Class Number 4 -Inverse of a Matrix by Gauss Jordan Method**
36	**Applications**

Classwork problems:

1. Examine if the following sets of vectors are linearly independent. If not, find a relation between the vectors:

(a) $\{(4,2,-1,3),(6,5,-5,1),(2,-1,3,5)\}$

(b) $\{sint, e^t, t^2\}$

(c) $\{t^2+t+2, 2t^2, 3t^2+2t+2\}$

(d) $\left\{\begin{pmatrix}1 & 2\\ 1 & 0\end{pmatrix}, \begin{pmatrix}2 & -1\\ 1 & 2\end{pmatrix}, \begin{pmatrix}3 & 1\\ 2 & 2\end{pmatrix}\right\}$ in $M_{2x2}(R)$

Answer: (a)dependent $2v_1=v_2+v_3$. (b) independent (c)independent

(d) dependent, $M_1+M_2=M_3$.

2	(a)Determine whether these vectors $\{(1,2,-2,1),(-3,0,-4,3),(2,1,1,-1),(-3,3,-9,6)\}$ form a basis of $\mathbb{R}^4$. If not, find the dimension of the subspace S they span. If S is a subspace of $\mathbb{R}^4$, extend the basis of S to a basis of $\mathbb{R}^4$. (b) Do these vectors $\{(2,2,3),(-1,-2,1),(0,1,0)\}$ span $\mathbb{R}^3$. **Answer:** (a)They do not form a basis of R^4. They span a subspace $S=\{(1,2,-2,1),(-3,0,-4,3)\}$ of dimension 2. (b)Yes. They span $\mathbb{R}^3$.
3	Reduce the following matrices to Row Reduced Echelon form and determine their ranks $\begin{pmatrix} 1 & 3 & -2 & 5 \\ 2 & 1 & 3 & 2 \\ 4 & 7 & -1 & 12 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & -2 & -1 \\ 1 & 2 & -1 & 3 \\ 3 & 7 & -8 & 3 \\ 4 & 5 & -7 & 0 \end{pmatrix}$. Identify the pivot variables and free variables. Find the special solutions to Ax=0. **Answer:** (i) (-11/5,7/5,1,0), (-1/5,-8/5,0,1); (ii) (0,-7/3,-5/3,1)
4	For the following vectors (i) $S=\{(0,0,1),(1,0,1),(0,1,1)\}$. Is u=(1, 1,1) in span of S. (ii) $S=\{v_1=2t^2+t+2, v_2=t^2-2t, v_3=5t^2-5t+2,\ v_4=-t^2-3t-2\}$. Does $u=t^2+t+2 \in \text{span}\{v_1,v_2,v_3,v_4\}$. Find a basis for the subspace W which spans S and what is its dimension. (iii) $\{(1,2,-1),(2,3,4),(0,0,1)\}$. Do these vectors span $\mathbb{R}^3$.(Home-work) **Answer:** (i) Yes (ii) No, $W=\{v_1,v_2\}$; $\dim W=2$ (iii)Yes, they span $\mathbb{R}^3$.
5	Let $U=\{(a,b,c)/a+b+c=0\}, V=\{(0,0,c)\}$. Show that $\mathbb{R}^3=U+V$. Is U+V a direct sum. **Answer:** Yes
6	Find a basis and dimension of U+V given; (i) $U=span\{(1,-1,-1.-2),(1,-2,-2,0),(1,-1,-2,-2)\}$. (ii) $V=span\{(1,-2,-3.0),(1,-1,-3,2),(1,-1,-2,2)\}$. What is the dimension of $U\cap V$.
7	Find the Column space and Null space for the following matrices: $\begin{pmatrix} 1 & 1 & -3 & -2 \\ 2 & 9 & 8 & 3 \\ 1 & -1 & 3 & 2 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 & -1 & 3 \\ 3 & 5 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ -1 & 0 & -2 & 7 \end{pmatrix}$ **Answer:** C(A) is a 3-d plane in $\mathbb{R}^3$and N(A) is a line in $\mathbb{R}^4$. C(A) is $\mathbb{R}^4$ and N(A) is origin in $\mathbb{R}^4$.

8	For which vector (a, b, c) does the following system Ax=b have a solution? x + 2y - 3z = a; 2x + 3y + 3z = b; 5x + 9y - 6z = c **Answer:** c-b-3a=0
9	Find a basis for the set of vectors in $\mathbb{R}^3$ in the plane x + y - 3z = 0. **Answer:** {(-1,1,0), (3,0,1)}
1	For which vector (b_1,b_2,b_3,b_4) is this system solvable? $\begin{pmatrix} 1 & 2 & -1 \\ 1 & 9 & -1 \\ -3 & 8 & 3 \\ -2 & 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$.(Home-work) **Answer:**. $b_3 - 2b_2 + 5b_1 = 0 \; and \; b_4 - b_2 + 3b_1 = 0$.
1	If the set of vectors {u,v,w} are linearly independent vectors, then show that the set $\{u + v - 2w,\ u - v - w,\ u + w\}$ is linearly independent.
1	Find a basis and dimension of the subspace W of V=M_{2x2} spanned by $\left\{\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}$. (Home-work) **Answer:** Basis of W =$\{M_1, M_2, M_5\}$ Dim=3
1	Find a basis and the dimension of the subspaces of $U = \{(a, b, c, d)/b - 2c + d = 0\}$ and $V = \{(a, b, c, d)/a = d, b = 2c\}$ in $\mathbb{R}^4$. Find basis and dim of $U \cap V$. **Answer:** Dim of U=3;Dim of V=2;Basis of $U \cap V = \{(0, 2, 1, 0)\}$ and Dim =1.
1	If the column space of A is spanned by the vectors (1,2,7,5), (-2,-1,-8,-7), (-1,3,3,0) find all those vectors that span the null space of A. What are the bases and dimensions of C(A^T) and N(A^T). **Answer:** $Basis\ of\ N(A) = (-\frac{7}{3}, -\frac{5}{3}, 1)$,Dim of C($A^T$)=2, Dim of N($A^T$)=1.
1	Obtain the four fundamental subspaces , their basis and dimension given $\begin{pmatrix} 1 & 3 & 1 & 2 & 1 \\ 1 & 9 & 4 & 5 & 2 \\ 4 & 12 & 8 & 8 & 7 \\ 0 & 0 & 3 & 1 & 4 \end{pmatrix}$. Also describe the four fundamental subspaces.
1	Find left / right inverse (whichever possible) for the following matrices: (i) $\begin{pmatrix} 3 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$ (ii) $\begin{pmatrix} 3 & 1 \\ 0 & 0 \\ 1 & -2 \end{pmatrix}$(Home-work).

1	Which of these transformations are not linear? Give reasons. $(i)T(x,y,z) = (x + y + z, 2x - 3y + 4z)$ $(ii)T(x,y) = (x + 3, 2y, x + y)$ (iii) $(iii)T(x,y) = (xy, x)$ Answer: (ii)T(0) is not equal to 0 (iii)T(kv) $\neq$ KT(v)
1	Find the image of these points after applying the transformation given: (i)Rotate (-1,3) counter clockwise through 30° line. (ii)Reflect (2,3) across 90° line and then project on x-axis. Answer: (i) $\left(\frac{-3-\sqrt{3}}{2}, \frac{-1+3\sqrt{3}}{2}\right)$ (ii) (-2,0)
1	For each of the following linear transformations T, find a basis and the dimension of the range and kernel of T: (i) $T: \mathbb{R}^3 \to \mathbb{R}^2$defined by T(x, y, z)=(2x+z, x+y) (ii) $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by T(x, y,z)=(x+2y+3z, -3x-2y-z, -2x+2z) Answer: (i) {(2,1),(0,1)} 2; {(-1/2,1/2,1)} 1 (ii) {(1,-3,-2),(2,-2,0)} 2; {(1,-2,1)} 1.
2	Find the matrix of the linear transformation T on $\mathbb{R}^3$ defined by T(x,y,z)=(2y+z, x-4y, 3x) with respect to (i)the standard basis (1,0,0),(0,1,0), ((0,0,1) (ii)the basis (1,1,1),(1,1,0), ((1,0,0) Answer: $(i)\begin{pmatrix} 0 & 2 & 1 \\ 1 & -4 & 0 \\ 3 & 0 & 0 \end{pmatrix}$ $(ii)\begin{pmatrix} 3 & 5 & -1 \\ -6 & -6 & -2 \\ 6 & 3 & 3 \end{pmatrix}$
2	Find a linear mapping $T: \mathbb{R}^4 \to \mathbb{R}^3$ whose Kernel is spanned by (-2,1,0,0) and (1,0,-1,1). Answer: T(x,y,z,t)=(x+2y+z, z+t, 0)
2	If $F: \mathbb{R}^3 \to \mathbb{R}^2$, $G: \mathbb{R}^3 \to \mathbb{R}^2$ and $H: \mathbb{R}^2 \to \mathbb{R}^2$ defined by F(x,y,z)=(y, x+z), G(x,y,z)=(2z, x+z-y), H(x,y)=(y, 2x) find F+G, 3F-2G, H∘F, F∘H, H∘ (F+G) and (H∘F+ H∘G) Answer: (F+G) (x,y,z)=(y+2z, 2x-y+z), (3F-2G) (x,y,z)=(3y+4z, x+2y+3z), (H∘F) (x,y,z)=(x+z, 2y), (H∘ (F+G)) (x,y,z)= (H∘F+ H∘G) (x,y,z)= (2x-y+z, 2y+4z), F∘H is not defined.
2	(i) If $T: \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation defined by T(x,y)=(3x+y,5x+7y, x+3y). Show that T is one-to-one. Is T onto? (ii) Is T invertible? Find a rule for T^{-1} like the one which defines T where T(x,y,z)=(3x,x-y, 2x+y+z) is a transformation in $\mathbb{R}^3$. Answer: (i)No (ii) Yes; T-1(x,y,z)=(x/3, x/3-y, -x+y+z)